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Miniworkshop: Mathematical Problems in the Nonlinear Elastodynamics of Rubber-Like Materials

November 3rd – November 9th, 2002

The aim of the mini-workshop was to focus on some open problems in the mathematics and mechanics of the dynamics of rubber-like solids and soft tissues. These are materials of fundamental importance in many technical and scientific applications described by challenging mathematical models that have their common roots in the dynamical nonlinear theory of elasticity. Indeed, it is well known that at any instant t the stress and the strain of such materials depend not just on the current strain and stress respectively, but also on their history. Therefore, to describe the mechanical behaviour of these materials we need models that account for phenomena like *creep*, *stress relaxation* and *hysteresis*. A convenient framework for capturing these time-dependent departures from hyperelasticity is to invoke the assumption that stress tensor is written as the sum of an elastic part and a viscoelastic part. For this reason a deep understanding of the basic mathematical issues of elastodynamics still plays a crucial role in the understanding of rubber-like materials and soft tissues.

The participants of the workshop focused their attention on some outstanding open problems in nonlinear elasticity and viscoelasticity, pointing out the needs for future research. Here we give a short summary of the crucial points that have been stressed in the talks and the long and animated subsequent discussions. R. Knops gave a careful and deep survey of the status of questions such as existence, uniqueness, regularity and stability of solutions in the dynamic theory of elasticity. At the present time it remains to establish global existence and uniqueness of solutions to initial boundary-values problems for these theories, an important step toward full understanding of the mathematical structure of dynamical theories. In this framework the result reported by W. Domanski concerning wave interactions in nonlinear elastodynamics have been very enlightening and precious. K. Rajagopal stressed the similarities between the solutions of some problems in fluid mechanics and solid mechanics, pointing out important analogies to be investigated in detail in the framework of viscoelasticity. I. Muller has shown, by careful experiments and a brilliant theoretical description, how the non-monotonic features of the stress-strain curve of rubber-like materials are responsible for interesting hysteresis phenomena.

Talks by M. Destrade, G. Rogerson and Y. Fu were devoted to discussion of surface waves. Various mathematical methods have been proposed to make major advances on this fundamental subject, some of which are well known in fields such as control theory and Hamiltonian mechanics. Their application in elasticity is quite unusual and therefore deserving of further investigation. Ph. Boulanger has pointed out a connection between elastic waves in deformed Mooney-Rivlin materials and electromagnetic waves in crystals, a connection that may be very deep. Lengthy discussion at the end of his talk has opened the possibility of new interesting results. M. Hayes showed that the polar decomposition theorem may be generalized by considering unsheared triads, a kinematical tool that may be very promising in many applications.

R. Ogden and G. Saccomandi presented some results about the propagation of small and finite amplitude waves in nonlinear viscoelasticity, a nearly virgin field in which there is an urgent need for systematic and complete investigations. The use of elastic waves in investigating phase transitions has been proposed by T. Pence, whereas the connection between molecular and continuum models was the concern of the talk by F. Theil. The possibility of the application of elastic theories to problems of molecular biological has been presented in the outstanding talk by J. Maddocks on DNA elastodynamics. An survey on advanced numerical methods suitable for application in nonlinear elasticity (LMF, MOF and other methods) has been provided by the talk of I. Sgura.

The mini-workshop provided an exceptional opportunity to highlight open problems in a classical, but still very active and crucial, field of applied mathematics. The mini-workshop was a unique opportunity to share ideas and to begin fruitful collaborations. In summary, is possible to say that in the collective opinion of the participants we need to concentration on the following problems in the next few years:

- the mechanical meaning of the mathematical conditions that ensure (global) existence;
- mathematical methods for the solution of surface waves problems (especially the Stroh formalism and its relationship with the Hamiltonian formalism);
- wave propagation in anisotropic media;
- finite amplitude and small amplitude wave propagation in viscoelastic media;
- possible applications of dynamic theories of continuum mechanics in biology.

Abstracts

Elastic waves in Mooney-Rivlin materials and electromagnetic waves in crystals

Philippe Boulanger, Bruxelles
(joint work with Michael Hayes)

An analogy is exhibited between results for electromagnetic waves in linear media which are both electrically and magnetically anisotropic (crystals) and results for finite amplitude elastic waves in Mooney-Rivlin materials which are maintained in a state of static homogeneous deformation. More exactly, it is shown that the results for elastic waves in deformed Mooney-Rivlin materials may be obtained from the results for electromagnetic waves in electrically and magnetically anisotropic crystals by appropriate formal substitutions. The anisotropic properties of electromagnetic waves in these crystals are described in terms of two tensors, the electric permittivity and magnetic permeability tensors. The anisotropic properties of elastic waves in deformed Mooney-Rivlin materials are described in terms of one tensor, the left Cauchy-Green strain tensor of the static homogeneous deformation of the material. The formal substitutions express both the electric permittivity and magnetic permeability tensors in terms of one tensor, the left Cauchy-Green strain tensor of the static deformation of Mooney-Rivlin materials. Hence, the results for elastic waves in deformed Mooney-Rivlin materials appear formally as a special case of the results for electromagnetic waves in crystals. The analogy is used to formulate the problem of finding the wave speeds and the polarization directions of the finite amplitude elastic waves as an eigenvalue problem.

Rayleigh waves and surface stability for compressed Bell materials; comparisons with rubber

Michel Destrade, Paris

The stability of a Bell-constrained half-space in compression is studied. To this end, the propagation of Rayleigh waves propagating on the surface of the material when it is maintained in a static state of triaxial prestrain is considered. The exact secular equation is established and compared to the secular equation for surface waves in deformed incompressible materials. As the speed of the waves tends to zero, the bifurcation criterion, or stability equation, is obtained. Then the analysis is specialized to specific forms of strain energy functions and prestrain, and comparisons are made with results previously obtained in the case of incompressible neo-Hookean or Mooney-Rivlin materials. It is found that these rubber-like incompressible materials may be compressed more than "Bell empirical model" materials, but not as much as "Bell simple hyperelastic" materials, before the critical stretch, solution of the bifurcation criterion, is reached. In passing, some classes of incompressible materials which possess a relative-universal bifurcation criterion are presented.

Wave interactions in nonlinear elastodynamics.

WLODEK DOMANSKI, WARSAW

Interactions of plane waves in nonlinear elasticity are investigated. We use an asymptotic method of weakly nonlinear geometric acoustics to derive evolution equations for the amplitudes of elastic waves . Both resonant and non-resonant cases are discussed. All coefficients of the asymptotic equations are calculated analytically. The tables of all interaction coefficients are displayed and their role in the behaviour of the solution of the asymptotic equations is analyzed.

A proof of the uniqueness of surface waves that is independent of the Stroh formalism

YIBIN FU, STAFFORDSHIRE

It is well-known in surface wave theory that the secular equation for the surface-wave speed v can be written as $\det M = 0$ in terms of the surface impedance matrix M. It has recently been shown by the present authors that M satisfies a simple algebraic Riccati equation. It will be shown at the talk that a purely matrix algebraic analysis of this equation suffices to prove that whenever a surface wave exists it is unique.

Unsheared Triads and extended Polar Decompositions in Finite Strain

MICHAEL HAYES, DUBLIN
(joint work with Philippe Boulanger)

In the theory of finite strain, a classical result due to Cauchy is that at every particle $\mathbf{X}$ in the body before deformation there exists a triad of mutually orthogonal infinitesimal material line elements whose members are also mutually orthogonal after deformation. Here we also consider oblique triads of material line elements at $\mathbf{X}$ which are unsheared in the deformation, that is triads such that their three mutual angles are unchanged in the deformation.

First, the concept of unsheared pairs of material line elements at a material point $\mathbf{X}$ is recalled. In general, corresponding to every material line element in a plane there is a unique "companion element" such that the pair is unsheared. However, there are two elements, neither of which has a companion forming with it an unsheared pair. These "limiting directions" are such that elements along them undergo the greatest shear in the plane.

Then the concept of unsheared triads at $\mathbf{X}$ is introduced. It is seen that there is an infinity of such triads. They may be constructed from unsheared pairs. If an unsheared pair of material line elements elements is given at $\mathbf{X}$, then, in general, a unique third material line element may be found such that the three material line elements form an unsheared triad.

We also consider the rigid rotations of the unsheared triads. There is indeed a unique proper orthogonal tensor $\mathbf{Q}$ which characterizes the rigid rotation of a given (non coplanar) unsheared triad. It is seen that the deformation gradient $\mathbf{F}$ may be written $\mathbf{F} = \mathbf{QG}$, where $\mathbf{G}$ is the tensor whose right eigenvectors are along the material elements of the triad *before* deformation, and whose eigenvalues are the stretches of these elements. Similarly, $\mathbf{F}$ may be written $\mathbf{F} = \mathbf{HQ}$, where the tensor $\mathbf{H}$ has the same eigenvalues as $\mathbf{G}$, but with right eigenvectors along the material elements of the triad *after* deformation. The decompositions $\mathbf{F} = \mathbf{QG} = \mathbf{HQ}$ (where $\mathbf{G}$ and $\mathbf{H}$ are not symmetric) are called "extended polar decompositions". Examples for the case of simple shear are presented.

Uniqueness of strong solutions to simple problems in nonlinear homogeneous elasticity.

Robin J. Knops, Edinburgh

Counterexamples in nonlinear elastostatics demonstrate the undesirability of unqualified uniqueness of strong solutions to both the displacement and traction boundary value problems. In elastodynamics, strong solutions are unlikely to exist globally in time, but then weak solutions are non-unique without a selection principle. Nevertheless, there are simple problems in both statics and dynamics for which it is reasonable to expect global uniqueness in the sense that the property holds for all solutions and not only those that are small in some spatial measure. Constitutive conditions introduced in such uniqueness proofs should be consistent with those required for existence and stability. Material properties are independent of qualitative ones.

Such topics are treated as introduction to a description of a uniqueness proof for smooth solutions to the affine displacement boundary problem of nonlinear homogeneous elastostatics on a bounded star-shaped region subject to a strain energy function that is rank-one convex and strictly quasiconvex. When the region is cone-like or contained in an infinite strip, the assumptions on the strain energy may be replaced by the single one of strict rank-one convexity but certain asymptotic behaviour must additionally be prescribed.

The proof in both cases relies upon integral identities derivable from the Eshelby tensor, and proceeds by way of contradiction.

Treatment of global uniqueness for smooth solutions to the affine displacement initial boundary value problem appeals to conservation laws analogous to those derived in statics from the Eshelby tensor. The proof constructs a certain second order differential inequality that again leads to a contradiction. The strain energy function is required to be rank-one convex and quasiconvex, and the region is supposed bounded and star-shaped.

Apart from that presented in [1] and [2], the work described is still in course of preparation.

[1] Knops, R.J. and Stuart, C.A., Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity. Archive for Rational Mechanics and Analysis,86 (1984) 233-249.

[2] On uniqueness in nonlinear homogeneous elasticity. In: Rational Continua,Classical and New. A collection of papers dedicated to Gianfranco Capriz. pp 55-73. Springer Verlag Italia. Milan 2002.

Multi-scale models of DNA tertiary structure

John H. Maddocks, Lausanne

I describe how a mixture of analysis, computation and experiment involving elastic models of DNA fragments can be used to probe the sequence-dependent mechanical properties of DNA at the biologically important length scales of 100 to 1,000 base pairs.

Stability of a balloon - one, two, many

Ingo Müller, Berlin

The most exciting feature of the pressure-radius characteristic of a rubber balloon is its non-monotonicity which implies an interesting stability behaviour. We study the stability of a single balloon first when it is in contact with a cylinder that is closed off by a piston. This investigation will provide us with a suggestive stability criterion which may then be extrapolated to the case of two interconnected balloons and subsequently to more and more interconnected balloons. An interesting experiment is the simultaneous inflation and deflation of many balloons. The corresponding pressure-filling characteristic is strongly hysteretic and approaches a pseudo-elastic hysteresis loop for very many balloons

Discussion of the dynamic properties of rubber

Ray Ogden, Glasgow

In this lecture we illustrate how the behaviour of rubber departs from the purely elastic; we examine stress softening associated with the Mullins effect, and the different degrees of stress softening for different rubbers are highlighted. Other inelastic effects such as hysteretic stress-strain cycling following pre-conditioning of the material (to remove the Mullins effect) and time and rate dependence are also described. Finally, some outstanding problems in the modelling of the inelastic behaviour of rubber are discussed with particular reference to viscoelasticity.

Waves in pre-stressed viscoelastic solids

Ray Ogden, Glasgow
(joint work with Giuseppe Saccomandi)

In this lecture we discuss the influence of viscoelasticity on the propagation of small amplitude waves in a finitely deformed material. The constitutive law is taken to be of rate form and for the most part, for simplicity of illustration, attention is restricted to plane strain deformations. First, plane waves are examined from the point of view of time decay and attenuation and then we consider in detail the propagation of surface waves on a pre-stressed homogeneously deformed half-space of incompressible isotropic. The influence of the viscoelastic part of the material model on surface waves is assessed in relation to corresponding results for an elastic material.

Interaction of plane waves with a mobile elastic phase boundary in anti-plane shear

Tom Pence, East Lansing
(joint work with Hungyu Tsai)

We shall discuss the dynamics of phase boundary motion exclusively in the context of anti-plane shear. By phase boundary we mean a surface across which the deformation is continuous but its gradient is not. As such it is a standard feature in configurations for a multi-phase elastic system. We shall consider an energy minimal equilibrium state containing such a phase boundary that is then perturbed by the introduction of small amplitude plane waves. The phase boundary is allowed to move, and the standard mathematical framework demands some sort of kinetics to determine this motion. In the context of anti-plane shear there appear to be two obvious standard settings for developing some insights: the plane wave of finite spatial extent with arbitrary shear profile, and the harmonic plane wave. For these cases we shall consider how the kinetic relation affects the resulting phase boundary motion, the reflected and transmitted waves, and the overall energy dissipation.

On the Deformations in Inhomogeneous Elastic Solids

Kumbakonam R. Rajagopal, College Station

In recent years, there has been a considerable amount of interest in the study of inhomogeneous and motions of nonlinear elastic material by using the semi-inverse method. This talk review the subject by stressing the analogies between solid mechanics and fluid mechanics, showing that many motions that have an important role in the framework of the mechanics of fluids have an analogue in nonlinear elasticity. Although the deformations found by the semi-inverse method usually are special solutions an non uniqueness is guaranteed, they are sufficient to reveal the very rich structure of nonlinear elasticity and they are an important guidance for more complete analytical and qualitative studies.

Low and high frequency long wave theories for motion in a pre-stressed, incompressible elastic layer

Graham Rogerson, Salford

The first part of this talk will present an asymptotically consistent two-dimensional theory, developed to help elucidate dynamic response in finitely deformed layers. The layers in question are composed of incompressible elastic material, with the theory appropriate for long wave motion associated with the fundamental mode. Leading order and refined higher order equations for the mid-surface deflection are derived. In the case of zero normal initial static stress and in-plane tension, the leading order equation reduces to the classical membrane equation, with its refined counterpart also being obtained. The theory is applied to a one-dimensional edge loading problem for a semi-infinite plate. In doing so the leading order and higher order governing equations are used as inner and outer asymptotic expansions, the latter valid within the vicinity of the associated quasi-front. A solution of the model problem is derived by using the method of matched asymptotic expansions.

The second part of the talk concerns the derivation of a similar model for high frequency long wave motion. This is motion within the vicinity of the cut-off frequencies, sometimes termed resonance frequencies. Asymptotic solutions for displacement components are determined in terms of the long wave amplitude. Leading and second order equations are obtained for the long wave amplitude and we observe that the leading order equation may become elliptic for certain forms of pre-stress. Some comments are given concerning the connection between this equation becoming elliptic and the possible existence of negative group velocity.

Finite amplitude waves in viscoelasticity

Giuseppe Saccomandi, Lecce
(joint work with Michael Hayes and K.R. Rajagopal)

We investigate the propagation of rectilinear and anti-plane shear waves superimposed to homogeneous and inhomogeneous motions in a special class of nonlinear viscoelastic solids of differential type. We determine several useful exact solutions for such models for initial data with compact and non-compact support. For a special material where it is assumed that the Cauchy stress tensor may be written as the sum of an elastic part and a viscous dissipative part, these exact solutions may be found by solving a system of linear differential equations. The elastic part of this special material is of the form of the stress corresponding to a Mooney-Rivlin material, whereas the dissipative part is linear in the first Rivlin-Ericksen tensor as in a Navier-Stokes fluid. The exact solutions here illustrated are not only useful in understanding the physics of the material or as benchmarks for numerical methods for more complicated problems, but can be related to the study of material stability issues and in some case to new classes of exact solutions for the Navier-Stokes equations. Moreover, we find some explicit examples of blow-up for boundary-value problems with smooth initial data, for a class of materials with a nonlinear dissipative part of the Cauchy stress tensor. These results show that the mathematical structure of the equation of nonlinear viscoelasticity is still not well understood.

BVM methods for partial differential equations

Ivonne Sgura, Lecce

The main concern of this presentation is the numerical approximation of differential equations arising in the modelling of nonlinear elastic materials by means of Boundary Value Methods (BVMs). The BVMs are a class of methods for ODEs recently introduced in the framework of the Linear Multistep Formulae (LMF). They are a generalization of LMF that allow to solve with the same schemes both IVP and BVP to find stable approximations of high order. We present some recent results obtained by using BVMs for solving highly nonlinear ODE-BVPs arising in modelling of axial-shear between two concentric cylinders. We discuss the advantages with respect to other well known techniques such as shooting and collocation methods. We show that by using the Method of Lines (MOL) it is possible to apply BVMs for the solution of hyperbolic PDEs in elasto-dynamics. We focus on the Transverse MOL that performs a semi-discretization of the time variable such that a high dimensional BVP-ODEs system has to be numerically solved in the space domain. Some preliminary results are shown for anti-plane shearing of nonlinear elastic rubber-like materials.

Surface energies in a two-dimensional mass spring lattice

Florian Theil, Coventry

The Cauchy-Born rule postulates that when a monatomic crystal is subjected to a small linear displacement of its boundary, then all atoms will follow this displacement. This rule is a crucial for the extraction of continuum energy densities from discrete systems. In the model case of a $2D$ cubic lattice interacting via harmonic springs we can show that a variant of the Cauchy-Born rule is actually a theorem and use it to discuss surface energy density functions $W(n)$ (n is the normal vector). Simple counterexamples show that for unfavourable spring parameters the Cauchy-Born rule fails. The main tool is a novel estimate based on the rigidity of maps $y : Z^2 \to R^2$ which have the property that the distance between nearest and second nearest neighbours is changed only to a small degree.

Edited by Giuseppe Saccomandi

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