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Theory of the Riemann Zeta and Allied Functions

September 16 – September 22, 2001

The present conference was organized by

Martin N. Huxley, Cardiff
Matti Jutila, Turku
Yoichi Motohashi, Tokyo

Over forty mathematicians accepted the invitation of the Institute, but the great tragedy in the U. S. on September 11 unfortunately prevented some of them from coming. The total number of participants was 33 representing 17 countries.

The main topics considered in the 29 lectures given in the conference and in the problem session were

- approaches to the Riemann Hypothesis and other conjectures on zeta- and L-functions
- applications of the spectral theory of automorphic functions and the representation theory of Lie groups
- L-functions connected with algebraic number fields
- related arithmetical problems.

The organizers and participants are grateful to the Land Baden-Württemberg, and to the director Prof. Kreck as well as to the staff of the Institute, for providing us with such a nice opportunity to hold this special workshop.

Abstracts

A Summation Formula on $\mathrm{SL}_2(\mathbb{Z})$ and the Fourth Power Moment of the Riemann Zeta–Function

JOHAN ANDERSSON, STOCKHOLM

We discuss a new summation formula

$$\sum_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} f(\gamma) \;=\; \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varrho_j(m)\,\varrho_j(n)\,F(n,m,k_j)$$
$$+ \text{ terms coming from Eisenstein series and holomorphic cusp forms}$$

for the full modular group, and how it relates to Motohashi's formula for the fourth power moment. The main idea is to expand a sum over the big cell as

$$\sum_{\substack{c\in\mathbb{Z}\\ c\neq 0}}\sum_{m,n\in\mathbb{Z}}\sum_{h\overline{h}\equiv 1 (\mathrm{mod}\ c)} f\begin{pmatrix} h+mc & * \\ c & \overline{h}+nc\end{pmatrix} = \sum_{\substack{c\in\mathbb{Z}\\ c\neq 0}}\sum_{m,n\in\mathbb{Z}}\sum_{h\overline{h}\equiv 1 \bmod c} f_c\big(m+\frac{h}{c}, n+\frac{\overline{h}}{c}\,\big)$$

and then using the double Poisson summation on f_c. Kloosterman sums appear and the Kuznetsov sum formula can be applied. The same trick can be used to get a simple proof for the fourth power moment.

Some Determinants Connected with RH

MICHEL BALAZARD

(joint work with Luis Báez-Duarte, Bernard Landreau and Eric Saias)

Let H be the Hilbert space $L_2(0,+\infty;t^{-2}dt)$, let $e_\alpha(t)=\{t/\alpha\}$ and $\chi=\chi_{(1,\infty)}$. Then e_α and χ are in H, and we are interested in the distance $d_n=\ \mathrm{distance}_H\big(\chi,\ \mathrm{Vect}(e_1,\dots,e_n)\big)$. It is well-known that RH follows from $d_n=o(1)$, but the converse is unknown. This is related to the Nyman–Beurling equivalent form for RH. Numerical experiments support the conjecture:

$$d_n^2 \sim \frac{2+\gamma-\log 4\pi}{\log n} \qquad (n\to\infty)$$

Burnol proved recently that

$$d_n^2 \geq \frac{2+\gamma-\log 4\pi + o(1)}{\log n}\,.$$

Let us try to compute d_n by the Gram formula

$$d_n^2 = \frac{\mathrm{Gram}\,(e_1,\dots,e_n,\chi)}{\mathrm{Gram}\,(e_1,\dots,e_n)}\,.$$

One is led to study the multiplicative self-correlation of the fractional part function

$$A(\lambda) = \int_0^\infty \{t\}\{\lambda t\}\,t^{-2}\,dt\,.$$

The local behavior of A near each rational can be accurately described. In particular

$$A(p/q+t)-A(p/q) \sim \frac{1}{2p}\,|t|\log|t| \qquad (t\to 0),$$

and A has a strict local maximum at each rational point. One can deduce from that, that putting $f_k = (k+1)e_{k+1} - ke_k$ is a first step in the orthogonalization of the e_k, and one gets some insight into the structure of the Gram matrix of the f_k.

As a first result about the asymptotic behavior of $G_n := \mathrm{Gram}\,(e_1, \ldots, e_n)$, we obtain the following

Proposition. There exists an absolute positive constant c such that

$$\exp\left(c\sqrt{n}\,(\log n)^{1/4}\log\log n\right) \le n!^{\,2}G_n \le \exp\left(n\log\log n + O(n/\log n)\right).$$

We conjecture that

$$n!^{\,2}\,G_n = \exp\left(n^{1+o(1)}\right).$$

On the Class Number 1 Problem for Special Real Quadratic Number Fields

András Biro, Budapest

For an odd positive integer p where p^2+4 is squarefree, we consider the quadratic field $K = \mathbb{Q}(\sqrt{p^2+4}\,)$. H. Yokoi conjectured that $h(K) > 1$ if $p > 17$. This is a real analogue of the famous problem of determining explicitly the imaginary quadratic fields with class number 1, since for our K the fundamental unit is small. So Siegel's Theorem implies the finiteness of such fields K with $h(K) = 1$, but this result is ineffective.

We give an effective upper bound for p if $h(K) = 1$, hence we prove Yokoi's conjecture. The proof applies ideas of a paper of J. Beck together with new ingredients. It is very likely that the same method applies to the similar conjecture of Chowla.

A Variational Approach to Weil's Explicit Formula

Enrico Bombieri, Princeton

We owe to André Weil the formulation of the Riemann Hypothesis (RH) as a statement about the positivity of a certain quadratic or hermitian functional, representing a far reaching extension of Riemann's celebrated formula for the number of primes up to a given bound. It is natural to study this problem as that of minimizing this functional in the unit sphere of a suitable Hilbert space, naturally associated to the problem. In this lecture it is shown that a minimum is attained, and we obtain some properties of the associated extremal. The associated kernel is highly singular and its regularization leads to the study of high order iterated kernels. It is shown that the pointwise positivity of iterates of sufficiently large order implies RH, together with some heuristic arguments suggesting that this may be the case. The lecture concludes with a reduction, based on heuristic arguments, of the problem to the study of a certain random walk on the real line, although no definite conclusion has been reached yet about the possible implications of these considerations for an attack to RH.

Primes Representable as Sums of k-th Powers

Jörg Brüdern, Stuttgart

(joint work with K. Kawada and T. Wooley)

Let $P(k)$ denote the smallest s such that infinitely many primes are the sum of s k-th powers of natural numbers. We prove

Theorem 1. *Assume GRH. Then* $P(k) \le \frac{8}{3}\, k$.

Unconditionally one only knows that $P(k) \le (1/2+o(1))\, k \log k$ although it is very likely $P(k) \le 3$ for all k. Similar results can be obtained unconditionally when the primes are replaced by other related sequences. For example, when $s > 2k$, there are infinitely many numbers with at most two prime factors in the numbers of the type $x_1^k + \ldots + x_s^k$. We also have

Theorem 2. *If* $s > (1/2 + \log 2)\, k$ *, then there are infinitely many sums of two squares that are the sum of* s k*-th powers.*

Automorphic Forms and the Zeta–Function

Roelof W. Bruggeman, Utrecht

The explicit formula of Y. Motohashi (see §4.7, Spectral Theory of the Riemann zeta-function, Cambridge Univ. Press, 1997) gives an explicit expression for

$$\int_{-\infty}^{\infty} \left|\zeta(\tfrac{1}{2} + it)\right|^4 g(t)\, dt$$

where g is a suitable test function in terms of automorphic forms for $\Gamma = \mathrm{SL}_2(\mathbb{R})$: holomorphic cusp forms, Maass cusp forms, and Eisenstein series. In the proof of the explicit formula, this relation between the Riemann zeta-function and modular forms arises from the use of the sum formula of Kuznetsov (Mat. Sb. 111, 1980). In the lecture various points of view of this fomula were discussed. The original approach of Kuznetsov is based on the spectral theory $L^2(\Gamma \backslash \mathcal{H})$, where $\mathcal{H}$ denotes the upper half plane. But Petersson's formula for Fourier coefficients of holomorphic Poincaré series implies that a more natural formulation of the sum formula should involve holomorphic cusp forms as well. This leads to the spectral decomposition in other even weights. A limit procedure gives the full sum formula. But this formulation is also a consequence of a representational approach, indicated by J. Cogdell and I. Pyatetskii-Shapiro. The last point of view should lead to a more direct relation between the zeta-function and automorphic forms, bypassing the sum formula of Kuznetsov.

Modular Forms, Fractal Sets and Differentiability Properties

Fernando Chamizo, Madrid

Given an elliptic curve with Hasse-Weil L-function $\sum a_n n^{-s}$, we consider the Fourier series $A_\alpha(x) = \sum a_n n^{-\alpha} \cos 2\pi n x$ and $B_\alpha(x) = \sum a_n n^{-\alpha} \sin 2\pi n x$. We prove that

(a) For $3/2 < \alpha < 2$ the functions A_α and B_α are differentiable at $x = x_0$, if and only if x_0 is rational.

(b) For $1 < \alpha < 2$ the graphs of A_α and B_α are fractal sets with Minkowski dimensions $3 - \alpha$.

In fact we state a more general theorem which applies to fractional integrals of modular forms.

A Conjecture for the $2k$-th Moment of the Riemann Zeta–Function

Brian Conrey

(joint work with D. Farmer, J. Keating, M. Rubinstein and N. Snaith)

Let

$$I_k(g) \;=\; \int_0^\infty g(t)\,\left|\zeta(\tfrac12+it)\right|^{2k}\,dt$$

where g is a reasonable test function. We have in mind $g(t)=\chi_{[0,T]}(t)$ or $g(t)=e^{-t/T}$. It may be conjectured that

$$I_k(g) \;\sim\; \int_0^\infty g(t)\,P_{k^2}\big(\log\tfrac{t}{2\pi}\big)\,dt$$

where P_{k^2} is a polynomial of degree k^2. Such formulas are known for $k=1$ and $k=2$ (Hardy and Littlewood, Ingham, Atkinson, Heath–Brown, Motohashi). The leading term of P_{k^2}, if it exists, seems to have the form

$$g_k\,a_k\,\frac{\big(\log\frac{t}{2\pi}\big)^{k^2}}{(k^2)!}$$

with

$$a_k \;=\; \prod_p \big(1-\tfrac1p\big)^{(k-1)^2}\sum_{j=0}^{k-1}\binom{k-1}{j}^2 p^{-j}$$

and where $g_1=1$ and $g_2=2$. It was conjectured by Conrey and Ghosh and by Conrey and Gonek that $g_4=24024$. These conjectures were based on Dirichlet polynomial considerations. Recently, Keating and Snaith used Random Matrix Theory to suggest that

$$g_k \;=\; (k^2)!\prod_{j=0}^{k-1}\frac{j!}{(j+k)!}\,.$$

It was unclear how a_k and g_k would mix in lower order terms. Combining number theoretic and random matrix theoretic techniques we are led to the following

Conjecture.

$$P_{k^2}(x) \;=\frac{(-1)^k}{(k!)^2}\frac{1}{(2\pi i)^{2k}}\int\!\cdots\!\int_{|z_i|=\varepsilon}\frac{G(z_1,\ldots,z_{2k})\,\Delta(z_1,\ldots,z_k,-z_{k+1},\ldots,-z^{2k})^2\,e^{\frac{x}{2}\sum z_i}}{z_1^{2k}z_2^{2k}\cdots z_{2k}^{2k}}\,dz_1\cdots dz_{2n}$$

where

$$G(z) \;=\; \prod_{i=1}^k\prod_{j=1}^k\zeta(1+z_i+z_j)\cdot\prod_p\prod_{i=1}^k\prod_{j=1}^k\big(1-p^{-1-z_i-z_j}\big)$$
$$\cdot\int_0^1\prod_{j=1}^k\Big(1-\frac{e(\theta)}{p^{1/2+z_j}}\Big)^{-1}\Big(1-\frac{e(-\theta)}{p^{1/2-z_{k+j}}}\Big)^{-1}d\theta\,,$$

and $\Delta(w_1,\ldots,w_r)=\prod_{1\le i<j\le r}(w_i-w_j)$ is the Vandermode.

We give numerical evidence for the conjecture.

Mean Value of Automorphic L–Functions and the Selberg Kernel Function

Shigeki Egami, Toyama

Let $A(z)$ be a holomorphic cusp form of weight k ($k \geq 12$ an even integer) for the full modular group, and let $H_A(s)$ be the automorphic Hecke L-function associated to A via Mellin transformation. In 1994 Y. Motohashi obtained the spectral resolution of the weighted mean square of $H_A(1/2+it)$, i.e.

$$\int_{-\infty}^{\infty} |H_A(\tfrac{1}{2}+it)|^2 \, g(t) \, dt \, ,$$

which is an analogue of his famous ζ^4–formula. In this talk I explain my attempt to give an alternative proof by using the Selberg kernel function and the Parseval identity for the Mellin transformation (see my paper in: *Number Theory and its Applications* by S. Kanemitsu and K. Györy, 101–110). Our method of the proof is to calculate

$$\int_0^{\infty} y^{s-1} \int_{\Gamma\backslash\mathcal{H}} K_\Gamma(iy, w) \, |A(w)|^2 \, I_m^{k-2} \, w \, dw \, dy$$

in two ways: first unfolding, second spectral resolution. In this talk I also described my recent attempt to generalize to the Hilbert modular case. After the talk R. Bruggeman suggested that there are infinitely many such identities if we apply the Maass operator repeatedly to $A(z)$.

Low–lying Zeros of Class Group L-Functions

Etienne Fouvry, Orsay

(joint work with H. Iwaniec)

We assume the Generalized Riemann Hypothesis. Let D be a squarefree integer, $D > 3$, $D \equiv 3 \pmod 4$, $K = \mathbb{Q}(\sqrt{-D}\,)$, ψ a character on the class group of ideals of O_K, and let $L(s,\psi)$ be the attached L-function.

We investigate the distribution of the zeros γ_ψ of $L(s,\psi)$ near the point $1/2$ and prove the following theorems.

Theorem 1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ even, smooth, such that $\operatorname{supp} \widehat{\phi} \subset (-1,1)$. Then for $D \to \infty$ we have*

$$\frac{1}{h(-D)} \sum_{\psi \in \widehat{\mathcal{C}\ell(K)}} \sum_{\gamma_\psi} \phi\left(\frac{\gamma_\psi}{2\pi} \log D\right) = \int_{-\infty}^{\infty} \phi(x) \, W(\mathrm{Sp})(x) \, dx \; + \; O\left(\frac{\log\log D}{\log D}\right)$$

where $W(\mathrm{Sp})(x)$ is the symplectic measure $W(\mathrm{Sp})(x) = 1 - \sin 2\pi x/(2\pi x)$.

The question is to prove Theorem 1 for functions ϕ with larger compact support of $\widehat{\phi}$ (Density Conjecture). This is achieved, on average only, for $\operatorname{supp} \widehat{\phi} \subset (-4/3, 4/3)$. More precisely, we prove

Theorem 2. *Suppose $\phi : \mathbb{R} \to \mathbb{R}$ even, smooth, such that $\operatorname{supp} \widehat{\phi} \subset (-\theta,\theta)$ with $0 < \theta < 4/3$. Let $\Delta \geq 3$ and $\mathcal{D}$ be any set of squarefree numbers $D \equiv 3 \pmod 4$, with $\Delta < D \leq 2\Delta$, of cardinality $|\mathcal{D}| \geq \Delta^{\theta - 1/3}$. Then for $D \to \infty$, we have*

$$\frac{1}{|\mathcal{D}|} \sum_{D \in \mathcal{D}} \left| \frac{1}{h(-D)} \sum_{\psi \in \widehat{\mathcal{C}\ell(K)}} \sum_{\gamma_\psi} \phi\left(\frac{\gamma_\psi}{2\pi} \log D\right) - \int_{-\infty}^{\infty} \phi(x) \, W(\mathrm{Sp})(x) \, dx \right| = O_{\theta,\phi}\left(\frac{\log\log \Delta}{\log \Delta}\right).$$

The proof of Theorem 2 requires a study (on average over D) of primes p of the form $m^2 + Dn^2 = 4p$, so called Euler primes, satisfying $p < \Delta^\theta$.

The Subconvexity Problem for Artin L–Functions

John B. Friedlander
(joint work with W. Duke and H. Iwaniec)

Let $k \geq 0$ be an integer, χ a primitive character modulo D satisfying $\chi(-1) = (-1)^k$, $u_j(z)$ a Hecke–Maass cusp form of weight k for the congruence group $\Gamma_0(D)$ with nebentypus χ, and Laplace eigenvalue $1/4 + t_j^2$. The attached L-function satisfies a functional equation which leads to the convexity bound $L_j(s) \ll |D|^{1/4+\varepsilon}$ on the critical line $\Re s = 1/2$, where the implied constant depends on k, t_j and s. We succeeded to prove a subconvexity bound in the D-aspect

$$L_j(s) \ll \left(|t_j| + |\varrho|\right)^{10} D^{1/4-\theta}$$

where now the implied constant depends only on k (which could also have been done in the previous bound) and $\theta = 1/23042$. In particular we deduce the bound

$$L(s, \varrho) \ll |s|^{10} D^{1/4-\theta}$$

for $\Re s = 1/2$ for those degree 2 Artin L–functions over $\mathbb{Q}$ which we know to be entire (i.e. all those not of icosahedral type) where D is the modulus of the primitive determinantal character. These in turn include the L–functions associated to the class–group of the real or imaginary quadratic field $\mathbb{Q}(\sqrt{d}\,)$ where $D = |d|$.

From the latter we obtain as corollary the existence of ideals having small norm in every coset of quotients of sufficiently small index of the class group. In the special case of the genus group only Dirichlet characters are required and A. Baker and A. Schinzel had derived such a result (quantitatively stronger) using the Burgess bound. Another corollary gives the existence of a generator of small norm in every cyclic subgroup of the class group.

Technical Improvements in the Bombieri–Iwaniec Method for Exponential Sums

Martin N. Huxley, Cardiff

The method uses the short interval structure of the sums

$$\sum_m e\left(f(m)\right) \tag{1}$$

$$\sum_h \sum_m e\left(hf'(m)\right) \tag{2}$$

$$\sum_h \sum_m e\left(f(m+h) - f(m-h)\right) \tag{3}$$

which depends on rational approximation to $f''(x)$. Means of short interval sums using the large sieve require estimates of the number of coincident pairs in each of two sets of four–dimensional vectors, the two spacing problems.

Watt has seen that the first spacing problem in (3) can be regarded as a perturbation of the corresponding problems for both (1) and (2) in different ranges. This work will lead to better bounds.

The second spacing problem is the same for all three sums given in (1), (2) and (3). A coincident pair corresponds to an integer-preserving affine map taking one region of the graph of $y = f'(x)$ to another. Affine maps with the same matrix part correspond rather precisely to integer points close to a "resonance curve" in a two-dimensional space dual to the space of $(x, f(x))$. The inclusion map in our space acts functorially as an affine map of resonance curves in the dual space. Swinnerton-Dyer's approach to integer points close to curves leads to slightly better estimates for the sums given above.

Table of exponents in the classical problems*

		Old	New	Limit of method	Target
(1)	Size of $\zeta(1/2+it)$	89/570	32/205	3/20	0
(2)	Divisor problem	23/73	131/416	5/16	1/4
(2)	Circle problem	46/73	131/208	5/8	1/2

* (up to ε)

On Some Conjectures and Results for the Riemann Zeta-Function and Hecke Series

Aleksandar Ivić, Belgrade

The lecture covered three related topics:

1. Mean values of $|\zeta(\frac{1}{2}+it)|$.
2. The Mellin transform zeta-function
$$Z_k(s) := \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k}\, x^{-s}\, dx\,,$$
where $k \in \mathbb{N}$, $\Re s > c(k) > 1$.
3. Some conjectures on $Z_k(s)$ and the Hecke series $H_j(s)$.

The asymptotic formula

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \;=\; T\,P_{k^2}(\log T) + E_k(T)\,, \tag{1}$$

$k \in \mathbb{N}$, was extensively discussed with the accent on known results and conjectures for $E_k(T)$. The function $Z_k(s)$, which is a natural tool for investigating the integral in (1), was studied. Recent results by M. Jutila, Y. Motohashi and the author were presented as well as some conjectures of the author on $Z_k(s)$ and $H_j(s)$. For example if the conjectured bound, the analogue of the Lindelöf Hypothesis for $Z_2(s)$,

$$Z_2(\sigma+it) \;\ll_\varepsilon\; |t|^\varepsilon$$

holds for all $\varepsilon > 0$ and $\sigma > 1/2$ fixed, then

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^8\, dt \;\ll_\varepsilon\; T^{1+\varepsilon}, \qquad E_2(T) \;\ll_\varepsilon\; T^{1/2+\varepsilon},$$

which is (up to ε) best possible.

The Class Number Problem and Spacing of Zeros of Hecke L-Functions

HENRYK IWANIEC, NEW BRUNSWICK

(joint work with Brian Conrey)

Let $K=\mathbb{Q}(\sqrt{-q}\,)$ be the imaginary quadratic field of discriminant $-q$ with $q>4$. Let $\psi\in\widehat{\mathcal{C}\ell}(K)$ be a class group character and

$$L(s,\psi)\;=\;\sum_{\mathfrak{a}}\psi(\mathfrak{a})\,(N\mathfrak{a})^{-s}\,,$$

where $\mathfrak{a}$ runs over the non–zero integral ideals, the corresponding L–function. We proved that if the gap between consecutive zeros of $L(s,\psi)$ on the critical line is somewhat smaller than the average for sufficiently many pairs, note that no Riemann Hypothesis is required, then the class number of K satisfies $h\gg\sqrt{q}\,(\log q)^{-A}$, where A is an absolute constant and the implied constant is effectively computable. In particular for the trivial character $\psi=1$ we have $L(s,\psi)=\zeta_K(s)=\zeta(s)\,L(s,\chi_q)$, and restricting our choice to the zeros of $\zeta(s)$ we obtain the following theorem.

THEOREM. *Let $\varrho=1/2+i\gamma$ be the zeros of $\zeta(s)$ on the critical line and $\varrho'=1/2+i\gamma'$ be the nearest zero to ϱ on the critical line, $\varrho'=\varrho$ if ϱ is a multiple zero. Suppose*

$$\#\Big\{\varrho:\;0<\gamma\le T,\;|\gamma-\gamma'|\le\frac{\pi}{\log\gamma}\Big(1-\frac{1}{\sqrt{\log\gamma}}\Big)\Big\}\;\gg\;T\big(\log T\big)^{4/5}$$

for any $T\ge 2001$. Then we have

$$L(1,\chi_q)\;\gg\;\big(\log q\big)^{-90}$$

where the implied constant is effectively computable.

REMARKS. The condition of the theorem requires gaps between consecutive zeros of $\zeta(s)$ to be only slightly smaller than the half of the average gaps. This condition follows from the Pair Correlation Conjecture.

Spectral Averages and Estimates for *L*–Functions Attached to Maass Wave Forms

MATTI JUTILA, TURKU

(joint work with Y. Motohashi)

Let

$$H_j(s)\;=\;\sum_{n=1}^{\infty}t_j(n)n^{-s}\qquad(\sigma>1)$$

be the L–function attached to the Maass wave form corresponding to the eigenvalue $\lambda_j=1/4+\kappa_j^2$ of the hyperbolic Laplacian. The functional equation of $H_j(s)$ and the convexity principle imply the bound

$$H_j(1/2+it)\ll(t+\kappa_j)^{1/2+\varepsilon}\qquad\text{for }\;t\ge1\,.$$

The estimates

$$H_j(1/2+it)\ll t^{1/3+\varepsilon}\qquad\text{for fixed }\kappa_j\;\text{ and }\;t\ge1$$

by T. Meurman 1987 and

$$H_j(1/2)\ll\kappa_j^{1/3+\varepsilon}$$

by A. Ivić 1999, suggest the following

Conjecture.

$$H_j(1/2+it) \ll (|t|+\kappa_j)^{1/3+\varepsilon}\,.$$

In recent joint work with Y. Motohashi, this was verified for $|t| \ll \kappa_j^{2/3-\varepsilon}$ as an immediate corollary of the following

Theorem. *Let $\alpha_j = |\varrho_j(1)|^2/\cosh(\pi\kappa_j)$ with $\varrho_j(n)$ the Fourier coefficients of the j-th Maass wave form. Then for $|t| \ll K^{2/3-\varepsilon}$ we have uniformly*

$$\sum_{|\kappa_j - K| \ll K^{1/3}} \alpha_j \left|H_j(1/2+it)\right|^4 \ll K^{4/3+\varepsilon}\,.$$

The main ingredients of the proof are:

- An approximate functional equation for $H_j^2(1/2+it)$
- Kuznetsov's trace formula
- Voronoï's sum formula with additive characters
- Motohashi's identity for the additive divisor problem
- The saddle point method
- The spectral large sieve

On the Structure of the Selberg Class

Jerzy Kaczorowski, Poznan
(joint work with A. Perelli)

We prove the following theorem.

Theorem. *Let $S_d^\#$ denote the set of all L-functions of the extended Selberg class of degree d. Then $S_d^\# = \emptyset$ if $1 < d < 5/3$.*

The proof depends on the theory of the hypergeometric Fox functions. To deal with bilinear forms of the form

$$\sum_m \sum_n a(m)\overline{a(n)}\, e(f(m,n,t))$$

where $a(n)$ are the coefficients of $F \in S_d^\#$, we apply the saddle point method and estimates of sums

$$\sum_{K \le n \le K+H} |a(n)|^2$$

over short intervals $\bigl(H \ge K^{2-1/(d-1)}\bigr)$. To this end we develop an analytic theory of the Rankin-Selberg convolution.

Quantum Ergodicity for Arithmetic 3-Manifolds

Shin-ya Koyama, Yokohama

Three topics were presented in the generalization of the quantum chaos theory developed by Sarnak, Luo and other people.

(1) The quantum ergodicity of Eisenstein series is valid for Bianchii manifolds. Namely,

$$\lim_{t\to\infty}\frac{\mu_t(A)}{\mu_t(B)}=\frac{\mathrm{vol}(A)}{\mathrm{vol}(B)}$$

for any compact Jordan measurable subsets A, B of the Bianchii manifold, where $\mu_t=|E(v,1+it)|^2dV$ with $E(v,s)$ being the Eisenstein series, and dV is the volume element.

(2) An improvement of the prime geodesic theorem is possible, if we assume the mean Lindelöf Hypothesis in the λ-aspect for automorphic L-functions. Precisely,

$$\pi_\Gamma(x)=\mathrm{li}(x^2)+O(x^{11/7+\varepsilon})$$

with Γ being the Picard group.

(3) A new estimate of the first eigenvalue will be obtained by using the recent results of Kim–Sarnak and the theorem of Kim.

A Limit Theorem for the Riemann Zeta–Function in the Space of Continuous Functions

Antanas Laurinčikas

We consider the value distribution of the Riemann zeta-function in the sense of the weak convergence of probability measures. Denote by $\mathcal{B}(\mathcal{S})$ the class of Borel sets of the space $\mathcal{S}$. Let γ be the unit circle on the complex plane and $\Omega=\prod_p\gamma_p$, where $\gamma=\gamma_p$ for each prime p. On $\big(\Omega,\mathcal{B}(\Omega)\big)$ there exists the probability Haar measure m_H, and we have a probability space $\big(\Omega,\mathcal{B}(\Omega),m_H\big)$. Let $\omega(p)$ be the projection of $\omega\in\Omega$ to γ_p, and $\omega(m)=\prod_{p^\alpha\|m}\omega^\alpha(p)$. Moreover, let $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$ with the topology of uniform convergence on compact sets. Let $d_a(m)$ be the coefficients of the Dirichlet series for $\zeta^a(s)$ in the half plane $\sigma>1$ $(s=\sigma+it)$, and let $\sigma_T=1/2+\theta\,\log\log^{3/2}T/\log T$, $\theta>\sqrt{2}/2$ fixed, $\kappa_T=(1/2\log\log T)^{-1/2}$. Then

$$\sum_{m\le T}\frac{d_{\kappa_T}\,\omega(m)}{m^{\sigma_T+it}}$$

converges uniformly in t on compact subsets of $\mathbb{R}$ for almost all $\omega\in\Omega$ to some function $\beta(t,\omega)$ as T tends to infinity. Therefore, $\beta(t,\omega)$ is a $C(\mathbb{R})$-valued random element defined on $\big(\Omega,\mathcal{B}(\Omega),m_H\big)$. Denote by P_β the distribution of $\beta(t,\omega)$, and let meas A denote the Lebesgue measure of the set A.

Theorem. *Under RH the probability measure*

$$\frac{1}{T}\,\mathrm{meas}\left\{\tau\in[0,T]:\ \zeta^{\kappa_T}(\sigma_T+it+i\tau)\in A\right\}$$

where $A\in\mathcal{B}\big(C(\mathbb{R})\big)$ *converges weakly to* P_β *for* $T\to\infty$.

The latter theorem continues the work of Bohr, Jessen, Wintner, Selberg, Joyner, Matsumoto, Montgomery, the author and others.

The Additive Divisor Problem

Tom Meurman, Turku

Let $d(n)$ denote the divisor function, and let

$$D(N;f) = \sum_{n\le N} d(n)\,d(n+f)\,.$$

We announce a new estimate for the mean square of the error term in the asymptotic formula of $D(N;f)$.

The Subconvexity Problem for Rankin–Selberg L–Functions

Philippe Michel, Paris

We consider the subconvexity problem for the rank 4 Rankin–Selberg L-functions $L(s,f\otimes g)$, where g is a fixed holomorphic or Maass cusp form, and f is a cusp form with level $q\to\infty$ and nebentypus $\chi_f(q)$. Two years ago, Kowalski, the speaker and Vanderkam proved

$$L(s,f\otimes g) \ll_\varepsilon q^{1/2\,-\,1/80\,+\,\varepsilon} \qquad \text{for } \Re s = 1/2$$

for holomorphic cusp forms f of weight larger than 1 and such that the conductor q^* of χ_f satisfies $q^* < q^\beta$ for some $\beta < 1/2$.

In this talk we explain how to remove these two assumptions on f and how to prove the bound

$$L(s,f\otimes g) \ll q^{1/2\,-\,1/300}$$

under the only hypothesis that $\chi_f^* \neq \chi_g^*$ where χ^* is the underlying primitive character of χ. Note that this hypothesis can be removed as well with more computations and that our method applies also when g is an Eisenstein series.

The key point is to estimate non-trivially shifted convolution sums

$$\sum_{h\neq 0} G_{\chi_f}(h;c) \sum_{\ell m - n = h} \lambda_g(m)\lambda_g(n)\,W(m,n)$$

when $m,n,c\approx q$, and in particular to incorporate the oscillations of the Gauss sums $G_{\chi_f}(h;c)$ as h varies. We treat the shifted sums, where $\ell m - n = h$, using a method of Sarnak based on spectral methods which allows us to reduce the problem to the subconvexity problem for twisted L-functions $L(s,\phi_j\otimes\chi)$ in the q aspect, where ϕ_j ranges over a basis of Maass forms for $\Gamma_0(\ell)$, a problem which was solved by Duke, Friedlander and Iwaniec more than a year ago.

We describe applications of our bound to the equidistribution problem of Heegner points on Shimura curves.

Upper bounds near $s=1$ for an Axiomatic Class of L–Functions

Giuseppe Molteni

Estimates of the form $L^{(j)}(s,A) \ll_{\varepsilon,j} R_A^\varepsilon$ in the range $|s-1| \ll 1/\log R_A$ for general L-functions, where R_A is a parameter related to the functional equation of $L(s,A)$, can be quite easily obtained if the Ramanujan Hypothesis is assumed. We prove the same estimates when the L-functions have an Euler product of polynomial type and the Ramanujan Hypothesis is replaced by a much weaker assumption about the growth of certain elementary symmetrical functions. As a consequence, we obtain an upper bound of this type for every $L(s,\pi)$, where π is an automorphic cusp form on $\mathrm{GL}(n,\mathbb{A}_K)$. We employ these results to obtain Siegel-type lower bounds for twists by Dirichlet characters of the third symmetric power of a Maass form.

Beyond Pair Correlation

Hugh L. Montgomery, Ann Arbor
(joint work with S. Gonek and U. Vorhauer; K. Soundararajan)

Let

$$F(\alpha,T) \;=\; \Big(\frac{T}{2\pi}\log T\Big)^{-1} \sum_{\substack{0<\gamma\le T\\ 0<\gamma'\le T}} T^{i\alpha(\gamma-\gamma')}\, w(\gamma-\gamma')$$

where $w(u) = 4/(4+u^2)$. If $r(t) = \int_{\mathbb{R}} \widehat{r}(\alpha)e(t\alpha)\,d\alpha$ then

$$\sum_{\substack{0<\gamma\le T\\ 0<\gamma'\le T}} r\Big((\gamma-\gamma')\frac{\log T}{2\pi}\Big)\, w(\gamma-\gamma') \;=\; \frac{T}{2\pi}\log T \int_{\mathbb{R}} F(\alpha,T)\,\widehat{r}(\alpha)\,d\alpha\,.$$

Assuming RH, the asymptotic size of F was determined in 1972 for $-1\le\alpha\le 1$. Thus the left hand side above can be estimated when $\operatorname{supp}\widehat{r}\subseteq[-1,1]$. In joint work with S. Gonek and U. Vorhauer, wider classes of kernels are allowed in which the sign of $\widehat{r}(\alpha)$ is specified when $|\alpha|\ge 1$. Since $F(\alpha,T)\ge 0$ for all α, this yields useful inequalities.

In joint work with K. Soundararajan, the prime k-tuple Conjecture is used to generate a heuristic argument that suggests that

$$\frac{1}{X}\int_0^X \big(\psi(x+h)-\psi(x)-h\big)^k\,dx \;=\; \big(c_k+o(1)\big)\big(h\log X/h\big)^{k/2}$$

when $X^{\varepsilon}\le h\le X^{1-\varepsilon}$. Here the case $k=2$ is equivalent assuming RH to the Pair Correlation Conjecture. The numbers c_k are the moments of the normalized normal distribution, so we are led to expect that $\psi(x+h)-\psi(x)$ is approximately normally distributed with mean h and variance $\sim h\log X/h$.

Trying to embed $\zeta(s)$ into $L^2(\Gamma\backslash G)$

Yoichi Motohashi, Tokyo

It is discussed how to *directly* prove the spectral decomposition of the fourth moment of $\zeta(s)$. Here *directly* means that it is wished to dispense with Kloostermania. To achieve this aim, it is suggested to use two main tools, which are well-known in the theory of the representation of Lie groups: A) the Kirillov model and B) the Bessel function of a representation.

Here we investigate the case $L^2(\Gamma\backslash G)$ with $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$ and $G=\mathrm{PSL}_2(\mathbb{R})$. The use of A), especially its unitaricity, is given, e.g. in the book by J. W. Cogdell, I. I. Pyatetskii-Shapiro: *The Arithmetic and Spectral Analysis of Poincaré Series*, Perspectives in Mathematics, 13, Academic Press, San Diego, California, 1990. We prove, however, the most essential point in their work with an alternative and elementary argument pertaining to an orthogonality among the relevant family of Whittaker functions.

B) is a concept due to Gel'fand and Formin. But we use it in the formulation due to Vilenkin-Klimyk (1991), which we prove, alternatively, as the Mellin inversion of the local functional equation attached to an irreducible representation, in the sense of Jacquet and Langlands. The functional equation itself is proved in an elementary and quick way.

An extension to the complex case e.g. the fourth moment of the Dedekind zeta-function of the Gaussian number field is also given. Here $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ and $G = \mathrm{PSL}_2(\mathbb{C})$. The tools A) and B) are extended to this setting of $L^2(\Gamma\backslash G)$ too. This part is a joint work with R. W. Bruggeman.

Further, a suggestion was made about the unified treatment of mean values of zeta, L- and Hecke L-functions. This is related to a joint project with M. Jutila.

Distribution Property of Residual Orders

Leo Murata, Tokyo
(joint work with K. Chinen)

Let a be a positive integer which is not a perfect k-th power with $k \geq 2$, and $Q_a(x;4,\ell)$ be the set of primes $p \leq x$ such that the residual order of $a(\mathrm{mod}\ p)$ in $\mathbb{Z}/p\mathbb{Z}^\times$ is congruent to ℓ modulo 4. When $\ell = 0, 2$, it is known that calculations of $\#\, Q_a(x;4,\ell)$ are simple, and we can get these natural densities unconditionally. On the contrary, when $\ell = 1, 3$, the distribution properties of $Q_a(x;4,\ell)$ are rather complicated. Here, under the assumption of the Generalized Riemann Hypothesis (GRH) we determine completely the natural densities of $\#\, Q_a(x;4,\ell)$ for $\ell = 0,1,2,3$. For example we proved the following result.

Theorem. *Let a_1 be the squarefree part of a. If $a_1 \equiv 1$ or $3 \pmod 4$ then*

$$\text{the natural density of } \#\, Q_a(x;4,\ell) \;=\; \begin{cases} 1/3 & \text{if } \ell = 0 \text{ or } 2 \text{ (unconditionally)} \\ 1/6 & \text{if } \ell = 1 \text{ or } 3 \text{ (under GRH)} \end{cases}$$

If $a_1 = 2$ then

$$\text{the natural density of } \begin{cases} \#\, Q_a(x;4,0) = 5/12 \\ \#\, Q_a(x;4,1) = 7/48 - C/8 \\ \#\, Q_a(x;4,2) = 7/24 \\ \#\, Q_a(x;4,3) = 7/48 + C/8 \end{cases}$$

where

$$C = \prod_{\substack{p \equiv 3 \pmod 4 \\ p \text{ prime}}} \left(1 - \frac{2p}{(p^2+1)(p-1)}\right) \approx 0.64365\,.$$

Experiments in Analytic Number Theory

Samuel J. Patterson, Göttingen

One problem that often arises is the following: Let a_n be an arithmetic function for which one knows or suspects that an asymptotic law of the type $\sum_{n \leq X} a_n \sim cX^\gamma$ holds. We shall suppose that γ is known but one wishes to determine c. It is often necessary to do this as accurately as possible for reasons of numerology. One case of particular interest is $a_n = S(f(x), n)$ where $f \in \mathbb{Z}[x]$ is a polynomial, for simplicity of degree 3. Here

$$S(f(x), n) \;=\; \sum_{j \pmod n} \exp\left(2\pi i \,\frac{f(j)}{n}\right).$$

In this case one suspects, and in some cases one can prove, that $\gamma = 4/3$. By comparison with classical examples, and also from statistical considerations it appears that in general one can find no decent estimate for c substantially better than $X^{-\gamma} \sum_{n \le X} a_n$ for the largest value of X available. It is, in general, unrealistic to expect more than a rough estimate for c by this method.

On an Asymptotic Formula of Srinivas Ramanujan

Ayyadurai Sankaranarayanan, Mumbai
(joint work with K. Ramachandra)

Let $d(n)$ denote the Dirichlet divisor function and let

$$E(x) \;=\; \sum_{n \le x} d^2(n) \;-\; x P_3(\log x)$$

where P_3 is a polynomial of degree three. We discussed upper bounds and Ω-results for $E(x)$ unconditionally and proposed a conjecture.

Mean Square of the Central Values of Automorphic *L*–Functions

Kai–Man Tsang, Hong Kong
(joint work with Yuk-Kam Lau)

Let χ be a real primitive character of conductor D. Any modular form

$$f(z) \;=\; \sum_{n=1}^{\infty} \lambda_f(n)\, n^{(k-1)/2}\, e(nz)\,,$$

$\lambda_f(1) = 1$, can be twisted by χ:

$$f_\chi(z) \;=\; \sum_{n=1}^{\infty} \lambda_f(n)\, \chi(n)\, n^{(k-1)/2}\, e(nz)\,.$$

The associated L-function is

$$L(s, f \otimes \chi) \;=\; \sum_{n=1}^{\infty} \lambda_f(n)\, \chi(n)\, n^{-s} \qquad (\Re s > 1)\,.$$

We investigate the mean square

$$\sum_{f \in \mathcal{F}_k} \omega_f\, L^2(1/2, f \otimes \chi)$$

where $\mathcal{F}_k$ is the Hecke basis in the space of holomorphic cusp forms of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ and $\omega_f = \Gamma(k-1)(4\pi)^{1-k}\|f\|^{-2}$ where $\|f\|$ is the Petersson norm. We prove that

$$\sum_{f \in \mathcal{F}_k} \omega_f\, L^2(1/2, f \otimes \chi) \;= 2(1 + \chi(-1)\, i^k)\,\Big(\log \frac{Dk}{4\pi} + \gamma + \sum_{p | D} \frac{\log p}{p-1}\Big) + O\big(D^3\big) + O\big(D^{15/2} k^{-1/4+\varepsilon}\big)\,,$$

as k tends to infinity, k even.

Corollary. We have

$$\#\{f \in \mathcal{F}_k : \ L(1/2, f \otimes \chi) > 0\} \gg \frac{k}{(\log k)^2}\,.$$

Greedy Sums of Distinct Squares

Ulrike M. A. Vorhauer, Kent
(joint work with Hugh L. Montgomery)

When a positive integer is expressed as a sum of squares, with each successive summand as large as possible, the summands decrease rapidly in size until the very end, where one may find two 4's, or several 1's. We find that the set of integers for which the summands are distinct, we call them greedy sums of distinct squares, does not have a natural density, but that the counting function oscillates in a predictable way. Let $A(v)$ be the number of greedy sums of distinct squares less than v, then

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} \frac{A\big(4\exp\left(2^{k+x}\right)\big)}{4\exp\left(2^{k+x}\right)} = f(x)$$

where f is a continuous, non-constant function with period 1.

Explicit Formulas for the n-th Prime

Dieter Wolke, Freiburg

The classical Riemann-von Mangoldt formula

$$\psi(x) = \sum_{p^k \le x} \log p = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} + O\Big(\frac{x \log^2 x}{T}\Big)$$

where $\varrho = \xi + i\eta$ runs through all non-trivial zeros of $\zeta(s)$ with $|\eta| \le T$, $2 \le T \le x$, allows one, roughly speaking, to calculate $\psi(x)$ up to an error of order x/T by using zeros $|\varrho| \le T$. We discussed the question whether a formula of the type

$$p_n = \mathrm{Li}^{-1}(n) + \sum_{\varrho, |\eta| \le T} f(n, \varrho) + \text{ Error} \tag{1}$$

for the n-th prime can be derived where $\mathrm{Li}\, x = \int_2^x dt/\log t$. The result is a bit disappointing and seems not to be appropriate for numerical or theoretical use. First, T has to be restricted by $2 \le T \le n^{5/12-\varepsilon}$. Secondly, p_n can be written in the form (1) by means of an iterative process for which $\le c(\varepsilon)\left(\log\log n\right)^2$ steps are sufficient.

Edited by Ulrike M. A. Vorhauer

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