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Interactions between Algebraic Geometry and Noncommutative Algebra

April 14th – April 20th, 2002

This meeting was organized by M. van den Bergh (Diepenbeek), D. Happel (Chemnitz), L. Small (La Jolla) and T. Stafford (Ann Arbor). There were 41 participants from 10 countries (Belgium, Canada, Denmark, France, Germany, Israel, Norway, Russia, UK and the US) and 22 lectures were presented during the five day period.

This meeting examined the applications of ideas and techniques from algebraic geometry to noncommutative algebra. While algebraic geometry has pervaded essentially all aspects of mathematics, its influence on noncommutative algebra is more recent. Therefore, a major objective was to bring together researchers from different geometrically influenced parts of noncommutative algebra, in order to make the various groups more aware of the others' work, and to thus stimulate cross-fertilization. Areas of concentration, which were represented were noncommutative algebraic geometry, representation theory of quivers, symplectic algebras, etc. The abstracts included in this report reflect this breadth of material.

Abstracts

Triangulated categories as a subject of noncommutative geometry

Alexei Bondal

Serious technical problems with proving that a given functor between derived categories yields an equivalence of triangulated categories appeals to a better understanding of the structure of triangulated categories. We propose a "stringy" interpretation of the extended axiomatics of triangulated categories, which goes as follows. Consider a 2-category $E\Lambda$ in the category of categories $\underline{\mathrm{Cat}}$ with objects (n) being $\mathbb{Z}$ considered as the category of the poset $\mathbb{Z}$ with the natural ordering of the elements. Let $\tau_n : \mathbb{Z} \to \mathbb{Z}$ be the shift $p \mapsto p+1$ in the nth $\mathbb{Z}$. 1–morphisms $(n) \to (m)$ in $E\Lambda$ are increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy $\tau_m^m f = f\tau_n^n$. The category $E\Lambda$ is 2–full in $\underline{\mathrm{Cat}}$.
A multi-triangulated category is a 2–functor $E\Lambda^{op} \to \underline{\mathrm{Cat}}$ satisfying conditions similar to those of Segal's delooping machine. More generally, we can introduce the "space argument" $\mathcal{F}$ in the derived categories $\mathcal{D}(\mathcal{F},\mathcal{E})$, where $\mathcal{F}$ is a Frobenius category and $\mathcal{E}$ is exact. In the particular case when $\mathcal{F}$ is the category of graded modules over $k[x]/x^n$ this yields the multi-triangulated structure described above. The axioms on the functor $E\Lambda^{op} \to \underline{\mathrm{Cat}}$ are analogous to properties of sheaves with respect to a Grothendieck topology on $\mathcal{F}$.

Flops and derived categories

Tom Bridgeland

It is well-known that any surface of non-negative Kodaira dimension has a unique non-singular minimal model obtained by blowing down all (-1) curves. The corresponding statement is false for threefolds: the minimal model may have singularities and even when it is non-singular it will not in general be unique. I discussed how non-commutative algebra may help with these problems. I gave a simple example where although the minimal model is singular there is a natural sense in which a certain stack or non-commutative variety is a non-singular minimal model. I then discussed a theorem which shows that distinct non-singular minimal models of a given threefold have equivalent derived categories. The proof proceeds by describing certain birational operations called flops via a moduli space construction.

Parabolic bundles and the Deligne-Simpson problem

Willian Crawley-Boevey

The talk centred on two conjectures. The first conjecture concerns parabolic bundles on the Riemann sphere. The question is, what is the possible numerical data of an indecomposable parabolic bundle - the degree and rank of the bundle and the dimensions of the vector spaces in the filtrations. The conjectural answer is that the rank and dimensions should form a root for an appropriate Kac-Moody Lie algebra.

The Deligne-Simpson problem asks for a characterization of the k-tuples of conjugacy classes in the general linear group with the property that one can find a matrix in each class in such a way that the product is the identity, and such that the matrices have no common invariant subspace. The second conjecture is an explicit answer for this problem in terms of root systems.

The main tool for studying the Deligne-Simpson problem is the middle convolution functor of Katz, simplified by Dettweiler and Reiter. It gives a proof of the second conjecture in the rigid case. By considering logarithmic connections and using the Riemann-Hilbert correspondence, we can use it to prove the first conjecture in many cases.

Noncommutative coordinate rings and stacks

Daniel Chan

(joint work with Colin Ingalls)

Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme with X a quasi-projective variety and let S be its coarse moduli scheme. We associate to the groupoid scheme a coherent sheaf of algebras $\mathcal{O}_{X/Y}$ on S which we call the noncommutative coordinate ring of the groupoid scheme. We show that when X is a smooth curve and the groupoid action is generically free, the noncommutative coordinate rings which can occur are, up to Morita equivalence, the hereditary orders on smooth curves. This gives a bijective correspondence between smooth one dimensional Deligne-Mumford stacks which are generically schemes and Morita equivalence classes of hereditary orders on smooth curves.

The classification of finite dimensional triangular Hopf algebras over an algebraically closed field of characteristic 0

Shlomo Gelaki

In 1998, P. Etingof and myself proved that a *semi-simple* triangular Hopf algebra H over $\mathbb{C}$ is obtained from the group algebra $\mathbb{C}[G]$ of a unique finite group G by twisting its co-multiplication in the sense of Drinfeld. The proof of this theorem relies in an essential way on a theorem of Deligne on Tannakian categories. Later on, we used this and the theory of Movshev on twisting in finite groups to completely classify such Hopf algebras H in terms of certain quadruples (G, A, V, u) of group-theoretical data.

However, for *non-semisimple* finite dimensional triangular Hopf algebras H over $\mathbb{C}$, Deligne Theorem cannot be applied. Nevertheless, in 1999 I classified minimal basic H. Such Hopf algebras H and the semisimple ones have the *Chevalley property* in common. In 2001, N. Andruskiewitsch, P. Etingof and myself proved that H has the Chevalley property if and only if it is obtained from a certain modification of the *supergroup* algebra $\mathbb{C}[G]$ of a finite *supergroup* G by twisting its co-multiplication. Later on, Etingof and myself used it to completely classify such Hopf algebras H in terms of certain septuples (G, W, A, Y, B, V, u) of group-theoretical data.

Very recently, Deligne has generalized his theorem on Tannakian categories to the super-case. This remarkable generalized theorem, combined with our results, thus lead to the complete and explicit classification of *all* finite dimensional triangular Hopf algebras over $\mathbb{C}$, and also allowed us to answer some questions about triangular Hopf algebras and symmetric rigid tensor categories.

Symplectic reflection algebras

Victor Ginzburg

To any finite group G of automorphisms of a symplectic vector space V we associate a new multi-parameter deformation, H_k, of the smash product of G with the polynomial algebra on V. The algebra H_k, called a symplectic reflection algebra, is related to the coordinate ring of a universal Poisson deformation of the quotient singularity V/G. If G is the Weyl group of a root system in a vector space h and $V = h \oplus h^*$, then the algebras H_k are 'rational' degenerations of Cherednik's double affine Hecke algebra.

Let $G = S_n$, the Weyl group of $g = gl_n$. We construct a 1-parameter deformation of the Harish-Chandra homomorphism from $D(g)^g$, the algebra of invariant polynomial differential operators on gl_n, to the algebra of S_n-invariant differential operators with rational coefficients on C^n. The second order Laplacian on g goes, under the deformed homomorphism, to the Calogero-Moser differential operator with rational potential. Our crucial idea is to reinterpret the deformed homomorphism as a homomorphism: $D(g)^g \to \{\text{spherical subalgebra in } H_k\}$, where H_k is the symplectic reflection algebra associated to S_n. This way, the deformed Harish-Chandra homomorphism becomes nothing but a description of the spherical subalgebra in terms of 'quantum' Hamiltonian reduction.

In the classical limit $k \to \infty$, our construction gives an isomorphism between the spherical subalgebra in H_∞ and the coordinate ring of the Calogero-Moser space. We prove that all simple H_∞-modules have dimension $n!$, and are parametrised by points of the Calogero-Moser space. The algebra H_∞ is isomorphic to the endomorphism algebra of a distinguished rank $n!$ vector bundle on this space.

Prime ideals generated by quantum minors

Kenneth R. Goodearl

Classical studies of determinantal varieties led to the result that determinantal ideals – ideals generated by all the minors of a given size – are prime ideals in the coordinate ring of $n \times n$ matrices. The analogous fact for quantum determinantal ideals was recently established by Lenagan and the speaker. In the classical setting, ideals generated by minors fitting certain combinatorial patterns, such as the ladder determinantal ideals, have also received much attention. Quantum analogs of such ideals, corresponding to many different patterns, must be understood in order to fully describe the prime spectrum of the coordinate ring of quantum matrices. We discuss recent progress in this direction by Lenagan and the speaker. In particular, in the 3×3 case it is now established that every prime ideal invariant under all winding automorphisms can be generated by a set of quantum minors, and explicit generating sets have been determined.

Poisson Orders and Symplectic Reflection Algebras

Iain Gordon
(joint work with K. A. Brown)

We introduce a class of k-algebras, called Poisson orders. Such an algebra is an affine noetherian k-algebra, say A, which is finite over a central subalgebra Z, and equipped with a k-linear map from Z to the k-derivations of A. Examples include enveloping algebras of Lie algebras in positive characteristic, quantum groups at roots of unity and symplectic reflection algebras. We study the geometry of $Max(Z)$, and in particular stratifications which are induced by the derivations. The coarsest stratification is by rank, and the finest by symplectic leaves. We present a theorem which shows that the representation theory of A is constant along the the symplectic leaves of Z (in the case they are algebraic).

We study in more detail the example of a symplectic reflection algebra. As initial data we have a symplectic vector space V, a finite subgroup Γ of the linear symplectic automorphisms of V, a complex parameter t, and an $Ad\Gamma$-invariant function c from the symplectic reflections of Γ to $\mathbb{C}$, written $H_{0,c}$. Writing this as $H_{t,c}$ we prove that in the case $t = 0$, $H_{0,c}$ is a Poisson order. We show there are a finite number of symplectic leaves in $M_c = Max(Cent(H_{0,c}))$, all of which are necessarily algebraic, and apply the above analysis. A particularly important question is whether there are values of c for which M_c is smooth. This is equivalent to M_c being symplectic. To attack this problem we introduce baby Verma modules and give examples of groups Γ where M_c is never the case. On the other hand, if M_c is smooth we present a relationship between the geometry of M_c and the irreducible representations of Γ.

Nilpotent class representations and the irreducible components of Lusztig's nilpotent variety

Lutz Hille

Let Q be a quiver without loops. We denote the quantized version of the enveloping algebra of the negative part of the corresponding Kac-Moody-Lie algebra by U^-. Lusztig has defined varieties $\mathcal{R}(\Pi(Q); d)_0$, also called Lusztig's nilpotent varieties, consisting of nilpotent representations of the pre-projective algebra $\Pi(Q)$ of Q of dimension vector d. It was shown by Kashiwara and Saito that the irreducible components of the various $\mathcal{R}(\Pi(Q); d)_0$, where d runs through all possible dimension vectors of Q, form the crystal of U^-.

The principal aim of this talk is to compute the number of irreducible components of $\mathcal{R}(\Pi(Q); d)_0$ using so-called nilpotent class representations (nc-representations) of Q with dimensions vector d. Informally, a nc-representation assigns to Q and d certain nilpotent classes, so that the generic nc-representations are in natural bijection with the irreducible components of $\mathcal{R}(\Pi(Q); d)_0$.

Furthermore we mention certain applications: the number of irreducible components in the intersection of a nilpotent class with the strictly upper-triangular matrices (in the general linear group) can be computed using nc-representations. Finally we relate $\mathcal{R}(\Pi(Q); d)_0$ for Q affine and d the imaginary root to the exceptional locus in the Kleinian singularity.

Representation theory of cyclics by means of Grassmannians

Birge Huisgen-Zimmermann

Given a finite dimensional algebra A over an algebraically closed field, we introduce and explore Grassmannians parametrizing the isomorphism types of representations with fixed square-free top (the cyclics, in case A is basic) and determine fibre structure (here a remarkably transparent pattern emerges), as well as fibre dimensions. The fibres of the representation map that accompanies any such Grassmannian – they are the orbits under the operation of the automorphism group Aut of a projective cover of the top – are intimately related to the GL-orbits and their closures in the classical module varieties, but more accessible in the Grassmannian setting, due to the presence of a big unipotent radical in the acting group Aut. In particular, the points of the Grassmannian orbit closures (accessible via an Aut-stable affine cover of the Grassmannian, which can be readily obtained from quiver and relations of A) correspond to degenerations in the classical sense. This leads to an overview of the "top-stable" degenerations of cyclics, complete with hierarchy of immediate successors in the degeneration order. One can roughly summarize the picture by saying that the Grassmannian is "as close as possible" to a geometric quotient of the pertinent subvariety of the classical module variety modulo its GL-action. In particular, such a geometric quotient exists if and only if the Grassmannian has a geometric quotient modulo its Aut-action, if and only if all Aut-orbits of the latter are reduced to points, i.e., precisely when the Grassmannian coincides with the geometric quotient of the classical variety, a situation can be characterized in terms of quiver and relations.

Two of the principal applications address the problems of when the category of all finite direct sums of cyclics (resp., that of all finite direct sums of local modules) has finite representation type and of classifying the pertinent cyclics (locals) in the representation-finite case.

Birational classification of orders over surfaces

Colin Ingalls

We classify orders over surfaces up to birational and Morita equivalence. We use the geometric ramification data of a maximal order on a surface to define a class of terminal orders. We compute all possible étale local structures of terminal orders. We use the ideas of Mori's minimal model program for log surfaces to show that terminal orders with non-negative Kodaira dimension have unique minimal models up to Morita equivalence. We describe the possible centres and ramification divisors of the minimal models of orders of negative Kodaira dimension.

Homological identities for differential graded algebras

Peter Jørgensen

Consider the following classical theorems:

(1): The Auslander-Buchsbaum theorem $\operatorname{depth} M = \operatorname{depth} A - \operatorname{pd} M$ for a noetherian local commutative ring A and a finitely generated A-module M with $\operatorname{pd} M < \infty$.

(2): The additivity formula $\operatorname{hd}_k P = \operatorname{hd}_k G + \operatorname{hd}_k X$ for a path connected topological monoid G and a G-Serre fibration $G \longrightarrow P \longrightarrow X$, where k is a field so that $\mathrm{H}_*(G;k)$, $\mathrm{H}_*(P;k)$, and $\mathrm{H}_*(X;k)$ are finite dimensional over k, and where $\operatorname{hd}_k X = \sup\{\, i \mid \mathrm{H}_i(X;k) \neq 0 \,\}$.

The talk shows that both are special cases of an Auslander-Buchsbaum theorem for differential graded algebras (DGAs). (1) arises from viewing A as a DGA, while (2) arises from the DGA $\mathrm{C}_*(G;k)$.

Sheets and the Topology of *Prim* $U(\mathfrak{g})$: $\mathfrak{g}$ semisimple

Anthony Joseph
(joint work with W. Borho)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We define and describe the sheets in the primitive spectrum $Prim\, U(\mathfrak{g})$ of its enveloping algebra $U(\mathfrak{g})$ and compare this to the sheets in the co-adjoint orbit space $\mathfrak{g}^*/G$. The results can be briefly summarized as follows.

Each sheet $\mathcal{S}$ in $Prim\, U(\mathfrak{g})$ has a dense open subset $\mathcal{S}_0$ such that the Goldie rank takes a value $\leq n_{\mathcal{S}}$ on $\mathcal{S}$ with equality on $\mathcal{S}_0$ and such that $\mathcal{S}_0$ is an unbranched covering of an irreducible algebraic variety of dimension $\leq rk\mathfrak{g}$ of degree $d_{\mathcal{S}} \in \{1,2,3\}$. The fibres of the map $\mathcal{S} \mapsto n_{\mathcal{S}}$ are finite.

It is conjectured that if $\mathcal{S}$ is a sheet for which $n_{\mathcal{S}} = 1$, then $\mathcal{S}$ is homeomorphic to a closed subvariety of a sheet in $\mathfrak{g}^*/G$. This is partly settled using a relative Dixmier map; difficult unresolved points being the construction of enough completely prime primitive ideals, well-defineness (in types D, E) and injectivity. Finally to each sheet $\mathcal{S}$ one may associate a simple Weyl group module $V_{\mathcal{S}}$. It is conjectured that $\mathcal{S}$ is a union of at most $dim\ V_{\mathcal{S}}$ algebraic varieties.

Twisted homogeneous coordinate rings over a commutative ring

Dennis S. Keeler

Let X be a scheme, proper over a commutative noetherian ring A. We introduce the concept of an ample sequence of invertible sheaves on X and generalize the most important equivalent criteria for ampleness of an invertible sheaf. We also prove the Theorem of the Base for X and generalize Serre's Vanishing Theorem. We then generalize results for twisted homogeneous coordinate rings which were previously known only when X was projective over an algebraically closed field. Specifically, we show that the concepts of left and right σ-ampleness are equivalent and that the associated twisted homogeneous coordinate ring must be noetherian.

Derived equivalences and higher structures on the Hochschild complex

Bernhard Keller

The Hochschild cohomology groups of an (associative, unital) algebra A over a field k may be interpreted as morphism spaces in the derived category of A-bimodules. This interpretation shows that Hochschild cohomology is preserved under derived equivalence as an algebra, a result due to D. Happel and J. Rickard. We show that the Gerstenhaber bracket on the Hochschild complex is also preserved by proving the following stronger statement: If X is a complex of A-B-bimodules whose associated total derived functor is an equivalence from the derived category of A to that of B, there is a canonical isomorphism Φ_X in the homotopy category of B_∞-algebras from the Hochschild complex of A to that of B.

The McKay Correspondence in Dimension 3

Alastair King

For a finite subgroup G of $\mathrm{SL}(V)$, the McKay correspondence relates the geometry/topology of the non-commutative space (or stack) $V_G = Spec(k[V] * G)$ to that of a commutative space Y, which is a crepant (morally: minimal) resolution of the quotient V/G. This is classical in the case $\dim V = 2$, where an observation of McKay was interpreted by Gonzalez-Sprinberg & Verdier. By looking closely at this classical case in parallel with the generalisation to $\dim V = 3$ by Bridgeland, King & Reid, we see how much remains to be understood.

Frobenius-Schur indicators for representations of semisimple Hopf algebras

Susan Montgomery
(joint work with Y. Kashina, G. Mason)

Let H be a finite-dimensional semisimple Hopf algebra over an algebraically closed field k of characteristic not 2; if char $k > 0$, assume also that H is co-semisimple. Let V be a simple (left) H-module with character χ, and let Λ be an integral of H with $\varepsilon(\Lambda) = 1$. Then the *indicator* of V is defined to be $\sum \chi(\Lambda_1 \Lambda_2)$. This specializes to the usual Frobenius-Schur indicator when $H = kG$ for G a finite group, and as in the classical situation determines whether or not V is self-dual, and whether, in the self-dual case, V has a non-degenerate bilinear H-invariant symmetric or skew form, depending on whether the indicator is non-zero, is $+1$, or -1, respectively . These facts are work of [M - Linchenko, JART 2000]. In the present talk we explicitly compute the indicator for representations of Hopf algebras which are abelian extensions, that is (in our case) of the form

$$(kG)^* \subset H \to kL$$

where G, L are groups. We are particularly concerned with when the indicator is 0 for all non-zero simple modules (this will be the case when dim H is odd), or, at the other extreme, when it is always one. In particular, we prove that this is the case for $D(S_n)$, the Drinfeld double of the symmetric group.

Quantum groups and cohomology of quiver moduli

Markus Reineke

This talk is a report on the preprint math.QA/0204059. Methods of Harder and Narasimhan from the theory of moduli of vector bundles are applied to moduli of quiver representations. Using the Hall algebra approach to quantum groups, an analog of the Harder-Narasimhan recursion is constructed inside the quantized enveloping algebra of a Kac-Moody algebra. This leads to a canonical orthogonal system, the HN system, in this algebra. Using a resolution of the recursion, an explicit formula for the HN system is given. As an application, explicit formulas for Betti numbers of the cohomology of quiver moduli are derived, generalizing several results on the cohomology of quotients in 'linear algebra type' situations.

Hereditary categories

Idun Reiten

The general setting for the talk was Ext–finite (connected) hereditary abelian categories $\mathcal{H}$ over a field k (which in the first part was assumed to be algebraically closed). Additional interesting conditions are having a tilting object, almost split sequences, or Serre duality. In the noetherian case these having a tilting object are mod Λ for a finite dimensional hereditary k–algebra Λ or coh($\mathbb{X}$) for a weighted projective line $\mathbb{X}$ in the sense of Geigle–Lenzing, as proved by Lenzing. Happel has shown that any $\mathcal{H}$ with a tilting object is derived equivalent to one of the noetherian ones. (A generalization to arbitrary fields is given in work with Happel.) Further we discussed the classification of the noetherian $\mathcal{H}$ with Serre duality from work with Van den Bergh [JAMS, April 2002]. We also mentioned work with Kleiner on connections with existence of almost split sequences for co-modules over path co-algebras.

In the second part we talked about recent work with Lenzing on noetherian $\mathcal{H}$ with Serre duality over an arbitrary field k. For a locally finite quiver Q where all paths are finite there is a naturally associated $\mathcal{H} = \mathcal{H}_Q$ with Serre duality and no nonzero projective objects. Then $\mathcal{H}_Q$ is noetherian if and only if the underlying (valued) graph of Q has an additive function. We gave various characterizations of these noetherian $\mathcal{H}_Q$ amongst all noetherian $\mathcal{H}$ with Serre duality and no nonzero projective object. Some of these equivalent conditions are for example that there is no cycle of nonzero maps between indecomposable objects of infinite length, or that each such object is exceptional.

Continued fractions, tilting modules and the construction of large indecomposable algebraically compact modules

Claus M. Ringel

One conjectures that a finite-dimensional k-algebra (k a field) is tame if and only if any non-zero algebraically compact module has an indecomposable direct summand. A complete classification of the indecomposable algebraically compact modules is known only in special cases; in addition to the well-known classes of finite dimensional, generic, Prüfer and adic modules, only indecomposable algebraically compact modules have been constructed which are given by combinatorial data ($\mathbb{N}$-word, $\mathbb{Z}$-words, related to paths in the quiver), namely in the case of special biserial algebras. The lecture has outlined a new kind of construction, valid for tubular algebras with 3 simple modules, starting with the continued fraction expansion of a positive non-rational real number and forming direct limits of suitable embeddings and inverse limits of suitable projections. The method is based on the observation that for these algebras the tilting modules correspond to Farey triples and provides a construction of all the exceptional modules parallel to the iterated mediant description of the positive rational numbers. For any real number w we obtain in this way a large rigid system of infinite dimensional modules with slope w. Its cardinality depends on the given base field k.

Noncommutative Mukai equivalence

Aidan Schofield

Let C be a smooth projective curve and E a vector bundle such that $\mathrm{End}(E) = k$. Let $E^{\perp} = \{F \in \mathrm{Coh}(C) \mid \mathrm{Hom}(E,F) = 0 = \mathrm{Ext}(E,F)\}$ the full subcategory of quasi-coherent sheaves on C right perpendicular to E.

Theorem $E^{\perp} = \mathrm{Mod}R$ for a suitable ring R. Under the equivalence, the vector bundles of rank r in $E^{\perp}$ correspond to the representations of rank r of R.

This is proved by noting that if $p \in C$ and S_p is the simple sheaf at p, then the functor $\mathrm{Ext}(S_p, -)|_{E^{\perp}}$ is exact and faithful and is also representable. All of these follow by considering a short exact sequence $0 \to E_p \to E \to S_p \to 0$ from which it follows that $\mathrm{Ext}_C(S_p, -) \simeq \mathrm{Hom}_C(E_p, -)$ on $E^{\perp}$ and since there is an induction functor $\uparrow \mathrm{QCoh}(C) \to E^{\perp}$, $\mathrm{Hom}_{E^{\perp}}(E_p \uparrow, -) \simeq \mathrm{Hom}_C(E_p, -) \simeq \mathrm{Ext}_C(S_p, -)$ on $E^{\perp}$. We take R_p to be $\mathrm{End}(E_p \uparrow)$.

Irreducible components of varieties of modules

Jan Schroer

(joint work with William Crawley-Boevey)

Let k be an algebraically closed field, and let A be a finitely generated k-algebra (associative, with 1). By $\mathrm{mod}_A^d(k)$ we denote the variety of d-dimensional A-modules. Given irreducible components $C_1 \subseteq \mathrm{mod}_A^{d_1}(k)$ and $C_2 \subseteq \mathrm{mod}_A^{d_2}(k)$ let

$$\mathrm{ext}_A^1(C_1, C_2) = \min\{\dim \mathrm{Ext}_A^1(M_1, M_2) \mid (M_1, M_2) \in C_1 \times C_2\}.$$

For irreducible components $C_i \subseteq \mathrm{mod}_A^{d_i}(k)$, $1 \le i \le t$, we consider all modules of dimension $d = d_1 + \cdots + d_t$, which are of the form $M_1 \oplus \cdots \oplus M_t$ with the M_i in C_i, and we denote by $C_1 \oplus \cdots \oplus C_t$ the corresponding irreducible subset of $\mathrm{mod}_A^d(k)$. The following theorem generalizes results of Kac and Schofield on representations of quivers.

Theorem. *If $C_i \subseteq \mathrm{mod}_A^{d_i}(k)$, $1 \le i \le t$, are irreducible components and $d = d_1 + \cdots + d_t$, then $\overline{C_1 \oplus \cdots \oplus C_t}$ is an irreducible component of $\mathrm{mod}_A^d(k)$ if and only if $\mathrm{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.*

Homological Properties of Noetherian Affine Hopf Algebras

Wu Quanshui

(joint work with James Zhang)

Many Hopf algebras, for example, enveloping algebras of finite dimensional restricted Lie algebras in positive characteristic, group algebras of finitely generated abelian-by-finite groups, quantised enveloping algebras of finite dimensional semisimple Lie algebras at roots of unity and quantised function algebras of simply connected semisimple Lie groups at roots of unity, are Noetherian affine PI Hopf algebras.

We show that

Theorem 1: Every Noetherian affine PI Hopf algebra has finite injective dimension.

This answers a question of K. A. Brown (Contemp. Math. 229, 1998).

Theorem 2: Let H be an involutory Hopf algebra over a field of characteristic 0. If H is either a semi-prime Noetherian affine PI algebra or is a finite module over its affine centre, then it is Auslander regular, Cohen-Macaulay and is a finite direct sum of prime rings.

This partially answers another question of K. A. Brown (Contemp. Math. 229, 1998)

Perverse sheaves and dualizing complexes over noncommutative ringed schemes

Amnon Yekutieli
(joint work with James J. Zhang)

In this talk I will discuss an attempt at Grothendieck Duality on noncommutative spaces. Since in the case of affine noncommutative spaces (i.e. rings) Grothendieck Duality is pretty well understood, and on the other hand we don't even know what is a noncommutative space in general, we consider an intermediate case: a noncommutative space Y that's an affine fibration over a commutative scheme X. That's a fancy way to say that $Y = (X, A)$ where A is a sheaf of quasi coherent noncommutative rings on X. We call (X, A) a quasi coherent ringed scheme. As usual in such circumstances, we encounter the problem of gluing. On each affine open set U in X we have a rigid dualizing complex for $A|_U$ from the ring construction, and these are compatible on intersections. But how to glue these complexes globally? One should note that Cousin complexes, the solution devised by Grothendieck for gluing dualizing complexes, will not work in the noncommutative world due to well known obstructions. Instead we choose to use perverse sheaves. This is a gluing method invented by Bernstein-Beilinson-Deligne-Gabber in the context of geometry of singular spaces. We discovered that the Auslander condition of dualizing complexes over noncommutative rings (a very algebraic property) is exactly what is needed to define perverse modules over a noncommutative ring. And furthermore using a few nice features of the theory, we can also extend the definition from rings to noncommutative ringed schemes. Finally it turns out that rigid dualizing complexes are themselves perverse sheaves of bimodules (namely on the product X^2), so we can glue the local pieces together. I will explain what are dualizing complexes and what they are good for (concentrating on the noncommutative side). Then I'll discuss perverse sheaves, the Auslander condition and how they interact. I'll finish by sketching our construction.

Conformal algebras of small Gelfand–Kirillov dimension

Efim Zelmanov

Let F be an algebraically closed field of zero characteristic. A conformal algebra C is a module over a polynomial algebra $F[\partial]$, equipped with a countable system of binary bilinear operations $0_n, n \geq 0$, such that for arbitrary $a, b \in C$, $n \geq 0(1), \partial(a \circ_n b| = \partial a \circ_n b + a \circ_n \partial b, (2)(\partial a)(n) + na(n-1) = 0(3)$, there exists an integer $N = N(a,b) \geq 1$ such that $a \circ_k b = 0$ for $n \geq N$.

For any conformal algebra C there exists a (universal) coefficient algebra Coeff (C) such that C can be realized as a conformal algebra of formal distributions over Coeff (C), $a \hookrightarrow \sum_{i \in Z} a(i) z^{-i-1}, a(i) \in \text{Coeff}(C)$. We say that C is associative (Lie) if Coeff (C) is an associative (Lie) algebra.

D'Andrea and Kac developed a structure theory of associative and Lie conformal algebras of finite type (that is ${}_{F[\partial]}C$ is a finitely generated module).

Let $W = \langle t, t^{-1}, d/dt \rangle$ be the algebra of differential generators, $A \in M_n(F)$, $A \otimes J_k = \sum_{i \in \mathbb{Z}} \Big(A \otimes t^i \big(\frac{d}{dt}\big(^k\Big) z^{-i-1}$. The formal distributions of the type $A \otimes J_k$ are pairwise local. Let $M_n(W)$ denote the associative conformal algebra of formal distributions generated by $A \otimes J_k$'s. If $P = \mathrm{diag}\Big(f_1\big(\frac{d}{dt}\big) \ldots, f_n\big(\frac{d}{dt}\big)\Big)$, $f_i \neq 0$, then $M_n(W)P$ denotes the conformal algebra that corresponds to the left ideal generated by P.

An element of an associative conformal algebra C is called an identity if $e \circ_k e = 0, U \geq s; e \circ_0 a = a$ for an arbitrary element $a \in C$.

Theorem (A. Retakh) Let C be a simple finitely generated associative conformal unital algebra of GKdim1. Then $C \cong M_n(W)$.

An element $e \in C$ is called an idempotent if $e \circ_k e = 0, U \geq 1; e \circ_0 e = e$.

Every idempotent gived rise to a Peirce decomposition; idempotents lift modulo nilpotent ideals etc.

Theorem Let C be a simple finitely generated associative conformal algebra of GKdim1, containing an idempotent. Then $P \cong M_n(W)P$, where $P = (1, f_2, \ldots, f_n)$.

Theorem Let C be a Lie conformal finitely, generated algebra of GKdim1 which contains $\mathrm{cur}(sl_2)$. Then $\cong (M_n(W)P)^{(-)}$ or $C \cong$ a conformal algebra of their symmetric elements of $M_n(W)P$ with respect to an involution.

Involutions in $M_n(W)P$ we classified by Boyallian, Kac, Liberati.

A-infinity Algebras

James Zhang

(joint work with Lu, Palmieri and Wu)

Let A be a connected graded locally finite A_∞-algebra and let $A^\#$ be the dual A_∞-algebra defined to be $\oplus_{inZ} Ext_A(k,k)$ where k is the trivial module over A. Let $D_{per+k}(A)$ denote the full triangulated subcategory of $D(A)$ generated by A and k.

Theorem: Let A be as above and let $E = A^\#$.

(a) $(A^\#)^\# \cong A$.

(b) $D_{per+k}(A)$ and $D_{per+k}(E)$ are contravariant equivalent.

(c) Suppose ${}_A k$ is a small object in $D(A)$ and k_E is a generator for $D(E)$. Then $D(A)$ is equivalent to $D(E)$.

This generalizes a result of Beilinson-Ginsburg-Soergel about Koszul algebras.

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