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Orders in Arithmetic and Geometry

February 3rd – February 9th, 2002

This conference, led by Jürgen Ritter (Augsburg) und Martin J. Taylor (UMIST, Manchester), was the third on the above subject held in Oberwolfach. It was dedicated to the memory of Ali Fröhlich who died on November 8, 2001. Below is a list of the presented talks and their abstracts. Topics of the meeting included Stark's, Rubin's, and the Coates-Sinnott conjectures, Fitting ideals of the minus class group of CM fields, Tamagawa numbers and values of L-functions, equivariant Iwasawa theory, deformations of Galois cohomology classes and orthogonal representations, Hilbert's Theorem 90 for formal groups, Hilbert's Theorem 132 and Galois module theory, Galois module classes to Steinitz classes for integers in tame extensions, quadratic forms on schemes, Hopf orders, group actions on polynomial rings. Ivan Fesenko spoke on recent work on ζ-functions of higher dimensional local fields (generalising Tate's thesis) and potential applications to arithmetic varieties. Lively discussions accompanied and completed the talks.

Monday

09:15-10:10	Victor Snaith *The Coates-Sinnott conjecture*
10:30-11:25	Martin Taylor *Galois invariants for curves in positive characteristic*
16:00-16:55	Bernhard Köck *Computing equivariant Euler characteristics on curves*
17:15-18:10	Mikhail Bondarko *Hilbert's Theorem 90 for formal groups and 'good' extensions of local fields*

Tuesday

09:15-10:10	David Solomon *Abelian Stark conjectures in* $\mathbf{Z}_p$*-extensions*
10:30-11:25	Adebisi Agboola *Twisted forms and relative* K*-theory*
16:00-16:55	Nigel Byott *Which Hopf orders are associated orders of valuation rings?*
17:15-18:10	Ivan Fesenko *Analysis on arithmetic schemes*

Wednesday

09:00-09:55	Eva Bayer-Fluckiger *Euclidean fields, Euclidean minima and Arakelov invariants*
10:10-11:05	Ted Chinburg *Deformations of Galois cohomology classes and orthogonal representations*
11:20-12:15	Leon McCulloh *From Galois module classes to Steinitz classes*

Thursday

09:15-10:10	Masato Kurihara *Fitting ideals of the minus class group of CM fields*
10:30-11:25	Al Weiss *Equivariant Iwasawa theory*
16:00-16:55	David Burns *Tamagawa numbers and values of* L*-functions*
17:15-18:10	Cristian Popescu *On the Rubin-Stark Conjecture in characteristic* p

Friday

09:15-10:10	Boas Erez *Invariants of tame coverings of schemes*
10:30-11:25	Philippe Cassou-Noguès *Quadratic forms on schemes*
15:30-16:25	Peter Symonds *Group actions on polynomial rings*
16:40-17:25	Marcin Mazur *Galois module structure of units in real biquadratic extensions of* $\mathbf{Q}$
17:40-18:25	Werner Bley *Hilbert's Theorem 132 and Galois module theory*

Abstracts

Twisted forms and relative K-theory

Agboola, Adebisi
(joint work with D. Burns)

In this talk we explained how combining ideas arising from the study of Galois structure invariants attached to finite group schemes with techniques from relative K-theory leads to a natural refinement of several different aspects of Galois module theory.
Let R be a Dedekind domain with field of fractions F. Write F^c for an algebraic closure of F, and let $\mathcal{A}$ denote a finitely generated R-algebra such that $\mathcal{A}_{F^c} := \mathcal{A} \otimes_R F^c \simeq F^c\Gamma$. Here Γ is a finite group acted upon by $\mathrm{Gal}(F^c/F)$.
Suppose that Λ is any extension of R. A categorical Λ-twisted $\mathcal{A}$-form is a triple (M, N, λ), where

$$M, N \text{ are finitely generated projective } \mathcal{A} - \text{modules},$$
$$\lambda : M \otimes_R \Lambda \xrightarrow{\simeq} N \otimes_R \Lambda \text{ is a } \mathcal{A}_\Lambda\text{-isomorphism.}$$

Such twisted forms may be classified by a relative algebraic K-group $K_0(\mathcal{A}, \Lambda)$.
Theorem If Λ is a field, then $K_0(\mathcal{A}, \Lambda)$ admits a Fröhlich-type Hom-description.
Suppose that F is a number field, and $\Lambda = F^c$. Then there is a natural surjection (defined in the first instance via Hom-descriptions) $\partial : K_0(\mathcal{A}, F^c) \to AC(\mathcal{A})$, where $AC(\mathcal{A})$ denotes the arithmetic class group of $\mathcal{A}$. (The group $AC(\mathcal{A})$ measures the structure of metrised locally free $\mathcal{A}$-modules; when $\mathcal{A}$ is abelian, $AC(\mathcal{A}) \simeq \widehat{\mathrm{Pic}}(\mathcal{A})$, the Arakelow divisor class group of $\mathcal{A}$.)
Theorem Suppose that (M, N, λ) is a twisted form with M, N locally free $\mathcal{A}$-modules. Then for any metric μ on N, we have

$$\partial(M, N, \lambda) = (M, \lambda^* \mu) - (N, \mu) \in AC(\mathcal{A}).$$

In the remainder of the talk, we discussed the following:
(a) A natural refinement of the class invariant homomorphism first studied by W. Waterhouse. This refined homomorphism takes values in a suitable relative K-group, is injective, and its image admits a precise functorial description in almost all cases of interest.
(b) We explained how results of L. McCulloh concerning realisable classes of rings of integers of tame extensions may be lifted to an appropriate relative K-group. In general, (unlike the case of realisable classes in locally free class groups) the collection of realisable classes in the relative K-group does not form a group.

Euclidean fields, Euclidean minima and Arakelov invariants

Bayer-Fluckiger, Eva

Let K be an algebraic number field, $N : K \to \mathbf{Q}$ the absolute value of the norm map and let $\mathcal{O}$ be the ring of integers of K. The Euclidean minimum of K is defined as

$$M(K) = \inf\{\mu \in \mathbf{R} \mid \forall x \in K \exists c \in \mathcal{O} \text{ s.t. } N(x - c) \le \mu\}.$$

If $M(K) < 1$, then K is Euclidean.
Let $n = [K : \mathbf{Q}]$ and let d_K be the absolute value of the discriminant of K. The following is called "Minkowski's conjecture":

Conjecture If K is totally real, then $M(K) \leq 2^{-n}\sqrt{d_K}$.
This conjecture was proved by Minkowski for $n = 2$, by Remak for $n = 3$, by Dyson for $n = 4$ and by Skubenko for $n = 5$. This leads to the following definition.
Definition A number field K is said to be a Minkowski field if $M(K) \leq 2^{-n}\sqrt{d_K}$.
(Note that many fields do not have this property: for instance, an imaginary quadratic field $\mathbf{Q}(\sqrt{d})$ is a Minkowski field if and only if $d = -1, -3$.)
This talk presented the following result:
Theorem Cyclotomic fields and their maximal totally real subfields are Minkowski fields.
For certain families of fields one gets much better bounds. For instance,
Proposition Let $K = \mathbf{Q}(\zeta_m), m = 2^r 5^s 7^t, r \geq 0, s, t \geq 1$. Then $M(K) \leq 2^{-n} 3^{-n/2}\sqrt{d_K}$.
We also study a class of Euclidean fields, called thin fields. The proofs use packing and covering invariants associated to hermitian line bundles over Spec($\mathcal{O}$).

Hilbert's Theorem 132 and Galois module theory

Bley, Werner

(joint work with D. Burns)

Let L/K be a finite Galois extension of number fields with group G. We formulate a conjectural equality between an element of the relative K-group $K_0(\mathbf{Z}[G], \mathbf{R})$ which is constructed from the equivariant Artin epsilon constant of L/K and a sum of invariants associated to L/K. The precise conjecture is motivated by the requirement that the Equivariant Tamagawa Number Conjectures (as formulated by Burns and Flach) for the Tate motives $h^0(\mathrm{Spec}(L))$ and $h^0(\mathrm{Spec}(L))(1)$ should be compatible with the functional equation of the associated L-functions. Our conjecture may also be seen as a refinement of Chinburg's second conjecture and, as a very special case, also recovers Hilbert's Theorem 132. The conjecture is known to be true modulo $K_0(\mathbf{Z}[G], \mathbf{R})_{tors}$ and is fully verified in the following cases:

a) $K = \mathbf{Q}, L/\mathbf{Q}$ abelian with odd conductor.
b) L/K tame.
c) Certain wildly ramified non-abelian extensions.

Hilbert's Theorem 90 for formal groups and 'good' extensions of local fields

Bondarko, Mikhail

We apply the long exact sequence of tensor powers of a faithfully flat algebra with filtration and prove that Amitsur cohomology groups for a formal group are zero. This implies an if and only if condition for the splitting of a 1-cocycle in a formal Galois module in terms of associated (additive) Galois modules. As a partial case we prove that $H^1 = \{0\}$ for a formally étale extension. Besides we obtain an if and only if condition for an extension to be Kummer for a formal group. Some of such extensions have very nice structure.

Tamagawa Numbers and values of L-functions

Burns, David

In the first part of our talk we described an algebraic formalism which, under suitable conditions, allows one to interpret the determinant of a perfect complex of modules over a commutative ring as an explicit relation between the fitting ideals of its homology groups.

In the second part of our talk we applied the above formalism in the context of the following special cases of the Equivariant Tamagawa Number Conjecture (ETNC).

Tate motives: In this context we show that the validity of the ETNC implies an explicit description of the Fitting ideal of certain algebraic K-groups in terms of the values of the Dirichlet L-functions at negative integers. This relation is finer than those given in recent works of Cornacchia and Østvaer, of Nguyen Quang Do and of Snaith and is also related to a recent result of Kurihara.

We also showed that the conjecture of Gross (and Tate's recent refinement of it) concerning congruences for the values at 0 of Dirichlet L-functions can in many cases be recovered as a consequence of the ETNC.

Recalling that the ETNC is known to be valid for Tate motives over absolutely abelian fields (by joint work with C.Greither) and over function fields (at least away from the residue characteristic) we obtain new insight concerning the above results and conjectures.

Elliptic curves: In this case we show that, under certain very special conditions, the ETNC for modular elliptic curves over $\mathbf{Q}$ implies congruences for modular elements which appear to be very similar to those predicted by the "refined conjecture of the Birch and Swinnerton-Dyer type" formulated by Mazur and Tate.

Which Hopf orders are associated orders of valuation rings?

Byott, Nigel

Let K be finite extension of $\mathbf{Q}_p$, G a finite abelian group, and $\mathcal{A} \subset K[G]$ a Hopf order. We sketch a proof of the following theorem.

Theorem Assume that $\mathcal{A}$ and its dual Hopf order $H = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{A}, \mathcal{O}_K)$ are connected (i.e. contain no idempotents $\neq 0, 1$). Then the following are equivalent:

(i) $\exists$ an extension L/K, Galois with group isomorphic to G, whose valuation ring $\mathcal{O}_L$ has $\mathcal{A}$ as its associated order: $\mathcal{A} = \{\alpha \in K[G] \mid \alpha\mathcal{O}_L \subseteq \mathcal{O}_L\}$.

(ii) H is monogenic as an $\mathcal{O}_K$-algebra.

When L exists, $\mathcal{O}_L$ is a free $\mathcal{A}$-module by the Childs-Hurley criterion, i.e. we have "tamed" the wild extension L/K using the Hopf order $\mathcal{A}$.

The proof (i) $\implies$ (ii) depends on the fact that $\mathrm{Spec}(\mathcal{O}_L)$ is a torso for the group scheme $\mathrm{Spec}(H)$; one then has an algebraic isomorphism $\mathcal{O}_L \otimes H \simeq \mathcal{O}_L \otimes \mathcal{O}_L$.

The proof (ii) $\implies$ (i) relies on the following

Embedding Theorem Let $H = \mathcal{O}_K[x_1, \dots, x_d]$ be a Hopf order as above, requiring d generators. Then $\mathrm{Spec}(H)$ can be obtained as the kernel of an isogeny $f : F \to F'$ between n-dimensional formal groups over $\mathcal{O}_K$ (with $n = |G| - 1$), where moreover $f_i(X) = X_i$ for all $i > d$.

Quadratic forms on schemes

Cassou-Noguès, Philippe

(joint work with B. Erez and M.J. Taylor)

In a sequel to the talk of B. Erez we presented a proof of the main results explained above in the light of "twisting". The sbs attached to $X \xrightarrow{\pi} Y$ involves the trace form. The map π gives the representation t of $\pi_1(Y)$ (or $\pi_1(Y)^{tame,odd}$) into the symmetric group on degree of π letters. Vaguely speaking the trace form is the twist of the standard form via t. Following Fröhlich we start with a sbs E on Y and show how to get a sbs E_t on Y by twisting via a representation t of π_1 in $O(E)$. Then we compare the Hasse-Witt invariants of E and E_t in degree 1 and 2. One interesting aspect of this work is the appearance of an algebraic class in $H^2(Y_{\text{ét}}, \mathbf{F}_2)$ which generalizes the class defined by Serre and only depends on ramification (which has this flavour: $\sum_\xi \frac{e_\xi^2-1}{8}[\xi]$).

Deformations of cohomology classes and orthogonal representations

Chinburg, Ted

(joint work with F. Bleher)

The first part concerned applications of the existence of versal deformations of complexes of Galois modules satisfying certain natural conditions. A class c in $H^j(\mathrm{Gal}(\bar{L}/L), N)$ for some field L and finite $k\Gamma$-module N with k a finite field defines a complex $V^\bullet$ of $k\Gamma$-modules which is the cone of c when one views c as a morphism in the derived category. One can thus view the versal deformations of $V^\bullet$ as the versal deformations of c. The natural problem is then to find explicit constructions of such deformations parallel to the known or conjectural constructions of the versal deformations of various Galois modules, e.g. of the module of p-torsion points on an elliptic curve. I discussed a conjectural construction of this kind when $L = \mathbf{Q}_l, l > 2$ is a prime, $N = \{\pm 1\}, k = \mathbf{Z}/2$ and c is the element of order two in $H^2(\mathrm{Gal}(\bar{L}/L), N)$.

The second part of the talk was about the fact that one has a parallel of Mazur's deformation theory on replacing Gl_n by an arbitrary smooth linear algebraic group $\mathcal{G}$ over $\mathbf{Z}_p$. When $\mathcal{G}$ is the orthogonal group $O(n)$, this leads to the problem of finding "universal" versions of the theorems of Serre, Fröhlich and Saito concerning Stiefel Whitney classes and Hasse-Witt invariants. Our result about this is that if one specializes a universal orthogonal representation at a geometric point, the associated second Stiefel Whitney class is independent of the point.

Invariants of tame coverings of schemes

Erez, Boas

(joint work with Ph. Cassou-Noguès and M.J. Taylor)

Jardine has defined Hasse-Witt invariants for symmetric bilinear spaces (sbs) over schemes Y such that $\frac{1}{2}$ is in $\mathcal{O}_Y$. We have presented joint work with Ph.Cassou-Noguès and M.J. Taylor in which we study these invariants for a sbs attached to a covering of schemes $X \to Y$ which is tamely ramified in the sense of Grothendieck and Murre and for which the ramification indices are odd. The case of étale coverings had been considered by Esnault-Kahn-Viehweg and Kahn. Our main contribution lies in showing how to reduce from the tame case to the étale. This we do by defining Z over Y such that the normalisation T of $T' := Z \times_Y X$ is étale over Z. To obtain the right properties for T/Z we use our knowledge of the local structure of tame coverings. The difference between the total Hasse-Witt invariants of the sbs for X/Y (pulled back to Z) and that of the sbs for T/Z is a product of expressions involving only linear invariants, namely Chern classes of auxiliary bundles which appear in a (necessary) factorisation of the normalisation map $T \to T'$. In fact this difference can be understood as follows: the difference of the sbs (for X/Y and T/Z) is metabolic as a symmetric complex and hence its invariant can be expressed in terms of Chern classes of the complex Lagrangian it contains. That is we view sbs inside the derived category etc.

From Galois module classes to Steinitz classes

McCulloh, Leon

Let G be a finite group of order n and exponent e. Corresponding to a tame Galois extension L/K of number fields with an isomorphism $\mathrm{Gal}(L/K) \simeq G$, one has associated the Galois module class $cl_{\mathcal{O}_K G}(\mathcal{O}_L)$ (resp., the Steinitz class $\mathrm{cl}_{\mathcal{O}_K}(\mathcal{O}_L)$) in the class group $\mathrm{cl}(\mathcal{O}_K G)$ (resp., $\mathrm{cl}(\mathcal{O}_K)$). For fixed number field K, the set of Galois module (resp., Steinitz) classes realized in $\mathrm{cl}(\mathcal{O}_K G)$ (resp., $\mathrm{cl}(\mathcal{O}_K)$) as L/K ranges over all tame Galois extensions with isomorphism $\mathrm{Gal}(L/K) \simeq G$ is denoted by $R(\mathcal{O}_K G)$ (resp., $R_t(\mathcal{O}_K, G)$). We show
Theorem a)

$$R_t(\mathcal{O}_K, G) \subseteq \prod_{m|e} \prod_{\substack{\bar{s}\in\bar{G} \\ |s|=m}} N(K(\bar{s})/K)^{\frac{n}{m}\frac{m-1}{2}}$$

unless the Sylow 2-subgroups of G are non-trivially cyclic, in which case
b)

$$R_t(\mathcal{O}_K; G)^2 \subseteq \prod_{m|e} \prod_{\substack{\bar{s}\in\bar{G} \\ |s|=m}} N(K(\bar{s})/K)^{\frac{n}{m}(m-1)},$$

where $N(K(\bar{s})/K) = N_{K(\bar{s})/K}(\mathrm{cl}(\mathcal{O}_{K(\bar{s})})) \subseteq \mathrm{cl}(\mathcal{O}_K)$.
Further explaining, $\bar{G}$ is the set of conjugacy classes of G endowed with a "cyclotomic" action of $\Omega(= \mathrm{Gal}(K^c/K))$ via $\kappa : \Omega \to \mathrm{Gal}(K(\zeta_e)/K) \hookrightarrow (\mathbf{Z}/e\mathbf{Z})^\times$ - explicitly, for $s \in G$ and corresponding $\bar{s} \in \bar{G}$ and $\omega \in \Omega$, $s^\omega = s^{\kappa^{-1}(\omega)}$ and $\bar{s}^\omega = \overline{s^\omega}$, where $\zeta_e^\omega = \zeta_e^{\kappa(\omega)}$. Finally, $K(\bar{s}) = (K^c)^{\Omega_{\bar{s}}}$, where $\Omega_{\bar{s}}$ is the Ω-stabilizer of $\bar{s}$.
The result is obtained from a known inclusion of $R(\mathcal{O}_K G)$ via $\mathrm{res}_1^G : \mathrm{cl}(\mathcal{O}_K G) \to \mathrm{cl}(\mathcal{O}_K)$.

Computing equivariant Euler characteristics on curves

Köck, Bernhard

Let X be a smooth projective curve over an algebraically closed field k and G a finite subgroup of $\mathrm{Aut}(X/k)$. We gave a new approach to known formulas (by Ellingsrud/Lønsted, Nakajima and Kani) for the equivariant Euler characteristic of a locally free Zariski G-sheaf on X. The basic idea is to use the coherent Lefschetz trace formula in conjunction with the classical Riemann-Roch formula. Applying the same method to a constructible G-sheaf $\mathcal{F}$ of $\mathbf{F}_l$-vector spaces on the étale site $X_{\text{ét}}$, i.e. replacing these formulas by the étale Lefschetz trace formula and the Grothendieck- Ogg-Shafarevich formula respectively, we then proved an equivariant Grothendieck-Ogg-Shafarevich formula, a formula which explicitly computes the equivariant Euler characteristic $\chi_{\text{ét}}(G, X, \mathcal{F})$ of $\mathcal{F}$ in the Grothendieck group $G_0(\mathbf{F}_l[G])$ of all finitely generated $\mathbf{F}_l[G]$-modules (if the order of G is prime to the characteristic of k). As a corollary we obtained that the sum of the wild conductors of $\mathcal{F}$ is divisible by the order of G.

Fitting ideals of the minus class group of CM fields

Kurihara, Masato

In this talk, for a finite abelian field F over $\mathbf{Q}$, I defined the Stickelberger ideal Θ_F by a slightly different method from Sinnott's, and proposed a conjecture that the Fitting ideal of the minus part of the ideal class group $\mathrm{Pic}(\mathcal{O}_F)$ would be equal to Θ_F^- except 2-components. I gave a theorem which says that this conjecture $\otimes\mathbf{Z}_p$ is true over the cyclotomic $\mathbf{Z}_p$-extension F_∞. I also gave a theorem (a weaker statement then the above theorem) for a general CM field F over a totally real field k, and gave an application on the structure of the χ-component of the p-component of the class group of F for some odd character χ of $\mathrm{Gal}(F/k)$.

Galois module structure of units in real biquadratic extensions of Q

Mazur, Marcin

(joint work with S. Ullom)

Let $N/\mathbf{Q}$ be a biquadratic, real number field, U the group of units of N and $V = U/\{\pm 1\}$. We investigate the structure of V as a module over the group ring of the Galois group $\Gamma = \mathrm{Gal}(N/\mathbf{Q})$. Building on earlier work of S.Kuroda and T.Kubota we observe that the $\mathbf{Z}\Gamma$-module V can be essentially of 4 different isomorphism types. We prove that each type occurs infinitely often among fields N with exactly r ramified primes, for each $r \geq 3$. In particular, there exist real quadratic fields with Minkowski unit and arbitrary many ramified primes. We describe a family of examples where the field N (with exactly 3 ramified primes) has a Minkowski unit exactly when the central class field of N is different from its genus field.

We investigate the fields of the form $N = \mathbf{Q}(\sqrt{p}, \sqrt{q})$ with p, q primes. We observe that in the most interesting case, when $p \equiv 1(8)$ and $q \equiv 1(4)$ and $\left(\frac{p}{q}\right) = 1$ the investigation of the Galois module V is related to the governing fields of Cohn and Lagarias. We give an explicit description of the minimal governing field for the divisibility by 8 of the class number of $\mathbf{Q}(\sqrt{pq})$, where $q \equiv 3(4)$ is fixed and $p \equiv 1(4)$ varies.

On the Rubin-Stark Conjecture in characteristic p

Popescu, Christian

We stated a refined version ("over $\mathbf{Z}$") of Stark's Main Conjecture, in the case of abelian L-functions of arbitrary order of vanishing at $s = 0$. This statement is in general weaker than a similar conjecture formulated by Rubin in 1994.

In the case of characteristic p function fields, we indicated how one can prove this conjecture, for all extensions K/k, where K is contained in the compositum $k_{p\infty}$ of the maximal pro-p abelian extension k_p and the maximal constant field extension k_∞ of k.

We stated a theorem which establishes the equivalence between Rubin's Conjecture and the Conjecture mentioned above for all extensions K/k, with $K \subseteq k_{p\infty}$. This way, a proof for Rubin's conjecture for all K/k with $K \subseteq k_{p\infty}$ was obtained.

The Coates-Sinnott conjecture

Snaith, Victor

In 1967 Brumer conjectured that if L/K is a Galois extension of number fields with abelian group G and K totally real

$$\Theta_{L/K,S}(1) \cdot \mathrm{ann}_{\mathbf{Z}[G]}(\mu(L)) \subseteq ann_{\mathbf{Z}[G]}(TorsK_0(\mathcal{O}_{L,S'}))$$

where, for $n = 1, 2, \ldots$ and each character χ, $\chi(\Theta_{L/K,S}(n)) = L(1-n, \chi^{-1})$ – the Artin L-function with Euler factors associated to the set of primes S removed. Here S is a finite set of primes of K including those which ramify in L/K and S' is the set of primes of L above S. In 1974 Coates-Sinnott suggested, at least for cyclotomic extensions, to replace $n = 1$ by $1 - r$ for $r = -1, -3, -5, \ldots$ and $\mu(L) = TorsK_1(\mathcal{O}_{L,S'}), TorsK_0(\mathcal{O}_{L,S'})$ by $TorsK_{1-2r}(\mathcal{O}_{L,S'}), K_{-2r}(\mathcal{O}_{L,S'})$. Of course, in 1890 Stickelberger published a proof of the case $n = 1$ for cyclotomic fields. Coates-Sinnott, apart from a small numerical factor, proved the cyclotomic case for $r = -1$.

Work of Voevodsky-Suslin implies that we may replace K-groups by étale cohomology. Let l be a prime and $(m, l) = 1$. Let $X_l = \mathrm{Spec}(\mathbf{Z}[\xi_{ml^{s+1}}][1/ml]), X_l^+ = \mathrm{Spec}(\mathbf{Z}[\xi_{ml^{s+1}}]^+[1/ml])$ denote the spectrum of the S_{ml}-integers of $\mathbf{Q}(\xi_{ml^{s+1}}), \mathbf{Q}(\xi_{ml^{s+1}})^+$, respectively. Here S_{ml} equals the set of primes dividing ml.

Theorem Let l be an odd prime. Let l, m and X_l^+ be as above and let $\Theta_{\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q}}(1-r)$ denote the higher Stickelberger element associated to the primes dividing ml and the Dirichlet L-function. Then, for $r = -1, -3, -5, \ldots$ and any positive integer s:

(i) There exists a chain of annihilator ideal relations for étale cohomology of the form

$$\begin{aligned}
&\{t^{m_0} \mid t \in \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^1_{\text{ét}}(X_l^+; \mathbf{Q}_l/\mathbf{Z}_l(1-r)))\} \\
&\subseteq \Theta_{\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q}}(1-r)\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^0_{\text{ét}}(X_l^+; \mathbf{Q}_l/\mathbf{Z}_l(1-r))) \\
&\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^1_{\text{ét}}(X_l^+; \mathbf{Q}_l/\mathbf{Z}_l(1-r))).
\end{aligned}$$

(ii) If l does not divide $m-1$ then the final in (i)

$$\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^1_{\text{ét}}(X_l^+; \mathbf{Q}_l/\mathbf{Z}_l(1-r)))$$

may be replaced by

$$F_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^1_{\text{ét}}(X_l^+; \mathbf{Q}_l/\mathbf{Z}_l(1-r))).$$

Here m_0 is the minimal number of generators of the $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]$-module $H^1_{\text{ét}}(X_l^+; \mathbf{Q}_l/\mathbf{Z}_l(1-r))$ and $F_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(M)$ denotes the Fitting ideal of M.

Theorem Let l be an odd prime. Let l, m and X_l^+ be as above. Let $H^1_{\text{cyclo}}(X_l^+; \mathbf{Z}_l(1-r))$ denote the cyclotomic elements of Beilinson-Soulé. Then, for $r = -2, -4, -6, \ldots$ and any positive integer s:

(i) There exists a chain of annihilator ideal relations for étale cohomology of the form

$$\{t^{m_0} \mid t \in \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^2_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r)))\}$$

$$\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}\left(\frac{H^1_{\text{ét}}(X_l^+;\mathbf{Z}_l(1-r))}{H^1_{\text{cyclo}}(X_l^+;\mathbf{Z}_l(1-r))}\right)$$

$$\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^2_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r)).$$

(ii) If l does not divide $m-1$ then in (i) the final

$$\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^2_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r)))$$

may be replaced by

$$F_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^2_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r))).$$

Here m_0 is the minimal number of generators of the $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]$-module $H^2_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r))$.

From this result I showed how to construct from the Stark regulator associated to the Borel regulator for higher K-groups a fractional ideal of the rational group-ring whose intersection with the $\mathbf{Z}_l$-group-ring lies in

$$\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]}(H^2_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r)).$$

Abelian Stark Conjectures in $\mathbf{Z}_p$-extensions

Solomon, David

Let K/k be an abelian extension of totally real number fields with Galois group G. Assume (for simplicity) that K/k is ramified at 2 or more finite primes of k. Then the Stark conjecture for complex characters of G can be formulated together in terms of the value of a function $\Phi_{K/k} : \mathbf{C} \to \mathbf{C}G$ as follows

$$C(K/k): \qquad \Phi_{K/k}(1) = \frac{2^r}{\sqrt{d_k}} R_{K/k}(\eta) \text{ for some } \eta \in \Lambda^r_{\mathbf{Q}G}\mathbf{Q} \otimes E(K).$$

(Here, $R_{K/k}$ is a group-ring-valued regulator inspired by Rubin's work.)

Let p be an odd prime number. "Removing the Euler factors above p" from $\Phi_{K/k}(s)$ gives a function $\Phi_{K/k,(p)} : \mathbf{C} \to \mathbf{C}G$. Interpolating this at $m \leq 0, m \equiv 1(p-1)$ gives $\Phi_{K/k,p} : \mathbf{Z}_p \to \mathbf{C}_pG$, a p-adic analogue of $\Phi_{K/k}$. (Holomorphic if, for example, the ramified primes in K/k do not divide p, which we assume from now on.) We have conjectured that an analogue of the equation in C(K/k) should hold simultaneously for $\Phi_{K/k,(p)}(1)$ and $\Phi_{K/k,p}(1)$ using the same η. (Call this $C^+(K/k,p)$.)

If p is also unramified in k, we consider $C(K_n/k)$, where K_n is the n-th layer in the cyclotomic $\mathbf{Z}_p$-extension over $K = K_0$. Assuming this conjecture holds for all $n \geq 1$, norm-compatability of Φ and R imply that $\underline{\eta} := (\eta_n)_n$ must lie in $\varprojlim \Lambda^r_{\mathbf{Q}G_n}\mathbf{Q} \otimes E(K_n)$ (limit with respect to norm maps). We investigate the stronger hypothesis

$$P1(K/k,p): \qquad \underline{\eta} \text{ is an element of } \Lambda^r_{\mathbf{Q}G_\infty}\mathbf{Q}\mathcal{E}(K)$$

(where $\mathcal{E}(K) = \varprojlim E(K_n)$ and $G_\infty = \mathrm{Gal}(K_\infty/k)$).

Combining P1 with $C^+(K_n/k,p), \forall n \geq 1$, implies

$$P2(K/k,p): \qquad \Phi_{K_n/k,p}(1) = \frac{2^r}{\sqrt{d_K}} R_{K_n/k}(e_n\beta_n(\underline{\eta})), \forall n \geq 1,$$

where $e_n = 1 - \frac{1}{p} \sum\limits_{\sigma \in \mathrm{Gal}(K_n/K_{n-1})} \sigma \in \mathbf{Q}G_n$, β_n is the natural map from $\Lambda^r_{\mathbf{Q}G_\infty}\mathbf{Q}\mathcal{E}(K)$ to $\Lambda^r_{\mathbf{Q}G_\infty}\mathbf{Q}E(K_n)$.

Next we use a generalisation of the t-th Coates-Wiles-homomorphism to define regulators

$$\mathcal{R}_{t,n} : \Lambda^r_{\mathbf{Q}G_\infty}\mathbf{Q}\mathcal{E}(K) \to \mathbf{C}_p G_n, \forall n \geq 1, \forall t \in \mathbf{Z}.$$

We show in particular that $P2(K/k,p)$ implies a "Higher Stark conjecture" at $s = m$ for any $m \in \mathbf{Z}$ as follows:

$$\Phi_{K_n/k,p}(m) = \frac{2^r}{\sqrt{d_k} < d_k >^{1-m}} \mathcal{R}_{1-m,n}(\underline{\eta}).$$

(For $m = 1$ this is simply $P2(K/k,p)$.)

Group actions on polynomial rings

Symonds, Peter

(joint work with D. Karagueuzian)

We consider a finite group G acting by linear substitutions on a polynomial ring S in n variables over a finite field k of characteristic p with q elements.

We want to understand S as a kG-module in as explicit a manner as possible.

The maximal p group that acts faithfully is U_n, the group of upper triangular matrices with 1's on the diagonal. The ring of invariants $S^{U_n} = k[d_1, \dots, d_n]$, $\deg(d_i) = q^{i-1}$. If G is any p-group acting we may assume that $G \subseteq U_n$ and:

Theorem As graded kG-modules

$$S \simeq \bigoplus_{J \subseteq I} k[d_i, i \in I \cup \{n\} \setminus J] \otimes_k \bar{X}_J(I),$$

where $\bar{X}_J(I)$ is a finite dimensional kG-module (and G acts trivially on $k[d_i]$) and $I = \{1, \dots, n-1\}$.

Corollary S, as a kG-module, only contains a finite number of isomorphism classes of indecomposable summands (any G, not necessarily a p-group).

Corollary For any G, the invariants S^G are generated as a ring by the elements in degrees less than or equal to $\frac{q^n-1}{q-1}(nq-n-1)$.

Galois invariants for curves in positive characteristic

Taylor, Martin

(joint work with T. Chinburg and G. Pappas)

Let $\pi : X \to Y$ be a tame G-cover of smooth projective curves over $k = \mathbf{F}_p^{sep}$; suppose there is a curve Y' over $\mathbf{F}_p$ such that $Y = Y' \times_{\mathbf{F}_p} k$; and let $\mathcal{F}$ be a coherent $G - X$ sheaf. Then $R\Gamma(X, \mathcal{F})$ is an element in the derived category of bounded kG-complexes; moreover, since the G-action is tame, it may be presented by a perfect complex $P^\bullet = P_0 \xrightarrow{\partial} P_1$ of length 2.

The projective Euler characteristic $\chi R\Gamma(\mathcal{F}) = (P_0) - (P_1) \in K_0(kG)$ may be calculated by Lefschetz-Riemann-Roch. In the case when $\mathcal{F} = \mathcal{O}_X$, this Euler characteristic can also be determined by the p-adic absolute values of the ϵ-constants associated to the cover X/Y (T. Chinburg). In this talk we considered the extension class in $\mathrm{Ext}^2_{kG}(H^1, H^0)$ given by the exact sequence

$$0 \longrightarrow H^0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow H^1 \longrightarrow 0 \qquad (*)$$

when $\mathcal{F} = \mathcal{O}_X, \Omega_X$ and $H^i = H^i(X, \mathcal{F})$.

By some relatively straightforward algebra, one reduces to the case when G is a cyclic group of order p; so that the cover X/Y is an étale cover of degree p.

Theorem If Y' is an elliptic curve, then the $(p-1)$st power of the extension class of $(*)$, with $\mathcal{F} = \Omega_X$, is given by the Hasse invariant (in $\mathbf{F}_p^\times$) of the curve Y'.

More generally, when $g(Y') > 1$, the $(p-1)$st power of this extension class was determined by the pairing

$$\mathrm{Hom}(\pi_1(Y), \mathbf{F}_p) \times H^0(\Omega_{Y'}) \longrightarrow H^1(X, \mathcal{O}_Y) \times H^0(\Omega_Y) \longrightarrow H^1(\Omega_Y) = k.$$

Equivariant Iwasawa theory

Weiss, Alfred
(joint work with J. Ritter)

Let l be an odd prime, K/k a finite Galois extension of totally real number fields, and S a sufficiently large finite set of primes of k containing $l\infty$. Put $G_\infty = \mathrm{Gal}(K_\infty/k)$, where K_∞/K is the cyclotomic $\mathbf{Z}_l$-extension, and $X_\infty = \mathrm{Gal}(M_\infty/K_\infty)$, where M_∞ is the maximal abelian l-extension of K_∞ which is unramified outside S. A natural goal for "equivariant Iwasawa theory" is a description of X_∞ as $\mathbf{Z}_l[[G_\infty]]$-module.

We construct a special class $\mho_S$ in the Grothendieck group $K_0T(\mathbf{Z}_l[[G_\infty]])$ of finitely generated torsion $\mathbf{Z}_l[[G_\infty]]$-modules with finite projective dimension over $\mathbf{Z}_l[[G_\infty]]$. This $\mho_S$ has the property that, under the localisation map $K_0T(\mathbf{Z}_l[[G_\infty]]) \to K_0T((\mathbf{Z}_l[[G_\infty]])_\wp)$, its image coincides with the class $[(X_\infty)_\wp]$, for every non-zero prime ideal $\wp$ of $\mathbf{Z}_l[[G_\infty]]$. Since X_∞ has infinite projective dimension we consider $\mho_S$ as a "smoothing" of X_∞.

When G_∞ is abelian, we can use the Deligne-Ribet power series $G_{\chi,S}(T)$ to define a unique element Θ_S of the total quotient ring of fractions Q of $\mathbf{Z}_l[[G_\infty]]$ by requiring $\chi(\Theta_S) = G_{\chi,S}(0)$ for all l-adic characters χ of G_∞. Since $K_1(Q) = Q^\times$ we consider Θ_S in $K_1(Q)$. Our main result is

Theorem If "$\mu = 0$" then the map $\partial : K_1(Q) \to K_0T(\mathbf{Z}_l[[G_\infty]])$ of the localization sequence takes Θ_S to $\mho_S$.

This refines (and depends on) the Main Conjecture of Iwasawa theory which was proved by Wiles. Generalizing to arbitrary μ is a refined goal for "equivariant Iwasawa theory".

Edited by Werner Bley

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