Mathematisches Forschungsinstitut Oberwolfach

Report No. 29/2001

Mini-Workshop:

Geometrization of Kazhdan's Property (T)

July 7th – July 14th, 2001

The present Mini-Workshop was organized by B. Bekka (Metz), P. de la Harpe (Geneva), and A. Valette (Neuchâtel). It was the first meeting ever, totally devoted to Property (T).

Property (T) is a representation-theoretic tool that was invented by D. Kazhdan in 1967. A locally compact group G has Property (T) if every continuous isometric action of G on an affine Hilbert space admits a globally fixed point. Kazhdan proved that simple, real or p-adic groups of split rank at least 2, as well as their lattices, have Property (T) (e.g. $SL_n(\mathbb{R})$, $SL_n(\mathbb{Z})$, $n \geq 3$). He also showed that, if Γ is discrete with Property (T), then Γ is finitely generated with finite abelianization (for some non-uniform lattices, this is the only known proof of finite generation!).

Over the years, the importance of Property (T) has been ever growing, and several applications have been found: to structure of normal subgroups in higher rank lattices, to uniqueness of invariant means, to structure of operator algebras, to construction of expander graphs ...

In the last decade, many new geometric constructions/characterizations of groups with Property (T) have been found, so that one may talk of a "Geometrization of Property (T)". The purpose of this workshop was to bring together experts in the field and researchers working in related areas in order to explore these new avenues. Special emphasis was given to:

- Kazhdan constants, which make Property (T) quantitative;
- Property (T) for infinite-dimensional Lie groups;
- characterizations of Property (T) via ergodic theory;
- cohomological and graph-theoretic characterizations of Property (T);
- relations with Gromov's theory of random groups;
- Property (T) for groups acting properly on simplicial complexes.

Possible generalizations of Property (T) were also discussed.

There were 17 participants, including the organizers, coming from the following countries: Switzerland (5), France (4), Australia (2), Israel (2), Germany (1), Italy (1), Poland (1), U.S.A. (1). Two of the participants were PhD students; six are less than 5 years after PhD.

On the whole week, there were 18 lectures, one hour each. This left ample time for discussions which, in the wonderful environment of Oberwolfach, were greatly appreciated.

A problem session was held: the problems proposed there, appear at the end of the present report.

The organizers

Abstracts

Kazhdan's Property (T) for some infinite dimensional Lie group.

Bekka, B.

A topological group G has Kazhdan's Property (T) if there exists a compact subset K of G and $\varepsilon > 0$ such that, for every strongly continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G without non-zero fixed vector, one has

$$\sup_{g \in K} \|\pi(g)\xi - \xi\| \geq \varepsilon, \qquad \forall \xi \in \mathcal{H}_\pi, \|\xi\| = 1.$$

In this talk, we gave a survey on the known examples of infinite dimensional Lie groups with this property. We discussed this property for the loop groups $L(SL_n(\mathbb{C}))$ of $SL_n(\mathbb{C})$ for $n \geq 3$, a result due to Y. Shalom. Another example is the unitary group $\mathcal{U}(\mathcal{H})$ of an infinite dimensional separable Hilbert space, with the strong operator topology.

Sp(1,n).

Cowling, M.

There is revived interest in the explicit construction of uniformly bounded representations of groups. The aim of this talk is to outline the construction of uniformly bounded representations of $Sp(1, n)$, following the presenter's paper "Unitary and uniformly bounded representations of some simple Lie groups" in *Harmonic Analysis and Group Representations*, CIME, 1980.

Let X be the usual unbounded flat realisation of the Furstenberg boundary of $Sp(1, n)$, and Q be its dimension as a Carnot–Caratheodory manifold. When $|\alpha| < Q/2$, there are natural homogeneous Sobolev spaces $H_\alpha(X)$ on X of functions f such that $\Delta^{\alpha/2} f \in L^2(X)$, which are related to Lebesgue spaces $L^p(X)$, where $1/p - 1/2 = \alpha/Q$, by the Sobolev embedding theorem.

The analytic continuation of the unitary principal series of $Sp(1, n)$ is naturally isometric on a space $L^p(X)$, for an appropriate p, at least when the representations have bounded matrix coefficients. The point of the talk is that the (extended) principal series representations which act isometrically on $L^p(X)$ act uniformly boundedly on the corresponding Sobolev space $H_\alpha(X)$.

Measure equivalence and Quasiisometries.

Furman, A.

Definitions (Gromov) : Two finitely generated groups Γ and Λ are *Topologically Equivalent* (TE) if there exist a locally compact space X with continuous commuting proper and cocompact actions of Γ and Λ.

Two finitely generated groups Γ and Λ are *Measure Equivalent* (ME) if there exists a measure space (Ω, m) with commuting actions of Γ and Λ, such that each of the groups has a fundamental domain of finite measure.

Gromov observed that two finitely generated groups are TE if and only if they are quasi-isometric (i.e. there exists a large scale bi-Lipschitz map between their Cayley graphs). Cocompact lattices in the same locally compact group G are TE and hence QI.

All lattices in the same locally compact group are ME.

In the talk recent results on ME invariants (such as Property (T), amenability, a-T-menability, ℓ^2-Betti numbers, etc...) were discussed and compared to QI invariants.

Measure Equivalence Rigidity for higher rank lattices.

Furman, A.

In the talk we discuss the following theorem

Theorem 1. *Let $\Gamma \subset G$ be a lattice in a simple real Lie group with $Rank_{\mathbb{Q}}(G) \geq 2$. Any countable group Λ which is ME to Γ is commensurable to a lattice in G. More precisely : there exist a finite index subgroup $\Lambda' \subset \Lambda$ and an homomorphism $\rho : \Lambda' \to G$ with $\Lambda_0 = Ker\ \rho$ finite and with $\Lambda_1 = \rho(\Lambda)$ a lattice in G.*

Furthermore, ME coupling ${}_{\Gamma}\Omega_{\Lambda}$ admits a (unique) measurable $\Phi : \Omega \to G$ such that $\Phi(\gamma\omega\lambda) = \gamma\Phi(\omega)\rho(\lambda)$ and $\Phi^* m$ is a combination of Haar measure and couting measure on a conjugate of Γ.

Applications of this theorem to Orbit Equivalence Rigidity and equivalence relations were discussed.

The geometry of the set of invariant measures and Property (T).

Glasner, E.

(joint work with Weiss, B.)

A Choquet simplex is called a Bauer simplex if $ext(Q)$ is closed and a Poulsen simplex when $ext(Q)$ is dense in Q. (Here Q is a convex compact set and $ext(Q)$ is the set of extreme points of Q.) For a compact metric space X with an action of a countable group Γ via homeomorphisms of X, let $\mathcal{M}_{\Gamma}(X)$ be the simplex of all Γ-invariant probability measures on X. I indicated the proofs of the following theorems.

Theorem 1. *If Γ has Property (T) then every $\mathcal{M}_{\Gamma}(X)$ is a Bauer simplex.*

Theorem 2. *If Γ is not a Kazhdan group then $\mathcal{M}_{\Gamma}(\Omega)$ is the Poulsen simplex (up to isometric isomorphism, there is a unique separable Poulsen simplex).*

Here $\Omega = \{0,1\}^{\Gamma}$ is the natural "Bernoulli" action of Γ on the "symbolic" space Ω.

Analogous theorems can be proved for a locally compact second countable topological group G.

Equivariant cohomology of buildings.

Januszkiewicz, T.

(joint work with Dymara, J.)

Two classes of simplicial complexes with simplicial group actions are defined : $B_t \supset B_+$.

B_t contains Tits-Kac-Moody buildings of (very) large thickness. For X to be in B_+ the entries of the Coxeter matrix of the Weyl group of X cannot be infinite.

For spaces in B_+ we give explicit formulae for the equivariant cohomology with coefficients in any unitary representation. For spaces in B_t a similar computation is performed for coefficients contained in $\oplus^{\infty} L^2(G)$. These formulae have many applications. They are used to prove that some groups have Property (T), that some are a-T-menable, to determine the L^2-cohomology of X in B_t and the continuous-cohomological dimension of G.

Furthermore for a subclass $B_t(p) \subset B_t$ containing buildings of even larger, depending on p, thickness, we compute L^p cohomology of X.

Groups acting on the circle.

Louvet, N.

Let T be a discrete countable group.

We present some rigidity results for actions of such groups by diffeomorphisms of class $\mathcal{C}^0$, $\mathcal{C}^1$, ... on the circle.

In particular, there is the following result of A. Navas:
If a group Γ has Kazhdan's Property (T), then any action of Γ by orientation preserving diffeomorphisms of the circle of class $\mathcal{C}^{1+\alpha}$, with $\alpha > 1/2$, has a finite image.

Kazhdan Constants for Compact Groups.

Neuhauser, M.

Let G be a locally compact group, Q a compact subset of G and $\widehat{G}$ the unitary dual of G. Denote by H_π the space of a representation $\pi \in \widehat{G}$ and by $H_{\pi,1}$ the set of unit vectors of H_π. Then the *Kazhdan constant* is defined by

$$\kappa_G(Q) = \inf_{\pi\in\widehat{G}\setminus\{1\}} \inf_{\xi\in H_{\pi,1}} \sup_{g\in Q} \|\pi(g)\xi - \xi\|.$$

Theorem 1. *Let now G be compact, Q a conjugacy class, $\chi_\pi(g) = \operatorname{tr}\pi(g)$ the character of $\pi \in \widehat{G}$, and $d_\pi = \chi_\pi(1) = \dim H_\pi$. Then*

$$\kappa_G(Q) \geq \inf_{\pi\in\widehat{G}\setminus\{1\}} \sqrt{2 - \frac{2}{d_\pi}\operatorname{Re}\chi_\pi(Q)}.$$

Theorem 2. *If Q moreover generates G, i. e. there is a ν such that $G = (Q \cup \{1\} \cup Q^{-1})^\nu$, then*

$$\inf_{\pi\in\widehat{G}\setminus\{1\}} \sqrt{2 - \frac{2}{d_\pi}\operatorname{Re}\chi_\pi(Q)} \geq \frac{1}{\nu}.$$

Theorem 3. *Let $G = \mathrm{SU}(n)$, $q_t = \operatorname{diag}(e^{it_1}, \ldots, e^{it_n})$, and Q_t the conjugacy class of q_t. Then*

$$\kappa_G(Q_t) \geq \sqrt{\frac{2}{n}} \max_{1\leq j,k\leq n} \left|\sin\left(\frac{t_j - t_k}{2}\right)\right|.$$

Theorem 4. *Let $G = \mathrm{SU}(2)$, $q_t = \operatorname{diag}(e^{it}, e^{-it})$, and Q_t the conjugacy class of q_t. Then for $0 \leq t \leq 1.4216$*

$$\kappa_G(Q_t) = 2\sin(t/2).$$

How robust is Property (T)?

Monod, N.

There is a natural strengthening of Property (T), called Property (TT). This is encoded algebraically by both H^1 and H_b^2. This new property fails to hold for any hyperbolic group, yet is proved for higher rank lattices. Using vanishing theorems for irreducible lattices (Burger & M.) and a construction of cocycles in negative curvature (Shalom &

M.), one obtains notably strong superrigidity theorems and an abstract strong rigidity for measure equivalences.

Vanishing theorems and Property (T).

Pansu, P.

This was a survey of a few vanishing theorems in differential geometry: Bochner, Hodge, Matsushima, Corlette, Garland, all sharing a common origin. Their (obvious) generalization to local systems leads to a proof of Property (T) for cocompact lattices in simple Lie groups (except $SO(n,1)$ and $SU(n,1)$) and cocompact isometry groups of $\tilde{A}_2$-buildings. A non linear generalization (fixed points for isometric actions on nonpositively curved spaces) was advertised.

Characterizing Property (T), for countable groups, by actions on measure spaces.

Robertson, G.

(joint work with Steger, T.)

A class of negative definite kernels is defined in terms of measure spaces. Using this concept, Property (T) for a countable group Γ is characterized in terms of measure preserving actions of Γ, as follows. If a set S is translated a finite amount by any fixed element of Γ, then there is a uniform bound on how far S is translated.

The ideas of the proof are examined explicitly in hyperbolic spaces. The geodesic distance between points in real hyperbolic space is a hypermetric, and hence is a kernel negative type. The proof given uses an integral formula for geodesic distance, in terms of a measure on the space of hyperplanes. An analogous integral formula, involving the space of horospheres, can be given for complex hyperbolic space. By contrast geodesic distance in a projective space is not of negative type.

Reduced cohomology of unitary representation

Shalom, Y.

In the series of three talks we discuss the notion of reduced cohomology and present various applications.

First talk : General existence results of representations with non vanishing reduced first cohomology for any group not having Property (T). Among the applications : Every finitely generated Kazhdan group is a quotient of a finitely presented Kazhdan group (a question by A. Zuk and S. Grigorchuk), every compactly generated non Kazhdan group G admits an irreducible representation π with $\overline{H^1}(G,\pi) \neq 0$ (a conjecture by Vershik-Karpushev) and others.

Second talk : Applications to uniformly bounded representations. For any simple Lie group G of rank ≥ 2 and any uniformly bounded representation π, $\overline{H^1}(G,\pi) = 0$.

In contrast, we show that this fails for all rank one Lie groups (and their discrete subgroups, including $Sp(n,1)$ and $F_4(-20)$).

Third talk : Applications to quasiisometries of amenable groups : We show that a group quasiisometric to $\mathbb{Z}^d$ is virtually isomorphic to $\mathbb{Z}^d$ without using Gromov's theorem of polynomial growth (answering a question of Ghys and de la Harpe) and we show that any group quasiisometric to a polycyclic solvable group must admit a finite index subgroup with infinite abelianization. The relation of reduced cohomology and quasiisometry comes via Gromov's topological characterization of quasiisometry and the amenability of the groups is used to produce an invariant measure on the topological coupling, thereby enabling us to transfer (via induction) certain cohomological properties between quasiisometric groups.

Vanishing of first cohomology and Property (T).

Valette, A

In this survey talk, I first gave a proof of the Delorme-Guichardet theorem, stating that Property (T) for a locally compact separable group G is equivalent to the following fixed point property : any affine isometric action of G on a Hilbert space, has a globally fixed point. Then I explained how the main properties of Kazhdan groups can be deduced from the fixed point property, namely :

- compact generation
- compact abelianization
- Property (T) goes down to lattices
- every action of a Kazhdan group on a tree fixes either an edge or a vertex.

Non-uniform Kazhdan constants.

Zuk, A.

We discuss the problem of dependence of Kazhdan constants for finitely generated groups on finite sets of generators. In particular we are interested in existence of Kazhdan constants which are independent of the choice of the generating system (uniform Kazhdan constants). We relate this question to other problems like uniform exponential growth. We show (joint work with T. Gelander) that for instance irreducible lattices in products of semi-simple Lie groups do not have uniform Kazhdan constants.

Generic groups and Property (T), I and II.

Zuk,A.

We present a simple condition which enables one to prove Property (T) and compute Kazhdan constants for a discrete group given its presentation. This condition applies to some lattices and gives an elementary proof of Property (T) for them. Using this condition one can construct new examples of Kazhdan groups and show that a generic group (in the sense of Gromov) is infinite, hyperbolic and has Property (T).

Open Problems Session

1. Is there an infinite Kazhdan group with uniform Kazhdan constants? Is there an infinite Kazhdan group Γ and an integer k such that Γ has generating sets of size k and such that there is a bound on Kazhdan constants of Γ with respect to subsets of size at most k?

   What about $SL_3(\mathbb{Z})$?

If an infinite Kazhdan group Γ has a dense homomorphic image in a connected topological group G which has a continuous unitary representation without non zero invariant vectors, then Γ does not have uniform Kazhdan constants. See T. Gelander and A. Zuk, Dependence of Kazhdan constant on generating subsets, Israel J. of Math., to appear.

2. Can one find $N > 0$ and generating subsets $B_n \in \mathrm{Sym}(n)$ with $\#B_n \leq N$ such that $\mathcal{G}(\mathrm{Sym}(n), B_n)$ is a family of expanders?

3. Does the equivalence $(FH) \Leftrightarrow (T)$ hold for more general classes of groups than σ-compact locally compact groups?
   What about locally compact groups? polish groups?
   The implication "$\Leftarrow$" is known to hold for every locally compact group.

4. Let G be a simple compact Lie group. Does the loop group $L(G)$ have (T)?
   For G as above, or more generally for G amenable, is $L(G)$ amenable?

5. Let $\mathcal{M}$ be a von Neumann algebra and let $U(\mathcal{M})$ be its unitary group, together with the strong topology. When does $U(\mathcal{M})$ have (T)?
   Is it true when $\mathcal{M}$ is injective?
   B. Bekka has shown that it is true if $\mathcal{M}$ is the algebra of all bounded operators on a separable Hilbert space $\mathcal{H}$, in which case $U(\mathcal{M}) = U(\mathcal{H})$ is the unitary group of $\mathcal{H}$.

6. Does the Robertson-Steger characterization of Property (T) for countable groups hold true for ergodic actions?
   More precisely is the following property of a group Γ equivalent to Property (T)?
   For every ergodic measure preserving action of Γ on a measure space $(\Omega, \mathcal{B}, \mu)$ and every set $S \in \mathcal{B}$ such that $\mu(S \,\Delta\, \gamma S) < \infty$ for all $\gamma \in \Gamma$, we have
   $$\sup_{\gamma \in \Gamma} \; \mu(S \Delta \gamma S) < \infty.$$
   See G. Robertson and T. Steger, Negative definite kernels and a dynamical characterization of property (T) for countable groups, Ergod. Th. and Dynam. Syst. 18 (1998) 247-253.

7. Does some Robertson-Steger characterization of (T) hold for locally compact groups?

8. Is the usual metric on a complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ an hypermetric?
   Recall that a hypermetric on a set X is a distance function $d : X \times X \to \mathbb{R}^+$ such that $\sum_{i,j=1}^n t_i t_j d(x_i, x_j) \leq 0$ for all $n \geq 0$, $x_1, \cdots, x_n \in X$ and $t_1, \cdots, t_n \in \mathbb{Z}$ with $\sum_{i=1}^n t_i = 1$.
   See G. Robertson, Crofton formulae and geodesic distance in hyperbolic spaces.
   What is the class of groups on which every hypermetric function is bounded?
   What is the class of groups on which there exist proper hypermetric functions?

9. Find vanishing theorems (in the sense of Pansu's lecture) for non-linear spaces of harmonic mapping $M \to N$. There are interesting candidates for the target space N, such that the manifold of all Riemannian metrics on a given finite-dimensional manifold. We should also consider cases where M is not a manifold, but is some kind of a complex, or a product of a complex and a manifold.

10. Is Property (T) invariant under rough isometries (i.e. quasiisometries where the multiplicative constants is 1)?

11. Does $SL_3(\mathbb{Z}[t])$ have Property (T)? Is it finitely presented ? Is it boundedly generated by subgroups $\{I + E_{i,j}(f) \mid f \in \mathbb{Z}[t]\}$ for $i, j \in \{1,2,3\}$ and $i \neq j$?

   See Y. Shalom, Bounded generation and Kazhdan's Property (T), Pub. Math. I.H.E.S. 90 (1999) 145-168. The third question above has been asked in the context of K-theory by W. van der Kallen.

12. Say a separable locally compact group G has Property (T_{strong}) if $H^1(G, \pi) = 0$ for every uniformly bounded representation π of G on a Hilbert space.

   What are the properties of (T_{strong})? Is it invariance under rough isometries? by measure equivalence? central extension (à la Serre) for which Property (T) is invariant? It is known that (T_{strong}) is not invariant by quasi-isometries from the same examples which show the analogous fact for (T).

13. Does there exist a locally compact group containing a uniform lattice Γ and a finitely generated non-uniform lattice Λ such that Λ and Γ are quasiisometric?

14. Shalom's conjecture : Any hyperbolic group Γ admits a uniformly bounded representation π with $\overline{H^1}(\Gamma, \pi) \neq 0$ together with a proper cocycle in $Z^1(\Gamma, \pi)$.

15. Is there an analogue of Delorme-Guichardet's theorem for uniformly bounded representations?

16. Is every finitely generated Kazhdan group a quotient of a finitely presented Kazhdan group with finiteness conditions stronger than finite presentation?

17. Do these groups have Property (T)?
   - Burnside groups $B(n, p)$,
   - mapping class group of surfaces of large genus, $MCG(\Sigma_g)$ for various g,
   - $\mathrm{Out}(\mathbb{F}_n)$ for $n \geq 4$,
   - $\mathrm{Aut}(\mathbb{F}_n)$ for $n \geq 4$,

   Do $(\mathrm{Aut}(\mathbb{F}_n) \ltimes \mathbb{F}_n, \mathbb{F}_n)$ and $\mathrm{MCG}(\Sigma_g) \ltimes \Sigma_g, \Sigma_g)$ have relative Property (T)?

18. If Γ has Property (T), is $\mathrm{Out}(\Gamma)$ finite ? There is partial result from F.Paulin : this is true for hyperbolic Kazhdan groups.

19. Is there a (non trivial) left orderable Kazhdan group ?

Edited by Pierre-Alain Cherix (Geneva)

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