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Geometric Stochastic Analysis

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The goal of this conference was to stimulate interaction between differential geometry, partial differential equations and stochastics. In order to explore this interplay, we brought together experts from various fields that are open for these connections, hoping to encourage further active research and new discoveries in this direction. The conference was attended by 43 participants. It was organized by K. David Elworthy (Warwick Univ.), Jürgen Jost (MPI Leipzig) and Karl-Theodor Sturm (Bonn Univ.). Unfortunately, J. Jost could not attend.

There had been 19 lectures and two introductory series of lectures. In these lectures, the speakers presented progress and problems in geometric analysis and/or recent results in stochastic analysis. In several talks, probabilistic approaches to problems in geometric analysis were described. Other speakers illustrated geometric aspects and methods in stochastic analysis.

Analysis on metric spaces (including geometric and stochastic aspects) and convergence of Riemannian manifolds (with emphasis on the question of convergence of spectra, heat kernels and processes) developed into some of the main topics of the conference. Other important topics were e.g. harmonic maps, heat kernels (on manifolds and graphs), Sobolev inequalities, stochastic methods in complex analysis, loop space analysis, spectral geometry and Yang-Mills equation.

Abstracts

Brownian motion and value distribution of minimal surfaces

Atsushi Atsuji

We wish to consider some relations between some value distributional properties of minimal surfaces and global behaviours of Brownian motion on them. Our first observation is related to Omori's observation in '67. He showed that if a minimal submanifold immersed in $\mathbf{R}^n$ of bounded sectional curvature cannot lie inside any non-degenerate cones. We can improve this easily by stochastic arguments as follows.

Theorem 1. *If a minimal submanifold immersed in an Hadamard manifold is stochastically complete, then it cannot lie inside domains which allow a concave majorant of $k(d(x_o, x))$ for some proper function $k : [0, \infty) \to [0, \infty)$.*

From this we expect that stochastic completeness should be some constraint on global behavior of minimal submanifolds. We already discussed on the value distribution of (classical) Gauss maps, but we found that there are many minimal surfaces whose Gauss maps have arbitrary value distribution. Then our first question is ;What global property of Briownian motion implies that the Gauss map cannnot omit a set of logarithmic capacity zero ?
We have already known that triviality of invariant $\sigma-$field ($= L^\infty$ Liouville property) implies it and we can show the similar result to Theorem 1 under the assumption of L^∞ Liouville property.

We are also interested in Bernstein type theorem. The famous Bernstein theorem is closely related to parabolicity of mionimal surface. Schoen and Simon gave a generalization of it as follows.

Theorem (Schoen-Simon '83) . *If M is a simply connected, complete and properly embedded minimal surface in $\mathbf{R}^3$ satisfying that $vol(B(r) \cap M) = O(r^2)$, then it is a plane, where $B(r) = \{x \in \mathbf{R}^3 : |x| < r\}$.*

We consider this type of result without properness condition. We can replace the above volume condition with finiteness of projective volume which is introduced by Tkachev. We define for a minimal surface $x : M \to \mathbf{R}^3$,

$$Q(M) = \int_M \frac{|x^\perp|^2}{|x|^4},$$

where $x^\perp$ is a normal part of x to a tangent vector of $x(M)$. We can replace properness on M with stochastic completeness. We have

Theorem 2. *Let M be a complete, stochastically complete and simply connected embedded minimal surface in $\mathbf{R}^3$. If $Q(M) < \infty$, then it is a plane.*

We remark that properness automatically implies stochastic completeness. For a properly immersed minimal surface M ($\not\ni o$), we have

$$Q(M) = C \lim_{r\to\infty} \frac{1}{\log r} \int_{B(r)\cap M} \frac{1}{|x|^2}.$$

Hence the above theorem includes Schoen-Simon's result.

On the Spectrum of Dirac Operators on Noncompact Manifolds

Werner Ballmann

Let M be a complete and connected Riemannian manifold and $\pi : E = E^+ \oplus E^- \to M$ be a Dirac bundle over M in the sense of Gromov and Lawson. The associated Dirac operator D on $C^\infty(E)$ is elliptic and essentially selfadjoint in $L^2(E)$. In the case where M is closed, the essential spectrum of D is empty and the Atiyah–Singer index theorem gives an explicit formula for the index of $D|C^\infty(E^+)$.

If M is noncompact and if one is interested in relations between the geometry of M and the spectrum of D, only natural bundles E can be dealt with. Then the geometry of the ends of M comes into play.

In their work on compact manifolds with cylindrical boundary (collar), Atiyah, Patodi and Singer show that their boundary value problem corresponds to the index problem for D on $L^2(E)$, where the ends are extended to (one sided) infinite cylinders. In a sense, this is the most simple geometry of ends.

In joint work with Jochen Brüning, I studied the case where M is a surface with finitely may ends of finite area and pinched negative curvature. We obtain explicit conditions on the bundle which are more or less equivalent to D being Fredholm, we also obtain an explicit index formula in the latter case. Our methods can also be applied in other situations. Independently and with similar methods, John Lott (Ann Arbor) obtained related results for the operator $d + d^*$ in arbitratay dimension.

Random holonomy and Yang-Mills fields

Robert Bauer

Yang-Mills fields on a vector bundle can be characterized by martingales: The connection A is Yang-Mills iff the vertical variation of the stochastic parallel transport under certain perturbations of the driving Brownian motion is a martingale. We give several applications of this result:

First, in radial gauge, the stochastic parallel transport $v(t)$ can be written as a double integral (relative to the time parameter and the perturbation parameter s) over the curvature

$$v(t) = v_1(t) = \int_0^1 v_s(t) \int_0^1 v_s^{-1}(r) F(sx(r)) < s\partial x(r), x(r) > v_s(r)\ ds,$$

with $x(t)$ Brownian motion on R^d and F the curvature of the vector bundle. The martingale characterization together with this formula can now be used to obtain a small-ball characterization of Yang-Mills fields: The net rotation of the stochastic parallel transport along Brownian motion until it exits a ball of radius ϵ is of the order ϵ^4 iff the connection is Yang-Mills.

Second, one can study weak convergence of the stochastic parallel transport for time $t \to \infty$ for a vector bundle over Euclidean space. For closed paths we show that there exists a gauge invariant notion of a weak limit of the random holonomy and we give conditions that insure the existence of such a limit. Then, we study the asymptotic behavior of the average of the random holonomy in the case of 't Hooft's 1-instanton.

For open paths there exists an almost sure limit under appropriate curvature assumptions. This result relies on the existence of good gauges.

Manifolds and graphs with slow heat kernel decay

Thierry Coulhon

This talk reports on a joint work with Martin Barlow and Alexander Grigor'yan. Let M be a complete non-compact Riemannian manifold, Δ its Laplace-Beltrami operator and $p_t(x,y)$ its heat kernel, i.e. the kernel of the heat semigroup $e^{t\Delta}$. Denote by $V(x,r)$ the Riemannian volume of the geodesic ball of center $x \in M$ and radius $r > 0$. The question is the direct relationship between the behaviour of

$$\sup_{x \in M} p_t(x,x)$$

as a function of $t \to +\infty$, and the volume growth of M, i.e. the behaviour of

$$v(r) = \inf_{x \in M} V(x,r)$$

as $r \to +\infty$. We show that

$$\sup_{x \in M} p_t(x,x) \leq \frac{C}{\gamma(ct)},$$

where γ is defined by

$$t = \int_0^{\gamma(t)} v^{-1}(s)\, ds,$$

and that this estimate is essentially sharp in the sense that for every suitable v there exists a manifold M with bounded geometry and volume growth at least v such that

$$\sup_{x \in M} p_t(x,x) \geq \frac{c}{\gamma_1(Ct)},$$

where γ_1 differs from γ from at most a logarithmic factor. If $v(r) \simeq r^D$, $\gamma(t) \simeq t^{\frac{D}{D+1}}$. We also show that there exists a manifold with bounded geometry satisfying

$$cr^D \leq V(x,r) \leq Cr^D$$

and

$$\sup_{x \in M} p_t(x,x) \geq ct^{-\frac{D}{D+1}}.$$

This example is inspired by the recent litterature on fractals. As a conclusion, we deduce the exact range of possible behaviours of $\sup_{x\in M} p_t(x,x)$ for manifolds with polynomial volume growth.

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Collapse of Random Hypergraphs: Pure Jump Processes with a Deterministic Limit

R. W. R. Darling, J. R. Norris

Context: Convergence of multidimensional pure jump Markov processes of pure jump type to the solution of a deterministic ODE. General but elementary results of this kind are proved in an appendix, using exponential martingales.
Specific application: Construct a random hypergraph Λ on a set of L vertices as follows: for $j \geq 0$ and for sets G of j vertices, the number of j-tuples covering G is Poisson, and these random variables are independent. A 1-tuple covering a vertex is called a patch. A 0-tuple covering the empty set is called debris. Collapse of Λ is a process in which a patch is selected uniformly at random, causing deletion of the vertex v it covers, and conversion of every every j-tuple on a set of vertices including v to a $(j-1)$-tuple on the set of remaining vertices; the patches on v become debris. Such selections occur at the jump times of a Poisson process.
Asymptotics: The three-dimensional jump process, consisting of vertices deleted, patches, and debris, is Markov, and when rescaled by converges at an exponential rate to the solution of an ODE. There is a critical regime in which small changes in model parameters cause abrupt changes in model behavior.

Spectral gaps on loop spaces : A counterexample

Andreas Eberle

Let M be a compact connected and simply connected Riemannian manifold, and let $LM = C(S^1, M)$ denote the space of continuous loops over M, endowed with the Bismut measure P.

We show that if there exists a closed geodesic $\gamma : S^1 \to M$ such that the curvature is constant and strictly negative in a neighbourhood of $\gamma(S^1)$, then a Poincaré inequality on LM w.r.t. the H^1 type metric and the measure P does not hold. A similar result holds on based loop spaces endowed with pinned Wiener measure, provided the base point is close to $\gamma(S^1)$.

The key point is that the closed geodesics γ_n obtained by winding around γ n times are local minima for the energy functional. This forces the Bismut measure to concentrate near these geodesics. Because of the negative curvature, the "concentration gets stronger for large n", which destroys the Poincaré inequality. The rigorous proof of the theorem is based on estimates for the concentration of Brownian bridges on hyperbolic spaces near the minimal geodesic connecting the endpoints of the bridge.

The Geometry of the First Steklov Eigenvalue

Jose Escobar

Let (M^n, g) be a compact Riemannian manifold with boundary and dimension $n \geq 2$. In this talk we discuss the first non-zero Steklov eigenvalue problem

$$\begin{aligned} \Delta\varphi &= 0 && \text{in} \quad M, \\ \frac{\partial\varphi}{\partial\eta} &= \nu_1\varphi && \text{on} \quad \partial M, \end{aligned}$$

where ν_1 is a positive real number.

Problem (1) is known as the Stekloff problem because it was introduced by him in 1902, for bounded domains of the plane. In this case the problem has applications in physics. The function φ represents the steady state temperature on a domain M and the flux on the boundary is proportional to the temperature.

I will discuss the relation of this problem to harmonic analysis and some areas of differential geometry, then we discuss upper and lower estimates of the eigenvalues ν_1 in terms of the geometry of the manifold (M^n, g). Some of the estimates I will discuss are: a sharp estimate for surfaces with non-negative Gaussian curvature which says that $\nu_1 \geq k_0$ where k_0 is the minimum of the geodesic curvature. An upper estimate for a convex manifold with non-negative Ricci curvature which is given in terms of the first non-zero eigenvalue for the Laplacian on the boundary. An estimate from below for a starshaped domain on a manifold whose Ricci curvature is bounded from below. A comparison theorem for simply connected domains in a simply connected manifold. We exhibit annuli domains for which the comparison theorem fails to be true. We introduced the isoperimetric constant $I(M)$ defined as

$$I(M) = \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}},$$

where $\Omega_1 = \Omega \cap \partial M$ is a non–empty domain with boundary in the manifold ∂M, $\Omega_2 = \partial M - \Omega_1$, and $\Sigma = \partial\Omega \cap \mathrm{int}(M)$, where $\mathrm{int}(M)$ is the interior of M. I will discuss a Cheger's type inquality that involves the isoperimetric constant I. Finally we will discuss upper and lower estimates for the constant I in terms of isoperimetric constants of the boundary of M.

Geodesic flow and diffusions on hyperbolic manifolds

Jacques Franchi

I have been working for these last years on the diffusions and the geodesic flow on hyperbolic manifolds.

An hyperbolic manifold is a smooth manifold with constant negative curvature, and is the quotient of the standard hyperbolic space by some discrete group of Möbius isometries. The geodesic flow on such a manifold is a basic example of unstable dynamical system, intensively studied, among others by Hadamard, Hopf, Sinaï, Patterson, Sullivan, Bowen, Margulis.

Y. Le Jan found six years ago how to obtain the central limit theorem for the geodesic flow on an hyperbolic manifold of finite volume by using the Brownian motion on this manifold.

This idea of reducing the geodesic flow case to the Brownian case was then developed to the asymptotic study of the singular windings about the cusps of the manifold, still in the finite volume case, by Enriquez-Le Jan in two dimensions and by myself in three dimensions.
It was so established that the geodesic flow winds around the cusps, under the Liouville measure, asymptotically at time t : with speed t following a Cauchy law in two dimensions, and with speed $\sqrt{t \log t}$ following a Gaussian law in three dimensions. This concluded such studies for the finite volume case.

The infinite volume case is much harder to handle. The Liouville measure has to be replaced by the singular Patterson-Sullivan measure, and the Brownian motion by the ground-state diffusion ; but the lift of this diffusion to the stable foliation and the geodesic flow do not have the same invariant measure anymore. It is then necessary to study the excursions in the cusps of some approximating other diffusion, above a level eventually going to infinity, and to perform a lot of estimates.
This work was recently done by Enriquez, Le Jan and myself in the two-dimensional case. The winding speed is some power of the time, and the asymptotic winding law is stable.

We also worked out, in any dimension, a simple construction of the fundamental diffusion, through its canonical lift to the frame bundel, and deduced its exit law from the universal cover.

Very recently, we obtained a central limit theorem in the case $\delta > d/2$, δ being the Hausdorff dimension of the limit set, and the manifold having dimension $d+1$.

Calculus on Metric Spaces, I and II

Juha Heinonen

Concepts and results from first order calculus have recently been extended to a certain class of metric measure spaces. The principal assumptions are the doubling condition on the measure and the validity of a Poincaré inequality defined in terms of "upper gradients". In these two talks, I explain the basic concepts in the area – in particular some of the recent work of J. Cheeger who has shown that one can construct a measurable cotangent bundle together with an exterior differential on doubling spaces that support a Poincaré inequality. A theory of Sobolev spaces based on upper gradients is instrumental in Cheeger's work. Finally, I explain some examples of spaces that fall under this theory; these spaces can be complicated both topologically and geometrically, they can have Hausdorff dimension any prescribed positive number larger than one, for example.

Convergence of Riemannian manifolds and Laplace operators

Atsushi Kasue

Riemannian manifolds are considered as metric spaces equipped with Riemannian distances and also Dirichlet spaces endowed with the Riemannian measures and the energy forms; on a family of compact Riemann ian manifolds, the Gromov-Hausdorff distance and the spectral distance induce uniform topologies. The former is concerning the metric structure and the latter is defined by the heat kernels.

A basic fact is that a family of compact, connected Riemannian manifolds whose heat kernels satisfy a uniform on-diagonal estimate from a bove is precompact with respect to the both metrics. In this talk, we discussed on (1) the structure of limit spaces of the family from the view-points of Gro mov-Hausdorff distance and the spectral distance; (2) the convergence of en ergy forms; (3) the spectral convergence of vector bundle Laplacians; (4) the convergence of harmonic maps of manifolds in the family to Riemannian manifolds of nonposit ive curvature.

Probability and Analysis on Euclidean Complexes

Yuri Kifer

A natural Brownian motion on Euclidean complexes was constructed by M.Brin and myself (to appear in Math. Z.) using a direct gluing procedure but a natural approach to it via Dirichlet forms should work, as well. The latter should provide also additional results like Cheeger's inequality between the top of spectrum of the natural Laplacian on such complexes. The next natural step is to introduce an appropriate notion of martingales on complexes (and on more general metric spaces) and connect them with with harmonic maps to metric spaces defined in recent years by Jost, Korevaar and Schoen. It would be nice to have a characterization of harmonic maps as those which map Brownian motins to matingales similarly to the smooth case. There is an old definition of martingales on metric spaces due to Benes and recently it was discussed in the language of barycenters by Es-Sahib and Heinich. It is not clear whether these martingales are appropriate for the study of harmonic maps and it would be interesting to check whether the natural Brownian motion on complexes and even graphs is a martingale in this sense.

Actually, the main point of our work with Brin was to understand the asymptotic behaviour of the Brownian motion on Euclidean complexes and to describe the spaces of harmonic functions there (Poisson and Martin boundaries) and the main motivation was to see how Alexandrov's definition of negatively curved metric spaces together with Gromov's hyperbolicity condition produces an asymptotic behaviour of the Brownian motion familiar from the case of negatively curved manifolds. Related ideas appear in the probabilistic approach to the little Picard theorem for harmonic maps so if a connection between martingales on complexes and harmonic maps is established we can extend the little Picard theorem to harminic maps to hyperbolic Euclidean complexes. An advantage of studying of specific nonsmooth objects like complexes vis-a-vis general metric spaces is that complexes possess a natural metric, a natural Brownian motion and a natural Laplacian and so they should serve as a testing ground for all constructions and definitions suggested for general metric spaces.

Analysis in metric spaces

Pekka Koskela

Heinonen and I have recently established a theory of quasiconformal mappings on Ahlfors regular Loewner spaces. These spaces are metric spaces that have sufficiently many rectifiable curves in a sense of good estimates on moduli of curve families. The Loewner condition can be conveniently described in terms of Poincaré inequalities for pairs of functions and upper gradients. Here an upper gradient plays the role that the length of the gradient of a smooth function has in the euclidean setting. For example, the euclidean spaces and Heisenberg groups and the more general Carnot groups admit the type of a Poincaré inequality we need. We describe the basics of spaces that admit a Poincaré inequality for pairs of functions and upper gradients and discuss the associated Sobolev spaces. We also discuss the concept of a Sobolev mapping between two metric spaces. In order to gain linear structure we embed the target space into a Banach space. One interesting point here is that the validity of a Poincaré inequality for maps of a metric doubling space into a (non-trivial) Banach space does not depend on the Banach space in question. In particular, if we have a Poincaré inequality for real valued maps, then we automatically have a Poincaré inequality for maps into any Banach space.

Sobolev spaces over map between metric spaces

Kazuhiro Kuwae

We construct the $(1,p)$-Sobolev spaces and energy functionals over L^p-maps between metric spaces for $p \geq 1$ under the condition so-called strong measure contraction property of Bishop-Gromov type (SMCPBG in short). Under this property, we also prove the existence of energy measures, and the weak Poincaré inequality, which extends some parts of the results of Korevaar-Schoen and Sturm. Alexandrov spaces are included in this formulation and we show that the constructed Sobolev spaces are compatible with $(1,p)$-Sobolev spaces over L^p-functions on Alexandrov spaces.

Stochastic pants on a Riemannian Manifold

Remi Leandre

With Z. Brzerzniak, we construct random pants over a manifold. We show that they realize an application from $E \otimes E$ into E where E is the space of continuous functions over the loop space. This gives an analoguous of one of Segal's axiom in conformal field theory, the Hilbert space of the theory being replaced by a Banach space. I define a Dirac-Taubes operator over the quotient of a loop group, and I justify the conjecture that its equivariant Index is equal to the Witten genus of the underlying homogeneous manifold, by doing an expansion in small time of the considered operator. I define a Dirac-Taubes operator over the quotient of a twisted loop group,. This allows me to repeat with this new model the arguments of Witten for the rigidity theorem.

Sobolev inequalities and rigidity theorems

Michel Ledoux

We review some joint works with D. Bakry on the geometric aspects of Sobolev inequalities with sharp constants on Riemannian manifolds. In particular, we show that the optimal logarithmic Sobolev inequality entails optimal heat kernel bounds. Furthermore, a Riemannian manifold with dimension n and non-negative Ricci curvature satisfying the sharp logarithmic Sobolev inequality is isometric to R^n. The corresponding result for the classical Sobolev inequality is also established. Some open problems on the Nash inequality and the compact case are discussed.

Special Itô maps and an L^2 Hodge theory for one forms on path spaces

Xue-Mei Li

This is a joint work with K. D. Elworthy. Let M be a smooth compact Riemannian manifold. For a point x_0 of M and a fixed $T > 0$, let $C_{x_0}M$ denote the space of continuous paths $\sigma : [0,T] \to M$ with $\sigma(0) = x_0$. There are the deRham cohomology groups $H^q_{deRham(r)}(C_{x_0}M)$. Each such group is equal to the singular cohomology group by a recent work of C. J. Artkin, even though $C_{x_0}M$ does not admit smooth partitions of unity, and so trivial for $q \geq 0$ since based path spaces are contractible. Contractibility need not imply triviality of the deRham cohomology group when some restriction is put on the spaces of forms. We are interested in a suitable L^2 theory. In finite dimensions the L^2 theory has especial significance because of its relationship with Hodge theory and its associated geometric analysis. For analogous analysis on infinite dimensional manifolds, we need a suitably defined Hilbert tangent subspace of 'admissible directions'. The Bismut tangent spaces $H^1_\sigma = \{v_t = //_t(\sigma)h_t \mid h. \in L_0^{2,1}(T_{x_0}M)\}$, where $//_t(\sigma) : T_{x_0}M \to T_{\sigma(t)}M$ is parallel translation of the Levi-Civita connection, has played an important role in the work of Jones-Léandre and Driver. To have a satisfying L^2 theory of differential forms on $C_{x_0}M$ the obvious choice would be to consider 'H-forms' i.e. for 1-forms these would be ϕ with $\phi_\sigma \in (H^1_\sigma)^*$, $\sigma \in C_{x_0}M$, and this agrees with the natural H-derivative df for $f : M \to \mathbb{R}$. For L^2 q-forms the obvious choice would be ϕ with $\phi_\sigma \in \wedge^q(H^1_\sigma)^*$. An L^2-deRham theory would come from the complex of spaces of L^2 sections

$$\cdots \xrightarrow{\bar{d}} L^2\Gamma \wedge^q (H^1_\sigma)^* \xrightarrow{\bar{d}} L^2\Gamma \wedge^{q+1} (H^1_\sigma)^* \xrightarrow{\bar{d}} \cdots$$

where $\bar{d}$ would be a closed operator obtained by closure from the usual exterior derivative. From this would come the deRham-Hodge-Kodaira Laplacians $\bar{d}\bar{d}^* + \bar{d}^*\bar{d}$ and an associated Hodge decomposition. However the brackets $[V^i, V^j]$ of sections of $H^1_\cdot$ are not in general sections of $H^1_\cdot$, and the formula for d does not make sense for ϕ_σ defined only on $\wedge^q H^1_\sigma$, each σ. The project fails at the stage of the definition of exterior differentiation. We first studied the Itô map by gradient flows. The Itô map and the technique of filtering of Elworthy-Yor are used to give an admissible space $\mathcal{H}^2$ of differential 2-forms. The corresponding Laplacian operator is then investigated leading to a Hodge decomposition theorem for differential 1-forms. Below is the result for differential 1-forms. Denote by $\mathcal{I} : C_0(\mathbb{R}^m) \to C_{x_0}M$ the Itô map by gradient stochastic differential equations, and H the Cameron-Martin space over $\mathbb{R}^m$. Let $\overline{T\mathcal{I}}()_\sigma : H \to \mathcal{H}^1_\sigma$ denote the map

$$h \mapsto \mathbb{E}\{T\mathcal{I}.(h.) \,|x.(\omega) = \sigma\}$$

defined for μ_{x_0} almost all $\sigma \in C_{x_0}M$, where $\mathcal{H}^1$ is the same as the Bismut tangent space with a different inner product induced by the Itô map. We shall denote by $L^2\Gamma\mathcal{H}^1$ the space of L^2 sections of $\mathcal{H}^1$-valued vector fields.

Theorem 1. *The map $h \mapsto \mathbb{E}\{T\mathcal{I}(h)\,|\mathcal{F}^{x_0}\}$ determines a continuous linear map*

$$\overline{T\mathcal{I}(-)} : L^2\left(C_0(\mathbb{R}^m); H\right) \to L^2\Gamma\mathcal{H}^1,$$

which is surjective. The pull back map $\mathcal{I}^*$ on 1-forms extends to a continuous linear map of H-forms:

$$\mathcal{I}^* : L^2\Gamma(\mathcal{H}^{1^*}_\cdot) \to L^2\left(C_0\left(\mathbb{R}^m\right); H^*\right),$$

which is the co-joint of $\overline{T\mathcal{I}(-)}$. It is injective with closed range.

Theorem 2. *The space $\mathcal{H}^2_\sigma$ consists of elements of $\wedge^2 T_\sigma C_{x_0}M$ of the form $V + Q(V)$ where $V \in \wedge^2\mathcal{H}^1_\sigma$ and $Q : \wedge^2\mathcal{H}^1_\sigma \to \wedge^2_\sigma C_{x_0}M$ is the continuous linear map determined by*

$$Q(V)_{(s,t)} = \frac{1}{2}\left(1 \otimes W_t(W_s)^{-1}\right) W_s^{(2)} \int_0^s (W_r^{(2)})^{-1} \mathcal{R}(V_{(r,r)})\, dr, \qquad 0 \le s \le t \le T$$

where (i) $W_t : T_{x_0}M \to T_{\sigma(t)}M$ *is the damped parallel translation satisfies:* $\frac{D}{\partial t}W_t(v) = -\frac{1}{2}Ric^{\#}(W_t(v))$, $W_0(v) = v$. *Here Ric denotes the Ricci curvature. (ii)* $W_t^{(2)} : \wedge^2 T_{x_0}M \to \wedge^2 T_{\sigma(t)}M$ *is the the damped translation of 2-vectors on* M *given by*

$$\begin{aligned} \frac{D}{\partial t}W_t^{(2)}(u) &= -\frac{1}{2}\mathcal{R}^{(2)}_{\sigma(t)}\left(W_t^{(2)}(u)\right) \\ W_0^{(2)}(u) &= u, \qquad\qquad u \in \wedge^2 T_{x_0}M, \end{aligned}$$

for $\mathcal{R}^{(2)}_{\sigma(t)}$ *the Weitzenböck curvature on 2-vectors and (iii)* $\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$ *denotes the curvature operator.*

Theorem 3. *The space* $L^2\Gamma(\mathcal{H}^{1^*})$ *of* $\mathcal{H}$ *1-forms has the decomposition*

$$L^2\Gamma(\mathcal{H}^{1^*}_{\cdot}) = Ker\Delta^1 \oplus Image\ \bar{d} \oplus \overline{Image\ d^{1*}}.$$

In particular every cohomology class in $L^2H^1(C_{x_0}M)$ *has a unique representative in* $Ker\Delta^1$.

Bundle of orthonormal frames : old and new

Paul Malliavin

Diffrential Geometry on based Path spce was founded in Cruzeiro-Malliabvim (JFA 1996) on two tools Parallelization of the Path Spaces obtained by Stochastic parallel transport Renormalisation of divegent sums through a systemeatik use of stochastic integrals A impoant forward step is made in a forthcoming paper in the JFA.

Fistly a reasonnable bundle of orthonormal frame id constructed above path space; its structural group is the paths spsce above the orthogonal group; this bundle of frame pemit to cinstruct a fully general cvarant derivative valid for general tensor fields. FRom this tensorial analysis is deduced easily the structural equqtion of the frame bundle.

Secondly a new type of renormalization is introduced . Renormalization by restriction inside the ADAPTED CATEGORY. Under thisful renormlization procedure it is shown that the Ricci tensor of a the Path space above a Ricci flat manifold vanishes and a corresponding Weitzenbock identity is obtained for diffrential 1-difffderntial forms coupled with adapted vector field.

Gradient flows on nonpositively curved metric spaces and harmonic maps

Uwe F. Mayer

The notion of gradient flows is generalized to nonpositively curved metric spaces in the sense of Alexandrov. The metric spaces considered are a generalization of Hilbert spaces, but without any linear structure or local compactness assumptions, and the properties of such metric spaces

are used to set up a finite-difference scheme of variational form. The proof of the Crandall–Liggett generation theorem is adapted to show convergence. The resulting flow generates a strongly continuous semigroup of Lipschitz-continuous mappings, is Lipschitz continuous in time for positive time, and decreases the energy functional along a path of steepest descent. In case the underlying metric space is a Hilbert space, the solutions resulting from this new theory coincide with those obtained by classical methods. As an application, the harmonic map flow problem for maps from a manifold into a nonpositively curved metric space is considered, the setting being the Sobolev space theory by Korevaar and Schoen. The existence of a solution to the initial boundary value problem is established.

Weak Convergence of Laws of Stochastic Processes on Riemannian Manifolds

Yukio Ogura

1. Introduction

The convergence of analytic items for a class of Riemannian manifolds is studied extensively in these several years. Among them, Kasue and Kumura [4] introduced a class of Riemannian manifolds with heat kernels uniformly bounded by a constant on each compact set of the time parameter and with bounded volumes, and obatined limit theorems for analytic items for the manifolds in this class(see also [2], [5]). In this note, we give limit theorems for stochastic processes in that class, that is, the weak convergence of the laws on the space of cadlag paths. We also give a convergence theorem for the Γ-martingales induced by the harmonic maps studied in [3].

2. Statement of Results

Let $\mathcal{M}$ be the collection of all connected compact Riemannian manifold (M, g) with the nornalized volume element $\mu_M = \mu_g/\mathrm{vol}_g(M)$. For each $\nu, \alpha > 0$, we denote by $\mathcal{M}(\nu, \alpha)$ the space of all $M = (M, g) \in \mathcal{M}$ satisfying

$$p^M(t, a, b) \le \frac{\alpha}{(t \wedge 1)^{\nu/2}}, \quad t > 0,\ a, b \in M,$$

where $p^M(t, a, b)$ is the heat kernel associated with $M = (M, g) \in \mathcal{M}$. Let $(Y^n(t), P_a)$ $(t > 0,\ a \in M_n)$ be the Brownian motion on $M_n = (M_n, g_n)$, that is the diffusion process on M_n determined by the transition function $p^{M_n}(t, a, b)\mu_n(db)$, where $\mu_n = \mu_{M_n}$.

Theorem 1. (i) *Suppose that* $\{M_n = (M_n, g_n)\} \subset \mathcal{M}(\nu, \alpha)$ *and there exist a compact Hausdorff space* X, *a continuous pseudo-distance function* δ *on* X *and measurable mappings* $f_n : M_n \to X$ *such that*

$$\sup_{a,b \in M_n} |d_{g_n}(a, b) - \delta(\pi \circ f_n(a), \pi \circ f_n(b))| < \varepsilon_n$$

holds for a positive sequence $\varepsilon_n \downarrow 0$. *Let* X_δ *be the quotient space* $X/\sim_\delta$, *where* $\sim_\delta$ *is the equivalent relation induced by the pseudo-distance function* δ, *and* π *the natural projection from* X *to the metric space* (X_δ, δ). *Then, for each* $\tau \in (0, 2\nu + 2)$, *the laws of processes* $\{(\pi \circ$

$Y^n(\varphi_n(t)), P_{a_n})\}$ *are tight in* $\mathcal{P}(D([0,\infty) \to (X_\delta, \delta)))$, *where* $\varphi_n(t) = [t/\varepsilon_n^\tau]\varepsilon_n^\tau$ *with the symbol* $[t]$ *being the greatest integer which is less or equal to* t.
(ii) *Suppose, in addition, that there exist a Radon measure* μ *on* X *and a continuous transition density* $p^X(t,x,y)$ $(t > 0,\ x, y \in X)$ *such that*

$$\lim_{n\to\infty} \sup_{a,b\in M_n} |p^{M_n}(t,a,b) - p^X(t, f_n(a), f_n(b))| = 0, \qquad t > 0,$$

the image measures $f_{n*}\mu_n$ *converge to* μ *weakly, and* $f_n(a_n)$ *converge to* x_0 *in* X. *Then the laws of* $(\pi\circ Y^n(\varphi_n(t)), P_{a_n})$ *converge to that of* $(\pi\circ Y(t), P_{x_0})$ *weakly in* $\mathcal{P}(D([0,\infty) \to (X_\delta, \delta)))$, *where* $(Y(t), P_x)$ *is the diffusion process on* X *determined by the transition function* $p^X(t,x,y)\mu(dy)$.

Theorem 2. *Suppose, in addition to the assumptions in Theorem* 1 (i), *that there exist a Radon measure* μ *on* X *and a transition density* $p^X(t,x,y)$ $(t > 0,\ x, y \in X)$ *such that the image measures* $f_{n*}\mu_n$ *converge to* μ *in* $B(X)^*$ *and*

$$\lim_{n\to\infty} \|p^{M_n}(t,*,*) - \ p^X(t, f_n(*), f_n(*))\|_{L^1(M_n\times M_n;\mu_n\times\mu_n)} = 0, \quad t > 0,$$

holds, where $B(X)$ *is the space of all bounded Borel measurable functions on* X. *Then the laws of* $(\pi \circ Y^n(\varphi_n(t)), P_{\mu_n})$ *converge to* $(\pi \circ Y(t),\ P_\mu)$ *weakly in* $\mathcal{P}(D([0,\infty) \to (X_\delta, \delta)))$, *where in general* P_ν *stands for the law of the process with the initial distribution* ν.

Let next $N \in \mathcal{M}$ have nonpositive sectional curvature. For each $M \in \mathcal{M}$, denote by $\Sigma_i(M,N)$, $i = 0, 1, \cdots \nu_{M,N}$ the connected components of hamonic maps from M to N oredered by the height of their energies.

Theorem 3. *Let* $\{M_n = (M_n, g_n)\} \subset \mathcal{M}$ *with* $\dim M_n = m$ *and* $Ric_{g_n} \geq -(m-1)$. *Suppose also that* $\mathrm{diam} M_n \leq D$ *for some* $D > 0$, *and assume the assumptions in Theorem* 1 (i) *and* (ii). *If* $\phi_{n,i} \in \Sigma_i(M_n, N)$ *and* $\phi_i \in \Sigma_i(X, N)$ *satisfy*

$$\lim_{n\to\infty} \sup_{a\in M_n} d_N(\phi_{n,i}(a), \phi_i(a)) = 0,$$

then the laws of processes $\{(\phi_{n,i}(Y^n(t)), P_{a_n})\}$ *converge to that of* $(\phi_i(Y(t)), P_{x_0})$ *weakly in* $\mathcal{P}(C([0,\infty) \to (N, d_N)))$.

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Existence and smoothness of harmonic maps with stochastic method

Jean Picard

In classical analysis, there are mainly two types of techniques which enable the construction of harmonic maps $h : M \to N$ between Riemannian manifolds:

- By taking the limit of the solution of the heat equation as time tends to infinity.
- By minimizing an energy functional associated to the Laplacian on M.

These techniques can be extended to a more general framework and have probabilistic counterparts. Consider for instance a manifold M with boundary ∂M, endowed with a diffusion X_t stopped at the exit τ of M, and a manifold N endowed with a connection; if g is a map from ∂M into N, the Dirichlet problem consists in finding a map $h : M \to N$ which is harmonic in M and coincides with g on ∂M. The probabilistic problem consists in finding a map h such that $h(X_t)$ is a N-valued martingale with final value $g(X_\tau)$. The analogues of the above analytical techniques are:

- Proving the existence of N-valued martingales with prescribed final value; this can be done under convexity assumptions on N and is now well known.
- When the diffusion is symmetric and N is Riemannian, one can minimize an energy functional associated to the Dirichlet form of the diffusion (this can be done under a weak non degeneracy condition on the diffusion); then one has to verify that the solution of this minimization problem is a solution of the probabilistic Dirichlet problem; to this end, an important tool is the theory of Dirichlet processes.

When we have found a probabilistic solution h, we want to know whether it is a solution in the classical sense; this is equivalent to proving the smoothness of h. This problem can be divided into:

- Proving the existence of a continuous modification of h.
- Proving the C^∞ smoothness of this modification.

For the first point, one needs some convexity conditions on N. The second point is much more technical, and one can prove the smoothness under Hörmander's condition on the diffusion; an important step is the study of martingales on the tangent bundle of N. Moreover it can be interesting to find a priori estimates for the derivatives of h; this problem can be solved when the diffusion is elliptic, but the general case seems to be much more difficult.

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Global analysis of foliations and group actions

Ken Richardson

We consider a generalization of the trace of the heat kernel to Riemannian foliations, and we will observe asymptotic behavior similar to the results for group actions. Suppose that a compact, Riemannian manifold M is equipped with a *Riemannian foliation* $\mathcal{F}$; that is, the distance from one leaf of $\mathcal{F}$ to another is locally constant. For simplicity, we assume that M is connected and oriented and that the foliation is transversally oriented.

A natural question to consider is the following: if we assume that the temperature is always constant along the leaves of $(M, \mathcal{F})$, how does heat flow on the manifold? To answer this question, we must restrict to the space of basic functions $C_B^\infty(M)$ (those that are constant on the leaves of the foliation) and more generally the space of basic forms $\Omega_B^*(M)$ (smooth forms ω such that given any vector X tangent to the leaves, $i(X)\omega = 0$ and $i(X)d\omega = 0$, where $i(X)$ denotes the interior product with X). The exterior derivative d maps basic forms to basic forms; let d_B denote d restricted to $\Omega_B^*(M)$. The relevant Laplacian on forms is the basic Laplacian $\Delta_B = d_B\delta_B + \delta_B d_B$, where δ_B is the adjoint of d_B on $L^2\left(\Omega_B^*(M)\right)$. The basic heat kernel $K_B(t,x,y)$ on functions is a function on $(0,\infty) \times M \times M$ that is basic in each M factor and that satisfies

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_{B,x}\right) K_B(t,x,y) &= 0 \\ \lim_{t\to 0^+} \int_M K_B(t,x,y)\, f(y)\, dV(y) &= f(x) \end{aligned}$$

for every continuous basic function f. The existence of the basic heat kernel allows us to answer the question posed at the beginning of this paragraph. The basic heat kernel on forms is defined in an analogous way. Many researchers have studied the analytic and geometric properties of the basic Laplacian and the basic heat kernel. A point of difficulty that often arises in this area of research is that the space of basic forms is not the set of all sections of any vector bundle, and therefore the usual theory of elliptic operators and heat kernels does not apply directly to Δ_B and K_B.

It is natural to try to prove the existence of asymptotic expansions for the basic heat kernel. We remark that the basic heat operator is trace class, since the basic Laplacian is the restriction of an elliptic operator on the space of all functions (see lower bounds for eigenvalues in [3] and [2]). In [4], it was shown that, as $t \to 0$, we have the following asymptotic expansion for any positive integer k:

$$K_B(t,x,x) = \frac{1}{(4\pi t)^{q_x/2}}\left(a_0(x) + a_1(x)t + \ldots + a_k(x)t^k + O\left(t^{k+1}\right)\right),$$

where q_x is the codimension of the leaf closure containing x and $a_j(x)$ are functions depending on the local transverse geometry and volume of the leaf closure containing x. The first two nontrivial coefficients were computed in [4]. In general, the power q_x may vary, but its value is minimum and constant on an open, dense subset of M. One might guess that the asymptotics of the trace of the basic heat operator could be obtained by integrating the expansion (), similar to the standard case. However, the functions $a_j(x)$ for $j \geq 1$ are not necessarily bounded or even integrable over the dense subset. Simple examples exhibit this precise behavior.

Despite these obstacles, we prove that an asymptotic expansion for the trace of the basic heat operator exists. Let $\overline{q}$ be the minimum codimension of the leaf closures of $(M, \mathcal{F})$. As $t \to 0$, the trace $K_B(t)$ of the basic heat kernel on functions satisfies the following asymptotic expansion for any positive integer J:

$$K_B(t) = \frac{1}{(4\pi t)^{\overline{q}/2}} \left(a_0 + \sum_{\substack{1 \leq j \leq J \\ 0 \leq k < K_0}} a_{jk} t^{j/2} (\log t)^k + O\left(t^{\frac{J+1}{2}} (\log t)^{K_0 - 1} \right) \right),$$

where K_0 is less than or equal to the number of different dimensions of leaf closures in $\mathcal{F}$, and where

$$a_0 = V_{tr} = \int_M \frac{1}{\mathrm{Vol}\left(\overline{L_x}\right)} \, dV(x).$$

The idea of proof is as follows. We rewrite the integral $K_B(t) = \displaystyle\int_M K_B(t,x,x)\, dV(x)$ in terms of an integral over $W \times SO(q)$, where W is the *basic manifold*, an $SO(q)$-manifold associated to $(M, \mathcal{F})$. Then, we apply the results of [1]. In addition, we obtain the Weyl asymptotic formula for the eigenvalues of the basic Laplacian.

We remark that these asymptotic expansions yield new results concerning the spectrum of the basic Laplacian. We can show that the eigenvalues of the basic Laplacian determine the minimum leaf closure codimension and the transverse volume V_{tr} of the foliation. The results also give more specific information in special cases. For example, if the leaf closure codimension is one, then the spectrum of the basic Laplacian determines the L^2 norm of the mean curvature of the leaf closure foliation. Therefore, the spectrum determines whether or not the leaf closure foliation is minimal.

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Central Gaussian semigroups on compact groups

Laurent Saloff-Coste (joint work with Alexander Bendikov)

On any compact connected group G with neutral element e, call Brownian motion any process $X = ((X_t)_{t>0}, \mathbf{P})$ such that $X_0 = e$, X has stationary independent increments, continuous paths and is symmetric, bi-invariant and non-degenerate. If G is a simple Lie group ($e.g., SO(n)$) there is essentially one such process up to change of time, and its generator is the Laplace-Beltrami operator for the Killing Riemannian metric on G.

Let μ_t be the law of X_t, $t > 0$, where X is a Brownian motion on G. These one-dimensional marginals form a central symmetric Gaussian convolution semigroup $(\mu_t)_{t>0}$. If G is a Lie group, μ_t admit a smooth positive density $x \mapsto \mu_t(x)$ w.r.t Haar measure and we have

$$\log \mu_t(e) \sim \frac{n}{2} \log(1/t) \text{ as } t \to 0$$

where n is the tological dimension of G.

Assume now that G is infinite dimensional. Can we find, on any compact connected group G, a Brownian motion having absolutely cointinuous one dimensional marginals μ_t, $t > 0$? The answer is no in this generality. Necessary conditions are that G be locally connected and have a countable basis for its topology. See [2]. We have the following converse statement.

Theorem *Let G be a compact connected locally connected group having a countable basis for its topology. Then there are Brownian motions on G whose one dimensional marginals are absolutely continuous and have continuous densities, for all $t > 0$.*

It is natural to investigate the behavior $\mu_t(e)$ as $t \to 0$ in such cases. It is not hard to see that if G is not finite dimensional,

$$\lim_{t\to 0} \frac{\log \mu_t(e)}{\log(1/t)} = +\infty.$$

Theorem *Let G be a compact connected locally connected group having a countable basis for its topology. Let ψ be a positive contiuous increasing function on $(0, +\infty)$. Then there exists a Brownian motion X on G whose one dimensional marginals are absolutely continuous, have continuous densities $x \mapsto \mu_t(x)$, $t > 0$, and satisfy*

$$\forall\, t \in (0,1), \quad \log \mu_t(e) \le \log(1 + 1/t)\, \psi(1/t)$$

In words, on any G as above, we can achieve a behavior which is very close, in some sense, to what happens on Lie groups. To go further, one would like to understand how the behavior of $\mu_t(e)$ for small t relates to the structure of G. We have results in this direction which imply the following.

(a) There are groups G as in the theorems above on which no Brownian motion X can satisfy $\forall\, t \in (0,1), \quad t^{-a} \le \log \mu_t(e) \le t^{-b}$ for any $0 < a < b < +\infty$.

(b) If G admits a Brownian motion X such that $\forall\, t \in (0,1), \quad c \le t^a \log \mu_t(e) \le C$, Then, for any $b \in (0, \infty)$, it also admits a Bronwian motion X^b such that $\forall\, t \in (0,1), \quad c_b \le t^b \log \mu_t(e) \le C_b$.

Details and other results can be found in [1].

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Convergence of spectral structures: A functional analytic theory and its applications to spectral geometry

Takashi Shioya (joint work with Kazuhiro Kuwae)

The classical perturbation theory of linear operators tells us that if we perturb a Riemannian metric on a fixed manifold, then the spectral objects such as the spectral measure, the spectrum of the Laplacian etc. are continuous in metrics with respect to a suitable topology. What if we perturb not only the metric but also the topology of a manifold? In this case, there are no more natural identification between L^2 spaces of Riemannian manifolds and so we cannot rely on the standard perturbation theory. Nevertheless, we obtain some asymptotic correspondence between them under convergence of Riemannian volume measures. In this direction, Fukaya first defined the measured Gromov-Hausdorff topology on the set of metric spaces with Radon measures, and studied the asymptotic behavior as $i \to \infty$ of the eigenvalues of the Laplacian of Riemannian manifolds M_i, $i = 1, 2, \dots$, with a uniform bound of sectional curvature, when M_i is convergent with respect to the measured Gromov-Hausdorff topology. After that, Kasue-Kumura introduced a natural distance, called the spectral distance, between Riemannian manifolds under a uniform bound (in some sense) of heat kernels. The spectral distance expresses how close the analytic structures are and is a powerful tool to study convergence of Riemannian manifolds and their analytic structures. In our current work, we present a systematic and functional analytic framework of topology on the set of spectral structures, which is much more general than the spectral distance. In particular, we do not need the existence of the integral kernels of the semigroups, and also the spectrum is not needed to be discrete. Our topologies are indeed defined on the set of spectral structures on general Hilbert spaces, so that it can also be applied to that defined on L^2 differential forms, L^2 sections of vector bundles, L^2 functions on graphs, etc. Our theory can be applied to shrinking, blowing-up, and degeneratiing sequences of (possibly noncompact, incomplete) Riemannian manifolds.

Large Time Asymptotics for the Heat Kernel on a Periodic Manifold

Toshikazu Sunada

The aim of this talk is to show asymptotic properties of the heat kernel on a Riemannian manifold X with a free co-compact abelian group action, including the local central limit theorem and the asymptotic expansion. Emphasis is put on the underlying ideas and concepts such as Albanese tori and Albanese maps which originate in classical algebraic geometry. This a joint work with Motoko Kotani.

Let $k(t,x,y)$ be the heat kernel on X. It is rather easy to establish the following theorem, by a perturbation technique for the eigenvalues of the twisted Laplacians.

Theorem 1 (Local central limit theorem) There are a constant $C(X)$ and a function $d_0(x,y)$ on $X \times X$ such that

$$(4\pi t)^{k/2} k(t,x,y) - C(X) \exp(-\frac{d_0(x,y)^2}{4t}) \to 0$$

as $t \uparrow \infty$, uniformly in x, y. Here k is the polynomial growth rate of the geodesic balls in X.

Theorem 2 (Asymptotic expansion)

(1)

$$k(t,x,y) \sim (4\pi t)^{-k/2} C(X) \sum_{i=0}^{\infty} a_i(x,y) t^{-i}$$

as $t \uparrow \infty$.

(2) $a_i(x,y) \sim \left(-\frac{d_0(x,y)^2}{4}\right)^i$ as $d(x,y) \uparrow \infty$ ($d(x,y)$ being the Riemannian distance).

In this talk, I shall discuss the geometric nature of $C(X)$ and $d_0(x,y)$ in terms of Albanese tori and Albanese maps associated with X.

The detail can be seen in M. Kotani and T. Sunada, *Albanese maps and off diagonal long time asymptotics for the heat kernel* in Comm. Math. Phys. **209**(2000), 633-670. For a similar result, see J. Lott, *Remarks about heat diffusion on periodic spaces*, Proc. A.M.S. **127**(1999), 1243-1249. A similar idea is applied to asymptotics of the transition probability of the simple random walk on a crystal lattice, a discrete version of periodic manifolds (see the above mentioned paper and A. Krámli and D. Szász, *Random walks with internal degree of freedom, I. Local limit theorem*, Z. Wahrscheinlichkeittheorie **63**(1983), 85-95).

Heat Equation Derivative Formulas for Vector Bundles

Anton Thalmaier (joint work with Bruce K. Driver)

We use martingale methods to give Bismut type formulas for differentials and co-differentials of heat semigroups on forms, and more generally for sections of vector bundles. The formulas are mainly in terms of Weitzenböck curvature terms, in most cases derivatives of the curvature are not involved. In particular, our results improve the formula in Driver (JMPA **76**, 1997) for logarithmic derivatives of the heat kernel measure on a Riemannian manifold. Our formulas also include the formulas in Elworthy and Li (CRAS **327**, 1998).

Let M be an n-dimensional oriented Riemannian manifold (not necessarily complete) without boundary and E a smooth Hermitian vector bundle over M. Denote by $\Gamma(E)$ the smooth sections of E. Further assume that L is a second order elliptic differential operator on $\Gamma(E)$ whose principle symbol is the dual of the Riemannian metric on M tensored with the identity section of $\mathrm{Hom}(E)$.

Our aim is to develop stochastic calculus formulas for $De^{tL}\alpha$ and $e^{tL}D\alpha$ where $\alpha \in \Gamma(E)$ and D is an appropriately chosen first order differential operator on $\Gamma(E)$.

As an example of the kind of formula found in our paper, let us consider one representative special case. Namely suppose that M is a compact spin manifold, $E = S$ is a spinor bundle over M, D is the Dirac operator on $\Gamma(S)$ and $L = -D^2$. Let scal denote the scalar curvature of M. Then

$$\begin{aligned}\big(e^{-tD^2/2}D\alpha\big)(x) &= \big(De^{-tD^2/2}\alpha\big)(x)\\ &= \frac{1}{t}\mathbb{E}\left[e^{-\frac{1}{8}\int_0^t \mathrm{scal}(X_s(x))ds}\,\gamma_{B_t}//_t^{-1}\alpha(X_t(x))\right],\end{aligned}$$

where $X_t(x)$ is a Brownian motion on M starting at $x \in M$, $//_t$ is stochastic parallel translation along $X_{\bullet}(x)$ in S relative to the spin connection, B_t is a T_xM-valued Brownian motion associated to $X_t(x)$ and γ_{B_t} is the Clifford multiplication of B_t on S_x.

It is also possible to deal with higher order derivatives. For instance, iterating a minor generalization of the previous formula gives a formula for $D^2e^{-tD^2/2}\alpha$. To formulate the simplest case, let $0 < t_1 < t$, then

$$\big(D^2e^{-tD^2/2}\alpha\big)(x) = \frac{1}{t_1(t-t_1)}\mathbb{E}\left[e^{-\frac{1}{8}\int_0^t \mathrm{scal}\,(X_s(x))ds}\,\gamma_{B_{t_1}}\gamma_{B_t - B_{t_1}}//_t^{-1}\,\alpha(X_t(x))\right].$$

Axiomatic Sobolev Spaces on Metric spaces [1]

Marc Troyanov (This is common work with V. Gol'dshtein)

Recent years have seen important activities devoted to geometric analysis on metric spaces. Motivations came from such fields as analysis on singular Riemannian manifolds and rectifiable sets; Carnot-Caratheodory geometries and Hörmander system of vector fields; weighted Sobolev spaces and applications to PDE; graphs and discrete groups, combinatorial Laplacian; analysis on fractal sets; Gromov hyperbolic spaces and their ideal boundaries.

What is particularly interesting is the fact that a number of analytical problems transit from one theory to another one. For instance the notion of p-capacity of a subset F of a metric space X is more or less defined as the infimum $p-$energy $\mathcal{E}_p(u) = \int_X |\nabla u|^p$ among all functions $u : X \to \mathbb{R}$ which vanish at the boundary of X (in some sense) and such that $u \geq 1$ on F. A precise definition can be given in each special case.

We may then consider a number of classical problems such as

1) Prove the existence and uniqueness of an extremal function for the p-capacity $Cap_p(F,X)$.
2) Prove that if $Cap_p(B,X) = 0$ for some ball $B \subset X$, then $Cap_p(F,X) = 0$ for every bounded subset $F \subset X$.
3) Prove that $Cap_p(\,)$ is a Choquet capacity.
4) Give necessary and sufficient geometric or capacitary conditions implying the embedding $W^{1,p}(X) \subset C(X)$.

[1] The paper is available at http://dmawww.epfl.ch/ troyanov/recents.html

A natural program is thus to obtain precise theorems and proofs which holds in *any* reasonable theory.

An important step was made in 1993 when Piotr Hajłasz introduced a notion of Sobolev space $W^{1,p}(X)$ which makes sense on all measured metric spaces. However Hajłasz's definition does not always coincide with more classic Sobolev spaces. In fact it is important to realize that given a measure metric space (X,d,μ), there is in general not one but several natural notions of Sobolev space $W^{1,p}(X)$. The corresponding notions of capacities will also differ.

To fulfill the proposed program, we develop an axiomatic construction of the Sobolev spaces $W^{1,p}(X)$ for any measure metric space X. This construction turns out to be wide enough to cover all known example and yet rich enough so that we can prove significant theorems.

In the metric measure space (X,d,μ), we select a Boolean ring $\mathcal{K}$ of bounded Borel subsets which generates the Borel σ-algebra. (typical examples are the ring of all bounded Borel subsets of X and the ring of all relatively compact Borel subsets if X is locally compact and separable.) The ring $\mathcal{K}$ and the measure μ will be asssumed to satisfy the following conditions:

M1) every ball $B \subset X$ is measurable and $0 < \mu(B) < \infty$ if B has positive radius;

M2) for every $A \in \mathcal{K}$ there exists a finite sequence of open balls $\{B_1, B_2, ...B_m\} \subset \mathcal{K}$ such that $A \subset \cup_{i=1}^m B_i$ and $\mu(B_i \cap B_{i+1}) > 0$ for $1 \leq i < m$.

We then associate (by some way) to each function $u : X \to \mathbb{R}$ a set $D[u]$ of functions called the *pseudo-gradients* of u; intuitively a pseudo-gradient $g \in D[u]$ is a function which exerts some control of the variation of u (for instance in the classical case $X = \mathbb{R}^n$, $D[u] := \{g \in L^1_{loc}(\mathbb{R}^n) : g \geq |\nabla u| \ a.e.\}$). A function u belongs then to $W^{1,p}(X)$ if there exists a pseudo-gradient $g \in D[u] \cap L^p(X)$. Depending on the type of control required the construction yields different versions of Sobolev spaces in metric spaces

The correspondence $u \to D[u]$ is supposed to satisfy the following six axioms:

Axiom A1 (Non triviality) *If $u : X \to \mathbb{R}$ is non negative and k-Lipschitz, then the function $g = k\,sgn(u)$ belongs to $D[u]$.*

Axiom A2 (Upper linearity) *If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$ and $g \geq |\alpha| g_1 + |\beta| g_2$ almost everywhere, then $g \in D[\alpha u_1 + \beta u_2]$.*

Axiom A3 (Leibniz rule) *If $u : X \to \mathbb{R}$ is any measurable function and $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function $g_1(x) = (\sup|\varphi| g(x) + Lip(\varphi)|u(x)|)$ belongs to $D[\varphi u]$.*

Axiom A4 (Lattice property) *Let $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ where $u_1, u_2 \in L^1_{loc}$. If $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$, then $g := \max\{g_1, g_2\} \in D[v] \cap D[u]$.*

Axiom A5 (Completeness) *Let $\{u_i\}$ and $\{g_i\}$ be two sequences of functions such that $g_i \in D[u_i]$ for all i. Assume that $u_i \to u$ in L^p_{loc} topology and $g_i \to g$ in L^p, then $g \in D[u]$.*

We now define the notion of Dirichlet and Sobolev spaces based on Axioms 1–5:

Definitions

i) The $p-$Dirichlet energy of a function u is defined by

$$\mathcal{E}_p(u) = \inf\left\{\int_X g^p d\mu : g \in D\left[u\right]\right\}$$

ii) The *p-Dirichlet space* is the space $\mathcal{L}^{1,p}(X)$ of functions $u \in L^p_{loc}(X)$ with finite p-energy.
iii) The *Sobolev space* is then defined as $W^{1,p}(X) = W^{1,p}(X, d, \mu, D) := \mathcal{L}^{1,p}(X) \cap L^p(X)$.

Theorem *$W^{1,p}(X)$ is a Banach space with norm*

$$\|u\|_{W^{1,p}(X)} = \left(\int_X |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p}.$$

Our final axiom states that if the energy of a function is small, then this function is not far from being constant.
Axiom A6 (Energy controls variation) *Let $\{u_i\} \subset \mathcal{L}^{1,p}(X)$ be a sequence of functions such that $\mathcal{E}_p(u_i) \to 0$. Then for any metric ball B there exists a sequence of constants $a_i = a_i(B)$ such that $\|u_i - a_i\|_{L^p(B)} \to 0$.*

Exterior differentiation on metric measure spaces

Nik Weaver

Our starting point is a definition of "bounded measurable vector field" that makes sense on any metric measure space (a metric space which is equipped with a σ-finite Borel measure).

In the classical case of a Riemannian manifold M, the smooth vector fields on M can be identified with the derivations

$$\delta : C^\infty(M) \to C^\infty(M),$$

that is, those linear maps δ which satisfy $\delta(fg) = f\delta(g) + \delta(f)g$ for all $f, g \in C^\infty(M)$. Such a map can be produced from a smooth vector field X by defining $\delta(f)(p)$ to be the derivative of f at p in the direction $X(p)$. Every derivation of $C^\infty(M)$ arises in this way.

This definition can be modified so as to make sense on a general metric measure space. By Rademacher's celebrated theorem, Lipschitz functions on a Riemannian manifold are differentiable almost everywhere, so that a smooth vector field X also gives rise to a derivation

$$\delta : \mathrm{Lip}(M) \to L^\infty(M).$$

Indeed, for this construction one only needs X to be a bounded measurable vector field. Conversely, every bounded, weak*-continuous derivation from $\mathrm{Lip}(M)$ to $L^\infty(M)$ arises from some bounded measurable vector field. This motivates the following definition:

Definition: Let M be a metric measure space. Then the module of bounded measurable vector fields on M is the set

$$\mathcal{X}(M) = \{\delta : \mathrm{Lip}(M) \to L^\infty(M) |\ \delta \text{ is a bounded, weak*-continuous derivation}\}.$$

Using general functional-analytic techniques, one can then define "tangent" and "cotangent" spaces at almost every point of M, such that $\mathcal{X}(M)$ is identified with the set of bounded measurable sections of the tangent bundle. We also define, for every $f \in \mathrm{Lip}(M)$, an exterior derivative df by the simple formula

$$(df)(\delta) = \delta(f).$$

Then df belongs to the dual module $\mathcal{X}(M)'$, and is identified with a bounded measurable section of the cotangent bundle.

A related constuction has recently been given by Cheeger [2]. That approach requires a doubling condition and a Poincaré inequality, and correspondingly it gives more detailed local structure. But our definition is more general, and makes sense, for instance, in infinite-dimensional spaces. When Cheeger's construction is defined the two agree.

The tangent bundle can be explicitly computed in a large number of examples. In the classical Riemannian case, or more generally on any Lipschitz manifold or rectifiable set, the result is standard; namely, the tangent space at almost every point is $\mathbf{R}^n$ where n is the (Hausdorff) dimension of M. For Carnot-Caratheodory spaces, Hilbert cubes, and Banach manifolds we again produce the expected result.

For those recently introduced exotic metric spaces with non-integral Hausdorff dimension but sufficient rectifiable curves to be geometrically interesting, it is less clear what the tangent space at a point ought to be. In this category are Laakso's tangled Cantor sets [4], the boundaries of hyperbolic buildings studied by Bourdon and Pajot [1], and Hanson and Heinonen's limit spaces [3]. But in all of these cases, the tangent bundle and exterior derivative can be computed using our definition.

Another large class of examples arise from Dirichlet spaces. It has long been known that local Dirichlet forms typically give rise to "intrinsic metrics," and the geometry of these spaces has been investigated by Sturm [6]. In a separate development, Sauvageot [5] bypassed the metric and defined an exterior derivative directly from the Dirichlet form. Our construction, via the intrinsic metric, does not exactly duplicate Sauvageot's, but does always contain it. Examples arising in this context include infinite-dimensional spaces such as Weiner space, where we recover the Gross-Sobolev derivative.

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