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Funktionentheorie

February 11 – February 17, 2001

Organizers: Kari Astala (Jyväskylä), Walter Bergweiler (Kiel), Reiner Kühnau (Halle).

The conference *Funktionentheorie* is being held bi-annually at Oberwolfach, with each meeting having emphasis on different aspects of function theory. In the present conference the emphasis was on conformal and quasiconformal mappings and their relations with different areas in mathematics.

The purpose of the meeting was to bring together specialists in a variety of different related directions to provide a fruitful interaction on the basis of common interests. This aim is also reflected in the diversity of the topics among the 22 talks given during the conference. We describe here in detail three of the highlights:

The solution for the long standing open problem on the conformal invariance of the continuum scaling limits of the critical site percolation processes was obtained by S. Smirnov. With harmonic conformal invariants and discrete harmonic maps Smirnov gave a rigorous proof for Cardy's formula predicted by physicists and conformal field theory.

Secondly, M. Bonk described his joint work with B. Kleiner towards Cannon's question if the boundary of a Gromov hyperbolic group is quasisymmetric to the standard two-sphere, whenever it is topologically a two-sphere. This is an important step in Thurston's hyperbolization conjecture. The work of Bonk and Kleiner shows that any metric Ahlfors 2-regular space which is linearly locally connected is quasisymmetrically equivalent to the two sphere.

A. Eremenko studied properties of rational maps, and showed that if a rational function f has all its critical points on a circle C, then f maps C into a circle. This result has several strong consequences to real enumerative geometry in real Grassman varieties and also, in a quite different direction, to questions on the output feedback of linear systems.

Other highlights include recent breakthroughs for the mappings of finite dilatation (Iwaniec, Koskela) related to questions in nonlinear elasticity, and the work of Jones on the Cauchy transform of measures, whose geometric approach has applications e.g. for image processing. In the more traditional directions of geometric function theory we had reports about a solution to an old problem of Teichmüller concerning uniqueness of extremal quasiconformal mappings and relations between Grunsky coefficients and universal Teichmüller space. Further talks were given on questions related e.g. to complex dynamics, value distribution theory and univalent functions.

In addition to the talks there were two problem sessions which led to lively discussions.

Again the pleasant atmosphere of Oberwolfach and the impressively friendly assistance of the staff of MFO contributed substantially to the success of the conference.

Abstracts of talks

Quasisymmetrically standard two-dimensional spheres

M. Bonk

In connection with Thurston's hyperbolization conjecture, Cannon asked the following question: Suppose G is a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is a topological 2-sphere. Is $\partial_\infty G$ quasisymmetric to the standard sphere $\mathbb{S}^2$? This leads to the following general recognition problem of the standard sphere $\mathbb{S}^2$ up to quasisymmetry. Suppose Z is a metric space homeomorphic to $\mathbb{S}^2$. Under what conditions is Z quasisymmetric to $\mathbb{S}^2$? In joint work with B. Kleiner we obtained the following sufficient condition. If Z is Ahlfors 2-regular and linearly locally connected, then Z is quasisymmetric to $\mathbb{S}^2$. This answers a question by Heinonen and Semmes.

Our methods can also be used to get positive results for spaces that are not 2-regular.

Hyperbolic and elliptic capacity

P. Duren

In collaboration with John Pfaltzgraff (1999), a connection was given between hyperbolic transfinite diameter (or hyperbolic capacity) and extremal length. Hyperbolic capacity of a closed set in the unit disk is defined in terms of the pseudohyperbolic metric $\rho(z,\zeta) = |(z-\zeta)/(1-\overline{\zeta}z)|$. In 1947, M. Tsuji defined elliptic (or spherical) capacity for closed sets in the extended complex plane $\widehat{\mathbb{C}}$, using the chordal metric, an unfortunate choice. Reiner Kühnau (1969) used the pseudoelliptic metric $[z,\zeta] = |(z-\zeta)/(1+\overline{\zeta}z)|$ to develop elliptic capacity for closed sets $E \subset \widehat{\mathbb{C}}$ disjoint from the antipodal set $E^* = \{-1/\overline{z} : z \in E\}$. In joint work with Kühnau, the theory is now extended to include an extremal length description of elliptic capacity, a "physical interpretation" of elliptic Robin capacity as a least energy computed via a Neumann kernel, and a characterization of elliptic Robin capacity of a subset $A \subset \partial\Omega$ as the minimum of elliptic capacity of the image under diametrically symmetric mappings of the domain Ω "between" $A \cup B$ and $A^* \cup B^*$. The last result depends on a new variational method for diametrically symmetric domains.

Extremal quasiconformal mappings and Schwarz's lemma

C. Earle

Let T be the universal Teichmüller space and Δ be the open unit disk in $\mathbb{C}$. We denote the Teichmüller metric on T and the Poincaré metric on Δ by d_t and d_Δ respectively. The Teichmüller norm of a tangent vector v to T is denoted by $\|v\|_T$. We prove the following δ-ϵ form of "Schwarz's Lemma" for T.

Theorem *For any t in Δ with $t \neq 0$ there exist functions $\hat{\delta}\colon (0,1) \to (0,\infty)$ and $\hat{\epsilon}\colon (0,\infty) \to (0,1)$ such that for any holomorphic map $f\colon \Delta \to T$ the following inequalities hold:*

$$\begin{aligned} 1 - \|f'(0)\|_t &\geq \hat{\epsilon}(\delta) &\quad \textit{if} \quad d_\Delta(0,t) - d_T(f(0),f(t)) &\geq \delta\ (>0), \\ d_\Delta(0,t) - d_T(f(0),f(t)) &\geq \hat{\delta}(\epsilon) &\quad \textit{if} \quad 1 - \|f'(0)\|_T &\geq \epsilon\ (>0). \end{aligned}$$

These inequalities imply and are implied by Gardiner's Principle of Teichmüller contraction, so they also imply the Hamilton-Krushkal-Reich-Strebel condition for extremality. They also follow, with sharp constants, from the classical Schwarz Lemma and Krushkal's recent theorem that the Teichmüller metric d_T is the Carathéodory metric on T.

Rational functions with real critical points

A. Eremenko

This is a report on the recent joint work with A. Gabrielov, on real Grassman varieties $G(m, m+2)$ of subspaces of codimension 2. We find the degree of the central projections $G(m, m+2) \to \mathbb{P}^{2m}$. The target space is non-orientable, while the domain may be orientable or not, so an appropriate notion of degree has to be used. It turns out that the original definition of Kronecker is suitable. We prove:

1. These projections have degrees, which are independent of the center of projection, and
2. If m is even, the degree is 0, while when M is odd it equals to the Catalan number $u((m+1)/2)$.

These results have several applications to real enumerative geometry and to the problem of pole placement for linear systems by static output feedback.

The main analytic tool is the following

Theorem *If f is a rational function whose critical points belong to a circle C, then f maps C into a circle.*

A brief sketch of the proof of this result is given.

Linearizability and the Brjuno condition

L. Geyer

An analytic function $f(z) = \lambda z + \ldots$ is *linearizable*, if there exists $\phi(z) = z + \ldots$ such that $\phi(f(z)) = \lambda\phi(z)$ near 0. Most interesting is the *irrationally indifferent* case $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. From the work of Rüssmann, Brjuno and Yoccoz it is known that every such f is linearizable if and only if the convergents (p_n/q_n) of α satisfy the *Brjuno condition* $\sum q_n^{-1} \log q_{n+1} < \infty$. The condition is sharp for quadratic polynomials and Douady raised the following conjecture:

Conjecture *If some $f(z) = \lambda z + a_2 z^2 + \ldots + a_{d-1} z^{d-1} + z^d$, $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $d \geq 2$, is linearizable, then α satisfies the Brjuno condition.*

Pérez-Marco showed that this conjecture is true in the generic case. We show that it holds for many non-generic cases, too. In particular, for $d = 3$ we obtain

Theorem *Assume that $f(z) = \lambda z + az^2 + z^3$, $\lambda = e^{2\pi i\alpha}$, is linearizable and one of the following holds:*

(i) *f has some indifferent periodic point $z_0 \neq 0$,*
(ii) *f satisfies a critical orbit relation $f^k(c_1) = f^l(c_2)$ where c_1 and c_2 are the critical points of f, or*
(iii) *f has a preperiodic critical point.*

Then α satisfies the Brjuno condition.

Doubling conformal densities

J. HEINONEN

I discuss the problem how to characterize those nonnegative, locally integrable functions w in $\mathbb{R}^2$ that are comparable to the Jacobians J_f of quasiconformal maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. Two conditions, the doubling property of the measure $w\,dx$ and the comparability of the quantity $d_w(x,y) = (\int_{B_{x,y}} w\,dx)^{1/2}$ (where $B_{x,y}$ denotes the disk with diameter points x and y) to a metric on $\mathbb{R}^2$, were thought of as possible candidates for characterizing conditions.

Recently, Tomi Laakso crushed these hopes. I described a joined work with Mario Bonk and Steffen Rohde, where we, in an appropriate sense, characterize the behavior of quasiconformal Jacobians on a circle.

Oscillation of quasiregular maps

H. HINKKANEN

This talk described joint work with J. M. Anderson, performed at Oberwolfach in the Research in Pairs programme in the summer of 2000.

We proved the following results. For a real-valued function u in a ball $B^n(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$, we define the oscillation by

$$\operatorname{osc}(u,r) = \sup\{u(x) : |x| < r\} - \inf\{u(x) : |x| < r\}.$$

We obtained the estimate

$$\operatorname{osc}(u,R) \le C(\operatorname{osc}(u,r))^{\alpha} \operatorname{osc}(u,1))^{1-\alpha}$$

for $0 < r < R < 1$, when u is harmonic and bounded in the unit disk $B^2(0,1)$, with $C = 563$ and $\alpha = (\log R)/(\log(r/2))$, and when $u = f_j$ is a component of a bounded $K-$quasiregular map $f = (f_1, \ldots, f_n)$ in $B^n(0,1)$, where $n \ge 2$, with C depending on n and K, and $\alpha \in (0,1)$ depending on n, K, r, and R. The proof uses the two-constant theorem of S. Rickman for quasiregular maps, and a counterpart of the Borel–Carathéodory inequality for quasiregular maps, which we proved following a method of C. Nolder. Oscillation estimates are of interest in the theory of partial differential equations.

Jacobians

T. IWANIEC

The lecture presents recent results of the speaker's joint papers with F. Giannetti, J. Onninen and A. Verde. We study mappings $f\colon \Omega \to \mathbb{R}^n$ of the Sobolev class $W^{1,n-1}(\Omega,\mathbb{R}^n)$ defined in an open region $\Omega \subset \mathbb{R}^n$, its differential $Df = [\partial f^i/\partial x_j]$ and the cofactor matrix $D^\sharp$.

Theorem *If $|D^\sharp f|^{n/(n-1)} \in L^1(\Omega)$ then the Jacobian determinant belongs to the Hardy space $H^1(\Omega)$, and we have the uniform estimate*

$$\|\det Df\|_{H^1(\Omega)} \le C(n) \int_\Omega |D^\sharp f|^{n/(n-1)}.$$

The key tool in the proof is the spherical maximal operator. Next, we take on stage the orientation preserving mappings, meaning that $\det Df(x) \ge 0$ almost everywhere. We want to impose the minimal integrability conditions on the cofactor matrix required to ensure that the Jacobian will be locally integrable

Theorem *Suppose* $|D^\sharp f|^{n/(n-1)}$ *lies in the Orlicz space* $L^P(\Omega)$ *where* $P\colon [0,\infty) \to [0,\infty)$ *satisfies the so-called divergence condition* $\int_1^\infty P(s)s^{-2}\,ds = \infty$*. Then, under additional minor technical assumptions on* P*, the Jacobian determinant is locally integrable and obeys the usual rule of integration by parts.*

As a consequence, we establish the so-called *weak compactness principle* for mappings with subexponentially integrable distortion.

On extremal decompositions

J. A. Jenkins

The study of extremal decompositions of Riemann surfaces was initiated in a paper which I published in 1957 to obtain the solution of a module problem for multiple curve families on a finite Riemann surface.

Let $\mathcal{R}$ be a finite Riemann surface, H_j, $j = 1, \dots, L$, a free family of homotopy classes on $\mathcal{R}$, a_j, $j = 1, \dots, L$, non-negative constants not all zero. The two problems are as follows.

$P(a_1, \dots, a_L)$. Let $\rho(z)\,|dz|$ be a conformally invariant metric on $\mathcal{R}$ with ρ measurable, non-negative and such that $\int_{\gamma_j} \rho\,|dz|$ exists for every γ_j rectifiable in H_j and satisfies $\int_{\gamma_j} \rho\,|dz| = a_j$, $j = 1, \dots, L$. Find the greatest lower bound $M(a_1, \dots, a_L)$ of $\int\int_\Omega \rho^2\,dA$.

$\mathcal{P}(a_1, \dots, a_L)$. Let D_j, $j = 1, \dots, L$, be non-overlapping doubly-connected domains of module M_j on $\mathcal{R}$, associated with H_j, $j = 1, \dots, L$. Find the least upper bound of $\sum_{j=1}^L {a_j}^2 M_j$.

The essential step is to prove the equivalence of these problems. Renelt considered another type of extremal decomposition:

$R(b_1, \dots, b_L)$. Let b_j be positive numbers, D_j non-overlapping doubly-connected domains of module M_j on $\mathcal{R}$ associated with H_j, $j = 1, \dots, L$. Find the greatest lower bound of $\sum_{j=1}^L {b_j}^2/M_j$.

In a paper published in 1993 I gave another treatment of the earlier results and also showed that the solution of problem $R(b_1, \dots, b_L)$ is an easy consequence of the solution of problem $\mathcal{P}(a_1, \dots, a_L)$. As a matter of fact both these problems are special cases of the following problem.

$X(a_1, \dots, a_N, b_{N+1}, \dots, b_L)$. Let a_j, $j = 1, \dots, N$, b_j, $j = N+1, \dots, L$, be non-negative constants not all zero with any b_j which actually occurs positive. For non-overlapping doubly-connected domains D_j of module M_j on $\mathcal{R}$ associated with H_j, $j = 1, \dots, L$. Find the least upper bound of $\sum_{j=1}^N {a_j}^2 M_j - \sum_{j=N+1}^L {b_j}^2/M_j$.

Cauchy transforms of measures

P. Jones

Let $\mu \geq 0$ be a probability measure on $[0,1]^n \subset \mathbb{R}^n$, and let d be an integer, $1 \leq d \leq n$. When does there exist a "good" d-dimensional surface $\mathcal{S}$ with d-area $H^d(\mathcal{S}) \leq A$, such that $\mu(\mathcal{S}) \geq \epsilon > 0$? We give a sharp answer in terms of L^2 based Beta Numbers. For a dyadic cube Q we define

$$\beta_2(Q) = \inf_P \left(\frac{\mathrm{dist}(x,P)^2}{l(Q)^2} \frac{d\mu(x)}{\mu(Q)} \right)^{1/2},$$

where the infimum is over all d-hyperplanes P, and where $l(Q)$ is the sidelength. (If $\mu(Q) = 0$, set $\beta_2(Q) = 0$).

Theorem [P. W. Jones, G. Lerman] *If for all x, $\sum_{Tr} \sum_{Q \ni x} \beta_2(Q)^2 \leq A$, then there is a "good" d "surface" $\mathcal{S}$ (it is not actually a surface, but a canonical gluing of surfaces) with $H^d(\mathcal{S}) \leq e^{cA}$ and $\mu(\mathcal{S}) \geq e^{-cA}$.*

The sharpest form allows exponential integrability ($d\mu$) of the above pointwise defined function. The $\sum_{Tr}$ indicates that one must translate the dyadic grid 2^n times by canonical translations. All results are best possible. This theorem is actually a descendent of L^2 estimates for the Cauchy Integral. As an application we prove every continuum in the plane supports a probability measure whose Cauchy Transform is bounded.

Mappings of finite distortion

P. Koskela

Let $\Omega \subset \mathbb{R}^n$ be a domain. A mapping $f\colon \Omega \to \mathbb{R}^n$ has finite distortion if

(1) $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$,
(2) $J_f \in L^1_{\mathrm{loc}}(\Omega)$,
(3) $|Df(x)|^n \leq K(\lambda) J_f(x)$ a.e. where $1 \leq K(x) < \infty$ is measurable.

When $K \in L^\infty(\Omega)$ we obtain the class of quasiregular mappings, also called mappings of finite distortion. In this case, $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, f is continuous and if non-constant, also open and discrete, $J_f(x) > 0$ a.e. and $|f(E)| = 0$ iff $|E| = 0$.

If we apriori assume that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, then all these conditions hold when $f \in L^p(\Omega)$, $p > n-1$, and the bound $n-1$ is critical. How much can the boundedness of K be relaxed when no stronger regularity assumptions than (1), (2) and (3) are made?

It turns out that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ if $\exp(\lambda K) \in L^1(\Omega)$ for some $\lambda > \lambda(n) > 0$ and that such a bound on λ is needed for this regularity conclusion. By the above, the mentioned properties of mappings of bounded distortion then follow. Different methods show that in fact all the good properties of mappings of bounded distortion hold except for $W^{1,n}$-regularity as soon as $\exp(\psi(K)) \in L^1(\Omega)$ for some increasing ψ with $\int_1^\infty \psi(t)/t^2\, dt = \infty$ and that this divergence condition is critical.

Complex geometry of the universal Teichmüller space and Grunsky coefficients

S. Krushkal

We show that all invariant distances on the universal Teichmüller space coincide and are determined by the Grunsky coefficients of the naturally related conformal maps. This fact has various important consequences. In particular, we obtain solutions for certain well-known problems in complex analysis and related fields.

Rational maps of manifolds and quasiregular mappings

G. Martin

A uniformly quasiregular mapping of a Riemannian manifold $\mathcal{M}$ is a mapping which is rational (conformal away from branch set) with respect to some measurable Riemannian structure. The Lichnerowicz conjecture, proved by J. Ferrand, asserts that among all compact Riemannian manifolds only the n-sphere admits a non compact conformal automorphism group. We consider the analogous problem of determining which compact Riemannian manifolds admit a non injective rational mapping. The semigroup of all rational mapping will be non compact. We prove among other things that all such manifolds are qr-elliptic. A concept developed by Gromov and Rickman and having various topological consequences (such as polynomial growth fundamental group). In three dimensions this condition is necessary and sufficient.

Harmonic and quasiconformal mappings

M. Mateljević

We state a version of Dirichlet's principle for harmonic mappings and generalize the classical area theorem in different directions. We study uniqueness of harmonic mappings using Dirchlet's principle and different versions of the main inequality.

Extremal problems for quasiconformal mappings are also subject of our investigations. In particular, we solve Teichmüller's problem posed around 1935. We generalize the argument principle to harmonic maps. More precisely, we express the number of zeros and poles in terms of total geodesic curvature and the number of zeroes of the Jacobian of the mapping in terms of the index of rotation and graphs.

Some extremal problems for univalent functions

O. Roth

Problem 1: For an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ its hyperbolic derivative Df is defined by $Df(z) = (1-|z|^2)/(1-|f(z)|^2)f'(z)$.

The following result includes and refines distortion theorems due to Koebe, Pick, Blatter, Kim and Minda, Ma and Minda, and Jenkins. The proof uses the Löwner differential equation and methods from optimal control theory.

Theorem *If $f\colon \mathbb{D} \to \mathbb{D}$ is univalent, and z_1, z_2 are two distinct points in $\mathbb{D}$, then we have, for any $p \geq 1$,*

$$\left(\left(\frac{|Df(z_1)|}{1-|Df(z_1)|} \right)^p + \left(\frac{|Df(z_2)|}{1-|Df(z_2)|} \right)^p \right)^{1/p} \leq \frac{(2\cosh(2p\rho))^{1/p}\sinh(2\rho')}{\sinh(2(\rho-\rho'))},$$

where $\rho = \mathrm{d}_{\mathbb{D}}(z_1, z_2)$ and $\rho' = \mathrm{d}_{\mathbb{D}}(f(z_1), f(z_2))$. Equality occurs if and only if f maps $\mathbb{D}$ onto $\mathbb{D}$ slit along a hyperbolic ray on the hyperbolic geodesic determined by $f(z_1)$ and $f(z_2)$. The inequality does not obtain for $0 < p < 1$.

Problem 2: (joint work with R. Greiner) Let $\mathcal{S}$ be the set of univalent functions $f(z) = z + a_2 z^2 + \ldots$ in $\mathbb{D}$ and let $k(z) = z/(1-z)^2 \in \mathcal{S}$ be the Koebe mapping. The following result disproves a conjecture of Bombieri.

Theorem $\liminf_{f \to k}(2 - \operatorname{Re} a_2)/(3 - \operatorname{Re} a_3) = (e-1)/(4e)$ *where the* $\liminf$ *is taken over all functions in* $\mathcal{S}$.

Meromorphic functions with shared limit values

A. Sauer

Let $f, g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be transcendental meromorphic functions. We say that f and g *share the limit value* $a \in \widehat{\mathbb{C}}$ if for all sequences $z_n \to \infty$

$$f(z_n) \to a \qquad \Longleftrightarrow \qquad g(z_n) \to a.$$

It is shown that $T(r, f) \leq q/(q-2)T(r, g) + S(r, f)$ if $q \geq 3$ limit values are shared. Examples with three and four shared limit values are given. Further we prove a weak analogue of R. Nevanlinna's five point theorem. We state the following conjecture:

Conjecture *If f and g share five limit values, then all limit values are shared.*

We investigate the generalized problem

$$f(z_n) \to a \in M \qquad \Longrightarrow \qquad g(z_n) \to \varphi(a) \in \varphi(M)$$

with some function φ on an open set M. We prove that φ is necessarily meromorphic. This shows that there is little hope to construct non-trivial examples by deformations of the form $\varphi \circ f \circ \psi$. In all proofs Ahlfors's theory of covering surfaces and normal families (in particular the Zalcman lemma) play a key role.

Critical percolation in the plane and conformal invariants

S. Smirnov

We introduce harmonic conformal invariants with the help of the critical site percolation on triangular lattice. As a corollary we obtain Cardy's formula and conformal invariance of the critical percolation, and construct its (continuum) scaling limit.

Results and questions in minimal area problems for conformal mappings

A. Solynin

A minimal area problem consists in minimizing the Dirichlet integral $D(f)$ among functions f from certain classes of analytic functions. We discuss two kinds of such problems.

One concerns the minimal area problem with analytic side conditions first studied by H. S. Shapiro. In particular, we present a solution to minimal area problems for a subclass of univalent functions from the standard class $\mathcal{S}$ which have prescribed second coefficient. This long standing problem was solved by D. Aharonov, H. Shapiro, and A. Solynin.

Another kind of minimal area problems initiated by A. W. Goodman deals with classes of analytic univalent functions with geometric constraints. We shall discuss such a problem for the Carathéodory class of analytic functions having positive real part in the unit disk. This part is based on our joint work with R. Barnard.

On the existence and computation of the Robin function

M. Stiemer

The Robin function R in a domain $\Omega \subset \widehat{\mathbb{C}}$, that contains ∞, is the solution of a homogeneous mixed boundary value problem for the Laplace operator with a logarithmic singularity at ∞, where Dirichlet conditions are given on one part of $\partial\Omega$ and Neumann conditions on the other part. Recent research of P. Duren and M. Schiffer has revealed that the Robin capacity $\rho(A) = \exp\left(-\lim_{z\to\infty}[R(z) - \log(z)]\right)$ of the Dirichlet boundary A is exactly the minimum of the logarithmic capacity $\operatorname{cap} f(A)$ over all conformal mappings of the given domain Ω, that are normalized by $f(z) = z + O(1)$, $z \to \infty$.

We present a new existence proof for the Robin function, that is constructive and needs only basic principles from complex analysis. Moreover, formulas for $\rho(A)$ were derived in special situations. Finally, approximations to $R(z)$ and $\rho(A)$ are calculated in the case that A is a subarc of a Jordan curve. We apply an extremal point system introduced by K. Menke. Its good distribution properties provide a geometrically shrinking error estimate.

Perturbations of isometries

J. Väisälä

A map $f\colon X \to Y$ between metric spaces is an s-*nearisometry* if

$$|x-y| - s \le |f(x) - f(y)| \le |x-y| + s$$

for all $x, y \in X$. We considered conditions under which an s-nearisometry can be approximated by an isometry $T\colon X \to Y$ such that $d(T,f) = \sup\{|T(x) - f(x)| : x \in X\}$ is small if s is small. A basic result is the theorem of Hyers-Ulam-Gruber-Gevirtz-Omladič-Šemrl: If X and Y are Banach spaces and if f is surjective, then there exists a surjective T with $d(T,f) \le 2s$. We discussed related results and gave a quantitative geometric characterization for bounded sets $X \subset R^n$ such that for $f\colon X \to R^n$ there is T with $d(T,f) \le cs$.

Phase-dependence of the Hausdorff dimension of Julia-Lavaurs set

M. Zinsmeister

This is a joint work with M. Urbanski. Let $d(c)$ be the Hausdorff dimension of the Julia set of the polynomial $z^2 + c$. It is known that d is not continuous at any point c for which there exists a parabolic cycle. It is not known however if this function, restricted to $\mathbb{R}$ has a limit from the right at $1/4$. This is equivalent to the following question: Is $\delta(\sigma)$, the Hausdorff dimension of the Julia-Lavaurs set $J(f_0, g_\sigma)$, constant for real σ? We show that $\delta(\sigma)$ is real-analytic in Σ, the set of σ's for which the critical point escapes in one step. We prove moreover that δ is continuous along any external ray landing on $\partial\Sigma$. As a consequence we prove that, if constant, $\delta > 4/3$.

Abstracts of problems

Rank-one convexity, quasiconvexity, and integrability of partial derivatives of quasiregular mappings

A. Baernstein

Let $f\colon B \to \mathbb{R}^n$ be a Lipschitz function in a ball $B \subset \mathbb{R}^n$, and $\epsilon > 0$.
Conjecture [Stretch Conjecture]

$$\int_B (|Df|^{n/2} - \epsilon)^+ \left(\frac{J}{|Df|^{n/2}} - \epsilon \right) dV \leq \int_{\partial B} \left((|Df|^{n/2} - \epsilon)^+ \right)^2 dS.$$

Here $|Df|$ and Jf denote maximal directional derivative and Jacobian determinant. Truth of the Stretch Conjecture would imply sharp integrability results for gradients of quasiregular mappings in $\mathbb{R}^n$. Falsity for $n = 2$ and some f vanishing on ∂B would confirm Morrey's conjecture that rank-one convex functions on the space of real 2×2 matrices need not be quasiconvex.

Critical points in parabolic basins

W. Bergweiler

Let f be rational, with $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ near 0, where $a_2 \neq 0$. The (unique) invariant domain U where the iterates of f tend to 0 contains at least one critical point of f. If U contains exactly one critical point, then $\mathrm{Re}(a_3/{a_2}^2) \leq 3/4$, as shown by Shishikura, Buff and Epstein, and Bergweiler. While $a_3/{a_2}^2 = 3/4$ for the Koebe function, Buff and Epstein have given improved estimates for polynomials. What is the sharp upper bound for $\mathrm{Re}(a_3/{a_2}^2)$, if f is a polynomial? For degree 3 the bound seems to be 1/4.

Two problems on the Wronskians

A. Eremenko

1. Suppose that the Wronskian determinant of several polynomials $f_1, \dots, f_n$ is a polynomial with only real zeros. Then there exists a matrix $M \in \mathrm{GL}(n, \mathbb{C})$, such that $(f_1, \dots, f_n)M$ is a vector of real polynomials. This is a special case of the so-called B. and M. Shapiro conjecture. Gabrielov and I proved this for $n = 2$, but our method does not seem to extend to the case $n \geq 3$.

2. The Wronskian determinant $W(f_1, f_2)$ of a pair of non-proportional polynomials of degrees at most d defines a map of the Grassman variety $G(d-1, d+1) \to \mathbb{P}^{2d-2}$, where the projective space $\mathbb{P}^{2d-2}$ is identified with the the set of all non-zero polynomials of degree at most $2d-2$, up to proportionality. Find the set of critical values (or critical points, whichever is easier) of this map. Almost nothing is known, except for $d = 3$, when this can be explicitly computed. The question was asked by Lisa Goldberg in 1990.

Background and References can be found on
`http://www.math.purdue.edu/~eremenko/newprep.html`

A Riemann surface arising from a problem of G. Mac Lane

W. Hayman

The following problem was raised by P. J. Rippon. Let a_n be a strictly increasing sequence of real numbers tending to $\pm\infty$ with n. A simply connected Riemann surface $\mathcal{R}$ is constructed by cutting, for each n, the plane from a_n to $a_n + i\infty$ and attaching a “half” $\mathcal{R}_n^+$ of the Riemann surface of $\log(z - a_n)$ to the cut as we approach it from the left, and a similar surface $\mathcal{R}_n^-$ as we approach the cut from the right. The question was, whether $\mathcal{R}$ could ever be hyperbolic.

Eremenko has pointed out that ideas and results of Volkoviskii can be used to give a positive answer. This should allow the construction of a locally univalent function in the Mac Lane class and having an arc tract.

Injectivity of certain integrals related to quasiconformal maps

S. Krushkal

It remains open a question posed by I. N. Vekua in 1961: are the iterations

$$w_n(z) = z + T\varrho_n(z) = z + T\mu(z) + T(\mu\Pi\mu)(z) + \dots$$

for a homeomorphic in $\mathbb{C}$ solution $w(z)$ of the Beltrami equation

$$\partial_{\bar{z}} w = \mu(z)\partial_z w,$$

$\mu \in L_\infty(\mathbb{C})$, $\|\mu\| = 1$ and $\mu(z) = 0$ in a neighborhood of infinity, also homeomorphisms of $\mathbb{C}$? Here $\varrho = \partial_{\bar{z}} w \in L_p(\mathbb{C})$ for some $p \in (2, p_0(\|\mu\|))$,

$$T\varrho(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\varrho(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta},$$

$\Pi = \partial_z T$ is the two-dimensional Hilbert transform, and ϱ_n denotes the nth partial sum of the series for ϱ. The only known here was a partial result of Belinkii on quasiconformality of variation w_1.

Recently, T. Iwaniec constructed a counterexample giving a non injective iteration w_1. A question is to establish the sufficient conditions, which ensure the injectivity of any iteration w_n.

Are the Teichmüller spaces holomorphically contractible?

S. Krushkal

The holomorphic contractibility of a Stein manifold means that the identity map is homotopic to a constant map in holomorphic category. The problem on such contractibility of Teichüller spaces $T(0, n)$ of the spheres $\mathbb{S}^2$ with n punctures was posed by E. Gorin in 1967 concerning the solving algebraic equations in Banach algebras; it relates to Oka's h-principle. The problem is open for all Teichüller spaces of dimension at least 2. The main obstruction is that, due to Hirschowitz and Zaidenberg-Lin, there are the domains in $\mathbb{C}^n$, $n > 1$ (even bounded), which are contractible but not holomorphically.

A coefficient problem for bounded non-vanishing functions

J. Krzyż

Suppose $\mathbb{D} = \{z : |z| < 1\}$ and $\mathcal{B}_0 = \{f \in \mathrm{Hol}(\mathbb{D}) : 0 < |f(z)| < 1\}$. Let $c_n = c_n(f) = f^{(n)}(0)/n!$, where $f \in \mathcal{B}_0$ and $n \in \mathbb{N}$. The following problem was posed in [Ann. Polon. Math. 20 (1967–68), p. 314]:

Problem *Prove (or disprove) that* $|c_n(f)| \le 2/e$ *for all* $f \in \mathcal{B}_0$ *and any* $n \in \mathbb{N}$ *with the sign of equality for* $f_n(z) = \exp(-(1+z^n)/(1-z^n))$.

It is easily verified that this holds for $n = 1, 2$. J. A. Hummel, St. Scheinberg and L. Zalcman [J. Anal. Math. 31 (1977), pp. 169-180] were able to find some formal similarity between $\mathcal{B}_0$ and the familiar class $\mathcal{S}$ of univalent functions. In particular, they established a Loewner type differential equation for functions $f \in \mathcal{B}_0$ which allowed them to verify the conjecture for $n = 3$. Another relevant result was established by C. Horowitz [Israel J. Math. 30 (1978), pp. 285–291]: $|c_n(f)| < 0.9998\ldots$ for any $n \in \mathbb{N}$ and any $f \in \mathcal{B}_0$. Several authors claim to have proved the conjecture for $n = 4$, however, some proofs are either incomplete, or relay on extensive computer calculations. A satisfactory proof is due to W. Szapiel [Ann. Univ. Mariae Curie-Skłodowska Sect. A 48 (1994), pp. 169–192], where also many bibliographical references can be found.

Extremal quasiconformal maps, laminations and critical phase phenomena

G. Martin

We discussed the issues of rank-1 convexity versus quasiconvexity for the linear distortion function. We announced that generically the linear distortion function is not rank-1 convex. However along the variety where a matrix has repeated singular values it is rank-1 convex. We asked the question if the linear distortion function is quasiconvex or even polyconvex at such points. It turns out that this is not the case, details will appear later.

An open question connected to a condition for extremal quasiconformal maps

E. Reich

In 1981 [Ann. Acad. Sci. Fenn., Ser. A I 6, pp. 289–301] I gave two sufficient conditions for unique extremality of quasiconformal mappings of Teichmüller type, referred to as "Theorem A" and "Theorem B" in that reference. As a consequence of the work of Bozin et al [J. Anal. Math., 75 (1998), pp. 299–338] we now know that the first of these conditions is also necessary in case of mappings of Teichmüller type. Theorem B has turned out to be very useful.

Question *Is it also necessary for unique extremality?*

Julia directions of f and f'

A. Sauer

Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function. It is a classical result of Julia, that f possesses a Julia direction if f has an asymptotic value. In general f need not have a Julia direction, as is shown by the example $f(z) := \prod_{n=0}^{\infty}(z-2^n)/(z+2^n)$. We pose the following

Problem *Does at least one of the functions f, f' have a Julia direction?*

The statement is true if f is $o(\sqrt{|z|})$ on a path to ∞. (The question originates from discussions with John Rossi.)

Can we recover a shape from measurements of capacity?

A. Solynin

For a continuum E in the closed unit disk containing the origin and $R > 1$, let $c(R)$ denote the capacity of a condenser with plates E and $\{|z| = R\}$. Let $R_1, R_2, \ldots$ be a sequence of distinct real numbers such that $R_i \geq R_0 > 1$.

1. Can we recover the shape of E (up to symmetry and rotation) from measurements $c(R_1)$,$c(R_2), \ldots$?
2. If "Yes", can we find an explicit procedure?
3. The same problem in $\mathbb{R}^n$.

C. Earle has proved recently that $c(R)$ is real analytic. Grötzsch's lemma implies that two measurements are enough to distinguish a circle and it follows from Solynin's and Laugesen's results that a sequence of measurements allows to distinguish a needle.

How the harmonic measure is distributed on sides of a hyperbolic polygon

A. Solynin

Let $D_n \ni 0$ be a hyperbolic n-gon with sides γ_k having all vertices on the unit circle $\mathbb{T}$. Let e_k be an arc on $\mathbb{T}$ corresponding to γ_k. Let u_k and v_k denote, respectively, the harmonic measure of e_k and γ_k w.r.t. the unit disk $\mathbb{U}$ and D_n evaluated at $z = 0$. Prove that $\{u_k\}$ is dominated (in the sense of Hardy-Littlewood-Pólya) by $\{v_k\}$. A recent result of A. Fryntov, conjectured by J. Velling, implies the first inequality conjectured here: $\max u_k \leq \max v_k$.

Holomorphic images of quadratic differentials on annuli

D. Schleicher

The following problem arises in the study of iteration of entire transcendental maps: Let $A := \{z \in \mathbb{C} : 1 < |z| < R\}$ for R large, and consider the quadratic differential $q = z^{-2}dz^2$ on A (extended by the zero differential outside of A). Let $\mathcal{F}_n$ be the class of entire maps of bounded type with at most n singular values (asymptotic or critical values) which have no critical point c with $|c| < 2R$.

Question *Is there a constant $c < 1$ (depending on n and R) so that every $f \in \mathcal{F}_n$ with $\|f_* q\|/\|q\| > c$ must be injective on $\{z \in \mathbb{C} : |z| < \sqrt{R}\}$? Here $\|q\|$ denotes the natural norm of q on A and f_* is the push-forward of q.*

Edited by Richard Greiner

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