MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 21/2000

Affine Algebraic Geometry

14.05.–20.5. 2000

Die Tagung fand unter der Leitung von Hubert Flenner (Bochum), Hanspeter Kraft (Basel) und Peter Russell (Montreal) statt. Die Vorträge bezogen sich auf die Gebiete

(1) Algebraische Varietäten mit einfacher Topologie ($\mathbb{Q}$-azyklische Flächen und exotische affine Räume);
(2) Automorphismen von affinen und verwandten Räumen (G-Vektorbündel, additive Operationen, Charakterisierungen des $\mathbb{A}^n$ via Automorphismengruppen);
(3) Log-algebraische Varietäten, insbesondere log-algebraische Flächen;
(4) Polynomiale Faserungen und Singularitäten im Unendlichen;
(5) Einbettungsprobleme, Kürzungsprobleme und das Komplement von Untervarietäten im projektiven und affinen Raum.

Zu den Gebieten 1-4 fanden 90-minütige Übersichtsvorträge von M. Zaidenberg, H. Kraft, M. Miyanishi und W. Neumann statt. Darüber hinaus gab es 21 Vortrge, die über aktuelle Fortschritte auf diesen Gebieten berichteten. Viele der Teilnehmer bereiteten Berichte ber ihre Ergebnisse vor, die ausgehängt wurden und die Möglichkeit zur Information über die Forschungsinteressen auch denjenigen boten, die keinen Vortrag hielten.

Shreeram Abhyankar

Characteristic sequences and approximate roots

Certain exponents in Newton's fractional power series expansion lead to characteristic sequences, whose importance was recognized by Smith in 1873 and Halphen in 1884. The theory of approximate roots gives a more direct approach to similar things. A combination of both these methods provides an effective tool for studying various questions of affine algebraic geometry such as the epimorphism theorem, the automorphism theorem, and the jacobian conjecture.

To introduce characteristic sequences, given any monic irreducible polynomial $f = f(X,Y)$ of degree n in Y with coefficients in the meromorphic series field $k((X))$ over an algebraically closed ground field k, by Newton's Theorem we can write $f(X^n, Y) = \prod_{1\le i\le n}[Y - z_i(X)]$ where $z_i = z_i(X) = \sum_{j\in\mathbb{Z}} z_{ij}X^j \in k((X))$ with $z_{ij} \in k$. Let $\mathrm{Supp}_X z_i$ be the X-support of z_i, i.e., the set of all integers j for which $z_{ij} \neq 0$, and note that this is independent of i. Let $m_0 = n$. Let $m_1 < \cdots < m_h$ be the sequence of integers augmented by $m_{h+1} = \infty$ and defined by putting $m_1 = \min(\mathrm{Supp}_X z_1)$ and $m_i = \min(\mathrm{Supp}_X z_1 \setminus m_0\mathbb{Z} + \cdots + m_{i-1}\mathbb{Z})$ for $2 \le i \le h+1$. Let $d_{h+2} = \infty$ and for $0 \le i \le h+1$ let $d_i = \mathrm{GCD}(m_0, \ldots, m_{i-1})$. The sequence $m = (m_i)_{0\le i\le h+1}$ is called the newtonian sequence of characteristic exponents of f relative to n, and the sequence $d = (d_i)_{0\le i\le h+2}$ is called the *GCD-sequence* of f. For $i = 0, 1, h+1$ let $q_i = m_i$ and for $2 \le i \le h$ let $q_i = m_i - m_{i-1}$. For $i = 0, h+1$ let $r_i = s_i = q_i$ and for $1 \le i \le h$ let $s_i = q_1d_1 + \cdots + q_id_i$ and $r_i = s_i/d_i$. The sequence $q = (q_i)_{0\le i\le h+1}$ is called the *difference sequence* of f, the sequence $s = (s_i)_{0\le i\le h+1}$ is called the *inner product sequence* of f, and the sequence $r = (r_i)_{0\le i\le h+1}$ is called the *normalized inner product sequence* of f.

The *approximates roots* of f are defined by generalizing the completing the square method of solving quadratic equations put forth by Shreedharacharya in 500 A.D., and it at once leads to the normalized inner product sequence $r = (r_i)_{0\le i\le h+1}$ of f which generates the semigroup of f.

Teruo Asanuma

Structure of $\mathbb{A}^1$-forms of characteristic p > 0

Let k be a field and let $\bar{k}$ be an algebraic closure of k. A commutative k-algebra A is called an $\mathbb{A}^1$*-form* if $A \otimes_k \bar{k}$ is $\bar{k}$-isomorphic to a polynomial ring $\bar{k}[x]$ in one variable. The purpose of the present article is to study the algebraic structure of $\mathbb{A}^1$-forms.

It is well-known that every $\mathbb{A}^1$-form A is purely inseparable, i.e., there exists a finite purely inseparable algebraic extension k'/k such that $A \otimes_k k' \cong k'[x]$, so that it is enough to consider only the case of purely inseparable $\mathbb{A}^1$-forms A of characteristic $p > 0$. We can find an integer $e \ge 0$ such that $A \otimes_k k^{p^{-e}} \cong k^{p^{-e}}[x]$. The smallest s of such integers is called the *height* of A and is denoted $\mathrm{height}\, A = s$.

Let k be a field of characteristic $p > 0$ and let e, ν, λ be positive integers such that $\nu\lambda \equiv 1 \pmod{p^e}$. We define a k-subalgebra B_ν of a polynomial ring $k^{p^{-e}}[x]$ by

$$B_\nu = k[x^{p^e}, u(x)^\nu v(x), u(x)v(x)^\lambda]$$

such that $u(x), v(x) (\in k^{p^{-e}}[x])$ satisfy the condition

$$\nu u(x)'v(x) + u(x)v(x)' = 1,$$

where $u(x)'$ and $v(x)'$ denote the standard derivative by x. We note that B_ν is uniquely determined by $e, \nu, f(x), g(x)$ as a k-algebra.

Theorem. *B_ν is an $\mathbb{A}^1$-form of* $\mathrm{height}\, B_\nu \le e$.

An $\mathbb{A}^1$-form A which is k-isomorphic to some B_ν is called an $\mathbb{A}^1$-form of *B_ν-type.*

Theorem. *Let k be a field of characteristic $p > 2$. Then every $\mathbb{A}^1$-form is of B_2-type.*

The structure of $\mathbb{A}^1$-forms of $\operatorname{ch} k = p = 2$ is quite different from the case of $\operatorname{ch} k = p > 2$. A polynomial $f(x) \in \bar{k}[x]$ is called a *p-polynomial* if $f(x)$ is of the form

$$f(x) = a_0 + x + a_1 x^p + a_2 x^{2p} + \cdots + a_n x^{np}.$$

Let us set

$$S = k[x^{p^e}, f(x)] \ (\subset k^{p^{-e}}[x])$$

for a p-polynomial $f(x) \in k^{p^{-e}}[x]$ and an integer $e \geq 0$. Then S is an $\mathbb{A}^1$-form and an $\mathbb{A}^1$-form which is k-isomorphic to S is called an $\mathbb{A}^1$-form of *p-polynomial type.*

Theorem. *Let k be a field of characteristic $p > 0$ and let A be an $\mathbb{A}^1$-form. Then the following three conditions are equivalent:*

(i) *A is generated by two elements over k, i.e.,* $\operatorname{Spec} A$ *is a plane curve.*
(ii) *$\Omega_k(A) \cong A$, where $\Omega_k(A)$ denotes the differential A-module of A.*
(iii) *A is of p-polynomial type.*
(iv) *A is of B_1-type.*

Tatiana Bandman and Leonid Makar-Limanov
Affine smooth surfaces with $\mathbb{C}^+$-actions

Definition. Let X be an affine variety and let $G(X)$ be the group generated by all $\mathbb{C}^+$-actions on X. Then $AK(X) \subset \mathcal{O}(X)$ is a subring of all the regular $G(X)-$ invariant functions on X.

In this note we are dealing with two problems: 1. To describe the surfaces S with $AK(S) \cong \mathbb{C}$; 2. To find a connection between $AK(S)$ and $AK(S \times \mathbb{C})$. If $AK(S) = \mathbb{C}$, then S is quasihomogeneous and may be obtained from a smooth rational projective surface by deleting a divisor of special form, which is called a "zigzag" ([4], [1]). We denote by $\mathcal{A}$ the set of all such surfaces, and by $\mathcal{H}$ those which have only three components in the zigzag.

Theorem 1. *For a smooth affine surface S with $AK(S) \cong \mathbb{C}$ the following statements are equivalent:*

1. *S is isomorphic to a hypersurface;*
2. *S is isomorphic to a hypersurface $S' = \{(x, y, z) \in \mathbb{C}^3 | xy = p(z)\}$, where p is a polynomial with simple roots only;*
3. *S admits a fixed-point-free $\mathbb{C}^+$-action;*
4. *$S \in \mathcal{H}$.*

If $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{A} \setminus \mathcal{H}$, then $S_1 \times \mathbb{C}^k \not\cong S_2 \times \mathbb{C}^k$ for any $k \in \mathbb{N}$.

To compute $AK(S \times \mathbb{C})$ one has to consider several cases. (A) $AK(S) \cong \mathcal{O}(S)$. Then $AK(S \times \mathbb{C}) \cong AK(S)$. (B) A surface S admits a $\mathbb{C}^+$-action. Then it possible to define a factor $F(S)$, which is an affine curve Γ of genus g with k punctures and n multiple points ([2], [3]). It does not depend on the $\mathbb{C}^+$-action.

It is known, [1] and [3], that any two surfaces admitting fixed-point-free $\mathbb{C}$-actions with the same factor have isomorphic cylinders. There are examples, showing that the fixed-point-free condition is essential.

Theorem 2. *Let S_1, S_2 be smooth affine surfaces such that $S_1 \times \mathbb{C} \cong S_2 \times \mathbb{C}$. Then either $S_1 \cong S_2$ or $AK(S_1) \not\cong \mathcal{O}(S_1), AK(S_2) \not\cong \mathcal{O}(S_2)$, and $F(S_1) \cong F(S_2)$.*

Theorem 3. *If $2g - 2 + k + n \geq 1$ for a factor $F(S)$ of a surface S, then $AK(S \times \mathbb{C}^k) \cong AK(S)$ for any $k \in \mathbb{N}$.*

Conjecture. *If for a smooth affine surface S the factor $F(S)$ is simple, then $AK(S \times \mathbb{C}) \cong \mathbb{C}$.*

[1] J. Bertin: Pinceaux de droites et automorphismes des surfaces affines. J. Reine und Angew. Math. 341, 32–53 (1983).
[2] W. Danielewski: On the cancellation problem and automorphism groups of affine algebraic varieties. Preprint, Warsaw, 1989.
[3] K.-H. Fieseler: On complex affine surfaces with $\mathbb{C}^+$-action. Comment. Math. Helvetici 69, 5–27 (1994).
[4] M. Gizatullin: Invariants of incomplete algebraic surfaces obtained by completions. Math. USSR Izvestiya 5, 503–516 (1971). Quasihomogeneous affine surfaces, Math. USSR Izvestiya 5, 1057–1081 (1971).

Daniel Daigle

Locally nilpotent derivations of affine domains

Let k be a field of characteristic zero. Recall that a k-derivation on a polynomial ring $D : k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_n]$ is (i) *locally nilpotent* if for each $f \in k[X_1, \ldots, X_n]$ there exists a positive integer s such that $D^s(f) = 0$; (ii) *triangular* if $DX_1 \in k$ and $DX_i \in k[X_1, \ldots, X_{i-1}]$ for $2 \leq i \leq n$. All triangular derivations are locally nilpotent. We discuss the following case of Hilbert's Fourteenth Problem:

(H14) *Let $n > 0$ be an integer. Is it the case that, given any locally nilpotent derivation $D : k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_n]$, the k-algebra* $\operatorname{kern}(D)$ *is finitely generated?*

Equivalently: *Does every algebraic action of G_a on $\mathbb{A}^n$ have a finitely generated ring of invariants?*

It is known (see Freudenburg's talk or [4], [1]) that the answer to (H14) is affirmative when $n < 4$ and negative when $n > 4$. In the case $n = 4$, what we currently know is summarized in:

Theorem A (Daigle & Freudenburg [2], [3])**.**

1. *The kernel of any triangular k-derivation of $k[X_1, X_2, X_3, X_4]$ is finitely generated as a k-algebra.*
2. *Given any integer $n \geq 3$, there exists a triangular k-derivation of the ring $k[X_1, X_2, X_3, X_4]$ whose kernel cannot be generated by fewer than n elements.*

In fact, the first part of Theorem A is a corollary of the following related result:

Theorem B (Daigle & Freudenburg [3])**.** *Assume that k is algebraically closed and let R be a Dedekind domain which is finitely generated as a k-algebra, or a localization of such a ring. Then every triangular R-derivation of $R[X, Y, Z]$ has a finitely generated kernel (as an R-algebra).*

We point out that Theorem B becomes false when R is replaced by a polynomial ring in two variables ([1] gives a counterexample). Note that one step in the proof of Theorem B requires a result of A. Sathaye [5], which he obtained as an application of his theory of Generalized Newton–Puiseux Expansions.

[1] D. Daigle and G. Freudenburg: A counterexample to Hilbert's Fourteenth Problem in dimension five. J. Algebra 221, 528–535 (1999).
[2] D. Daigle and G. Freudenburg: A note on triangular derivations of $k[x_1, x_2, x_3, x_4]$. To appear in Proc. Amer. Math. Soc.
[3] D. Daigle and G. Freudenburg: Triangular derivations of $k[x_1, x_2, x_3, x_4]$. Preprint.

[4] G. Freudenburg: A counterexample to Hilbert's Fourteenth Problem in dimension six. Transformation Groups 5, 61–71 (2000).
[5] A. Sathaye: An application of generalized Newton Puiseux Expansions to a conjecture of D. Daigle and G. Freudenburg. Preprint.

Alexandru Dimca
On the topology of polynomial functions: a D-module approach

In my talk I have reported on some joint work with Morihiko Saito on the topology and geometry of polynomials. Let Ω^k denote the global polynomial k-forms on $\mathbb{C}^n$ and d the usual exterior differential. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non constant polynomial. Then we have the following.

Theorem. *The cohomology of the general fiber F of f can be computed purely algebraically in terms of f. More precisely $H^{k+1}(\Omega^*, D_f) = \tilde{H}^k(F, \mathbb{C})$ where $D_f\omega = d\omega + df \wedge \omega$.*

Assume now that $f : \mathbb{C}^n \to \mathbb{C}$ has only isolated singularities and that F is $(n-2)$-homologically connected. Let $G^0 = \Omega^n / df \wedge d\Omega^{n-2}$ denote the algebraic Brieskorn module associated to f. It is a $\mathbb{C}[t]$-module in a natural way and C. Sabbah have shown that G^0 is of finite type over $\mathbb{C}[t]$ iff f satisfies strong regularity conditions at infinity. We have related the topology of the fiber $F_c = f^{-1}(c)$ to the Kernel and Cokernel of the multiplication by $t - c$ on G^0. For details see my web page at: www.math.u-bordeaux.fr.

Gene Freudenburg
Triangular Actions of G_a on $\mathbb{C}^n$ with Non-Finitely Generated Rings of Invariants

To date, every known counterexample to Hilbert's Fourteenth Problem can be realized as the ring of invariants of a triangular action of G_a^m on $\mathbb{C}^n$ for some $m \geq 1$ and $n \geq 5$. For example, Nagata's famous first counterexample is the fixed ring of a linear triangular G_a^{13}-action on $\mathbb{C}^{32}$. More recently, R. Steinberg produced a counterexample as the fixed ring of a linear triangular G_a^6-action on $\mathbb{C}^{18}$. So far, 18 is the smallest dimension for any *linear* counterexample. A family of non-linear counterexamples to H14 was published by P. Roberts in 1990. The importance of these examples lay not only in lowering the dimension of known counterexamples (G_a^1-actions on $\mathbb{C}^7$), but also in providing counterexamples which were relatively simple to describe.

We imitate Roberts' methods to construct further counterexamples in dimension 5 and 6. In general, an algebraic G_a-action on $\mathbb{C}^n$ is obtained by exponentiating a locally nilpotent derivation on the ring of regular functions $\mathcal{O}(\mathbb{C}^n)$. The fixed ring of such an action is the kernel of the corresponding derivation.

Theorem 3. *Let $B = \mathbb{C}[x, y, s, t, u, v]$ be the polynomial ring in 6 variables over $\mathbb{C}$, and let D be the triangular derivation on B defined by*

$$D = (x^3)\frac{\partial}{\partial s} + (y^3 s)\frac{\partial}{\partial t} + (y^3 t)\frac{\partial}{\partial u} + (x^2 y^2)\frac{\partial}{\partial v} \ .$$

Then the kernel of D is not finitely generated as a $\mathbb{C}$-algebra.

Theorem 4 (with D. Daigle). *Let $A = \mathbb{C}[a, b, x, y, z]$ be the polynomial ring in 5 variables over $\mathbb{C}$, and let d be the triangular derivation on A defined by*

$$d = a^2\frac{\partial}{\partial x} + (ax + b)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \ .$$

Then the kernel of d is not finitely generated as a $\mathbb{C}$-algebra.

In fact, d is conjugate to the quotient of D modulo the invariant $(y-1)$.

Marat Gizatullin

The plane curves producing automorphisms of the three-dimensional affine space

Plan of the talk:

1. A parallelism between some additive subgroups of the group $\mathrm{Aut}(\mathbb{P}^2 - C)$ (where C is a plane irreducible curve such that $\dim \mathrm{Aut}(\mathbb{P}^2 - C) = \infty$) (*) and some additive subgroups of $\mathrm{Aut}(\mathbb{A}^3)$.
2. Another characterization of the curves having the property (*).
3. A classification of these curves.
4. A conjectural description of generators of the group $\mathrm{Aut}(\mathbb{A}^3)$.

1. There is a way of transformation of formulas for action of the additive group $\mathbb{G}_a$ on $\mathbb{P}^2 - C$ (where C satisfies (*)) into formulas for action of the additive group on $\mathbb{A}^3$. The algorithm needs verifying of some cancellations, that is some maps written with the help of rational functions can be written in terms of polynomials.

2. We start with a definition.

Definition. Let C be an irreducible reduced algebraic curve on the projective plane. The curve C is said to be *exceptionable* if there exists a birational morphism $\Sigma : X \to \mathbb{P}^2$ such that

(i) the proper preimage $C^* = (\Sigma)^{-1}[C]$ is an exceptional curve of the first kind,
(ii) the reduced full preimage $D =$ supp $(\Sigma^{-1}(C))$ has strong normal crossings,
(iii) the set $C^* \cap (D - C^*)$ is a point $Q \in X$, and the open curve $C^* - Q$ is biregularly isomorphic to the affine line $\mathbb{A}^1$.

Theorem 1. *Let $C \subset \mathbb{P}^2$ be a plane irreducible curve. C has the property (*) if and only if C is exceptionable.*

Theorem 2. *If C is exceptionable and the group $\mathrm{Aut}_0(\mathbb{P}^2 - C)$ is not soluble, then the affine surface $\mathbb{P}^2 - C$ is completable by a zigzag, this surface is quasi-homogeneous with respect to the group $\mathrm{Aut}(\mathbb{P}^2 - C)$, the number $\deg(C)$ belongs to the set of Fibonacci's numbers with odd indices:*

$$\deg(C) \in \{\ 1,\ \ 2,\ \ 5,\ \ 13,\ \ 34,\ \ 89,\ \ 233,\ \ 610, \ldots\ \},$$

every curve of this family is defined by its degree up to a projective transformation. Name of a member of the family is a Fibonacci curve.

3. *A classification.* For every Fibonacci curve C, one can fix a finite set $S(C)$ of special exceptionable curves such that $\#S(\text{line}) = 1$, $\#S(\text{conic}) = 2$, $\#S(C) = 4$, if $\deg(C) > 2$.

Theorem 3. *Let E be an exceptionable curve. Then there exist a Fibonacci curve C, a curve $C_0 \in S(C)$ and an automorphism $g \in \mathrm{Aut}(\mathbb{P}^2 - C)$, such that $E = g(C_0)$.*

4. *The conjecture.* The group $\mathrm{Aut}(\mathbb{A}^3)$ is generated by linear affine transformations and by all the additive subgroups corresponding to the additive subgroups of all the groups $\mathrm{Aut}(\mathbb{P}^2 - E)$, where E runs through the set of all exceptionable curves.

Rajendra Gurjar

Normal surfaces dominated by $\mathbb{P}^2$

We will discuss some results proved in collaboration with C. Pradeep and D.-Q. Zhang. The main result is as follows.

Theorem. *Let V be a Gorenstein normal surface such that there is a non-constant morphism $f : \mathbb{P}^2 \to V$. Then V is isomorphic to a quotient $\mathbb{P}^2/G$, where G is a finite group of automorphisms of $\mathbb{P}^2$. We can, in fact, describe all such surfaces. If the fundamental group of $V - \operatorname{Sing} V$ is trivial then V is either $\mathbb{P}^2$, the quadric cone $Q := \{X^2 + Y^2 + Z^2 = 0\} \subset \mathbb{P}^3$, or a surface with exactly two singular points of Dynkin type A_1 an d A_2. If the fundamental group of $V - \operatorname{Sing} V$ is not trivial then V is a quotient of $\mathbb{P}^2$ or Q such that the quotient map is unramified over the smooth locus of V.*

In our proof we use some very general properties of Gorenstein log del Pezzo surfaces of rank 1. Except for this, our proof is self-contained. We also give a proof of the complete classification of Gorenstein log del Pezzo surfaces of rank 1 whose smooth locus is simply-connected. Our proof of the main result suggests a natural conjecture about normal surfaces (which are not necessarily Gorenstein) dominated by $\mathbb{P}^2$.

Shulim Kaliman

Polynomials with general $\mathbb{C}^2$–fibers are variables.

Suppose that X' is a smooth affine algebraic variety of dimension 3 with $H_3(X') = 0$ which is a UFD and whose invertible functions are constants. Suppose that Z is a Zariski open subset of X' which has a morphism $p : Z \to U$ into a curve U such that all fibers of p are isomorphic to $\mathbb{C}^2$. We prove that X' is isomorphic to $\mathbb{C}^3$ iff none of irreducible components of $X' \setminus Z$ has non-isolated singularities. Furthermore, if X' is $\mathbb{C}^3$ then p extends to a polynomial on $\mathbb{C}^3$ which is linear in a suitable coordinate system. As a consequence we obtain the fact formulated in the title of the paper.

Hideo Kojima

Affine surfaces with logarithmic Kodaira dimension zero

Let S be a smooth affine surface defined over an algebraically closed field k of $\operatorname{char}(k) \geq 0$ and let (X, B) be an SNC-completion of S. Let (W, C) be an almost minimal model of (X, B). By contracting (-1)-curves E with $(E \cdot C) \leq 1$ successively, we obtain a birational morphism $\mu : W \to V$ such that $(F \cdot \mu_*(C)) > 1$ for any (-1)-curve F on W. Put $D := \mu_*(C)$ and $S' := V - \operatorname{Supp} D$. Then S' is an affine open subset of S and $\overline{P}_n(S') = \overline{P}_n(S)$ for any $n > 0$. We call S' a *strongly minimal model* of S. A nonsingular affine surface S is said to be *strongly minimal* if there exists a strongly minimal model S' of S such that $S' = S$. We obtain the following result.

Theorem 1. *Let $S = \operatorname{Spec} A$ be a strongly minimal smooth affine surface with $\overline{\kappa}(S) = 0$. Then S is one of the surfaces in the following table, where $m(S)$ is the least positive integer such that $\overline{P}_m(S) > 0$.*

Type	$m(S)$	$\mathrm{Pic}(S)$	$\mathrm{rank}_{\mathbb{Z}} A^*/k^*$	$\pi_1(S)$ $(k = \mathbb{C})$
$(*)$	1	$\mathbb{Z}/(3)$	0	$\mathbb{Z}/(3)$
$*(8)$	1	$\mathbb{Z} \oplus \mathbb{Z}/(2)$	0	$\mathbb{Z}/(2)$
$O(8)$	1	$\mathbb{Z} \oplus \mathbb{Z}/(2)$	0	$\mathbb{Z}/(2)$
$O(n+4, -n)(n \geq 0)$	1	$\mathbb{Z}/(n+2)$	0	$\mathbb{Z}/(n+2)$
$O(4,1)$	1	0	1	$\mathbb{Z}$
$O(2,2)$	1	$\mathbb{Z}$	1	$\mathbb{Z}$
$O(1,1,1)$	1	0	2	$\mathbb{Z}^2$
$X[2]$	2	$\mathbb{Z} \oplus \mathbb{Z}/(4)$	0	$\mathbb{Z}/(4)$
$H[0,0]$	2	$\mathbb{Z}$	1	$\mathbb{Z}$
$H[n,-n](n \geq 1)$	2	$\mathbb{Z}/(4n)$	0	$\mathbb{Z}/(4n)$
$H[-1,0,-1]$	2	$\mathbb{Z}/(2)$	1	$< y, t > /(yty^{-1}t)$
$Y\{3,3,3\}$	3	$\mathbb{Z}/(9)$	0	$\mathbb{Z}/(9)$
$Y\{2,4,4\}$	4	$\mathbb{Z}/(8)$	0	$\mathbb{Z}/(8)$
$Y\{2,3,6\}$	6	$\mathbb{Z}/(6)$	0	$\mathbb{Z}/(6)$

By Theorem 1, we obtain the following results.

Theorem 2. *Let $B \subset \mathbb{P}^2$ be a reduced projective plane curve defined over $\mathbb{C}$ such that $\overline{\kappa}(\mathbb{P}^2 - B) = 0$. Then $\overline{P}_g(\mathbb{P}^2 - B) = 1$. In particular, $\pi_1(\mathbb{P}^2 - B)$ is abelian.*

Theorem 3. *Let $T = \mathrm{Spec}\, A$ be a normal affine surface defined over k. Assume that $\overline{\kappa}(T) = 0$. Then T is a rational surface. Further, $\mathrm{rank}_{\mathbb{Z}} A^*/k^* \leq 2$ and the equality holds if and only if $T \cong \mathbb{A}^1_* \times \mathbb{A}^1_*$.*

Mariusz Koras and Peter Russell

Contractible surfaces with quotient singularities

We proved the following :

Theorem. *Let S be a normal, contractible, affine surface with only quotient singularities. Assume that the logarithmic Kodaira dimension $\overline{\kappa}(S) = -\infty$. Then $\overline{\kappa}(S - \mathrm{Sing}(S)) = -\infty$, and either*

(i) *S is isomorphic to a quotient $\mathbb{C}^2//G$ where G is a finite group; or*
(ii) *$S - \mathrm{Sing}(S)$ is affine ruled, all singularities are cyclic, $\#(\mathrm{Sing}\, S) > 1$. In this case the structure of S can be described.*

The theorem is a generalization of the theorem we proved in [1]:

Theorem. *For any algebraic action of $\mathbb{C}^*$ on $\mathbb{C}^3$ the quotient $\mathbb{C}^3//\mathbb{C}^*$ is isomorphic to $\mathbb{C}^2//\omega_a$, where ω_a is a cyclic group.*

In the proof we consider 4 cases according to value of $\overline{\kappa}$: $-\infty$, 0 or 1, 2. The hardest case is the last one. In this case we prove that in "generic" case there exists a curve L in S such that $S - L$ is smooth, L is isomorphic to a line $\mathbb{C}^1$. But this is contradiction since $S - L$ has the Euler characteristic 0 and the logarithmic Kodaira dimension $-\infty$. It is impossible by Sakai, Kobayashi inequality.

[1] M. Koras, P. Russell: $\mathbb{C}^*$-actions on $\mathbb{C}^3$: the smooth locus is not of hyperbolic type. Journal of Alg. Geometry, 603–694 (1999).

Hanspeter Kraft

Automorphisms of affine n-space

The group $\mathrm{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of affine n-space $\mathbb{A}^n$ is still a challenging object for research, even for $n = 2$ where we have Structure Theorem (due to van der Kulk) saying that $\mathrm{Aut}(\mathbb{A}^2)$ is an amalgamated product of the subgroup of affine transformations and the so-called Jonquière subgroup. This result has a number of interesting consequences for algebraic group actions on affine 2-space $\mathbb{A}^2$, e. g. that every reductive group action on $\mathbb{A}^2$ is linearizable (Kambayashi, 1979). The question was raised whether every reductive group action on affine n-space is linearizable.

After a few positive results by Popov-Kraft and Panyushev for semi-simple group actions in dimension 3 and 4 (in 1984/85) Schwarz discovered in 1989 the first counter-examples, namely non-linearizable action of $\mathcal{O}(2)$ on $\mathbb{A}^4$ and of $\mathrm{SL}(2)$ on $\mathbb{A}^7$. These examples appeared in the joint work of Kraft and Schwarz on reductive group actions with one-dimensional quotient, following a strategy suggested by Luna. The positive results are:

Theorem 1 (Kraft-Schwarz). *Assume that a reductive group G acts on affine n-space $\mathbb{A}^n$ such that $\dim \mathbb{A}^n /\!\!/ G \leq 1$. Then this action is linearizable in the following cases.*

(1) *G is a torus;*
(2) *G^0 is a simple group;*
(3) *$n \leq 3$;*
(4) *The action is semifree (i.e. the generic orbit is closed and isomorphic to G);*
(5) *The tangent representation in a fixed point is selfdual.*

In case of torus action Koras and Russell recently proved a long-standing conjecture, using very interesting new ideas and methods:

Theorem 2 (Koras-Russell). *Every $\mathbb{C}^*$-action on $\mathbb{A}^3$ is linearizable.*

Using the results of Kraft-Schwarz and Theorem 2 one obtains the following:

Corollary. 1. *Assume that G is reductive, non-finite and that G is connected in case $G^0 \simeq \mathbb{C}^*$. Then every G-action on $\mathbb{A}^3$ is linearizable.*
2. *Assume that G is reductive, non-finite and that G^0 is not a torus. Then every G-action on $\mathbb{A}^4$ is linearizable.*

The counter-examples are based on ideas of Bass and Haboush who studied G-vector bundles over representation spaces. It turned out that there exist non-trivial G-vector bundles and that the underlying G-action is non-linearizable. Masuda, Moser-Jauslin and Petrie introduced a new method to construct non-trivial vector bundles and were able to produce the first families of non-linearizable actions and also non-linearizable actions for certain finite groups. On the other hand the proved the following result:

Theorem 3 (Masuda-Moser-Jauslin-Petrie). *If G is commutative reductive group, then every G-vector bundle over a representation space is trivial.*

There are no non-linearizable actions of a commutative reductive group known so far!

The construction of counter-examples via non-trivial G-vector bundles has as a consequence that all these actions are *holomorphically* linearizable. However, there is a very nice idea due to Asanuma, using non-rectifiable embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ which finally lead to the following result:

Theorem 4 (Derksen-Kutzschebauch). *For every reductive complex Lie group G there are faithful non-linearizable actions on $\mathbb{A}^n$ for all $n \geq N_G$.*

There are more results in other settings, i.e. in positive characteristics (Asanuma), for real algebraic actions (Asanuma), for special actions (Jurkievich), for Jonquière-actions (Kraft-Kutzschebauch), for actions on quadrics (Doebeli, M. Masuda) and, of course, in the $\mathcal{C}^\infty$-setting.

Frank Kutzschebauch

Holomorphic automorphisms of $\mathbb{C}^n$

Our talk was concentrated around the following two problems:

Holomorphic Linearization Problem: *Let G be a complex reductive group acting holomorphically on $\mathbb{C}^n$. Can one conjugate the resulting subgroup of the holomorphic automorphism group of $\mathbb{C}^n$ by a single automorphism into the general linear group $GL_n(\mathbb{C}) \subset \mathrm{Aut}_{hol}(\mathbb{C}^n)$, i.e., is every holomorphic action of a reductive group on $\mathbb{C}^n$ linearizable?*

Holomorphic Embedding Problem: *How many equivalence classes of proper holomorphic embeddings of $\mathbb{C}^k$ into $\mathbb{C}^n$ $(0 < k < n)$ do exist?*

Here we call two proper holomorphic embeddings $\phi_{1,2} : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ equivalent if there are two holomorphic automorphisms α of $\mathbb{C}^n$ and β of $\mathbb{C}^k$ such that $\phi_1 \circ \beta(z) = \alpha \circ \phi_2(z) \quad \forall z \in \mathbb{C}^k$.

In the talk we explained the method of Asanuma (which works in any category) to construct counterexamples to the holomorphic linearization problem using embeddings non-equivalent to the standard embedding. Using this method and the existence of non-standard holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ we were able to prove the following theorem (see [1]):

Theorem. *Let G be a complex reductive group (not the trivial group). Then there exists an N such that for all $n \geq N$ there is a nonlinearizable effective action of G on $\mathbb{C}^n$.*

Together with an overview over the known results about holomorphic embeddings we illustrated the proof of the above theorem in the cases $G = \mathbb{C}^*$ and $G = \mathbb{Z}/2\mathbb{Z}$. Also we announced the following result proving the existence of families of embeddings leading to families of non-conjugate group actions.

Theorem. *There is a holomorphic map $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^3$ such that for each fixed $w \in \mathbb{C}$ the map $F(w, \cdot) : \mathbb{C} \to \mathbb{C}^3$ is a proper holomorphic embedding and for different w those embeddings are non-equivalent.*

[1] H. Derksen, F. Kutzschebauch: Nonlinearizable holomorphic group actions. Math. Ann. 311, 41–53 (1998).

Vladimir Lin

Holomorphic self-mappings of non-degenerate binary forms

We study holomorphic self-mappings $F\colon \mathcal{F}_n \to \mathcal{F}_n$ of the non-singular affine algebraic variety $\mathcal{F}_n$ consisting of all complex projective binary forms $f(x,y) = z_0x^n + z_1x^{n-1}y + \cdots + z_{n-1}xy^{n-1} + z_ny^n$ with non-zero discriminant (*projective* means that we identify proportional forms). There is the natural one-to-one correspondence $\mathcal{F}_n \ni f \leftrightarrow Z_f \subset \mathbb{CP}^1$, where $Z_f = \{[x : y] \in \mathbb{CP}^1 \mid f(x,y) = 0\}$ is the zero set of f, $\# Z_f = n$.

Definition. A map $F\colon \mathcal{F}_n \to \mathcal{F}_n$ is said to be *tame* if there exists a holomorphic map $T\colon \mathcal{F}_n \to \mathrm{Aut}\,\mathbb{CP}^1 = \mathrm{SL}\,(2, \mathbb{C})/\{\pm I\}$ such that $Z_{F(f)} = T(f)(Z_f)$ for each $f \in \mathcal{F}_n$. For instance, any holomorphic matrix function $A = (a_{ij})_{i,j=1}^2 \in \mathrm{SL}(2, \mathcal{O}(\mathcal{F}_n))$ determines the tame map $\mathcal{F}_n \ni f \mapsto f^A \in \mathcal{F}_n$, where the form f^A is obtained from f by the unimodular linear change of variables x, y with the *variable* coefficients $a_{ij}(f)$ depending holomorphically on f itself, i. e., $f^A(x,y) = f(a_{11}(f)\,x + a_{12}(f)\,y,\, a_{21}(f)\,x + a_{22}(f)\,y)$.

A continuous map $F\colon X \to Y$ of arcwise connected topological spaces is called *cyclic* if the image $F_*(\pi_1(X))$ of the induced homomorphism $F_*\colon \pi_1(X) \to \pi_1(Y)$ of the fundamental groups is a cyclic group.

Theorem. *For $n > 4$ all non-cyclic holomorphic mappings $\mathcal{F}_n \to \mathcal{F}_n$ are tame.*

Remarks. Every $\mathcal{F}_n$ ($n \geq 2$) admits non-tame *cyclic* non-constant regular self-mappings; however for $n \geq 3$ all such mappings are, in a sense, degenerate. The fundamental group $\pi_1(\mathcal{F}_n)$ is known to be the sphere braid group $B_n(S^2)$ on n strings. $B_2(S^2) \cong \mathbb{Z}_2$; $B_n(S^2)$ is non-commutative for $n \geq 3$ and infinite for $n \geq 4$. V. Danilov's and M. Gizatullin's results on biregular automorphisms of the complement $\mathbb{CP}^2 \setminus C$ of a non-singular conic $C \subset \mathbb{CP}^2$ imply that every biregular automorphism of $\mathcal{F}_2 \cong \mathbb{CP}^2 \setminus C$ is tame (and certainly cyclic, for all continuous self-mappings of $\mathcal{F}_2$ are such). For $n \geq 3$ every homeomorphism of $\mathcal{F}_n$ is non-cyclic. I have learned from M. Gizatullin amazing Gotthold Eisenstein's example (1844) of a *non-tame* involutive biregular automorphism of the space $\mathcal{F}_3$. Seemingly, $\mathcal{F}_4$ admits non-tame non-cyclic regular self-mappings (to my best knowledge no examples of such kind are known).

Leonid Makar-Limanov

Locally nilpotent derivations of affine domains

We show that any locally nilpotent derivation of an affine domain is equivalent to a restriction of a Jacobian type derivation of a polynomial ring.

Let $\mathbb{C}$ be the field of complex numbers and let $\mathbb{C}_n$ be the ring of polynomials over $\mathbb{C}$ in n variables. Any $n-1$ elements $f_1, ..., f_{n-1}$ of $\mathbb{C}_n$ determine the Jacobian derivation $\epsilon_{f_1,...,f_{n-1}}$ where $\epsilon_{f_1,...,f_{n-1}}(g)$ is equal to the determinant $J(f_1, ..., f_{n-1}, g)$ of the Jacobi matrix of the elements $f_1, ..., f_{n-1}, g$.

Let I be a prime ideal of $\mathbb{C}_n$, let $R = \mathbb{C}_n/I$, and let ∂ be a locally nilpotent derivation of R. Let us denote by π the projection of $\mathbb{C}_n$ on R. Finally let us recall that two locally nilpotent derivations are called equivalent if their kernels coincide. Then the following is true. It is possible to find elements $P_i \in I$ and $r_i \in \mathbb{C}_n$ so that the derivation $\pi(\epsilon(r))$, where $\epsilon(r)$ is the Jacobian derivation $J(P_1, \ldots, P_k, r_{k+1}, ..., r_{n-1}, r)$, is defined on R and is equivalent to ∂.

Kayo Masuda and Masayoshi Miyanishi

Étale endomorphisms of algebraic surfaces with G_m-actions

Let X be an $\mathbb{A}^1_*$-fiber space, namely, a smooth algebraic surface defined over the complex field $\mathbb{C}$, endowed with a surjective morphism $\rho : X \to B$ such that B is a smooth projective curve and the general fiber of ρ is isomorphic to $\mathbb{A}^1_* = \mathbb{C}^*$. The famous Jacobian Conjecture is equivalent to ask whether or not an étale endomorphism φ of the affine plane $\mathbb{A}^2$ is an automorphism. We generalize this as follows:

Generalized Jacobian Problem. *Let X be an $\mathbb{A}^1_*$-fiber space and let $\varphi : X \to X$ be an étale endomorphism. Then is φ an automorphism?*

We consider the generalized Jacobian problem for the following three $\mathbb{A}^1_*$-fiber spaces:

(1) Platonic $\mathbb{A}^1_*$-fiber spaces,
(2) A weighted hypersurface $x_1^{m_1} + x_2^{m_2} + x_3^{m_3} = 0$ in the affine 3-space $\mathbb{A}^3$ with its singular point $(0,0,0)$ deleted off,
(3) An affine algebraic surface with an unmixed G_m-action and its unique fixpoint deleted off.

Of particular interest is the case of Platonic $\mathbb{A}^1_*$-fiber spaces. A Platonic $\mathbb{A}^1_*$-fiber space is written as $\mathbb{A}^2/G$ with the unique singular point deleted off, where G is a non-cyclic small finite subgroup of $\mathrm{GL}(2,\mathbb{C})$, and the generalized Jacobian problem is equivalent to asking whether or not an étale endomorphism $\psi : \mathbb{A}^2 \to \mathbb{A}^2$ is an automorphism whenever ψ commutes with a linear action of the finite group G on $\mathbb{A}^2$. The problem in the Platonic case is unsolved at present, however, we can show that φ is an automorphism in most cases above. But, the generalized Jacobian problem does not necessarily have a positive answer. In fact, there exists a counterexample: we can construct an $\mathbb{A}^1_*$-fiber space X and an étale endomorphism φ which preserves the $\mathbb{A}^1_*$-fibration but is not an automorphism.

Masayoshi Miyanishi
Log algebraic surfaces

A *log algebraic surface* (or more precisely, *log projective surface*) is a pair $(\overline{V}, \overline{\Delta})$ of a normal projective surface $\overline{V}$ and a reduced effective (Weil) divisor $\overline{\Delta}$ on $\overline{V}$ satisfying the following conditions:

(1) $K_{\overline{V}} + \overline{\Delta}$ is a $\mathbb{Q}$-Cartier divisor.
(2) If $f : V \to \overline{V}$ is the minimal resolution of singularities, then the proper transform Δ of $\overline{\Delta}$ is a divisor with simple normal crossings and

$$K_V + \Delta = f^*(K_{\overline{V}} + \overline{\Delta}) + \sum_{j=1}^{n} a_j E_j$$

with $a_j \in \mathbb{Q}$ and $0 \geq a_j > -1$, where $\{E_j\}_{1 \leq j \leq n}$ is the set of irreducible exceptional curves of f.

Let $\Delta = \sum_{i=1}^{r} C_i$ be the irreducible decomposition and let $D = \sum_{i=1}^{r} C_i + \sum_{j=1}^{n} E_j$. Then the above conditions are equivalent to the following conditions:

(i) D is a divisor with simple normal crossings.
(ii) $\overline{V}$ has only quotient singularities.
(iii) If $\overline{V}$ has a singular point on $\overline{\Delta}$, say P, then the dual graph of exceptional curves of a minimal resolution of $(\overline{V}, P)$ is a linear chain such that Δ meets only one of the end components of the linear chain, the intersection being at a single point and transverse.

We call $f : (V, \Delta) \to (\overline{V}, \overline{\Delta})$ the *minimal resolution* of $(\overline{V}, \overline{\Delta})$. An irreducible curve $\overline{C}$ on a log projective surface $(\overline{V}, \overline{\Delta})$ is called a *log exceptional curve of the first kind* if $(K_{\overline{V}} + \overline{\Delta} \cdot \overline{C}) < 0$ and $(\overline{C})^2 < 0$. This is an analogy of an exceptional curve of the first kind, i.e., (-1) curve, on a nonsingular projective surface. A log projective surface $(\overline{V}, \overline{\Delta})$ is called *relatively minimal* if there are no log exceptional curves of the first kind on $\overline{V}$.

Given a log projective surface $(\overline{V}, \overline{\Delta})$, there exists a birational morphism $\overline{\mu} : \overline{V} \to \overline{W}$ onto a normal projective surface $\overline{W}$ such that

(1) $(\overline{W}, \overline{\Gamma})$ is a log projective surface, where $\overline{\Gamma} = \overline{\mu}_*(\overline{\Delta})$;
(2) $(\overline{W}, \overline{\Gamma})$ is relatively minimal.

The construction of relatively minimal models is done by the theory of peeling applied to the minimal resolution (V, D). In the lecture, we state structure theorems, especially in the case of log Kodaira dimension $-\infty$, for relatively minimal log projective surfaces.

Lucy Moser-Jauslin
Relative cohomology of polynomial maps from $\mathbb{C}^n$ to $\mathbb{C}^q$

(On the work of Ph. Bonnet, thesis student of L. Moser-Jauslin and R. Moussu at the Université de Bourgogne.) Consider the following theorm due to Gavrilov:

Theorem (Gavrilov 1998). *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a non constant polynomial such that all fibers are connected and having only isolated singularities. Suppose ω is a polynomial 1-form on $\mathbb{C}^2$ which is relatively exact (i.e., it is exact on all generic fibers). Then ω is of the form $dR + Qdf$ for some polynomials R and Q.*

Bonnet (1999) gave an algebraic proof of this result. This proof allows one to generalize to a result when there are non-isolated singularities as well. Bonnet and Dimca then studied this result by giving the module structure of the first cohomology group of a truncated de Rham complex. A generalization to higher dimensions is given by

Theorem. *Suppose $f : \mathbb{C}^n \to \mathbb{C}^q$ is a polynomial application such that*
(i) *The complement of the image is of codimension ≥ 3 in $\mathbb{C}^q$;*
(ii) codim$\{\lambda \in \mathbb{C}^q | f^{-1}(\lambda)$ *is disconnected*$\} \geq 2$;
(iii) *the codimension of the singular set ≥ 2 in $\mathbb{C}^n$.*
Then any relatively exact form ω is of the form $\omega = dR + \sum_{i=1}^q h_i df_i$, where $f = (f_1, \dots, f_q)$.

A consequence of this result is: If f satisfies the hypotheses of the theorem and if there exist $n-q$ relatively exact 1-forms ω_i such that $\omega_1 \wedge \ldots \wedge \omega_{n-q} \wedge df_1 \wedge \ldots \wedge df_q = dx_1 \wedge \ldots \wedge dx_n$, then there exist polynomials $R_1, \dots, R_{n-q}$ such that $dR_1 \wedge \ldots \wedge dR_{n-q} \wedge df_1 \wedge \ldots \wedge df_q = dx_1 \wedge \ldots \wedge dx_n$.

András Némethi
The monodromy representation of polynomial maps (the Thom–Sebastiani construction)

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial map. If $\Lambda_f = \{c_1, \dots, c_s\}$ denotes the set of bifurcation points, then $f : f^{-1}(\mathbb{C} \setminus \Lambda_f) \to \mathbb{C} \setminus \Lambda_f$ is a locally trivial fibration. Fix $c_0 \in \mathbb{C} \setminus \Lambda_f$ and set $\mathbf{F} = f^{-1}(c_0)$. Our main object is the homological monodromy representation

$$\rho_f : \pi_1(\mathbb{C} \setminus \Lambda, c_0) \to Aut(H_*(\mathbf{F})).$$

If we fix a star of f with endpoints $\{c_1, \dots, c_s\}$ and basepoint c_0, then this provides in a natural way a generator set $\{\gamma_1, \dots, \gamma_s\}$ for $\pi_1(\mathbb{C} \setminus \Lambda_f, c_0)$ and also a direct sum decomposition $H_*(\mathbf{F}) = \oplus_i V_*(c_i)$, where $V_*(c_i)$ denotes the group of vanishing cycles associated with the non-regular value $c_i \in \Lambda_f$. With respect to this decomposition, $\rho_f(\gamma_i)$ has the form:

$$\rho_f(\gamma_i) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ m_{i1} & \cdots & m_{ii} & \cdots & m_{is} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad (*)$$

The monodromy at infinity $\rho_f(\gamma_s) \circ \cdots \circ \rho_f(\gamma_1)$ is denoted by $m_\infty(f)$.

If we have two polynomials $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^m \to \mathbb{C}$, then one can consider $h = f \oplus g : \mathbb{C}^{n+m} \to \mathbb{C}$ defined by $h(x, y) = f(x) + g(y)$. If $\mathbf{H}$, $\mathbf{F}$, $\mathbf{G}$ are the generic fibers of h, f, g respectively, then using an earlier result of the author, we have:

a) $\Lambda_h \subset \Lambda_f + \Lambda_g$,
b) $\mathbf{H} = \mathbf{F} * \mathbf{G}$ (homotopy type), in particular (over a field)

$$H_{q+1}(\mathbf{H}) = \oplus_{r+s=q} H_r(\mathbf{F}) \otimes H_s(\mathbf{G}).$$

(We will use the notation $\otimes$ for this graded tensor product.)
c) $m_\infty(h) = m_\infty(f) \otimes m_\infty(g)$.

In this talk we discussed how one can recover ρ_h from ρ_f and ρ_g. For example, if $\Lambda_g = \{0\}$, hence $\Lambda_h = \Lambda_f$, and we fix a tautologically identical pair of stars for f and h, then with respect to these stars (and with the notation $(*)$ of $\rho_f(\gamma_i)$):

$$\rho_h(\gamma_i) = \begin{pmatrix} 1\otimes 1 & & & & & \\ & \ddots & & & & \\ & & 1\otimes 1 & & & \\ m_{i1}\otimes 1 & \cdots & m_{i,i-1}\otimes 1 & m_{ii}\otimes m & \cdots & m_{is}\otimes m \\ & & & & \ddots & \\ & & & & & 1\otimes 1 \end{pmatrix},$$

where m is the monodromy operator of g $(= \rho_g(\text{positive generator of } \pi_1(\mathbb{C}\setminus\Lambda_g)))$. This result is obtained in a joint work with A. Dimca.

WALTER NEUMANN
Polynomial fibrations and singularities at infinity

This talk surveyed results on the topology of polynomial maps $f\colon \mathbb{C}^n \to \mathbb{C}$. Most of the results apply if $\mathbb{C}^n$ is replaced by a homologically acyclic variety. A convenient reference for most of the content of the talk is the paper "Unfolding singularities at infinity" by Paul Norbury and the speaker, which will appear in Math. Annalen.

The talk was in three sections. The first part described results valid with no extra assumptions. The underlying result is due to Broughton in the 1980's, who described a splitting of the homology of a regular (i.e., generic) fiber of f as the sum of the groups of vanishing cycles for the irregular fibers. This splitting has a strong relationship with monodromy, that has been explored by several authors. An attractive consequence of this has been observed by Dimca and Nemethi: the splitting together with the monodromy at infinity together determine the complete homological monodromy.

If f has only isolated singularities then more can be said. There is a natural concept of "Milnor Fiber at infinity" at a point at infinity where a fiber is irregular. I think this concept appears first in Suzuki's paper proving the Abhyankar-Moh-Suzuki theorem. The Broughton splitting of (co)homology refines to a sum of groups $H_*(F, \partial F)$, summed over all Milnor fibers F for f (both Milnor fibers at infinity and Milnor fibers of singularities of fibers). Again, this splitting connects with monodromy. In addition, it relates to the "Seifert linking form at infinity" in a way that is useful for computation.

Finally, when $n = 2$, the Milnor fibers at infinity together with the local monodromy that appears there can be read off from the splice diagrams at infinity for irregular fibers in a way that is similar to the description of the local topology of plane curve singularities by splice diagrams. But a complete description of the monodromy (and hence of the global topology) of $f\colon \mathbb{C}^2 \to \mathbb{C}$ still needs an efficient way of relating the descriptions of the monodromies around each irregular fiber to a fixed reference fiber. We illustrated this by the example of for the Briançon polynomial. In this case, since there are two irregular fibers, computing the complete geometric monodromy reduces to the problem of solving an equation in the mapping class group of the fiber of the form $h_1h_2 = h_\infty$, where the conjugacy classes of h_1, h_2, h_∞ are described explicitly. We have a solution to this equation which is conjecturally the only solution, and therefore the solution that actually describes the global topology for Briançon's polynomial.

Stepan Orevkov
What is the Milnor fiber for nonisolated singularities?

Peter Russell
Some formal aspects of the theorems of Mumford–Ramanujam

One can formally define the "fundamental group" $\pi(\Gamma)$ of a weighted tree, e.g. of

$*$ the dual graph Γ_D of a normal crossing divisor without circuits
and all components isomorphic to P^1 on a non-singular algebraic surface X

by using the presentation given by Mumford. $\pi(\Gamma)$ is invariant under formal blowing up and down of Γ (corresponding to blowing up and down on X in $(*)$), and under a dual formal anti blowing up and anti blowing down. (This operation has a geometric meaning for 4-manifolds, but not for algebraic surfaces. It is nevertheless quite useful in the formal approach.)

- We give a fairly explicit description of normal forms for trees with $\pi(\Gamma) = 1$, allowing both blowing up and down and anti blowing up and down.
- We give a quite precise classification, allowing blowing up and down only, of trees with $\pi(\Gamma) = 1$ and such that (the intersection form of) Γ satisfies the Hodge index condition.

We recuperate

Mumford's theorem: *If $\pi(\Gamma) = 1$ and Γ is negative definite, then Γ blows down to the empty tree.*

On an algebraic surface, the Hodge index theorem holds, and 0-curves move in a pencil. Hence:

- If $\Gamma = \Gamma_D$ is as in $(*)$, then Γ satisfies the Hodge index condition and (Ramanujam): If v in Γ has non-negative weight, then any branching vertex in Γ with weight -1 is a neighbour of v.

As an immediate consequence we obtain

Ramanujam's theorem: *If $\Gamma = \Gamma_D$ is as in $(*)$, connected, not negative definite and with $\pi(\Gamma) = 1$, then Γ is equivalent, under blowing up and blowing down, to the tree with a single vertex of weight 1.*

The following results are now valid in arbitrary characteristic.

Theorem: *With X and D as in $(*)$, put $S = X \backslash D$ and suppose $\pi(\Gamma_D) = 1$. If S is affine, or if $\operatorname{Pic} S = 0$ and D is connected, then $S \simeq \mathbb{A}^2$.*

Over $\mathbb{C}, \pi(\Gamma_D)$ is the fundamental group at infinity of S (Mumford). It can be shown in general that $\pi(\Gamma)$ is residually finite. Hence $\pi(\Gamma)$ could throughout be replaced by its profinite completion $\hat{\pi}(\Gamma)$. $\hat{\pi}(\Gamma_D)$ can be interpreted geometrically in some cases as an algebraic fundamental group by results of Grothendieck–Murre.

Fumio Sakai

Defining equations of rational cuspidal curves with one place at infinity (after K. Tono)

I reported on recent results of my student Keita Tono. Let $C \subset \mathbb{P}^2$ be a rational cuspidal plane curve. We say that C has *one place at infinity* if there is a line ℓ meeting C at one point P. Such a curve C is of *AMS type* (resp. *LZ type*), if $C \setminus P$ is smooth (resp. if $C \setminus P$ is singular). Let us introduce a polynomial affine transformation:

$$\tau_a : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y + \sum_{i=1}^{k} a_i x^i \\ x \end{pmatrix}$$

Theorem 1. *Let C be a rational cuspidal plane curve with one place at infinity. Then, up to the projective equivalence, C is defined by the equation:*

$$F \circ \tau_{a_1} \circ \cdots \circ \tau_{a_s},$$

where (i) $F = x$ (if C is of AMS type) and $F = y^n + x^m$ (if C is of LZ type) with relatively prime integers n, m such that $n > m \geq 2$, (ii) $s \geq 0$, (iii) $\deg(\tau_{a_i}) \geq 2$ except possibly if $i = 1$ and if C is of LZ type.

Set $k_i = \deg(\tau_{a_i})$, and $d_i = \deg(F) \prod_{j=1}^{i} k_j$. Then $\deg(C) = d_s$ and the curve defined by the above equation has a cusp with the following multiplicity sequence: $((k_s - 1) d_{s-1}, (d_{s-1})_{2(k_s-1)}, \dots)$

Corollary. *Let C be a rational cuspidal curve with one place at infinity of degree d. Then, one has the inequality: $d \leq 2\nu$, where ν is the maximal multiplicity of cusps on C.*

Theorem 2. *A rational bicuspidal curve C is of LZ type if and only if the log-Kodaira dimension of the complement of C is equal to one.*

Stefan Schröer

Quotient presentations for toric varieties

This is a joint work with A. A'Campo–Neuen and J. Hausen. The projective space $\mathbb{P}^n$ is the quotient of $\mathbb{A}^n - 0$ by the diagonal $\mathbb{G}_m$-action. Introducing the concept of quotient presentations, we generalize this to arbitrary toric varieties.

Roughly speaking, a *quotient presentation* for a toric variety X is a quasiaffine toric variety $\hat{X}$, together with a surjective affine toric morphism $q : \hat{X} \to X$ inducing a bijection on invariant Weil divisors. From another point of view, this amounts to the choice of *homogeneous coordinates* $S = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$.

We show that quotient presentations correspond to subgroups $\Lambda \subset \mathrm{WDiv}^T(X)$ of invariant Weil divisors. These subgroups are not arbitrary. Rather, they generalize the notions of ample sheaves or ample families of sheaves. The special cases that Λ is the group of all invariant Weil divisors or the group of all invariant Cartier divisors were studied by Cox and Kajiwara, respectively.

Relating quotient presentations to geometric invariant theory, we show that X is a free quotient of $\hat{X}$ with respect to a suitable group action precisely if the subgroup Λ comprises invariant Cartier divisors. Geometric quotient correspond to $\mathbb{Q}$-Cartier divisors. At this point, nonseparated toric varieties enter the scene, because quotients by group actions tend to be nonseparated.

As an application of quotient presentations, we express quasicoherent sheaves on toric varieties in terms of multigraded modules over homogeneous coordinate rings. Furthermore, we describe the set of morphisms into a given toric variety in terms of sheaf data related to homogeneous coordinates.

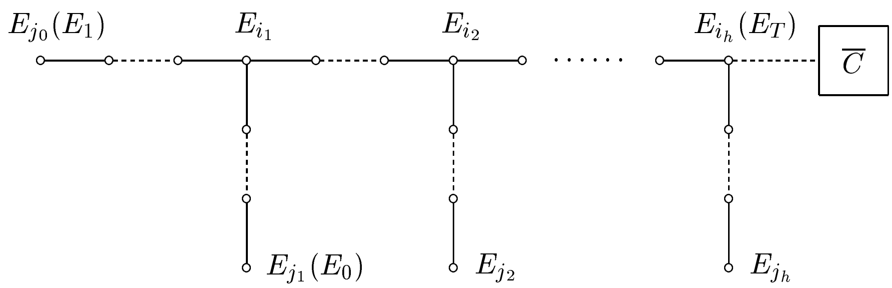
Masakazu Suzuki

Inverse of the semi-group theorem on the affine plane curves with one place at infinity

This is a joint work with M. Fujimoto. Let C be an irreducible affine algebraic curve with one place at infinity defined by a polynomial equation $f(x,y)=0$ in the complex affine plane $\mathbb{C}^2$ of degree m and n with respect to x and y respectively. It is known that, by a transformation of the coordinates by a polynomial automorphism in $\mathbb{C}^2$, it reduces to one of the following two cases:

(A) $m = 1,\ n = 0$ (In this case, C is a line);
(B) $m = pd,\ n = qd,\ (p,q) = 1,\ 1 < q < p$.

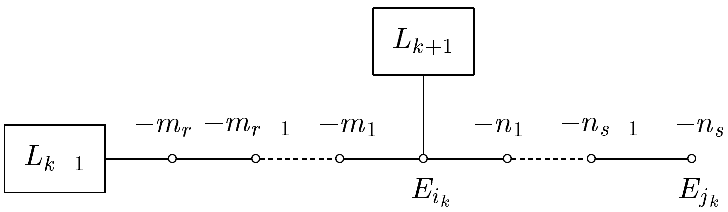
In the followings, we consider the latter case (B). A pair (M,E) of a projective surface M and a curve E on M such that $M - E \cong \mathbb{C}^2$ is called a compactification of $\mathbb{C}^2$ and E the boundary. The unique intersection point P of the closure $\overline{C}$ in M with E is called the point at infinity of C in M. Starting from the compactification (M_0, E_0) with $M_0 = \mathbb{P}^2,\ E_0 =$the line at infinity, blow up the point at infinity of C successively, until the closure $\overline{C}$ intersects the boundary curve E transversely at a regular point of E. Let us denote by (M,E) the compactification thus obtained, and by E_i $(1 \le i \le T)$ the proper image of the curve appeared by i-th blowing up in M. Then, the dual graph $\Gamma(E)$ of E is of the form:



Denote by δ_k the order of the pole of f on E_{j_k} for $0 \le k \le h$ and call $\{\delta_0,\ \delta_1,\ \cdots,\ \delta_h\}$ the δ-*sequence* of C (or of f). Note that we have $\delta_0 = n, \quad \delta_1 = m$. On the other hand, setting

$$L_k = \bigcup_{i_{k-1} < i \le i_k} E_i, \quad (1 \le k \le h),$$

we call L_k (resp. $\Gamma(L_k)$) the *k-th branch* of E (resp. of $\Gamma(E)$), where $i_0 = -1$.



(k-th branch L_k),

where $L_0 =$ the closure of the y-axis, $L_{h+1} = \overline{C}$ and $-m_i$, $-n_i$ are the self-intersection number of the corresponding irreducible component of E in M. Set $p_1 = p$, $q_1 = q$. For $2 \le k \le h$, define the positive integers p_k, a_k, q_k, b_k such that $(p_k, a_k) = 1,\ (q_k, b_k) = 1, \quad 0 < a_k < p_k,\ 0 < b_k < q_k$, by the following continuous fractions :

$$p_k/a_k = (m_1 - 1/(m_2 - 1/(\cdots - 1/m_r)\cdots)), \quad q_k/b_k = (n_1 - 1/(n_2 - 1/(\cdots - 1/n_s)\cdots)).$$

In case there is no vertex between the two branching vertices corresponding to $E_{i_{k-1}}$ and E_{i_k} ($k>1,\ r=0$), we set $p_k=1,\ a_k=0$. The sequence $(p_1,q_1),\cdots,(p_h,q_h)$ thus obtained from the dual graph $\Gamma(E)$ is called the (p,q)*-sequence* of C (or of f). Then, setting $d_k=\gcd\{\delta_0,\delta_1,\cdots,\delta_{h-1}\}(1\le k\le h+1)$, we have the following relations:

(1) $q_k=d_k/d_{k+1},\ d_{h+1}=1$,

(2) $d_{k+1}p_k=\begin{cases}\delta_1 & (k=1)\\ q_{k-1}\delta_{k-1}-\delta_k & (2\le k\le h),\end{cases}$.

Thus, the δ-sequence is determined by the weighted dual graph $\Gamma(E)$, and conversely it determines the dual graph with weights ([3]). Now, the semi-group theorem of Abhyankar-Moh (An algebro-geometric proof is also given in [3]) asserts that

(3) $q_k\delta_k\in\mathbb{N}\delta_0+\mathbb{N}\delta_1+\cdots\mathbb{N}\delta_{k-1}\ (1\le k\le h)$.

Conversely, Sathaye and Stenerson ([2]) proved that, a given sequence of $h+1$ natural numbers $\delta_0,\ \delta_1,\ \delta_2,\ \cdots,\ \delta_h\ (h\ge 1)$, is the δ-sequence of an affine plane curve with one place at infinity, if it satisfies the conditions : $q_k\ge 2(1\le k\le h)$, $d_{h+1}=1$, $\delta_k<q_{k-1}\delta_{k-1}\ (2\le k\le h)$ and the above condition (3), where $d_k=\gcd\{\delta_0,\cdots,\delta_{k-1}\}\ (1\le k\le h)$, $q_k=d_k/d_{k+1}\ (1\le k\le h)$, By this criterion, one can generate a list of the δ-sequences of the curves with one place at infinity, of a given genus for example, by computer ([3]).

Nakazawa and Oka ([1]) gave the classification of defining equations of the curves with one place at infinity of genus ≤ 16, calculating by hand.

We give a new proof to the Nakazawa-Oka's result, giving algorithms to compute the numbers of the parameters of the defining equations of the curves with one place at infinity corresponding to each δ-sequence, and to determine the family of defining equations of these curves using these parameters. We implemented the algorithm on a computer algebra system Risa/Asir. The parameter space is of type $(\mathbb{C}^*)^\lambda\times\mathbb{C}^\mu$ for each δ-sequence.

Example. The image of a non-constant polynomial mapping from $\mathbb{C}$ to $\mathbb{C}^2$ is called a *polynomial curve*. Sathaye-Stenerson ([2]) conjectured that there is no polynomial curve which has the δ-sequence $\{6,22,17\}$. The parameter space of the curves with one place at infinity for this δ-sequence is $(\mathbb{C}^*)^2\times\mathbb{C}^{34}$. The defining equation of these curves is of the form :

$f=(g_2^2+a_{2,1,0}x^2y)+c_{5,0,0}x^5+c_{4,0,0}x^4+c_{3,0,0}x^3+c_{2,0,1}x^2g_2+c_{2,0,0}x^2+c_{1,1,0}xy+c_{1,0,1}xg_2+c_{1,0,0}x+c_{0,1,0}y+c_{0,0,1}g_2+c_{0,0,0}$, where
$g_2=(y^3+a_{11,0}x^{11})+c_{10,0}x^{10}+c_{9,0}x^9+c_{8,0}x^8+(c_{7,1}y+c_{7,0})x^7+(c_{6,1}y+c_{6,0})x^6+(c_{5,1}y+c_{5,0})x^5+(c_{4,1}y+c_{4,0})x^4+(c_{3,2}y^2+c_{3,1}y+c_{3,0})x^3+(c_{2,2}y^2+c_{2,1}y+c_{2,0})x^2+(c_{1,2}y^2+c_{1,1}y+c_{1,0})x+c_{0,2}y^2+c_{0,1}y+c_{0,0}$,

$a_{***}\in(\mathbb{C}^*)$ and $c_{***}\in\mathbb{C}$. The genus of the regular curves in this family is 28.

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A Remark on Hilbert's 14th problem

The fourteenth of Hilbert's famous problems [5] is the following.

Let K/L and L/k be field extensions, and $A \subset K$ be a finitely generated k-algebra. Does this imply that $A \cap L$ is also a finitely generated k-algebra?

This problem was motivated by the following special case:

Let k be a field and $G \subset GL(n,k)$ a subgroup. Is the ring of invariants $k[x_1, \dots, x_n]^G$ a finitely generated k-algebra?

For reductive groups this is indeed the case. This was already shown by Hilbert. However, for non-reductive groups there is the celebrated counterexample of Nagata ([6]). Popov deduced from Nagata's example that for every non-reductive algebraic group G there exists an affine G variety such that the ring of invariants is not finitely generated [9]. In 1990, a new counterexample was found by Roberts [11]. Later, further counterexamples were obtained by Deveney and Finston [3] and by A'Campo-Neuen [1]. Recently, Daigle and Freudenburg constructed examples in dimension 6 and 5 ([2],[4]).

Reformulated in a more geometric fashion, Hilbert's 14th problem ask whether the ring of invariant functions is necessarily isomorphic to the ring of regular functions on some affine variety.

From this point of view it is maybe not too surprising that the answer is negative in general. Quotients of affine varieties by actions of (non-reductive) algebraic groups are often quasi-affine without being affine, and for arbitrary quasi-affine varieties the ring of regular functions is not necessarily finitely generated (see e.g. [7],[8],[10]). Thus even if the ring of invariants is not isomorphic to the ring of regular functions on an affine variety it nevertheless may be isomorphic to the ring of regular functions on a quasi-affine variety. Indeed, this is always the case. We can show that the a k-algebra occurs as the ring of invariants for some affine G-variety if and only if it is isomorphic to the algebra of regular functions on some quasi affine variety.

Theorem. *Let k be a field and R an integrally closed k-algebra.*

Then the following properties are equivalent:

- *There exists an irreducible, reduced k-variety V and a subgroup $G \subset Aut_k(V)$ such that $R \simeq k[V]^G$.*
- *There exists a quasi-affine irreducible, reduced k-variety V such that $R \simeq k[V]$.*
- *There exists an affine irreducible, reduced k-variety V and a regular action of $G_a = (k, +)$ on V defined over k such that $R \simeq k[V]^{G_a}$.*

If $char(k) = 0$, these properties are furthermore equivalent to the following:

- *There exists a finitely generated, integrally closed k-algebra A and a locally nilpotent derivation D on A such that $R \simeq \operatorname{kern} D$.*

This result is based on the following more general theorem.

Theorem. *Let k be a field, V an irreducible, reduced, normal k-variety, and L a subfield of the function field $k(V)$, containing k. Let $R = k[V] \cap L$.*

Then there exists a finitely generated k-subalgebra R_0 of R such that

- *The quotient fields of R and R_0 coincide.*
- *For every prime ideal p of height one in R the prime ideal $p \cap R_0$ of R_0 also has height one.*
- *There is an open k-subvariety $\Omega \subset \operatorname{Spec}(R_0)$ such that $R = k[\Omega]$ (as subsets of $Q(R)$).*

These results can be used to construct some "quasi-affine" quotient for a group action on an algebraic variety.

Theorem. *Let k be a field, V an irreducible, reduced, normal k-variety and $G \subset Aut(V)$.*

Then there exists a quasi-affine k-variety Z and a rational map $\pi : V \to Z$ such that

- *The rational map π induces an inclusion $\pi^* : k[Z] \subset k[V]$.*

- *The image of the pull-back $\pi^*(k[Z])$ coincides with the ring of invariant functions $k[V]^G$.*
- *For every affine k-variety W and every G-invariant morphism $f : V \to W$ there exists a morphism $F : Z \to W$ such that $F \circ \pi$ is a morphism and $f = F \circ \pi$.*

We may also translate our results in the language of category theory and deduce the following.

Theorem. *For a field k let $\mathcal{V}_k$ denote the category whose objects are irreducible reduced normal k-varieties and whose morphisms are those dominant rational maps for which the pull-back of every regular function is again regular. Let $\mathcal{Q}_k$ denote the full sub-category whose objects consist of all* quasiaffine *such varieties.*

Then for every object $V \in \mathcal{V}_k$ and every subgroup $G \subset \mathrm{Aut}_{\mathcal{V}_k}(V)$ the functor $\mathrm{Mor}_{\mathcal{V}_k}(V, \cdot)^G$ is representable in the category $\mathcal{Q}_k$.

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Mikhail Zaidenberg

Affine varieties with simple topology

The survey covers the following subjects:

I. Acyclic surfaces:

1) Generalities: main examples, affines (Fujita's Lemma), recognizing acyclicity by completion; rationality (Gurjar–Shastri theorem), examples of Pham–Brieskorn nonrational contractible surfaces; tom Dieck–Petrie line arrangements.

2) Characterizations of $\mathbb{C}^2$: Ramanujam's Theorem; existence of a cylinder.

3) Classification according to log-Kodaira dimension: Miyanishi–Sugie–Fujita Theorem, Fujita's classification ($\bar\kappa = 0$); Gurjar–Miyanishi's list of surfaces with $\bar\kappa = 1$.

4) Acyclic curves on acyclic surfaces: Theorems of Abhyankar–Moh–Suzuki, Lin–Zaidenberg, Zaidenberg, and Gurjan–Miganishi; characterization of $\mathbb{C}^2$, of surfaces with $\bar\kappa = 1$; tom Dieck–Petrie examples; Kaliman-Makar-Limanov embedding theorem.

5) Deformations: Log-general type surfaces: the rigidity conjecture (Flenner–Z.)

6) Rational Cuspidal Plane curves: $\mathbb{P}^2 \setminus C$ is a $\mathbb{Q}$-acyclic $\Longleftrightarrow$ C is a rational cuspidal curve; the rigidity conjecture; (d, d-2), (d, d-3), (d, d-4)-curves (Flenner–Z., Fenske); other series (unicuspidal, 2-cuspidal curves, tom Dieck's list); Orevkov's estimates.

7) Abhyankar-Suthaye Embedding Problem for planes in $\mathbb{C}^3$: Sathaye–Wright and Kaliman-Z. Theorem, Russell–Sathayer's Theorem.

II. Acyclic 3- and 4-folds:

1) Affine modifications: definitions; main properties; Davis' presentation. Examples: tom Dieck–Petrie surfaces, Russell's cubic, Kaliman-Z. hypersurfaces $u^m v - p\bar{x} = 0$. Kaliman-Z. Theorem on preservation of topology. Dimca–Ramanujam's version of h-cobordism theorem.

2) Exotic $\mathbb{C}^n$'s: definition; examples's; Makar-Limanov's theorem on exoticity of Russell's cubic; Iitaka–Fujita's Strong Cancellation Theorem.

3) Characterization of $\mathbb{C}^3$: Miyaniski's Theorem; Kaliman-Z. Theorem on existence of a cylinder; Kaliman's Theorem on a variable and conditions of exoticity of an acyclic 3-fold.

4) Makar-Limanov's Invariant: Locally nilpotent derivations and $\mathbb{C}_+$-actions; $\mathrm{Aut}_+(X)$; Kaliman–Z. Theorem on infinite transitivity of the action of $\mathrm{Aut}_+(X)$ on $\mathrm{Reg}(X)$ with $X = (uv - p\bar{x} = 0)$.

5) Exotic examples of exotic 4-folds:

Theorem. *(Kaliman-Z.) The 4-fold $X_{k,l,m} \subset \mathbb{C}^5$ with the equation*

$$u^m v - \frac{(xz+1)^k - (yz+1)^l - z}{z} = 0$$

is an exotic $\mathbb{C}^4$, where $m \geq z$, $k > l \geq 3$, $(k,l) = 1$.

Corollary. *Miyenishi's characterization of $\mathbb{C}^3$ cannot be extended to dimension 4.*

This leads to the following question: Is the 4-fold $X_{k,l,1}$ with $(k,l) = 1, k,l \geq 2$ an exotic $\mathbb{C}^4$?

D.-Q. Zhang

Part I: Open Algebraic Varieties and their Fundamental Groups

Part II: Automorphisms of Finite Order on Rational Surfaces

We assume that X is a normal projective variety with at worst log terminal singularities defined over $\mathbb{C}$. Set $X^0 := X \setminus SingX$.

Problem 1. Suppose that $dimX = 2$, X is simply connected and X has only one quotient singularity and no other singularity. Is $\pi_1(X^0) = (1)$?

Remark 2. (1) The answer to **2** is yes if X is a rational surface and the logarithmic Kodaira dimension $\kappa(X^0) \leq 1$ (cf. [Gurjar and Z., Math. Ann.]).
(2) There are examples of rational surfaces with two or more quotient singularities so that $\pi_1(X^0)$ is infinite.

Suppose that X is $\mathbb{Q}$-Fano, i.e., the anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Conjecture 3. If X is $\mathbb{Q}$-Fano, then $\pi_1(X^0)$ is finite.

Remark 4. (1) When $dimX = 2$, the answer to **3** is yes; (cf. [Gurjar and Z., J. Math. Sci. U Tokyo]). There are other proofs by Fujiki-Kobayashi-Lu and by Keel-McKernan.
(2) When the Fano index $r(X) > dimX - 2$, **3** is proven; (cf. [Z.; Osaka J. Math.]).

(3) There are counter-examples to **3** when X has log canonical singularities [Z., Trans AMS, 1996].

Clearly, **3** would imply the following

Theorem 5 (Takayama, JAG, 2000). If X is $\mathbb{Q}$-Fano, then $\pi_1(X)$ is finite.

Remark 6. Campana and Kollar–Miyaoka–Mori have shown independently that **5** would follow from the following, which they proved when $dimX \leq 3$.

Conjecture 7. If X is $\mathbb{Q}$-Fano, then X is rationally connected.

In a paper with I. Shimada [Nagoya Math. J.] and a paper with J. Keum, we have determined $\pi_1(X^0)$ where X is either a K3 surface or an Enriques surface with at worst Du Val singularities.

Corollary. Suppose that X has at worst c singularities of type A_{p-1} with p a fixed prime number. Then $\pi_1(X^0)$ is soluble. It is finite unless $(p, c) = (2, 8), (2, 16)$ or $(3, 9)$.

In a paper with an Appendix by I. Dolgachev, we apply the latest Mori theory of extremal ray in $\overline{N}E(X)^G$ and obtain a very short list of all minimal pairs (X, G) of a smooth projective rational surface with an effective finite group G-action, especially for groups of prime order. The first study of such G was done by Kantor over one hundred years ago, and later by Manin, Gizatullin, Iskovskih, et al.

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