Mathematisches Forschungsinstitut Oberwolfach

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Locally Symmetric Spaces

September 28th – October 4th, 2003

The conference was organized by Birgit Speh (Ithaca) and Jürgen Rohlfs (Eichstätt).

During the five days of the conference 23 main talks were given and 34 mathematicians participated. Thus there was enough time for discussions and additional work. The conference focused on recent results concerning analytical, topological and number theoretical aspects in the field of locally symmetric spaces.

The organizers and participants thank the 'Mathematisches Forschungsinstitut Oberwolfach' for making this conference possible.

The abstracts are listed in alphabetical order of the names of the speakers. At the beginning the schedule of the conference is given with the list of speakers and the title of their talk.

Conference schedule

Monday

J. Schwermer: On the automorphic cohomology of arithmetic groups.
U. Bunke: Eisenstein series for geometrically finite groups.
F. Shahidi: Functoriality: From symmetric powers to non-classical groups.
S. Kudla: Cycles of surfaces and modular forms.
F. Williams: Trace formula and asymptotics for p–form coefficients on real hyperbolic space.

Tuesday

J.P. Labesse: Geometric approach to trace formula stabilization.
A. Juhl: On scattering theory on geometrically finite hyperbolic manifolds.
L. Ji: Large scale geometry of arithmetic groups and the integral Novikov conjecture.
J. Millson: Cycles and homology with local coefficients.
T.N. Venkataramana: Construction of certain generalised modular symbols.

Wednesday

J. Hilgert: Period functions for congruence subgroups.
L.D. Saper: Cohomology of locally symmetric spaces and their compactifications.
M. Olbrich: Cohomology of convex co-compact groups.

Thursday

T. Oda: All the contiguous relations in the principal series $(\mathfrak{g}, K)$–modules of $\mathrm{Sp}(2, \mathbb{R})$.
W. Müller: On the cuspidal spectrum for GL_n.
J. Funke: Cycles with coefficients and automorphic forms.
M. Belolipetsky: On volumes of arithmetic locally symmetric spaces.
S. Böcherer: Explicit versions of the (global) Gross–Prasad conjecture.

Friday

R. Stanton: Complex methods for real Lie-groups.
S. Bullock: Gaussian weighted L_2–cohomology.
A. Nair: Intersection cohomology, Shimura varieties and motives.
J. Mahnkopf: Cohomology of arithmetic groups, parabolic subgroups and the special values of L-functions on GL_n.
G. Harder: Congruences between modular forms of genus 1 and of genus 2.

Abstracts

On volumes of arithmetic locally symmetric spaces

M. Belolipetsky

Let G be a semi–simple algebraic group defined over a number field k, $K \subset G_{\mathbb{R}}$ – a maximal compact subgroup, $\Gamma \subset G(k)$ – an arithmetic subgroup of G. We are interested in computing the volumes of the locally symmetric spaces

$$\mu(\Gamma \backslash G_{\mathbb{R}}/K)$$

with respect to a naturally normalized Haar measure μ on the group G. This problem has a long history which goes back to the work of Minkowski and Siegel.

There are two approaches which can give closed formulas for the volumes in a general situation. One is to use Eisenstein series, concerning this we would like to mention the papers of G. Shimura from 1997 – 1999. The second approach uses the Tamagawa measure on $G(\mathbb{A})$ and Bruhat–Tits theory. A recent breakthrough in this direction is due to G. Prasad who gave a closed formula for the co-volume of a principal arithmetic subgroup of G in his article in Publ. IHES, 1989.

The aim of my lecture is to discuss the second approach and to show how to use Prasad's volume formula for the actual computations. As an application we present the following result:

Theorem (B). *For any $n = 2r \geq 4$ there exists a* unique *compact orientable arithmetic hyperbolic n–orbifold O^n_{min} of the smallest volume. It is defined over $k = \mathbb{Q}[\sqrt{5}]$ and has Euler characteristic*

$$|\chi(O^n_{min})| = \frac{\lambda(r)}{N(r)4^{r-1}} \prod_{i=i}^{r} |\zeta_k(1-2i)|$$

where ζ_k is the Dedekind zeta function of k, $N(r) \in \mathbb{Z}$ and $\lambda(r) \in \mathbb{Q}$ are constants such that $1 \leq N(r) \leq 4$, $\lambda(r) = 1$ for even r and $1 \leq N(r) \leq 8$, $\lambda(r) = 2^{-1}(4^r - 1)$ for r odd.

The methods of Bruhat–Tits theory and results of G. Prasad appear to be effective in proving this theorem. We do not know how to get it using the first approach. Considering this, it could be interesting to see how far one can proceed with this method. In particular, we would like to ask whether it is possible to apply it to the problems of counting integer points on the *affine* symmetric spaces.

Explicit versions of the (global) Gross–Prasad conjecture

S. Böcherer

(joint work with M. Furusawa, R. Schulze-Pillot)

Very little is known so far about the Gross-Prasad conjecture, so it seems worthwhile to study explicitly low-dimensional cases (beyond those in the papers by Gross/Prasad) using isomorphisms of low-rank orthogonal groups with other linear groups.

We consider the case $SO(5) \supset SO(4)$, when translated into a $Sp_2 \subset SL_2 \times SL_2$ - setting. We study integrals

$$\iint_{(\Gamma\backslash\mathbb{H})^2} F\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \overline{f_1}(z_1)\overline{f_2}(z_2) d^*z_1 d^*z_2 \, ,$$

where F is a holomorphic Siegel modular form and f_1, f_2 are elliptic cusp forms. The Gross–Prasad conjecture in this version would predict that the nonvanishing of this integral (for Hecke-eigenforms) should be related to the nonvanishing of the L-function $L(\mathrm{Spin}(F) \times f_1 \times f_2, s)$ at the centre (modulo local conditions).

We verify this for the special case that F is a Yoshida lift, using results on central values of triple L-functions (by Gross/Kudla for weight 2 and by Böcherer/Schulze-Pillot for higher weights).

In the case at hand we can write down an explicit relation connecting the square of the integral above to the central L-value in question.

Gaussian weighted L_2–cohomology

S. Bullock

Let $\mathcal{M} = \Gamma\backslash G/K$ a locally symmetric space, complete, finite volume, noncompact, with $K \leq 0$. Let $X = G/K$, $\mathbb{E} = E \times_\Gamma X \cong (\Gamma\backslash G) \times_K E$ a coefficient system, and $H^\bullet_{(2),w}(\mathcal{M}, \mathbb{E})$ the unreduced w-weighted L_2 cohomology for $w : \mathcal{M} \to (0, \infty)$.

Let P be a minimal $\mathbb{Q}$ parabolic of G. We recall Franke's construction (*AENS*, vol 31, pg. 181) of $w_\lambda \in O(a^\lambda)$ on each Siegel set:

- $P = UMA$ the $\mathbb{Q}$–Langlands decomposition
- $H(-)$ the height function: $(H = \log \circ \pi_A) : [G = (UMA)K] \to \mathfrak{a} \cong \mathbb{R}^\ell$ for $\ell = \mathrm{rank}_\mathbb{Q} G$
- $\langle -, - \rangle$ the pairing of $\mathfrak{a}$ to Killing dual $\check{\mathfrak{a}}$
- $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth cutoff of bounded gradient, 0 on $(-\infty, T-1]$ and 1 on $[T, \infty)$
- $\sum^{++}(\mathfrak{g}, \mathfrak{a})$ the simple restricted $\mathbb{Q}$–roots
- $\{q_i\}_{i=1}^c$ is a set of representatives of $\Gamma\backslash G(\mathbb{Q})/P(\mathbb{Q})$.

Then for $\lambda \in \check{\mathfrak{a}}$ and careful choice of $T(\S 2.1)$, Franke defines

$$w_\lambda(\Gamma g K) = \sum_{i=1}^{c} \sum_{[\gamma]\in(\Gamma\cap UM)\backslash UM} \Big\{ \exp\big(\langle \lambda, H(\gamma {q_i}^{-1} g)\rangle\big) \prod_{\alpha \in \sum^{++}(\mathfrak{g},\mathfrak{a})} \chi\big(\langle \alpha, H(\gamma {q_i}^{-1} g)\rangle\big) \Big\}.$$

Suppose instead $Q \in \check{\mathfrak{a}} \otimes \check{\mathfrak{a}}$ a quadratic form, $-Q < 0$. Relabel $Q(H, H)$ as $-\langle Q, H\rangle$ for emphasis. Put

$$w_{-Q}(\Gamma g K) = \sum_{i=1}^{c} \sum_{[\gamma]\in(\Gamma\cap UM)\backslash UM} \Big\{ \exp\big(-\langle Q, H(\gamma {q_i}^{-1} g)\rangle\big) \prod_{\alpha \in \sum^{++}(\mathfrak{g},\mathfrak{a})} \chi\big(\langle \alpha, H(\gamma {q_i}^{-1} g)\rangle\big) \Big\}.$$

Henceforth let $-Q$ replace w_{-Q} in all notations.

Lemma. *On the reductive Borel–Serre compactification $\overline{\mathcal{M}}$, let $\mathcal{L}^\bullet_{-Q}(E)$ be the sheafification of the w_{-Q} - L_2 presheaf and $\mathcal{A}^\bullet(E)$ the special form presheaf (*Inventiones*, vol*

116, pg 139). For $j : \mathcal{M} \to \overline{\mathcal{M}}$ open inclusion, $j_*\Omega^\bullet(\mathbb{E}) \leftarrow Sh(\mathcal{A}^\bullet(E)) \to \mathcal{L}^\bullet_{-Q}(E)$ is a quasi-isomorphism of sheaves.

Theorem. *For $-Q < 0$, $H^\bullet_{(2),-Q}(\mathcal{M}, \mathbb{E}) \cong H^\bullet(\mathcal{M}, \mathbb{E})$ the de Rham cohomology.*

We may choose $-Q$ as $B|_{\mathfrak{a}\times\mathfrak{a}}$ on each Siegel set. Then for the Gaussian e^{-r^2}, $w_{-Q} \in O(e^{-r^2})$.

Corollary. *$\mathcal{M}$ as above, Gaussian weight w. Then $H^\bullet_{(2),w}(\mathcal{M}) \cong H^\bullet(\mathcal{M}, \mathbb{E})$.*

Eisenstein series for geometrically finite groups

U. Bunke

We first introduced the class of geometrically finite subgroups Γ of Liegroups of real rank one G. The Plancherel theorem has the general form $L^2(\Gamma\backslash G) = \int_{G_u} M_\pi \otimes V_\pi p(d_\pi)$. The main objection of the construction of Eisenstein series is to understand the spaces $M_\pi \subset {}^\Gamma V_{\tilde{\pi}}^{-\infty}$. We gave an overview of the geometric scattering theory framework involving the objects π_∞^Γ, ext^Γ and res^Γ. The main result reported on is that ext^Γ and π_∞^Γ have meromorphic continuations as operators acting on appropriate functions spaces related to the family of (non-)unitary principal series representations.

Cycles with coefficients and automorphic forms

J. Funke

(joint work with J. Millson)

Let X be an arithmetic quotient of the symmetric space associated to $O(p,q)$. Then Kudla–Millson constructed a map from $H_c^{(p-n)q}(X)$, the (de Rham) cohomology of X with compact support, to the space of classical holomorphic Siegel modular forms of genus n and weight $(p+q)/2$. Moreover, the Fourier coefficients are given by periods over certain special cycles in X, arising naturally from embedded $O(p-n,q)$. This map is realized by an explicit construction of a theta-function as kernel.

This project is concerned with the generalization of this work to non-trivial coefficient systems. As coefficient systems we consider the tensor powers with n factors of $\mathrm{Sym}^*(V)$, where V is the underlying rational quadratic space of signature (p,q). Note the irreducible constituents of these representations have highest weight vectors with at most n nonzero entries. The special cycles naturally define homology classes with these coefficients. We develop an analogous theory to the scalar-valued one of Kudla–Millson. In particular, we explicitly construct a theta kernel with values in the coefficient system W, giving rise to a new lift from $H_c^{(p-n)q}(X, \tilde{W})$ to (in general) vector–valued Siegel modular forms. If the input takes (pluri)–harmonic values, the lift is holomorphic, while the general case can be described in terms of the raising operators for the symplectic group. The Fourier coefficients of the lift are given by periods over the cycles with coefficients.

This work will play a major role in the extension of the original Kudla–Millson lift to the full cohomology, as the restriction of the scalar-valued theta kernels to the Borel–Serre boundary of X gives rise to the above theta kernels with coefficients (for smaller orthogonal groups).

Congruences between modular forms of genus 1 and of genus 2

G. Harder

This was a report on some computer experiments. We consider the two modular cusp forms f_{22} and f_{26} of weight 22 (resp. 26) for $SL_2(\mathbb{Z})$. Since in both cases the dimension of the space of cusp forms is 1 they are unique if the first coefficient is normalized to one, so they have the expansion

$$f = q + a_2 q^2 + a_3 q^3 \dots .$$

For the same reason they are eigenforms for the Hecke operators, we have $T_p(f) = a_p f$ in both cases.

We consider their L-function

$$L(f,s) = \frac{\Gamma(s)}{(2\pi)^2} \sum \frac{a_n}{n^s} = \frac{\Gamma(s)}{(2\pi)^2} \prod_p \frac{1}{1 - a_p p^{-s} + p^{2k-1-2s}}$$

here $k = 22$ or 26.

By a theorem of Manin we know that we can define two periods $\Omega_+(f)$, $\Omega_-(f)$ such that for $\nu \in [\frac{k}{2}, \dots, k-1]$ the values

$$\frac{L(f,\nu)}{\Omega_{\varepsilon(\nu)}(f)} \in \mathbb{Q}$$

and after a suitable normalization of the periods we can arrange that the values

$$\left\{ \frac{L(f,\nu)}{\Omega_+(f)} \right\}_{\nu \text{ even}} \qquad \left\{ \frac{L(f,\nu)}{\Omega_-(f)} \right\}_{\nu \text{ odd}}$$

are coprime each.

We look at large primes ℓ ($\ell > k$) and $\ell \nmid \zeta(1-k)$ which divide such an L-value and find the following divisibilities

$$41 \mid \frac{L(f_{22},14)}{\Omega_+(f_{22})}, \qquad 43 \mid \frac{L(f_{26},23)}{\Omega_-(f_{26})}, \qquad 97 \mid \frac{L(f_{26},21)}{\Omega_1(f_{26})}, \qquad 29 \mid \frac{L(f_{26},19)}{\Omega_-(f_{26})} .$$

For any such incident, we define the numbers m, t by

$$k = t + m + 4, \quad \nu = t + 3$$

and consider the spaces $S_{t-m,m+3}$ of vector valued, holomorphic modular cusp forms for $Sp_2(\mathbb{Z})$ (they are attached to the representation $\mathrm{Sym}^{t-m}(\mathbb{C}^2) \otimes \det^m$). In all four cases we have $\dim S_{t-m,m+3} = 1$ and we have the sequences (indexed by the primes) of eigenvalues $\lambda_1(F,p)$, $\lambda_2(F,p)$ for the two Hecke operators $T_p^{(1)}$, $T_p^{(2)}$ on $Sp_2(\mathbb{Z})$.

For some reasons I believe that we must have congruences

$$\lambda_1(F,p) \equiv p^{m+1} + a_p(f) + p^{t+2} \mod \ell$$

$$\lambda_2(F,p) \equiv a_p(f)(p^{m+1} + p^{t+2}) \mod \ell.$$

Using the programs written by C. Faber and G. v.d. Geer these congruence have been checked in the four cases above for all $p \leq 37$.

For a more detailed exposition I refer to my homepage in www.math.uni-bonn.de. In my ftp-directory in the folder Eisenstein one finds a file kolloquium.ps or kolloquium.pdf.

Period functions for congruence subgroups

J. Hilgert

(joint work with D. Mayer and H. Movasati)

We report on a surprising relation between the transfer operators for the congruence subgroups $\Gamma_0(nm), n, m \in \mathbb{N}$, and some kind of Hecke operators on the space of vector valued period functions for the groups $\Gamma_0(n)$. For this we study special eigenfunctions of the transfer operators for the groups $\Gamma_0(nm)$ with eigenvalues ∓ 1 which are also solutions of the Lewis equations for these groups and which are determined by eigenfunctions of the transfer operator for the congruence subgroup $\Gamma_0(n)$. In the language of the Atkin-Lehner theory of old and new forms one should hence call them old eigenfunctions or old solutions of the Lewis equation for $\Gamma_0(n)$. It turns out that certain linear combinations of the components of these old solutions for the group $\Gamma_0(nm)$ determine for any m a solution of the Lewis equation for the group $\Gamma_0(n)$ and hence also an eigenfunction of the transfer operator for this group.

Our construction gives linear operators $\tilde{T}_n$ in the space of vector valued period functions for the group $\Gamma_0(n)$ which are rather similar to the Hecke operators. Indeed, in the case of the group $\Gamma_0(1) = \mathrm{SL}(2, \mathbb{Z})$ these operators are just the well known Hecke operators on the space of period functions for the modular group derived previously using the Eichler-Manin-Shimura correspondence between period polynomials and modular forms for this group and its extension to Maass wave forms by Lewis and Zagier.

Large scale geometry of arithmetic groups and the integral Novikov conjecture

L. Ji

The original Novikov conjecture concerns the homotopy invariance of the higher signature, which is equivalent to the rational injectivity of the assembly map in the algebraic surgery theory, i.e., the L-theory. There are also assembly maps in other theories, such as the algebraic K-theory, C^*-algebras. In each such theory, the rational injectivity of the assembly map is called the Novikov conjecture, and the integral injectivity is called the integral Novikov conjecture.

In this talk, we proved the K-theoretic integral Novikov conjecture for torsion free arithmetic subgroups of any linear algebraic group defined over $\mathbb{Q}$ which is not necessarily reductive. Besides the class of arithmetic groups, another important one is the class of S-arithmetic groups. We also proved the integral Novikov conjecture in both K-, L-theories for torsion free S-arithmetic subgroups of reductive algebraic groups G defined over a number field k with the k-rank less than or equal to 1.

To prove the conjecture for arithmetic groups, we showed that they have finite asymptotic dimension and finite classifying spaces; and the conjecture follows from a result of Bartels, Carlsson–Goldfarb that the integral Novikov conjecture in K-theory holds for groups of finite asymptotic dimension and finite classifying spaces.

To prove the conjecture for S-arithmetic groups Γ, we used the approach of Carlsson–Pedersen, refined by Goldfarb, via a suitable compactification of the universal covering $E\Gamma = \widetilde{B\Gamma}$ of a compact classifying space $B\Gamma$ of Γ.

On scattering theory on geometrically finite hyperbolic manifolds

A. Juhl

Let $\Gamma \subset SO(1,n)^\circ$ be a discrete subgroup without (non-trivial) elliptic elements such that the quotient $X^n = \Gamma\backslash\mathbb{H}^n$ ($\mathbb{H}^n$ = real hyperbolic n-space) is a geometrically finite hyperbolic manifold. The lengths of closed geodesics in X^n and their monodromies define the Selberg zeta function of X. The natural perspective here is to consider ζ as being associated to the periodic orbits c of the geodesic flow $\Phi_t : SX \to SX$ on the sphere bundle of X. In these terms

$$\zeta(s) = \prod_c \prod_{N\geq 0} \det(\mathrm{id} - S^N(P_c^-)e^{-s|c|}), \Re(s) > \delta(\Gamma),$$

where $|c|$ is the period of the prime periodic orbit c and P_c^- its monodromy on the contracting part of the tangent bundle of SX. The central problem is to prove the meromorphy of ζ on $\mathbb{C}$ *and* to find *uniform* cohomological characterizations of its divisor (zeros and poles with multiplicities). Such results are now known for cocompact and convex-cocompact Γ. However, the geometrically finite case is still far from being understood.

With the latter problem as motivation we haven undertaken a detailed analysis of Eisenstein series on cylindrical $(n+1)$-spaces X^{n+1} with a finite volume hyperbolic manifold (with at least 1 cusp) X^n as cross-section. X^{n+1} then only has cusps of lower (i.e., non-maximal) rank n. We prove that the Eisenstein kernel

$$E^{n+1}(x,\zeta;\lambda) = \sum_{\gamma\in\Gamma} e^{\lambda\langle\gamma x,\zeta\rangle},\ \Re(\lambda) > \delta(\Gamma),\ x \in \mathbb{H}^n,\ \zeta \in \partial_\infty\backslash\Lambda(\Gamma)$$

has a meromorphic continuation to $\mathbb{C}$. The proof provides explicit information on poles and singular parts in terms of data associated to X^n. It rests on spectral theory of X^n and hypergeometric functions. Moreover, we determine a complete asymptotics of the above kernel for ζ near the cusps. The structure of the latter asymptotics is crucial in connection with the theory of Eisenstein series associated to Γ-invariant functions on the proper set $\Omega(\Gamma)$. Although the results show that generic poles of the Eisenstein kernel correspond to generic zeros of $\zeta(s)$ the complete relation is more involved and not yet understood. The results will serve as test cases for future general theories.

Cycles of surfaces and modular forms

S. Kudla

(joint work with J. Millson and S. Rallis (section 1),
and with M. Rapoport and T. Yang (section 2))

1. Complex surfaces

For V, $(\ ,\ )$ a quadratic space over $\mathbb{Q}$ of signature (2,2), let $G = \mathrm{SO}(V)$, $D \simeq \mathfrak{H} \times \mathfrak{H}$, the symmetric space associated to $G(\mathbb{R})$ and, for $K \subset G(\mathbb{A}_f)$, a compact open subgroup, $S_K = G(\mathbb{Q})\backslash\big(\ D \times G(\mathbb{A}_f)/K\big)$. Suppose that V is anisotropic over $\mathbb{Q}$ and that K is sufficiently small. Then S_K is a smooth projective surface over $\mathbb{C}$. We consider two generating functions for algebraic cycles on S_K. For $r = 1$, let $\tau = u + iv \in \mathfrak{H}_1$, $q = e(\tau) = e^{2\pi i\tau}$ and let $\varphi \in S(V(\mathbb{A}_f))^K$ be a K–invariant weight function. Define the generating function

$$\phi_1(\tau,\varphi) = \sum_{t\geq 0} [\, Z(t,\varphi)\,]\, q^t,$$

where $Z(t,\varphi)$ is a combination of curves on S_K analogous to Hirzebruch–Zagier curves, and $[Z(t,\varphi)] \in H^2(S_K)$ is the corresponding cohomology class. Then $\phi_1(\tau)$ is a modular form of weight 2. The modularity of $\phi_1(\tau,\varphi)$ results from the fact that

$$\phi_1(\tau,\varphi) = [\,\theta_1(\tau,\varphi)\,] \in H^2(S_K),$$

where $\theta_1(\tau,\varphi)$ is a nonholomorphic theta function valued in closed $(1,1)$–forms on S_K. For $r=2$, there is an analogous theta function $\theta_2(\tau,\varphi)$ where, now, $\tau = u+iv \in \mathfrak{H}_2$, the Siegel space of genus 2, $\varphi \in S(V(\mathbb{A}_f)^2)^K$, valued in $(2,2)$–forms on S_K. Then

$$\phi_2(\tau,\varphi) = [\,\theta_2(\tau,\varphi)\,] = \sum_{T\geq 0} [\,Z(T,\varphi)\,]\, q^T,$$

for $q^T = e(\mathrm{tr}(T\tau))$, $T \in \mathrm{Sym}_2(\mathbb{Q})$, and, for $T>0$, $Z(T,\varphi)$ is either empty or is a 0–cycle on S_K. Thus $\phi_2(\tau)$ is a Siegel modular form of weight 2 and genus 2. Finally, there is an Eisenstein series $E_2(\tau,s,\varphi)$ of weight 2 and genus 2 such that

$$E(\tau,\tfrac{1}{2},\varphi) = \phi_2(\tau).$$

These objects are related by an cup product identity

$$\langle \phi_1(\tau_1,\varphi_1), \phi_1(\tau_2,\varphi_2)\rangle \;=\; \phi_2\Big(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}, \varphi_1\otimes\varphi_2\Big) \;=\; E_2\Big(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}, \tfrac{1}{2}, \varphi_1\otimes\varphi_2\Big).$$

2. Arithmetic surfaces

The main point is that there are analogues of the identities of the previous section for cycles of codimensions 1 and 2 on the arithmetic surface associated to a Shimura curve over $\mathbb{Q}$.

Let B be an indefinite division quaternion algebra over $\mathbb{Q}$. Let O_B be a maximal order in B and let $C(\mathbb{C}) = C_B(\mathbb{C}) = O_B^\times\backslash(\mathfrak{H}^+\cup\mathfrak{H}^-)$ be the (complex points of the) Shimura curve over $\mathbb{Q}$. Let $\mathcal{S}$ be the Drinfeld model of C over $\mathrm{Spec}\,(\mathbb{Z})$. For $r=1$ and 2, we construct generating functions $\hat{\phi}_1(\tau)$ valued in $\widehat{\mathrm{CH}}^1(\mathcal{S})$, the first arithmetic Chow group of $\mathcal{S}$, and $\hat{\phi}_2(\tau)$ valued in $\widehat{\mathrm{CH}}^2(\mathcal{S}) \simeq \mathbb{C}$, the second arithmetic Chow group of $\mathcal{S}$, and, in addition, a normalized Eisenstein series $\mathcal{E}_2(\tau,s;B)$ of weight $\frac{3}{2}$ and genus 2. Our recent results are the following:

Theorem 1: *$\hat{\phi}_1(\tau)$ is a modular form of weight $\frac{3}{2}$ valued in $\widehat{\mathrm{CH}}^1(\mathcal{S})$*

Theorem 2:

$$\hat{\phi}_2(\tau) = \mathcal{E}_2'(\tau,0;B).$$

In particular, $\hat{\phi}_2(\tau)$ is a Siegel modular form of weight $\frac{3}{2}$ valued in $\widehat{\mathrm{CH}}^2(\mathcal{S})$.

Finally, there is a height pairing identity:

Theorem 3:

$$\langle \hat{\phi}_1(\tau_1), \hat{\phi}_1(\tau_2)\rangle \;=\; \hat{\phi}_2\Big(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}\Big) \;=\; \mathcal{E}_2'\Big(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}, 0; B\Big),$$

where $\langle\ ,\ \rangle$ is the Gillet–Soulé height pairing on $\widehat{\mathrm{CH}}^1(\mathcal{S})$.

References

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Geometric approach to trace formula stabilization

J.P. Labesse

The stabilization program was started 30 years ago by Langlands with in view the study of the Zeta function of Shimura varieties. The Jacquet-Langlands correspondence between automorphic forms for quaternion algebras and automorphic forms on $GL(2)$ can be seen as the first instance of this program: the transfer between groups and their inner forms. Next was the study by Langlands and myself of the stabilization of the trace formula for $SL(2)$. We were trying to detect representations π of $GL(2)$ such that $\pi \otimes \omega \simeq \pi$ for some character ω. Another instance of stabilization (in the modern sense) is the base change for cyclic Galois extensions; there one tries to detect representations s.t. $\pi \circ \theta \simeq \pi$.

The general setting is as follows. One considers a twisted space L over a group G i.e. a principal homogeneous space L under G together with a G-equivariant map from L to $\mathrm{Aut}(G)$. On a twisted space one has a left and a right G-action, hence one can consider conjugacy etc... If L is defined over a number field F one can define a right action of $L(\mathbb{A}_F)$ on $G(F)\backslash G(\mathbb{A}_F)$ provided there is a rational point $\delta_0 \in L(F)$. Now, given $f \in \mathcal{C}_c^\infty(L(\mathbb{A}_F))$ the contribution of elliptic elements to geometric side of the trace formula is the integral

$$T_e(f,\omega) = \int_{\mathfrak{A}_G G(F)\backslash G(\mathbb{A}_F)} K_e(x,x)dx$$

where

$$K_e(x,x) = \omega(x) \sum_{\delta \in L_e} f^1(x^{-1}\delta x) \qquad \text{with} \qquad f^1(x) = \int_{\mathfrak{A}_G} f(zx)dz$$

and L_e is the subset of elliptic elements in $L(F)$. Then, using abelianized Galois hypercohomology, one can show that

$$T_e(f,\omega) = \sum_{\mathcal{E} \in \underline{\mathcal{E}}(\omega)} a(\mathcal{E}) ST_e^{\mathcal{E}}(f_{\mathcal{E}})$$

under the transfer assumption. This is the stabilization of the elliptic terms of the geometric side twisted trace formula. This is a generalization of results of Kottwitz and Shelstad that have treated only the strongly regular elliptic terms. The next step will be to remove the ellipticity condition and to stabilize both sides (geometric and spectral) of the full trace formula.

Cohomology of arithmetic groups, parabolic subgroups and the special values of L-functions on GL_n

J. Mahnkopf

Let M/E be a motive; a conjecture of Deligne determines the values of the L–function $L(M,s)$ at points n which are critical for M as element of $\mathbb{C}^*/E^*$ (i.e. up to multiplication by E^*). Conjecturally, $L(M,s)$ coincides with an automorphic L-function $L(\pi,s)$ attached to an *algebraic* automorphic representation π of $GL_n(\mathbb{A})$. We explained an analogue of Deligne's conjecture for L–functions attached to the smaller class of *cohomological* cuspidal automorphic representations $\pi=\pi_f\otimes\pi_\infty$ on $GL_n(\mathbb{A})$. More precisely let $\mu\in X^+(T_n)$ be a dominant weight and denote by $Coh_0(\mu)$ the set of cuspidal automorphic representations on $GL_n(\mathbb{A})$ whose relative Lie–Algebra cohomology with respect to the coefficient system given by the highest weight module $M_\mu^\vee$ does not vanish. Let $\mathrm{Crit}(\pi)$ denote the set of all idele class characters, which are critical for π_∞ , i.e. the values $L(\pi_\infty\otimes\chi_\infty,0)$ and $L(\pi_\infty^\vee\otimes\chi_\infty^{-1},1)$ are regular. We denote by E_π the field of definition of π_f (a finite extension of $\mathbb{Q}$). There are complex "numbers" $\Omega(\pi,\chi_\infty)\in\mathbb{C}^*/E_\pi^*$ such that the following holds

Theorem. *Assume* $\mu\in X^+(T_n)$ *dominant and regular and let* $\pi\in Coh_0(\mu)$.

1.) *For all* $\chi\in Crit(\pi)$
$$L(\pi\otimes\chi,0)=\Omega(\pi,\chi_\infty)\mod E_\pi^*(\chi).$$
Moreover the ratios $\Omega(\pi,\chi_\infty)/\Omega(\pi,\chi'_\infty)$ *do not depend on* π.

2.) *For all* $\sigma\in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$
$$\left(G(\chi)^{[\frac{n}{2}]}\frac{L(\pi\otimes\chi,0)}{\Omega(\pi,\chi_\infty)}\right)^\sigma=G(\chi^\sigma)^{[\frac{n}{2}]}\frac{L(\pi^\sigma\otimes\chi^\sigma,0)}{\Omega(\pi^\sigma,\chi_\infty)}$$
($G(\chi)=$ *Gauß-sum attached to* χ, $[\ ]=$ *Gauß–bracket).*

The theorem is only valid under a certain local *non–vanishing assumption.*

The proof is by induction over the rank n and combines the method of Zeta–Integrals with the method of Langlands–Shahidi and the cohomology theory of arithmetic groups.

Cycles and homology with local coefficients

J. Millson

We prove by an elementary geometric argument that $H^p(\Gamma,W)\neq 0$ for certain $\Gamma\subset SO(n,1)$ a cocompact lattice, W an irreducible finite dimensional representation of SO(n,1) and p in a certain interval determined by the highest weight of W. We obtain all non-vanishing results for local coefficients compatible with the vanishing results of Vogan and Zuckerman [VZ].

The main Theorem

To make the above statement precise, let m be a square-free positive integer and let Γ be a congruence subgroup of the group of units of the quadratic form $f(x)=x_1^2+x_2^2+\cdots x_m^2-\sqrt{m}x_{n+1}^2$. Let W be the irreducible representation with highest weight $\mu=(a_1,\cdots,a_l)=\sum_{i=1}^l a_i\epsilon_i$, $l=[\frac{n+1}{2}]$. Here we use the notation of [Bo], pg. 252–253, for the coordinates of a weight (so the second fundamental weight has coordinates $(1,1,0,\cdots,0)$).

Definition. We define $i(\mu)$ to be the number of nonzero entries in μ.

The main theorem of [M2] is the following

Theorem.

Let W be the irreducible representation of $SO(n,1)$ with highest weight μ.

(1) *If $n = 2m-1$ and all entries of μ are nonzero then $H^p(\Phi, W) = 0$ for all cocompact lattices Φ and all p.*
(2) *Suppose either n is even or $i(\mu) > 0$. Then $p < i(\mu)$ or $p > n - i(\mu) \Rightarrow$ for all cocompact lattices Φ we have $H^p(\Phi, W) = 0$.*
(3) *Let $p \in \{i(\mu), i(\mu)+1, \cdots, n-i(\mu)\}$. Then exists an ideal $\mathfrak{b}$ in the integers of $\mathbb{Q}(\sqrt{m})$ depending on W and p such that if Γ is the congruence subgroup of level $\mathfrak{b}$ of the group of units of the form f then we have*

$$H^p(\Gamma, W) \neq 0.$$

The theorem is proved by extending the techniques of [MR] which were used to prove nonvanishing results in the *trivial coefficient* case. We replace the technique of intersecting pairs of totally-geodesic submanifolds considered there by intersecting the same submanifolds but now each is equipped with a *local coefficient*, i.e. a parallel section of the restriction of the associated local coefficient system $\widetilde{W}$ restricted to the cycle (or equivalently a nonzero vector in W fixed under the fundamental group of the submanifold).The key fact that it can be arranged that the two manifolds intersect in a single connected component we borrow from the earlier papers [MR], [JM] and [FOR]. The remaining problems are to find for which W the required local coefficients exist and then to verify that the coefficient pairings applied to the local coefficients are nonzero. The first we solve by an elementary argument from finite-dimensional representation theory and the second by a (Zariski) density argument.

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On the cuspidal spectrum for GL_n

W. Müller

Let G be a connected reductive algebraic group over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. An important problem in the theory of automorphic forms is the question of existence and the construction of cusp forms for Γ.

In this talk we will address the problem of existence for $G = \mathrm{SL}_n$, $n \geq 2$. Let Γ be a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$. Let $L^2_{\mathrm{cus}}(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}))$ be the closure of the span of cusp forms for Γ. Let (σ, V_σ) be an irreducible unitary representation of $\mathrm{SO}(n)$. Set

$$L^2(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}), \sigma) = \left(L^2(\Gamma\backslash \mathrm{SL}_n(\mathbb{R})) \otimes V_\sigma\right)^{\mathrm{SO}(n)},$$

and define $L^2_{\mathrm{cus}}(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}), \sigma)$ similarly. Let $\Omega \in \mathcal{Z}(\mathfrak{sl}(n, \mathbb{C}))$ be the Casimir element of $\mathrm{SL}_n(\mathbb{R})$. Then $-\Omega \otimes \mathrm{Id}$ induces a self-adjoint operator Δ_σ in the Hilbert space $L^2(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}), \sigma)$ which is bounded from below. The restriction of Δ_σ to the subspace $L^2_{\mathrm{cus}}(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}), \sigma)$ has a pure point spectrum consisting of eigenvalues $\lambda_0(\sigma) < \lambda_1(\sigma) < \cdots$ of finite multiplicity. Let $\mathcal{E}(\lambda_j(\sigma))$ be the eigenspace corresponding to the eigenvalue $\lambda_j(\sigma)$. For $\lambda \geq 0$ set

$$N^\Gamma_{\mathrm{cus}}(\lambda, \sigma) = \sum_{\lambda_j(\sigma) \leq \lambda} \dim \mathcal{E}(\lambda_j(\sigma)).$$

Then our main result is the following theorem.

Theorem *For $n \geq 2$ let $X_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Let $d_n = \dim X_n$. For every principal congruence subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$ and every irreducible unitary representation σ of $\mathrm{SO}(n)$ we have*

$$N^\Gamma_{\mathrm{cus}}(\lambda, \sigma) \sim \dim(\sigma) \frac{\mathrm{vol}(\Gamma\backslash X_n)}{(4\pi)^{d_n/2}\Gamma(d_n/2 + 1)} \lambda^{d_n/2}$$

as $\lambda \to \infty$.

This is Weyl's law for principal congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$. Especially, this result establishes the existence of cusp forms for the full modular group $\mathrm{SL}_n(\mathbb{Z})$ for all $n \geq 2$, which was not known before.

For the trivial representation $\sigma = 1$, the above theorem was announced by the author on a previous Oberwolfach conference. The extension to arbitrary σ is based on recent results of E. Lapid, B. Speh and the author on the absolute convergence of the Arthur trace formula for GL_n.

We also note that for $n = 2$ and the trivial representation Weyl's law was first proved by Selberg. For $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and σ the trivial representation, Weyl's law was proved by S. Miller.

It has been conjectured by Sarnak and the author that Weyl's law holds for every arithmetic subgroup of a reductive group G. The above theorem supports this conjecture.

Intersection cohomology, Shimura varieties and motives

A. Nair

Main results. Let (G, X) be a pair satisfying Deligne's axioms for Shimura data. For a small enough compact open subgroup $K \subset G(\mathbb{A}_f)$ the Shimura variety $S_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is a smooth projective variety defined over a number field E independent of K. Its (Betti, de Rham, étale) cohomology has an action of the Hecke algebra $\mathcal{H}_K = C_c^\infty(G(\mathbb{A}_f)//K)$. Assume that S_K is noncompact. The *minimal (or Satake-Baily-Borel) compactification* $\widehat{S}_K$ is a normal (but usually singular) projective variety over E in which S_K is Zariski-dense and open. Combining results of Looijenga–Saper-Stern and Borel-Casselman we have a natural Hecke-equivariant isomorphism

$$\mathrm{IH}^i(\widehat{S}_K, \mathbb{Q}) \otimes \mathbb{C} \cong \bigoplus_\pi m(\pi)\ \pi_f^K \otimes \mathrm{H}^i(\mathbf{g}, K_\infty, \pi_\infty).$$

The sum is over π appearing discretely in $L^2(G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A}))$ with multiplicity $m(\pi)$. So intersection cohomology is the cohomology of the discrete L^2 spectrum of G.

Mumford et. al. constructed noncanonical desingularizations of $\widehat{S}_K$. For each choice of combinatorial datum Σ, they construct a *toroidal compactification* $S_K \hookrightarrow \widetilde{S}_{K,\Sigma}$ with boundary a normal crossings divisor and defined over E (Pink). There is a projective morphism $\pi : \widetilde{S}_{K,\Sigma} \to \widehat{S}_K$ extending the identity. The possible $\widetilde{S}_{K,\Sigma}$ form a directed set and the direct limit (=union) $\varinjlim_\Sigma \mathrm{H}^i\big(\widetilde{S}_{K,\Sigma}, \mathbb{Q}\big)$ is a Hecke module.

Theorem 1. *There is a canonical $\mathcal{H}_K$-module decomposition*

$$\varinjlim_\Sigma \mathrm{H}^i\big(\widetilde{S}_{K,\Sigma}, \mathbb{Q}\big) \ \cong\ \mathrm{IH}^i\big(\widehat{S}_K, \mathbb{Q}\big) \oplus \mathrm{W}_K^i.$$

It is compatible with Poincaré duality. The induced decomposition in étale cohomology is $\mathcal{H}_K \times \mathrm{Gal}\big(\overline{\mathbb{Q}}/E\big)$-equivariant, and with $\mathbb{C}$-coefficients is a decomposition of $\mathbb{Q}$-Hodge-structures.

For a fixed suitable Σ, this gives a canonical decomposition $\mathrm{H}^i\big(\widetilde{S}_{K,\Sigma}, \mathbb{Q}\big) \cong \mathrm{IH}^i\big(\widehat{S}_K, \mathbb{Q}\big) \oplus \mathrm{W}_{K,\Sigma}^i$ with $\mathrm{W}_{K,\Sigma}^i = \mathrm{W}_K^i \cap \mathrm{H}^i\big(\widetilde{S}_{K,\Sigma}, \mathbb{Q}\big)$. For each $g \in G(\mathbb{A}_f)$ the algebraic cycle $\widetilde{C}_g :=$ (closure of C_g in $\widetilde{S}_{K,\Sigma} \times \widetilde{S}_{K,\Sigma}$) gives an endomorphism $\widetilde{C}_g^* : \mathrm{H}^i(\widetilde{S}_{K,\Sigma}, \mathbb{Q}) \to \mathrm{H}^i(\widetilde{S}_{K,\Sigma}, \mathbb{Q})$. (These do *not* always generate an action of the Hecke algebra on $\mathrm{H}^i(\widetilde{S}_{K,\Sigma}, \mathbb{Q})$.)

Theorem 2. *There exist $g_1, \ldots, g_n \in G(\mathbb{A}_f)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$ such that $p = \sum_i \lambda_i\ \widetilde{C}_{g_i}$ gives a projector in the Betti (or de Rham or ℓ-adic for any ℓ) cohomology of $\widetilde{S}_{K,\Sigma}$ with image the Betti (or de Rham or ℓ-adic for any ℓ) intersection cohomology of $\widehat{S}_K$.*

Theorem 2 implies that there is a Grothendieck motive $ih\big(\widehat{S}_K\big)$ over E with coefficients in $\mathbb{Q}$ with intersection cohomology realizations, i.e. intersection cohomology is a motive. The motive has an action of the Hecke algebra $\mathcal{H}_K$ by endomorphisms. There is a decomposition $ih\big(\widehat{S}_K\big) = \bigoplus_\sigma ih\big(\widehat{S}_K\big)[\sigma]$ (with σ running over irreducible $\mathcal{H}_K$-modules) where $ih\big(\widehat{S}_K\big)[\sigma]$, in any realization, is the σ-isotypic component of intersection cohomology. Arthur's conjectures on the discrete spectrum describe the decomposition.

The proofs of theorems 1 and 2 use perverse sheaves and the Beilinson-Bernstein-Deligne-Gabber decomposition theorem, Deligne's theory of weights, and the explicit geometry of compactifications of Shimura varieties. The methods generalize to Pink's setting of mixed Shimura varieties and to homogeneous coefficient systems. Theorems 1 and 2 have several applications to the arithmetic and geometry of Shimura varieties. Two applications are: (1) To associate Grothendieck motives to non-CAP non-endoscopic cohomological cusp forms for GSp(4), improving slightly work of Taylor, Harder, Laumon, Weissauer (1990s) who associated Galois representations (2) To injectivity results for Oda restriction maps (in ordinary cohomology) to Shimura subvarieties in the noncompact setting.

All the contiguous relations in the principal series $(\mathfrak{g}, K)$-modules of $\mathrm{Sp}(2,\mathbb{R})$

T. Oda

Firstly we reviewed the state of arts on the generalized spherical functions on $G = Sp(2,\mathbb{R})$ and other related groups. Among others Ishii's result on the class one Siegel–Whittaker functions and Moriyama's application to Spinor L–functions of automorphic forms $GSp(4)$ are reviewed. An application of the branching rule of cohomological representations by Toshiyuki Kobayaski: a vanishing theorem of certain Hodge component is also reviewed. We reviewed also our construction of Green currents for modular cycles, which is a joint work with Masao Tsuzuki. A remarkable fact is that this procedure resembles to the theory of Eisenstein cohomology.

As shown in the first part of the talk, special functions, or (generalized) spherical functions on semisimple Lie groups play an important role in the theory of automorphic forms. There are two reasons at least:

1) the theory of automorphic L–functions at infinite places;
2) the seeds (= the secondary spherical function) for construction of Green currents for modular symbols by the procedure of regularisation using Poincaré series.

Here the secondary spherical functions are "bad" spherical functions, which satisfy the same differential equations on the Lie groups as the usual spherical functions, but have some singularities along the subgroup R to define the spherical model.

As a case study the affine symmetric space $(G \times G, \triangle G)$ with $G = Sp(2,\mathbb{R})$ seems to be quite interesting, to consider a higher rank symmetric pair.

Recently the author found an explicit description of $(\mathfrak{g}, K)$–module structure of the principal series representations of G : these consist of 12 different contiguous relations, which are an abstract system of differential–difference equations. As an application of this, we can embed a large discrete series representation of G into a generalized principal series representation associated with the maximal parabolic subgroups P_J corresponding to the long root $2\varepsilon_2$ in the simple roots $\{\varepsilon, -\varepsilon_2, 2\varepsilon_2\}$ of $Sp(2,\mathbb{R})$. By this, we can reduce the problem of *determination* (i.e. integral expression, powerseries expression, etc.) of the spherical function with minimal K–type in the large discrete series, to a similar problem for the spherical function with the corner K–type in the P_J–principal series.

Cohomology of convex cocompact groups

M. Olbrich

Let $Y = \Gamma\backslash G/K$ be a Riemannian locally symmetric space of the non–compact type, and let F be a finite-dimensional irreducible representation of G. One of the central questions in the theory of automorphic forms is to describe the cohomology groups

$$H^*(\Gamma, F) \cong H^*(Y, \underline{F})$$

in terms of automorphic forms on Y. This amounts to develop a type of Hodge theory. This question has been intensively investigated for finite volume quotients.

In the talk I described an answer to this question for a class of infinite volume hyperbolic manifolds and indicated implications for the singularities of twisted Selberg zeta functions. This work was motivated by conjectures of Patterson (presented at a talk in Warwick in 1993). Complete results can be found in M. Olbrich, "Cohomology of convex cocompact groups ...", arXiv:math.DG/0207301.

Let us describe one of the main results. Let G/K be the real hyperbolic space, and let $\Gamma \subset G$ be a convex cocompact discrete subgroup. The role of automorphic forms is played by the space

$$C^{-\infty}(\Lambda, V^+(\sigma_\lambda))^\Gamma$$

of Γ–invariant distribution vectors of (thickend) principal series representations which are supported on the limit set. It has a distinguished subspace $E^+_\Lambda(\sigma_\lambda)$ formed by boundary values of singular parts of Eisenstein series. Then

a) $\dim C^{-\infty}(\Lambda, V^+(\sigma_\lambda))^\Gamma < \infty$.
b) If the infinitesimal character $\chi_{\sigma,\lambda} : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is not regular integral, then $C^{-\infty}(\Lambda, V^+(\sigma_\lambda))^\Gamma = E^+_\Lambda(\sigma_\lambda)$.
c) If $\chi_{\sigma,\lambda}$ is regular and integral, then (σ, λ) determines an irreducible finite-dimensional representation F of G and an integer $p \in \{1, \dots, n\}$ such that there is an exact sequence

$$0 \longrightarrow E^+_\Lambda(\sigma_\lambda) \longrightarrow C^{-\infty}(\Lambda, V^+(\sigma_\lambda))^\Gamma \longrightarrow H^p(\Gamma, F) \longrightarrow 0\ .$$

Cohomology of locally symmetric spaces and their compactifications

L.D. Saper

Let $X = \Gamma\backslash D$ be a locally symmetric space, where $D = G/K$ is a symmetric space, G is the group of real points of a reductive algebraic group defined over $\mathbb{Q}$, K is a maximal compact subgroup, and $\Gamma \subset G$ is an arithmetic subgroup. (To simplify this report we assume that G has no $\mathbb{Q}$-split torus in its centre.) Let $\widehat{X}$ be the reductive Borel-Serre compactification of X introduced by Zucker (1982) and let X^* be a Satake compactification (1960). In the case where D is Hermitian symmetric an important example is the Baily-Borel Satake compactification (1966). Any Satake compactification X^* may be realized as a quotient of $\widehat{X}$ (Zucker, 1983), but it is easier to do local computations on $\widehat{X}$ due to its relatively simple local structure in distinction to the more singular X^*.

Let $\mathbb{E}$ is a local coefficient system on X associated to a representation E of G. We study the relationships between three cohomology theories: the ordinary cohomology $H(X; \mathbb{E})$, Goresky-MacPherson's (1980, 1983) intersection cohomology $I_pH(\widehat{X}; \mathbb{E})$ and $I_pH(X^*; \mathbb{E})$

for either middle perversity $p = m$ or n, and Goresky-Harder-MacPherson's weighted cohomology (1994) $W^\eta(\widehat{X};\mathbb{E})$ for either middle weight profile $\eta = \mu$ or ν.

Theorem 1. Assume that all real boundary components of X^* are equal-rank locally symmetric spaces. (In particular this applies if X^* is the Baily-Borel-Satake compactification.) Then $I_pH(\widehat{X};\mathbb{E}) \cong I_pH(X^*;\mathbb{E}) \cong W^\eta H(\widehat{X};E)$.

The first isomorphism was conjectured in the Hermitian case by Goresky-MacPherson (1988) and Rapoport (1986). The second isomorphism was proved in the Hermitian case by Goresky-Harder-MacPherson (1994). The full result was proved in (S., 2001).

Theorem 2. If E is irreducible with regular highest weight, then $H^i(X;\mathbb{E}) = 0$ for $i < \frac{1}{2}(\dim X - (\mathbb{C}\text{-rank}\, G/A_G - \operatorname{rank} K))$.

In the equal-rank case, this was proved in (S., 2001); as noted in (S., 2003), the same proof applies in general. The result has also been proven by completely different methods by Li and Schwermer (2003).

Theorem 3. If $E^* \cong \overline{E}$ and the $\mathbb{Q}$-root system of G does not have any factor of type D_n, E_n, or F_4, then $I_mH(\widehat{X};\mathbb{E}) \cong W^\mu H(\widehat{X};E)$ and $I_nH(\widehat{X};\mathbb{E}) \cong W^\nu H(\widehat{X};E)$.

This result is recent (2003) and not yet written up. We expect the condition on the $\mathbb{Q}$-root system will be removed. If $E^* \ncong \overline{E}$, there is a more complicated result that expresses $I_pH(\widehat{X};\mathbb{E})$ as the direct sum of weighted cohomology groups of certain singular strata of $\widehat{X}$ with certain coefficients.

All these results may be proved using the theory of $\mathcal{L}$-modules and their micro-support (S., 2001); an *$\mathcal{L}$-module* $\mathcal{M}$ is a combinatorial analogue of a constructible complex of sheaves on $\widehat{X}$ and its cohomology is determined by its micro-support.

On the automorphic cohomology of arithmetic groups

J. Schwermer

The cohomology $H^*(\Gamma, E)$ of an arithmetic subgroup Γ of a connected reductive algebraic group G defined over some algebraic number field F can be interpreted in terms of the automorphic spectrum of Γ. With this frame work in place there is a sum decomposition of the cohomology into the cuspidal cohomology (i.e. classes represented by cuspidal automorphic forms for G) and the so called Eisenstein cohomology constructed as the span of appropriate residues or derivatives of Eisenstein series attached to cuspidal automorphic forms on the Levi components of proper parabolic F–subgroups of G. The talk had two objectives: (1) a discussion of the regular Eisenstein cohomology classes attached to cuspidal automorphic representations whose archimedean component is tempered. It was indicated that the cohomological degree of these classes is bounded from below by the constant $q_0(G(\mathbb{R})) = (1/2)[\dim X_{G(\mathbb{R})} - (\operatorname{rk} G(\mathbb{R}) - \operatorname{rk} K)]$ where K denotes a maximal compact subgroup of the real Lie group $G(\mathbb{R}), X_{G(\mathbb{R})}$ the associated symmetric space. One of the consequences of this result is a vanishing theorem for the cohomology $H^*(\Gamma, E)$ in the generic case (i.e. where the representation determining the coefficient system E has regular highest weight). This is a sharp bound only depending on the underlying real Lie group $G(\mathbb{R})$: one has $H^j(\Gamma, E) = 0$ for $j < q_0(G(\mathbb{R}))$. This result is supplemented by a qualitative structural result in the description of the cohomology in higher degrees by means of regular Eisenstein cohomology classes. (2) A discussion of some recent vanishing theorems for the cuspidal cohomology of congruence subgroups in classical groups. The approach is based on R. Howe's notion of rank for irreducible representations of $G(F_v), F_v$ the local field attached to the place v of F, a global property of cuspidal automorphic representations and explicit calculations of the rank for irreducible unitary representations of $G(F_v), v$ an archimedean place. [(1), (2) are joint work with Jian-shu Li (Hong Kong)].

Functoriality: From symmetric powers to non-classical groups

F. Shahidi

This is a report on new cases of Langlands Functoriality conjecture. While the results on symmetric powers for $GL(2)$ were published a year ago, there are quite recent results on functoriality for groups whose L–groups are of classical type, i.e. either classical or similitude classical (${}^LG = Sp$, SO, GSp, GSO).

In the case that the second group is GL_m, Langlands Functoriality conjecture demands that every homomorphism $\rho : {}^LG \longrightarrow GL_m(\mathbb{C}) \times Gal(\overline{F}/F)$, where $\mathbf{G}$ is a connected reductive group over a number field F with L–group ${}^LG$, must lead to a correspondence sending an automorphic representation $\pi = \otimes_v \pi_v$ of $\mathbf{G}(\mathbb{A}_F)$ to $\pi' = \otimes_v \pi'_v$, an automorphic representation of $GL_m(\mathbb{A}_F)$, in such a way that the Hecke–Conjugacy classes in $GL_m(\mathbb{C})$ which parametrize unramified components π'_v are generated by the images of those in ${}^LG$ parametrizing π_v's.

When ρ is either Sym^3 or Sym^4 representation of $GL_2(\mathbb{C})$, this conjecture has been fully established by Kim and Shahidi in papers published recently. They are consequences of establishing functoriality for $\rho = \rho_2 \otimes \rho_3 : GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ and $\rho = \wedge^2 : GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ sending cusp forms on $GL_2(\mathbb{A}_F) \times GL_3(\mathbb{A}_F)$ and $GL_4(\mathbb{A}_F)$ to automorphic forms on $GL_6(\mathbb{A}_F)$, respectively, which were established in the same papers. Due to the work of Harris–Taylor and Henniart, the candidate at every other finite place as well as archimedean ones (Langlands), are well–defined and our transfers agree with the local Langlands correspondence everywhere. The estimates 1/9 and 7/64 which are due to Kim–Shahidi ad Kim–Sarnak for forms over arbitrary number fields and $\mathbb{Q}$, respectively, are proved as exponents for estimates for Hecke–eigenvalues towards Ramanujan–Selberg conjectures at every prime as consequences of these results. The best estimate for Selberg's conjecture on the smallest positive eigenvalue of Laplacian on a hyperbolic Riemann surface at present is $\cong 0.238\ldots$ and follows from 7/64.

When $\mathbf{G}$ is a classical group, either split or quasi split unitary, and $\rho : {}^LG \hookrightarrow GL_N(\mathbb{C}) \times Gal(\overline{F}/F)$ is the natural embedding, the functoriality is proved by Cogdell–Kim–Piatetski-Shapiro–Shahidi for the generic spectrum of $\mathbf{G}(\mathbb{A}_F)$. (The unitary case is the subject matter of a forthcoming paper of Kim and Krishnamurthy). As a consequence CKPSS show that the Ramanujan Conjecture for the generic spectrum of split classical groups reduces to that of $GL_m(\mathbb{A}_F)$ for all $m \leq N$. When the Langlands–Shahidi method and these transfers are fully developed for function fields, the Ramanujan Conjecture for classical groups will immediately follow from Lafforgue's results.

When ${}^LG = GSO_{2n}(\mathbb{C})$ or $GSp_{2n}(\mathbb{C})$ and ρ is their natural embeddings in $GL_{2n}(\mathbb{C})$, the transfer of generic cusp forms from the corresponding $GSpin$–groups has now also been established by Asgari–Shahidi.

The main technical obstacle in establishing these last two results, that of stability of root numbers under highly ramified twists, has now been established of by combining a recent result of Shahidi with asymptotics of Bessel functions due to Cogdell–Piatetski–Shapiro. All these results are obtained by applying converse theorems of Cogdell–Piatetski-Shapiro to analytic properties of automorphic L–functions proved by Langlands–Shahidi method.

Complex methods for real Lie-groups

R. Stanton

(joint work with B. Krötz)

This is a report on two joint papers.

Let G be a connected semisimple Lie group and assume it is contained in a complexification $G_{\mathbb{C}}$. Take an Iwasawa decomposition of G, denoted $G = KAN$. Define the open set

$$\mathfrak{A}_{\pi/2} = \big\{X \in \mathfrak{A} \;\big|\; |\alpha(X)| < \pi/2,\ \alpha \text{ a restricted root}\big\}.$$

Theorem: *Let $(\pi, \mathcal{H}_\pi)$ be an irreducible Banach representation of G and $v \in \mathcal{H}_\pi$ a K-finite vector. Then the map*

$$g \longmapsto \pi(g)v$$

has a holomorphic extension to $G \exp i\,\mathfrak{A}_{\pi/2} K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$.

One can consider $\Xi := G \exp i\,\mathfrak{A}_{\pi/2} K_{\mathbb{C}}/K_{\mathbb{C}} \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$ as a natural domain for the holomorphic extension of harmonic analysis on G/K. About the geometry of Ξ we show

(i) Ξ is Stein;
(ii) Ξ has a large supply of Kähler metrics with associated Riemannian metric complete;
(iii) Ξ contains a subdomain biholomorphic but not isometric to an Hermitian symmetric space. We give necessary and sufficient conditions for the subdomain to agree with Ξ.

An important application of the holomorphic extension is to obtain estimates of Fourier coefficients of automorphic forms and triple products. This we do by generalizing and extending ideas on invariant Sobolev norms, introduced by Bernstein–Reznikov '99. Our representation theoretic techniques allow us to obtain results on triple products for $\mathrm{SL}(n, \mathbb{R})$.

Construction of certain generalised modular symbols

T.N. Venkataramana

(joint work with B. Speh)

We consider a pair (H, G), H a semi-simple $\mathbb{Q}$-subgroup of a semi-simple algebraic group G, with an inclusion of the symmetric spaces $X_H \hookrightarrow X_G$. Let $\Gamma \subset G(\mathbb{Q})$ be a congruence subgroup and consider the finite (proper) map $S_H(\Gamma) = \Gamma \cap H\backslash X_H \to \Gamma\backslash X_G$. We get a special cycle, a fundamental class $[S_H(\Gamma)] \in \mathrm{H}^{D-d}(\Gamma\backslash X_G)$, where d, D are the dimensions of X_H and X_G respectively. Consider the compact dual spaces $\widehat{X}_H \hookrightarrow \widehat{X}_G \in \widehat{X}$. We get a class $[\widehat{X}_H] \in \mathrm{H}^{D-d}(\widehat{X}_G)$. There exists the Borel map $j : \mathrm{H}^*(\widehat{X}_G) \to \mathrm{H}^*(\Gamma\backslash X_G)$ which is injective if and only if G is anisotropic over $\mathbb{Q}$. We prove

Theorem 1: *The class* $j[\widehat{X}_H]$ *is a span of Hecke translates of the class* $[S_H(\Gamma)]$. *In particular, if* $j[\widehat{X}_H] \neq 0$, *we obtain the nonvanishing of the generalised symbol* $[S_H(\Gamma)]$.

The proof uses the fact (proved by J. Franke) that the space of $G(\mathbb{A}_f)$ invariants of the $G(\mathbb{A}_f)$-module $\mathrm{H}^*(S_G)$, which is the direct limit as Γ runs through congruence subgroups of $G(\mathbb{Q})$ of the cohomology groups $\mathrm{H}^*(S(\Gamma))$, is a direct summand of $\mathrm{H}^*(S_G)$ as a $G(\mathbb{A}_f)$-module. The proof also uses the description of the $G(\mathbb{A}_f)$-invariants as a cohomology group of $\mathrm{H}^*(U)$ for a certain open subset U of $\widehat{X}$. As a consequence we deduce

Theorem 2: *The modular symbol* $[S_H(\Gamma)] \neq 0$ *for some congruence subgroup* $\Gamma \subset G(\mathbb{Q})$, *for the pairs* (G, H) *below*

(1) $G = R_{E/\mathbb{Q}}(\mathrm{SL}_{2n})$, $H = R_{E/\mathbb{Q}}(\mathrm{Sp}_{2n})$ *with* $E/\mathbb{Q}$ *an imaginary quadratic extension, and* $R_{E/\mathbb{Q}}$ *being the Weil restriction of scalars.*
(2) $G = R_{E/\mathbb{Q}}(\mathrm{SL}_{2n+1})$, $H = R_{F/\mathbb{Q}}(\mathrm{SL}_{2n+1})$, *where* E/F *is a totally imaginary quadratic extension of a totally real number field.*
(3) $G = \mathrm{Sp}_{2n}$, $H = \prod_{i=1}^{r} \mathrm{Sp}_{2n_i}$, *with* $\sum n_i = n$.

Trace formula and asymptotics for p–form coefficients on real hyperbolic space

F. Williams

Using the trace formula we analyze the spectral zeta function for twisted p–forms on a compact real hyperbolic space. Several applications are presented.

(1) Small–time asymptotics of the heat kernel where we compute explicitly all Minakshisundaram–Pleijel coefficients (= a new proof of results in R. Miatello's thesis),
(2) a new proof of Weyl's law for $L^2(\Gamma\backslash G)$–multiplicities,
(3) computation of the multiplicative anomaly for 2 Laplace operators,
(4) a formula that relates the spectral zeta function to the Selberg zeta function and
(5) various applications to physics — including a proof that the topological component of the vacuum energy of a hyperbolic Kaluza–Klein space–time is *always* negative.

Edited by Cornelia Busch

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