Mathematisches Forschungsinstitut Oberwolfach

Report No. 40/2001

Complex Geometry: Interactions between Algebraic, Differential, and Symplectic Geometry

August 26th – September 1st, 2001

Die Tagung fand unter der Leitung von A. Beauville (Nice), F. Catanese (Göttingen), E. Looijenga (Utrecht) und Ch. Okonek (Zürich) statt.

Wie schon bei früheren Tagungen über komplexe Geometrie in Oberwolfach haben auch dieses Jahr viele bedeutende Mathematiker aus verschiedenen Ländern an der Tagung teilgenommen. So fiel es nicht schwer, ein interessantes Tagungsprogramm zusammenzustellen.

Viele Vorträge bezogen sich auf wichtige komplex-geometrische Themen wie z.B. Modulräume von Kurven und Vektorbündeln, Hilbertschemata, Gromov-Witten Invarianten und Quantencohomologie. Behandelt wurden auch moderne Entwicklungen und neueste Resultate in der komplexen Geometrie, etwa: projektive Kontaktmannigfaltigkeiten, komplex symplektische Mannigfaltigkeiten, Calabi-Yau Varietäten, Invarianz der Plurigeschlechter. Darüber hinaus wurden auch Anwendungen von Methoden aus der Eichtheorie und aus der symplektischen und fast komplexen Geometrie dargestellt.

Abstracts

Estimated transversality in symplectic geometry and projective maps

D. Auroux

If (X^{2n}, ω) is a compact symplectic manifold, then it carries a compatible almost-complex structure, but this almost-complex structure is usually not integrable. Still, Donaldson has shown that constructions from complex geometry can be performed using sections that are only approximately holomorphic: if $L \to X$ is a line bundle with $c_1(L) = \frac{1}{2\pi}[\omega]$, then $L^{\otimes k}$ has "many" approximately holomorphic sections, some of which present estimated transversality properties. In general, one has the following:

Theorem 1. *Let $\mathcal{S}_k$ be finite stratifications of the jet bundles $J^m(\mathbb{C}^{r+1} \otimes L^{\otimes k})$ by approximately holomorphic submanifolds (+ geometric bounds). Then for $k >> 0$ there exist approximately holomorphic sections s_k of $\mathbb{C}^{r+1} \otimes L^{\otimes k}$, such that the m-jets $j^m s_k$ are uniformly transverse to $\mathcal{S}_k$, i.e. $\exists \eta > 0$ independent of k such that $j^m s_k$ either avoids strata by a distance $> \eta$ or intersects them transversally with angle $> \eta$. Moreover, this construction is canonical up to isotopy for k large enough.*

This makes it possible to construct in particular projective maps, i.e. to find $s_k = (s_k^0, \dots, s_k^r)$ such that $f = [s_k^0, \dots, s_k^r] : X \setminus base \to \mathbb{CP}^r$ behaves like a generic complex map, locally. In particular for maps to $\mathbb{CP}^2$ the only generic local models near critical points are

$$(z_1, \dots, z_n) \longmapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$$

and

$$(z_1, \dots, z_n) \longmapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n).$$

In particular, the branch curve $D \subset \mathbb{CP}^2$ is a symplectic curve with only (complex) cusps, and nodes (complex or anticomplex), as singularities. So for a given symplectic manifold we get invariants consisting of a plane curve and a monodromy morphism (or rather a sequence of such data for $k >> 0$). Conversely, these data allow one to reconstruct (X, ω) up to symplectomorphism: these are complete invariants.

When X is a symplectic 4-manifold, the monodromy data is just a homomorphism $\theta : \pi_1(\mathbb{CP}^2 \setminus D) \to S_N$, where $N = deg(f)$; the interesting information is therefore encoded in the isotopy class of the curve D, up to creation or cancellation of pairs of nodes. The curve D can be investigated using a complete invariant: its braid monodromy (studied in detail in the algebraic case by Moishezon and Teicher). On the other hand, a less complete but more manageable invariant in the complex case is $\pi_1(\mathbb{CP}^2 \setminus D)$. In the symplectic case, node cancellations affect this group by quotienting $\pi_1(\mathbb{CP}^2 \setminus D)$ by certain commutators; the resulting quotient G_k is a symplectic invariant for large enough k (joint result with L. Katzarkov, M. Yotov and S. Donaldson). It is worth mentioning that, in all known examples, for large enough k, one has $G_k \cong \pi_1(\mathbb{CP}^2 \setminus D)$ (stabilization does not affect the group). Also there exists an exact sequence $1 \to G_k^0 \to G_k \to S_N \times \mathbb{Z}_d \to \mathbb{Z}_2 \to 1$, where $d = degD$ and the maps from G_k to S_N and $\mathbb{Z}_d$ are respectively the monodromy and the linking number.

In all known examples, G_k^0 is solvable, with $[G_k^0, G_k^0]$ of order at most 4; $Ab(G_k^0)$ has been thought to be a very powerful invariant, but it turns out that this is maybe not the case; in particular, it cannot distinguish the so-called Horikawa surfaces, and there is evidence suggesting that it may be a purely homological invariant.

Szpiro inequalities for hyperelliptic pencils

Bogomolov

(joint work with T. Pantev and L. Katzarkov)

Let X^g be a symplectic family of curves over S^2 with double singular points only. We assume that all the vanishing cycles are nonseparating and the monodromy is a subgroup of $Map_{hyp(g)}$ - the hyperelliptic mapping class group of genus g. This group coincides with a centralizer of a hyperelliptic involution in the mapping class group $Map(g)$.

Denote by N the total number of singular points in singular fibres and by D the number of singular fibres.

Theorem 1. *Under the above assumptions we have the inequality*

$$N \leq (4g+2)D.$$

For $g = 1$ we obtain the classical Szpiro inequality and for $g > 1$ we obtain Lockhardt's conjecture in the case of projective fibrations.

The proof uses the monodromy considerations only.

BPS states of curves in Calabi-Yau 3-folds

J. Bryan

The Gopakumar-Vafa conjecture gives a reformulation of Gromov-Witten theory in terms of (conjecturally) integer invariants obtained in physics by counting BPS states. We give an approach to the conjecture that involves studying the contribution of isolated curves to the Gromov-Witten invariants and the corresponding BPS states. We prove this "local version" of the Gopakumar-Vafa conjecture in a variety of cases.

In the last few minutes we describe roughly how physics predicts the BPS invarints are defined. They are the multiplicities of certain representations of sl_2 on the cohomology of a moduli space of D-branes. The precise nature of the D-brane moduli space and the sl_2 action is not understood.

Multiple fibres and classification theory

F. Campana

X = projective manifold $/\mathbb{C}$ $f : X \longrightarrow Y$ a fibration (i.e. onto connected, smooth, projective Y)

Define $\Delta(f) := \sum(1 - \frac{1}{m_i})\Delta_i$, where Δ_i runs over all irreducible divisors of Y, and for any Δ_i, $m_i := m(f, \Delta_i) := gcd(m_{ij})$, where j runs over all irreducible components D_{ij} of $f^{-1}(\Delta_i)$ mapped onto Δ_i by f, with $f^*(\Delta_i) = (\sum m_{ij}D_{ij}) + Rest$. The pair $Y, \Delta(f))$ is the "preorbifold" structure on Y defined by f.

Define $K_Y + \Delta(f)$ (a $\mathbb{Q}$-divisor) to be the canonical "bundle" of this structure (a fundamental group can be defined also).

Define $\kappa(Y, f) := inf\{\kappa(Y', K_{Y'} + \Delta(f'))\}$, where $f' : X' \to Y'$ runs over all fibrations which are equivalent to f, equivalent means: $\exists u : X' \to X$ and $v : Y' \to Y$, bimeromorphic, such that $fu = vf'$. (If $\kappa(Y) \geq 0$, this infimum is not needed).

Define

1. $f : X \to Y$ is a fibration of general type if $\kappa(Y, f) = \dim Y > 0$.
2. X is special if $\nexists f; X \to Y$ of general type. We note: $X \in \mathcal{S}$, then

3. $f : X \to Y$ is a special fibration if so is its generic fibre.

Examples of special manifolds: curves of genus 0,1; Rationally connected manifolds; manifolds X with $\kappa(X) = 0$. Also by a generalization to our context of Kobayashi-Ochai's extension theorem, an X is special if there exists a dominating meromorphic map $\phi : \mathbb{C}^n \to X$, and in particular if the universal covering of X is $\cong \mathbb{C}^n$. Special surfaces and threefolds are classified, too.

Conjecture 1. *X special $\implies \pi_1(X)$ is almost abelian.*

This is true up to $\dim X = 3$ (except 2 special cases), for rationally connected X, and also if $\pi_1(X)$ is either solvable torsion free or is linear (i.e. $\pi_1(X) \subset Gl(N, \mathbb{C})$).

Conjecture 2. *X special $\iff d_X \equiv 0$ (where $d_.$ is the Kobayashi pseudometric).*

This essentually should reduce to the case where $\kappa(X) = 0$ (where one should prove that $d_X \equiv 0$) and to the case where X is of general type (where one should prove Lang's conjecture). But the reduction itself should be highly non-trivial (assume $f : X \to Y$ is a fibration such that $d_F \equiv 0$ on the fibres; there are no multiple fibres, and $d_Y \equiv 0$. One has to show that $d_X \equiv 0$).

We next construct a core $c_X : X \to C(X)$ with general fibre special and the largest special subvarieties sitting inside X.

Conjecture 3. *The core c_X is a fibration of general type.*

This is true up to dimension 3.

By conjecture 2, d_X vanishes on the fibres of c_X, There exists thus a pseudometric $d^{\cdot}_{C(X)}$ on $C(X)$ such that $d_X = (c_X)^*(d^{\cdot}_{C(X)})$.

Conjecture 4. *: $d^{\cdot}_{C(X)}$ is a metric outside some proper algebraic subset $A \subset C(X)$*

This generalizes Lang's conjecture, which is the case when $\kappa(X) = \dim X$, so that $C(X) = X$.

A more precise version of Conjecture 4 can be given: $d^{\cdot}_{C(X)}$ should be the Kobayashi pseudometric of the preorbifold $(C(X), \Delta(c_X))$, naturally defined. Such a preorbifold of general type has a Kobayashi pseudometric which is a metric outside some A as above.

Finally let us say that if X is defined over a (finitely generated over $\mathbb{Q}$) field $K \subset \mathbb{C}$, then $c_X : X \to C(X)$ is defined over K. Mordell-Lang's Conjecture should then extend and say:

The K-rational points of X lie over finitely many of the K-rational points of $C(X)$.

Numerical characterization of the Kähler cone of a compact Kähler manifold

J.P. Demailly
(joint work with M. Paun)

The goal of this work is give a precise numerical description of the Kähler cone of a compact Kähler manifold. Our main result states that the Kähler cone depends only on the intersection form of the cohomology ring, the Hodge structure and the homology classes of analytic cycles: if X is a compact Kähler manifold, the Kähler cone $\mathcal{K}$ of X is one of the connected components of the set $\mathcal{P}$ of real $(1,1)$ cohomology classes $\{\alpha\}$ which are numerically positive on analytic cycles, i.e. $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$. This result is new even in the case of projective manifolds, where it can be seen as a generalization of the well-known Nakai-Moishezon criterion, and it also extends previous results by Campana-Peternell and Eyssidieux. The principal technical step is to show that every nef class $\{\alpha\}$ which has positive highest self-intersection number $\int_X \alpha^n > 0$ contains a Kähler current; this is done by using the Calabi-Yau theorem and a mass concentration technique for Monge-Ampère equations. The main result admits a number of variants and corollaries, including a description of the cone of numerically effective $(1,1)$ classes and their dual cone. As an application, D. Huybrechts recently obtained a description of the Kähler cone of a very general hyperkähler manifold; a slightly more precise result by S. Boucksom states that the Kähler cone consists precisely of the $(1,1)$-classes in the positive quadratic cone defined by the Beauville-Bogomolov quadratic form, which are positive on every rational curve (as there are no such curves for a very general hyperkähler manifold, the Kähler cone then just coincides with the quadratic cone). Another important consequence is the fact that for an arbitrary deformation $\mathcal{X} \to S$ of compact Kähler manifolds, the Kähler cone of a very general fiber X_t is "independent" of t, i.e. invariant by parallel transport under the $(1,1)$-component of the Gauss-Manin connection.

Published on arXiv as math.AG/0105176

Contact structures and quaternionic geometry

J. Geiges
(joint work with J. Gonzalo)

A <u>contact circle</u> on a (real!) 3-manifold $M$is apair of contact forms (α_1, α_2) such that any nontrivial combination $\lambda_1\alpha_1 + \lambda_2\alpha_2$ with constant coefficients is again a contact form. We call (α_1, α_2) <u>taut</u> if the volume form $(\lambda_1\alpha_1 + \lambda_2\alpha_2) \wedge (\lambda_1 d\alpha_1 + \lambda_2 d\alpha_2)$ is the same for all $(\lambda_1, \lambda_2) \in S^2 \subset \mathbb{R}^2$. Similarly one defines a (<u>taut</u>) <u>contact sphere</u>.

We have shown earlier that the closed 3-manifolds which admit a taut contact circle are exactly those of the form $M = \Gamma \setminus G$ with G equal to $SU(2)$, $\tilde{SL}_2$, or $\tilde{E}_2$ (the universal cover of the symetry groups of the three 2-dimensional space forms), and $\Gamma \subset G$ discrete and cocompact. Taut contact spheres exists exactly on left quotients $\Gamma \setminus SU(2)$.

In this talk I describe how taut contact circles and spheres relate to complex and hyperkähler geometry. This is then used to describe the moduli of these families of contact structures. For instance, up to diffeomorphism and an obvious conformal equivalence, the taut contact spheres on $S^3 \subset \mathbb{H}$ are given in quaternionic notation by

$$i\alpha_1 + j\alpha_2 + k\alpha_3 = \frac{1}{4}(dq \cdot \bar{q} - q \cdot d\bar{q}) - \frac{\nu}{2}d(qi\bar{q})$$

with $\nu \in \mathbb{R}_0^+$. The key to this classification is the result that a taut contact sphere on a <u>closed</u> manifold M gives rise to a <u>flat</u> hyperkähler metric on $M \times \mathbb{R}$ (this is false, in general, if M is not closed).

Families of rationally connected varieties

T. Graber
(joint work with J. Harris and J. Starr)

If X is a smooth, projective variety $/\mathbb{C}$, we say X is rationally connected if any 2 points in X can be joined by a rational curve. We prove the following result:

Theorem 1. *Let $\pi : X \longrightarrow B$ be a morphism of smooth projective varieties, where B is an irreducible curve. If a general fiber of π is rationally connected, then π has a section.*

Our proof is by studying the local structure of the induced morphism $\bar{M}_g(X, \beta) \to \bar{M}_g(\mathbb{P}^1, d)$ of moduli spaces of stable maps together with some elementary properties of spaces of branched covers.

2 Kähler manifolds and special langrangien fibrations

M. Grassi

In this talk we introduce s-Kähler manifolds, which are a generalization of Kähler manifolds in which s forms are involved. One way of defining them is as follows: You have s forms $\omega_1, ..., \omega_s$ of degree 2 on M, and a Riemanniann metric $\mathbf{g}$ so that all the forms ω_j are constant. Moreover, over any $p \in M$ you have an orthonormal coframe $dx_1, ..., dx_n, dy_1^1, ..., dy_n^s$ so that $(\omega_j)_p = \sum_{i=1}^n dx_i \wedge dy_i^j$.
The easy way to build examples is with flat tori, but the motivation for their introduction is their conjectural relation with mirror symmetry for Calabi-Yau manifolds. Namely, in the Strominger-Yau-Zaslow approach, the mirror partner of a CY 3-fold X is the moduli space of special lagrangian tori in X endowed with a flat connection. We show that the "universal" or total space of such a moduli space is a 9-dimensional *almost* 2-kähler manifold, away from the singular fibres. We conjecture that this space admists a compactification M which can be endowed with a globally defined 2-Kähler structure, extending that coming from X at the level of forms (but not necessarily at the metric level). Such a structure would induce a representation of the Lie algebra $\mathbf{sl}(3, \mathbb{R})$ on the real cohomology of M, and this representation in turn should shed light on the mirror phaenomenon, at least at the cohomological level. The existence of this representation guarantees also the existence of a (Hard) Lefschetz theorem, which can be also used to put cohomological restrictions which guarantee that some manifold cannot be s-Kähler.

Symplectic sums and relations in $H^*(\bar{\mathcal{M}}_{g,n}, \mathbb{Q})$

E. Ionel

We show that any degree g monomial in descendant or tautological classes vanishes on $\mathcal{M}_{g,n}$ (where $g > 0$). This generalizes a result of Looijenga and proves a version of Getzler's conjecture. The method we use is the sudy of relative Gromov-Witten invariants of $\mathbb{CP}^1$ relative to r points to give a correspondence between $\bar{\mathcal{M}}_{g,n}$ and $\bar{\mathcal{M}}_{0,r}$. Pulling back relations in $H^*(\bar{\mathcal{M}}_{0,r})$ together with the degeneration formula for Gromov-Witten invariants of symplectic sums proves the result stated at the beginning (The degeneration formula for GW invariants is joint work with T. Parker from Michigan State University).

Uniqueness of contact structures

S. Kebekus

A complex manifold X of odd dimension $2n + 1$ is said to be a "contact manifold" if there exists a sequence

$$0 \to F \to T_X \to L \to 0$$

where L is a line bundle on X, and where the skew-symmetric map

$$N : F \otimes F \to L$$

which is induced by the Lie-bracket, is everywhere non-degenerate. It is conjectured that a projective contact manifold which satisfies $b_2(X) = 1$, is homogeneous. In the present talk, we prove a weaker statement which was conjectured by LeBrun: If X is a projective manifold with $b_2(X) = 1$ then either $X \cong \mathbb{P}_{2n+1}$, or the contact structure is unique.

The proof is based on a result of Demailly, who has shown that X is necessarily Fano. Using a recent characterization of the projective space, it follows that either $X \cong \mathbb{P}_{2n+1}$, or that X is covered by rational curves ℓ which intersect the line bundle L with multiplicity one. A detailed study of the deformations of these curves reveals that at a general point $x \in X$, the subspace $F|_x \subset T_X|_x$ is spanned by the tangent spaces of the minimal rational curves which contain x. In particular, $F|_x$ is canonically given and therefore the subbundle F is unique.

Complete sets of relations for the cohomology of moduli spaces of bundles over a compact Riemann surface

F. Kirwan

The cohomology of the moduli space of stable bundles of rank 2 and odd degree d over a compact Riemann surface is extremely well understood from the work of many people; in particular generators and a complete set of relations ("Mumford relations") among these generators are known. For bundles of rank n and degree d, where n and d are coprime (so that the moduli space is a nonsingular projective variety) and $n > 2$, a set of generators for the cohomology ring has been known for two decades or more, but the obvious generalisation of the Mumford relations in rank 2 is not a complete set when $n > 2$. This talk describes joint work with R. Earl, which provides a modified generalization of the rank 2 Mumford relations, and sketches a proof that they form a complete set of relations for any $n > 2$.

Quantum cohomology without moduli spaces

A. Kresch

We discuss methods for computing genus zero Gromov-Witten invariants of manifolds, without using moduli spaces.

We outline A. Buch's simplified proof of the quantum Giambelli formula for the Grassmannian $G = G(k,n) = \{V \subset \mathbb{C}^n \,|\, \dim V = k\}$. Let $\{\sigma_\lambda\}$, indexed by partitions $\lambda \subset (n-k)^k$, denote the usual basis of Schubert classes for $H^*(G)$. Buch establishes

Theorem 1. *Given any two partitions $\lambda, \mu \subset (n-k)^k$, with the sum of the lengths of λ and μ less than or equal to k, the quantum product $\sigma_\lambda * \sigma_\mu$ in $QH^*(G)$ is equal to the classical product $\sigma_\lambda \cup \sigma_\mu$.*

The quantum Giambelli formula, which was originally provd by Aaron Bertram and which states that Giambelli's determinant in special Schubert classes, evaluated in $QH^*(G)$, is equal to σ_λ is an immediate consequence of this theorem. The proof uses the notion of span of a map $\varphi : \mathbb{P}^1 \longrightarrow G$, $t \longmapsto V_t$, defined as follows:

$$Span(\varphi) = Span\{V_t | t \in \mathbb{P}^1\}.$$

We have

Proposition 2. *The span of any degree d map $\varphi : \mathbb{P}^1 \longrightarrow G$ has dimension $\leq k+d$.*

A dimension-counting argument, involving span of maps, establishes the theorem.

On the cohomology ring of Hilbert schemes of surfaces with $K = 0$

M. Lehn

Let X be a smooth projective surface $/\mathbb{C}$. By a classical result of Fogarty, for all $n \geq 0$, the Hilbert scheme $X^{[n]} = \{\zeta \subset X \text{ subscheme } | dim\zeta = 0,\ l(\zeta) = n\}$ is a smooth variety. Its Betti numbers were computed by Göttsche, and Nakajima proved that

$$\bigoplus_{n=0}^{\infty} H^*(X^{[n]})t^{-n} \cong S^*\left(H^*(X) \otimes t^{-1}\mathbb{C}[t^{-1}]\right)$$

as representations over the Heisenberg Lie algebra

$$\mathfrak{h} = H^*(X) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c.$$

We report on joint work with Chr. Sorger to determine the ring structure of $H^*(X^{[n]})$ in terms of the ring structure of $H^*(X)$.

Let $A = \bigoplus_{i=-d}^{d} A_i$ be a graded Frobenius algebra, d an even natural number. A ring structure is defined on $A\{S_n\} := \bigoplus_{\pi \in S_n} A^{\otimes \pi \backslash \{1,\dots,n\}}\pi$ in such a way, that the subring of invariants $A^{[n]} := A\{S_n\}^{S_n}$ with respect to the conjugation action of S_n is again a Frobenius algebra. We apply this construction to the case $A = H^*(X)[2]$.

Theorem 1. *If X is a smooth projective surface with $K_X \sim 0$,*

$$(H^*(X)[2])^{[n]} \cong H^*(X^{[n]})[2n].$$

The proof is based on calculations in Nakajima's vertex algebra structure on $\mathbb{H} = \bigoplus_{n=0}^{\infty} H^*(X^{[n]})$ and geometric results relating this structure with multiplication operators $m(\alpha) : \mathbb{H} \to \mathbb{H}$, where $m(\alpha)|_{H^*(X^{[n]})} = \alpha^{[n]} \cup -$, and $\alpha^{[n]} = p_*\left(ch\mathcal{O}_\Theta \otimes q^*(\alpha \cdot td\ X)\right)$, $\Theta \subset X^{[n]} \times X$ denoting the universal family.

A degeneration of the moduli of stable morphisms

J. Li

We study the following problem: Let $\pi : W \to C$ be a projective family of schemes. We assume that for $t \neq 0 \in C$ W_t are smooth and that the special fiber W_0 consists of two smooth components intersecting transversally along a smooth divisor D. Here we assume the total space W to be smooth. Let (g, n) be a pair of integers and β the degree of the stable morphisms to be studied, also fixed. Then it is well understood that the GW-invariants of W_t, $t \neq 0$, are constructed based on

1.) The moduli of stable morphisms $\mathfrak{M}_{g,n}(W_t, \beta)$;
2.) The virtual moduli cycle $[\mathfrak{M}_{g,n}(W_t, \beta)]^{virt}$ and
3.) $GW(W_t)(\cdot) = \int_{[\mathfrak{M}_{g,n}(W_t,\beta)]^{virt}} \cdot$.

The goal of this project is to define the GW-invariants of W_0, define the relative GW-invariants of the two components Y_1 and Y_2 of W_0, and then establish a degeneration formula of the form

$$\begin{array}{rrcl} & GW(W_t) & = & GW(W_0) \\ (*) & & = & \text{an expression in } \{\ rel - GW(Y_1, D),\ rel - GW(Y_2, D)\ \} \end{array}$$

The main ingredient of this project is to introduce a new set of stable morphisms to $W[n]_0$ and introduce

$$\mathfrak{M}_{g,n}(\mathcal{W}, \beta) = \mathfrak{M}_{g,n}(W \setminus W_0, \beta) \coprod \coprod_{n \geq 0} \mathfrak{M}_{g,n}(W[n]_0, \beta)^{st}/(\mathbb{C}^*)^n$$

We then prove that

Theorem 1. *$\mathfrak{M}_{g,n}(\mathcal{W}, \beta)$ is naturally a separated proper Deligne-Mumford stack with perfect obstruction theory.*

The part of the project, that was not mentioned in the lecture, but completed, includes the following steps: construct the moduli space of relative stable morphisms; construct their perfect obstruction theories and define two relative GW invariants; prove a degeneration formula of the form $(*)$.

A generalization of the Grauert Mumford criterion for semismall maps

L. Migliorini
(joint work with M. de Cataldo)

Let X be a nonsingular complex variety. A map $f : X \to Y$ is said to be semismall if for every subvariety $S \subset X$ $2dim(S) - dim\ f(S) \leq dimX$. The main source of semismall map is given by contraction of holomorphic symplectic varieties. Suppose X projective. A line bundle $\mathcal{L}$ is said to be LEF if some multiple is globally generated and the map it defines is semismall. LEF line bundles behave in many ways just as ample line bundles. For instance, the Kodaira Akizuki Nakano theorem holds for LEF bundles; a related statement is that the Hard Lefschetz theorem holds for a LEF bundle: $c_1(\mathcal{L}) : H^{n-k}(X) \longrightarrow H^{n+k}(X)$ is an isomorphism for all $k = 0, \ldots, n$.

This result has some interesting consequences for the topology of semismall maps. The first corollary is the following: let Y_i be a stratification of Y such that $f^{-1}(Y_i) \to Y_i$ is a topologically locally trivial fibration. Suppose $2dimf^{-1}(Y_i) - dimY_i = dimX$. Let $y \in Y_i$ and D be a local transversal slice to Y_i. The intersection form defined by the irreducible components of $f^{-1}(y)$ (computed in $f^{-1}(D)$) is definite ($(-1)^{\frac{dimD}{2}}$-definite). This is a generalization of the Grauert Mumford criterion for contractibility of curves on a surface. A corollary of this statement is the following statement: $Rf_*\mathbb{Q}_X[n] = \bigoplus \mathcal{JH}_{\bar{Y}_i}(L_i)$, which is a special case of the decomposition theorem of BeilinsonBernsteinDeligne. Here $\mathcal{JH}_{\bar{Y}_i}(L_i)$ is the intersection cohomology complex of the local system L_i on Y_i defined by the monodromy of the irreducible components of the fibres of $f|_{f^{-1}(Y_i)}$. The statement about nondegeneracy of intersection forms allows also the definition of a projector $P \in A_n(X \times X)_{\mathbb{Q}}$ such that (X, P) is the motive of intersection cohomology of Y. This is consistent with the recent mostly conjectural theory of motivic decomposition given by Corti-Hanamura.

Homology of moduli spaces of sheaves on a $K3$ surface

H. Nakajima

Let (X, H) be a polarized $K3$ surface $/\mathbb{C}$. Let $(H^*(X,\mathbb{Z}), \langle , \rangle)$ be the Mukai lattice, i.e. $\langle x, y\rangle = \langle x_1 \cup y_1 - x_0 \cup y_2 - x_2 \cup y_0, [X]\rangle$ where x_i, y_i are the $H^{2i}(X,\mathbb{Z})$-components of x, y. For a coherent sheaf E on X, let us define the Mukai vector $v(E)$ by $ch(E)\sqrt{tdX} \in H^*(X,\mathbb{Z})$. Let $M_H(v)$ be the moduli space of H-stable sheaves E on X with $v(E) = v$. If v is primitive and H is generic, it is known that $M_H(v)$ is a nonsingular projective variety with a natural symplectic structure (Mukai). Harvey-Moore proposed the following

Problem 1. *Study the structure of $\bigoplus_v H^*(M_H(v))$. Relate it to the representation theory of an ∞-dimensional Lie algebra, a kind of a generalized Kac-Moody Lie algebra (Borcherds).*

The structure should come fom the following subvariety in the triple product:

$$Clos\{(E_1, E_2, E_3) | 0 \to E_1 \to E_2 \to E_3 \to 0)\} \subset M_H(v_1) \times M_H(v_2) \times M_H(v_3).$$

We are intersted in a special case when $M_H(v_1)$ or $M_H(v_2)$ is a point, in other words, the corresponding sheaf is a <u>rigid</u> sheaf. When $E_2 = i_*\mathcal{O}_C(d)$ where C is a (-2)-curve, I have proved that the above operator gives a representation of $\hat{sl_2}$, an affine lie algebra of sl_2.

By a philosophy coming from the duality in the string theory, we should expect the whole construction has a symmetry under $O(H^*(X,\mathbb{Z}))$. It implies, that we could have a similar construction for any rigid sheaf.

Based on a work of Yoshioka and Markman, we can construct a representation of sl_2 (finite dimensional!) by the above correspondence with a rigid sheaf E_1. Her we must put a certain technical conditon on H and v_2, v_3.

Examples of irreducible symplectic varieties

K. O'Grady

We describe two new examples of irreducible symplectic varieties (of dimension 6 and 10). These manifols are not not deformation equivalent to the other known irreducible symplectic manifolds (Hilbert schemes of points on a $K3$ surface and generalized Kummer varieties). First one constructs a symplectic compactification of $\mathcal{M}^{st}$, the moduli space of rank-2 stable torsion-free sheaves with $c_1 = 0$ and $c_2 = 2k$ on a projective surface S with $K_s \sim 0$, where $k = 2$ if S is a $K3$, and $k = 1$ if S is an abelian surface. The moduli space $\mathcal{M}^{st}$ is an open subset of the singular moduli space $\mathcal{M}$ of semistable sheaves (with the same rank and Chern classes): one blows up $\mathcal{M}$ twice and then one contracts an extremal ray to get the symplectic desingularization $\tilde{\mathcal{M}}$. If S is a $K3$ then $\tilde{\mathcal{M}}$ is a new symplectic variety of dimension 10: it is "new" because $b_2(\tilde{\mathcal{M}}) \geq 24$, and all known examples have $b_2 \leq 23$. If S is an abelian surface, let $\tilde{\mathcal{M}}_0$ be the fibre over $(0, \hat{0})$ of the fibration $\tilde{\mathcal{M}} \to S \times \hat{S}$ which maps the point $[F]$ to $(Alb(c_2(F)), \det F)$ (strictly speaking we first map $\tilde{\mathcal{M}}$ to $\mathcal{M}$, and then apply this map). Then $\dim \tilde{\mathcal{M}}_0 = 6$, $b_2(\tilde{\mathcal{M}}_0) = 8$, and $\tilde{\mathcal{M}}_0$ is irreducible symplectic; since $b_2 = 7$ or $b_2 = 23$ for all other 6-dimensional examples, $\tilde{\mathcal{M}}_0$ is a "new" irreducible symplectic manifold. The topological computations needed to prove that $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_0$ are irreducible symplectic and that $b_2(\tilde{\mathcal{M}}_0) = 8$ ($b_2(\tilde{\mathcal{M}}) \geq 24$ is easy) are done by applying Goreski-MacPherson's LHT to a high power of the determinant line bundle over $\tilde{\mathcal{M}}$ (or $\tilde{\mathcal{M}}_0$).

The Rozansky-Witten topological quantum field theory

J. Sawon
(joint work with J. Roberts and S. Willerton)

About five years ago Rozansky and Witten wrote a paper in which they described a topological sigma model based on a path integral over the space of all maps from a 3-manifold to a hyperkähler manifold X. The topological output of this theory is twofold:

1. Fixing the hyperkähler manifold, the partition function of the theory is a topological invariant of the 3-manifold. For example, if X is the Atiyah-Hitchin manifold (a certain monopole moduli space), the 3-fold invariant is the Casson invariant.
2. A Feynman diagram expansion of the partition function gives coefficients which depend on the hyperkähler manifold (and on a Feynman diagram). These are a kind of a generalized "Chern numbers", and are constant under deformations of the hyperkähler structure.

In my talk I investigated the associated 3-dimensional TQFT.

In fact, instead of the usual TQFT, we look at the extended TFT. This involves extending the category of 3-bordisms to a 2-category (objects are 1-manifolds, morphisms are 2-bordisms, 2-functors are 3-bordisms with corners). Then an ETFT is a 2-functor from this 2-category. In other words, it takes 1-manifolds to categories, 2-bordisms to functors, and 3-bordisms to natural transformations.

In our construction the images of 1-manifolds are derived categories $\mathcal{D}(pt)$, $\mathcal{D}(X)$, $\mathcal{D}(X \times X)$, etc. and the functors are integral transforms constructed from the kernels $\mathcal{O}_{diagonal} \subset X \times \cdots \times X$. Only at the 3-bordism level do we need X to be hyperkähler (instead of just a complex manifold). Then the natural transformations are constructed using a generalization of the coeffizients from 2) above. This gives us a completely combinatorial description of the (extended) TQFT.

Quasi-projectivity of moduli spaces of polarized projective varieties

G. Schumacher

Theorem 1 (H. Tsuji-G. Schumacher)**.** *Any coarse moduli space of polarized, (smoth) projective varieties is quasi-projective.*

Corollary 2. *Moduli spaces of polarized non-uniruled varieties are quasi-projective.*

Known was the quasi-projectivity for moduli spaces of polarized varieties such that an m-canonical bundle is globally generated (E. Viehweg 89/90), including the case of canonically polarized varieties.

We construct a line bundle on the compactified moduli space with a positive hermitian metric where Lelong numbers vanish such that the curvature is strictly positive in the interior, as a current. The method is based upon the curvature formula for the Quillen metrics on determinant line bundles (Bismut-Gillet-Soulé), Griffith's theory about period mappings, and moduli of framed manifolds. In a second step, an embedding theorem is proved.

Deformational invariance of plurigenera

Y.T. Siu

Let X be a holomorphic family of compact complex projective algebraic manifolds with fibers X_t over the open unit 1-disk Δ. Let K_{X_t} and K_X be respectively the canonical line bundles of X_t and X. We prove that, if L is a holomorphic line bundle over X with a (possibly singular) metric $e^{-\varphi}$ of semipositive curvature current on X such that $e^{-\varphi}|_{X_0}$ is locally integrable on X_0, then for any positive integer m, any $s \in \Gamma(m\,K_{X_0}+L)$ with $|s|^2e^{-\varphi}$ locally bounded on X_0 can be extended to an element of $\Gamma(X, m\,K_X + L)$. In particular, $dim\Gamma(X_t, m\,K_{X_t} + L)$ is independent of t for φ smooth. The case of trivial L gives the deformational invariance of the plurigenera. The method of proof uses an appropriately formulated effective version, with estimates, of the argument in the speaker's earlier paper on the invariance of plurigenera for general type. A delicate point of the estimates involves the use of metrics as singular as possible for $p\,K_{X_0} + a_p\,L$ on X_0 to make the the dimension of the space of L^2 holomorphic sections over X_0 bounded independently of p, where a_p is the smallest integer $\geq \frac{p-1}{m}$. These metrices are constructed from s. More conventional metrices, independent of s, such as generalized Bergman kernels, are not singular enough for the estimates.

Gauge theoretical equivariant Gromov-Witten invariants

A. Teleman

(joint work with Ch. Okonek)

Let (F, ω, J) be an almost Kähler manifold, $\alpha : \hat{K} \times F \longrightarrow F$ an action of $\hat{K}$ which lets J invariant and K a closed subgroup of $\hat{K}$, which lets ω invariant ($\hat{K}$ is a compact Lie group).

We introduce, using gauge theoretical methods, Gromov-Witten type invariants for such triples $(F, \alpha, \hat{K})$. The adiabatic limit conjecture states that these invariants can be related to the (twisted) Gromov-Witen invariants of the corresponding symplectic quotient of F with respect to K.

We state a "universal" Kobayashi-Hitchin correspondence (generalizing previous results by Mundet i Riera) which gives a complex geometric interpretation of the moduli spaces associated with any triple (F, ω, J) with F Kähler. Using this Kobayashi-Hitchin correspondence, we describe explicitely the moduli spaces associated with triples of the form

1. $(Hom(\mathbb{C}^r, \mathbb{C}^{r_0}), \alpha_{can}, U(r))$, where α_{can} is the natural action of $U(r) \times U(r_0)$.
2. $(\mathbb{C}^r, \alpha_{can}, K_w)$, where α_{can} is the natural action of $[S^1]^r$ on $\mathbb{C}^r$ and $K_w = ker(w)$, where w is an epimorphism $w : [S^1]^r \longrightarrow [S^1]^m$.

An explicit formula for the invariants for the first case (when $r = 1$) and applications are discussed.

Vanishing comjectures for the moduli space of curves

R. Vakil

(joint work with T. Graber)

There are many "vanishing" conjectures and theorems on the moduli space $\bar{\mathcal{M}}_{g,n}$ of (genus g, n-pointed) curves, which roughly state, that certain cohomology classes vanish on some large open subset. I will motivate a new conjecture, Conjecture $*$ (dimension i classes vanish away from the locus of curves with at least $2g-2+n-i$ rational components; codimension j classes vanish away from the locus of curves with at least $j-g+1$ rational components), from which the other vanishing conjectures and theorems, and more, easily follows.

In some sense, one should think, that the fundamental geometrical fact is Conjecture $*$, and that the other vanishing results are combinatorial consequences. This conjecture also suggests that the coarse stratification of $\bar{\mathcal{M}}_{g,n}$ by number of genus 0 components is useful.

In the last few minutes, I will sketch a proof of at least part of of the conjecture, giving most of the desired consequences. The key idea is to consider "Hurwitz classes", and use both degeneration and virtual localization arguments on J. Li's (algebraic) moduli space of relative stable maps (see p. 8)

Symplectic Hilbert schemes and a conjecture of Ruan

C. Voisin

If X is a complex surface, it is known by Fogarty that the punctual Hilbert scheme $Hilb^k(X)$ is smooth. By the Hilbert-Chow morphism $c : Hilb^k(X) \to X^{(k)}$, it is a desingularization of the symmetric product. We first explain the construction of a similar desingularization $c : Hilb^k(X) \to X^{(k)}$, when X is now an almost complex fourfold. The map c is continuous, is a diffeomorphism over $X_0^{(k)}$, the open set parametrizing k-uples of distinct points, and its fibers are isomorphic to the fibers of the Hilbert-Chow morphism in the complex case. We show also that if (X, J) is compact symplectic , that is J is an almost complex structure ompatible with ω, $Hilb^k(X)$ is also symplectic. Following Göttsche-Siergel, we explain the computation of the cohomoloy of $Hilb^k(X)$ in the complex case. We also show it is still valid in our symplectic situation. It follows from this that $H^*(Hilb^k(X))$ is canonically isomorphic to the orbifold cohomology $H^*_{orb}(X^{(k)})$ defined by Ruan-Chen. Chen and Ruan have a construction of a ring structure on orbifold cohomology. Ruan conjectures that under the additive isomorphism above, the product on $H^*(Hilb^k(X))$ is deduced from the product on orbifold cohomology, modified by quantum correction associated to the Gromov-Witten invariants $\Phi_{A,0,n}$, where $A \in ker(H_2(Hilb^k(X), \mathbb{Z}) \stackrel{c_*}{\to} H_2(X^{(k)}, \mathbb{Z})) \cong \mathbb{Z}$.

Small contractions of complex symplectic 4-folds

J. Wisniewski

(joint work with J. Wierzba)

Let $\varphi : X \to Y$ be a local contraction in the sense of M & P: Assume that φ is birational and contracts the exceptional locus locus E to an isolated point $y \in Y$. We asssume moreover that X admits a symplectic (holomorphic) form (hence φ is a crepant contraction).

This talk reports on an approach to prove

Theorem 1 (Wierzba-)**.** *In the above situation, if* $\dim X = 4$, *then* φ *is analytically equivalent to contracting the zero section in the cotangent bundle of the projective plane.*

Edited by Markus Dürr

Participants

Prof. Dr. Michele Audin
maudin@math.u-strasbg.fr
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Dr. Denis Auroux
auroux@math.polytechnique.fr
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Prof. Dr. Paolo de Bartolomeis
deba@dma.unifi.it
Dipart. di Matematica Applicata
Universita degli Studi di Firenze
Via S. Marta, 3
I-50139 Firenze

Prof. Dr. Arnaud Beauville
beauville@math.unice.fr
Laboratoire J.A. Dieudonne
Universite de Nice
Parc Valrose
F-06108 Nice Cedex 2

Prof. Dr. Fedor A. Bogomolov
Bogomolo@cims.nyu.edu
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
USA

Prof. Dr. Jim Bryan
jbryan@math.ubc.ca
Dept. of Mathematics
University of British Columbia
1984 Math. Road
Vancouver, BC V6T 1Z2
CANADA

Prof. Dr. Frederic Campana
campana@iecn.u-nancy.fr
Dept. de Mathematiques
Universite de Nancy I
B.P. 239
F-54506 Vandoeuvre-les-Nancy Cedex

Prof. Dr. Fabrizio Catanese
catanese@uni-math.gwdg.de
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Jean-Pierre Demailly
demailly@fourier.ujf-grenoble.fr
Institut Fourier
UMR 5582 du CNRS
BP 74
100 rue de Maths
F-38402 Saint-Martin d'Heres

Dr. Markus Dürr
mduerr@math.unizh.ch
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Hubert Flenner
Hubert.Flenner@ruhr-uni-bochum.de
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Hansjörg Geiges
geiges@math.leidenuniv.nl
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
NL-2300 RA Leiden

Dr. Lothar Göttsche
gottsche@ictp.trieste.it
International Centre for
Theoretical Physics, International
Atomic Energy Agency, UNESCO
P. O. B. 586 Miramare
I-34100 Trieste

Prof. Dr. Tom Graber
graber@math.harvard.edu
Dept. of Mathematics
Harvard University
1 Oxford Street
Cambridge, MA 02138
USA

Dr. Michele Grassi
grassi@dm.unipi.it
Dipartimento di Matematica
Universita di Pisa
Via Buonarroti, 2
I-56127 Pisa

Prof. Dr. Peter Heinzner
peter.heinzner@cplx.ruhr-uni-bochum.de
Fachbereich Mathematik
Ruhr-Universität Bochum
44780 Bochum

Dr. Rafael Herrera
herrera@math.ucr.edu
Dept. of Mathematics
University of California
Riverside, CA 92521-0135
USA

Dr. Claus Hertling
hertling@mpim-bonn.mpg.de
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Eleny Ionel
ionel@math.wisc.edu
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison WI, 53706-1388
USA

Prof. Dr. Ludmil Katzarkov
lkatzarkov@math.uci.edu
Dept. of Mathematics
University of California at Irvine
Irvine, CA 92697-3875
USA

Dr. Stefan Kebekus
stefan.kebekus@uni-bayreuth.de
Mathematisches Institut
Universität Bayreuth
95440 Bayreuth

Prof. Dr. Frances C. Kirwan
kirwan@maths.ox.ac.uk
Mathematical Institute
Oxford University
24 - 29, St. Giles
GB-Oxford OX1 3LB

Dr. Andrew Kresch
kresch@math.upenn.edu
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
USA

Dr. Manfred Lehn
manfred.lehn@mi.uni-koeln.de
Mathematisches Institut
Universität zu Köln
50923 Köln

Prof. Dr. Jun Li
jli@math.stanford.edu
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Prof. Dr. Eduard J.N. Looijenga
looijeng@math.uu.nl
Faculteit Wiskunde
Rijksuniversiteit Utrecht
Postbus 80.010
NL-3508 TA Utrecht

Dr. Martin Lübke
lubke@math.leidenuniv.nl
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
NL-2300 RA Leiden

Prof. Dr. Luca Migliorini
luca@alpha.science.unitn.it
Dipartimento di Matematica
Universita di Trento
Via Sommarive 14
I-38050 Povo (Trento)

Prof. Dr. Hiraku Nakajima
nakajima@kusm.kyoto-u.ac.jp
Dept. of Mathematics
Kyoto University
Sakyo
Kyoto 606-8502
JAPAN

Prof. Dr. Kieran O'Grady
ogrady@mat.uniroma1.it
Dipt. di Matematica
G. Castelnuovo
Universita di Roma "La Sapienza"
P.le A. Moro 5
I-00185 Roma

Prof. Dr. Christian Okonek
okonek@math.unizh.ch
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Dr. Roberto Paoletti
rp@dimat.unipv.it
Dipartimento di Matematica
Universita di Pavia
Via Abbiategrasso 215
I-27100 Pavia

Prof. Dr. Victor Pidstrigach
pidstrig@uni-math.gwdg.de
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen

Dr. Justin Sawon
sawon@maths.ox.ac.uk
Mathematical Institute
Oxford University
24 - 29, St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Georg Schumacher
schumac@mathematik.uni-marburg.de
Fachbereich Mathematik
Universität Marburg
35032 Marburg

Dr. Bernd Siebert
Bernd.Siebert@ruhr-uni-bochum.de
Fachbereich Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Yum-Tong Siu
siu@math.harvard.edu
Dept. of Mathematics
Harvard University
1 Oxford Street
Cambridge, MA 02138
USA

Prof. Dr. Mina Teicher
teicher@sunglight.cs.biu.ac.il
Dept. of Mathematics
Bar-Ilan University
52 900 Ramat-Gan
ISRAEL

Prof. Dr. Andrei Teleman
teleman@cmi.univ-mrs.fr
Centre de Mathematiques et
d'Informatique
Universite de Provence
39, Rue Joliot Curie
F-13453 Marseille Cedex 13

Prof. Dr. Ravi Vakil
vakil@math.stanford.edu
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Prof. Dr. Claire Voisin
voisin@math.jussieu.fr
U.F.R. de Mathematiques
Case 7012
Universite de Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Jaroslaw Wisniewski
jarekw@mimuw.edu.pl
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND