Mathematisches Forschungsinstitut Oberwolfach

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Calculus of Variations

July 2nd – July 8th 2000

The meeting was organized by Gianni del Maso (Trieste), Gero Friesecke (Oxford) and Frédéric Hélein (Cachan - ENS). Variational Methods continue to occupy an important position at the frontier of research in differential geometry and partial differential equations. The participants represented a broad spectrum employing and developing variational techniques in different fields such as mathematical physics, homogenization theory, optimization and control, nonsmooth analysis and image analysis.

Abstracts

DNA molecules as elastic rods with self-contact

Heiko v. d. Mosel, Bonn

We model the mechanical behaviour of DNA molecules (bending, twisting) including knotting and supercoiling in the framework of the Casserat theory of elastic rods. Within the context of the calulus of variations, we have developed an existence theory for general nonlinear stored energy densities. As a special case, we obtain an existence theorem for ideal knots. To implement topological constraints such as a prescribed knot class, we restrict our minimization problem to framed curves with a possible lower bound on the global radius of curvature, a geometry condition that corresponds to a given thickness of the centerline. This nonsmooth side condition requires the Clarke calculus of generalized gradients to obtain the necessary Lagrange multipliers. The analysis of the Euler-Lagrange equations leads to a qualitative description of the contact forces at points of self-contact.

Gradient flow for the Willmore functional

Ernst Kuwert, Freiburg

(joint work with Rainer Schätzle)

For a closed immersed surface in $\mathbb{R}^d$, we consider the L^2 gradient flow for the total squared curvature, i.e. the Willmore energy. For this fourth order geometric evolution equation, it is not known whether singularities occur in finite time. We show that, if the maximal time of existence is finite, then the curvature must concentrate in L^2 by an a priori quantity $\epsilon_0 > 0$. Furthermore, a suitable rescaling converges to a possibly noncompact Willmore surface. If the initial surface is sufficiently round in an L^2 sense, then the flow exists globally and converges to a round sphere.

New regularity results for the inverse mean curvature flow

Gerhard Huisken, Tübingen

(joint work with Tom Ilmanen, ETH)

Let $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a family of hypersurfaces moving in normal direction ν with speed given by the inverse of the mean curvature $H > 0$: $\frac{d}{dt}F(p,t) = (1/H) \cdot \nu(p,t)$ for $p \in M^n$, $t \in [0,T)$. This is a parabolic system increasing the area exponentially at each point, with applications in General Relativity.

We prove that a smooth solution can be extended as long as the mean curvature is bounded away from zero, improving a result of Smozcyk for $n = 2$.

In addition, we prove that in the class of strictly starshaped surfaced in $\mathbb{R}^{n+1}$ the mean curvature satisfies a lower bound on H of the form $H \geq c_0 t^{1/2}\, exp\,(-c_1 t)$, which is independent of the initial data. The estimate applies to show that weak (level-set) solutions of the flow are smooth from the first time onward where a level set is starshaped . All results can be extended to sufficiently starshaped hypersurfaces of asymptotically flat Riemannian manifolds, ensuring that solutions of the flow constructed in earlier work on the Penrose inequality are smooth outside some compact set.

Non-Lipschitz minimizers for stongly convex functionals

Xiaodong Yan, University of Minnesota

(joint work with Vladimir Šverák)

We consider variational integrals of the form $I(u) = \int_\Omega f(\nabla u)$ where Ω is a smooth bounded domain in $\mathbb{R}^n, u = \Omega \to \mathbb{R}^m$. f is a smooth strongly convex function defined on $M^{m\times n}$ (i.e. $f_{p^i_\alpha p^j_\beta}(x)\zeta^i_\alpha\zeta^j_\beta \geq \nu|\zeta|^2 \,\forall x, \zeta \in M^{m\times n}$.)

It's well known for $n = Z, m \geq 1$ or $n \geq 2, m = 1$, that every minimizer of $I(u)$ is smooth provided f is smooth. However, the best general regularity results known for the vector case $(n \geq 3, m > 1)$ is $W^{2,2+\delta}_{\infty c}$. In this paper, we construct smooth strongly convex functionals which have minimizers that are only Hölden continuous for $n \geq 3, m \geq 5$ and $n = 4, m = 3$. In fact, one could even construct minimizers that are unbounded when $n \geq 5m \geq 14$.

Quantization results for harmonic maps

F.H. Lin and T. Rivière, Cachan

Following a sequence of stationary harmonic maps u_m between two Riemannian manifolds (M^m, g) and (N^n, h) $(\partial N = \emptyset$ and N noncompact), provided that this sequence has uniformly bounded energy, we may always extract a subsequence $u_{n'}$ so that the defect energy converges as a Radon measure. This defect energy measure is in fact carried by a $m-2$ rectifiable subset of M^m and we prove that the amount of energy concentrating at a point of this rectifiable subset is quantized and given by the energy of a harmonic 2-sphere of S^2. This result extends to non conformal dimension the famous result of Sacks Uhlenbeck Jost Parker Ding Tian in $2D$.

Singularities of first kind in the harmonic map heat flow

Andreas Gustel, Düsseldorf

Solutions to the harmonic map heat flow equation $\Delta u + A_N(u)(D_u, D_n) = \delta_r u(u : [a,b) \times \mathbb{R}^n \to N$ with second fundamental from $A_N)$ in general can blow up in finite time, implying that there is no long-time existence of smooth solutions. At blow-up time, $\lim\limits_{t\nearrow T} \overset{swp}{\underset{x}{}} |\nabla u(t,x)|^2 = \infty$, and a singularity at (T, x_0) is called "of first kind" if $|\nabla u(t,x)|^2 \sim \frac{c}{T-t}$ as $t \nearrow T$. We construct examples of first kind singularities in maps $(-\infty, 0) \times \mathbb{R}^m \to S^n$ (which have an isolated singularity at $T = 0$) which are explicit up to solving an o.d.e. This works for any $m \geq 3$, provided n is chosen large enough. The example generalizes an ansatz by Ilmanen/Fan, which worked for $m = 3 > 6$.

By a similar construction, we also get examples of "first kind" singularities for the Yang-Mills Heat Flow in dimensions $5 \leq M \leq 9$. Here it has been unknown whether such singularities exist.

Complete surfaces of constant mean curvature in Euclidean 3-space

DAN POLLACK, SEATTLE

We focus on surfaces with $H = 1$ which are complete and embedded in Euclidean 3-space. The fundamental building blocks for such surfaces are the classical Delannay unduloids of revolution. The asymptotics theorem of Korevaar, Kusner and Solomon show that these are the exact models at infinity for such "CMC" surfaces. We survey recent results on the existence of such surfaces and the moduli space, $\mathcal{M}_{g,K}$, of all such CMC sufaces with fixed topology. The gluing techniques used to produce such surfaces fall into two categories: those which try to join surfaces with similar geometries and those which join together surfaces with differing geometries. The former include Kapaleas' original construction as well as constructions of Mazzeo-Pacard and Ratzkin. The latter includes recent constructions of the author with Mazzeo and Pacard. Outlines of these constructions, together with applications, were presented.

Mathematiccal Models for nematic elastomers

GEORG DOLZMANN, LEIPZIG

We consider the free energy density $E(F,n) = r^{1/3}(|F|^2 - \frac{r-1}{r}|F^T n|^2| - 3$ for $\det F = 1, E(F,1) = +\infty$ otherwise, describing nematic elastomers. Here $r > 1, F \in M^{3\times 3}$ is the deformation gradient, $n \in S^2$ the director field. Minimization over n leads to $\tilde{E}(F) = r^{1/3}(\frac{1}{r}Z_1^2 + Z_2^2 + Z_3^2) - 3$ for $\det F = 1$ where $Z_1 \geq Z_2 \geq Z_3$ are the regular values of F (i,e. the eigenvalues of $(F^T F)^{1/2}$.) It follows that $\tilde{E}(F) = 0$ if and only if $F \in \bigcup_{nGJ^2} SO|3)(r^{1/3} n\otimes n + \frac{1}{r^{1/6}}(I - n\otimes n)) =: K$. An explicit characterization of the quasiconvex hull of K (the set of all affine boundary conditions for which the infinum of the energy is zero) and the relaxation $\tilde{E}^{ge}$ of the density $\tilde{E}$ are given.

Micromagnetics: Exact Solutions and Relaxation

IRENE FONSECA, CARNEGIE MELLON UNIVERSITY

Minima of the energy for large magnetic bodies with vanishing induced magnetic fields, where the energy is given by

$$E(m) := \int_\Omega (\varphi(m) - < h_e, m >)dx + \frac{1}{2}\int_{R^3} |h_m|^2 dx,$$

are completely characterized in terms of the anisotropic energy density φ and the applied external magnetic field $h_e \in \mathbb{R}^3$. More generally, one considers an energy functional for a large ferromagnetic body of the form

$$F(m) := \int_{\mathbb{R}^N} f(x, \chi_\Omega(x)mx, u(x), \nabla u(x))dx$$

where $(\chi_\Omega m, \nabla u)$ satisfies Maxwell's equations, i.e. $u \in H^1(\mathbb{R}^N)$ is the unique solution of $\triangle u + div(\chi_\Omega m) = 0$ in $\mathbb{R}^N$. It is shown that if f is a Carathéodory function satisfying very mild growth conditions then the relation at F with respect to L^∞ weak* convergence in $L^\infty(\Omega;\overline{B(0,1)})$ is given by

$$F(m)=\int_\Omega Q_M f(x,m(x),u(x),\nabla u(x))dx+\int_{\mathbb{R}^N\setminus\Omega} f(x,0,u(x),\nabla u(x))dx$$

where $Q_M f$ is the quasiconvex envelope of f relative to the underlying partial differential equations. This class of integrands includes those of the type

$$f(x,m,u,h)=\varphi(x,m,u)+\Psi(x,u,h)$$

with $\Psi(x,u,\cdot)$ non convex. The first part of this work was undertaken in collaboration with Bernard Dacorogna and the relaxation results were obtained jointly with Giovanni Leoni.

Remarks on Loewy spaces

Luc Tartar, Carnegie Mellon University, Pittsburgh

I described some classical properties of Lorentz spaces, which are interim spaces between $L^1(\Omega)$ and $L^\infty(\Omega)$ and seem to be quite useful for some fine questions considering solutions of partial differential equations. Some of these properties follow from more precise embedding theorem for Sobolev spaces, following an idea of Jaak Peetre and general theorems due to Jacques Louis Lions and Jaak Peetre. Some other properties follow from an improvement of results of Guido Stampacchia and some examples involve compensation properties for Jacobians, where advantages and defects of using Hardy spaces (following ideas of Ronald Coifmann, Pierre Louis Lions, Yves Meyer and Stephen Semmes) for Jacobians arise.

Some questions involving Ginzburg Landau equations seem a good field for applying some of the techniques mentioned.

Quasiconvex hulls and some nonlinear pdes

B. Dacorogna, EPFL., Lausanne

(joint work with P. Marcellini and C. Tanteri)

I have discussed several examples of first order pdes satisfying Dirichlet boundary condition. One of them considers singular values (recall that for an $n\times n$ matrix $\xi, 0\le\lambda_1(\xi)\le\ldots\le\lambda_n(\xi)$ denote the singular values, i.e. the eigenvalues of $\sqrt{\xi^T\xi}$). The problem is

$$\begin{cases}\lambda_i(Du(\pi))=a_i(x,u(x)) & a.e. x\in\Omega\\ u(x)=\varphi(x) & x\in\partial\Omega\end{cases}$$

This problem has a $\mathbb{W}^{1,\infty}(\Omega;\mathbb{R}^n)$ solution if $\varphi\in C^1(\bar\Omega;\mathbb{R}^n)$ and

$$\prod_{i=\nu}^n\lambda_i(D\varphi(x))<\prod_{i=\nu}^n a_i(x,\varphi(\pi)\quad\forall x\in\Omega.$$

On crystalline motion by mean curvature in $3D$

G. Bellettini, Univ. of Roma "Tor Vergata"

(joint work with M. Novaga and M. Paolini)

I have discussed how to define a notion of "smooth" hypersurface in $(\mathbb{R}^n, \varphi)$, where φ is a novum such that $\{\varphi \leq 1\}$ is a convex polytope. Then I have shown how to define a notion of mean curvature of such smooth surfaces. This definition is consistent with the current definition of curvature when the dimension $n = 2$. These are preliminary steps to define motion by crystalline curvature, which is a largely open problem in $3D$. Finally we discuss the problem of facet-breaking under motion by crystalline curvature, and we give a necessary and sufficient condition for a facet not to break during the subsequent evolution.

The Einstein equation and geodesic motion

D.M.A. Stuart, University of Cambridge

The gravitational field in general relativity is described by a pseudo-Riemannian metric. A standing physical assumption is that a "rest particle" (i.e. a localised distribution of matter, small in size and energy) introduced onto the space-time will move along a geodesic (to highest order). There are mnay possible corresponding mathematical problems corresponding to different energy-momentum tensors (which appear in the Einstein equation). I give an analytically rigorous derivation of the geodesic motion in the case when the role of the particle is played by a complex-valued solitary wave solution to a semi-linear wave equation. The relevant class of solitary waves are called non-topological solutions. The construction of non-topological solutions on pseudo-Riemannian manifolds in which the energy is concentrated on a geodesic is given. This is achieved as an outgrowth of a new modulational approach to stability for solitary wave solutions on flat Minkowski space. The analysis is then extended to incorporate coupling to the einstein equation and a solution is constructed for the initial value problem of the full system.

The abstract setting for the modulational stability analysis is the following: in the stable case there is a symplectic submanifold foliated by integral curves of the pde. It is shown that for small perturbations of the Cauchy data the solutions remain uniformly close to the integral curve of a deformed vector field. A crucial point is that the Hession of the augmented energy is strictly positive on the sympletic normal subspace in the stable case.

Asymptotic Analysis of Emden-Fowler equations in 2-dim.

Feng Zhou, East China Normal University, Shanghai/ ENS de Cachan

(joint work with Dong Ye)

We study the asymptotic behaviour of two-dimensional Emden-Fowler equations in divergence form with exponential nonlinearity. More precisely, let Ω be a bounded regular domain of $\mathbb{R}^2$ and $a(x)$ be a smooth positive function over Ω. Let $\{U_x\}$ be a sequence of solutions of $-div\,(a\nabla u) = \lambda a e^u$ in Ω such that $\int \lambda e^{u_\lambda} = O^{(1)}$ and $||u_\lambda||_{L^\infty} \longrightarrow +\infty$ as $\lambda \to 0$, then we prove that (for a subsequence, if necessary) either $u_x \longrightarrow 0$ uniformly or any compact subset of Ω; in particular, if $\Lambda = \{x \in \Omega | \nabla a(x) = 0\} = \emptyset$, this case must occur, or there exists a finite set $\Delta = \{x, \ldots, x_k\} \subset \Omega$ (the blow up set) such that $u_x \longrightarrow u^*$ weakly in $W^{1,p}(\Omega)\forall p \in (1,2)$, where u^* satisfies the equation $-div\,(a\nabla u^*) = 8\pi\Sigma m_i a(x_i)\delta_{x_i}$ with Dirichlet boundary conditions $m_i \in \mathbb{N}$ and $\mathcal{S} \subset \Lambda \neq \emptyset$. This is the "pinning" phenomenon. We discus also the inverse problem, i.e. the construction of the singular limits in the symmetric case.

Limiting energy for a family of functionals related to micromagnetics

Sylvia Serfaty, ENS Cachan

In a joint work with Tristan Rivière, we studied the asymptotic limit as $\epsilon \to 0$ of the family of functionals $E_\epsilon(u) = \int_2 \epsilon|\nabla u|^2 + \frac{1}{\epsilon}\int_{\mathbb{R}^2}|H|^2$, where Ω is a domain in $\mathbb{R}^2$, (simply connected, smooth), and u is constrained to be S^1 valued, while H is derived from

$$\begin{cases} div\,(\chi_\Omega u + H) = 0 & \text{in}\,\mathbb{R}^2 \\ curl\,H = 0 \end{cases}.$$

The vector-field u represents the local magnetization in the sample, while H is the induced demagnetizing field. Solutions of the minimizing problem or families of bounded energy will tend to exhibit jumps along line singularities as $\epsilon \to 0$. First, we prove a compactness result for sequences u_ϵ satisfying $E_\epsilon(u_\epsilon) \leq C$: we prove that they converge strongly in $\cap L^q$ to some limiting u (up to extraction), hence the constraint $|u| = 1$ is preserved in the limit.

Then, after studying the corresponding one-dimensional profiles, we show that E_ϵ has a kind of Γ-limit that we exhibit: this Γ-limit is the mass of a measure concentrated along the singular set of admissible limiting configurations (which are solution of $\left(\begin{cases} div(u\chi_\Omega) = 0 \\ |u| = 1 \end{cases}\right)$. This Γ-limit reflects some ??? of the jump set of the configuration. We prove that its minimal value is the perimeter of the domain and exhibit a minimizer.

On nonconvex symmetric functions

Johannes Zimmer, TU München

We start with a problem arising in nonlinear elasticity (in situations with phase transition): how should one construct functions depending on the deformation gradient and the temperature which have to satisfy symmetry conditions like crystalline symmetry and frame invariance? The presented method gives a formal description of all C^∞-functions being invariant under a compact Lie group. This method uses techniques which are well known in group and representation theory, like Hilbert's theorem on the $\mathbb{R}$-algebra of invariant polynomials. The results are discussed in the context of the calculus of variations, a number of open questions concerning different notions of convexities are addressed.

Lipschitz Functions and a mysterious vector-field

David Preiss, London

Given a null set in the plane, there is a Lipschitz deformation of the plane which is differentiable at no point of the given set. This is a surprising statement since (1) is would be false if we replaced deformations by real-valued functions and (2) it implies the existence of the "mysterious vector-field": For every null set N one has a (Borel) vector field $e(x)^{(x\in N)}$ which is determined uniquely up to a purely unrectifiable set by the property that every Lipschitz inversion is differentiable in the direction $e(x)$ on all $x \in N$ except a purely unrectifiable set. (A set is purely unrectifiable if it is of 1-dimensional measure zero on every rectifiable curve.) It follows that, if μ is a singular measure in the plane, which can be written as integral of 1-rectifiable measures on curves, when the direction of these curves is determined uniquely μ - almost everywhere: this, as far as I know, was the first proved instance of this phenomenon and is due to G. Alberti.

Singularities in Sobolev spaces between manifolds

M. Reza Pakzad, ENS, Cachan

(joint work with T. Rivière)

For a Sobolev space $W^{1,p}(M,N)$, M, N smooth compact manifolds, we search for defining a suitable obstruction "object",
$varrho_u$, for any map $u \in W^{1,p}/(M,N)$, which characterizes the approximability of u by smooth maps in the space. This question is of great importance in the study of variational problems (p-harmonic maps and relaxed energies) and also is related to the sequentially weak density of smooth maps in these spaces. The last result would be that for any map $u \in W^{1,p}(M,N)$, with p an integer greater than 2 and N a p-connected smooth manifold W there exists $u_n \in c_0^\infty(M,N)$ such that $u_n \to u$ is the weak topology. Also for some specific values of p we can describe its topological singularity (obstruction) as a $\pi_p(N)$ flat chain in the domain (based on the theory of Federer and Fleming on G-chains.

On the validity of the Euler Lagrange equation

Arrigo Celina, Milano

We consider minimization problems of the kind

$$min \quad \int_\Omega (f(\nabla u(x)) + g\,(u(x))dx$$

$$u|_{\partial\Omega} = u|_{\partial\Omega} \quad \text{and} \quad |\nabla u(x),| \leq 1.$$

Two cases are presented: The case where $f =$ indicator function of one unit ball "any" g (some monotonicity required) and the case where $f = \frac{1}{2}|\nabla u(x)|^2$, $g = 0$. We show the validity of the Euler Lagrange equation in the form

$$\exists \quad P(x) \in \partial f(|\nabla u(x)\Psi|)$$

$$\text{such that} \quad div\, P(x), = g'_u(u(x)).$$

A variational formulation of phase transformation problems with rate independent dissipation

Florian Theil, University of Oxford, U.K.

A model for the evolution of an elastically deformable body, which undergoes martensitic phase transformations is presented. The movement of the phase boundaries is hindered by dry friction. The model consists in an energy functional $I(t,\cdot)$ which varies with time, a dissipation functional $D(\cdot,\cdot)$ and a state space P. Due to the lack of compactness the existence of solutions cannot be expected, minimizing sequences of the variational problems associated to the time discretized systems generate microstructures. To overcome the difficulty a notion of relaxation is introduced. For important examples it is possible to show that they satisfy the relaxation assumptions, i.e. they admit solutions and the relaxed solutions can be approximated within the original system. The proof uses the idea of H-measures, introduced by L. Tartar in 1990.

Variational elliptic systems with nowhere smooth solutions

STEFAN MÜLLER, MPI LEIPZIG

Consider a variational integral

$$I(u) = \int_\Omega f(\nabla u) \quad , \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m .$$

We say that u is a minimizer of I if $I(u+\varphi \geq I(u) \quad \forall \varphi \in C_0^\infty$. For such minimizers there is a well-developed regularity theory, as long as f satisfies suitable ellipticity or convexity conditions.

Theorem (Evans, 1986) Suppose that

$$(*) \begin{cases} f \in C_0^\infty, \quad |D^2 f| \leq G, f \text{ is uniformly quasiconvex, i.e.} \\ \int_u f(A + \nabla_\psi - f(A) \geq C \int |\nabla_\psi|^2, c > 0, \forall_\psi \in C_0^\infty, \forall A \in M^{m\times n} \end{cases}$$

and for all (smooth) open sets U. Then for every minimizer u of I there exists an open set $\Omega_0 \subset \Omega$ of full measure such that $u \in C^\infty(\Omega_0)$.

In this talk we show that the situation is dramatically different if one considers general solutions of the Euler-Lagrange equation instead of minimizers.

Theorem (M.- Šverák) Let $n = m = 2$. Then there exist f satisfying $(*)$ and a Lipschitz u, which solves

$$-div\, \nabla f(\nabla u) = 0$$

such that u is nowhere C^1.

One can also construct Lipschitz solutions with compact support. As a consequence of these results one also obtains various counterexamples for linear elliptic systems

$$-div\, A(x)\nabla v = 0$$

with L^∞ coefficients which satisfy the Legendre-Hadamard condition

$$A_{ij}^{\alpha\beta}\, a^\alpha a^\beta b_i b_j \geq C\, |a|^2 |b|^2 .$$

Hamiltonian formalism with several variables and field theory

F. HÉLEIN (CMLA, ENS, CACHAN)

During the thirties, several generalizations of the classical Hamiltonian formalism were proposed by H. Weyl, T. De Donda, Carathéodary, etc., for varational problems with several variables. At this time (1935), H. Weyl and M. Born pointed out the possibility of using these formalisms for quantizing fields. But these theories were not developed enough and another possibility was offered by physicists, namely the use of infinite dimensional symplectic manifolds (by slicing space-time into constant time hypersurfaces). The success of this second approach was so big that the approach using a "covariant Hamiltonian formalism" was forgotten for a long time (Beside work by Lepage, Dedecker...). After 1970, theoretical physicists were more interested in this approach and the formalism developed again. in the 90's Konratschikov (assuming contributions of many other people) proposed an almost satisfying picture with a good definition of Poisson bracket.

In a joint work with J. Konreiher, we generalized, and improved these constructions in order to build a formalism with a satisfactory definition of Poisson bracket (among other things).

Paper is available on my internet page (you can find it by going to www.cmla.ens-cachan.fr)

Convergence for functionals of Ginzburg-Landau type

Giovanni Alberti, Pisa

The results presented in this talk have been obtained in collaboration with S. Baldo (Potenza) and G. Orlandi (Verona). We have studied the variational convergence in the limit $\epsilon \to 0$ of functionals of type

$$F_\epsilon(u) = \frac{1}{|\log \epsilon|} \int_\Omega |\nabla|^2 + \frac{\perp}{\epsilon^2} W(u)$$

where $u = (u_1, u_2) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^2, W : \mathbb{R}^2 \to [0, +\infty]$ and $W(u) = 0 \Leftrightarrow |u| = 1$. More precisely we have shown the following:

(i) [compactness] if $F_\epsilon(u^\epsilon \leq C < +\infty$, then, up to subseq., the Jacobians $Ju^\epsilon := du_1^\epsilon, \wedge du_2^\epsilon$ converge to an integral current T of codimension 2 without boundary in Ω (in the flat sense).

(ii) In this case, we also have $\lim_{\epsilon \to 0} F_\epsilon(u^\epsilon) \geq C||T||$

(iii) and the convergence is sharp for suitably chosen seq. (u^ϵ). This is essentially a Γ-convergence result, only the limit of the energies is now a functional of T, the limit of Jacobians, rather than a functional of u, due to a lack of compactness for the sequence (u^ϵ). From this result we deduce that in case of local minimizers u^ϵ of $F_{epsilon}$, then the limit $|T|$ correspond to the concentration of energies, and T is a local minimizer of the area (or mass). Similar results are proved for more general functionals with the same method. We recall that the first complete description of the asymptotic behaviour of minimizers in dimension 2 was carried out by F. Bethuel, H. Brezis, F. Hélein, and generalized to higher dimension by T. Rivière, F.H. Liu (and others). A variational approach quite similar to ours has been proposed independently by R. Jerrard and H.M. Soner.

A convexification approach for calibrations

Guy Bochitte, University of Toulon

In the talk, we present a general convexification argument which enables to write possibly non convex functional $F : u \in L^1(\Omega) \to [0, \infty]$ in the form $F(u) = G \circ \varphi(u)$ with G convex and $\varphi : L^1(\Omega) \to y$ continuous embedding. Here $y = L^\infty(\Omega \times \mathbb{R})$ and $\varphi : u \to 1_u(x,t) = \begin{cases} 1 & \text{if } u(x) > t \\ 0 & \text{otherwise.} \end{cases}$

Writing the Euler equation for G at $\varphi(u)$ provides **necessary** and **sufficient** conditions for minimality of u which can be stated as existence of calibrations. This work is in collaboration with A. Chambolle (Paris Céremade).

In a second part of the talk, we present a calibration method for free-disconuity functionals (in particular Mumford Shah functional) which leads to **sufficient** conditions for minimality. Examples are given namely the triple junction configuration conjecture by De Giorgi. This part has been in collaboration with G. Alberti (Pisa) and G. Dal Maso (Trieste).

Curvature flows on surfaces

Michael Struwe, ETH Zürich

Inspired by a recent new approach of Xiuxiong Chen to the Calabi flow of metrics $g = g(t) = e^{Zu(t)} g_0$ on a closed, compact surface (M, g_0), we give an elementary proof for global existence and exponential convergence of solutions to the Hamilton-Ricci flow

$$\frac{\partial g}{\partial t} = (r - R)g,$$

where $R = 2K, r = 2K_0$ (with no loss of generality) are the scalar curvature and its mean.

Our proof avoids the use of the maximum principle and instead is based only on estimates for the Liouville and Calabi energies

$$F(u) = \frac{1}{2} \int_M (|\nabla u|^2 + 2K_0 u) d\mu_0, \; Ca(g) = \int_M |K - K_0|^2 d\mu_g$$

and concentration - compactness results for conformal metrics with bounded Calabi energy and volume.

Deformations with finitely many gradients

Bernd Kirchheim, MPG Leipzig

How does the range of the gradient map of a Liptschitz mapping (defined on a domain) look like? Here we focus on finite sets as candidates for the range. In particular, one wants to understand if such sets are necessarily trivial, e.e. contain a rank one connection.

We construct the first counterexamples and give a precise estimate of their minimal cardinality.

In fact, we show that in distinction to the Gradient Young Measure problem, we have rigidity for exact solutions if 4 non-rank one connected gradients are considered. The construction of our counterexample for 5 gradients is presented in a new framework unifying and simplifying the present convex integration and Baire category approaches to existence results for partial differential inclusions. The work was to a large extent joint with M. Chebik (Bratislava) and D. Preiss (London).

Probabilistic approach to some problems in conformal geometry

Michael Kiessling, Rutgers University

It is shown that Nirenberg's problem of prescribed Gauss curvature on S^2 and all its higher-dimensional variants of Paneitz type, finds some answers in form of a law of large numbers of a Hamiltonion system of point particles with logarithmic pair interactions on S^2 (resp.S^n) in statistical equilibrium. This characterization is (currently) restricted to Gauss (Paneitz) curvature functions that do not change sign; also, only those metrics can be found which maximize a (relative) entropy functional.

Passage from discrete to continuous variational problems

Andrea Braides, SISSA, Trieste

(joint work with M.S. Gelli and L. Truskinovsky)

We first consider K-neighbour interaction discrete energies on the real line with $K \in \mathbb{N}$,

(0)

$$E_n(u) = \sum_{j=1}^{k} \sum_{c=0}^{n-k} \lambda_n \Psi_j^n \left(\frac{u_{i+j} - u_i}{j\lambda_n}\right),$$

where $\lambda_n = L/n$, $u : I_n \to \mathbb{R}$, where $I_n = \{0 \dots n\}$. Each function u is identified with a function (still denoted by u) defined by $u(i\lambda_n) = u_i$ on $\{0, \lambda_n, 2\lambda_n, \dots, L\}$ and as its precise-affine interpolation elsewhere. We then may view E_n as functionals defined on (subsets of) $L^1(O, L)$, and study their Γ-convergence as $n \to \infty$. We single out two main "principles" which describe the limit: "clustering" and "separation of scales".

A. Clustering. Note preliminarily that if $K = 1$ then the Γ-limit is given by:

(1)

$$F(u) = \int_{(O,L)} \Psi(u')dt$$

(suppose all functions are of p-growth), where $\Psi = \lim_n \Psi_1^{n**}$ (which exists upon extracting a subsequence). If all functions are of p-growth and $K > 1$ then the "clustering principle" reads as follows: There exists $N \in \mathbb{N}$ and $\bar{\Psi}_n$ such that, set $\eta_n = N\lambda_n$ and

(2)

$$\bar{E}_n(u) = \sum_{i=0}^{[u/N]-1} \eta_n \bar{\Psi}_n(\frac{u_{i+1}-u_i}{\eta_n})$$

Then $\Gamma - \lim_n E_n = \Gamma - \lim_n \bar{E}_n$(up to a small error). By the observation above, the limit can be represented as (1) with $\Psi = \lim_1 \bar{\Psi}_n^{**}$. The general formula to compute $\bar{\Psi}_n$ is of "homogenization type":

(3)

$$\bar{\Psi}_n(z) = \inf \mu f \left\{ \frac{I}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \Psi_j^n(\frac{u(i+j)-u(i)}{j}) : \begin{array}{l} u : \{0, \dots N\} \to \mathbb{R} \\ u(i) = iz \\ i = 0 \dots K \,\text{and}\, i \in N-K, \dots, N \end{array} \right\}$$

B. Separation of scales. If Ψ_j^n are not of p-growth, the limit energy is not of the form (1) but

(4)

$$F(u) = \int_{O,L} f(u')dt + \sum_{S(u)} g(u^+ - u^-)$$

defined on precise -$W^{1/p}$ functions and (if necessary) relaxed to BV. Consider for simplicity the case $K = 1$ and that $\psi_1^n(z) = +\infty$ if $z \leq 0$. Then the way to compute f and g reads as follows: choose T_n with the property $T_n \to \infty$ and $\lambda T_n \to 0$, set

(5)

$$f_n(z) = \begin{cases} \Psi_1^n(z) & \text{if}\, z \leq Tn \\ +\infty & \text{otherwise} \end{cases} \qquad g_n(z) = \begin{cases} \lambda_n(\Psi_1^n(\frac{z}{\lambda_u}) - \min \Psi_1^n) & z \geq \lambda_n T_n \\ +\infty & \text{otherwise} \end{cases}$$

Then

(6)

$$f = \Gamma - \lim_n f_n^{**} \qquad g = \Gamma - \lim_n (sub\, g_n) \qquad (\text{sub} = \text{"subadditive envelope"})$$

The case $K > 1$ is obtained by combining the two "principles above".
Finally, we remark that the case $K = K_n \to \infty$ cannot be treated directly in this way since we may have limit energies of the form

(7)

$$F(u) = \int f(u')dt + \sum_{S(u)} g(u^+ - u^-) + \int\int \Psi(u(x) - u(y))d\mu(x,y)$$

Some results extend to lattice engergies in $\mathbb{R}^n$.

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