MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 14 / 1999

Arbeitsgemeinschaft "Hyperbolic groups."

04.04-10.04.1999

Introduction.

Hyperbolic groups were introduced in 1987 by Gromov. In itself, this subject constitutes a significant proportion of the geometric theory of groups. Geometric group theory has grown into a major field over the last two decades or so, though its roots can be traced back to the early years of this century. For example, Dehn made use of hyperbolic geometry to solve the word problem for surface groups. Out of this developed small cancellation theory, which is still important today, and which might be viewed as a kind of precursor to the theory of hyperbolic groups. Since then, hyperbolic geometry has been one of the principal sources of ideas in combinatorial group theory. One should also mention the work of Aleksandrov, Busemann and Toponogov starting in the late 40s, who developed synthetic methods in Riemannian geometry. These techniques have exerted a strong influence on the evolution of the subject.

Another important source of ideas has been low-dimensional topology, in particular 3-manifold theory. These were translated into group theory through the work of Stalling, Dunwoody etc. Since the work of Thurston towards the end of the 70s, 3-dimensional topology and hyperbolic geometry have become intimately linked. This convergence of ideas produced a flurry of activity which has never lost pace. Among the more recent developments to spring from this source has been the theory of group actions on $\mathbf{R}$-trees, introduced by Morgan and Shalen, and developed by Rips, Bestvina, Feighn, Levitt, Paulin and others. It is now one of the most powerful tools in the subject.

It was against this background that Gromov published his article on hyperbolic groups. As a typical example of such a group, one might consider the fundamental group of a compact negatively curved manifold. The entire subject is founded on an axiom, commonly termed the "thin triangles" axiom, which is a coarse version of a comparison axiom introduced by Aleksandrov. It is remarkable how such a simple axiom can capture so much of the large scale geometry of negatively curved spaces. The large number of equivalent natural formulations of this notion bear witness to its central importance. Today there is an extensive literature on the subject, both elaborating on ideas already set out by Gromov, and exploring many new directions.

We have had to be very selective in the choice of topics : clearly, many important results where not even mentionned.

B. Bowditch T. Delzant

Lecture 1. B.Bowditch.
Introduction.

A survey lecture: what will and what will not be done in the programme:
-Small cancellation theory and hyperbolic geometryg Olshanskii's work on the Burnside problem, generic groups.
-The geodesic flow and its (potential) applications (e.g. bounded cohomology).
-The Novikov conjecture and related problems.
-Symbolic dynamics of the geodesic flow and zeta function for hyperbolic groups.
-Decision problems in group theory (word problem, conjugacy problem and isomorphism problem).

Lecture 2. A. von Heydebreck, Franz Degenhardt.
Definitions, first examples.

(a) Equivalent definitions of hyperbolicity. (b) Some examples, finite trees, hyperbolic n-space $\mathbf{H}^n$. (c) The lemma of approximation by finite trees. (d) As a corollary, if Γ acts properly discontinuously on a (e) Definition of a hyperbolic group as one which acts properly discontinuously Corollary of (d): A hyperbolic groups has finitely many conjugacy classes of finite subgroups.

Lecture 3. E. Lebeau.
CAT(χ) spaces.

(a) Definition of the CAT(χ) property. (b) The Cartan-Hadamard Theorem: (c) Theorem :

Let (X, ρ) and (Y, d) be two metric spaces, and suppose $\lambda \geq 1$, $K \geq 0$. A "(λ, K)-quasi-isometry" between X and Y is a map $f : X \longrightarrow Y$ such that for all $x, x' \in X$

$$\lambda^{-1}\rho(x, x') - K \leq d(f(x), f(x')) \leq \lambda\rho(x, x') + K$$

and

$$(\forall y \in Y)(\exists x \in X)(d(f(x), y) \leq K).$$

The notion of quasiisometry thus gives an equivalence relation between metric spaces.
A fundamental property of hyperbolicity is its quasi-isometry invariance (for geodesic spaces). The most direct way to prove this is to use quasigeodesics. In particular, hyperbolicity of a finitely generated group is independent of the generating set.

Lecture 5. B. Priwitzer.
Isoperimetric inequalities and hyperbolicity.

One can characterise hyperbolicity in terms of isoperimetric inequalities. Let M be a compact Riemannian manifold. i) The universal cover of M (or equivalently $\pi_1(M)$) is hyperbolic. ii) There is some $K > 0$ such that $A(L) \leq KL$ for all L.

Lecture 6. J. Winckelman.
The Rips complex.

To any finitely generated group, Γ, together with a finite generating set, and any natural number, d, is associated a (finite-dimensional) simplicial complex, $P_d(\Gamma)$, which admits a discrete cocompact action of Γ. If Γ is hyperbolic, and d is large enough (in relation to the hyperbolicity constant), then $P_d(\Gamma)$ is contractable. Moreover, it satisfies a inear isometric inequality in each dimension. This construction has many corollaries, for example: Corollary:

If Γ is a torsion-free hyperbolic group, then Γ has a finite $K(\Gamma, 1)$. In particular, Γ has finite cohomological dimension.

Lecture 7. A. Zuk.
Boundaries.

(a) Definition of the boundary, ∂X, of a proper hyperbolic space, X (for example, as parallel classes of (quasi)geodesic rays). The topology of ∂X ; ∂X is compact. The connected components of ∂X are the topological ends of X.
(b) Invariance of ∂X by quasiisometry. Definition of the boundary, $\partial\Gamma$, of a hyperbolic group, Γ. If Γ is not virtually cyclic, then $\partial\Gamma$ is perfect (no isolated points).
(c) By Stallings's theorem, $\partial\Gamma$ is connected if and only if Γ does not split over any finite subgroup.
(d) Kaimanovich has shown that $\partial\Gamma$ is the same as the Poisson boundary.

Lecture 8. Udo Baumgartner.
Convergence groups.

(a) Let M be a perfect compact metrisable space, and suppose that Γ acts by homeomorphism on M. Definition: Γ is a (uniform) convergence group if the induced action on the space of distinct triples of M is properly discontinuous (and cocompact). (b) The definition of convergence group is equivalent to the original definition of Gehring and Martin in terms of convergent subsequences. (c) Classification of elements of a convergence group (elliptic, parabolic, loxodromic). (d) If Γ acts properly discontinuously (cocompactly) on a proper hyperbolic space, X, then it acts as a (uniform) convergence group on ∂X. (e) Thus, a hyperbolic group Γ acts as a uniform convergence group on $\partial\Gamma$. Remark: There is a converse due to Bowditch.

Lecture 9. S. Rogmann.

Classification of isometries and elementary subgroups.

(a) Let X be any hyperbolic geodesic space. Any non-identity isometry of X is one of three types, elliptic, parabolic or loxodromic (or hyperbolic). If X is proper, and Γ acts properly discontinuously on X, then this agrees with the classifaction of elements of Γ as a convergence group acting on $\partial\Gamma$. (b) If Γ acts cocompactly on X, then Γ cannot contain any parabolic element. Put another way, every infinite order element of a hyperbolic group is loxodromic.

(c) Definition of the translation length of an element of a hyperbolic group, Γ. Given Γ, there is some natural number, L, such that the translation length of each element of Γ lies in $\frac{1}{L}\mathbf{N}$.

Lecture 10. G. Arjantseva.

Free and quasiconvex subgroups of hyperbolic groups.

(a) The Tits alternative for a hyperbolic space: Suppose that Γ is neither finite nor two-ended, then it contains a nonabelian free subgroup. (b) If g is a loxodromic isometry of X, and h doesn't fix either of the fixed points of g, then there is some natural number, n, such that g^n and hg^nh^{-1} generate a free subgroup. If X happens to be Cayley graph of a hyperbolic group, Γ, then n can be chosen to be independent of g. (c) Definition of a quasiconvex subgroup. Any quasiconvex subgroup of a hyperbolic group is hyperbolic. (d) Generics subgroups of generic groups are free quasi-convex.

Lecture 11. M. Puschnigg.

The theorem of Bestvina and Mess I.

Let Γ be a hyperbolic group, and let P be a (finite-dimensional) contractable simplicial complex on which Γ acts

properly discontinuously and cocompactly (for example, the Rips complex, $P_d(\Gamma)$ for sufficiently large d). Thus, $P \cup \partial\Gamma$ has a natural topology as a compact metrisable space.

The main result of Bestvina and Mess states that this is, in fact, a Z-set compactification (in the sense of shape theory). In particular, it follows that $P \cup \partial\Gamma$ is contractible. It has a number of important corollaries. For example, for all $n \geq 1$, we have $H^n(\Gamma; \mathbf{Z}\Gamma) \cong \check{H}^{n-1}(\partial\Gamma)$, where $\check{H}$ denotes the Čech cohomology.

Lecture 12. T. Kuessner.

The theorem of Bestvina and Mess II.

(a) Let M be a closed irreducible 3-manifold such that $\Gamma = \pi_1(M)$ is hyperbolic. Then the universal cover $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$, and $\tilde{M} \cup \partial\Gamma$ is a closed 3-ball.

Remark: This potentially represents a major step towards Thurston's Hyperbolisation conjecture. (b) If Γ is a hyperbolic group such that $\partial\Gamma$ is connected and has no (global) cut point, then $\partial\Gamma$ is locally connected.

Lecture 13. G. Kleineidam.

The Besvina-Paulin theorem; first applications.

(a) Equivariant Gromov-Hausdorff convergence for metric spaces. (b) Theorem: Let Γ be a given finitely generated group with a preferred system of generators $\{g_i \mid 1 \leq i \leq n\}$, and let (X_n, ρ_n) be a sequence of $\delta-$hyperbolic metric spaces each with an action of a Γ. Suppose that $\lambda_n = \min_{x \in X_n} \max_{1 \leq i \leq n} \rho_i(g_i x, x)$ is a not bounded. Then the family of spaces $\frac{1}{\lambda_n} X_n$ converges in the Gromov-Haudorff equivariant topology to an $\mathbf{R}$-tree without any global fixed point. This fact has many corollaries. For example: (b) We say that a group is "rigid" if it does not admit a non-trivial action on an $\mathbf{R}$-tree. (An action of an $\mathbf{R}$-tree is, rather confusingly, termed "trivial" if it has a global fixed point.)

Lecture 14. Volker Braungardt.

Bass-Serre theory, and the Rips machinery I.

(a) Serre's theorem: The following are equivalent for a group, Γ:

(1) Γ is isomorphic to an amalgam $\Gamma \cong A *_C B$ (respectively an HNN extension $\Gamma \cong A *_C$), and (2) Γ acts on a simplicial tree, T, with just one edge orbit, and no edge inversions, such that for some edge $e = [a, b]$ we have $\Gamma_a = A$, $\Gamma_b = B$ and $\Gamma_e = C$, and with a and b in different (respectively the same) Γ-orbit. (Here Γ_x denotes the stabiliser of x.) (b) Bass-Serre theory is a generalisation of Serre's theorem to the case where T/Γ might be any connected graph.

(c) The "Rips machinery" can be viewed as a generalisation of Bass-Serre theory to group actions on $\mathbf{R}$-trees. This is indeed more general. In particular, surface groups can act freely on **R**-trees. Some discussion of actions of surface groups, for example, via singular measured foliations on surfaces.

Lecture 15. Stefan Kuehnlein.
Bass-Serre theory and the Rips machinery II.

(a) Theorem (Rips). Any finitely generated group acing freely isometrically on an **R**-tree can be expressed as a free product of surface groups and free abelian groups. (Note that any finitely generated free abelian group acts freely on the real line.) (b) Definition of a stable action of a group on an **R**-tree. Statement of the theorem of Bestvina and Feighn concerning splittings of finitely presented groups acting stably on **R**-trees. (c) Foliated 2-complexes. The resolution of a finitely presented group acting on an **R**-tree. A sketch of the some of the ideas involved in the proof of (a) in the case of finitely presented groups.

Lecture 16. U. Hamenstaedt.
The topological characterisation of fuchsian groups.

The celebrated theorem of Tukia, Gabai and Casson and Jungreis states that if a group, Γ, acts as a convergence group on the circle, then it also admits a properly discontinuous action on the hyperbolic plane (such that the induced action on the circle at infinity is topologically conjugate to the original). The converse is an immediate consequence of Lecture 7.

Remark: The detailed analysis of Tukia left open the case of (putative)

virtual semitriangle groups, and this case was resolved independently by Gabai and by Casson and Jungreis.

Lecture 17. G. Levitt.
The JSJ splitting.

The JSJ splitting of a one-ended hyperbolic group is a canonical representation of Γ as (the fundamental group of) a finite graph of groups with two-ended edge groups, such that every splitting of Γ over a two-ended subgroup can be read off from this picture. It was first described by Sela (for splitting over cyclic groups) who made use of the Rips machinery. It was inspired by the analogous characteristic submanifold construction for irreducible 3-manifolds due to Waldhausen, Johannson, Jaco and Shalen. A topological construction of the JSJ splitting of an hyperbolic group has been given by Bowditch, using the local topological properties of the boundary of the group.

Lecture 18. D. Gaboriau.

Cut points an local connectivity.

The cut point theorem (Bowditch) states that if Γ is a one-ended hyperbolic group, then $\partial\Gamma$ has no global cut point. It follows from Bestvina-Mess (Lecture 12) that $\partial\Gamma$ is locally connected. The proof is by contradiction. Suppose that $\partial\Gamma$ has a global cut point. The steps are as follows:

(a) Use the separation properties of the set of cut points to construct n $\mathbf{R}$-tree on which Γ acts by homeomorphism. (b) Use this to construct another $\mathbf{R}$-tree on which Γ acts by isometry. (c) Use the Rips machinery to derive a contradiction.

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