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Numerical Methods for Singular Perturbation Problems

April 9th – April 14th

This week was organized by H-G. Roos (Dresden), M. Stynes (Cork) and P.W. Hemker (Amsterdam). During the meeting it seemed that the topic of "numerical methods for singular perturbations" is more alive than ever. The subject deals with the numerical solution of problems that change their character if a parameter approaches a critical value. It gets its momentum not only from the need to solve such partial differential equations (PDEs) in science and technology, but also because there exists an intrinsic interest in fundamental questions that arise in the study of these numerical problems. Although several books on the subject now have appeared and some essential issues have been thoroughly resolved, many problems still remain. In practice the numerical solution of PDEs with significantly varying behaviour in different parts of the domain of definition is still a real challenge. Often a singular perturbation parameter is responsible for this type of behaviour. The changing character of the differential equation results in many intriguing theoretical questions.

During the week a most interesting exchange of ideas took place. A total of 29 presentations were given. The subjects discussed concentrated on: parameter-uniform convergence, types of meshes used and ways to construct (self-)adaptive meshes, aspects of discretisation methods (Galerkin least squares, discontinuous Galerkin, non-conforming FEM, stabilization by bubble functions, cross-wind diffusion, and domain decomposition), a-priori and a-posteriori error bounds, shock or boundary layer capturing and fitting, and solution methods for the resulting discrete large linear systems.

A significant number (about 40%) of new people were invited, compared with the previous meeting organized by the same group at Oberwolfach in 1998. This resulted in valuable new viewpoints. The number of proposed presentations was much larger than the number than could be accommodated. To promote the dissemination of new ideas, people who gave a talk at the 1998 meeting generally had a lower priority when the organizers selected the lectures to be given.

Many people made progress in ongoing cooperative projects and also some new collaborations were started. Although most joint work aimed at developing scientific ideas and writing joint research papers, some people took the opportunity to plan new proposals for international scientific cooperation in the framework of the European Research Training Networks.

It became apparent that some of the scientific questions raised during the meeting still require much serious research. The "hot" topics are: self-adaptive mesh adaptation (including the corresponding a-posteriori error estimators) in more than one dimension, and the proper use of high-order methods.

Many participants expressed their enjoyment of the meeting to the organizers and most found it extremely beneficial for their scientific work. The meeting made clear that, since the last meeting at the Institute in 1998, significant progress has been made in many of the topics mentioned above. At the present meeting many new ideas were put forward, and fresh questions were raised that provide a challenge for the years to come.

Abstracts

The Operator Splitting Method to solve Singularly Perturbed Problems

Lilliam Alvarez Diaz

The Operator Splitting Method is presented as an efficient numerical technique to solve singularly perturbed two-point boundary value problems.

The original problem is decomposed in a series of simpler initial value problems of certain standard structure, by using the roots of the operator characteristic equation. Some special quadrature formulae, well adapted to solve numerically the correspondent IVPs, are constructed. The convergence study of the operator splitting technique shows that, the stiffer the problem the better convergence.

Some numerical illustrations are presented for some classical tests, the Holdt problem and some Sturm-Liouville type problems as Schrodinger and Orr-Sommerfeld equations.

Diminution of numerical crosswind smear in finite volume methods for convection-diffusion problems

Lutz Angermann

A well-known but undesirable effect in the application of standard upwind finite volume schemes to convection-dominated diffusion problems is that interior layers (resulting, for example, from discontinuous data on the inflow boundary) are perceptibly smeared in directions perpendicular to the characteristics (crosswind directions).

To overcome this disadvantage, finite volume methods with a certain "fitted" choice of control volumes are presented. The key idea is to build control volumes of which the shape depends on the the convective field (so-called *aligned control volumes*). In recent years, several *aligned finite volume methods* (AFVM) have been developed: AFVM with linear interpolation (Angermann '96), AFVM with exponential interpolation (Angermann and Gadau '00), AFVM with flux approximation using flux-density elements (Angermann '01),

An analysis of the artificial diffusion terms shows that, for a *fixed* interpolation grid, the first two of the proposed schemes yield a diminished crosswind diffusion in comparison with other standard upwind finite volume schemes. Currently, the investigation of the third scheme is in progress.

Numerical examples for upwind finite volume methods with standard control volumes and for different aligned finite volume methods were given.

Discontinuous Galerkin Finite Element Methods for Advection–Diffusion Problems

Markus Bause

Transport processes in porous media are often strongly convection-dominated. Lagrange-Galerkin-methods (MMOC) are appropriate approximation schemes for such kind of problems. In existing error analyses of these methods the arising constants depend normally on norms of the solution or even reciprocally on the small diffusion coefficient. Therefore, the error constants may explode in the hyperbolic limit and the error estimates have no meaning. New error estimates in which the constants depend on norms of the data and not of the solution and do not tend to infinity in the hyperbolic limit are presented. For the time discretization uniform convergence with respect to the diffusion parameter of order

$O(k/t)$ is shown for initial values in L^2 and $O(k)$ for initial values in H^2. For the spatial discretization with linear finite elements uniform convergence of order $O(h^2+\min\{h, h^2/k\})$ is proved for data in H^2. By interpolation of Banach spaces suboptimal convergence rates are derived under less restrictive assumptions. The analysis is heavily based on a priori estimates, uniform in the diffusion parameter, for the solution of the continuous and the semidiscrete problem. They are derived in a Lagrangian framework by transforming the Eulerian coordinates completely into subcharacteristic coordinates.

On the numerical solution of singularly perturbed PDEs using moving mesh methods

George Beckett

Numerical experiments are described that illustrate some important features concerning the performance of moving mesh methods for solving singularly perturbed partial differential equations (PDEs). The particular method considered here is an adaptive finite difference method based on one of the moving mesh methods proposed by Huang, Ren and Russell [SIAM J. Numer. Anal. 31 (1994) pp.709-730]. This talk shows how the accuracy of the computations on adaptive meshes depends on the choice of monitor matrix, and a monitor matrix is presented which yields an optimal rate of convergence. An algorithm for the solution of the moving mesh system is presented in which the mesh equations are uncoupled from the physical PDE. The accuracy of various formulations of the algorithms are considered and a robust time-step control mechanism is presented.

Uniform Domain Decomposition Methods for Convection-Dominated Diffusion Problems

Igor Boglaev

We are interested in iterative domain decomposition methods for solving the advection-diffusion problem with parabolic layers

$$\varepsilon\Delta u + b(x,y)u_x - c(x,y)u = f(x,y), \text{ in } \Omega = \{(x,y) : 0 < x < 1, 0 < y < 1\},$$
$$u = g \text{ on } \partial\Omega, \quad b(x,y) \geq \beta_* > 0, \; c(x,y) \geq c_* > 0,$$

where ε is a small positive parameter, β_* and c_* are constants and $\partial\Omega$ is the boundary of Ω.

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures for solving singularly perturbed problems have received much attention for their remarkable speed and parallelizability, see, for example, [1-3] and references cited there.

In [2], for solving the singularly perturbed advection-diffusion problem in the presence of only the elliptic boundary layers at $x = 1$ and $y = 1$ and in the case of domain decomposition into two subdomains, the classical Schwarz alternating method and some variants of it were analysed.

In [1], on the basis of asymptotic criteria, representations of optimal interface positions for the Schwarz alternating procedure were derived. In general, this approach leads to nonuniform in the perturbation parameter convergent domain decomposition procedures.

In this paper, we introduce the multidomain modification of the Schwarz alternating method from [3]. In this approach, the domain is partitioned into many nonoverlapping subdomains with interface Γ. Small interfacial subdomains are introduced near the interface Γ, and approximate boundary values computed on Γ are used for solving problems on nonoverlapping subdomains. Thus, this approach may be considered as a variant of a block Gauss-Seidel iteration (or in the parallel context as a multicoloured algorithm) for the subdomains with a Dirichlet-Dirichlet coupling through the interface variables. Finite difference schemes on subdomains are based on classical upwind schemes and piecewise uniform meshes of Shishkin-types.

Our purpose is to study the multidomain decomposition algorithms applied to finite difference approximations of the problem. We consider two domain decomposition algorithms: the first one is based on decomposition of the computational domain into nonoverlapping vertical subdomains, and second uses decomposition into nonoverlapping horizontal subdomains. We show that these algorithms converge uniformly and possess load balancing. The last property is very important for implementation of the iterative algorithms on parallel computers, since it avoids loss of efficiency due to any processors being idle. Our numerical results demonstrate that the proposed algorithms are efficient, since they require few iterations of the domain decomposition algorithms and sufficiently small sizes of the interfacial subdomains and still maintain stable approximations.

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A posteriori error analysis for singular perturbation problems

Claudio Canuto

In this survey lecture, I consider the abstract setting of a family of topological isomorphisms $A_\varepsilon : X \to X'$ (X Hilbert space) depending on a (small) parameter $\varepsilon > 0$. Given the problem $A_\varepsilon u = f$ and approximations u_h of u in X and f_h of f in X', I first relate the error norm $\|u - u_h\|_X$ to the residual norm $\|f_h - A_\varepsilon u_h\|_{X'}$, discussing the dependence on ε. Next, in the relevant case of u_h given by a Galerkin approximation, I give a representation of $\|f_h - A_\varepsilon u_h\|_{X'}$ in terms of a multilevel Riesz decomposition of the space X. This approach leads to a multilevel interpretation of classical error estimators used in finite elements (based on the solution of local problems, on the Zienkiewicz-Zhou gradient projection, on local residuals).

The multilevel paradigm allows us to give an interpretation of the appropriate scalings within these estimators, when they are applied to singularly perturbed problems. Thus, for a model reaction-diffusion problem, the natural scaling yields robust bounds for the effectivity index, as proven by Verfürth (1998) and Ainsworth and Babuska (1999). On the other hand, for the convection-diffusion problem, the bound for the energy norm is non-robust, as shown by Verfürth (1998). We indicate a possible fix for that, based on the use of a stabilized norm, which controls an anisotropic order -1/2 norm of the streamline derivative. We refer to the contribution by A. Tabacco at the meeting for more details on this norm.

Tracking of sub-scales in finite element methods for the transient convection–diffusion equation

Ramon Codina

In this talk we describe and analyze a finite element formulation to solve the transient convection-diffusion-reaction problem

$$\begin{aligned} \partial_t u + \mathcal{L} u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u &= u_0 \quad \text{on } \Omega \times \{0\}, \end{aligned}$$

where $\mathcal{L}$ is the convection-diffusion-reaction operator $\mathcal{L}u := \boldsymbol{a} \cdot \nabla u - \nu \Delta u + \sigma u$, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, $\boldsymbol{a}$ is a divergence-free velocity field, $\nu > 0$ is the diffusion coefficient and $\sigma \geq 0$ is the (constant) reaction coefficient.

The variational formulation of the problem that we need reads: find $u \in L^2(0, T; H_0^1(\Omega))$ such that

$$\begin{aligned} &(\partial_t u, v) + b(u, v) = l(v) \quad \forall\, v \in H_0^1(\Omega), \\ &b(u, v) := \nu(\nabla u, \nabla v) + (\boldsymbol{a} \cdot \nabla u, v) + \sigma(u, v), \quad l(v) := \langle f, v \rangle, \end{aligned} \tag{1}$$

where $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product and $\langle \cdot, \cdot \rangle$ for the duality pairing in $H^{-1}(\Omega) \times H^1(\Omega)$.

Many schemes can be employed to discretize in time. Since our main concern is the space discretization, we will use the simple backward Euler approximation. Our ideas extend easily to any other algorithm.

Our aim is to present a finite element method free of the oscillations associated to the standard Galerkin formulation when diffusion is small. Many of such methods have been developed during the last two decades, among which the SUPG, the GLS, the characteristic Galerkin, the Taylor Galerkin, the residual free bubble stabilization, and others, can be named.

Our starting point will be the decomposition of the unknown into resolvable and sub-grid scales, of the form $u = u_h + u'$. The former are intuitively associated to the component of the solution which can be represented by the finite element mesh, whereas the latter can not be captured. Nevertheless, its effect onto the resolvable scale needs to be accounted for. When introduced in (1), they appear in two places, namely, in $b(u', v_h)$ *and* in the discretized form of $(\partial_t u', v_h)$. The approach we will follow is similar to that presented by Hughes et al. However, it is interesting to note that other methods share very similar concepts, as for example the nonlinear Galerkin method.

The main conceptual ingredients of the formulation to be presented here are the following. First, we let the sub-scales vary in time, and thus they need to be tracked. Second, we propose a closed-form expression for them at each time step. This is necessarily heuristic. However, we describe a Fourier analysis that provides some rationale to our proposal. A very important point is that we take the sub-scales orthogonal to the finite element space. Finally, we add some additional approximations that lead to a method which is computationally feasible, in the sense that its cost is similar (very often smaller) than that of other stabilization methods.

Apart from a thorough description of the method, we also provide some results of the numerical analysis, which show in particular that the final scheme is stable and optimally convergence under the usual assumptions on the mesh and the continuous solution.

Iterative Methods for Convection-Diffusion Problems

Howard Elman

We discuss numerical algorithms for solving the linear and nonlinear algebraic systems that arise from discretization of the steady-state convection-diffusion equation. We demonstrate that algorithms based on splitting or preconditionings corresponding to relaxation or incomplete factorization are very effective for such problems, provided that some account of the character of the underlying flow is taken into account. We give both empirical and analytic justification for this statement, and demonstrate its implications for realistic models. In addition, we show that the choice of discretization has a strong impact on performance of solvers, Discretizations under consideration include various stabilizing methods based on upwinding. We compare and contrast the capabilities of these strategies for computing accurate solutions efficiently.

Discontinuous Galerkin Methods for Conservation Laws

Joseph E. Flaherty

We describe a discontinuous Galerkin (DG) method for solving hyperbolic systems of conservation laws that may be used for singularly-perturbed parabolic systems. We review several aspects of the (DG) method including basis construction, flux evaluation, solution limiting, and *a posteriori* error estimation. Focusing on error estimation, we show that the leading term of the spatial discretization error of one-dimensional problems using piecewise-polynomial approximations of degree p is proportional to a Radau polynomial of degree $p+1$ on each element. We also prove that the local and global discretization errors are $O(\Delta x^{2(p+1)})$ and $O(\Delta x^{2p+1})$ at the downwind point of each element. This strong superconvergence enables us to show that local and global discretization errors converge as $O(\Delta x^{p+2})$ at the remaining roots of Radau polynomial of degree $p+1$ on each element. Convergence of local and global discretization errors to the Radau polynomial of degree $p+1$ also holds for smooth solutions as $p \to \infty$. These results are used to construct asymptotically correct *a posteriori* estimates of spatial discretization errors that are effective for linear and nonlinear conservation laws in regions where solutions are smooth.

We extend these *a-posteriori* estimates of spatial discretization errors to multi-dimensional problems and, doing so, define a multi-dimensional Radau-type polynomial. Error estimates obtained in terms of this polynomial are shown to be (asymptotically) correct for linear hyperbolic problems. We indicate the presence of a superconvergence phenomena on unstructured meshes of triangular elements and speculate about its structure.

The results of serial and parallel computations are presented for unsteady model and compressible flow problems in one, two, and three dimensions. These include mixing and other complex instabilities as well as shock problems. Solutions obtained by adaptive h- and p-refinement are compared and contrasted.

Analysis of a Defect Correction Method on Shishkin–Type Meshes

Anja Fröhner

(joint work with Torsten Linß and Hans-Görg Roos)

We consider a linear singular perturbed boundary value problem on the unit interval and the unit square. For small values of the pertubation parameter, it is well known that standard numerical methods become unstable and fail to give accurate results. On the other hand are several stabilized methods like upwinding restrictet to first order convergence. Already in the early 1980's Hemker proposed the use of defect correction methods when solving singularly perturbed problems. By means of defect correction methods a low–order stabilized scheme is combined with a less stable higher–order scheme resulting in a higher order (iterative) method with the advantage that only well conditioned discrete problems have to be solved.

Numerical experiments indicate that defect correction on layer adapted meshes is a suitable method to achieve uniform (almost) second order convergence. We present the main ideas of the proofs of almost second order convergence, uniform with respect to the pertubation parameter, of the defect correction scheme combining upwind and central differencing in one and two dimensions.

A Survey of Stabilized Finite Element Methods for Computational Fluid Dynamics

Thomas J.R. Hughes

Before the meeting, I was asked by one of the organizers, Martin Stynes, to present a survey lecture on stabilized methods. I interpreted this to mean an overview of the subject that would provide an introduction to more specialized talks by others which would follow.

I began with the scalar, steady advection-diffusion equation and described first the Galerkin method and its well-known stability problems. Then I presented two stabilized methods, SUPG and GLS, and contrasted them with the classical upwind/artificial diffusivity method. I discussed their stability and accuracy properties, error estimates, and numerical convergence tests for a model problem. I then generalized the setting to the unsteady case, and presented a stabilized, discontinuous Galerkin method with respect to time on space-time slabs. I again reviewed known error estimates and mentioned some superconvergence results. I presented some numerical convergence studies for time- prismatic elements and slanty, characteristic directed elements, and I also presented some phase and amplitude results derived from Fourier analysis for the pure Galerkin and stabilized versions. I then described symmetric advective-diffusive systems and again quoted known error estimates for the stabilized steady and unsteady cases. I next showed how the incompressible Navier-Stokes equations could be written as a quasilinear symmetric advective-diffusive system. In passing, I identified the special case of Stokes flow which, as an advective-diffusive system stabilized in the usual way, is convergent for all combinations of velocity and pressure interpolations, circumventing the usual Babuska-Brezzi requirements of the Galerkin formulation. I then showed how, through the use of entropy variables, the compressible Navier-Stokes equations could be written as a symmetric advective-diffusive system. This leads to the stabilized finite element method automatically satisfying the entropy production inequality, which is the second law of thermodynamics and the fundamental nonlilnear stability condition of the theory. I also described the roles played by stabilization, discontinuous Galerkin, and residual- based shock capturing operators on entropy production. I argued and presented numerical results supporting the view that shock-capturing operators are essential to attain clean shock profiles without spurious over-and under-shoots. The numerical results presented were for structured and unstructured meshes and a broad range of Mach numbers, including chemically reacting hypersonic flows at Mach numbers in the range 17 to 25. I then described calculations involving turbulent boundary layers and I presented comparisons of results for structured and unstructured meshes with very large element aspect ratios up to 275,000. I then commented on the types of graded meshes that are used by engineers in industry to calculate turbulent boundary layer flows. I closed with some brief, introductory remarks on current research thrusts in stabilized methods, namely, multiscale and residual-free bubbles techniques, and the variational multiscale formulation of Large Eddy Simulation.

On Higher Order FE Discretizations and Multigrid Solvers for the 3D Navier–Stokes Equations

Volker John

This talk presents experiences in using higher order finite element discretizations in the three–dimensional Navier–Stokes equations. Numerical studies of the flow through a channel around a cylinder show that the drag and lift coefficient at the cylinder can be computed much more accurate with isoparametric higher order discretizations than, e.g., with stable non–conforming discretizations of lowest order, see Figure 1. However, stable non–conforming discretizations of lowest order have the advantage that the arising discrete systems can be solved efficiently by multigrid methods. We present a new multigrid approach for higher order discretizations which is based on multigrid methods for stable lowest order non–conforming discretizations. This new multigrid approach proves to be an efficient solver for the discrete systems arising in higher order discretizations. The robustness of the solver is enhanced by using the multigrid method as preconditioner in a flexible GMRES method.

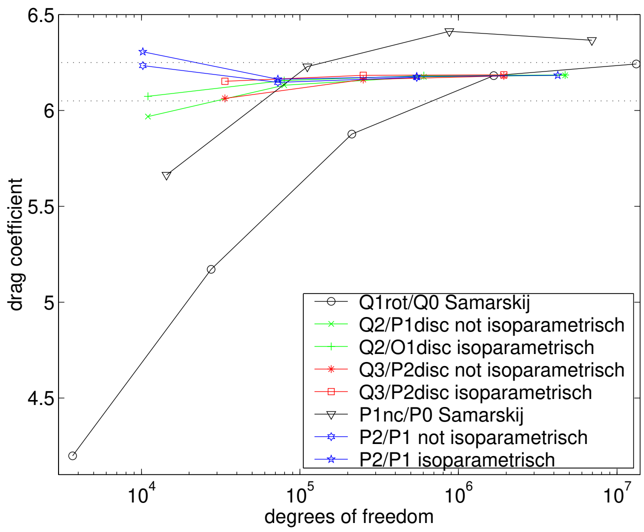

Figure 1. Drag coefficients at the cylinder computed with different discretizations

Model reduction in chemical kinetics

Hans G. Kaper

(joint work with Tasso J. Kaper)

Realistic models of combustion processes and atmospheric chemistry involve tens or hundreds of chemical species and tens or hundreds of reactions. The reactions are played out on widely disparate time scales, and the evolution of the system proceeds along complicated pathways, which are often difficult to predict. *We are interested in methods to reduce large systems of equations in chemical kinetics.*

Model reduction methods fall into several categories: systematic use of the classical equilibrium and steady-state approximations, geometric methods, singular perturbation techniques, and approaches based directly on a restatement of the chemical mechanism, such as lumping and sensitivity analysis. *We are interested in geometric methods for model reduction.*

The methods use ideas from dynamical systems theory and assume a separation of the motion into fast and slow modes. In our talk we discuss algorithms proposed by Fraser and Roussel, Maas and Pope, and Davis and Skodje for model reduction in chemistry. *We illustrate with a simple model from classical enzyme kinetcs (Michaelis–Menten–Henri).*

Nonconforming finite elements for convection dominated problems

Petr Knobloch

The talk will be devoted to nonconforming finite element discretizations of convection–diffusion equations stabilized by means of the streamline diffusion method. We shall be mainly interested in cases when convection dominates diffusion.

We shall present both theoretical and numerical results showing that the nonconforming piecewise linear Crouzeix–Raviart element is not suitable for solving convection–diffusion equations in the convection dominated regime. Therefore, we shall introduce a new class of nonconforming finite element spaces obtained by modifying the Crouzeix–Raviart element using suitable nonconforming bubble functions. The aim of these modifications is to obtain finite element functions satisfying a patch test of a higher order than usual. We prove that the new finite elements lead to optimal error estimates which are uniform with respect to the perturbation parameter and we shall demonstrate various numerical result showing that the resulting discretizations are very robust and that the discrete solutions are much more accurate than for the Crouzeix–Raviart element.

We shall also discuss the application of these new finite elements to the numerical solution of problems describing incompressible materials and we shall further explain how higher order noncoforming finite elements leading to optimal convergence results can be derived.

Robust error estimation for a singularly perturbed model problem on anisotropic tetrahedral meshes

Gerd Kunert

We consider a singularly perturbed reaction–diffusion model problem which is discretized by a standard FEM on *anisotropic* meshes.

As an important ingredient of adaptive algorithms, three *a posteriori* error estimators are presented. Two estimators are of residual type while the third estimator is obtained by solving a local problem. For all three estimators a stringent analysis yields upper and lower error bounds which are *uniform* w.r.t. the small perturbation parameter ε.

The error bounds are *uniform* also w.r.t. the the stretching ratio of the anisotropic elements provided that the anisotropic finite element mesh corresponds to the anisotropic problem. This correspondence is measured by a so–called *matching function*.

Hence the error estimation is *reliable*, *efficient* and *robust* for suitable anisotropic meshes. A numerical example supports the anisotropic error analysis.

Numerical Analysis of Large Eddy Simulation

W Layton

One very common claim of large eddy simulation (LES) is that it can , using centered methods, simulate turbulent flow with complexity independent of the Reynolds number. This talk describes work beginning mathematical support for this claim/conjecture. The mathematical analysis is delicate because of the reduced regularity in time that may be assumed of turbulent velocity fields. In two cases we prove such an error estimate of the usual (centered) FEM for a LES model. In the general case the analysis just fails. The talk describes the physical basis of the model, the near wall modeling , developed by N. Sahin, we use and the proofs.

Layer-adapted meshes for convection-diffusion problems

Torsten Linß

We consider the model convection-diffusion problems

$$-\varepsilon u'' - bu' + cu = f \quad \text{in} \quad (0,1)$$

and

$$-\varepsilon \Delta u - b^T \nabla u + cu = f \quad \text{in} \quad (0,1)^2,$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter.

We derive a general classification for layer-adapted meshes in 1D based on a hybrid stability inequality that involves a negative norm.

In the second part of talk an overview of previous results for a number of numerical methods will be given. This includes finite difference scheme, finite volume and finite element methods, derivative approximation and problems with turning-point boundary layers.

A Sequential Regularisation Formulation for Incompressible Navier-Stokes Equations

Ping Lin

Many mathematical models arising in Science and Engineering, including constrained mechanical systems in robotics and vehicle simulation, incompressible fluid flows, lead to differential equations with constraints (or differential-algebraic equations (DAEs)). The direct discretization of such models in order to solve them numerically is typically fraught with difficulties. We thus need to reformulate the original problem to obtain a better behaved problem before discretization.

In this talk, we will consider regularization methods. The regularized problem is often a singular singularly perturbed problem. We will propose a method called the sequential regularization method (SRM) via a relatively simple example. An important improvement of the SRM over usual regularization methods is that the problem after regularization is less stiff and explicit time integration can be used for the regularized problem. The asymptotic expansion technique is used to analyse the convergence of the method. The SRM is then applied to the incompressible Navier-Stokes equations

where the incompressibility condition can be seen as a constraint. The SRM formulation keeps benefits of the penalty method, that is, velocity and pressure can be obtained separately and no pressure-Poisson equation is involved. It provides a completely explicit procedure with theoretical justification so that the computation is extremely simple. No linear or nonlinear system needs be solved. Finite element and finite difference spatial discretization are analysed. In the case that a large number of time steps are needed a domain-decomposition based parallelization is used to reduce the computational time. With this completely explicit procedure the parallel implementation and its message passing are very simple as well. Other applications of the formulation to a large mechanical chain with many closed loops and to miscible displacement in porous media in reservoir simulation will be briefly mentioned.

Stabilized finite element methods with shock-capturing foradvection-diffusion problems

G. Lube

(joint work with T. Knopp and G. Rapin)

The numerical solution of the scalar advection-diffusion-reaction model is still a challenge when advection is strongly dominant. Among the many stabilized finite element methods that partly cure the failure of the standard Galerkin method, we mention the streamline-diffusion (SUPG) method, the Galerkin/ Least-squares (GLS) method, and the algebraic subgrid scale (ASGS) method. Nearly optimal global and local a-priori error estimates are available for such methods, cf. [3]. Nevertheless, such stabilized methods do not preclude local oscillations (over- and undershoots) in the vicinity of sharp gradients of the solution. These perturbations are very often not desirable in applications since they may deteriorate the solution of nonlinear problems.

The small oscillations appearing in standard stabilized methods can be traced backed to the fact that these methods are neither monotone nor monotonicity preserving. As a remedy, it is possible to add a certain amount of artificial crosswind diffusion. Such linear methods have been proposed for linear elements e.g. in [2, 7] leading to nearly optimal pointwise error estimates in subregions away from discontinuities. A refined linear method has been proposed in [4, 5]. Quite promising results were obtained using piecewise linear or bilinear finite elements. Unfortunately, the extension to higher-order elements is less appropriate since the artificial crosswind diffusion is added in a non-consistent way.

The only feasible way to obtain high order accuracy with strongly reduced local oscillations is to design a nonlinear method. The basic idea of shock-capturing (or discontinuity-capturing) methods is to introduce artificial crosswind diffusion in a consistent or residual (but still nonlinear) way. For a review of such methods we refer to [1] or [5]. The numerical analysis of such nonlinear finite element methods is seemingly not complete yet. Some particular results can be found in [6] or [5].

The goal of the present paper is to contribute to the analysis of a certain class of nonlinear shock-capturing methods. This class contains the discontinuity-capturing/ crosswind-dissipation (DC/CD) method [1] which guarantees a proper balance of the crosswind diffusion with the artificial diffusion of the SUPG method. Here we prove an existence result and discuss the question of uniqueness.

We proceed with a-priori estimates. First we show strong H1-convergence of the method in case of weak solutions. Then we prove global a-priori error estimates for smooth solutions including the effect of additional crosswind stabilization. Then we present a-posteriori error estimates. First we show that the standard upper estimate for a residual based error estimator for the streamline-diffusion method can be extended to the shock-capturing scheme. Then we prove such an estimate which measures the error between the DC/CD-and the basic stabilized method (here SUPG) via control of weighted local residual in the vicinity of strong gradients.

The last part is devoted to the iterative solution of the nonlinear discrete problem. We propose to use a very simple defect correction method or method of frozen nonlinear terms. This approach is very cheap as opposed to the approach [5] where nonlinear GMRES methods are considered.

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Maximum norm a posteriori error estimates and a robust adaptive method for a $1D$ convection-diffusion problem

Natalia Kopteva

A quasi-linear conservative convection-diffusion two-point boundaryvalue problem is discretized on arbitrary nonuniform meshes. We give first- and second-order maximum norm a posteriori error estimates that are based on difference derivatives of the numerical solution and hold true uniformly in the small parameter. One of these estimates give a theoretical framework for the following adaptive method based on an upwind difference scheme. The mesh used has a fixed number $(N+1)$ of nodes, and is initially uniform, but its nodes are moved adaptively using equidistribution of the arc-length of the current computed piecewise linear solution. It is proved that a mesh exists that equidistributes the arc-length along the polygonal curve and the corresponding computed solution is first-order accurate uniformly in ϵ. In the case when the problem is linear, it is shown that after $O(\log(1/\epsilon)/\log N)$ iterations, the computed solution is first-order accurate in the $L^\infty[0,1]$ norm uniformly in ϵ.

Navier-Stokes Equations in Rotation Form: a Robust Multigrid Solver for the velocity Problem

Arnold Reusken

(joint work with Maxim A. Olshanskii)

The topic of this presentation is motivated by the Navier-Stokes equations in *rotation* form:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\operatorname{curl}\mathbf{u})\times\mathbf{u} + \nabla P &= \mathbf{f} \quad \text{in} \quad \Omega\times(0,T], \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega\times(0,T]. \end{aligned}$$

Linearization and application of an implicit time stepping scheme results in a linear stationary problem of Oseen type. In well-known solution techniques for this problem such as the Uzawa (or Schur complement) method, a subproblem consisting of a coupled nonsymmetric system of linear equations of diffusion-reaction type must be solved to update the velocity vector field. In this talk we analyse a standard finite element method for the discretization of this coupled system and we introduce and analyse a multigrid solver for the discrete problem. Both for the discretization method and the multigrid solver the question of robustness with respect to the amount of diffusion and variation in the convection field is addressed. We prove stability results and discretization error bounds for the Galerkin finite element method. We present a convergence analysis of the multigrid method which shows the robustness of the solver. Results of numerical experiments are presented which illustrate the stability of the discretization method and the robustness of the multigrid solver.

On adaptive meshes for a singularly perturbed reaction-diffusion equation with a moving concentrated source

Grigorii I. Shishkin

We consider an initial value problem on an axis $(-\infty, +\infty)$ for a singularly perturbed parabolic reaction-diffusion equation in the presence of a moving concentrated source. For such a problem, classical finite difference schemes converge only when $\varepsilon \gg N^{-1} + N_0^{-1}$, where N and N_0 define the number of nodes in the grids with respect to x (on segments of unit length) and t. In the case of piecewise uniform and patched (in the nearest neighbourhood of the moving source) grids, we show that on a rectangular stencil there does not exist a finite difference scheme convergent for $\varepsilon \leq N^{-2} + N_0^{-2}$. We discuss conditions under which the discrete solutions on a non-rectangular stencil (in the nearest neighbourhood of the moving source) are ε-uniformly convergent.

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Discontinuous Galerkin Finite Element Methods for Advection–Diffusion Problems

Endre Süli

In this talk we review some recent developments which concern the error analysis of the discontinuous Galerkin finite element method. After some historical notes, we begin by discussing the analysis of the method, as applied to the multi-dimensional linear advection equation. In particular, we highlight the fact that, unlike its continuous counterpart, the discontinuous Galerkin finite element method does not require streamline-diffusion stabilisation. Error bounds for the hp-version of the method are then derived which are optimal, both with respect to the local mesh-size h and the local polynomial degree p.

We then turn our attention to second-order partial differential equations with dominant hyperbolic behaviour, following a recent paper by Houston, Schwab and Süli (2000). Operating within the general class of PDEs with non-negative characteristic form, we consider the *hp*-version of the discontinuous Galerkin finite element method which stems from a discretisation of the diffusion term due to Baumann (1997); we shall relate Baumann's discretisation to earlier work by Wheeler (1978) and Arnold (1982). We then consider the error analysis of the method and show a series of numerical experiments to illustrate the error bounds. The examples include advection–dominated diffusion equations, problems of mixed type, and degenerate elliptic equations (such as Grushin's equation). We review the work of Wihler and Schwab (2000) on robust exponential convergence of the discontinuous Galerkin method on two-element meshes in the singular perturbation of limit of the diffusion coefficient converging to zero. Finally, we comment on alternative discretisations of advection–diffusion equations, such as the Local Discontinuous Galerkin method of Cockburn and Shu, following the recent papers by Castillo, Cockburn, Schötzau, and Schwab (2000) and Cockburn and Shu (2000).

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Anisotropic Multilevel Stabilization of Convection-Diffusion Problems

Anita Tabacco

A new functional framework for consistently stabilized discrete approximations to convection-diffusion problems was recently proposed by S. Bertoluzza, C. Canuto and the author. The key ideas are the evaluation of the residual in an inner product of the type $H^{-1/2}$ and the realization of this inner product via explicitly computable multilevel decompositions of function spaces (such as those given by wavelets or hierarchical finite elements).

Here we provide further developments for such approach, obtained in collaboration with C. Canuto. We want to take into account the anisotropic nature of the convection-diffusion operator and derive uniform (in the diffusion parameter) anisotropic estimates. To this end, we develop a functional framework involving anisotropic Sobolev spaces which depend on the velocity field. We represent the norm of a function in such spaces as infinite weighted ℓ^2-sum of its wavelet coefficients; in this setting it is natural to use anisotropic biorthogonal wavelets.

We propose a variational formulation of the exact problem which is obtained by considering a variable order inner product. It behaves like an $H^{-1/2}$-inner product where the advective part of the operator is dominant, whereas it behaves like an H^{-1}-inner product where the diffusion part of the operator is dominant. The coercivity and the continuity estimates of the associated bilinear form involve anisotropic norms which are very close to each others.

When we apply a standard Galerkin method to this formulation we obtain stability and error estimates, which imply a uniform control on the $H^{1/2}$ norm in the streamline direction.

Stability of the RFB-Approach for Bilinear Elements

Lutz Tobiska

(joint work with L. P. Franca)

We consider a convection-diffusion problem of following type

$$-\varepsilon\Delta u + a \cdot \nabla u = f \text{ in } \Omega, \quad u = 0 \quad f \text{ on } \partial\Omega,$$

with ε a small, positive parameter. The standard Galerkin method with piecewise linear elements enriched with cubic bubbles upon eliminating the bubbles results in the streamwise-diffusion (or SUPG) method [3]. The concept of residual free bubbles has been developed to recover the correct asymptotic behaviour of the streamwise - diffusion parameter in both the convection dominated and the diffusion dominated case [4, 2, 1]. It turns out that the stabilizing term arising in the residual free bubble approach is identical to that, in the streamline diffusion method in the piecewise linear case but differs from it for polynomials of degree $k > 1$.

Much less is known in the case of quadrilateral elements. We study the nature of the stabilizing term arising in the residual free bubble approach for piecewise bilinear functions on rectangular grids. We show, that on the subspace of piecewise linear functions the stabilizing term is identical to that in the streamwise diffusion approach. However, on the space of piecewise bilinear functions there is a case in which the stabilizing term is weaker compared to the term used in the streamline dlffusion method. In the particular case when the direction of the convection is parallel to a diagonal of the quadrilateral, control is lost over the mixed derivatives iin the convection -dominated limit.

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Higher–Order Schemes on Improved Shishkin Meshes

Relja Vulanović

The simplicity of Shishkin piecewise equidistant meshes cannot be accepted as the only reason for preferring them over Bakhvalov meshes, since the latter produce much better results. If one wants to use modified Shishkin meshes to improve numerical results, then those meshes become as complicated as the Bakhvalov ones. However, there are modifications that are still piecewise equidistant. In this talk, we consider some of those and show that they are suitable for higher–order finite–difference schemes which are too complicated to apply on the fully non–equidistant Bakhvalov meshes.

More specifically, some third–order schemes are constructed for the quasilinear problems,

$$-\varepsilon u'' - \mu b(x,u)u' + c(x,u) = 0, \quad x \in (0,1), \quad u(0) = u(1) = 0,$$

where $0 < \varepsilon << 1$ and b and c are sufficiently smooth functions with $c_u(x,u) \geq \gamma > 0$. Three special cases are discussed:

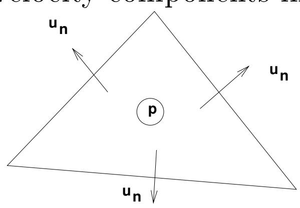
1. $\mu = \varepsilon^p$, $p > 1/2$
2. $\mu = \varepsilon^{1/2}$, $b = b(x)$
3. $\mu = 1$, $b = b(x) \geq \beta > 0$.

The third–order schemes are four–point schemes and are used only on equidistant parts of the mesh, simpler standard second–order schemes being applied elsewhere.

Mach-uniform computing methods in computational fluid dynamics and unstructured staggered finite volume methods

Pieter Wesseling

Motivated by the so-called homogeneous equilibrium method for cavitating flow in hydrodynamics (involving a nonconvex hyperbolic system), we are interested in Mach-uniform numerical methods (unification of compressible and incompressible flow computation methods). Ways to do this are described in Chapter 14 of Principles of Computational Fluid Dynamics, Springer, 2001, by Pieter Wesseling. To facilitate grid generation, we use unstructured grids, following the contemporary trend in industry. To have good numerical behavior for low Mach, we generalize the classic structured quadrilateral MAC (Marker-and-Cell scheme (Richardson (1922), Harlow and Welch (1965)) to unstructured triangular finite volume grids. Questions to be answered are: What should the control volume for u_n be? How to construct tangential velocity components in cell face centers?



For Mach= $O(1)$, do we obtain genuine weak solutions that satisfy the entropy condition? What does conservation and monotonicity mean on unstructured staggered grids? Guidelines are derived from special cases, such as a uniform Courant triangulation and a one-dimensional row of cells on which Riemann problems can be solved. Numerical results indicate satisfactory accuracy for fully compressible flows involving shocks and satisfaction of the entropy condition. Realistic-looking results are obtained with the homogeneous equilibrium model for cavitating flow around a hydrofoil, in which the Mach number varies between 0.001 and 20.

Superconvergent Approximation for Problems with Boundary Layers

Zhimin Zhang

Numerical approximations of problems with exponential boundary layers are considered in both one and two dimensional settings. Standard finite element methods are applied with Shishkin type meshes. Gaussian quadrature is used on the diffusion term to define a discrete energy norm. With this discrete norm, superconvergence results are established under proper regularity assumptions.

In the one dimensional case, superconvergent rates of $(N^{-1}\ln N)^{p+1}$ type in the discrete energy norm are proved for both reactor-diffusion and convection-diffusion equations, where p is the polynomial degree of the finite element space. In the two dimensional case, a similar superconvergent error bound $(N^{-1}\ln N)^2 + \epsilon N^{-1.5}\ln N)$ is obtained for bilinear finite element approximation to convection-diffusion equations.

As by-products, the same convergence rates are valid in the L_2-norm and much improved convergence rate are also obtained in the L_∞-norm.

All error bounds are uniformly valid with respect to the singular perturbation parameter. Numerical tests indicate that the error bound $(N^{-1}\ln N)^{p+1}$ is sharp in the sense that the logarithm term cannot be removed.

The analysis and numerical experiments indicate that as long as boundary layers can be resolved by anisotropic meshes, superconvergence phenomenon occurs.

Edited by P.W. Hemker

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