Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2003

Partielle Differentialgleichungen

August 3rd – August 9th, 2003

The program covered a wide range of topics in nonlinear PDE theory. Among them were problems from geometric analysis and geometric flows, regularity results for fully nonlinear equations and elliptic systems as well as analytic problems from physics such as elasticity theory, micromagnetics or fluid flows.

In total there were 25 talks, among them an evening session where recent developments concerning the Ricci flow were presented. The participants enjoyed very much the good working atmosphere at the institute and there were a lot of scientific discussions and exchange also between the talks.

The following abstracts are in chronological order.

Abstracts

Surgery for mean curvature flow

Gerhard Huisken

(joint work with Carlo Sinestrari)

Let M_t^n be a family of smooth, closed hypersurface immersions moving by mean curvature flow in $\mathbb{R}^n$. We show that for 2-convex surfaces, i. e. surfaces where the sum of the lowest two principal curvatures is positive the flow can be combined with finitely many surgeries to exist on a finite time interval until all the area is extinct. As a corollary we derive that such $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, 2-convex, is diffeomorphic either to S^n or a connected sum of $S^2 \times S^1$'s. A key new ingredient in the result are a priori estimates on the second fundamental form and its gradient, together with a surgery construction for "necks".

Some properties of the Born-Infeld equations

Yann Brenier

The Born-Infeld system is a nonlinear version of Maxwell's equations. We first show that, by using the energy density and the Poynting vector as additional unknown variables, the BI system can be augmented as a 10x10 system of hyperbolic conservation laws. The resulting augmented system has some similarity with Magnetohydrodynamics (MHD) equations and enjoy remarkable properties (existence of a convex entropy, galilean invariance, full linear degeneracy). In addition, the propagation speeds and the characteristic fields can be computed in a very easy way, in contrast with the original BI equations. Then, we investigate several limit regimes of the augmented BI equations, by using a relative entropy method going back to Dafermos, and recover, the Maxwell equations for low fields, some pressureless MHD equations for high fields, and pressureless gas equations for very high fields.

Epiperimetric inequalities and the regularity of almost-complex cycles

Tristan Rivière

The epiperimetric inequality introduced by E. R. Reifenberg gives a rate of decay at one point for the decreasing k-density of area of an area minimizing integral k-cycle. While dilating the cycle at that point, this rate of decay holds once the configuration is close to a tangent cone configuration and *above* the limiting density corresponding to that configuration. This is why such an inequality could be named "upper-epiperimetric" inequality. A direct consequence of this "upper-epiperimetric" inequality is the statement that says that any point passes a unique tangent cone. The "upper-epiperimetric" inequality was proved by B. White for area minimizing 2-dimensional integer multiplicity rectifiable cycles in $\mathbb{R}^n$. We present in this talk the notion of "lower-epiperimetric" inequality. This inequality gives this time a rate of decay for the decreasing k-density of area of an area minimizing integral k-cycle, while dilating the cycle at a point, once the configuration is close to a tangent cone configuration – in flat norm – but this time *below* the limiting density corresponding to that configuration. We present a proof of this lower-epiperimetric inequality for area minimizing 2-dimensional integer multiplicity rectifiable cycles in $\mathbb{R}^n$ we explain how this inequality plays a central role in the proof of the regularity of almost-complex 2-cycles.

Branch points of Willmore surfaces

Reiner Schätzle

(joint work with Ernst Kuwert)

We prove that point singularities of Willmore surfaces with finite density consist in codimension one of finitely many branching singularities each over a unique tangent plane and with second fundamental form estimated by

$$|A(x)| \leq C_\varepsilon |x|^{-1+1/\theta_0-\varepsilon} \quad \forall \varepsilon > 0,$$

where θ_0 is the maximal branching order assuming the point singularity being at 0. This estimate is optimal up to $\varepsilon > 0$, as examples of minimal surfaces with branch points show.

Constructing convex solutions via Perron's method

John McCuan

(joint work with Mikhail Feldman)

We show the existence of convex solutions (in the weak-viscosity sense) for fully nonlinear elliptic equations $F(x, u, Du, D^2u) = 0$ under the Alvarez-Lasry-Lions structure condition that for fixed p the map $(x, z, A) \longrightarrow F(x, z, p, A^{-1})$ is convex on $\Omega \times \mathbb{R} \times \mathcal{S}^+_{n\times n}$. We require no comparison principle, and this solutions constructed may be non-unique. The main idea is that given an element of the closure of the subject of a convex function (probably not smooth) which gives the strict subsolution inequality in the operator, one can find a local modification resulting in a convex function that is larger on some neighbourhood and admitting estimates on derivatives up to second order on that neighbourhood. (See related works by Ishii, Alvarez-Lasry-Lions, and Andrews-Feldman.)

Symmetry in anisotropic media

Bernd Kawohl

Imagine a carpet woven out of one-dimensional elastic strings whose deformation energy is given by $\int_\Omega \sum_i |u_{x_i}|^p \, dx$. The first eigenvalue $\widetilde{\lambda}_p$ of the pseudo-p-Laplace operator $\widetilde{\Delta}_p u = \sum_i \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right)$ is characterized as minimizing $\int_\Omega \sum_i |u_{x_i}|^p \, dx$ on $W_0^{1,p}(\Omega) \cap \{|u|^p \, dx = 1\}$. In contrast to the p-Laplace operator $\Delta_p u = \sum_i \frac{\partial}{\partial x_i}(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i})$, the pseudo-$p$-Laplace operator is not rotation-invariant. Consequently its first eigenfunction is not radially symmetric on a ball. In the lecture I prove uniqueness, symmetry and log concavity results of these eigenfunctions, a Faber-Krahn type inequality, and study the limit cases $p \to \infty$ and $p \to 1$. Most of the results were obtained in cooperation with M. Belloni from Parma and V. Ferone from Napoli.

Functionals with a gap: regularity and relaxation

Giuseppe Mingione

Functionals with (p, q)-growth conditions are variational integrals that are coercive in a Sobolev space which is strictly lager than the one in which they are finite:

$$\int_\Omega f(x, Du(x))\, dx, \quad |z|^p \leq f(x, z) \leq L(|z|^q + 1), \quad p < q, \quad \Omega \subset \mathbb{R}^n.$$

Functionals with such a gap have been the object of intensive investigation during the last years; the main issues are: regularity of minimizers (in the case the energy density is convex or quasiconvex) and representation results for the relaxed functional. The aim of this talk is to present a unified approach to these issues discovering that a unique, unexpected assumption, namely $x \to f(x, z)$ if Hölder continuous with exponent $\alpha \in (0, 1]$ and

$$\frac{q}{p} < 1 + \frac{\alpha}{n} \tag{1}$$

leads to obtain both regularity results for minimizers and lower semicontinuity/relaxation results, for most of such functionals ([1], [2], [4]). This reveals a subtle interplay between the regularity of f with respect to x and the gap between growth and coercivity/ellipticity; such a phenomenon cannot be observed in the standard case $p = q$, of course. The assumption (1) is also sharp in that failure causes irregularity of minimizers. Finally, in such a case, I shall present an example of a scalar, convex and smooth functional of the type

$$u \in W^{1,p}(\Omega) \to \int_\Omega |Du|^p + a(x)|Du|^q\, dx \quad 0 \leq a(x) \in C^{0,\alpha}$$

(with $\alpha \in (0, +\infty)$) admitting a wild minimizer with a singular set being a Fractal set with Hausdorff dimension nearly maximal, that is $\geq n - p - \epsilon$, for ϵ small at will ([3]).

References

[1] E. Acerbi, G. Mingione: Regularity results for a class of functionals with nonstandard growth, *Arch. Rational Mech. Anal.* 156 (2001), pp.121–140

[2] L. Esposito, F. Leonetti, G. Mingione: Sharp regularity for functionals with (p, q) growth, *Journal of Differential Equations*, to appear

[3] I. Fonseca, J. Maly, G. Mingione: Scalar minimizers with fractal singular sets, preprint CNA 02-CNA-004, March 2003, Carnegie-Mellon-University

[4] G. Mingione, D. Mucci: Integral functionals and the gap problem: sharp bounds for relaxations and energy concentration, preprint Dip. Mat. Univ. Parma, 2003

Solutions of the isoperimetric problem and of the Cahn-Hilliaird equation

Frank Pacard

Some nonlinear problems (depending on a small parameter ε) have solutions whose energy density concentrates along submanifolds (of dimension ≥ 1) as ε tends to 0. We give examples of such concentration sets.

1) Constant mean curvature hypersurfaces in Riemannian manifolds are shown to arise as concentration sets for smooth families of critical points of $E_\varepsilon(u) = \varepsilon \int |\nabla u|^2 + \frac{1}{\varepsilon} \int W(u)$ under the constraint $V(u) = \infty$, as $\varepsilon \to 0$ (joint work with M. Ritoré).

2) Geodesics are shown to arise as condensation sets of sequences of constant mean curvature embeddings of $S^1 \times S^{n-1}$ in $(n+1)$-dimensional Riemannian manifolds as $\varepsilon := \frac{1}{H}$ tends to 0 (joint work with R. Mazzeo).

Rigidity of immersed stable minimal hypercones

Neshan Wickramasekera

This talk presents a proof and applications of the following theorem: If a cone that arises as the weak (i.e. varifold) limit of a sequence of smooth stable minimal hypersurfaces immersed in a Euclidean space (of arbitrary dimension) has vertex density strictly between 1 and 3, and its support is weakly close to a pair of hyperplanes (two transverse hyperplanes or a multiplicity 2 hyperplane), then it is equal to a pair of hyperplanes.

Some results about Bose-Einstein condensates

Robert Jerrard

This talk deals with the relationship between the Gross-Pitaevsky functional G_ε, which characterizes the wave function of a Bose-Einstein condensate, and a reduced functional E, proposed as a model for the energy of a vortex filament in such a condensate. I prove the existence of large numbers of local minimizers of E, when the condensate is subjected to suitable rotational forcing, and I show that for any such local minimizer J, when a scaling parameter ε is sufficiently small, there exists a local minimizer of G_ε whose vorticity is close to J, in an appropriate sense. I also present results (joint work with A. Aftalion) describing various properties of local minimizers of the reduced functional E.

Ginzburg-Landau-type vortices for thin ferromagnetic films

Roger Moser

We consider variational problems from the theory of micromagnetics for thin films of ferromagnetic materials, represented by domains of the form $\Omega_\delta = \Omega' \times (0, \delta)$ with $\Omega' \subset \mathbb{R}^2$. The corresponding energy functional is

$$E_\delta(m) = \frac{1}{2\delta} \int_{\Omega_\delta} |\nabla m|^2 \, dx + \frac{1}{2\delta^2} \int_{\mathbb{R}^3} |\nabla u_\delta(m)|^2 \, dx$$

for magnetization vector fields $m \in H^1(\Omega_\delta, S^2)$, where $u_\delta \in H^1(\mathbb{R}^3)$ is determined by

$$\Delta u_\delta(m) = \operatorname{div} m \quad \text{in } \mathbb{R}^3.$$

Here m is extended by 0 outside of Ω_δ. We consider first minimizers, and then solutions of an evolution equation belonging to E_δ, the so-called Landau-Lifshitz-Gilbert equation. We determine their asymptotic behaviour for $\delta \searrow 0$. The analysis is a generalization of the classical Ginzburg-Landau asymptotic analysis of F. Bethuel, H. Brezis, and F. Hélein. In particular we observe the development of vortices in the limit. In contrast to the classical problem, however, the vortices appear on the boundary of Ω', not in the interior.

Partial regularity for minimizers of degenerate quasi-convex variational integrals

Frank Duzaar
(joint work with G. Mingione)

We present a new method to prove regularity of solutions to certain degenerate elliptic problems. The method is based on the p-harmonic approximation lemma, recently proved by the author and G. Mingione (Parma), that allows to approximate functions with p-harmonic functions as the classical harmonic approximation lemma (going back to De Giorgi) does via harmonic functions. The method presented also bypasses certain difficulties arising when treating degenerate and singular problems with a weak structure, such as degenerate and singular quasi-convex variational integrals, and provides transparent and elementary proofs.

Sharp rigidity estimates for nearly umbilical surfaces

Camillo De Lellis
(joint work with Stefan Müller)

A classical theorem in differential geometry states that if $\Sigma \subset \mathbf{R}^3$ is a closed connected smooth surface and the surface is umbilical (i.e. the principal curvatures are equal at every point of Σ), then Σ must be a round sphere, and therefore its second fundamental form must be a constant multiple of the identity.

We give a sharp quantitative generalization of this theorem. Namely, we prove the existence of a universal constant C such that, for every smooth closed connected surface $\Sigma \subset \mathbf{R}^3$, the following inequality holds

$$\min_{\lambda\in\mathbf{R}} \int_\Sigma |A - \lambda \mathrm{Id}|^2 \;\leq\; C \int_\Sigma \left| A - \frac{\operatorname{tr} A}{2}\mathrm{Id}\right|^2 . \tag{2}$$

The exponent 2 in the inequality cannot be replaced by $p < 2$. When the right hand side of (2) is smaller than a geometric constant, Σ is diffeomorphic to a sphere. In this case we can estimate the $W^{2,2}$ distance of Σ to a round sphere with a universal constant times the right hand side of (2).

The proof is divided into three steps. First, combining a result of Müller and Sverak with elementary geometric considerations, we find conformal coordinates which enjoy good estimates (i.e. these estimates depend only on the right hand side of (2)). Then we use Codazzi–Mainardi equations and PDE estimates from the classical theory of the Laplacian to prove that

$$\min_{\lambda\in\mathbf{R}} \|A - \lambda \mathrm{Id}\|_{L^{2,\infty}(\Sigma)} \;\leq\; C \left(\int_\Sigma \left| A - \frac{\operatorname{tr} A}{2}\mathrm{Id}\right|^2 \right)^{1/2} .$$

Finally, with the aid of Gauss–Bonnet Theorem and Wente–type estimate on suitable wedge products, we are able to improve the $L^{2,\infty}$ bound to the desired L^2 bound (2).

Recent developments for the harmonic map flow

Peter Topping

We describe some recent results about the harmonic map flow from 2D domains. These concern questions of uniqueness (and reverse bubbling), rate of blow-up, regularity of the flow u(T) at a singular time T, and their interrelations.

Gradient estimates for convex functions and applications

Oliver Schnürer
(joint work with Hartmut Schwetlick)

We consider convex functions on a bounded domain that fulfil a Neumann boundary condition and prove a gradient estimate that does not depend on the oscillation. Our estimate generalizes a gradient estimate of Lions, Trudinger and Urbas.

We obtain that solutions to the parabolic flow equation

$$\begin{cases} \frac{\partial}{\partial t}u &= \log\det D^2u - \log f(x, Du) \\ D_\nu u &= \varphi \end{cases}$$

converge to translating solutions.

Aspects of the Hamilton-Perelman Ricci flow

Klaus Ecker
(joint work with Gerhard Huisken)

Thurston's geometrization conjecture states that any closed 3-manifold M can be decomposed along 2-spheres and incompressible 2-tori into pieces which are modelled by the 8 homogeneous geometries given by S^3, $\mathbb{H}^3$, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\overline{\mathrm{SL}(2,\mathbb{R})}$ and Solv and their quotient by discrete, properly discontinuous groups. Of these 8 models which are all simply connected, only S^3 is closed. Therefore, if M is simply connected it has to correspond to S^3. This is the Poincaré conjecture.

In the early 80's, Hamilton proposed the Ricci flow as a method of attacking Thurston's conjecture. This flow evolves a metric on M by the equation

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$$

where R_{ij} is the Ricci curvature of g_{ij}, or by a normalized version of it which keeps the volume of M fixed during the evolution. The curvature operator of g_{ij} which in 3 dimensions is determined by the Ricci tensor satisfies a reaction–diffusion system with quadratic nonlinearity. One therefore expects singularities to form during the flow due to the reaction term with the diffusion resulting in a particularly simple structure of the singularity. For instance after rescaling near a singularity the curvature operator becomes nonnegative.

Hamilton's programme now proceeds as follows: Run normalized Ricci flow with some initial metric. Singularities which form in finite time (near points where the curvature blows up) should locally look like $S^2 \times \mathbb{R}$ or S^3. In infinite time, further singularities may form which look locally like $T^2 \times \mathbb{R}$ cusps. In other words, Ricci flow is expected to find exactly the right places in the 3-manifold where a decomposition should be performed. Near the $S^2 \times \mathbb{R}$ singularities one performs a surgery by removing an $S^2\times$ interval neck, capping off the resulting boundary components by spheres in a quantitatively very specific way. After that one re-starts the flow. Near the $T^2 \times \mathbb{R}$ singularities, the manifold is separated into "thick" pieces on which the metric converges to a hyperbolic one and "thin" pieces where it collapses. The latter pieces correspond to Seifert fibred pieces and therefore admit a geometrization type decomposition.

There are however a number of technical problems which have hindered the completion of Hamilton's programme. One of them is the question whether on any finite time interval only finitely many singularities can arise. Related to this issue are difficulties in controlling the ratio of injectivity radius to square root of maximum curvature of the evolving metric locally near a singularity. This information is essential for ensuring that a rescaling limit of solutions of Ricci flow near a singularity exists in the first place, but also for ruling out undesirable limits such as $\Sigma \times \mathbb{R}$ where Σ is the 2-dimensional translating solution of Ricci flow called the cigar solution. Near such a singularity, it is not clear how to perform surgery. The metric on $\Sigma \times \mathbb{R}$ has the property that there are families of geodesic balls $V_{r_k} = B_{r_k}(p_k)$ for which $|\text{curv}| \leq \frac{1}{r^2}$ on B_{r_k} but

$$\frac{\text{vol}(B_{r_k})}{r_k^3} \xrightarrow{k \to 0} 0.$$

This is where Perelman's recent work has led to a major breakthrough: Perelman has established that a certain entropy functional on M increases during Ricci flow. This functional can essentially be controlled from above by $\log\left(\frac{\text{vol}(B_r)}{r^n}\right)$ on balls B_r on which $|\text{curv}| \leq \frac{1}{r^2}$ and therefore has the value $-\infty$ on $\Sigma \times \mathbb{R}$. On the other hand it is bounded below on any manifold M. Combining these facts leads for such solutions to a lower bound on the volume ratio in balls on which the curvature is controlled. Since this information is scaling invariant, $\Sigma \times \mathbb{R}$ can therefore not arise as a rescaling limit near singularities. There are many other important new geometric/analytic tools provided in Perelman's work which have resulted in major progress in Hamilton's programme but there are also many technical points in his work which still need to be checked.

Stability of travelling waves in the Fermi-Pasta-Ulam model

Gero Friesecke

The FPU model (1947) describes the evolution of a 1D chain of atoms interacting via an interatomic potential,

$$q(j,t)_{tt} = V'\big(q(j+1,t) - q(j,t) - V'(q(j,t) - q(j-1,t)\big)$$

e. g. $V(r) = \big((1+r)^{-\alpha} - 1\big)^2$, $\alpha > 0$.

Formally, for small amplitude slow-time long-wave solutions to the system of conservation laws of electrodynamics,

$$q_t t = V'(q_x)_x \quad \text{or } (r := q_x, p := q_x) \quad \begin{pmatrix} r \\ p \end{pmatrix}_t + \begin{pmatrix} -p \\ -V(r) \end{pmatrix}_x = 0.$$

Whereas for conservation laws, existence/uniqueness for the Cauchy pb, is difficult and shock dynamics is easy, for FPU the Cauchy problem is trivially well posed but the analogon of shock dynamics (travelling wave dynamics) is difficult.

Main result (joint with R. Pego, Maryland):
Small-amplitude FPU travelling waves are stable globally in time (start close, stay close),

and in fact asymptotically stable (start close, approach unique nearby travelling wave exponentially fast) in a suitable weighted norm.

(Ideas of pf: 1) Find a conserved simple dz form for the FPU equations, use to decompose any nearby state uniquely into a travelling wave part and a "radiation" part. Show both parts decouple at linearized level and infer that decay of radiation under nonlinear FPU equations follows from decay under linearized equation. 2) Reduce decay of linearized equation to stable continuous and discrete Floquet spectrum. 3) Rule out unstable continuous spectrum by explicit computation and unstable discrete spectrum by deforming equation into a KdV continuum limit, and "lifting" in)o about KdV back onto the lattice. 4) Figure out the asymptotic profile of FPU travelling waves, which enters the equation for Floquet modes as "coefficients").

Connections and differences between FPU and a backward-discretization scheme for conservation laws studied recently (2003) by Bianchini were indicated.

Geometry and nonconvex variational problems

Shankar Venkataramani

We consider the variational problem for the elastic energy

$$\mathcal{E} = \int \|Du^T \cdot Du - g\|^2 + \sigma^2 \|A^2\|$$

where $u : (S,g) \to \Omega$ is a $W^{1,4} \cap W^{2,2}$ immersion of a n-dimensional Riemannian manifold (S,g) into $\Omega \subseteq \mathbb{R}^d$ equipped with the standard Euclidean metric. We present a heuristic relation between the singularity structure of the minimizers as $\sigma \to 0$ with the geometry of the isometric immersions $\phi : (S,g) \to \Omega$. We flesh out this relation rigorously for a single line singularity of a 2-sheet immersed in $\mathbb{R}^3$.

Long-time behaviour of kinetic equations

Cédric Villani

(joint work with Laurent Desvillettes)

Kinetic equations describe the time-evolution of the distribution $f(x,v)$ of particles in position-velocity space, and are typically of the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \text{ force term } = \text{ collisions } + \text{ diffusion } + \text{ drift },$$

where the operators on the right-hand side only act on the velocity dependence. A simple but fundamental example is the kinetic (collisionless) Fokker-Planck equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv),$$

where V is a confining potential, satisfying certain growth conditions at infinity.

Because there is no diffusion in the x direction, regularization effects and rates of convergence to equilibrium are interesting and nontrivial problems. The whole thing is to understand how the first-order terms on the left combine with the second-order operator on the right, so as to produce a diffusion-like behaviour in the x variable.

In this talk, I presented two general methods for controlling rates of convergence to equilibrium for such equations. The first one is based on commutators and limited to linear equations; it is reminiscent of Hörmander's hypoelliptic regularity theorem (it is in fact a "hypocoercivity" theorem). The second one, based on first-order and second-order differential inequalities in time, coupled with functional inequalities of log Sobolev or Poincaré type, carries over to nonlinear models, and in particular enables to prove convergence to equilibrium for nonlinear Boltzmann or Vlasov-Fokker-Planck equations, as soon as one has strong smoothness/moment a priori estimates, uniform in time.

Partial regularity of almost minimizers of quasi-convex variational integrals with subquadratic growth

Joseph F. Grotowski
(joint work with Frank Duzaar and Manfred Kronz)

We prove a small excess regularity theorem for almost minimizers of a quasi-convex variational integral of subquadratic growth. The proof is direct, and it yields an optimal modulus of continuity for the derivative of the almost minimizer. The result is new for general almost minimizers, and in the case of absolute minimizers it considerably simplifies the existing proof.

Ricci flow of L^∞ metrics on three manifolds

Miles Simon

We consider the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2Ricci(g)_{ij},$$

of Riemannian metrics whose initial value $g_0 = g(0)$ is not necessarily smooth but which is controlled by a smooth background metric, in the sense that

$$\frac{1}{c} h \leq g_0 \leq ch,$$

for some smooth metric h. In particular we prove the following theorems.

Theorem 1. Let $(M^n, g(t))_{t\in[0,T)}$ be a smooth solution to the Ricci-flow, where $\frac{1}{c}h \leq g(\cdot, t) \leq ch$, for all $t \in [0, T)$. Then the solution may be extended to $(M, g(t))_{t\in[0,T+\epsilon)}$ for some small $\epsilon > 0$.

As an application we obtain the following theorem.

Theorem 2. Let $(M^3, {}^i g)$,$i \in \mathbb{N}$ be a family of smooth metrics which satisfy $\frac{1}{c}h \leq {}^i g \leq ch$, for some constant c independent of i, and $sec({}^i g) \geq -\epsilon(i)$ where $\epsilon(i) \to 0$ as $i \to \infty$. Then there exists a smooth metric g' on M^3 such that $sec(g') \geq 0$ and so M^3 may be differentially/topologically classified using the theorem of R. Hamilton [Ha].

[Ha] Hamilton, R. *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. 5 (1997), no. 1, pp. 1–92.

One-dimensional domain wall models in thin-film micromagnetics

Christof Melcher

Most mathematical models for interfaces and transition layers in materials science exhibit sharply localized and rapidly decaying transition profiles. We show that this behaviour can largely change when non-local interactions dominate and internal length scales fail to be determined by dimensional analysis, as in reduced models for micromagnetic Neel walls: The main analytical feature in this model is that the variational principle gives uniform control only in $H^{1/2}$ but incorporates pointwise constraints. We show that critical estimates based on the associated Euler-Lagrange equation imply the existence of uniform logarithmic tails of transition profiles.

Viscosity solutions and problems in classical mechanics

Diogo Gomes

It is well known that smooth solutions to the Hamilton-Jacobi equation

$$H(P + D_x u, x) = \bar{H}(P)$$

yield, at least formally, a change of coordinates $X(p, x)$ and $P(P, x)$ defined by the equations

$$p = P + D_x u \quad X = x + D_p u, \tag{1}$$

which simplifies the Hamilton dynamics

$$\dot{x} = -D_p H(p, x) \quad \dot{p} = D_x H(p, x)$$

into

$$\dot{P} = 0 \quad \dot{X} = -D_p \bar{H}(P).$$

Unfortunately, this procedure may fail. However, by considering viscosity solutions, instead of classical solutions, one may recover some weaker versions of these statements.

In fact, viscosity solutions yield important information concerning certain measures, the Mather measures, which are invariant under the Hamiltonian flow. For instance, for each such measure, μ, there is a value P for which μ is supported on the graph $(x, P + D_x u)$.

Of particular interest is the study of integrable Hamiltonian systems under small perturbations. We consider Hamiltonians of the form

$$H(p, x) = H_0(p) + \epsilon H_1(p, x),$$

in which ϵ is a small parameter. We prove a weak form of the KAM theory for viscosity solutions of Hamilton-Jacobi equations. In particular, we show that under suitable hypothesis approximated solutions, obtained through formal expansion are needed in order to obtain uniform control both for the function and its derivatives. Therefore there are no convergence issues involved. An application of these results is to show that Diophantine invariant tori and Mather sets are stable under small perturbations.

Geometric Paneitz-Branson operator: concentration phenomenon and fourth-order PDE's

Frédéric Robert
(joint work with Michael Struwe)

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. The Paneitz-Branson operator (that we will denote P_g^n) is a geometric operator, the principal part of which is the square of the Laplacian. For a Euclidean metric, P_g^n reduces to the square of the Euclidean Laplacian.

The operator P_g^n is conformally invariant and is attached to a specific curvature, namely the $Q-$curvature. This $Q-$curvature plays the same role for P_g^n as the scalar curvature plays for the conformal Laplacian. An important issue in the domain is to prescribe this $Q-$curvature in a conformal class. When $n \geq 5$ and given $f \in C^\infty(M)$, this amounts to finding $u \in C^\infty(M)$, $u > 0$, solution to the equation $P_g^n u = \frac{n-4}{2} f u^{2^\sharp - 1}$, where $2^\sharp = \frac{2n}{n-4}$: in this case, f is the $Q-$curvature for the metric $\tilde{g} = u^{\frac{4}{n-4}} g$. The exponent $2^\sharp$ is critical from the viewpoint of second-order Sobolev embeddings. Needless to say that one of the difficulties is the lack of compactness of the Sobolev embedding involved in our problem. We will then be mainly interested in describing the concentration phenomenon attached to it. This will allow us to prescribe the $Q-$curvature in many situations.

Among the specificities of the bi-Laplacian, let us mention that the regularity issues and the lack of comparison principle make the study more intricate.

The hyperbolic Monge-Ampère equation in the plane

Bernd Kirchheim

We discuss the structure of solutions to

$$\det(\nabla^2 u) = -1, u \in W^{2,\infty}(\Omega, \mathbb{R}^2),$$

more precisely, the possible singularities of u. The "singular transformation"

$$(\nabla u)_\pm(z) = \nabla u(z) \pm \mathrm{i} z$$

provides local coordinates which describe the vector field ∇u, its singular branch points and shows that they are discrete. It allows also a exhaustive analysis of the situation when $\det(\nabla^2 u) = 0$.

On the other hand, the "partial Legendre transform" allows a good understanding of the solutions away from singular branch points and here also for the Hessian determinant close to but not necessarily equal to minus one.

In this way, Chaudhuri-Müller could prove that quasi- and rank-one convexity agree on

$$\{X \in \mathbb{R}^{2\times 2}_{\text{symm}} \ ; \ \det X = -1 \text{ and } X_{11} > 0\}$$

and we can extend this result to the three dimensional surface

$$\{X \in \mathbb{R}^{2\times 2} \ ; \ \det X = 0 \text{ and } X_{11} > 0\}.$$

Edited by Tobias Lamm

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