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Miniworkshop: Dynamics and Applications of Stochastic Partial Differential Equations

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This mini-workshop was organized by Igor Chueshov, Jinqiao Duan and Björn Schmalfuß.

It has long been clear that it is indispensable to take nonlinearity into account, in order to better understand complex systems. There is a growing recognition of a role for the inclusion of stochastic terms in the modeling of complex, multi-scale phenomena. The addition of such terms has led to interesting new mathematical problems at the interface of probability, dynamical systems, bifurcation, topology, and numerical analysis.

Randomness is ubiquitous in scientific and engineering systems. Taking stochastic effects into account is of central importance for the development of mathematical models of complex phenomena in engineering and science. Macroscopic models in the form of ordinary/partial differential equations contain such randomness as stochastic forcing, uncertain parameters, random sources or inputs, and random initial and boundary conditions. Nonlinear stochastic ordinary/partial differential equations are appropriate models for randomly influenced nonlinear systems.

It has become more and more evident recently that randomness may have delicate impact (such as stochastic bifurcation, stochastic resonance and noise-induced pattern formation) on the dynamical evolution of natural and engineering systems. In some systems, noise may be a key for understanding certain phenomena (such as turbulence, phase transition, dynamic instability, chaos). There has been active recent research on stochastic approaches to fluid flows, climate dynamics, and materials modeling. We observe a wide spread interest in the engineering and applied sciences as well as mathematics community in the study of random dynamical systems generated by SPDEs.

The theory of random dynamical systems provides useful tools to treat SPDEs; see Ludwig Arnold's book *Random Dynamical Systems*, Springer-Verlag, 1998. Moreover, new mathematical approaches are needed for investigating engineering systems driven by non-Gaussian or non-Markovian noises, and some very fundamental mathematical issues related to these models are not well understood. The tools of random dynamical systems are also useful for these approaches. It is now the very time to bring new mathematical approaches to the investigation of fluid flows, climate dynamics, and materials under uncertainty with the hope of finding and providing new stimulus in the general theory of SPDEs.

Our main motivation for organizing this mini-workshop is two-fold: (i) discuss mathematical challenges and mathematical interests from SPDE point of view, (ii) investigate physical relevance of many SPDE models arising from mathematical modeling of climate dynamics, fluid flows, materials science and other areas.

Abstracts

Stochastic Burgers equations with boundary forcing

Dirk Blomker

(joint work with Jinqiao Duan, Hongjun Gao and Christoph Gugg)

We consider the stochastic Burgers equation with body-forcing or with boundary forcing. In a transient regime we observe that important statistical quantities like mean energy or correlation functions are the same for the two types of forcing. Nevertheless the solutions behave completely different.

The effects produced by random terms in the long-time behaviour of dynamical systems

Toms Caraballo

The main objective of this talk is to discuss about the behaviour that a dynamical system may exhibit when some random terms are considered in the equations of the model. In fact, we will point out that the interpretation given to the random term may provide different results. Roughly speaking, we shall show that when the stochastic perturbation is interpreted in Stratonovich's sense, it may not affect the behaviour provided that a certain commutativity hypothesis is assumed. Unlikely, if the random term is considered in Ito's sense, it may produce stabilization and destabilization effects, even if the commutativity assumption holds.

Monotone Random Dynamical Systems

Igor Chueshov

The main goal in this lecture is to present the basic ideas and methods for order-preserving (or monotone) random dynamical systems which have been developed over the past few years. We focus on the qualitative behaviour of monotone random systems and our main objects are equilibria and attractors. The study of this objects allow us to discover possible scenarios of the long-term behaviour in many dissipative random systems which are important from applied point of view. Our consideration relies crucially on the concepts of sub- and super-equilibria and on comparison principles for random and stochastic ordinary differential equations. Sub- and super-equilibria are random analogs of sub- and super-solutions of ordinary differential equations. These semi-equilibria are proved to be very useful in the description of random attractors for the systems considered. In particular their use allow us prove that random attractors of monotone systems lie between two equilibria. We also prove a theorem on the existence of equilibria between two ordered sub- and super-equilibria. For sublinear random systems we establish a random limit set trichotomy, stating that in a given part either (i) all orbits are unbounded, (ii) all orbits are bounded but their closure reaches out to the boundary of the part, or (iii) there exists a unique, globally attracting equilibrium. Several examples, including Markov chains and affine systems, are given.

Measure attractors

Hans Crauel

There have been several approaches to find a generalization of the notion of an attractor for random dynamical systems. One of these approaches introduces attractors for the Markov semigroup induced by a stochastic (partial) differential equation. We compare this notion with the notion of random attractors, defined as a compact-set-valued random variable.

Uniqueness of the invariant measure: a coupling approach

Martin Hairer

We consider parabolic stochastic partial differential equations driven by white noise in time. We prove exponential convergence of the transition probabilities towards a unique invariant measure under suitable conditions. These conditions amount essentially to the fact that the equation transmits the noise to all its determining modes. Several examples are investigated, including some where the noise does not act on every determining mode directly.

A rigorous derivation of Smoluchowski's equation in the moderate limit

Christian Klingenberg

Smoluchowski's equation is a macroscopic description of a many-particle system with coagulation and shattering interactions. We give a microscopic model of the system from which we derive this equation rigorously. We prove the law of large numbers for the empirical processes. This approach can be justified in the regime of high temperatures and particle densities, which is of special interest in astrophysical studies.

Some remarks on the discretization of attractors of stochastic partial differential equations

Peter E. Kloeden

In this talk I sketched the basic ideas and issues on the discretization of attractors for autonomous differential equations. I then outlined the process and skew-product formalisms for nonautonomous systems and discussed their pullback attractors, explaining how a pullback attractor can be defined for a discretized system. I then discussed my results with Grecksch and Shott on the error bounds for strong Taylor schemes for Galerkin SDE approximations of stochastic reaction–diffusion type equations.

Flows and attractors for parabolic SPDEs with multiplicative noise

Hannelore Lisei

The purpose of this paper is to transform a nonlinear stochastic partial differential equation of parabolic type with multiplicative noise into a random partial differential equation by using a bijective random process. A stationary conjugation is constructed, which is of interest for asymptotic problems. The conjugation is used here to prove the existence of the stochastic flow, the perfect cocycle property and the existence of the random attractor, all nontrivial properties in the case of multiplicative noise.

Structural Stability, Invariant Manifolds, and Invariant Foliations for Infinite Dimensional Dynamical Systems

KENING LU

The investigation of a particular dynamical system or a family of dynamical systems usually can be traced to an evolving physical system whose behavior one would like to understand and possibly predict. And so one of the main goal of the study of dynamical systems is to understand the long term behavior of states in the systems. In the original modeling process, empirical laws, simplifying assumptions, and even conjectured relationships are used to derive a dynamical system in the hope of then being able to approximately describe physical reality. Therefore, to have a better understanding of the physical phenomena being modeled, one needs to investigate not only the mathematical model but also the perturbations of the model. One also needs to study how the qualitative properties of the perturbed models are related to the qualitative properties of the original model.

Although bifurcation theory, static and dynamic, are natural outcomes of this line of inquiry, for physical systems being modeled, one hopes that all the flows associated with small perturbations of the given system exhibit the same qualitative behaviors. When such is the case the dynamical system is said to be structurally stable, and the study of structural stability and its necessary or sufficient conditions has been a particularly fruitful field of study.

If a dynamical system is not structurally stable, one may want to know when part of the qualitative properties are preserved under small perturbation. This leads to the fundamental problem of the existence and the persistence of invariant manifolds under perturbation and to the study of the qualitative properties of the flow near invariant manifolds

In this talk, I shall report some recent work on the structural stability, the existence and persistence of normally hyperbolic invariant manifolds and invariant foliations for infinite dimensional dynamical systems. A related topic is the Floquet theory for parabolic equations. I shall also report a recent work on overflowing and inflowing manifolds, and approximated normally hyperbolic invariant manifolds. I should mention that the infinite dimensional dynamical systems generated by, for example, parabolic equations are not reversible and the phase spaces are not locally compact. This characteristic creates difficulties not encountered in the study of finite dimensional dynamical systems and new methods need to be developed to understand the nature of these systems.

Non-Markovian Dynamics of SPDE's

BOHDAN MASLOWSKI

Recent results on stochastic PDE's driven by the fractional Brownian motion (fBm) obtained jointly with T.E.Duncan, B.Pasik-Duncan, D.Nualart and B.Schmalfuss are reviewed. Stochastic integral with respect to the fBm is defined for deterministic integrands similarly to the classical Ito integral; for stochastic integrands pathwise fractional integration is used. For semilinear stochastic parabolic equations with non-additive noise existence and uniqueness of solution is established. For linear equations large time behaviour is also studied. The concept of "invariant measure" is discussed and weak and strong (variational) convergence to a limiting measure are shown under appropriate conditions.

Topological methods for random dynamical systems

Gunter Ochs

(joint work with Konstantin Mischaikow)

The Conley index is a topological tool for the qualitative analysis of dynamical systems. It provides informations about the existence and the structure invariant sets. In this talk we discuss the notion of a Conley index for random dynamical systems, which serve as a model for dynamics influenced or perturbed by probabilistic noise. We also give an example, which shows that topological fixed point theorems do not hold the random case and hence not all results in the deterministic theory have a canonical generalization to our situation.

Invariant manifolds for stochastic partial differential equations

Björn Schmalfuss

(joint work with Jinqiao Duan and Kening Lu)

Invariant manifolds provide the geometric structures for describing and understanding dynamics of nonlinear systems. The theory of invariant manifolds for both finite and infinite dimensional autonomous deterministic systems, and for stochastic ordinary differential equations is relatively mature. In this paper, we present a unified theory of invariant manifolds for infinite dimensional *random* dynamical systems generated by *stochastic* partial differential equations. We first introduce a random graph transform and a fixed point theorem for non-autonomous systems. Then we show the existence of generalized fixed points which give the desired invariant manifolds.

Dynamics of gravitationally interacting sticky gas with random initial velocities

Toufic Suidan

We study a one dimensional gravitationally interacting sticky gas with random initial velocities. We identify three continuum limit regimes for the evolution of the mass distribution. We understand the mass distribution by means of functional of Brownian motion. The problem is motivated by work on variational principles for systems of conservation laws.

Edited by Jinqiao Duan

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