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Curvature and Dispersion Effects in Nonlinear Partial Differential Equations

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Introduction

The conference was organized by Carlos E. Kenig (Chicago), Herbert Koch (Dortmund) and Daniel Tataru (Berkeley) with participants from France, Germany, Japan, Spain and the USA. The talks represented fast recent developments ranging from aspects of modelling and asymptotic equations, over studies of particular prominent dispersive equations, precise asymptotics of blow up to the study of rough initial data in particular for nonlinear wave equations including the Einstein equations.

The contributions of J.C. Saut and F. Merle give an overview on some aspects of modelling and asymptotics of blow up for the Schrödinger equation.

The organizers and the participants of this conference are grateful to the Oberwolfach institute for providing a stimulating atmosphere and a very pleasant environment for the conference.

Abstracts

Global estimates in space-time for some classes of oscillatory integrals

Matania Ben-Artzi
(joint work with H. Koch and J. C. Saut)

We give global estimates for three classes of oscillatory integrals, related to nonlinear evolution equations:

I)

$$I_\varepsilon(x,t) \int_{\mathbb{R}^d} e^{it(|\xi|^4-\varepsilon|\xi|^2)+ix\xi} d\xi, \qquad \varepsilon = \pm 1, 0,$$

which is the fundamental solution for

$$i\psi_t + \Delta^2\psi + \varepsilon\Delta\psi = 0 \qquad x \in \mathbb{R}^d.$$

Theorem 1.

a) $|\partial^\alpha I_0(x,t)| \le ct^{-\frac{d+|\alpha|}{4}}(1+\frac{|x|}{t^{\frac{1}{4}}})^{-\frac{1-|\alpha|}{3}}$, $\alpha \in \mathbb{N}^d$,

b) *(short time or long time $t \le 1$ or $\frac{|x|}{t} \ge 1$)*
$|\partial^\alpha I_0(x,t)| \le ct^{-\frac{d+|\alpha|}{4}}(1+\frac{|x|}{t^{\frac{1}{4}}})^{-\frac{1-|\alpha|}{3}}$, $\alpha \in \mathbb{N}^d$,

c) *(long time, large space, $t \ge 1$, $|x| \le t$)*
$|\partial^\alpha I_{-1}(x,t)| \le ct^{-\frac{d+|\alpha|}{2}}(1+t^{\frac{1}{2}}|x|)^{|\alpha|}$, $\alpha \in \mathbb{N}^d$,
$|\partial^\alpha I_1(x,t)| \le ct^{-\frac{1}{2}}(1+|x|)^{-\frac{d-1}{2}} + ct^{-\frac{d}{2}+\frac{1}{6}}\chi(\frac{|x|}{t})$, $\chi = \chi_{[\frac{2\sqrt{6}}{9}-\delta, \frac{2\sqrt{6}}{9}+\delta]}$, $\delta > 0$.

II)

$$I_\alpha(x,t) = \int_0^\infty s^\alpha e^{i(tp(s)-sx)} ds, \;\; t > 0, \;\; 0 \le \alpha \le m-2, \;\; p(s) = s^m + a_{m-1}s^{m-1} + ..$$

Theorem 2.

a) *Large time $t \ge \delta$: $|I_\alpha(x,t)| \le ct^{-\frac{1}{n}}$.*

b) *Short time $0 < t \le \delta$: $|I_\alpha(x,t)| \le ct^{-\frac{\alpha+1}{m}}$.*

III) Full estimates for

$$\int_{\mathbb{R}^2} e^{i(tp(s)+sx)} ds, \qquad x \in \mathbb{R}^2,$$

where p is any real polynomial of third order.

Smoothing effects of dispersive equations

Hiroyuki Chihara

This talk is concerned with well-posedness and smoothing effects of solutions to the initial value problem for dispersive-type pseudo-differential equations of the form

(1) $$D_t u - a(x, D_x)u = f(t,x) \quad \text{in} \quad \mathbb{R}^{1+n},$$

(2) $$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^n,$$

where $u(t,x)$ is a complex-valued unknown function of $(t,x) \in \mathbb{R}^{1+n}$, $i = \sqrt{-1}$, $D_t = -i\partial/\partial t$, $D_x = -i\partial/\partial x$, and $a(x, D_x)$ is a pseudo-differential operator of order $m \ge 2$, whose principal symbol is real-valued. Our main results are the following:

1. If the principal symbol is real principal type, the lower order term satisfies some integrability condition, and the Hamiltonian flow generated by the principal symbol is never trapped, then the IVP (1)-(2) is L^2-well-posed, and the local smoothing effect occurs, that is, the solution u becomes $(1-\Delta)^{(m-1)/4}$-smoother than u_0 and $(1-\Delta)^{(m-1)/2}$-smoother than f.
2. If the Hamiltonian flow has attractive points in the cotangent bundle $T^\star\mathbb{R}^n$, then the micro-local smoothing effect fails in the conic neighborhood of the attractive points.

Ill-posedness for defocusing NLS and applications

James E. Colliander

(joint work with M. Christ and T. Tao)

I describe a construction of an approximate solution to the cubic defocusing nonlinear Schrödinger equation inspired by the pseudo-conformal transformation. The solution obtained, first found by Ozawa, contains a $\log t$ term in its phase which distinguishes it from the linear solutions as $t \to \infty$. These modified wave operator solutions are combined with scaling and galilean invariance to prove ill-posedness in L^2-based Sobolev spaces $H^s,\ s < 0$. A parabolic dispersion approximation to solutions of the mKdV equation via solutions of cubic defocusing NLS allows us to transport the ill-posedness result to the mKdV setting in the range $s < \frac{1}{4}$. Finally, the Miura transform transports these ill-posedness properties to real-valued KdV in the range $s < -\frac{3}{4}$. The results described here complement earlier ill-posedness examples obtained in collaborations among Birnir, Kenig, Ponce, Svanstedt and Vega and reveal that the H^s local well-posedness theory for the generalized KdV equations is essentially complete.

Some recent progress on the problem of strong cosmic censorship

Mihalis Dafermos

This talk describes a characteristic initial value problem for the Einstein-Maxwell-scalar field equations under the assumption of spherical symmetry. Initial data are defined on an event horizon and a conjugate characteristic ray. For a suitable class of such data, the Reissner-Nordstrom Cauchy horizon is found to be stable in the C^0-sense and unstable in C^1. The initial data corresponds to that which is thought to be induced by the gravitational collapse of a scalar field. The result is then related to the strong cosmic censorship conjecture in general relativity.

A local well-posedness result for the quasi-linear wave equation in $\mathbb{R}^{2+1}$

Dan Geba

In this paper we consider the following Cauchy problem

$$(1)\qquad \begin{gathered} \partial_t^2\phi - g^{ij}(\phi)\partial_i\partial_j\phi = N(\phi,\partial\phi), \\ \phi(0,x) = \phi_0(x),\ \partial_t\phi(0,x) = \phi_1(x),\ (t,x)\in\mathbb{R}^{1+n}. \end{gathered}$$

where the metric $g(u) = (g_{ij}(u))_{i,j=1,\dots,n}$ is a smooth, uniformly positive definite matrix and the nonlinearity $N(\phi,\partial\phi)$ is quadratic in $\partial\phi$. Assuming that the initial data satisfies

$(\phi_0, \phi_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, we will be interested in the local well-posedness of the initial value problem (1), e.g. for what values of s there exists a unique local solution

$$\phi \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n)).$$

There are two aspects that one should consider when studying equation (1). The first one is scaling, meaning that (1) is invariant under the transformation

$$\phi \to \phi_\lambda = \phi(\frac{t}{\lambda}, \frac{x}{\lambda}).$$

The Sobolev space $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ is conserved under this scaling, which, heuristically, implies the restriction

$$s \geq \frac{n}{2}$$

The other aspect is a geometrical one and it has to do with the concentration of null rays. According to Lindblad, in order to avoid focusing we have to take

$$s > \frac{n+5}{4}$$

These considerations lead us to the following conjecture:
Conjecture *The initial value problem (1) in $\mathbb{R}^{1+n}$ is locally well-posed in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, for s satisfying:*

$$s > \max\{\frac{n}{2}, \frac{n+5}{4}\} \tag{2}$$

Recently, Klainerman and Rodnianski have proved this conjecture in the particular case of the Einstein vacuum equations ($n = 3$), while Smith and Tataru have resolved it in the case of the dimensions $n = 2, 3$ for general systems of quasi-linear wave equations.

We investigate the case when $n = 2$. Our work extends the geometric methods pioneered by Klainerman and Klainerman-Rodnianski for the same problem in $\mathbb{R}^{3+1}$. The main new ingredient of our argument is the use of two new vectorfields, the scaling vectorfield S and the angular momentum vectorfield Ω, which complement the decay information provided by the Morawetz vectorfield K. Compared to the proof of Smith and Tataru for this problem, this is a totally different approach.

Our main result is the following:

Theorem 1. *The Cauchy problem (1) with the metric g^{ij} satisfying the assumptions below, is locally well posed in $H^s \times H^{s-1}$ for any $s > s_0 = \frac{7}{4} + \frac{5-\sqrt{22}}{4}$. Moreover, ϕ satisfies the following Strichartz type estimate*[1]

$$\|\partial\phi\|_{L^4_{[0,T]}L^\infty_x} \leq c\, T^{s-s_0} \|\phi[0]\|_{H^s}. \tag{3}$$

Conditions satisfied by the coefficients g^{ij} : *The metric $g^{ij} = g^{ij}(z)$ is smooth and uniformly positive definite with respect to bounded values of the parameter $z \in \mathbb{R}$. Namely, there exist positive constants M_0, A_0 such that for a sufficiently large integer k*

$$\sup_{|z|\leq A_0} |\left(\tfrac{d}{dz}\right)^l g^{ij}| \leq M_0, \quad \forall\ 0 \leq l \leq k \tag{4}$$

$$M_0^{-1}|\xi|^2 \leq g^{ij}(z)\xi_i\xi_j \leq M_0|\xi|^2, \quad \forall |z| \leq A_0, \tag{5}$$

$$N(\phi, \partial\phi) = \textstyle\sum_{\alpha,\beta} N^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi, \quad \sup_{|z|\leq A_0} |\left(\tfrac{d}{dz}\right)^l N^{\alpha\beta}(z)| \leq M_0, \quad \forall\ 0 \leq l \leq k. \tag{6}$$

[1] We denote the initial data for the equation (1) by $\phi[0]$ and say that $\phi[0] \in H^s$ if $(\phi_0, \phi_1) \in H^s \times H^{s-1}$.

Long range scattering and modified wave operators for the wave-Schrödinger system

Jean Ginibre
(joint work with G. Velo)

We study the theory of scattering for the system consisting of a Schrödinger equation and a wave equation with a Yukawa type coupling in space dimension 3. We prove in particular the existence of modified wave operators for that system with no size restrictions on the asymptotic states and we determine the asymptotic behaviour in time of solutions in the range of the wave operators. The method consists in solving the wave equation and substituting the result into the Schrödinger equation, which then becomes both nonlinear and non local in time. The Schrödinger function is parametrized in terms of an amplitude and a phase satisfying a transport/Hamilton-Jacobi system, and the Cauchy problem for that system, with infinite initial time and prescribed asymptotic behaviour determined by the asymptotic state, is solved by an energy method, thereby leading to solutions of the original system with prescribed asymptotic behaviour in time.

Local well-posedness for NLS below L^2: Quartic nonlinearities in one space dimension

Axel Grünrock

The Fourier restriction norm method is applied to prove local well-posedness of the Cauchy problem

$$u_t - iu_{xx} = N(u), \qquad\qquad u(0) = u_0 \in H^s_x$$

for

1. $N(u) = |u|^4$, $s > -\frac{1}{8}$,
2. $N(u) = u^4,\ u^3\overline{u},\ u\overline{u}^3,\ \overline{u}^4$, $s > -\frac{1}{6}$.

New tools are: A bilinear estimate for solutions of the linear Schrödinger equation and a trilinear refinement of the onedimensional L^6-Strichartz-estimate. Similar arguments apply to the gKdV-3 equation and lead to local well-posedness of the Cauchy problem for $s > -\frac{1}{6}$ and to global well-posedness for $s \geq 0$.

Almost global existence for quasi-linear wave equations in three space dimensions

Markus Keel
(joint work with H. Smith and C. Sogge)

We discuss almost global existence for solutions of quadratically quasi-linear systems of wave equations in three space dimensions. The approach here uses only the classical invariance of the wave operator under translations, spatial rotations, and scaling. Using these techniques we can handle wave equations in Minkowski space or Dirichlet-wave equations in the exterior of a star-shaped obstacle. We can also apply our methods to systems of quasi-linear wave equations in Minkowski space having different wave speeds.

The problem of evolution for the Einstein equations in general relativity
Sergiu Klainerman

We start by a review of the Cauchy problems for the Einstein equations in vacuum. We formulate a few simplified open problems such as the $2+1$ reduced Einstein equations, Wave maps, Yang Mills. We then discuss the issue of optimal well posedness, some of the recent results for Wave maps and Yang-Mills and end with a discussion of the recent result obtained in collaboration with I. Rodnianski concerning H^s-well-posedness, $s > 2$, for the general $3+1$ reduced Einstein equations.

Existence and description of solutions blowing up in finite time in the energy space for the critical generalized KdV equation
Yvan Martel
(joint work with F. Merle)

We present new results in collaboration with Frank Merle concerning a class of solutions that blow up in finite time in the energy space for the generalized critical KdV equation $u_t + (u_{xx} + u^5)_x = 0$. Blow up in finite or infinite time is first proved for solutions that are close to the family of solitons and that have negative energy. This result relies partly on a classification of bounded solutions around the family of solitons. Second, imposing an additional decay assumption on the right in space on the solution, we are able to study directly the dynamics of the blow up. This study allows us to prove blow up in finite time and an upper bound on the blow up rate for a subsequence of time.

Blow up Dynamic and Upper Bound on the Blow up Rate for critical nonlinear Schrödinger Equation
Frank Merle
(joint work with P. Raphael)

We consider the critical nonlinear Schrödinger equation

$$(NLS) \qquad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^N \to \mathbb{C} \end{cases}$$

with $u_0 \in H^1 = H^1(\mathbb{R}^N)$, in dimension $N \geq 1$. This equation is locally well-posed in H^1. The problem we address is the one of formation of singularities for solutions to (NLS). Note that from the conservation of the mass and the energy (from the Hamiltonian formulation) and Gagliardo-Nirenberg inequality, the power of the nonlinearity is the smallest one for which blow-up may occur. We will see that this criticality makes the problem global. In the energy space H^1, (NLS) admits three conservation laws: L^2-norm, Energy, Momentum, and four fundamental symmetries: Space-time translation, Phase, Scaling and Galilean invariances. At the critical power, special regular solutions play an important role. They are the so called solitary waves and are of the form $u(t,x) = e^{i\omega t}W_\omega(x)$, $\omega > 0$, where W_ω solves

$$\Delta W_\omega + W_\omega |W_\omega|^{\frac{4}{N}} = \omega W_\omega. \tag{1}$$

Equation (1) is a standard nonlinear elliptic equation. There is a unique positive solution up to translation $Q_\omega(x)$. Q_ω is in addition radially symmetric. Letting $Q = Q_{\omega=1}$, scaling properties and Pohozaev identity yield $|Q_\omega|_{L^2} = |Q|_{L^2}$ and $E(Q_\omega) = 0$. In particular, none of the three conservation laws in H^1 sees the variation of size of the W_ω stationary solutions. These two facts are deeply related to the criticality of the problem. Weinstein used the variational characterization of the ground state solution Q to (1) to derive the explicit constant in the Gagliardo-Nirenberg inequality

$$\forall u \in H^1 \quad , \quad \frac{1}{2+\frac{4}{N}} \int |u|^{\frac{4}{N}+2} \leq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(\frac{\int |u|^2}{\int Q^2} \right)^{\frac{2}{N}} , \tag{2}$$

so that for $|u_0|_{L^2} < |Q|_{L^2}$, the solution is global in H^1. In addition, blow up in H^1 has been proved to be equivalent to "blow up" for the L^2 theory from concentration in L^2.

On the other hand, for $|u_0|_{L^2} \geq |Q|_{L^2}$, blow up may occur. Indeed, for this special power nonlinearity, (NLS) admits another symmetry which is *not* in the energy space H^1, the so called pseudo conformal transformation: if $u(t,x)$ solves (NLS), then so does

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} \overline{u}(\frac{1}{t}, \frac{x}{t}) e^{i\frac{|x|^2}{4t}}.$$

This additional symmetry yields for $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$:

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 = 4 \frac{d}{dt} Im(\int x \nabla u \overline{u})(t,x) = 16 E(u_0). \tag{3}$$

Now since $E(Q) = 0$ and $\nabla E(Q) = -Q$, there exists $u_{0\epsilon} \in \Sigma$ with $|u_{0\epsilon}|_{L^2} = |Q|_{L^2} + \epsilon$ and $E(u_{0\epsilon}) < 0$, and the corresponding solution must blow up from viriel identity (3).

The case of critical mass $|u_0|_{L^2} = |Q|_{L^2}$ has been studied by Merle. The pseudo-conformal transformation applied to the stationary solution $e^{it}Q(x)$ yields an explicit solution

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q(\frac{x}{t}) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \tag{4}$$

which blows up at $T = 0$. Note that $|S(t)|_{L^2} = |Q|_{L^2}$. It turns out that $S(t)$ is the unique minimal mass blow up solution in H^1 in the following sense: let $u(-1) \in H^1$ with $|u(-1)|_{L^2} = |Q|_{L^2}$, and assume that $u(t)$ blows up at $T = 0$, then $u(t) = S(t)$ up to the symmetries of the equation.
Another fact suggested by numerical simulations, see Landman, Papanicolaou, Sulem-Sulem, is the existence of solutions blowing up as

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\ln(|\ln|t||)}{|t|}}. \tag{5}$$

These appear to be stable with respect to perturbation of the initial data. In this frame, for $N = 1$, Perelman proves the existence of one solution which blows up according to (5). Note that such solutions are stable with respect to perturbation of the initial data from numerics, but are known to be structurally unstable. Indeed, in dimension $N = 2$, if we consider the next term in the physical approximation leading to (NLS), we get Zakharov equation, and finite time blow up solutions to Zakharov equation are known to satisfy $|\nabla u(t)|_{L^2} \geq \frac{C}{|T-t|}$.

Our approach to study blow up solutions to (NLS) is based on a qualitative description of the solution. We focus on the case when the nonlinear dynamic plays a role and interacts with the dispersive part of the solution. This last part will be proved to be small in L^2 for initial conditions which satisfy

$$\int Q^2 < \int |u_0|^2 < \int Q^2 + \alpha_0 \quad \text{and} \quad E(u_0) < 0 \tag{6}$$

where α_0 is small. Indeed, under assumption (6), from the conservation laws and the variational characterization of the ground state Q, the solution $u(t,x)$ remains close to Q in H^1 up to scaling and phase parameters, and also translation in the non radial case. We then are able to define a regular decomposition of the solution of the type

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\epsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}$$

where $|\epsilon(t)|_{H^1} \leq \delta(\alpha_0)$ with $\delta(\alpha_0) \to 0$ as $\alpha_0 \to 0$, $\lambda(t) > 0$ is a priori of order $\frac{1}{|\nabla u(t)|_{L^2}}$, $\gamma(t) \in \mathbf{R}$, $x(t) \in \mathbf{R}^N$.

In particular, we derive a control from *above* of the blow rate for such solutions. More precisely, we claim the following assuming a spectral property on a potential related to Q and checked in dimension $N = 1$.

Theorem 1. *There exists $\alpha^* > 0$ and C^* such that the following is true. Let $u_0 \in H^1$ such that $0 < \int |u_0|^2 - \int Q^2 < \alpha^*$, $E_0 < \frac{1}{2}\left(\frac{|Im(\int \nabla u_0 \overline{u_0})|}{|u_0|_{L^2}}\right)^2$. Let $u(t)$ be the corresponding solution to (), then:*

(i) $u(t)$ blows up in finite time $T < +\infty$ in H^1.

(ii) Moreover, there holds for t close to T, $|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\ln(|\ln(T-t)|)}{T-t}\right)^{\frac{1}{2}}$.

Comments on the result

1. *Blow up rate*: Assume that u blows up in finite time. By scaling properties, a known *lower* bound on the blow up rate is $|\nabla u(t)|_{L^2} \geq \frac{C^*}{\sqrt{T-t}}$.

The problem here is to control the blow up rate from *above*. Our result is the first of this type for critical NLS. No upper bound on the blow up rate was known, not even of exponential type. Note indeed that there is no Lyapounov functional involved in the proof of this result, and that it is purely a dynamical one with all dynamical controls exhibited in H^1 and not in Σ.
We first prove an upper bound on the blow up rate as $|\nabla u(t)|_{L^2} \leq \frac{C^*}{\sqrt{|E_0|(T-t)}}$. This bound is optimal for NLS in the sense that there exist blow up solutions with this blow up rate, explicitly $S(t)$ of (4). Note nevertheless that these solutions have strictly positive energy.

In our setting of strictly negative energy initial conditions, no solutions of this type is known, and we indeed are able to improve the upper bound by excluding any polynomial growth between the pseudo-conformal blow up and the scaling estimate. It says in particular that there is a large open set of initial data which blow up with a control from above of the blow-up rate suggested by numerics. This bound is conjectured to be optimal.

We would like to point out that this improvement of blow up rate control heavily relies on algebraic cancellations deeply related to the degeneracy of the linear operator around Q which are unstable with respect to "critical" perturbations of the equation. Indeed, recall for example that all strictly negative energy solutions to Zakharov equation satisfy the lower bound $|\nabla u(t)|_{L^2} \geq \frac{C}{|T-t|}$. On the other hand, we expect the first argument to be structurally stable in a certain sense.

2. *Blow up result*: In the situation $\int |u_0|^2 \leq \int |Q|^2 + \alpha_0$, we show that blow up is related to a local in space information, and we do not need the additional assumption $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$.

Modified wave operators for Hartree equations

Kenji Nakanishi

We study the Hartree equation of the form

$$2i\dot{u} - \Delta u + V(u)u = 0, \tag{1}$$

where the potential $V(u)$ depends on the charge density as

$$V(u) = \lambda |x|^{-\nu} * |u|^2, \tag{2}$$

where $\lambda \in \mathbb{R}$ and $\nu > 0$ are given constants, and $*$ denotes the spatial convolution. It is well known that the potential has long-range effect when $\nu \leq 1$, and any nontrivial solution can not be approximated by the free evolution for large time, but we need some oscillating modifier. Then modified wave operators give us a class of dispersive solutions with prescribed asymptotic behaviour at time infinity involving such modification from the free evolution.

Existence of wave operators can be thought as "well-posedness at time infinity". But from this viewpoint, the available results for the modified wave operators were rather unsatisfactory, since the constructed solutions did not have as much regularity as the prescribed asymptotic data, did not converge at time infinity in the strong topology, and continuous dependence on data was given only in weak senses. These discrepancies would cause serious problems when we study the inverse of the wave operators, the scattering operator and further developments of the scattering theory.

We show that this problem can be resolved at least in the case $\nu > 1/2$, where the Dollard-type first order modification suffices. We construct modified wave operators in the weighted space $(1 + |x|)^{-s}L^2$ such that we have strong convergence at time infinity and bi-continuous dependence on the data. The lower bound of the weight $s > 1 - \nu/2$ is also sharp from the scaling argument. We have not obtained the result in the lower dimensional case $n \leq 2$ for $\nu < 1$. Ginibre and Velo had constructed modified wave operators for any $\nu > 0$ and in any dimension, but in smaller spaces with the discrepancy mentioned above.

The main ingredients of the proof are proper choice of the modifier which does not cause derivative loss, iteration scheme and associated energy estimates which allow us to derive time decay essentially only from the convolution factor in the potential, and some bilinear estimates to have cancellation between several phase modifiers that appear in the equation for the modified field.

Global solutions for the Klein-Gordon-Schrödinger system with rough data

Hartmut Pecher

Consider the Cauchy problem for the (3 + 1)-dimensional KGS system with Yukawa coupling

$$\begin{aligned} &i\psi_t + \Delta\psi = -\phi\psi \\ &\phi_{tt} + (-\Delta + 1)\phi = |\psi|^2 \\ &\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \ \phi_t(0) = \phi_1. \end{aligned}$$

This system with data $(\psi_0, \phi_0, \phi_1) \in H^{s,2}(\mathbb{R}^3) \times H^{m,2}(\mathbb{R}^3) \times H^{m-1,2}(\mathbb{R}^3)$ is shown to have a unique global solution, if $1 \geq s, m > \frac{7}{10}$ and $s + m > \frac{3}{2}$. The proof uses the Bourgain method of splitting the data into low and high frequency parts and a bilinear refinement of a Strichartz type estimate.

On unique continuation for the nonlinear Schrödinger equations

Gustavo Ponce

(joint work with C. E. Kenig and L. Vega)

This talk is concerned with uniqueness properties of solutions of nonlinear Schrödinger equation of the form

(1) $$i\partial_t u + \Delta u + F(u, \overline{u}) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

More precisely, we shall consider the following question :

Q : Let u_1, u_2 be solutions of the equation (1) with $(x,t) \in \mathbb{R}^n \times [0,1]$, belonging to an appropriate class X and such that for some domain $D \subset \mathbb{R}^n$, $D \neq \mathbb{R}^n$

(2) $$u_1(x,0) = u_2(x,0), \ \text{ and } \ u_1(x,1) = u_2(x,1), \quad \forall\, x \in D.$$

Is $u_1 \equiv u_2$?

B-Y. Zhang answered question Q in the case

(3) $$n = 1, \ \ F = \alpha|u|^2 u, \ \alpha \in \mathbb{R}, \ \ u_2 \equiv 0, \ \ D = (-\infty, a) \ \ (\text{or}\, D = (a, \infty)),$$

for some $a \in \mathbb{R}$. His proof is based on the inverse scattering theory (IST). It is not clear to us if in the case (3) the IST can be applied to obtain the desired result for any pair of solutions.

Other unique continuation results have been obtained under analyticity assumptions on the data, and under appropriate assumptions on the form of the non-linearity

(4) $$F = F(u, \overline{u}, \nabla_x u, \nabla_x \overline{u}).$$

It is not clear that such results extend to pairs of (analytic) solutions), or under analitycity assumptions on the non-linearity F, without analyticity of the data, but under the stronger assumption that $supp\, u(\cdot, t)$ is compact for all $t \in [0,1]$.

Our main result is the following.

Theorem 1. *Let u_1, $u_2 \in C([0,1] : H^s(\mathbb{R}^n))$, $s \geq max\{n/2^+; 2\}$ be two solutions of the equation*

(5) $$i\partial_t u + \Delta u + F(u, \overline{u}) = 0,$$

where $F \in C^{[s]+1}(\mathbb{C} : \mathbb{C})$ with

(6) $$|F(u, \overline{u})| \leq c(|u|^{p_1} + |u|^{p_2}), \ \ p_1, \, p_2 > 1,$$

and

(7) $$|\nabla F(u,\overline{u})| \leq c(|u|^{p_1-1} + |u|^{p_2-1}), \;\; p_1, p_2 > 1,$$

If there exists Γ *a convex cone strictly contained in a half-space such that*

(8) $$u_1(x,0) = u_2(x,0), \quad u_1(x,1) = u_2(x,1), \quad \forall\, x \notin \Gamma + y_0, \;\; y_0 \in \mathbb{R}^n.$$

Then $u_1 \equiv u_2$.

Remarks

a) In the one dimensional case our assumption on the complement of $\Gamma + y_0$, i.e. $(\Gamma + y_0)^c = D$ reduces to a semi-line (a,∞) (or $(-\infty,a)$). Also we observe that the class of nonlinearities F considered is very general. In particular, it does not contain any analyticity hypothesis on F.

b) We do not know if the result of Theorem 1 is still valid for the case where Γ_{x_0} is just a semispace.

Time decay for solutions of the Schrödinger equation with rough/time dependent potentials

Igor Rodnianski

(joint work with W. Schlag)

This talk describes the joint work with W. Schlag on the dispersive and Strichartz estimates for solutions of the Schrödinger equation with potential. In particular, we show that under the assumptions

$$\sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}dy < 4\pi, \qquad \int_{\mathbb{R}^6} \frac{|V(x)V(y)|}{|x-y|^2}dy < (4\pi)^2$$

we have the dispersive estimate

$$\|\Psi(t)\|_{L^\infty} \leq ct^{-\frac{3}{2}}\|\Psi_0\|_{L^1}.$$

We also settle the conjecture of Journé-Soffer-Sogge showing Strichartz estimates for potentials $|V(x)| \leq \langle x\rangle^{-2-\varepsilon}$.

Miscelleanous problems arising in the theory of water waves

Jean-Claude Saut

The lecture will start by a long quotation of V. Zakharov emphasizing the relevance of dispersive waves in most realistic wave phenomena, in particular in the propagation of hydrodynamical waves.

The full water wave problem (Euler system with free-boundary) is too complicated to expect a description of its long time dynamics and of significant wave phenomena (there have been on the other hand a lot of significant contributions on solitary waves solutions to the Euler system).

Starting with the work of Boussinesq, Korteweg and de Vries, a fruitful approach is to derive simpler asymptotic models which keep track of a some basic physical properties of the wave (e.g. long wave, weak nonlinearity,...). They are obtained by (mostly) formal asymptotic expansions using a right scaling. We dress a list of the more popular, both for surface and internal waves. They fall into two categories : long waves models (Boussinesq, Benney-Luke, Kadomstev-Petviashvili (KP I and KP II), Korteweg- de Vries (KdV), Benjamin-Bona-Mahony (BBM), Benjamin- Ono,...) or "wave packets models (nonlinear Schrodinger, Dysthe, Benney-Roskes, Davey-Stewartson,...). Most of them occur in other physical contexts (under similar scalings) and thus appear as normal forms of various complicated dispersive wave systems. Except for a few notable examples (KdV), there are no rigorous derivations of those models from the full Euler system.

We emphasize that some interesting mathematical properties of the models are irrelevant with respect to the Physics of the water waves problem (eg the "smoothing" effects, the "dispersive blow-up phenomenon",...). Also the question of looking for rough solutions is largely irrelevant in this context, except when one looks for the local well-posedness of the Cauchy problem in the natural "energy space" where GLOBAL well-posedness can be proven. Moreover the energy space is the right one to use in order to state the orbital stability of "ground states" solutions. This has been achieved for instance by Kenig, Ponce and Vega for the KdV equation and by Bourgain for the KPII equation.

The technical part of the talk is devoted to present joint works with Luc Molinet and Nikolay Tzvetkov which display a serious obstruction to the solvability of the local Cauchy problem for the KP II and the Benjamin- Ono equations in their respective energy space. Namely it is proven that one cannot solve those Cauchy problems by a Picard iterative method on the (Duhamel) integral formulation, for data in any Sobolev natural classes. As a consequence, the flow map cannot be smooth. This is in strong contrast with what happens in the KP II or the KdV equation (see above). The proof relies in particular on a careful analysis of the interaction of small and large frequencies. On the other hand we have been able to prove the first global well-posedness result for the KP I equation, by using a rather involved compactness method for smooth enough initial data.

Models for the 2 D water wave problem

Guido Schneider
(joint work with C. E. Wayne)

The so called 2D water wave problem consists in finding the irrotational flow of an inviscid, incompressible fluid in an infinitely long canal of finite depth subject to gravitational force and surface tension. We show that in certain limit situations the problem can be described by the KdV-, the NLS-, or the TWI-system by proving estimates between the associated formal approximation and exact solutions of the water wave problem.

Wave packets and Strichartz estimates for low regularity wave equations

Hart Smith
(joint work with D. Tataru)

We discuss joined work with D. Tataru in introducing wave packets adapted for quasilinear wave equations with data belonging to H^s, $s > \frac{n+1}{2}$. For the constant coefficient wave equation, the wave packet construction is related to the second dyadic decomposition of phase space. In this setting, wave packets retain their size at all times t. For quasilinear equations, it is necessary to adapt the construction to take into account the failure of plane wave surfaces to be C^2. Nevertheless, for dimensions $n \leq 5$, we are able to obtain a construction which satisfies an orthogonality condition sufficient to prove local existence.

On global well-posedness of nonlinear Schrödinger equations

Hideo Takaoka

(joint work with J. Colliander, M. Keel, G. Staffilani and T. Tao)

We consider the problem of obtaining sharp global well-posedness results below the energy norm for the derivative nonlinear Schrödinger equations. The results follow from the method of exploiting the a priori estimate on H^s norm from almost conserved energies. We can also apply same argument to the well-posedness of quintic nonlinear Schrödinger equations.

On the nonlinear Schrödinger equation on a plane domain

Nikolay Tzvetkov

(joint work with N. Burq and P. Gérard)

We study the cubic nonlinear Schrödinger equation (NLS) posed on a bounded domain of $\mathbb{R}^2$ with Dirichlet boundary conditions. In the case when the domain is a disc, we prove that the Cauchy problem is ill posed in the following sense: the flow map is not uniformly continuous on bounded sets of the Sobolev space H^s, $s < \frac{1}{3}$, contrary to what is known on the square (recall that the scale invariant Sobolev space for the cubic NLS in $2D$ is L^2).

Vortex filaments and nonlinear Schrödinger equations

Luis Vega

(joint work with J. Rivas and S. Gutiérrez)

In the talk I gave some explicit solutions to the flow of curves in $\mathbb{R}^3$

(1) $$X_t = X_s \times X_{ss} \qquad s, t \in \mathbb{R}$$

(2) $$X(0, s) = X_0(s)$$

obtained in collaboration with J. Rivas and S. Gutiérrez. This PDE was proposed by Da Rios in 1906 as an approximation to the dynamics according to Euler equations of vortex tubes of infinitesimal cross section. These solutions are of selfsimilar type and can develop corners in finite time or more generally logarithmic spirals. Examples of lack of uniqueness of weak solutions were also given.

Edited by Axel Grünrock und Herbert Koch

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