Mathematisches Forschungsinstitut Oberwolfach

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Schnelle Löser für partielle Differentialgleichungen

May 27th – June 2nd, 2001

The meeting was organized by Randolph E. Bank (La Jolla, USA), Wolfgang Hackbusch (Leipzig), and Gabriel Wittum (Heidelberg).

The main topic of the conference was the analysis and the application of fast solution methods for partial differential equations, emphasizing

- Multigrid methods
- Domain Decomposition methods.

The presentations were divided into different sessions with special topics. Some sessions were devoted to the convergence analysis of fast solvers, other sessions on special solver aspects such as algebraic multigrid methods, Mortar elements, wavelets, anisotropic meshes, etc.

A further part of the meeting was concerned with the application of fast solvers, e. g. to flow problems, image processing, electrodynamics, solid mechanics, remediation problems, optimization problems, and Einstein equations.

The meeting confirmed by the high quality and the large variety of the presented results that fast solvers for partial differential equations are an important research field with a large impact on many challenging applications.

Abstracts

Multigrid Methods for Anisotropic Edge Refinement

Thomas Apel
(joint work with Joachim Schöberl)

We consider partial differential equations in non-smooth three-dimensional domains. A finite element method with optimal convergence requires anisotropic mesh refinement towards edges with large internal angles. Such meshes contain elements with a small mesh size perpendicularly to the edge (in the direction of the rapid variation of the solution) and a much larger mesh size in the direction of the edge.
In this talk we suggest and analyze a new multigrid scheme combining semi-coarsening and line smoothers to obtain a solver of optimal algorithmic complexity for anisotropic meshes along edges.

Robust coarse space construction for the Neumann-Neumann algorithm

Petter Bjørstad

This lecture explains the algebraic construction of an appropriate coarse space assembled from the low energy eigenvectors from individual substructures. The method is tested by complex examples. The method is adaptive in selecting a different number of equations from each substructure until a tolerance is met. Preliminary results are promising and the method compare well with the so-called cross point method.

Tensor-Product Multigrid

Steffen Börm

We consider a multigrid algorithm for anisotropic problems of the form

$$-\operatorname{div}\begin{pmatrix} \alpha(x) & \\ & \beta(x) \end{pmatrix} \operatorname{grad} u(x,y) = f(x,y)$$

on a domain $\Omega \subseteq \mathbb{R}^n$.
The anisotropy is aligned with the coordinate axes, so a row-block method is the natural choice for the smoothing iteration. Since the coefficients depend only on one coordinate, a semicoarsened grid with constant resolution with respect to this coordinate is used.
For domains of the form $\Omega = \Omega_x \times \Omega_y$ and tensor-product finite element spaces, the discrete problem can be split into a family of problems of the form $-\Delta + c$ on Ω_y. We show that a multigrid analysis of our algorithm only requires the analysis of these well-known simple problems.
The idea of splitting the space into subspaces on which the operator is reduced to a simple form can be extended to problems in $H_0(\mathrm{curl})$ and $H_0(\mathrm{div})$.

A Cascadic Multigrid Method for Mortar Elements

Dietrich Braess

Cascadic multigrid methods require a severe analysis since errors that are done on some level cannot be simply compensated later. Specifically the transfer between the grids has to be considered carefully. An estimate of the norm of the transfer operator by a

constant C is not sufficient since C^{level} would enter in the final result. An estimate by $1 + C/(\text{number of iterations})$ is possible by a duality argument. The basis of the cg-algorithm (on each level) is the restricted Richardson iteration. Formally the Lagrange multiplier from the preceding step has no impact on the iterates, but this is misleading. Reasonable values are required to have small residues in an auxiliary problem. A shifted optimality property is helpful which together with another observation on the inverses of block matrices gives rise to an efficient imbedding into a cg-iteration.

An additive theory for V-cycle algorithms

Susanne C. Brenner

In this talk an additive convergence theory for the V-cycle algorithm will be presented. This theory is effective for establishing the asymptotic behavior of the contraction number of the V-cycle algorithm as the number of smoothing steps is increased.
The following applications of the additive theory will be discussed:
(1) a complete generalization of the classical V-cycle convergence theorem of Braess and Hackbusch to the case of less than full elliptic regularity,
(2) convergence of V-cycle and F-cycle algorithms for nonconforming methods with a sufficiently large number of smoothing steps.

Fast Solvers for Nonlinear PDEs Arising in Image Processing

Tony F. Chan

The use of nonlinear partial differential equations (PDE) in image processing has attracted a lot of attention recently. It offers new and complementary ideas and techniques to traditional approaches such as transform and stochastic techniques and brings the vast knowledge base of nonlinear PDEs (and CFD) to bear on image problems. Many of these PDE image models give rise to challenging computational problems, including the solution of large and ill-conditioned linear systems and the associated preconditioning techniques, linearization and iterative methods for highly nonlinear systems, and optimization problems involving non-differentiable objectives. In this talk, I'll give a survey of some of the ideas and techniques that have been proposed, with some emphasis on work done at our group at UCLA.

An adaptive solver for saddle point problems

Wolfgang Dahmen
(joint work with S. Dahlke, K. Urban)

An adaptive wavelet scheme for saddle point problems is presented which has asymptotically optimal work/accuracy balance when compared with the number of wavelets needed to approximate the solution within some given target accuracy. The basic strategy is to (i) establish first a certain mapping property of the operator induced by the underlying variational problem, (ii) characterize the corresponding "energy function" spaces in terms of norm equivalences induced by suitable wavelet bases, (iii) combine (i) and (ii) to transform the original problem into an equivalent one on ℓ_2 which is well conditioned, (iv) conceptually apply an iterative scheme to the full infinite dimensional ℓ_2-problem by adaptively evaluating the action of the involved infinite dimensional matrices on finitely supported

vectors within dynamically updated accuracy tolerances. As a consequence one can prove that compatibility constraints such as the LBB condition do not arise. This and the quantitative performance of the scheme are illustrated by some numerical experiments.

A FETI-DP method for the mortar discretization of elliptic problem

Maksymilian Dryja
(joint work with Olof Widlund)

The dual-primal FETI (FETI-DP) method for solving discrete problems arising from the approximation of the Dirichlet problems defined on a union of substructures Ω_i is discussed. The discretization is obtained by the mortar method on nonmatching triangulation across $\partial\Omega_i$. As in all other iterative substructuring methods the unknowns corresponding to interior nodal points are eliminated; in this dual-primal FETI method unknowns at the vertices of Ω_i are eliminated as well. The remaining Schur complement system is solved by the FETI method.
It is proved that the discussed method is convergent with a rate of convergence almost optimal. The method is well suited for parallel computations.

New Advances in Algebraic Multigrid

Robert D. Falgout

AMG is well-suited for solving unstructured grid problems, and works remarkably well over a wide variety of applications. However, for some problems, AMG is not effective without certain problem-specific modifications or careful parameter tuning. To address this, CASC researchers are developing a new class of algorithms for finite element problems called AMGe. As a departure from standard AMG that only requires the system matrix, AMGe also assumes access to local element stiffness matrices. These stiffness matrices are used to construct a local measure derived from multigrid theory to determine a local representation of algebraically "smooth" error. This representation provides the basis for constructing effective interpolation and coarsening procedures.
In this talk, we present the latest developments in AMGe research, including a new spectral AMGe method. We also discuss recent findings relating Achi Brandt's compatible relaxation ideas to the AMGe measure.
This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract no. W-7405-Eng-48.

Using Krylov-subspace iterations in discontinuous Galerkin methods for nonlinear reaction-diffusion equations

Roland W. Freund
(joint work with Donald J. Estep)

Discontinuous (in time) Galerkin methods for the numerical solution of time-dependent ordinary and partial differential equations have a number of advantages over their more common continuous counterparts. These advantages include an increased convergence order at the time nodes, better behavior for long-time integration, and in the case of PDEs, the easy implementation of spatial grids that change with time. On the other hand, discontinuous Galerkin methods can be significantly more expensive than continuous Galerkin methods, especially when no attention is paid to the linear algebra problems that actually

dominate the computational costs. In this talk, we discuss the linear algebra issues arising in discontinuous Galerkin methods and demonstrate how this class of methods can be made competitive.

Multigrid methods for gridless discretizations

Michael Griebel
(joint work with A. Schweitzer)

We present a multigrid method for the Partition of unity (PUM) discretization of elliptic PDEs. There, a sequence of coarser particle sets is constructed with help of an octree approach. The next step is to discretize the PDE on all these sets. This leads to a sequence of non-nested PUM-spaces. Consequently the intergrid transfer operators have to be set up via L_2 projections between the spaces. Here, the PUM property helps to simplify things substantially. We obtain cheap local operators. We show the resulting convergence rates for the V and W cycle. They exhibit convergence rates of 0.25 independent of the number of particles. For higher polynomial degrees p of the PUM the rates grow with p.

A General Concept for the Construction and Parallelization of Algebraic Multigrid methods

Gundolf Haase
(joint work with Ulrich Langer and Stefan Reitzinger)

Solving huge systems of equations requires an optimal solver, i.e., the memory requirements and the time for solving should be proportional to the number of unknowns. Recent research has enhanced multigrid methods which are now fulfilling these requirements for many problem classes, i.e., 3D Maxwell equations. In contrast to geometric multigrid methods, the algebraic multigrid method (AMG) presented requires only system matrix information. This is sufficient for constructing the typical multigrid hierarchy. For certain problem classes, the knowledge on the differential operator makes the method robust.
The classical algebraic multigrid (AMG) method requires the M-matrix property of the given sparse matrix. This requirement is often violated in practical applications. We present an approach for this case by constructing an auxiliary matrix from the original matrix. This auxiliary matrix is an M-matrix by construction. The application of a standard AMG coarsening on the auxiliary matrix results in the set of coarse nodes used for defining the inter grid transfer operators of the original system matrix. Both matrices will be projected to the coarse space by the Galerkin approach. The coarse auxiliary matrix is again an M-matrix. Promising numerical results are presented for anisotropic operators and for coupled systems.
The parallelization needs some modifications in the coarsening process such the the inter grid transfer operators fulfill a certain condition on the pattern of the interpolation/restriction. This guarantees that the parallel AMG is only a simple modification of the sequential AMG. The presented parallelization strategy for AMG results in very good speedups.
Discretized differential equations have to be solved several thousand times inside the solution process of an inverse problem. We got for this special application of AMG a significant gain in CPU time (factor 4 and more) due to additional acceleration of our code PEBBLES by simultaneous handling of several data sets, cache aware programming and by merging

of multigrid subroutines. Together with a parallelization, the solution time of the original code was accelerated from 8 days to 5 hours on a 12 processor parallel computer.

Multigrid in $\boldsymbol{H}(\mathrm{curl};\Omega)$: Latest News

Ralf Hiptmair

Multigrid methods for $\boldsymbol{H}(\mathrm{curl};\Omega)$-elliptic problems discretized by means of conforming edge elements can be viewed as subspace correction schemes that also involve nodal splittings of discrete potentials spaces. Up to now proofs of optimal asymptotic convergence have relied on discrete and continuous Helmholtz decompositions. However, for general domains the components of the Helmholtz decomposition lack the essential $H^1(\Omega)$-regularity required for the current proofs.
The idea is to use decompositions

$$\boldsymbol{H}_0(\mathrm{curl};\Omega) = \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathrm{curl};\Omega) + \mathrm{grad}\, H^1_0(\Omega)$$

that are no longer $\boldsymbol{L}^2(\Omega)$-orthogonal, but retain regularity. They can be used to establish the uniform stability of the multilevel decomposition by resorting to known results about multilevel splittings of standard Lagrangian finite element spaces.

Adaptive solution of the Einstein constraints using partition of unity and parallel computers

Michael Holst

In this talk we consider a coupled nonlinear elliptic system representing the Hamiltonian and momentum constraints in the Einstein equations. This system must be solved exactly or numerically to produce consistent initial data for general relativistic simulations of black hole and neutron star collisions. Moreover, the constraints must hold at all times in dynamical situations. Well-posedness of the system on connected compact Riemannian manifolds with Lipschitz boundaries was previously unstudied, and therefore in the first part of the talk we establish that the constraints have unique weak solutions under minimal smoothness assumptions on the data. The proof technique (Riesz-Schauder theory and convex analysis) allows for some degree of negative conformal scalar curvature. We also establish two quasi-optimal a priori error estimates for Galerkin approximations, and derive an a posteriori error estimate which leads to two distinct error indicators for adaptive simplex subdivision algorithms (one estimator is purely residual based, the other involves a linearized dual problem).
In the second part of the talk we outline an implementation of the adaptivity techniques using the adaptive finite element software package "MC", which is designed to adaptively solve general nonlinear systems of tensor equations on manifolds. MC is an adaptive multilevel finite element package which employs a posteriori error estimation, simplex subdivision, algebraic multilevel methods, global inexact Newton methods, and numerical continuation methods, for the solution of coupled elliptic systems on 2- and 3-manifolds. MC employs a new low-communication approach in parallel adaptive finite element methods, developed jointly with R. Bank at UCSD. We derive a global H^1-error bound for the solutions produced by the new parallel adaptive method by reinterpreting it as a partition of unity method, and by employing Xu and Zhou's generalizations of the Nitsche-Schatz interior estimates.

Efficient Solvers in Electromagnetic Field Computation

Ronald H.W. Hoppe

We consider multigrid and domain decomposition techniques in the numerical computation of electromagnetic fields based on curl-conforming edge element discretizations. In particular, we focus on domain decomposition approaches on nonmatching grids also known as mortar edge element methods. The macro-hybrid variational formulation of the problem involving a subtle analysis of the trace spaces of vector fields in $H(\mathrm{curl})$ is the clue how to impose continuity constraints on the skeleton of the decomposition and how to realize them by means of appropriate Lagrange multipliers. In the discrete regime, the specification of the multiplier space is discussed in detail which is crucial to prove an LBB condition for the resulting saddle point problem. For the numerical solution of the saddle point problem a multigrid algorithm is proposed featuring a distributive smoothing process involving a defect correction on the subspace of irrotational vector fields that takes care of the non-trivial kernel of the discrete curl-operator. A posteriori error estimation is addressed as well.

Numerical simulation of density driven flow

Klaus Johannsen

The numerical simulation of density driven flow in porous media is still a challenging problem. A number of benchmarks has been designed to validate the mathematical model and to compare numerical codes. But most of them are 2D test cases and there is still a lack of 3D problems. Recently new laboratory experiments have been performed which can be used to define a 3D benchmark. We present new 3D test cases derived from model calibrations for these experiments. For the well-defined mathematical model we show the numerical results to match the experiments within a reasonable accuracy. The test cases are analyzed numerically and error bounds of the discrete solution are given in different norms. Furthermore the sensitivity of the (numerical) results w.r.t. some of the parameters is investigated. The simulations have been done using the program package d^3f, a simulator for density driven flow in porous media for unstructured grids running on MIMD parallel computers. It is based on a second order finite volume / finite differences discretization in space / time using a full coupled / full implicit solving strategy.

Approximate elliptic solutions and Poincare-Steklov operators

Boris Khoromskij
(joint work with W. Hackbusch and M. Melenk)

We consider the problem of a data-sparse approximation of elliptic solutions and Poincare-Steklov operators. Typical examples of nonlocal operators are given by boundary/volume potential operators, by the fundamental solutions of elliptic operators, Green functions and Poincare-Steklov operators, operator exponentials, and fractional powers of elliptic operators. In the case of constant coefficients we discuss the data-sparse approximation to the Greens function by a sum of $\mathcal{H}$-matrices and low complexity matrices which can only be implemented with matrix-vector multiplication. The latter is realized by using boundary concentrated FEM which shows the linear logarithmic complexity w.r.t. the number of boundary degrees of freedom. The preconditioning issues have been also addressed.

Uniform Finite Element Error Estimates for Differential Equations with Jumps in Coefficients

Andrew V. Knyazev
(joint work with Olof B. Widlund)

We consider a parametric family of boundary value problems for the diffusion equation with the diffusion coefficient equal to a small constant in a subdomain. Such problems are not uniformly well-posed when the constant gets small. However, in a series of papers, Bakhvalov and Knyazev have suggested a natural splitting of the problem into two well-posed problems. Using this idea, we prove a uniform regularity of the solution and a uniform finite element error estimate for our model problem in the standard parameter-independent Sobolev norm. We consider a traditional finite element method with only one additional assumption, namely, that the boundary of the subdomain with the small coefficient does not cut any finite element.
The talk is based on the following paper: A. V. Knyazev and Olof Widlund, Lavrentiev Regularization + Ritz Approximation = Uniform Finite Element Error Estimates for Differential Equations with Rough Coefficients. A revised version accepted to Math. Comp., 2001. Published as a technical report UCD-CCM 132, 1998, at the Center for Computational Mathematics, University of Colorado at Denver (Viewgraphs of the talk are available at http://www-math.cudenver.edu/ aknyazev/research/confs/prism01.ps.gz).

Local Estimates for Algebraic Multigrid

Jan Mandel

1) AMG is based on heuristics guided by numerical sensors. The weak approximation property (WAP)

$$\forall u \exists v \ \|u - Pv\| \leq c \ u^T A u$$

is one important sensor. Convergence of smoothed aggregation AMG can be established under such assumption even in in multilevel case. WAP can be localized if $A = \sum A_i$, $A_i \geq 0$ and the footprint of A_i is related to nonzeros of P. We prove that in general given A and the footprints such A_i do not exists. But estimates on approximation $A = \tilde{A}_i +$ rest, $A_i \geq 0$, rest ≥ 0 are useful to access the difficulty of $Ax = b$, and we describe the construction of such pseudo-local $\tilde{A}_i$ with a given footprint.
2) Given $A = \sum A_i$, footprint A_i an element, one tries to aggregate the elements so that $\sum_{\text{aggregate}} A_i$ is well conditioned (modulo the rigid body modes subspace common to all A_i). It turns out that aggregation including strongly coupled elements are performable. Coupling of elements i and j is defined as $\lambda_7(A_i + A_j)$, the 7th smallest eigenvalue of common footprint of A_i and A_j. A computer visualization of such algorithm was demonstrated.

Algebraic Multilevel Methods and Sparse Approximate Inverses

Volker Mehrmann

We introduce a new approach to algebraic multi level methods and their use as preconditioners for the solution of positive definite linear systems. The multilevel process and in particular the coarsening process is based on the construction of sparse approximate inverses and their augmentation with corrections of smaller size. We present comparisons of the effectiveness of the resulting multilevel technique and numerical results.

Preconditioned eigensolvers

Klaus Neymeyr

The discretization of eigenvalue problems for self-adjoint and coercive elliptic partial differential operators leads to generalized matrix eigenvalue problems for sparse symmetric positive definite matrices. Preconditioned gradient type eigensolvers, which use standard preconditioners as designed for the solution of boundary value problems, can be applied to determine a modest number of the smallest eigenvalues and eigenvectors with multigrid efficiency.
We present a geometric convergence theory which interprets such a scheme as a perturbation of inverse iteration. The analysis provides sharp estimates for the Rayleigh quotient and guarantees that the convergence properties of the preconditioner transfer to the preconditioned eigensolver, which makes possible grid-independent convergence. The results hold for the generalized eigenvalue problem and for an associated subspace iteration scheme. The new estimates stimulate the search for new proof techniques that could be used to analyze such practically important improved preconditioned eigensolvers as the Locally Optimal Block Preconditioned Conjugate Gradient method. Numerical results are given for the Laplacian, which demonstrate the effectiveness of these preconditioned eigensolvers for mesh eigenproblems.

A genetic search for optimal multigrid components within a Fourier analysis setting

Cornelis W. Oosterlee
(joint work with R. Wienands)

An important analysis tool for multigrid methods is Fourier analysis. The two-grid analysis is the basis for the classical asymptotic multigrid convergence estimates. Furthermore, the local Fourier analysis (also called local mode analysis) is, in fact, the main multigrid analysis possibility for nonsymmetric problems.
Recently, we generalized the classical analysis in two ways. First of all, we have provided a Fourier analysis framework for analyzing the use of multigrid as a preconditioner for restarted GMRES quantitatively. Secondly, we have proposed the so-called three-grid analysis. Compared to the usual two-grid analysis, the three-grid analysis can yield additional insights especially for singularly perturbed and nonelliptic equations. An issue that can be evaluated in more detail by three-grid analysis is the coarse grid correction. In the two-grid analysis, one assumes an exact solve of the coarse grid problem on the first coarser grid level. It can, however, occur that an operator on the coarse grids is not favorable for the smoothing method applied. This cannot be taken into account by a two-grid Fourier analysis, but it can be by the three-grid analysis. An important example of such a coarse grid discretization is the Galerkin discretization. Its entries are not known in advance and depend on the fine grid discretization and the transfer operators in use. If a large difference in convergence factors occurs between the two-grid and the three-grid convergence factors from Fourier analysis, this is an indication for a problematic coarse grid correction.
The Fourier analysis tools can give a good insight into possible multigrid convergence difficulties from the smoother or from the coarse grid correction. It is, however, not easy in the case of convergence troubles to find from the analysis improved or even optimal multigrid components for the PDE under consideration. This is even more true if the Galerkin coarse grid operators are employed, since it is difficult to oversee their effect for nonsymmetric, singularly perturbed or indefinite problems.

The new idea in this paper is that we will employ an optimization technique help us searching for optimal multigrid components. We use a genetic algorithm in a three-grid analysis setting to find improved combinations of smoothers, coarse grid correction components and relaxation parameters for PDE problems that are known to be difficult for standard multigrid. The reason for choosing the genetic optimization is that the objective function, i.e., obtaining the best three-grid convergence factors, is not at all a smooth function in the "parameters" of the optimization, like different smoothers, transfer operators and the coarse grid discretization. Therefore, conventional calculus-based local optimization methods cannot be used for our purpose. On the other hand, it is also not possible to just apply an optimization by enumeration of all possibilities, as we can have a search space of about 2^{38} possibilities with about 10 varying parameters. So, the genetic algorithm is the method of choice here: The fitness function is well-defined. Genetic algorithms usually work with a binary coding of the underlying parameter set. They work with a population of a number of different parameter sets in the search space and base on probabilistic genetic rules, in which recombination and mutation operators play an important role. The selection of an individual parameter set is based on the fitness of the individual related to this set. A strong, i.e. fit individual spreads its genes into the next generation of individuals.
The research will be oriented towards sets of satisfactory components, say those that guarantee a three-grid convergence factor of less than a certain value instead of just looking for the optimal components. This directs towards robustness of the multigrid method, as it becomes possible to observe trends within the different parameters. The robustness of the multigrid algorithms can also be improved by considering multi-objective fitness functions: the algorithm should converge well for different PDEs.

Multigrid for the Navier-Stokes equations in rotation form

Arnold Reusken
(joint work with Maxim A. Olshanskii)

The topic of this presentation is motivated by the Navier-Stokes equations in *rotation* form:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \nabla P &= \mathbf{f} \quad \text{in} \quad \Omega \times (0,T], \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in} \quad \Omega \times (0,T]. \end{aligned}$$

Linearization and application of an implicit time stepping scheme results in a linear stationary problem of Oseen type. In well-known solution techniques for this problem such as the Uzawa (or Schur complement) method, a subproblem consisting of a coupled nonsymmetric system of linear equations of diffusion-reaction type must be solved to update the velocity vector field. In this talk we analyze a standard finite element method for the discretization of this coupled system and we introduce and analyze a multigrid solver for the discrete problem. Both for the discretization method and the multigrid solver the question of robustness with respect to the amount of diffusion and variation in the convection field is addressed. We prove stability results and discretization error bounds for the Galerkin finite element method. We present a convergence analysis of the multigrid method which shows the robustness of the solver. Results of numerical experiments are presented which illustrate the stability of the discretization method and the robustness of the multigrid solver.

Grid Generation, Stable Interpolation and Multigrid

Stefan Sauter
(joint work with N. Frauböse)

We will study the convergence of multi-grid methods for solving linear systems as they arise from finite element discretisations of elliptic boundary value problems on complicated domains. "Composite finite elements" are employed for the construction of the sequence of coarse-level discretisations, where the minimal dimension of the coarsest linear system is very small, independent of the number and size of geometric details in the domain. The convergence of the corresponding multi-grid method is proved in the framework of geometric multi-grid methods while the emphasis is on the "robustness" with respect to the geometric details in the domain.

Parameter Identification in Flow Problems

Volker Schulz

In this talk we present simultaneous optimization approaches for two parameter identification problems in flow models of different structure.
In the first problem we investigate special reduced SQP techniques for the numerical determination of material properties in Bingham flow. These techniques allows a computational effort bound of factor two compared with the effort necessary for the solution of the flow simulation problem alone. The practical problem stems from an application together with the firm Braun, Friedrichshafen. In addition to that also optimum experimental design approaches have to be applied to improve the measurement device used. Here, the underlying flow problem is considered stationary.
Furthermore we investigate parameter identification for instationary multiphase flow. There, a special reduced Gauss-Newton technique in combination with a multiple shooting approach is presented which allows a robust solution of the identification problem. This approach also enables a reuse of most of the numerical state of the art for multiphase flow as manifested in the package MUFTE-UG [Bastian/Helmig].

Preconditioning techniques for elliptic variational problems

Olaf Steinbach

For preconditioning the stiffness matrix appearing from the Galerkin discretization of an elliptic operator $A : V \to V^*$ we use an appropriate operator of opposite order, $B : V^* \to V$. The Galerkin discretization of B^{-1} then yields an almost optimal preconditioner. However, for an efficient application we need to have a computable approximation available. Beside the original trial space $V_h \subset V$ we need to introduce a trial space $V_h^* \subset V^*$ satisfying a certain stability condition. We will discuss these conditions in detail for different choices of trial spaces. Applications are preconditioning strategies for boundary and finite element methods.

A stable direct solver of the gradient equation

Rob Stevenson

Multi-level finite element discretizations of the equation grad $p = \mathbf{g}$ are presented. The discretizations lead to invertible systems that can be solved directly, requiring a number of operations proportional to the number of unknowns. We give optimal error estimates,

and furthermore show that the methods are stable with respect to perturbations in the right-hand side. Applications to the Stokes equations and the Poisson equation in mixed form are discussed.

Fast pressure calculation for two- and three-dimensional time dependent incompressible flow

Pieter Wesseling

A black box geometric multigrid preconditioner is described for second order elliptic partial differential equations in two and three dimensions. The number of cells in a block is not restricted to multiples of 2(+1), as in standard geometric multigrid, but completely arbitrary. The code can be used in a multiblock environment. The code is applied to pressure calculations in a pressure correction method for the incompressible Navier-Stokes equations. A comparison with conventional preconditions is made regarding wall-clock times. Details can be found in: J. van Kan, C. Vuik, P. Wesseling, Fast pressure calculation for 2D and 3D time dependent incompressible flow Numerical Linear Algebra with Applications 7:429–447 2000. The multigrid code is freely available at http://dutita0.twi.tudelft.nl/Ftp/nw/vankan/multigrid.

An Algebraic Multilevel Method – Development and Application to a remediation problem

Christian Wagner

The basic idea of the presented algebraic multilevel approach is to determine for each node a set of nodes (usually two) which allow an optimal interpolation of the considered node. The selected set of nodes is called suitable set of parent nodes. For some nodes e.g. due to symmetry, several sets of suitable parent nodes can be constructed. A theoretical analysis shows that the problem of finding these parent nodes for a node i can be reduced to a minimization problem of the form

$$\text{minimize } ||S^T z|| \quad \text{subject to } (z,t) = 0, \tag{1}$$

where S is a smoothing operator, $-z$ is the i-th row of the projection operator $I - P R_{inj}$ and t is a given test vector. Except for $z_i = -1$, the non-zero entries in z define the weights in the prolongation matrix P. Hence, solving the minimization problem (1) under the additional constraint that only a certain number, e.g. 2, non-zero entries z_j, $j \neq i$, are allowed, leads to the suitable sets of parent nodes and the interpolation weights. As (1) can be solved locally, the construction scheme is relatively cheap.
After the possible sets of parent node have been determined, the nodes are labeled as C- and F-nodes such that each F-node can be interpolated using these suitable sets of parent nodes and the already computed coefficients. Additionally, a simple heuristic algorithm tries to minimize the number of C-nodes and the number of edges in the coarse grid graph. The algorithm has been parallelized and generalized to systems of reaction transport equations. Realistic numerical experiments with up to 14 mil. unknowns confirm the efficiency of the presented algorithm. In addition, a biochemical groundwater remediation problem is presented as real live test case.

An ELLAM-MFEM solution technique for accurate and efficient simulation of compressible subsurface flow

Hong Wang

As an intermediate step in developing a fully coupled, accurate and efficient numerical simulation technique for multiphase, multicomponent fluid flow or compositional models, we develop an ELLAM-MFEM solution technique for a single-phase, multicomponent compressible fluid flows in compressible porous media with point sources and sinks. An Eulerian-Lagrangian localized adjoint method (ELLAM), which was shown to be very competitive with many widely used and well regarded methods in the context of linear transport partial differential equations (PDEs), is presented to solve the transport equations for the concentrations. Since accurate fluid velocities are crucial in numerical simulations, a mixed finite element method (MFEM) is used to simultaneously solve the pressure PDE as a system of first-order PDEs for the pressure and mass flow rate. This minimizes the numerical difficulties occurring in standard methods caused by differentiation of the pressure and then multiplication by rough coefficients.

While optimal-order error estimates were derived for the ELLAM schemes, computational experiments show that the developed ELLAM-MFEM solution technique can accurately simulate compressible fluid flows in porous media with coarse spatial grids and very large time steps, which are one or two orders of magnitude larger than those used in many numerical methods. The ELLAM-MFEM solution technique symmetrizes the governing PDEs, eliminates nonphysical oscillation and/or excessive numerical dispersion in many large-scale simulators. It conserves mass and treats boundary conditions in a natural manner. It can treat large adverse mobility ratios, discontinuous permeabilities and porosities, anisotropic dispersion in tensor form, compressible fluid, heterogeneous media, and point sources and sinks.

Furthermore, the ELLAM formulation can be utilized much like a "preconditioner" or "preprocessor" in the development of domain decomposition methods (DDMs) or multigrid methods (MGs) for unsteady-state advection-diffusion transport PDEs. Because of their hyperbolic nature, many DDMs or MGs that work very well for elliptic and parabolic equations could perform less promising for these problems. The fundamental reason is that these DDMs or MGs are aimed at elliptic PDEs and do not necessarily respect the hyperbolic nature of unsteady-state advection-diffusion transport PDEs.

Extensive research was carried out on the development of DDMs or MGs for unsteady-state advection-dominated PDEs. For example, in the context of DDMs, Cai developed multilevel additive and multiplicative Schwartz preconditioners. The Adaptive Dirichlet-Neumann (ADN) and Adaptive Robin-Neumann (ARN) nonoverlapping DDMs introduced by Gastaldi *et al.* and Ciccoli choose the interface matching conditions to be adapted to the local flow direction. However, the underlying numerical methods used in these DDMs (and other MGs) are often conventional methods possibly with certain upwinding, which tend to generate solutions with nonphysical oscillations or numerical dispersion unless very fine spatial grids and time steps are used. This leads to very large, strongly nonsymmetric coefficient matrices at each time step, and thus requires significant computational effort/cost in numerically solving the discrete systems even though very advanced DDM or MG techniques are used.

Rannacher and Zhou and Tai *et al.* developed overlapping DDMs based on the modified method of characteristics (MMOC). Due to the intrinsic difficulty of characteristic methods in treating general boundary conditions and in conserving mass, there is no Eulerian-Lagrangian type of nonoverlapping DDMs reported in the literature. We present

an ELLAM-based, nonoverlapping DDMs for unsteady-state advection-dominated PDEs by carefully choosing the subdomain interface matching conditions to respect the hyperbolic nature of the problems. The DDMs allow MGs to be used as an inner solver on each subdomain. The use of the characteristic (ELLAM or MMOC) methods in all these DDMs generate symmetric and positive-definite coefficient matrices with greatly reduced sizes that need to be solved at much less number of time steps.

Balancing Neumann-Neumann methods for incompressible Stokes equations

Olof B. Widlund
(joint work with Luca F. Pavarino)

Balancing Neumann-Neumann methods are introduced and studied for incompressible Stokes equations discretized with mixed finite or spectral elements with discontinuous pressures. After decomposing the original domain of the problem into nonoverlapping subdomains, the interior unknowns, which are the interior velocity component and all except the constant pressure component, of each subdomain problem are implicitly eliminated.
The resulting saddle point Schur complement is solved with a Krylov space method with a balancing Neumann-Neumann preconditioner based on the solution of a coarse Stokes problem with a few degrees of freedom per subdomain and on the solution of local Stokes problems with natural and essential boundary conditions on the subdomain boundary. This preconditioner is of hybrid form in which the coarse problem is treated multiplicatively while the local problems are treated additively.
The condition number of the preconditioned operator is independent of the number of subdomains and is bounded from above by the product of the square of the logarithm of the local number of unknowns in each subdomain and the inverse of the inf-sup constants of the discrete problem and of the coarse subproblem.
Numerical results show that the method is quite fast; they are also fully consistent with the theory. A discussion is given on the performance of the method for the spectral element case as well as some genuine experiments on parallel computing systems using an implementation in PETSc by Paulo Goldfeld, a graduate student at the Courant Institute.

Multigrid methods for finite elements

Christian Wieners
(joint work with Nicolas Neuss)

Applications. We present different applications for multigrid methods for finite elements: a standard conforming method ($\mathbf{Q}_1$ elements for finite plasticity), a stabilized method ($\mathbf{Q}_1/\mathbf{P}_0$ elements for infinitesimal plasticity, nonnested, fits to the 1. Strang Lemma), a mixed method ($\mathbf{Q}_2/\mathbf{Q}_1$ Taylor-Hood elements for a hybrid-viscoplastic soil model), a nonconforming method ($\mathbf{Q}_1 - \mathbf{Q}_1$ mortar coupling for linear elasticity, fits to the 2. Strang Lemma), and a hybrid method ($\mathbf{RT}_1$ elements for computing error bounds, implemented via interelement Lagrange multipliers).
Proofs. For providing a unified multigrid analysis for these applications, we extend the abstract framework for nonconforming finite elements by Brenner: we provide a full set for criteria guaranteeing the approximation property which can be applied to a large class of nonnested finite element spaces including nonconforming elements on curved boundaries.

Problems. Finally, we present numerical experiments showing effects with respect to regularity, local smoothing, and interface robustness which up to now cannot be explained by multigrid analysis.

A sharp Convergence Theory for the Method of Subspace Corrections

Jinchao Xu

The method of subspace corrections refers to a class of methods for finding the solution of a linear (or nonlinear) equation in a Hilbert space by approximately solving equations restricted to a number of subspaces the make up the entire space. In this talk, we first discuss the close relationship of this method with another class of methods, namely the method of alternating projections (which refers to a class of iterative methods for determining the best approximation to a given point in a Hilbert space from the intersection of a finite number of subspaces by alternatively computing the best approximations from the individual subspaces which make up the intersections). We then present a sharp convergence theory for these methods based on a new identity for estimating the norm of the product of nonexpansive operators in the Hilbert space. Finally, we discuss how this new theory may be used in the design of algebraic multigrid methods.

Edited by Christian Wieners

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