MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Elliptic and parabolic problems of higher order

Miniworkshop, July 8 - 14, 2001

This workshop was organized by Filippo Gazzola (Alessandria, Italy), Hans-Christoph Grunau (Utrecht, Netherlands and Bayreuth, Germany) and Guido Sweers (Delft, Netherlands). It was intended to focus on the pecularities of elliptic and parabolic problems of order at least 4 and to bring together people, working on various subfields of higher order PDE. The main subjects were:

- Eigenvalue problems
- Nonlinear evolution equations like surface diffusion and thin film equations
- Numerical methods
- Higher order PDE in Differential Geometry
- Critical growth problems
- Positivity preserving properties

Every participant explained in detail (60-75 minutes) recent results, interesting open problems and possible strategies. Most talks were followed by extensive discussion rounds.

The abstracts were collected by Hans-Christoph Grunau; Filippo Gazzola wrote the summary of the open problem session.

Abstracts

On the Polya-Szegö conjecture for the buckling of a clamped plate

Mark S. Ashbaugh

This talk surveys the history of Polya and Szegö's conjecture concerning the first buckling eigenvalue of a clamped plate, as well as certain other fourth order eigenvalue problems "of Faber-Krahn type" (most notably Rayleigh's 1877 conjecture for the fundamental frequency of vibration of a clamped plate, which is still open in 4 and more dimensions). The Polya-Szegö conjecture for the buckling problem asserts that for a clamped plate of given area the shape that buckles most readily under uniform compressive loading of its edges is the circular one, all other physical parameters being equal. The talk presents our current knowledge of this and related problems.

On the first eigenvalue for the buckling of the clamped plate

Friedemann Brock

Let Ω a bounded domain in $\mathbb{R}^n$, and let $\Lambda_1(\Omega)$ be the first eigenvalue of the buckling of the clamped plate, that is, $\Lambda_1(\Omega) = \inf\{v \in W_0^{2,2}(\Omega), v \not\equiv 0 : \int_\Omega (\Delta v)^2 / \int_\Omega |\nabla v|^2\}$. We prove some relations for the solution u of the associated *Schiffer* problem:

(P) $$-\Delta u = \Lambda_1(\Omega) u \text{ in } \Omega, \partial u/\partial \nu = 0,\ u = c \text{ on } \partial\Omega,$$

where ν is the exterior normal and c is some constant. We also show the following:
If problem (**P**) has a nontrivial solution, and if the isoperimetric inequality

$$\lambda_{n+1}(G) \geq \lambda_2(B)$$

(G: domain in $\mathbb{R}^n$, B: ball with same measure as G, $\lambda_k(G)$: k-th Dirichlet eigenvalue of the Laplacian in G), holds true, then Ω must be a ball.

On the isoperimetric inequality for the buckling of a clamped plate: a variational approach

Dorin Bucur

In two dimensions, we prove the existence of a domain minimizing the buckling load of a clamped plate among all domains of given measure, i.e.which solves the following minimization problem

$$\min_{|\Omega|=c} \min_{u \in W_0^{2,2}(\Omega), u \neq 0} \frac{\int_\Omega |\Delta u|^2 dx}{\int_\Omega |\nabla u|^2 dx}.$$

In order to prove that the optimal set is the ball, we discuss possible ways of applying an idea of Willms and Weinberger, if the regularity of the optimal set is not a priori assumed. We observe that the only point in their proof where the smoothness of the minimizing domain is necessary is to perform the shape derivative and to get a boundary integral identity which gives that Δu is constant on the boundary of the domain, u being the first normalized eigenvector on the optimal domain.

This is a joint work with Mark Ashbaugh.

Some higher order and fully non-linear elliptic equation in conformal geometry

Sun-Yung Alice Chang

Elliptic equations have always been an important tool in the study of problems in geometry. In the recent decades, non-linear second order elliptic equations with critical exponents have played a special role in the solutions of several important problems in conformal geometry; e.g. the problem of prescribing Gaussian curvature and the Yamabe problem. In this talk, I describe some recent effort to extend the role played by second order equations to higher order ones. First, I describe properties of a class of conformal covariant operators– in particular a 4-th order operator with its leading symbol the bi-Laplace operator, discovered by Paneitz in 1983 –; then I describe the relations of these operators to some natural functionals (e.g. the zeta-functional determinant for the Laplace operator) and elliptic equations (e.g. the Monge-Ampere equation). I discuss questions of existence, uniqueness and regularity of the associated nonlinear equations. As applications, I describe a problem on prescribing the Ricci curvature on 4-dimensional manifolds; and an extension of the Cohn-Vossen inequality to 4-dimension, relating the Euler number to the total integral of a 4-th order curvature invariant.

Moving boundary problems of higher order

Joachim Escher

The purpose of my talk is to give a survey over some recent developments in the theory of classical solutions to elliptic and parabolic problems of higher order involving moving surfaces. Problems of this type do not satisfy a superposition principle for solutions and, hence, carry an inherent nonlinear structure. In fact, it turns out that most of the equations describing the evolution of surfaces are of quasilinear or even of fully nonlinear type. Additionally, these equations are often of a nonlocal nature. We present examples of first, second, third, and fourth order parabolic moving boundary problems steming from different applications: flows through porous media, mean curvature flows, Mullins-Sekerka models, surface diffusions flows, and Willmore flows. Besides existence and uniqueness results, we also study the stability of equilibira and some singular perturbations.

From second order to fourth order semilinear elliptic equations

Filippo Gazzola

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ be an open bounded smooth domain, let $f \in C^1(\mathbb{R})$ and consider the following semilinear biharmonic equation

$$\Delta^2 u = f(u) \qquad \text{in } \Omega \tag{1}$$

either with Navier boundary conditions ($u = \Delta u = 0$ on $\partial\Omega$) or with Dirichlet boundary conditions ($u = \nabla u = 0$ on $\partial\Omega$).

We are interested in extending to (1) some well-known results relative to the second order equation $-\Delta u = f(u)$. In particular, we focus our attention on the critical growth case $f(u) = |u|^{8/(n-4)}u$ and on the radial symmetry of *positive* solutions for general f when Ω is the unit ball. This extension is not always possible: we give both positive and negative answers, together with a number of open problems.

Positivity in clamped plate equations
– State of the art and open problems –

Hans-Christoph Grunau

We are interested in positivity results for Dirichlet problems like

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ u = \nabla u = \ldots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We would like to find an answer to the following positivity question:

For which $\Omega \subset \mathbb{R}^n$ does $f \geq 0$ always imply $u \geq 0$?

From an old result by Boggio (1905), it is known that the answer is affirmative, provided $\Omega = B$ is a ball. Numerous counterexamples show that, in general, such a positivity preserving property will not hold true.

Our contributions are mainly concerned with a perturbation theory for Boggio's positivity result, and even here, still some fundamental problems remain to be solved. With respect to a global understanding of positivity in (2), we are still at the beginning.

The state of the art is briefly summarized, and many open problems are mentioned. Here is a selection of them:

- Is the positivity preserving property of (2) in balls preserved under domain perturbations in any space dimension n? Such a result was shown for the two-dimensional case.
- In long thin ellipses, our positivity question has to be denied (Garabedian, 1951). What can be said about the sign/uniqueness of the corresponding first eigenfunction? Perhaps, support from numerical analysis will be needed here.
- How can the transition from positivity to sign change – e.g. when the disk is smoothly deformed into long ellipses – be characterized? We feel that this will be first observed at the boundary $\partial\Omega$ via degeneracies in the boundary behaviour of Green functions.
  In simpler one dimensional situations such a criterion was found by J. Schröder.
  Could one perhaps find the critical ratio of half axes rigorously, beyond which change of sign can be observed first?

Most of the work, I did in this field, is joint work with Guido Sweers.

Anti-eigenvalues and their dependence on the geometry of the domain

Bernd Kawohl

It is well known, that the problem

$$(\Delta)^2 u + \lambda u = f \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega$$

is positivity preserving in the sense that $f \geq 0$ implies $u \geq 0$, provided $-\lambda_1(\Omega)^2 < \lambda \leq \lambda_c(\Omega)$, where λ_1 is the first Dirichlet Laplace eigenvalue and λ_c is some positive number. A conjecture of McKenna and Walter about the isoperimetric dependence of λ_c, namely that $\lambda_c(\Omega) \leq \lambda_c(\Omega^*)$ is discussed, put into context with related problems and disproved. This is joint work with Guido Sweers.

Thin film equations: Film rupture, and new energies

Richard S. Laugesen

The model *thin fluid film* equation is $h_t = -(h^n h_{xxx})_x$, where $n > 0$ is given and $h(x,t) \geq 0$. The cases $n = 1, 2, 3$ arise in various situations. For example, $n = 3$ arises when h is the thickness of a fluid film on a flat substrate, with the film evolving under the influence of surface tension but not gravity.

The equation $h_t = -(h^n h_{xxx})_x$ dissipates the energy $\int h_x^2 \, dx$ (provided we assume periodic boundary conditions in x, say). I show that one can do better for $\frac{1}{2} < n < 3$, obtaining dissipation of $\int (h^p)_x^2 \, dx$ for certain exponents $p = p(n) < 1$.

I speculate on connections to proving that "films cannot rupture" for certain values of n (known already for $n \geq 3.5$ by work of Bertozzi *et al.*).

Direct and inverse problems for linear integro-differential equations as generators of high-order evolution equations

Alfredo Lorenzi

We study first-order operator integro-differential equations of the form

$$(*) \qquad u'(t) - A_1 u(t) - \int_0^t h(t-s) A_2 u(s) \, ds = f(t), \qquad t \in [0, T],$$

related to a Hilbert space H and to a bounded time interval $[0, T]$, under the assumption that operator A_2 dominates A_1 and $h(0) \neq 0$.

We prove existence and uniqueness results for the solutions of both direct and inverse problems related to $(*)$. The inverse problem consists of recovering, in addition to u, also the kernel h appearing in the integral term. The results so found are applied to integro-differential partial equations governing thermal materials with memory related to spatial cylindrical domains.

When $h(0) = 0$ and $A_1 = 0$, $A_2 = A$, equation $(*)$ easily reduces to the convolution second-order integro-differential equation

$$(**) \qquad u''(t) - \int_0^t k(t-s) A_2 u(s) \, ds = f(t), \qquad 0 \leq t \leq T,$$

in the Hilbert space H, where $k = h'$. Unlike the usual situation the Cauchy data $u(0) = u_0$, $u'(0) = u_1$ are replaced by the *mixed* condition $u(0) = u_0$, $u(T) = u_2$. We analyze the two cases $k(0) \neq 0$ and $k(0) = 0$, $k'(0) \neq 0$, which lead to the study of differential and integro-differential equations of the third and fourth order in time, respectively. Under suitable assumptions we solve the inverse problem consisting of finding the pair (u, k) when the additional (measured) data $g(t) := (u(t), \varphi_0)$ are available, φ denoting some eigenvector of A.

On higher order nonlinear evolution equations and systems

Enzo Mitidieri

We discuss some recent results obtained jointly with Stanislav Pohožaev (Steklov Mathematical Institute-Moscow) on blow-up solutions for higher order (in time and space) evolution problems. The technique we use is the classical "test function method".

Computer-assisted proofs for boundary value and eigenvalue problems

Michael Plum

The lecture is concerned with computer-assisted methods for obtaining tight enclosures (and simultaneously proving existence) of solutions of problems of the form

$$(+) \qquad L_0[u] + \mathcal{F}(u) = 0,$$

with L_0 and $\mathcal{F}$ denoting a closed linear and a Fréchet-differentiable (nonlinear) operator, respectively, in an appropriate Banach space setting.

For this purpose, an approximate solution ω of (+) is computed by some numerical method, and (+) is transformed into a fixed-point equation involving the defect $L_0[\omega] + \mathcal{F}(\omega)$ of ω, and the inverse of the linearized operator $L := L_0 + \mathcal{F}'(\omega)$.

For the application of Schauder's or Banach's Fixed-Point-Theorem (yielding the desired results), there is need, in particular, for *explicit a priori bounds* for L (in suitable norms). For obtaining these, *eigenvalue enclosures* (for self-adjoint eigenvalue problems) play a crucial role.

Particular emphasis is put on the application of this abstract approach to elliptic boundary value problems. Here, the required eigenvalue bounds can be obtained by computer-assisted methods of their own, based on variational principles.

The method has mainly been applied successfully to many second-order boundary value problems (where $L_0 = -\Delta$), but also for fourth-order problems, convincing results have been obtained by Ch. Wieners. As a further application to a fourth order (ODE) problem, an instability proof for the Orr-Sommerfeld equation from hydrodynamics is reported on.

Existence of minimizers for the Willmore functional

Leon Simon

This talk gives an exposition of the author's 1993 proof of the existence of genus 1 minimizers of the Willmore functional $\mathcal{F}(\Sigma) = \int_\Sigma H^2\, dA$, where Σ is an embedded surface in $\mathbb{R}^3$ and $H = \frac{\kappa_1+\kappa_2}{2}$ is the mean curvature of Σ. The talk focusses in particular on the fourth order PDE aspects of the proof.

Anti-maximum principles

Guido Sweers

For second order elliptic boundary value problems such as $-\Delta u = \lambda u + f$ in Ω and $u = 0$ on $\partial\Omega$ one finds that for any reasonable bounded domain in $\mathbb{R}^n$ a first eigenvalue λ_1 exists and moreover, for any $\lambda < \lambda_1$ a positive source term f implies that the solution u is positive. Clément and Peletier showed that for λ in a right neighborhood opposite behaviour occurs, a phenomenon which they named the *anti-maximum principle.* Roughly such behaviour is explained by the pole of the resolvent at λ_1 and the positive sign of the corresponding eigenfunction φ_1 :

$$u = \frac{1}{\lambda_1 - \lambda} \langle \varphi_1, f \rangle \varphi_1 + R_\lambda f$$

with $R_\lambda f$ the remaining regular part.

For higher order elliptic boundary value problems in general neither a maximum principle exists nor is positivity of the first eigenfunction guaranteed. The corresponding conjectures, named respectively after Boggio-Hadamard and Szegö, have many counterexamples by now. One of the cases where $f \geq 0$ does imply $u \geq 0$, and hence a positive eigenfunction exists, has been presented by Boggio: on balls in $\mathbb{R}^n$ any polyharmonic Dirichlet problem is positivity preserving. For those cases also an anti-maximum principle exists.

What is maybe more interesting is the fact that the steps leading from a positive eigenfunction (inside an appropriate cone) to a *uniform anti-maximum principle* also lead to a maximum principle in the left neighborhood of λ_1. Numerical evidence shows that this might hold for the clamped plate problem on stadium-like domains.

The anti-maximum principle for higher order elliptic boundary value problems is joint work with Ph. Clément. Sharp conditions for the uniform anti-maximum principle appear in a joint work with H.-Ch. Grunau.

Efficient and reliable numerical methods for higher-order problems

Christian Wieners

For fourth-order eigenvalue problems and for weakly nonlinear fourth-order elliptic problems methods exist for proving the existence of a continuous solution $u \in H^2(\Omega)$ near a very close numerical approximation $\tilde{u} \in H^2(\Omega)$. In particular, based on explicit computations of embedding constants, pointwise error bounds can be derived.

For this purpose very efficient numerical methods for computing approximations are required. We present numerical examples realized with the software system *UG* which supports parallel multigrid methods. In particular, we consider biharmonic eigenvalue problems

$$\Delta^2 u - \lambda u = 0, \qquad u \in H_0^2(\Omega),$$

and

$$\Delta^2 u + \lambda \Delta u = 0, \qquad u \in H_0^2(\Omega).$$

Furthermore we investigate the fourth-order analog of the Bratu-Gelfand problem

$$\Delta^2 u = \lambda \exp(u), \qquad u \in H_0^2(\Omega),$$

and a fourth-order streamline-vorticity formulation of the Navier-Stokes equation.

Our examples focus on special properties of solutions which cannot be studied by purely analytic considerations. In some cases we could already achieve rigorous verifications, i. e., it can be proved that the numerical observations reflect the actual behavior of the continuous solution.

Positivity of the Paneitz operator

Paul Yang

In this survey, I introduce the 4th order Paneitz operator in conformal geometry. I motivate the question for positivity criteria by reviewing its importance in dimension four. I recall the criteria developed by Gursky in this dimension, and discuss the extension of such criteria to higher dimensions.

Open problem session

- Problem suggested by Friedemann Brock:

Can the well-known symmetry result by Gidas-Ni-Nirenberg be extended to fourth order semilinear elliptic equations? In other words, if B denotes the unit ball of $\mathbb{R}^n$ $(n \geq 2)$ and f a C^1 function, can we deduce that any smooth solution of the problem

$$\begin{cases} \Delta^2 u = f(u) & \text{in } B \\ u > 0 & \text{in } B \\ u = \nabla u = 0 & \text{on } \partial B \end{cases}$$

is radially symmetric and radially decreasing about the origin?

- Problem suggested by Filippo Gazzola:

Let B denote the unit ball in $\mathbb{R}^n$ $(n > 2)$ and $\lambda > 0$: in fact, n may be seen as *any real number* since we are interested in radially symmetric solutions of the second order critical growth problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{4/(n-2)} u & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1}$$

The *nonresonant dimensions* are the "space dimensions" n for which the parameter λ tends to the first eigenvalue λ_1 of $-\Delta$ *from the right* (i.e. $\lambda \to \lambda_1^+$) whenever the L^∞ norm of the radial solutions with one node of (1) tends to blow-up. It is known (Gazzola-Grunau, Analysis 2000) that the nonresonant dimensions are $4 \leq n \leq 2 + 2\sqrt{2}$. Is it possible to show that if $4 \leq n \leq 2 + 2\sqrt{2}$ and $\lambda = \lambda_1$, then (1) admits no radial solutions with one node? Further, previous independent work by Gazzola and Grunau gives some evidence to the fact that nonresonant dimensions for the polyharmonic problem

$$\begin{cases} (-\Delta)^k u = \lambda u + |u|^{4k/(n-2k)} u & \text{in } B \\ D^\alpha u = 0 \quad (|\alpha| \leq k-1) & \text{on } \partial B \end{cases}$$

are $4k \leq n \leq (2 + 2\sqrt{2})k$. Can one show this fact rigorously, according to the previous definition?

- Problem suggested by Hans-Christoph Grunau:

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ be a smooth star-shaped domain and consider the semilinear fourth order critical growth elliptic equation

$$\begin{cases} \Delta^2 u = |u|^{8/(n-4)} u & \text{in } \Omega \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases}$$

Differently from the corresponding second order problem, it seems not possible to obtain non-existence results for *any* nontrivial solution of this problem just by using Pohozaev-type identities and unique continuation properties. Nevertheless, is it possible to show non-existence results?

- Problem suggested by Bernd Kawohl:

Is it possible to partially extend Talenti's comparison principle (Ann. Sc. Norm. Sup. Pisa 1976) to fourth order equations? In other words, let B denote the unit ball of $\mathbb{R}^n$ $(n \geq 5)$ and let $f \in L^{2n/(n+4)}(B)$ be a nonnegative function. Let u and v be the solutions of the problems

$$\begin{cases} \Delta^2 u = f & \text{in } B \\ u = \nabla u = 0 & \text{on } \partial B \end{cases} \qquad \begin{cases} \Delta^2 v = f^* & \text{in } B \\ v = \nabla v = 0 & \text{on } \partial B \end{cases}$$

where f^* denotes the Schwarz symmetrization of f. Then, is it true that $u^* \leq v$?

- Problem suggested by Richard Laugesen:

For the eigenvalue problem of the clamped plate under tension

$$(*) \qquad \begin{cases} \Delta^2 u - a\Delta u = \Gamma \; u & \text{in } \Omega \\ u = \nabla u = 0 & \text{on } \partial\Omega \end{cases}$$

the isoperimetric question is posed:

$$\Gamma(\Omega) \geq \Gamma(\Omega^*)?$$

Here, $-a\Delta u$ with the parameter $a \geq 0$ is the tension term, $\Omega \subset \mathbb{R}^n$ is a bounded domain, Ω^* is the ball having the same measure as Ω and $\Gamma(\Omega)$ denotes the smallest eigenvalue of $(*)$. Only when $a = 0$ –the classical clamped plate eigenvalue problem– and $n = 2, 3$, the question has already been answered, in fact in the affirmative (Nadirashvili, Ashbaugh-Benguria). See Ashbaugh-Laugesen for an approximate version in any space dimension.

- Problem suggested by Guido Sweers:

The solution operator for the biharmonic Dirichlet problem is positivity preserving on a ball. Lack of the positivity preserving property on general domains for the biharmonic operator Δ^2 suggests to try modifying this fourth order operator with one which "takes into account" the geometry of the domain. For instance, it is known that Δ^2 with Dirichlet boundary conditions is not positivity preserving in the square $(0, 1)^2$. One may consider the following question. Is the inverse of the linear fourth order operator $Lu = u_{xxxx} + u_{yyyy}$ in Ω with $u = |\nabla u| = 0$ on $\partial\Omega$ positivity preserving on the square $\Omega = (0, 1)^2$? And if not, maybe the solution operator for $L - \lambda$ is positivity preserving when $\lambda \in (\lambda_1 - \varepsilon, \lambda_1)$ and ε is sufficiently small. The hope is based on the fact that the first eigenfunction of this operator L is of one sign on the square.

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