Mathematisches Forschungsinstitut Oberwolfach

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Mini-Workshop: Wavelets and Frames

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Introduction by the Organisers

The workshop was centered around two important topics in modern harmonic analysis: "Wavelets and frames", as well as the related topics "time-frequency analysis" and "operator algebras". [1]

The theory of *frames*, or stable redundant non-orthogonal expansions in Hilbert spaces, introduced by Duffin and Schaeffer in 1952, plays an important role in *wavelet theory* as well as in *Gabor (time-frequency) analysis* for functions in $L^2(\mathbb{R}^d)$. Besides traditional and relevant applications of frames in signal processing, image processing, data compression, pattern matching, sampling theory, communication and data transmission, recently the use of frames also in numerical analysis for the solution of operator equation by adaptive schemes is investigated. These important applications motivated the study of frames as decompositions in classical Banach spaces, e.g. Lebesgue, Sobolev, Besov, and modulation spaces. Fundamental concepts on operator theory, as well as on the theory of representations of groups and algebras are also involved and they have inspired new directions within frame theory with applications in pseudodifferential operator and symbolic calculus and mathematical physics.

[1]Partial travel funding was provided by grants in Austria, Germany, and the USA. The three organizers from the US, are part of a Focused Research Group (FRG), funded by the US National Science Foundation (NSF), and two other participants are in this FRG group, Professors Chris Heil, GATECH, USA, and Akram Aldroubi, Vanderbilt University, USA. The organizers thank the US NSF for partial support.

Any element f of the Hilbert space $\mathcal{H}$ can be expanded as a series with respect to a frame $\mathcal{G} = \{g_n\}_{n\in\mathbb{Z}^{2d}}$ in $\mathcal{H}$, and the coefficients of such expansion can be computed as scalar products of f with respect to a *dual frame* $\tilde{\mathcal{G}} = \{\tilde{g}_n\}_{n\in\mathbb{Z}^{2d}}$:

$$f = \sum_{n\in\mathbb{Z}^{2d}} \langle f, \tilde{g}_n\rangle g_n, \text{ for all } f \in \mathcal{H}. \tag{1}$$

In particular, $\mathcal{G}$ is a frame if (and only if) the so called *frame operator*

$$Sf = \sum_{n\in\mathbb{Z}^{2d}} \langle f, g_n\rangle g_n,$$

is continuous and continuously invertible on its range. Then there exists a *canonical* choice of a possible dual frame (delivering the minimal norm coefficient) defined by the equation

$$S\tilde{\mathcal{G}} = \mathcal{G}.$$

The existence of a dual frame makes the expansion (1) work. On the other hand, it may be a hard problem to predict properties of the *canonical dual frame* since it is only implicitly defined by the previous equation, and not always is there an efficient way of computation approximations at hand. This motivated the so called *localization theory for frames*, making use of well-chosen Banach $*$-algebras of infinite matrices. They allow to deduce relevant properties of the canonical dual and to extend the Hilbert space concept of frames to *Banach frames* which characterize corresponding families of Banach spaces.

Another problem within frame theory concerns structured families of functions, depending perhaps on several parameters, and the question of whether such a family constitutes a frame for $L^2(\mathbb{R}^d)$. Classical examples are the following ones. Gabor frames are frames in $L^2(\mathbb{R}^d)$ constructed by *modulations* and *translations*: given a square-integrable function g our sequence is $g_{nm}(x) = e^{2\pi i(m,x)} g(x-n)$, $(n,m) \in \Lambda$, where Λ is a discrete subset of $\mathbb{R}^{2d}$. The wavelet frames are constructed using *dilations* and *translations*: given a set $\Delta \subset \mathrm{GL}(d,\mathbb{R})$ and $\Gamma \subset \mathbb{R}^d$, as well as a suitable square integrable function ψ, we set $\psi_{D,\gamma}(x) = |\det D|^{1/2}\psi(Dx+\gamma)$ for $D \in \Delta$ and $\gamma \in \Gamma$. They are canonically related to Besov spaces. The reader can find several interesting questions and problems related to those concepts in the following abstracts.

We would like to exemplify here two simple existence problems. If the density of the points in Λ is too small, then a Gabor frame cannot be constructed, and if the density is too large, then one can construct a frame, but not a basis. Suitable definitions of density and their relations with respect to the existence of frames in one of the current relevant topics in the frame theory.

In the wavelet case, an interesting problem has been the construction of *wavelet sets*. Given the set Δ and Λ, find the measurable subsets $\Omega \subset \mathbb{R}^d$ of positive, and finite measure, such that, with $\hat{\psi} = \chi_\Omega$, the sequence $\{\psi_{D,\gamma}\}_{D\in\Delta,\gamma\in\Gamma}$, is an orthogonal basis for $L^2(\mathbb{R}^d)$. Such a set is called a *wavelet set*. This line of work includes both geometry (tilings of $\mathbb{R}^d$) and analysis (the Fuglede conjecture). More general question is when $\{\psi_{D,\gamma}\}_{D\in\Delta,\gamma\in\Gamma}$ can be a frame.

Other more general frames, called *wave packets*, can be constructed as combinations of modulations, translations and dilations to *interpolate* the time-frequency properties of analysis of Gabor and wavelet frames. Interesting problems related to density and existence of such frames are an important direction of research and connections with new Banach spaces (for example *α-modulation spaces*), Lie groups (for example the *affine Weyl-Heisenberg group*), and representation theory (for example the *Stone-Von Neumann representation*) are currently fruitful fields of investigation. All these families of frames are generated by the common action of translations. *Shift invariant spaces* and *their generators* constituted the main building blocks from which to start the construction of more complicated systems. They showed relevant uses in engineering, signal and image processing, being one of the most prominent branch in the applications.

Rather than formal presentations of recent advances in the field, this workshop tried instead to aim at outlining the important problems and directions, as we see it, for future research, and to discuss the impact of the current main trends. In particular, the talks were often informal with weight on interaction between the speaker and the audience, both in form of discussion and general comments. A special problem session was organized by D. Larson one afternoon. Another afternoon session was devoted to talks and informal discussions of further open problems, new directions, and trends.

The topics that emerged in these discussions included the following general areas:

(1) Functional equations and approximation theory: wavelet approximation in numerical analysis, PDE, and mathematical physics. At the meeting, we discussed some operator theoretic methods that resonate with what numerical analysts want, and questions about localizing wavelets. We refer to the abstracts by M. Frank and K. Urban for more details. Two workshop lectures covered connections to numerical analysis and PDE.

(2) Gabor frames: We had many discussions, much activity, and several talks on aspects of this. H. Feichtinger explained some important results and discussed some open problems involving frames and Gelfand triples. K. Gröchenig gave a lecture on new formulations and results generalizing Wiener's inversion theorems, in particular for twisted convolution algebras and Gabor frames. The applications are striking in that they yield sharper frame bounds. And they involve non-commutative geometry and other operator algebraic tools. C. Heil discussed the basic properties of frames which are not bases, and in particular he discussed the current status of the still-open conjecture that every finite subset of a Gabor frame is linearly independent. A related problem is that there do not exist any explicit estimates of the frame bounds of finite sets of time-frequency shifts.

(3) Continuous vs. discrete wavelet transforms: We had several talks at the Oberwolfach workshop where the various operations, translation, scaling, phase modulation, and rotation, get incorporated into a single group. H.

Führ and G. Ólafsson gave talks, where links to Lie groups and their representations were discussed. This viewpoint seems to hold promise for new directions, and for unifying a number of current wavelet constructions, tomography, scale-angle representations, parabolic scaling, wavelet packets, curvelets, ridgelets, de-noising ... Wavelets are usually thought of as frames in function spaces constructed by translations and dilations. Much less is understood in the case of compact manifolds such as the n-dimensional sphere, where both "dilations" and "translations" are not obviously defined. The talk by Ilgewska–Nowak explained some of her joint work with M. Holschneider on the construction of discrete wavelet transforms on the sphere.

(4) Harmonic analysis of Iterated Function Systems (IFS): Several of the participants have worked on problems in the area, and P. Jorgensen spoke about past work, and directions for the future. The iterated function systems he discussed are closely related to the study of spectral pairs and the Fuglede problem. Recent work by Terence Tao makes the subject especially current.

(5) Multiplicity theory, spectral functions, grammians, generators for translation invariant subspaces, and approximation rates: We had joint activity at the workshop on problems in the general area, and we anticipate joint papers emerging from it. A. Aldroubi lectured on the engineering motivations. In particular he discussed translation invariant subspaces of $L^2(\mathbb{R})$ where two lattice-scales are involved, and issues about localizing the corresponding generating functions for such subspaces. O. Christensen presented an equivalent condition for two functions generating dual frame pairs via translation. The result lead to a way of finding a dual of a given frame, which belongs to a prescribed subspace. Several open questions related to this were discussed.

(6) Decompositions of operators and construction of frames: D. Larson discussed the problem of when is a positive operator a sum of finitely many orthogonal projections, and related it to frame theory. Problems and some recent results and techniques of D. Larson and K. Kornelson were discussed in this context, involving other related types of targeted decompositions of operators. In response, H. Feichtinger and K. Gröchenig pointed out that similar techniques just may lead to progress on a certain problem in modulation space theory. There are plans to follow up on this lead.

(7) Wave packets: We had two talks at the workshop about this broad research area. G. Kutyniok gave a talk about the role of the geometric structure of sets of parameters of wave packets for the functional properties of associated systems of functions. In this context some recent results of D. Speegle, G. Kutyniok, and W. Czaja were discussed. M. Fornasier presented the construction of a specific family of wave packet frames for $L^2(\mathbb{R})$ depending on a parameter $\alpha \in [0, 1)$, as a mixing tuner between

Gabor and wavelet frames. These more classical and well-known frames arise as special and extreme cases.

The organizers:

H. Feichtinger, P. Jorgensen, D. Larson, and G. Ólafsson

Mini–Workshop on Wavelets and Frames

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Abstracts

Almost Translation Invariant Spaces
Akram Aldroubi

Shift invariant spaces that are considered are of the form

$$V^2(\Phi) = \Big\{ \sum_{j\in\mathbb{Z}} D(j)^T\Phi(\cdot - j) : D \in (\ell^2)^{(r)} \Big\} \tag{1}$$

for some vector function $\Phi = (\phi_1, \ldots, \phi_r)^T \in (L^2)^{(r)}$, where $D = (d_1, \ldots, d_r)^T$ is a vector sequence such that $d_i := \{d_i(j)\}_{j\in\mathbb{Z}} \in \ell^2$, i.e., $D \in (\ell^2)^{(r)}$. Thus $\sum_{j\in\mathbb{Z}} D(j)^T\Phi(\cdot - j) = \sum_{i=1}^r \sum_{j\in\mathbb{Z}} d_i(j)\phi_i(\cdot - j)$.

We also assume that the Gramian satisfies

$$G_\Phi(\xi) := \sum_{k\in\mathbb{Z}} \widehat{\Phi}(\xi + k)\overline{\widehat{\Phi}(\xi + k)}^T = I, \quad a.e.\, \xi, \tag{2}$$

where I is the identity matrix.

An important and prototypical space is the space of band-limited functions where $r = 1$, $\phi = \frac{\sin(\pi x)}{\pi x}$. This space is translation invariant for all translates. This feature is important in applications since it allows the construction of signal/image processing algorithms that are invariant under time or space translations. However, band-limited functions are analytic and are not always well suited as signal models or for computational purposes. For this reason, we wish to investigate spaces that are almost translation invariant, thereby allowing for almost reproducibility and origin independence of the algorithms without the limitation of analyticity and the computational complexity of band-limited function space.

Let T_a be the translation operator by a factor a, i.e., $(T_af)(x) = f(x - a)$, then obviously $T_1V = V$. We would like to characterize the generators Φ such that $T_{1/n}V = V$ for some fixed integer n. This problem has been studied and Φ characterized for a particular case by Weber in [3] and for the general case by Chui and Sun in [1]. For the case $r = 1$ and $n = 2$ we have the following useful characterization:

Let $E_0 := \{\xi \in [0, 1) : \phi(\xi + 2j) \neq 0$ for some $j \in \mathbb{Z}\}$, $E_1 := \{\xi \in [0, 1) : \phi(\xi + 2j + 1) \neq 0$ for some $j \in \mathbb{Z}\}$, then $T_{1/n}V = V$ if and only if $E_0 \cup E_1 = [0, 1)$, and $E_0 \cap E_1 = \emptyset$. We conjecture that a similar characterization which is not an easy or direct consequence of [1] can be obtained for the general case.

Another direction that we will investigate is the problem of ϵ-$1/n$ translation invariant: Given $\epsilon > 0$ we wish to study the set A_ϵ of generators Φ such that

$$\sup\{\|f(\cdot - 1/n) - Pf(\cdot - 1/n)\|,\ f \in V, \|f\| = 1\} \le \epsilon$$

where P is the orthogonal projection on V. This problem is related to the problem discussed in [2]. The problems under considerations are currently investigated in collaboration with C. Heil, P. Jorgensen, K. Kornelson, and G. Olafsson.

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Trends in Frame Theory

Ole Christensen

The increased flexibility (compared to orthonormal bases) is often an argument for the use of frames. However, in most cases we also want our frames to have some structure, and there are cases where this additional constraint limits (or removes) the freedom. For this reason we seek to extend classical frame theory by allowing duals belonging to a different space than the frame.

Given a frame for a subspace W of a Hilbert space H, we characterize the set of oblique dual frame sequences (i.e., dual frame sequences that are not constrained to lie in W). We then consider frame sequences in shift invariant spaces, and characterize the translation invariant oblique dual frame sequences. For a given translation invariant frame sequence an easily verifiable condition on another shift-invariant frame sequence implies that its closed linear span contains a generator for a translation invariant dual of the frame sequence we start with; in particular, this result shows that classical frame theory does not provide any freedom if we want the dual to be translation invariant. In the case of frame sequences generated by B-splines we can use our approach to obtain dual generators of arbitrary regularity.

Some open problems were presented during the lecture:

- It is well known that the canonical dual of a wavelet frame does not necessarily have the wavelet structure. Which conditions on the generator implies that the canonical dual has wavelet structure? The answer is known for quasi-affine systems, cf. [1].
- Frazier et. al have characterized all dual wavelet frame pairs for $L^2(R)$. How can this be extended to frames for subspaces?
- Is it possible to construct a tight Gabor frame for which the generator g as well as $\hat{g}$ decay exponentially and g is given explicitly in closed form as a linear combination of elementary functions?

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Banach frames, Banach Gelfand Triples, and Wiener Amalgam Spaces

Hans G. Feichtinger

The theory of *frames* is usually described in the context of Hilbert spaces. One may consider frames as those sequences in a Hilbert spaces which rich enough to allow the representations of all the elements in a given Hilbert space, using a series expansion with square summable coefficients. Equivalently to the standard definition one can say that the coefficient mapping $C : f \mapsto (\langle f, f_n \rangle)_{n \in N}$ establishes an isomorphism between the Hilbert space and a its closed range in ℓ^2. The natural inversion (the Moore-Penrose inverse to the coefficient mapping) - we will call it R - is defined on all of ℓ^2, projecting a given sequence onto the range of the coefficient mapping and then back to the uniquely determined function having the given coefficients. As a matter of fact R is realized by the usual (canonical) *dual frame*, called $(\tilde{f}_n)$, via $\mathbf{c} \mapsto \sum_n c_n \tilde{f}_n$. Obviously one has $R \circ C = Id_H$, which is just another form of describing the standard frame expansion for $f \in H$. Since R is bounded and any sequence in ℓ^2 is the norm limit of its finite sections, the convergence of the series is unconditional as well.

From a more abstract point one can say that the pair (C, R) establishes a **retract** between the Hilbert space H and the sequence space ℓ^2, making H isomorphic to a subspace of ℓ^2 (via C) and at the same time to a quotient of ℓ^2 (namely $\ell^2/null(R)$).

The established notion of a *Banach frame* (as formalized by K. Gröchenig in [Grö91]) extends some aspects of this situation to the case where H is replaced be some Banach space and ℓ^2 by some Banach space of sequences (such as a weighted mixed-norm ℓ^p-space). We would like to suggest to add to these assumptions that the Banach space of sequences is also solid (i.e. $|x_n| \leq |y_n|$ for some sequence y, and all n should imply that $\|x\|_B \leq \|y\|_B$). This would imply unconditional convergence of the reconstruction process (which is not granted by the standard terminology). [2] We will see in a moment that this is not merely an abstract generalization of the frame concept but contributes very much to the actual usefulness of Gabor or wavelet frames.

It is however true that this is only part of the story. Wavelet and Gabor systems would not be so useful for applications if aside from the fact that their coefficients have a specific "meaning" in terms of time, frequency or scale they would not be useful to characterize various functions spaces (for example Besov-Triebel-Lizorkin spaces, with wavelet coefficients in suitable weighted mixed-norm spaces). So, in a way, the Banach frames for individual couples (one Banach space of functions and its corresponding Banach space of sequences) are just continuous extension of the corresponding mappings C and R defined on the smaller spaces. While "Banach frames for compatible families of Banach spaces" are an important mathematical concept they are not so easy to explain to non-experts, and therefore we discuss *Banach Gelfand Triples*: Given a Banach space $(B, \|\cdot\|_B)$ and some Hilbert space $\mathcal{H}$ are forming a *Banach Gelfand Triple* $(B, \mathcal{H}, B')$ if the following is true:

[2] We suggest the term *"unconditional Banach frame"*.

- $B \hookrightarrow \mathcal{H} \hookrightarrow B'$;
- B is norm dense in $\mathcal{H}$ and w^*-dense in B'.

The prototypical example consists of the sequence space $(\ell^1, \ell^2, \ell^\infty)$. For many applications in Gabor Analysis the (minimal TF-shift invariant) Segal algebra S_0 (cf. [Fei81]) plays an important role. Together with it's dual it establishes a Banach Gelfand triple (S_0, L^2, S_0'), the S_0-GT.

There is a natural concept of *"Gelfand triple morphism"*: bounded linear mappings at each level, mapping the corresponding "small spaces" into each other, also the corresponding Hilbert spaces, and finally the dual spaces with respect to two topologies, their standard norm topologies and their w^*-topologies respectively. If such a mapping is unitary at the level of Hilbert spaces we will call it a "unitary Gelfand triple isomorphism".

A really basic example of such a unitary GT-isomorphism is the Fourier transform, acting on (S_0, L^2, S_0'). While Plancherel's theorem takes care of the L^2 case, this statement includes the fact that the Fourier transform maps S_0 into itself, but also extends to the (not too large) dual space $S_0'(G)$. At the S_0 level one can use ordinary Riemannian integrals while at the $S_0'-$level one finds that "pure frequencies" are mapped into point-measures (i.e. Dirac Deltas). This is the correct analogue of the "linear algebra situation" (connected with the DFT or FFT), describing it simply as a (orthogonal) change of bases. Moreover, due to the w^*-density of the linear span of pure frequencies resp. discrete measures in S_0' the Fourier transform is uniquely determined by these properties as a unitary Gelfand triple isomorphism.

There are plenty of other Gelfand triple isomorphisms resp. Gelfand triple Banach frames (i.e. retracts between GTs of functions to sequence spaces GTs): Any Gabor frame of the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$, with a Gabor atom in $S_0(R^d)$, and some lattice $\Lambda = \mathbf{A} * Z^{2d}$, for some non-singular $2d \times 2d$ matrix $\mathbf{A}$ has the property (as shown by Gröchenig and Leinert in their recent paper) that the canonical dual window $\tilde{g}$ also belongs to $S_0(R^d)$, and therefore the mappings establishing the standard frame diagram, $C(f) = V_g f(\lambda)$ and $R(\mathbf{c}) = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\tilde{g}$ extend to a retract between the Gelfand triples $(S_0, L^2, S_0')(R^d)$ and the GT $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$. Wilson bases built from S_0 atoms are in fact establishing unitary GT isomorphisms between the same GTs (this is the perfect analogue to the statement of linear algebra: bases are in a one-to-one correspondence to isomorphisms between a finite dimensional vector space and its canonical version R^k).

In connection with operators (relevant for time-frequency analysis) one should point at various representations of operators. While we know from linear algebra that linear mappings from R^n to R^m can be uniquely determined by their matrices (with respect to given matrices) we have to look for a GT analogue in the case of non-finite groups. As already in the case of the finite groups (e.g Z_N, the cyclic group of order N) we have different choices by just making use of the standard basis (of "unit vectors" which then turn into Diracs, resp. pure frequencies, for example). We mention here only the the most important ones (one can find many applications in [FK98]).

Writing $\mathcal{L}$ for the space of bounded linear operators one finds that $\mathcal{L}(S_0', S_0)$ is identified with "smooth kernels", i.e. any such operator T has a nice (continuous and integrable) kernel $K = K(x, y)$ such that for functions f as input one has $Tf(x) = \int K(x, y)f(y)dy$. Just as one would identify the matrix of a linear mapping by realizing its columns as the images of the unit vectors, one expects that $K(x, y) = T(\delta_y)(x)$, which makes sense, because $\delta_y \in S_0'$ while $T(\delta_y)$ is a continuous function in S_0. Of course it is important to see that this functional connection can be extended to a unitary GT isomorphism. The Hilbert space (of operators) being now the space of Hilbert-Schmidt operators $\mathcal{H}S$. Since they are exactly the integral operators with kernel $K \in L^2(G \times G)$, acting on $L^2(G)$ they are also contained in $\mathcal{L}(S_0, S_0')$, which makes $(\mathcal{L}(S_0', S_0), \mathcal{HS}, \mathcal{L}(S_0, S_0'))$ a GT. The kernel theorem can be interpreted as a unitary GT-isomorphism between this triple and their kernels in $(S_0, L^2, S_0')(G \times G)$. The so-called *spreading symbol* of operators. It can be characterized as the uniquely determined unitary GT isomorphism between the GT of operator spaces given above to the S_0-GT over phase space (i.e. $G \times \hat{G}$), which identifies the pure time-frequency shifts $\pi(\lambda) = M_\omega T_t$ for $\lambda = (t, \omega)$ with δ_λ. An often used argument in Gabor analysis is the fact that Gabor frame operators commute with TF-shifts from a given TF-lattice Λ and therefore have a so-called Janssen representation: they can be written as an infinite series of the form $T = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \pi(\lambda^\circ)$ can be seen as a consequence of the following GT statement. Here Λ° is the "adjoint lattice" to Λ, which in the case of $aZ^d \times bZ^d$ equals $(1/b)Z^d \times (1/a)Z^d$. The operators in $\mathcal{L}(S_0, S_0')$ which commute with TF-shifts from Λ are exactly the ones having a Janssen representation. Moreover, the mapping between the operators in $\mathcal{HS}-$ GT of operator spaces with this extra property is isomorphic to the GT $(\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$ through the mapping from T to it's Janssen coefficients (c_λ).

While the spreading function is an important tool in communication theory, because it is used to model slowly time-variant channels occurring in wireless communication, the Kohn-Nirenberg symbol of an operator is more popular in the context of pseudodifferential operators. However, it is not difficult to show that the symplectic Fourier transform, which is another unitary Gelfand triple isomorphism onto itself establishes in a natural link between spreading symbol and KN-symbol of a linear operator. Needless to say that, as a consequence of the statements above, the membership of the KN-symbol in the GT (S_0, L^2, S_0') is again equivalent to the membership of the operator in the corresponding member of the $\mathcal{HS}$-GT. It turns out to be also an appropriate tool to establish a connection between the theory of *Gabor multipliers* and the theory of spline type (resp. principal shift invariant) spaces. The most interesting case for Gabor multipliers, i.e. operators of the form $Tf = \sum_{\lambda \in \Lambda} m_\lambda P_\lambda f$, with $P_\lambda(f) = \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$ arises when these operators form a Riesz basis for their closed linear span within $\mathcal{HS}$, which is the case if and only if the Λ- Fourier transform of the function $|V_g g(\lambda)|^2$ is free of zeros. One can show that in this case there is a canonical bi-orthogonal family (Q_λ) in their closed linear span $\mathcal{GM}_2$ (within $\mathcal{HS}$). Hence the orthogonal projection of $\mathcal{HS}$ onto $\mathcal{GM}_2$ takes the form $T \mapsto \sum_\lambda \langle T(\pi(\lambda)g), \pi(\lambda)g \rangle Q_\lambda$. If the atom g is in $S_0(R^d)$ then one can also show that $P_\lambda \in \mathcal{L}(S_0', S_0)$ and that that orthogonal projection extends to a bounded GT-mapping from the $\mathcal{HS}$-GT onto the Gelfand triple of Gabor multipliers $(\mathcal{GM}_1, \mathcal{GM}_2, \mathcal{GM}_\infty)$ with coefficients in the GT triple $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Finally we mention that *Wiener amalgam spaces* are at the technical level an important tool. It can be used to show the boundedness of coefficient operators (between suitable couples of Banach spaces), respectively the corresponding synthesis operators, but we cannot go into details here. A report on the use of Wiener amalgam spaces in the context of Gabor analysis is under preparation.

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Building a Bridge between Gabor and Wavelet Worlds

Massimo Fornasier

The theory of *frames* or stable redundant non-orthogonal expansions in Hilbert spaces, introduced by Duffin and Schaeffer [DS52] plays an important role in wavelet theory [Dau92, Kai94] as well as in Gabor analysis [Grö02, FS98, FS03]. Many relevant contributions describe Gabor and wavelet analysis as two parallel theories with similar, but different structures and typically different applications. In [FGr88, FGr89, FGr89I, Grö91] Feichtinger and Gröchenig presented a unified approach to Gabor and wavelet analysis, which can*not* be used to describe any *intermediate* theory. Therefore, as a further [HN03, Tor91, Tor92] answer to the theoretical need of a common interpretation and framework between Gabor and wavelet frames, the author has recently proposed [FF04] the construction of frames, which allows to ensure that certain family of Schwartz functions on $\mathbb{R}$ obtained by a suitable combination of translation, modulation of dilation

$$\begin{aligned}&T_x(f)(t) = f(t-x),\\ &M_\omega(f)(t) = e^{2\pi i\omega\cdot t} f(t),\\ &D_a(f)(t) = |a|^{-1/2} f(t/a), \quad x,\omega,t \in \mathbb{R}, a \in \mathbb{R}_+,\end{aligned}$$

form Banach frames for the family of L^2-Sobolev spaces of any order. In the construction a parameter $\alpha \in [0,1)$ governs the dependence of the dilation factor on the frequency parameter. The well-known Gabor and wavelet frames arise as special case ($\alpha = 0$) and limiting case ($\alpha \to 1$) respectively. One example of such intermediate families is given as follows. Consider the two functions

$$p_\alpha(j) := \operatorname{sgn}(j)\left((1+(1-\alpha)|j|)^{\frac{1}{1-\alpha}} - 1\right), \quad s_\alpha(j) := (1+(1-\alpha)(|j|+1))^{\frac{\alpha}{1-\alpha}},$$

and g_0 is the Gaussian function. Then the family$\{g^\alpha_{j,k} := M_{p_\alpha(j)} D_{s_\alpha(j)^{-1}} T_{ak} g_0\}_{j,k\in\mathbb{Z}}$ is in fact a frame for $H^s(\mathbb{R})$ for $s > 0$ and for $a > 0$ small enough. The parameter α functions as a tuning tool of the mixture of the modulation and dilation operators, like “walking on a bridge” between the Gabor and wavelet worlds. Moreover, to frames endowed with intrinsic localization properties [FoGr04], i.e. the Gramian of the frame has nice off-diagonal decay, one can associate natural Banach spaces [Grö04] defined as the spaces of the frame series expansions with coefficients in suitable corresponding Banach sequence spaces. The associated spaces to Gabor and wavelet frames are the well-known families of modulation [Fei89I, Fei03] and Besov spaces [FJ85] respectively. A natural question arises: which are the associated spaces to the intermediate α-Gabor-wavelet frames? An answer to this question has been given in [For02, For04I], where it has been shown that the associated spaces are in fact the so called *α-modulation spaces*, introduced by Gröbner in 1992 [Grö92] in his Ph.D. thesis (see also [PS88]), as an intermediate family of spaces between modulation and Besov spaces This family is appearing also in other contributions and we refer to [For04I] for an extended literature. Let us just mention here that Borup [Bor04], Holschneider, and Nazaret [HN03] have recently described the mapping properties of pseudodifferential operators on α-modulation spaces as an extension of the earlier work of Cordoba and Fefferman [CF78]. From this, relevant open problems for applications arise, for example, on the behaviour of the spectrum of matrices $\left(\langle T g^\alpha_{j,k}, g^\alpha_{j',k'}\rangle\right)_{j,k,j',k'\in\mathbb{Z}}$, depending on $\alpha \in [0,1)$, associated to symmetric operators T acting on H^s. Anyway, even the more simple and related problem of discussing the behaviour of the *frame bounds* depending on $\alpha \in [0,1)$ might be indeed quite difficult. Also applications in best n-term approximation of functions with respect to the dictionary $\{g^\alpha_{j,k}\}_{j,k\in\mathbb{Z},\alpha\in[0,1)}$ might be investigated [DT01]. In particular the different approximation properties of such α-expansions can characterize different classes of functions, may be related by inclusions to α-modulation spaces.

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Frames for Hilbert C*-Modules
Michael Frank

There is growing evidence that Hilbert C*-module theory and the theory of wavelets and Gabor (i.e. Weyl-Heisenberg) frames are tightly related to each other in many aspects. Both the research fields can benefit from achievements of the other field. The goal of the talk given at the mini-workshop was to give an introduction to the theory of module frames and to Hilbert C*-modules showing key analogies, and how to overcome the existing obstacles of Hilbert C*-module theory in comparison to Hilbert space theory.

The theory of module frames of countably generated Hilbert C*-modules over unital C*-algebras was discovered and investigated studying an approach to Hilbert space frame theory by Deguang Han and David R. Larson [7]. Surprisingly, almost all of the concepts and results can be reobtained in the Hilbert C*-module setting. This has been worked out in joint work with D. R. Larson in [4, 5, 6]. Complementary results have been obtained by T. Kajiwara, C. Pinzari and Y. Watatani in [8] using other techniques and motivations. Frames have been also used by D. Bakić and B. Guljaš in [1] calling them quasi-bases. Meanwhile, the case of Hilbert C*-modules over non-unital C*-algebras has been investigated by I. Raeburn and S. J. Thompson [14], as well as by D. Bakić and B. Guljaš discovering standard frames even for this class of countably generated Hilbert C*-modules in a well-defined larger multiplier module. However, many problems still have to be solved.

How to link core C*-theory to wavelet theory was first observed by M. A. Rieffel in 1997, cf. [15]. His approach has been worked out by J. A. Packer and M. A. Rieffel [12, 13], and by P. J. Wood [16, 17] in great detail. As major results a framework in terms of Hilbert C*-modules has been obtained sharing most of the basic structures with generalized multi-resolution analysis for key classes of wavelet and Gabor frames. The Gabor case has been investigated by P. G. Casazza, M. A. Coco and M. C. Lammers [2, 3], and by F. Luef [11] obtaining an adapted to the Gabor situation variant of the Hilbert C*-module approach. In particular, the results by J. A. Packer and M. A. Rieffel in [13] indicate that the described operator algebraic approach to the wavelet theory in $L^2(\mathbb{R}^2)$ is capable to give new deep insights into classical wavelet theory.

To give an instructive example how to link a particular case of generalized multi-resolution analysis to Hilbert C*-module theory we explain one of the core ideas of M. A. Rieffel by example: Assume the situation of a wavelet sequence generated by a multi-resolution analysis in a Hilbert space $L_2(\mathbb{R}^n)$. Denote the mother wavelet by $\phi \in L_2(\mathbb{R}^n)$, $\|\phi\|_2 = 1$, and consider $\mathbb{R}^n$ as an additive group. The second group appearing in the picture is $\Gamma = \mathbb{Z}^n$ acting on $L_2(\mathbb{R}^n)$ by translations in the domains of functions, i.e. mapping $\phi(x)$ to $\phi(x-p)$ for $x \in \mathbb{R}^n$ and $p \in \mathbb{Z}^n$. The mother wavelet ϕ has to be supposed to admit pairwise orthogonal $\mathbb{Z}^n$-translates, i.e. $\int_{\mathbb{R}^n} \overline{\phi(x-q)}\phi(x-p)\,dx = \delta_{qp}$ for any $p, q \in \mathbb{Z}^n$. Introducing the group C*-algebras $A = C^*(\mathbb{Z}^n)$ of the additive discrete group $\mathbb{Z}^n$ into the picture and interpreting the set of all $\mathbb{Z}^n$-translates of ϕ as elements of the $*$-algebra $C_c(\mathbb{R}^n)$ we obtain a right action of A on $C_c(\mathbb{R}^n)$ by convolution and an A-valued inner product there defined by $\langle \phi, \psi \rangle_A(p) := \int_{\mathbb{R}^n} \overline{\phi(x)}\psi(x-p)\,dx$ for $\phi, \psi \in C_c(\mathbb{R}^n)$ and $p \in \mathbb{Z}^n$, (see below for details). The completion of $C_c(\mathbb{R}^n)$ with respect to the norm $\|\phi\| := \|\langle \phi, \phi \rangle_A\|_A^{1/2}$ is a (right) Hilbert C*-module $\mathcal{H} = \overline{C_c(\mathbb{R}^n)}$ over A.

Considering the dual Fourier transformed picture things become mathematically easier. The C*-algebra $A = C^*(\mathbb{Z}^n)$ is transformed to the C*-algebra $B = C(\mathbb{T}^n)$ of continuous functions on the n-torus. The right action of A on $\mathcal{H}$ by convolution becomes a right action of B on $\mathcal{H}$ by pointwise multiplication. Moreover, $\mathcal{H} = \overline{C_c(\mathbb{R}^n)}$ coincides with the set $B\phi$, i.e. it is a singly generated free B-module with B-valued inner product $\langle \phi, \psi \rangle_B(t) := \sum_{p \in \mathbb{Z}^n} (\overline{\phi}\psi)(t-p)$ for $t \in \mathbb{R}^n$. The set $\{\phi\}$ consisting of one element is a module frame, even a module Riesz basis. However, for $n \geq 2$ there exist non-free B-modules that are direct orthogonal summands of $\mathcal{H} = B$, cf. [12] for their construction. For them module Riesz bases might not exist, and module frames consist of more than one element. In a similar manner multi-wavelets give rise to Hilbert B-modules B^k of all k-tuples with entries from B and coordinate-wise operations. Since norm-convergence and weak convergence are in general different concepts in an infinite-dimensional C*-algebra B (whereas both they coincide in $\mathbb{C}$). Some more investigations have to be carried out to treat Gabor analysis, for example.

A *pre-Hilbert C*-module* $\mathcal{H}$ over a (unital) C*-algebra A is a (left) A-module equipped with an *A-valued inner product* $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to A$ such that (i) $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{H}$, (ii) $\langle x, x \rangle = 0$ if and only if $x = 0$, (iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for any$x, y \in \mathcal{H}$, and (iv) $\langle .,. \rangle$ is A-linear in the first argument. The induced norm $\|.\| = \|\langle .,. \rangle\|^{1/2}$ opens up the opportunity to restrict attention to norm-closed A-modules of this kind, i.e. to *Hilbert* A-modules. The A-module $\mathcal{H}$ is algebraically finitely generated if there exists a finite set $\{x_i\}_{i=1}^N \subset \mathcal{H}$ such that $\mathcal{H} = \operatorname{span}\{a_i x_i : a_i \in A\}$. A Banach A-module is countably generated if there exists a finite or countable set $\{x_i\}_{i \in I} \subset \mathcal{H}$ such that $\operatorname{span}\{a_i x_i : a_i \in A\}$ is norm-dense in $\mathcal{H}$. For a comprehensive account to Hilbert C*-module theory we refer the reader to [10].

For unital C*-algebras A a finite or countable set $\{x_i\}_{i \in I} \subset \mathcal{H}$ is said to be a *frame* for the Hilbert C*-module $\mathcal{H}$ if there exist two real constants $C, D > 0$ such that the inequality $C \cdot \langle x, x \rangle \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \cdot \langle x, x \rangle$ is valid for any $x \in \mathcal{H}$. The frame is called *standard* if the sum in the middle of the inequality converges in norm in A. A frame is *normalized tight* if $C = D = 1$. A sequence $\{x_i\}_{i \in I} \subset \mathcal{H}$ is a *standard Riesz basis of* $\mathcal{H}$ if it is a standard frame for $\mathcal{H}$ with the additional property that $\sum_{i \in S \subseteq I} a_i x_i = 0$ if and only if $a_i x_i = 0$ for any $i \in S$. Two frames $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ for a Hilbert A module $\mathcal{H}$ are *unitarily equivalent* (resp., *similar*) if there exists a unitary (resp., invertible adjointable) A-linear bounded operator T on $\mathcal{H}$ satisfying $T(x_i) = y_i$ for any $i \in I$. By Kasparov's stabilization theorem and by tensor product constructions one can easily see that standard (normalized tight) frames for Hilbert C*-modules over unital C*-algebras exist always and in abundance. For canonical examples of Hilbert C*-modules standard Riesz bases are found not to exist, and so orthogonal Hilbert bases often may not exist.

As the crucial result that makes the entire theory work one obtains two *reconstruction formulae* for standard (normalized tight) frames $\{x_i\}_{i \in I}$ of finitely or countably generated Hilbert C*-modules $\mathcal{H}$ over unital C*-algebras A. If $\{x_i\}_i$ is a standard normalized tight frame for $\mathcal{H}$ then the following reconstruction formula always holds for every $x \in \mathcal{H}$:

$$x = \sum_{i \in I} \langle x, x_i \rangle x_i \, .$$

The sum converges with respect to the norm of $\mathcal{H}$. If $\{x_i\}_{i \in I}$ is merely a standard frame for $\mathcal{H}$ then there exists a positive invertible A-linear bounded operator S on $\mathcal{H}$, the *frame operator*, such that the reconstruction formula

$$x = \sum_{i \in I} \langle x, S(x_i) \rangle x_i$$

is valid for any $x \in \mathcal{H}$. The sequence $\{S(x_i)\}_{i \in I}$ is a frame for $\mathcal{H}$ again, and it is said to be the *canonical dual frame* of for the frame $\{x_i\}_{i \in I}$. The key point of the proofs is the existence of the *frame transform* $\theta : \mathcal{H} \to l_2(A)$, $\theta(x) = \{\langle x, x_i \rangle\}_{i \in I}$, and its properties which can be found to be guaranteed in *any* situation - boundedness, A-linearity, and, most important, adjointability. The frame operator S can be expressed by $S = (\theta\theta^*)^{-1}$, and for every standard frame $\{x_i\}_{i \in I}$ the frame $\{S^{1/2}(x_i)\}_{i \in I}$ turns out to be a standard normalized tight one.

Starting from this point similarity of standard frames and the image of their frame transform can be investigated, leading to similar results about the canonical and alternate duals as in the Hilbert space situation. In the same manner as for Hilbert spaces results for complementary frames and inner sums of frames can be obtained giving rise to several types of disjointness of pairs of frames. Standard frames turn out to be precisely the inner direct summands of standard Riesz bases for Hilbert A-modules A^N, $N < \infty$, or $l_2(A)$. whereas standard normalized tight frames are the inner direct summands of orthonormal Hilbert bases of A^N or $l_2(A)$.

Establishing this key point of the theory of standard modular frames of countably Hilbert C*-modules over unital C*-algebras A one (re-)obtains an whole collection of frame theory results in this setting: Every standard frame of a countably generated Hilbert A-module is a set of generators. Every standard Riesz basis $\{x_i\}_{i \in I}$ with normalized tight frame bounds has the property $\langle x_j, x_k \rangle = \delta_{jk} \cdot \langle x_j, x_k \rangle^2$ for any $j, k \in I$, i.e. it is orthogonal and "normalized" in some sense. Every finite set of algebraic generators of a finitely generated Hilbert A-module is a frame for it. If the equality $x = \sum_{i \in I} \langle x, y_i \rangle x_i$ holds for any $x \in \mathcal{H}$ and for some standard frame $\{y_i\}_{i \in I}$ for $\mathcal{H}$ then this *alternate dual frame* fulfills the inequality

$$\sum_{i \in I} \langle x, S(x_i) \rangle \langle S(x_i), x \rangle < \sum_{i \in I} \langle x, y_i \rangle \langle y_i, x \rangle$$

for any $x \in \mathcal{H}$.

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Frame Generators and Traces on the Commuting Algebra

Hartmut Führ

Given a representation $(\pi, \mathcal{H}_\pi)$ of a unimodular, separable locally compact group G, we want to discuss the existence and characterization of vectors giving rise to coherent state expansions on $\mathcal{H}_\pi$.

For this purpose, a vector $\eta \in \mathcal{H}_\pi$ is called **bounded** if the coefficient operator

$$V_\eta : \mathcal{H}_\pi \to \mathrm{L}^2(G) \ , \ (V_\eta \varphi)(x) = \langle \varphi, \pi(x)\eta \rangle$$

is a bounded map. A pair of bounded vectors (η, ψ) is called **admissible** if $V_\psi^* V_\eta = \mathrm{Id}_{\mathcal{H}_\pi}$. This property gives rise to the weak-sense inversion formula

$$z = \int_G \langle z, \pi(x)\eta \rangle \ \pi(x)\psi \ d\mu_G(x) \ ,$$

which can be read as a continuous expansion of z in terms of the orbit $\pi(G)\psi \subset \mathcal{H}_\pi$. A single vector η is called admissible if (η, η) is an admissible pair. It is obvious from the definition that (η, ψ) is admissible iff (ψ, η) is. In such a case η is called the **dual vector** of ψ.

If G is a discrete group, the notions of bounded vectors and admissible pairs can be reformulated in terms of frames: Rewriting the inversion formula as

$$z = \sum_{x \in G} \langle z, \pi(x)\eta \rangle \ \pi(x)\psi \ ,$$

we see that the (η, ψ) are an admissible pair iff the systems $\pi(G)\eta$ and $\pi(G)\psi$ are a dual frame pair of $\mathcal{H}_\pi$. We want to discuss representation-theoretic criteria for frame generators. The study of discrete groups necessitates to go beyond the so-called *discrete series* or *square-integrable* representations [3], but also beyond the type I groups studied in [2].

It turns out that more general statements are possible by use of a particular trace on the right von Neumann algebra $VN_r(G)$, which is the commutant of the left regular representation λ_G on $\mathrm{L}^2(G)$. Indeed, the following observations can be made:

1. Up to unitary equivalence, any representation π having an admissible pair can be realized as a subrepresentation of λ_G, acting on some leftinvariant closed subspace $\mathcal{H} \subset \mathrm{L}^2(G)$. In particular, the projection onto $\mathcal{H}$ is in $VN_r(G)$.
2. Defining $f^*(x) = \overline{f(x^{-1})}$, the coefficient operators acting on $\mathrm{L}^2(G)$ (or subspaces) can be written as $V_f g = g * f^*$.
3. $VN_r(G)$ carries a natural faithful normal, semifinite trace defined for positive operators S

$$t_r(S) = \begin{cases} \|f\|_2 & : S = V_f^* V_f \text{ for a suitable bounded vector } f \in \mathrm{L}^2(G) \\ \infty & : \text{otherwise} \end{cases}$$

   Polarisation of the definition yields for bounded vectors

$$t_r(V_g^* V_f) = \langle g, f \rangle \ .$$

4. If G is discrete, any $T \in VN_r(G)$ is uniquely determined by its "impulse response" $T(\delta_e)$. In this case t_r is finite and given by

$$t_r(T) = T(\delta_e)(e) \ .$$

Given a particular trace tr on a von Neumann algebra $\mathcal{A}$, we call a pair of elements (η, ψ) of the underlying Hilbert space **tracial** if

$$\forall T \in \mathcal{A}^+ : tr(T) = \langle T\eta, \psi \rangle \ .$$

Then we have

Theorem 1. *Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a closed, leftinvariant subspace, with associated leftinvariant projection p, and let π denote the restriction of λ_G to $\mathcal{H}$.*

(a) There exists an admissible pair for $\mathcal{H}$ iff $t_r(p) < \infty$.

(b) For all pairs $(\eta, \psi) \in \mathcal{H} \times \mathcal{H}$ of bounded vectors: (η, ψ) is admissible iff (η, ψ) is tracial for $\pi(G)'$.

We shortly sketch two applications. The first concerns the central decomposition of λ_G. Let $\check{G}$ denote the space of quasi-equivalence classes of factor representations of G, and let

$$\lambda_G \simeq \int_{\check{G}}^{\oplus} \rho_\sigma d\nu_G(\sigma)$$

denote the central decomposition. This also provides the direct integral decompositions

$$\begin{aligned} VN_r(G) &\simeq \int_{\check{G}}^{\oplus} \mathcal{A}_\sigma d\nu_G(\sigma) \\ t_r(T) &= \int_{\check{G}} tr_\sigma(T_\sigma) d\nu_G(\sigma) \ , \end{aligned}$$

where A_σ is the commuting algebra of ρ_σ, $(T_\sigma)_{\sigma \in \check{G}}$ denotes the operator field corresponding to T under the central decomposition and tr_σ is a suitable faithful normal, semifinite trace on the factor $\mathcal{A}_\sigma$. Standard direct integral arguments then yield:

Proposition 2. *Let π denote the restriction of λ_G to a closed, leftinvariant subspace $\mathcal{H} \subset \mathrm{L}^2(G)$. Let P denote the projection onto $\mathcal{H}$, then P decomposes into a measurable field of projections $\widehat{P}_\sigma$, and $\pi(G)'$ decomposes under the central decomposition into the von Neumann algebras $\mathcal{C}_\sigma = \widehat{P}_\sigma \mathcal{A}_\sigma \widehat{P}_\sigma$.*

(a) For bounded $\eta, \psi \in \mathcal{H}$, we have

$$(\eta, \psi) \text{ is admissible for } \mathcal{H} \Leftrightarrow (\widehat{\eta}_\sigma, \widehat{\psi}_\sigma) \text{ is tracial for } \mathcal{C}_\sigma \ (\nu_G a.e.)$$

(b) $\mathcal{H}$ has an admissible pair of vectors iff $\int_{\check{G}} tr(\widehat{P}_\sigma) d\nu_G(\sigma) < \infty$. In particular, almost all C_σ are finite von Neumann algebras.

Generally the representations of interest are not realized as acting by left translations on subspaces of $\mathrm{L}^2(G)$. Therefore, applying Theorem 1 requires first to embed the representation into λ_G. The following corollary sketches an alternative approach. Roughly speaking, it derives a criterion for admissible pairs based on **one** explicitly known admissible pair.

Corollary 3. *Suppose we are given*

- *A family $(T_i)_{i \in I} \subset \pi(G)'$ spanning a weak-operator dense subspace of $\pi(G)'$.*
- *An admissible pair (η_0, ψ_0).*

Then for a pair of bounded vectors (η, ψ) we have the following equivalence:

$$(1) \qquad (\eta, \psi) \text{ is admissible } \Longleftrightarrow \forall i \in I \ : \ \langle T_i \eta, \psi \rangle = \langle T_i \eta_0, \psi_0 \rangle \ .$$

The criterion is explicit as soon as the T_i and the admissible pair (η_0, ψ_0) are known explicitly. Using results from [1] it can be shown that the Wexler-Raz criteria for Gabor frames can be derived this way, thus yielding **explicit** criteria for a whole family of type-II representations.

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Frames, Operators, and Banach Algebra Techniques

Karlheinz Gröchenig

Symbolic Calculus. A symbolic calculus is a mapping from a class of *symbols* to a class of *operators* acting on some Hilbert space (or subspace thereof):

$$\sigma \longrightarrow \mathrm{Op}(\sigma)$$

In many areas of mathematics one finds manifestations of the following principle.

Metatheorem. *If the symbol σ is nice and* $\mathrm{Op}(\sigma)$ *is invertible on Hilbert space, then* $\big(\mathrm{Op}(\sigma)\big)^{-1} = \mathrm{Op}(\tau)$ *for nice τ.*

An important consequence is the following extension principle.

Meta-Corollary. $\big(\mathrm{Op}(\sigma)\big)^{-1} = \mathrm{Op}(\tau)$ *is bounded on large class of Banach spaces.*

We give several examples of a symbolic calculus drawn from different fields of mathematics. Usually a symbolic calculus is proved by means of some "hard analysis", but we will emphasize the role of Banach algebra techniques in the analysis of symbolic calculi. A second aspect is the role of weights. Weighted versions of symbolic calculus can usually be derived from the corresponding unweighted versions and the growth properties of the weights.

1. Convolution Operators on Groups.

The prototype of a symbolic calculus is **Wiener's Lemma.** In its standard form asserts the following: If f has an non-vanishing absolutely convergent Fourier series, then so does $1/f$.

Wiener's Lemma can be recast as a statement about convolution operators defined by $T_{\mathbf{a}}\mathbf{c} = \mathbf{a} * \mathbf{c}$ for two sequences $\mathbf{a}, \mathbf{c}$ on $\mathbb{Z}^d$. In this case the symbol is the sequence $\mathbf{a}$ and the operator is $T_{\mathbf{a}}$. "Nice" symbols are sequences in the weighted ℓ^1 space $\ell^1_v(\mathbb{Z}^d)$ by the norm $\|\mathbf{a}\|_{\ell^1_v} = \sum_{k\in\mathbb{Z}^d} |c_k| v(k)$. The weight is always assumed to satisfy $v(0) = 1, v(k) = v(-k)$, and $v(k+l) \le v(k)v(l), k, l \in \mathbb{Z}^d$.

Theorem 1. *Assume that*

(a) $\mathbf{a} \in \ell^1_v(\mathbb{Z}^d)$,
(b) $T_{\mathbf{a}}$ *is invertible on* $\ell^2(\mathbb{Z}^d)$ *and*
(c) $\lim_{n\to\infty} v(nx)^{1/n} = 1, \forall x \in \mathbb{Z}^d$ *(GRS-condition).*

Then $T_{\mathbf{a}}^{-1} = T_{\mathbf{b}}$ *for* $\mathbf{b} \in \ell^1_v(\mathbb{Z}^d)$ [2].

Let $\sigma_{\ell^p_m}(\mathbf{a})$ be the spectrum of the convolution operator $T_{\mathbf{a}}$ on the weighted ℓ^p-space $\ell^p_m(\mathbb{Z}^d)$. Then we have

Corollary 2. *If* $m(x+y) \le Cv(x)m(y)$, *then*

$$\sigma_{\ell^p_m}(\mathbf{a}) = \sigma_{\ell^2}(\mathbf{a})$$

The role of the GRS condition is illuminated by the following statement.

Theorem 3.

$$\sigma_{\ell^1_v}(\mathbf{a}) = \sigma_{\ell^2}(\mathbf{a})$$

if and only if v *satisfies the GRS-condition* $\lim_{n\to\infty} v(nx)^{1/n} = 1, \forall x \in \mathbb{Z}^d$.

Similar types of a symbolic calculus can be shown for "twisted convolution", for the rotation algebra [4], and for convolution operators on groups of polynomial growth [1, 6].

2. Matrix Algebras.

The second type of example concerns matrix algebras. In this case the "symbol" is an infinite matrix A, the associated operator is obtained simply by the action of A on a sequence c. "Nice" matrices are determined by their decay off the diagonal.

Theorem 4. [3, 5]. *Assume that* u *is a radial weight function on* $\mathbb{Z}^d$ *satisfying the GRS-condition and that* $v(x) = u(x)(1+|x|)^s$ *for some* $s > d$. *If the matrix* A *invertible on* $\ell^2(\mathbb{Z}^d)$ *and if*

$$|A_{kl}| \le Cv(k-l)^{-1}\,,$$

then

$$|(A^{-1})_{kl}| \le C' v(k-l)^{-1}$$

and

$$\sigma_{\mathcal{A}^1_v}(A) = \sigma(A) \qquad \forall A \in \mathcal{A}^1_v$$

where $\sigma(A)$ *is the spectrum of* A *as an operator on* ℓ^2.

As a consequence, A and A^{-1} are bounded on many weighted ℓ^p-spaces.

Theorems of this type are important in numerical analysis because they are used in error estimates, when infinite-dimensional matrix equations are approximated by finite-dimensional models (finite section method).

3. Self-Localized Frames.
In the final example the "symbols" are frames $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ and the associated operator is the frame operator $Sf = S_{\mathcal{E}}f = \sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x$. In the context of symbolic calculus, "nice" frames are frames with a localization property.

Definition: A frame $\{e_x : x \in \mathcal{X}\}$ is intrinsically s-self-localized, if

$$|\langle e_y, e_x\rangle| \le C(1 + |x - y|)^{-s} \qquad \forall x, y \in \mathcal{X}\,.$$

Theorem 5 (Fornasier, Gröchenig, 2004)**.** *If $\{e_x : x \in \mathcal{X}\}$ is s-self-localized, then so is the canonical dual frame $\{\tilde{e}_x\}$, i.e.,*

$$|\langle \tilde{e}_y, \tilde{e}_x\rangle| \le C(1 + |x - y|)^{-s} \qquad x, y \in \mathbb{R}^d$$

and

$$|\langle e_y, \tilde{e}_x\rangle| \le C(1 + |x - y|)^{-s} \qquad x, y \in \mathbb{R}^d$$

This statement has wide applications in sampling theory, time-frequency analysis, and wavelet theory.

As further examples of a symbolic calculus we mention pseudodifferential operators and their spectral invariance on various function spaces, and new classes of matrix algebras that are dominated by a convolution operator.

All the above examples can be viewed as statements about the symmetry and inverse-closedness of the Banach algebra under discussion.

An involutive Banach algebra $\mathcal{A}$ is symmetric, if $\sigma(a^*a) \subseteq [0, \infty)$ for all $a \in \mathcal{A}$ (if and only if $\sigma(a) \subseteq \mathbb{R}$ for all $a = a^* \in \mathcal{A}$). Theorems 3 and 4 assert that $(\ell^1_v, *)$ and $\mathcal{A}_v$ are symmetric Banach algebras.

Another central concept is inverse-closedness. Let $\mathcal{A} \subseteq \mathcal{B}$ be two Banach algebras with a common identity. Then $\mathcal{A}$ is said to be inverse-closed in $\mathcal{B}$, if

$$a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \quad \Longrightarrow \quad a^{-1} \in \mathcal{A} \quad .$$

Other terminology frequently used is that of a Wiener pair, a spectral subalgebra, or of spectral invariance. Theorems 1 and 4 state that $(\ell^1_v, *)$ is inverse-closed in ℓ^1 and $\mathcal{B}(\ell^2)$, and that $\mathcal{A}_v$ is inverse-closed in $\mathcal{B}(\ell^2)$.

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The Zero Divisor Conjecture for the Heisenberg Group

Christopher Heil

The following conjecture was introduced in the paper [HRT96], and is still open today.

Conjecture 1. If $g \in L^2(\mathbf{R})$ is nonzero and $\{(\alpha_k, \beta_k)\}_{k=1}^N$ is any set of N distinct points in $\mathbf{R}^2$, then $\{e^{2\pi i \beta_k x} g(x - \alpha_k)\}_{k=1}^N$ is a linearly independent set of functions in $L^2(\mathbf{R})$.

The composition $M_b T_a g(x) = e^{2\pi i b x} g(x - a)$ of translation $T_a g(x) = g(x - a)$ and modulation $M_b g(x) = e^{2\pi i b x} g(x)$ is called a time-frequency shift of g, and the analysis and application of these operators is time-frequency analysis. A beautiful introduction to time-frequency analysis can be found in [Grö01]. Conjecture1 has many connections, to harmonic analysis, representation theory, functional analysis, the geometry of Banach spaces, and even more unexpected areas such as ergodic theory.

Today Conjecture 1 sometimes goes by the name of the *HRT Conjecture* or the *Zero Divisor Conjecture for the Heisenberg Group.* Despite attacks by a number of groups, the only published results specifically concerning the conjecture appear to be [HRT96], [Lin99], and [Kut02], which can be summarized as follows.

The paper [HRT96] introduced the conjecture and obtained some partial results, including the following.

(a) If a nonzero $g \in L^2(\mathbf{R})$ is compactly supported, or just supported on a half-line, then the independence conclusion holds for any value of N.

(b) The independence conclusion holds for any a nonzero $g \in L^2(\mathbf{R})$ if $N \leq 3$.

(c) If the independence conclusion holds for a particular $g \in L^2(\mathbf{R})$ and a particular choice of points $\{(\alpha_k, \beta_k)\}_{k=1}^N$, then there exists an $\varepsilon > 0$ such that it also holds for any h satisfying $\|g - h\|_2 < \varepsilon$, using the same set of points.

(d) If the independence conclusion holds for one particular $g \in L^2(\mathbf{R})$ and particular choice of points $\{(\alpha_k, \beta_k)\}_{k=1}^N$, then there exists an $\varepsilon > 0$ such that it also holds for that g and any set of points in $\mathbf{R}^2$ within ε of the original ones.

Another partial advance was made by Linnell in [Lin99]. He used C^*-algebra techniques to prove that if the points $\{(\alpha_k, \beta_k)\}_{k=1}^N$ are a subset of some translate of a lattice in $\mathbf{R}^2$, then the independence conclusion holds for any g (a lattice is a set of the form $A(\mathbf{Z}^2)$, the image of $\mathbf{Z}^2$ under an invertible matrix A). Note that any three points in the plane always lie on a translate of some lattice, so this recovers and extends the partial result (b) mentioned above. However, given four arbitrary points in the plane it is not always possible to find a translate of a lattice that contains those points. Indeed, the case $N = 4$ of the conjecture is still open. In fact, the following special case seems to be open.

Conjecture 2. If $g \in L^2(\mathbf{R})$ is nonzero then

$$\{g(x),\, g(x-1),\, g(x-\sqrt{2}),\, e^{2\pi i x} g(x)\}$$

is a linearly independent set of functions in $L^2(\mathbf{R})$.

Conjecture 2 remains open even if we impose the condition that g be continuous. The real-valued version obtained by replacing $e^{2\pi i x}$ by $\sin 2\pi x$ is likewise open.

One motivation for Conjecture 1 comes from looking at *frames*, which are possibly redundant or over-complete collections of vectors in a Hilbert space which nonetheless provide basis-like representations of vectors in the space. Thus a frame "spans" the space in some sense, even though it may be "dependent." However, in infinite dimensions there are many shades of gray to the meanings of "spanning" and "independence." Some of the most important frames are "dependent" taken as a whole even though have the property that every finite subset is linearly independent. One motivation for Conjecture 1 is the question of whether the the special class of *Gabor frames* have this property that every finite subset is independent.

Gabor frames are related to the Schrödinger representation of the Heisenberg group. If we instead use the affine group and the standard representation induced from dilations and translations, we obtain wavelets. However, the analogue of Conjecture 1 for wavelets fails in general. For example, a compactly supported refinable function φ satisfies an equation of the form

$$\varphi(x) = \sum_{k=0}^{N} c_k\, \varphi(2x-k).$$

This is an expression of linear dependence among the time-scale translates of φ. In particular, the box function $b = \chi_{[0,1)}$ satisfies the refinement equation

$$b(x) = b(2x) + b(2x-1).$$

The more general analogue of Conjecture 1 for the case of other groups is related to the Zero Divisor Conjecture in algebra; we refer to [Lin99] and the references therein for more on this connection.

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Poisson Wavelet Frames on the Sphere

Ilona Ilgewska–Nowak
(joint work with Matthias Holschneider)

People would like to create a mathematical repesentation of the Earth's magnetic field and how it is changing. One of the most broadly used possibilities is to represent the magnetic field in terms of spherical harmonics. This method has some disadvantages. One of them is a poor localization: coefficients obtained in Europe have influence on the representation of the field over Africa. On the other hand, it is difficult to distinguish the big–scale field component from the core and the small–scale field component from the crust. Moreover, changing the truncation level of spherical harmonics changes all the coefficients, according to spatial aliasing of the higher–order harmonics.

Another possibility is to use a wavelet representation of the magnetic field. This would solve some of the problems mentioned above. Here, we would like to introduce Poisson wavelets and give some ideas how frames of such wavelets could be constructed.

Note that there exists no natural dilation operator on the sphere, hence, we do not have a group structure of the wavelet coefficients. Here, the scales are defined in a more or less *ad hoc* way, but so that the wavelets behave like wavelets over the plane. The definition we use goes back to [2], in this talk we base on the simplified definition given in [3].

If Σ denotes the unit two–dimensional sphere, $\hat{e}$ the unit vector in direction of the north–pole, then Poisson wavelets are defined to be

$$g_a^n(x) = \sum_{l=0}^{n} (al)^n e^{-al} Q_l(x),$$

where $Q_l(x) = \frac{2l+1}{4\pi} P_l(x \cdot \hat{e})$, P_l – l-th Legendre polynomial. They are equal to the electromagnetic field caused by a sum of multipoles inside the unit ball:

$$g_a^n = a^n (2\Psi_{e^{-a}}^{n+1} + \Psi_{e^{-a}}^{n}), \quad \text{where } \Delta\Psi_\lambda^n = (\lambda\partial_\lambda)\delta_{\lambda\hat{e}}$$

(therefore the name Poisson wavelets.)

We obtain explicit expressions in terms of finite sums of Legendre polynomials if we develop g_a^n around the point $e^{-a}\hat{e}$:

$$g_a^n(x) = a^n \sum_{k=1}^{n+1} k!(2C_k^{n+1} + C_k^n) e^{-ka} P_k(\cos\chi) \frac{1}{|x - e^{-a}\hat{e}|^{k+1}},$$

where χ is the angle between $\hat{e}$ and $x - e^{-a}\hat{e}$, and C_k^n are constants defined through

$$(\lambda\partial_\lambda)^n = \sum C_k^n \lambda^k \partial_\lambda^k.$$

For small scales a the Euklidean limit holds:

$$\lim_{a\to 0} a^2 g_a^n(\Phi^{-1}(ax)) = g(x) \quad \text{for some } g \in \mathcal{L}^2(\mathbb{R}^2),$$

where Φ is the stereographic projection of the sphere onto the plane. This means that g_a^n are scaling like wavelets over $\mathbb{R}^2$ assymptitically for small a.

The wavelet transform of a function s is given by

$$\mathcal{W}_{g^n} s(x,a) = \int_\Sigma g_a^n(x\cdot y)s(y)d\sigma(y)$$

and the inverse wavelet transform is given by

$$\mathcal{M}_{g^n} r(x) = \int_{\mathbb{R}_+}\int_\Sigma r(y,a) g_a^n(y) d\sigma(y)\frac{da}{a}.$$

The following holds:

$$\mathcal{M}_{g^n}\mathcal{W}_{g^n} s = cs$$

for some constant $c = c(g^n)$, i.e., g^n build a continuous frame.

Remark: the wavelet transform with respect to this family can also be obtained as follows: take s as Dirichlet boundary data for the interior problem. Then apply a suitable radial derivative to the harmonic extension inside the unit ball.

The image of $\mathcal{W}$ is a Hilbert space with reproducing kernel. This reproducing kernel can be written in terms of the wavelets:

$$P_{g^n}(x,a;y,b) = \left(\frac{ab}{(a+b)^2}\right)^n g_{a+b}^{2n}(x\cdot y)$$

(if we identify $g(x)$ with $g(x\cdot\hat{e})$ for zonal functions g.)

In applications in geophysics this continuous family has to be discretized over some grid. We consider the following grid $\Lambda = \{(x,a)\}$ in $\Sigma\times\mathbb{R}_+$: for a fixed scale $a \in \{n\cdot 2^{-j},\ j\in\mathbb{N}_0\}$ (n – order of the wavelet) we take a cube centered with respect to the sphere , divide each of its six sides into 4^j similar squares and project the centers of the faces onto the sphere in order to define positions x. Question: is $A = \{g_{x,a},\ (x,a)\in\Lambda\}$ a frame for $\mathcal{L}^2(\Sigma)$ (for some set of weights $\mu(x,a)$)? Some approaches we have considered are:

(1) based on the atomic space decomposition of [1]: if

$$\begin{aligned}&|\sum_{(y,b)\in\Lambda} P_{g^n}(x,a;y,b)P_{g^n}(y,b;z,c)\mu(y,b)\\ &-\int_{\mathbb{R}_+}\int_\Sigma P_{g^n}(x,a;y,b)P_{g^n}(y,b;z,c)d\sigma(y)\frac{db}{b}| \le \frac{1}{c^2} f\left(\frac{x\cdot z}{c},\frac{a}{c}\right)\end{aligned}$$

for some function f which is $\mathcal{L}^2$-integrable with respect to $\theta d\theta da/a$, then A is a frame;

(2) transform the unit ball onto the upper half–plane (essentially by the Kelvin–transform) such that harmonic functions remain harmonic functions; consider the image of Poisson wavelets under this map and check if they build a frame of the weighted $\mathcal{L}^2(\mathbb{R}^2)$;

(3) based on quasi-frames: locally around each point of the sphere we obtain a quasi-frame; these however have to be patched together to a global frame.

In view of the remark above, having proven that $\{g_\lambda,\ \lambda \in \Lambda\}$ and alike grids build a frame for $\mathcal{L}^2(\Sigma)$, we automatically obtain some interesting results for harmonic functions (e.g. density of local maxima, sets of uniqueness, ...).

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Duality Principles in Analysis

Palle E. T. Jorgensen

Several versions of spectral duality are presented. On the two sides we present (1) a basis condition, with the basis functions indexed by a frequency variable, and giving an orthonormal basis; and (2) a geometric notion which takes the form of a tiling, or a Iterated Function System (IFS). Our initial motivation derives from the Fuglede conjecture, see [3, 6, 7]: For a subset D of $\mathbb{R}^n$ of finite positive measure, the Hilbert space $L^2(D)$ admits an orthonormal basis of complex exponentials, i.e., D admits a Fourier basis with some frequencies L from $\mathbb{R}^n$, if and only if D tiles $\mathbb{R}^n$ (in the measurable category) where the tiling uses only a set T of vectors in $\mathbb{R}^n$. If some D has a Fourier basis indexed by a set L, we say that (D, L) is a spectral pair. We recall from [9] that if D is an n-cube, then the sets L in (1) are precisely the sets T in (2). This begins with work of Jorgensen and Steen Pedersen [9] where the admissible sets $L = T$ are characterized. Later it was shown, [5] and [10] that the identity $T = L$ holds for all n. The proofs are based on general Fourier duality, but they do not reveal the nature of this common set $L = T$. A complete list is known only for $n = 1$, 2, and 3, see [9].

We then turn to the scaling IFS's built from the n-cube with a given expansive integral matrix A. Each A gives rise to a fractal in the small, and a dual discrete iteration in the large. In a different paper [8], Jorgensen and Pedersen characterize those IFS fractal limits which admit Fourier duality. The surprise is that there is a rich class of fractals that do have Fourier duality, but the middle third Cantor set does not. We say that an affine IFS, built on affine maps in $\mathbb{R}^n$ defined by a given expansive integral matrix A and a finite set of translation vectors, admits Fourier duality if the set of points L, arising from the iteration of the A-affine maps in the large, forms an orthonormal Fourier basis (ONB) for the corresponding fractal μ in the small, i.e., for the iteration limit built using the inverse contractive maps, i.e., iterations of the dual affine system on the inverse matrix A^{-1}. By "fractal in the small", we mean the Hutchinson measure μ and its compact support, see [4].

(The best known example of this is the middle-third Cantor set, and the measure μ whose distribution function is corresponding Devil's staircase.)

In other words, the condition is that the complex exponentials indexed by L form an ONB for $L^2(\mu)$. Such duality systems are indexed by complex Hadamard matrices H, see [9] and [8]; and the duality issue is connected to the spectral theory of an associated Ruelle transfer operator, see [1]. These matrices H are the same Hadamard matrices which index a certain family of quasiperiodic spectral pairs (D, L) studied in [6] and [7]. They also are used in a recent construction of Terence Tao [11] of a Euclidean spectral pair (D, L) in $\mathbb{R}^5$ for which D does not a tile $\mathbb{R}^5$ with any set of translation vectors T in $\mathbb{R}^5$.

We finally report on joint research with Dorin Dutkay where we show that all the affine IFS's admit wavelet orthonormal bases [2] now involving both the $\mathbb{Z}^n$ translations and the A-scalings.

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Minimal Generator Sets for Finitely Generated Shift Invariant Subspaces of $L^2(\mathbb{R}^n)$

Norbert Kaiblinger

(joint work with Marcin Bownik)

Given a family of functions $\phi_1, \ldots, \phi_N \in L^2(\mathbb{R}^n)$, let $S = S(\phi_1, \ldots, \phi_N)$ denote the closed subspace of $L^2(\mathbb{R}^n)$ generated by their integer translates. That is, S is the closure of the set of all functions f of the form

$$f(t) = \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}^n} c_{j,k}\ \phi_j(t-k), \qquad t \in \mathbb{R}^n, \tag{1}$$

where finitely many $c_{j,k} \in \mathbb{C}$ are nonzero. By construction, these spaces $S \subset L^2(\mathbb{R}^n)$ are invariant under shifts, i.e., integer translations and they are called finitely generated shift-invariant spaces. Shift-invariant spaces play an important role in analysis, most notably in the areas of spline approximation, wavelets, Gabor (Weyl-Heisenberg) systems, subdivision schemes and uniform sampling. The structure of this type of spaces is analyzed in [1], see also [2, 3, 4, 9]. Only implicitly we are concerned with the dependence properties of sets of generators, for details on this topic we refer to [7, 8]. [3]

The minimal number $L \leq N$ of generators for the space S is called the length of S. Although we include the case $L = N$, our results are motivated by the case $L < N$. In this latter case, there exists a smaller family of generators $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R}^n)$ such that

$$S(\phi_1, \ldots, \phi_N) = S(\psi_1, \ldots, \psi_L), \qquad \text{with } L < N.$$

Since the new generators $\psi_1, \ldots, \psi_L$ belong to S, they can be approximated in the L^2-norm by functions of the form (1), i.e., by finite sums of shifts of the original generators. However, we prove that at least one reduced set of generators can be obtained from a linear combination of the original generators without translations. In particular, no limit or infinite summation is required. In fact, we show that almost every such linear combination yields a valid family of generators. On the other hand, we show that those combinations which fail to produce a generator set can be dense. That is, combining generators can be a sensitive procedure.

Let $M_{N,L}(\mathbb{C})$ denote the space of complex $N \times L$ matrices endowed with the product Lebesgue measure of $\mathbb{C}^{NL} \cong \mathbb{R}^{2NL}$.

Theorem. *Given $\phi_1, \ldots, \phi_N \in L^2(\mathbb{R}^n)$, let $S = S(\phi_1, \ldots, \phi_N)$ and let $L \leq N$ be the length of S. Let $\mathscr{R} \subset M_{N,L}(\mathbb{C})$ denote the set of those matrices $\Lambda = (\lambda_{j,k})_{1 \leq j \leq N,\, 1 \leq k \leq L}$ such that the linear combinations $\psi_k = \sum_{j=1}^{N} \lambda_{j,k}\, \phi_j$, for $k = 1, \ldots, L$, yield $S = S(\psi_1, \ldots, \psi_L)$.*

(i) *Then $\mathscr{R} = M_{N,L}(\mathbb{C}) \setminus \mathscr{N}$, where $\mathscr{N}$ is a null-set in $M_{N,L}(\mathbb{C})$.*

(ii) *The set $\mathscr{N}$ in* (i) *can be dense in $M_{N,L}(\mathbb{C})$.*

Remark. (i) The conclusions of the Theorem also hold when the complex matrices $M_{N,L}(\mathbb{C})$ are replaced by real matrices $M_{N,L}(\mathbb{R})$.
(ii) We note that our results are not restricted to the case of compactly supported generators.

We illustrate the Theorem by an example in the special case of $N = 2$ given generators for a principal shift-invariant space, i.e., $L = 1$. In this case, $M_{N,L}(\mathbb{C})$

[3] Our results presented here are available in more detail in the form of a preprint.

reduces to $\mathbb{C}^2$. We use the following normalization for the Fourier transform,

$$\widehat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi i t x} dt, \qquad x \in \mathbb{R}.$$

Example. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer less or equal x. We define a discretized version of the Archimedean spiral by $\gamma\colon [0,1) \to \mathbb{Z}^2$,

$$\gamma(x) = (\lfloor u \cos 2\pi u \rfloor, \lfloor u \sin 2\pi u \rfloor), \qquad u = \tan \tfrac{\pi}{2} x, \quad x \in [0,1).$$

Next, let

$$\gamma^\circ(x) = \begin{cases} \gamma(x)/|\gamma(x)|, & \text{if } \gamma(x) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad x \in [0,1).$$

Now define $\phi_1, \phi_2 \in L^2(\mathbb{R})$ by their Fourier transforms, obtained from $\gamma^\circ = (\gamma_1^\circ, \gamma_2^\circ)$ by

$$\widehat{\phi}_j(x) = \begin{cases} \gamma_j^\circ(x), & x \in [0,1), \\ 0, & x \in \mathbb{R} \setminus [0,1), \end{cases} \qquad j = 1, 2.$$

Let $S = S(\phi_1, \phi_2)$. Then S is principal. In fact, the function $\psi = \lambda_1 \phi_1 + \lambda_2 \phi_2$ is a single generator, $S = S(\psi)$, if and only if λ_1 and λ_2 are rationally linearly independent. So here the set $\mathscr{N}$ of the Theorem is

$$\mathscr{N} = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \colon \lambda_1 \text{ and } \lambda_2 \text{ rationally linear dependent}\}.$$

In particular, any rational linear combination of ϕ_1, ϕ_2 fails to generate S. This example illustrates the Theorem for the case of real coefficients, cf. Remark (i). Namely, $\mathscr{N} \cap \mathbb{R}^2$ is a null-set in $\mathbb{R}^2$ yet it contains $\mathscr{Q}^2$, so it is dense in $\mathbb{R}^2$.

Open Problem. It is interesting to ask whether the Theorem also holds for finitely generated shift-invariant subspaces of $L^p(\mathbb{R}^n)$, where $1 \leq p \leq \infty$ and $p \neq 2$. For a few properties of these spaces we refer to [5, 6]. Since the proof of the Theorem relies heavily on fiberization techniques for $p = 2$ and on the characterization of shift-invariant spaces in terms of range functions, this question remains open for $p \neq 2$.

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Geometry of Sets of Parameters of Wave Packets

Gitta Kutyniok
(joint work with Wojciech Czaja and Darrin Speegle)

The goal of our project is to describe completeness properties of wave packets via geometric properties of the sets of their parameters. Our research is motivated by the simple observation that for $L^2(\mathbb{R})$ the sets of parameters of Gabor and wavelet systems form discrete subsets of 2–dimensional linear subspaces in $\mathbb{R}^3$ and that there exists an abundance of sets of parameters which give rise to Gabor or wavelet frames. On the other hand, it is known that systems associated with either translations, dilations, or modulations of a single function do not form frames nor Riesz bases in $L^2(\mathbb{R})$, cf., [7] and [3] for systems consisting of translations (and equivalently modulations) of a single function, and see [4] for systems of dilations. Furthermore, it is known that systems associated with full lattices of translations, dilations, and modulations are infinitely over-complete. Therefore, we shall investigate the role of the geometric structure of sets of parameters of wave packets for the functional properties of associated systems of functions.

1. Wave packets. In [1], Córdoba and Fefferman introduced "wave packets" as those families of functions, which consist of a countable collection of dilations, translations, and modulations of the Gaussian function. Here we will generalize this definition to collections of dilations, translations, and modulations of an arbitrary function in $L^2(\mathbb{R})$.

Definition. Given a function $\psi \in L^2(\mathbb{R})$ and a discrete set $\mathcal{M} \subset \mathbb{R}^+ \times \mathbb{R}^2$, we define the discrete *wave packet* $\mathcal{WP}(\psi, \mathcal{M})$, associated with ψ and $\mathcal{M}$, to be:

$$\mathcal{WP}(\psi, \mathcal{M}) = \{D_x T_y M_z \psi : (x, y, z) \in \mathcal{M}\},$$

where D_x, T_y, and M_z are the $L^2(\mathbb{R})$ unitary operators of dilations, translations, and modulations, respectively:

$$D_x(f)(t) = \sqrt{x} f(xt), \quad T_y(f)(t) = f(t - y), \quad M_z(f)(t) = e^{2\pi i t z} f(t).$$

With this definition, Gabor systems ($\mathcal{M} = \{1\} \times \Lambda$, $\Lambda \subset \mathbb{R}^2$) as well as wavelet systems ($\mathcal{M} = \mathcal{B} \times \{0\}$, $\mathcal{B} \subset \mathbb{R}^+ \times \mathbb{R}$) are thus special examples of wave packets.

2. Density and Dimension. A successful approach to study Gabor frames utilizes the notion of *Beurling density* of the collection of parameters Λ. If $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, Rieffel proved in 1981 that an associated Gabor system is complete only if $ab \leq 1$. This result has been further extended and generalized, and Ramanathan and Steger in [8] proved that if a Gabor system associated with an arbitrary set Λ is a frame then the lower Beurling density of Λ satisfies $D^-(\Lambda) \geq 1$. Moreover, if this frame is a Riesz basis then $D^+(\Lambda) = D^-(\Lambda) = 1$. We refer to [3] for further results in this area and for additional references.

An analogous approach has been undertaken in [4] to study wavelet systems in terms of an appropriately redefined notion of density that is suitable for the structure of the affine group associated with the sets of parameters of wavelet systems. Using these notions, the authors were able to obtain necessary conditions for the existence of wavelet frames in $L^2(\mathbb{R})$.

Our approach shall be an analogue of the two above described methods of characterizations of special wave packets. We introduce a notion of density with respect to the geometry of the affine Weyl–Heisenberg group, which is the appropriate setting for sets of parameters of wave packets. Since the results for wavelet systems indicate that we cannot expect to have a critical density for general wave packets, cf., [4], we need to develop another tool to correlate the geometric properties of the sets of parameters of wave packets with their functional properties. Based on density considerations and motivated by the definition of Hausdorff dimension, we therefore introduce a notion of upper and lower dimension $\dim^{\pm}(\mathcal{M})$ for discrete subsets $\mathcal{M} \subset \mathbb{R}^+ \times \mathbb{R}^2$. The following result shows some basic properties of this notion.

Theorem. Let $\mathcal{M}$ be a subset of $\mathbb{R}^+ \times \mathbb{R}^2$. Then,

(i) $\dim^+(\mathcal{M}) \in [0,3] \cup \{\infty\}$.
(ii) $\dim^-(\mathcal{M}) \in \{0\} \cup [3,\infty]$.

Moreover, by just employing the definition, we obtain the relation $\dim^-(\mathcal{M}) \leq \dim^+(\mathcal{M})$.

3. General Results. Although the situation we consider is more general than that considered in [4], an analogous necessary condition for the existence of an upper frame bound for wave packets $\mathcal{WP}(\psi, \mathcal{M})$ still holds.

Theorem. Let $\psi \in L^2(\mathbb{R})$ and let $\mathcal{M}$ be a discrete subset of $\mathbb{R}^+ \times \mathbb{R}^2$. If $\mathcal{WP}(\psi, \mathcal{M})$ possesses an upper frame bound, then

$$\mathcal{D}^+_A(\mathcal{M}) < \infty \quad \text{for all } A \geq 3. \tag{1}$$

This immediately leads to necessary conditions on the upper and lower dimension of sets of parameters of wave packets, since (1) implies that $\dim^-(\mathcal{M}) \in \{0,3\}$ and $\dim^+(\mathcal{M}) \in [0,3]$. Thus if $\mathcal{WP}(\psi, \mathcal{M})$ has an upper frame bound, then there are only two possible values for $\dim^-(\mathcal{M})$. Wavelet frames and Gabor frames are examples of wave packet frames that satisfy the condition $\dim^-(\mathcal{M}) = 0$. We conjecture that this is the only value, which can be attained by sets of parameters of frames in general.

Conjecture. Let $\psi \in L^2(\mathbb{R})$ and let $\mathcal{M}$ be a discrete subset of $\mathbb{R}^+ \times \mathbb{R}^2$. If $\mathcal{WP}(\psi, \mathcal{M})$ possesses an upper frame bound, then $\dim^-(\mathcal{M}) = 0$.

We can answer this question when the sets of parameters of wave packets have the special form $\mathcal{M} = \mathcal{B} \times \mathbb{Z}$. Wave packets with such sets of parameters have been recently studied by Guido Weiss and his collaborators, see, e.g., [5, 6].

4. Case of Integer Modulations. In this situation we can prove the conjecture and also obtain additional restrictions for the upper dimension.

Theorem. Let $\psi \in L^2(\mathbb{R})$ and let $\mathcal{B}$ be a discrete subset of $\mathbb{R}^+ \times \mathbb{R}$. If $\mathcal{WP}(\psi, \mathcal{B} \times \mathbb{Z})$ possesses an upper frame bound, then

$$\dim^-(\mathcal{B} \times \mathbb{Z}) = 0 \quad \text{and} \quad \dim^+(\mathcal{B} \times \mathbb{Z}) \in [1, 3].$$

It is natural to ask, whether each value in $[1, 3]$ is indeed attained. To study this question we need to construct wave packet frames with prescribed dimensions. Our investigation of this problem leads to multiple examples of non–standard wave packets.

We split our study into two cases. If $1 \leq d \leq 2$, we are even able to construct orthonormal wave packet bases, not only just wave packet frames. It turns out that the most difficult examples to construct are for large dimensions. In this situation by using a highly technical construction, we obtain wave packet frames but no orthonormal wave packet bases so far.

We obtain the following results:

Theorem.

(i) For every $1 \leq d \leq 2$, there exists a discrete subset $\mathcal{B} \subset \mathbb{R}^+ \times \mathbb{R}$ such that $\dim^+(\mathcal{B} \times \mathbb{Z}) = d$ and $\mathcal{WP}(\chi_{[0,1]}, \mathcal{B} \times \mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$.

(ii) For every $2 < d \leq 3$, there exists a discrete subset $\mathcal{B} \subset \mathbb{R}^+ \times \mathbb{R}$ such that $\dim^+(\mathcal{B} \times \mathbb{Z}) = d$ and $\mathcal{WP}(\chi_{[-\frac{1}{2}, \frac{1}{2}]}, \mathcal{B} \times \mathbb{Z})$ is a frame for $L^2(\mathbb{R})$.

Thus we obtain a full description of which values the upper and lower dimension associated with a frame wave packet can attain in the case $\mathcal{M} = \mathcal{B} \times \mathbb{Z}$ under consideration.

For more detailed information on this project we refer to [2].

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Decomposition of Operators and Construction of Frames

David R. Larson

The material we present here is contained in two recent papers. The first was authored by a [VIGRE/REU] team consisting of K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower, and E. Weber, with the title *Ellipsoidal Tight Frames*, and is to appear in Illinois J. Math. This article started as an undergraduate research project at Texas A&M in the summer of 2002, in which Dan Freeman was the student and the other five were faculty mentors. Freeman is now a graduate student at Texas A&M. The project began as a solution of a finite dimensional frame research problem, but developed into a rather technically deep theory concerning a class of frames on an infinite dimensional Hilbert space. The second paper, entitled *Rank-one decomposition of operators and construction of frames*, is a joint article by K. Kornelson and D. Larson, and is to appear in the volume of Contemporary Mathematics containing the proceedings of the January 2003 AMS special session and FRG workshop on *Wavelets, Frames and Operator Theory*, which took place in Baltimore and College Park.

We will use the term *spherical frame* for a frame sequence which is *uniform* in the sense that all its vectors have the same norm. Spherical frames which are tight have been the focus of several articles by different researchers. Since frame theory is essentially geometric in nature, from a purely mathematical point of view it is natural to ask: Which other surfaces in a finite or infinite dimensional Hilbert space contain tight frames? In the first article we considered ellipsoidal surfaces.

By an *ellipsoidal surface* we mean the image of the unit sphere S_1 in the underlying Hilbert space H under a bounded invertible operator A in $B(H)$, the set of all bounded linear operators on H. Let E_A denote the ellipsoidal surface $E_A := AS_1$. A frame contained in E_A is called an *ellipsoidal frame*, and if it is tight it is called an ellipsoidal tight frame (ETF) for that surface. We say that a frame bound K is *attainable* for E_A if there is an ETF for E_A with frame bound K.

Given an ellipsoidal surface $E := E_A$, we can assume $E = E_T$ where T is a positive invertible operator. Indeed, given an invertible operator A, let $A^* = U|A^*|$ be the polar decomposition, where $|A^*| = (AA^*)^{1/2}$. Then $A = |A^*|U^*$. By taking $T = |A^*|$, we see tht $TS_1 = AS_1$. Moreover, it is easily seen that the positive operator T for which $E = E_T$ is unique.

The starting point for the work in the first paper was the following Proposition. For his REU project Freeman found an elementary calculus proof of this for the real case. Others have also independently found this result, including V. Paulsen, and P. Casazza and M. Leon.

Proposition 1. *Let E_A be an ellipsoidal surface on a finite dimensional real or complex Hilbert space H of dimension n. Then for any integer $k \geq n$, E_A contains a tight frame of length k, and every ETF on E_A of length k has frame bound $K = k\left[trace(T^{-2})\right]^{-1}$.*

We use the following standard definition: For an operator $B \in H$, the *essential norm* of B is:

$$\|B\|_{ess} := \inf\{\|B - K\| \ : \ K \text{ is a compact operator in} B(H)\}$$

Our main frame theorem from the first paper is:

Theorem 2. *Let E_A be an ellipsoidal surface in an infinite dimensional real or complex Hilbert space. Then for any constant $K > \|T^{-2}\|_{ess}^{-1}$, E_T contains a tight frame with frame bound K.*

So, for fixed A, in finite dimensions the set of attainable ETF frame bounds is finite, whereas in infinite dimensions it is a continuum.

Problem. If the essential norm of A is replaced with the norm of A in the above theorem, or if the inequality is replaced with equality, then except for some special cases, and trivial cases, no theorems of any degree of generality are known concerning the set of attainable frame bounds for ETF's on E_A. It would be interesting to have a general analysis of the case where $A - I$ is compact. In this case, one would want to know necessary and sufficient conditions for existence of a tight frame on E_A with frame bound 1. In the special case $A = I$ then, of course, any orthonormal basis will do, and these are the only tight frames on E_A in this case. What happens in general when $\|A\|_{ess} = 1$ and A is a small perturbation of I?

We use elementary tensor notation for a rank-one operator on H. Given $u, v, x \in H$, the operator $u \otimes v$ is defined by $(u \otimes v)x = \langle x, v\rangle u$ for $x \in H$. The operator $u \otimes u$ is a projection if and only if $\|u\| = 1$.

Let $\{x_j\}_j$ be a frame for H. The standard frame operator is defined by: $Sw = \sum_j \langle w, x_j\rangle x_j = \sum_j (x_j \otimes x_j)\, w$. Thus $S = \sum_j x_j \otimes x_j$, where this series of positive rank-1 operators converges in the strong operator topology (i.e. the topology of pointwise convergence). In the special case where each x_j is a unit vector, S is the sum of the rank-1 projections $P_j = x_j \otimes x_j$.

For A a positive operator, we say that A has a *projection decomposition* if A can be expressed as the sum of a finite or infinite sequence of (not necessarily mutually orthogonal) self-adjoint projections, with convergence in the strong operator topology.

If x_j is a frame of unit vectors, then $S = \sum_j x_j \otimes x_j$ is a projection decomposition of the frame operator. This argument is trivially reversible, so a positive invertible operator S is the frame operator for a frame of unit vectors if and only if it admits a projection decomposition $S = \sum_j P_J$. If the projections in the decomposition are not of rank one, each projection can be further decomposed (orthogonally) into rank-1 projections, as needed, expressing $S = \sum_n x_n \otimes x_n$, and then the sequence $\{x_n\}$ is a frame of unit vectors with frame operator S.

In order to prove Theorem 2, we first proved Theorem 3 (below), using purely operator-theoretic techniques.

Theorem 3. *Let A be a positive operator in $B(H)$ for H a real or complex Hilbert space with infinite dimension, and suppose $\|A\|_{ess} > 1$. Then A has a projection decomposition.*

Suppose, then, that $\{x_n\}$ is a frame of unit vectors with frame operator S. If we let $y_j = S^{-\frac{1}{2}}x_j$, then $\{y_j\}_j$ is a *Parseval* frame (i.e. tight with frame bound 1). So $\{y_j\}_j$ is an ellipsoidal tight frame for the ellipsoidal surface $E_{S^{-\frac{1}{2}}} = S^{-\frac{1}{2}}S_1$. This argument is reversible: Given a positive invertible operator T, let $S = T^{-2}$. Scale T if necessary so that $\|S\|_{ess} > 1$. Let $S = \sum_j x_j \otimes x_j$ be a projection decomposition of S. Then $\{Tx_j\}$ is an ETF for the ellipsoidal surface TS_1. Consideration of frame bounds and scale factors then yields Theorem 2.

Most of our second paper concerned *weighted* projection decompositions of positive operators, and resultant theorems concerning frames. If T is a positive operator, and if $\{c_n\}$ is a sequence of positive scalars, then a weighted projection decomposition of T with weights $\{c_n\}$ is a decomposition $T = \sum_j P_j$ where the P_j are projections, and the series converges strongly. We have since adopted the term *targeted* to refer to such a decomposition, and generalizations thereof. By a *targeted decomposition* of T we mean any strongly convergent decomposition $T = \sum_n T_n$ where the T_n is a sequence of *simpler* positive operators with special prescribed properties. So a weighted decomposition is a targeted decomposition for which the scalar weights are the prescribed properties. And, of course, a projection decomposition is a special case of targeted decomposition.

After a sequence of Lemmas, building up from finite dimensions and employing spectral theory for operators, we arrived at the following theorem. We will not discuss the details here because of limited space. It is the *weighted* analogue of theorem 3.

Theorem 4. *Let B be a positive operator in $B(H)$ for H with $\|B\|_{ess} > 1$. Let $\{c_i\}_{i=1}^{\infty}$ be any sequence of numbers with $0 < c_i \le 1$ such that $\sum_i c_i = \infty$. Then there exists a sequence of rank-one projections $\{P_i\}_{i=1}^{\infty}$ such that $B = \sum_{i=1}^{\infty} c_i P_i$.*

We refer the interested reader to the *Open Problems* section of this report for more on targeted decompositions. In the first problem, we raised the question of which positive operators admit *finite* projection decompositions. The second problem related to a completely different type of targeted decomposition than discussed in this abstract, or considered in the two papers we presented. It was motivated by talks and discussions in this Workshop, and just may be relevant to the theory of modulation spaces and Gelfand triples. We plan to pursue this further.

Groups, Wavelets, and Function Spaces

Gestur Ólafsson

In the talk we discussed several connections between the following topics:

(1) Representation theory of Lie groups;
(2) Linear action of Lie groups on $\mathbb{R}^d$;
(3) Wavelets and wavelet sets;
(4) Besov spaces associated to symmetric cones.

Let $H \subseteq \mathrm{GL}(d,\mathbb{R})$ be a closed subgroup, and hence a Lie group. Let G be the group of affine linear maps $(x,h)(t) := h(t) + x$, $h \in H$, $x, y \in \mathbb{R}^d$. Then G is the semi-direct product of $\mathbb{R}^d$ and H, $G = \mathbb{R}^d \times_s H$. Define a unitary representation of G on $L^2(\mathbb{R}^d)$ by

$$\pi(x,h)f(t) = |\det(h)|^{-1/2} f((x,h)^{-1}(t)) = |\det(h)|^{-1/2} f(h^{-1}(t-x))\,.$$

It is quite often useful to have an equivalent realization of π in frequency space. Define for $F \in L^2(\widehat{\mathbb{R}}^d)$

$$\widehat{\pi}(x,h)F(\omega) = |\det h|^{1/2} e^{-2\pi x\cdot\omega} F(h^T(\omega))\,.$$

Then the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\widehat{\mathbb{R}}^d)$, $\mathcal{F}(f)(\omega) = \int f(t) e^{-2\pi t\cdot\omega}\, dt$ is a unitary intertwining operator. Here, and elsewhere, we write $\widehat{\mathbb{R}}^d$ to underline, that we are looking at $\mathbb{R}^d$ as the *frequency domain*.

For $\psi \in L^2(\mathbb{R}^d)$ define $W_\psi : L^2(\mathbb{R}^d) \to C(G)$, by

$$W_\psi(f)(g) := (f, \pi(g)\psi) = |\det h|^{-1/2} \int f(t)\overline{\psi(h^{-1}(t-x))}\, dt \quad g = (x,h) \in G\,.$$

Note, that W_ψ depends on our choice of wavelet function ψ. In particular, if $\psi \in S(\mathbb{R}^d)$, then W_ψ extends to a linear map on $S'(\mathbb{R}^d)$, the space of tempered distributions. It is an important question in analysis to study spaces of functions or distribution using the wavelet transform. In particular, for a given weight function w on G, one can, if the representation π is integrable, define a Banach space of distribution by $\{f \in S'(\mathbb{R}^d) \mid W_\psi(f) \in L^p(G, w d\mu_G)$, with norm $\|f\| = \|f\|_{L^p(G,wd\mu_G)}$. Here $d\mu_G$ denotes a left invariant measure on G. Using the structure of G as a semi-direct product, one can even define mixed $L^{p,q}$-norm. This is related to the Feichtinger-Gröchenig co-orbit theory for group representations, which for the Heisenberg group has become quite important through the theory of Modulation spaces, [11, 12, 19].

The simples case is $p = 2$ and $w = 1$. A simple calculation shows, that

$$\|W_\psi(f)\|^2_{L^2(G)} = \int_{\widehat{R}^d} |\mathcal{F}(f)(\omega)|^2 \left(\int_H |\mathcal{F}(\psi)(h^T\omega)|^2\, d\mu_H(h) \right) d\omega\,.$$

It follows that $W_\psi(f) \in L^2(G)$ if and only if $\int_H |\mathcal{F}(\psi)(h^T(\omega))|^2\, d\mu_H(h) < \infty$ for almost all $\omega \in \widehat{R}^d$. Furthermore, $W_\psi : L^2(\mathbb{R}^d) \to \mathrm{Im}(W_\psi) \subset L^2(G)$ is an unitary isomorphism onto its image if and only if

$$\int_H |\mathcal{F}(\psi)(h^T(\omega))|^2 \, d\mu_H(h) = 1$$

for almost all $\omega \in \widehat{R}^d$. In this case we have $f = W_\psi^*(W_\psi(f))$, or

$$f = \int_G W_\psi(f)(g)\pi(g)\psi \, d\mu_G(g)$$

as an weak integral. We refer to [3, 8, 9, 10, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 26] for more detailed discussion.

Let $\Delta \subset H$ and $\Lambda \subset \mathbb{R}^d$ be countable subsets. Let $\Gamma := \Lambda \times \Delta \subset G$. Define a sequence of functions ψ_γ, $\gamma \in \Gamma$, by

$$\psi_{\lambda,\delta}(t) := \pi((\lambda, \delta)^{-1})\psi(t) = |\det \delta|^{1/2}\psi(\delta(t) + \lambda) .$$

Then ψ is a (subspace) *wavelet* if the sequence $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a orthonormal basis for its closed linear span. A measurable set $\Omega \subset \widehat{\mathbb{R}}^d$, $0 < |\Omega| < \infty$ is a (subspace) *wavelet set* if $\psi = \mathcal{F}^{-1}(\chi_\Omega)$ is a (subspace) wavelet. For discussion on wavelet sets see [1, 4, 5, 6, 7, 23, 24, 25]. A special class of groups H was studied in [10, 22, 23]. Here it was assumed, that H has finitely many open orbits $\mathcal{O}_1, \dots, \mathcal{O}_r \subset \widehat{\mathbb{R}}^d$ of full measure, i.e., $(H, \widehat{\mathbb{R}}^d)$ is a *pre-homogeneous vector space*. We set

$$L_j^2 = L_{\mathcal{O}_j}^2(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \mid \mathrm{Supp}(\mathcal{F}(f)) \subseteq \overline{\mathcal{O}_j}\} .$$

As an example, take $H = \mathbb{R}^+\mathrm{SO}(d)$. There is only one open orbit $\mathcal{O}_1 = \widehat{\mathbb{R}}^d \setminus \{0\}$. In particular, $L_1^2 = L^2(\mathbb{R}^d)$. Let $F \subset \mathrm{SO}(d)$ be a finite subgroup and $\lambda > 1$. Let $\Delta = \{\lambda^n R \mid n \in \mathbb{Z}, R \in F\}$, and let $\Lambda \subset \mathbb{R}^d$ a lattice. Then there exists a $\Gamma \times \Delta$-wavelet set for $L^2(\mathbb{R}^d)$. This follows from Theorem 1 [6] as was pointed out to me by my student M. Dobrescu. We get a more complicated example by taking $H = \mathbb{GL}(n, \mathbb{R})_o$ and $\mathbb{R}^d = \mathrm{Sym}(n, \mathbb{R})$, $d = n(n+1)/2$, the space of symmetric $n \times n$-matrices. The group H operates on $\mathbb{R}^d$ by $h \cdot X = gXg^T$. The open orbits are $\mathcal{O}_{p,q} = H \cdot I_{p,q}$. Here $\mathcal{O}_{p,q}$ stands for the open set of regular matrices of signature $(p, q = n - p)$, and $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. The set $\mathcal{O}_{n,0}$ is an open symmetric cone. It is well known, that the group S of upper triangular matrices acts transitively on C. Let A be the group of diagonal matrices with positive diagonal elements, and let N be the group of upper triangular matrices (x_{ij}), with $x_{ii} = 1$, $i = 1, \dots, n$. Then $S = AN = NA$. In [22, 23] a special choice for Δ was made. This set Δ is closely related to the structure of H. In our example this construction can be explained by taking $\Delta_N = \{(x_{ij} \in N \mid x_{ij} \in \mathbb{Z}\}$, and $\Delta_A = \{d(\lambda_1^{k_1}, \dots, \lambda_n^{k_n}) \mid (k_1, \dots, k_n) \in \mathbb{Z}^n\}$, where $\lambda_j > 1$. Then we set $\Delta = \Delta_A\Delta_N$. It follows by [23], Theorem 4.5, that, if Λ is a lattice in $\mathbb{R}^d$, then there exists a $L^2(\mathbb{R}^d)$ wavelet set for $\Lambda \times \Delta$. But it is an open problem, if there exists a $L_C^2(\mathbb{R}^d)$ wavelet set for $\Lambda \times \Delta$. One can even complicate this by adding a finite group of rotations that centralize A and normalize N.

It was also shown in [22, 23] that, for our special choice of Δ, there is always a set $\Omega \subset \mathcal{O}_j$ such that $\psi = \mathcal{F}^{-1}(\chi_\Omega)$ generates a tight frame for L_j^2. It is clear, that we can replace χ_Ω by a compactly supported function $\phi \geq 0$, $\phi|_\Omega = 1$, and get a frame generator that is rapidly decreasing. But it is an open problem if we can in fact get a rapidly decreasing function that generates a *tight frame*. A private note by D. Speegle indicates, that this might in fact be possible.

One of the reason I discuss the last example is, that this is just an example of H being the automorphism group of a symmetric cone $C \subset R^d$, i.e., $H = \mathrm{GL}(C) := \{h \in \mathrm{GL}(d, \mathbb{R}) \mid h(C) = C\}$. The general philosophy is, that wavelets are associated to Besov spaces. In fact, one sees easily, that the Besov spaces in [2] can also be defined by using the continuous wavelet transform. It is therefore a natural question, which we pose here as a third open problem, to study the Besov spaces, introduced in [2], using the theory of co-orbit spaces and the discrete wavelet transform using the results from [22, 23] applied to the group $\mathrm{GL}(C)$.

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Adaptive Wavelet Methods for the Numerical Solutions of Operator Equations

Karsten Urban

We review recent results on the construction, analysis and realization of adaptive wavelet methods for the numerical solution of operator equations. The theoretic results are mainly based on on work by Cohen, Dahmen and DeVore, [4, 5, 6, 7].

Elliptic Operators. We consider (just for the sake of simplicity) the boundary value problem on a bounded, open domain $\Omega \subset \mathbb{R}^n$ determining $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u(x) = f(x),\ x \in \Omega, \qquad u_{|\partial\Omega} = 0, \tag{1}$$

for a given function $f : \Omega \to \mathbb{R}$. The variational formulation reads: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := (\nabla u, \nabla v)_0 = (f, v)_0 \text{ for all } v \in H_0^1(\Omega) \tag{2}$$

for a given function $f \in H^{-1}(\Omega)$, where $(\cdot, \cdot)_0$ denotes the standard L_2-inner product on Ω. Introducing the differential operator

$$A : H_0^1(\Omega) \to H^{-1}(\Omega), \qquad \langle Au, v\rangle := a(u, v), \quad u, v \in H_0^1(\Omega), \tag{3}$$

we can rewrite (2) as an operator equation

$$Au = f \tag{4}$$

in the Sobolev space $H_0^1(\Omega)$. Note that (4) is an infinite-dimensional operator equation in a function space. We always assume in the sequel, that A is *boundedly invertible*, i.e.

$$\|Au\|_{-1} \sim \|u\|_1, \qquad u \in H_0^1(\Omega), \tag{5}$$

where we use the notation $A \sim B$ in order to abbreviate the existence of constants $0 < c \leq C < \infty$ such that $cA \leq B \leq CA$. At this point also general elliptic operators on Hilbert spaces are included.

Wavelet Characterization of Sobolev Spaces. The first step is to transform (4) into a (well-conditioned) problem in sequence spaces. This is done with the aid of a bi-orthogonal wavelet bases. Assume

$$\Psi := \{\psi_\lambda : \lambda \in \mathcal{J}\} \tag{6}$$

is a Riesz basis for $L_2(\Omega)$. Here $\mathcal{J}$ is an infinite set of indices and we always think of an index $\lambda \in \mathcal{J}$ as a pair (j, k), where $|\lambda| := j \in \mathbb{N}$ always denotes the *scale* or *level* and k (which possibly is a vector) contains information on the localization of ψ_λ (e.g. the center of its support). We assume that Ψ admits a characterization of a whole scale of Sobolev spaces in the sense, that the following estimates hold:

$$\left\| \sum_{\lambda \in \mathcal{J}} d_\lambda \, \psi_\lambda \right\|_s \sim \left(\sum_{\lambda \in \mathcal{J}} |d_\lambda|^{2s|\lambda|} \right)^{1/2}, \tag{7}$$

for $s \in (-\tilde{\gamma}, \gamma)$ and $\gamma, \tilde{\gamma} > 1$ depend on the properties of Ψ such as polynomial exactness and order of vanishing moments. Using the short hand notations

$$\boldsymbol{d}^T \Psi := \sum_{\lambda \in \mathcal{J}} d_\lambda \, \psi_\lambda, \quad \boldsymbol{d} = (d_\lambda)_{\lambda \in \mathcal{J}}, \qquad \boldsymbol{D} = \mathrm{diag}\,(2^{|\lambda|})_{\lambda \in \mathcal{J}} \tag{8}$$

we can rephrase (7) in the following way

$$\|\boldsymbol{d}^T \Psi\|_s \sim \|\boldsymbol{D}^s \boldsymbol{d}\|_{\ell_2(\mathcal{J})}. \tag{9}$$

Note that nowadays there are criteria known in order to ensure (9) and also constructions of wavelets also on complex domains are on the market, [2, 3, 10, 11].

Then, the Riesz Representation Theorem guarantees the existence of a *bi-orthogonal* wavelet basis $\tilde{\Psi} = \{\tilde{\psi} : \lambda \in \mathcal{J}\}$ such that

$$\|\boldsymbol{d}^T \tilde{\Psi}\|_{-s} \sim \|\boldsymbol{D}^{-s} \boldsymbol{d}\|_{\ell_2(\mathcal{J})}. \tag{10}$$

An equivalent well-conditioned problem in ℓ_2. This implies for any $u = \boldsymbol{u}^T \Psi$

$$\begin{aligned}
\|\boldsymbol{u}\|_{\ell_2(\mathcal{J})} &\overset{(9)}{\sim} \|(\boldsymbol{D}^{-1} \boldsymbol{u})^T \Psi\|_1 \overset{(5)}{\sim} \|A((\boldsymbol{D}^{-1} \boldsymbol{u})^T \Psi)\|_{-1} \\
&= \|(A((\boldsymbol{D}^{-1} \boldsymbol{u})^T \Psi), \Psi)_0 \, \tilde{\Psi}\|_{-1} \overset{(10)}{\sim} \|\boldsymbol{D}^{-1}(A((\boldsymbol{D}^{-1} \boldsymbol{u})^T \Psi), \Psi)_0\|_{\ell_2(\mathcal{J})} \\
&= \|\boldsymbol{D}^{-1} \, (A\Psi, \Psi)_0 \, \boldsymbol{D}^{-1} \, \boldsymbol{u}\|_{\ell_2(\mathcal{J})},
\end{aligned}$$

which shows that $\|\boldsymbol{u}\|_{\ell_2(\mathcal{J})} \sim \|\boldsymbol{A}\boldsymbol{u}\|_{\ell_2(\mathcal{J})}$ for $\boldsymbol{A} := \boldsymbol{D}^{-1} \, (A\Psi, \Psi)_0 \, \boldsymbol{D}^{-1}$. In other words, $\boldsymbol{A} : \ell_2(\mathcal{J}) \to \ell_2(\mathcal{J})$ is a *boundedly invertible* operator on the sequence space $\ell_2(\mathcal{J})$. Defining $\boldsymbol{f} := (f, \Psi)_0$, we are led to the equivalent discrete problem

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}. \tag{11}$$

An infinte-dimensional convergent adaptive algorithm. Ignoring for a minute that an infinite ℓ_2-sequence can not be represented in a computer, we aim at constructing an iterative solution method for the discrete problem (11). This is done by a *Richardson*-type iteration: Given an initial guess $\boldsymbol{u}^{(0)} \in \ell_2(\mathcal{J})$ and some $\alpha \in \mathbb{R}^+$, we define

$$\boldsymbol{u}^{(i+1)} := \boldsymbol{u}^{(i)} + \alpha(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}^{(i)}) = (\boldsymbol{I} - \alpha\boldsymbol{A})\boldsymbol{u}^{(i)} + \alpha\boldsymbol{f}. \tag{12}$$

The convergence of this algorithm is easily seen:

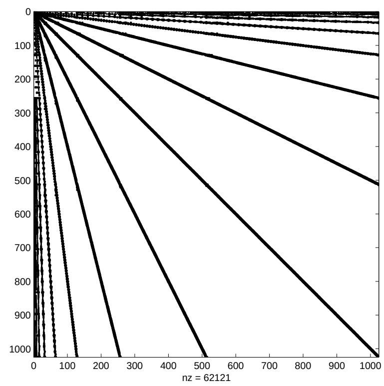
$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}^{(i+1)}\|_{\ell_2(\mathcal{J})} &= \|\boldsymbol{u} + \alpha\underbrace{(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u})}_{=0} - \boldsymbol{u}^{(i)} - \alpha(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}^{(i)})\|_{\ell_2(\mathcal{J})} \\ &= \|(\boldsymbol{I} - \alpha\boldsymbol{A})(\boldsymbol{u} - \boldsymbol{u}^{(i)})\|_{\ell_2(\mathcal{J})} \\ &\le \|\boldsymbol{I} - \alpha\boldsymbol{A}\|_{B(\ell_2(\mathcal{J}))}\|\boldsymbol{u} - \boldsymbol{u}^{(i)}\|_{\ell_2(\mathcal{J})}, \end{aligned}$$

i.e., this iteration converges if $\rho := \|\boldsymbol{I} - \alpha\boldsymbol{A}\|_{B(\ell_2(\mathcal{J}))} < 1$. This condition, in turns, can be guaranteed e.g. if $\boldsymbol{A}$ is s.p.d. which holds e.g. for wavelet representations of elliptic partial differential operators.

Approximate Operator Applications. Using the locality and the vanishing moment properties of wavelets, on can show that the wavelet representation of a large class of operators is *almost sparse*, i.e., one has

$$|a_{\lambda,\lambda'}| \le C\, 2^{-\left||\lambda|-|\lambda'|\right|\sigma}(1 + d(\lambda, \lambda'))^{-\beta}, \tag{13}$$

where $d(\lambda, \lambda') := 2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'})$ for some parameters σ and β. Roughly speaking this means that one has a decay in the level difference as well as in the spatial distance of wavelets. A typical structure is the well-known *finger structure*, see figure right.



For such kind of operators, an approximate application APPLY was constructed. Replacing any multiplication with $\boldsymbol{A}$ by the routine, yields a convergent adaptive method that moreover was proven to be asymptotically optimal in the sense that the rate of convergence stays proportional to the decay of the best N-term approximation at optimal cost.

Numerical results are shown for the Laplace [1] and the Stokes problem, [8, 12].

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Orthogonal Frames for Encryption

Eric Weber
(joint work with Ryan Harkins and Andrew Westmeyer)

There are several encryption algorithms in which randomness plays a role in the encryption process. The first example is the one time pad, which is an unconditionally secure cipher and is optimal in terms of key length. The process of the one time pad is the following: take a message m, expressed in some binary format, choose at random a binary sequence of the same length, and bitwise add the message to the random sequence. The recipient, knowing the random sequence, then adds the sequence to the cipher text again to recover the message. We remark here that this is actually the basis for quantum cryptography. The other example is the McEliece cipher. The encryption here is based on error correcting codes: choose a code which corrects N errors, encode the message, and introduce N randomly chosen errors. The cipher text then is the encoded message with the errors. The decryption then is to decode the ciphertext which corrects the errors. It is possible to actually alter this slightly to make it a public key encryption system.

Both ciphers have drawbacks: the one time pad is a private key system, and the key must change every time a message is encrypted. The McEliece cipher requires a prohibitively large key size compared to the size of the message.

We propose here a third encryption algorithm which utilizes randomness in the encryption process based on Hilbert space frames.

Let H be a separable Hilbert space over the field $\mathbb{F}$ with scalar product $\langle \cdot, \cdot \rangle$, where $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. A frame for H is a sequence $\mathbb{X} := \{x_n\}_{n \in \mathbb{Z}}$ such that there exist constants $0 < A \leq B < \infty$ such that for all $v \in H$,

$$A\|v\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle v, x_n\rangle|^2 \leq B\|v\|^2. \tag{1}$$

If $A = B = 1$, the frame is said to be *Parseval*, and then for all $v \in H$,

$$v = \sum_{n\in\mathbb{Z}} \langle v, x_n\rangle x_n.$$

For elementary frame theory, see [Han et al. 2000, Casazza 2000].

Let H be a finite dimensional Hilbert space. A finite frame is a frame $\mathbb{X} := \{x_i\}_{i=1}^M$ for H, where M is necessarily no smaller than the dimension of H. The analysis operator for $\mathbb{X}$ is given by

$$\Theta_{\mathbb{X}} : H \to \mathbb{F}^M : v \mapsto (\langle v, x_1\rangle, \langle v, x_2\rangle, \ldots, \langle v, x_M\rangle).$$

Definition 1. Let H and K be finite dimensional Hilbert spaces. Two frames $\mathbb{X} := \{x_n\}_{n=1}^M \subset H$ and $\mathbb{Y} := \{y_n\}_{n=1}^M \subset K$ are *orthogonal* if for all $v \in H$, $\sum_{n=1}^M \langle v, x_n\rangle y_n = 0$. Equivalently, $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal if $\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}} : H \to K$ is the 0 operator, where $\Theta_{\mathbb{Y}}^*$ denotes the Hilbert space adjoint.

Our encryption scheme, which is similar to a subband coding scheme, is an effort to approximate the One-Time Pad. The (private) key for this encryption scheme is two orthogonal Parseval frames $\{x_n\}_{n=1}^M \subset H$ and $\{y_n\}_{n=1}^M \subset K$. Let $\Theta_{\mathbb{X}}$ and $\Theta_{\mathbb{Y}}$ respectively denote their analysis operators. Suppose $m \in H$ is a message; let $g \in K$ be a non-zero vector chosen at random. The ciphertext $c \in \mathbb{F}^M$ is given as follows:

$$c := \Theta_{\mathbb{X}} m + \Theta_{\mathbb{Y}} g.$$

To recover the message, we apply $\Theta_{\mathbb{X}}^*$:

$$\begin{aligned}
\Theta_{\mathbb{X}}^* c &= \Theta_{\mathbb{X}}^*\Theta_{\mathbb{X}} m + \Theta_{\mathbb{X}}^*\Theta_{\mathbb{Y}} g \\
&= \sum_{n=1}^M \langle m, x_n\rangle x_n + \sum_{n=1}^M \langle m, y_n\rangle x_n \\
&= m + 0 = m.
\end{aligned}$$

Our experiments show that this encryption algorithm is robust against a brute force attack. However, the encryption algorithm is vulnerable to a chosen-plaintext attack.

A chosen-plaintext attack is an attack mounted by an adversary which chooses a plaintext and is then given the corresponding ciphertext. For convenience, assume that $H = K = \mathbb{R}^N$ and $M = 2N$. The attack on our scheme is as follows:

Step 1. Determine the range $\Theta_{\mathbb{Y}}(\mathbb{R}^N)$. Choose any plaintext m of size N. Encode the plaintext twice, with output, say, e_0 and e_1. Compute $e_1 - e_0 = \Theta_{\mathbb{X}} m + \Theta_{\mathbb{Y}} g_1 - (\Theta_{\mathbb{X}} m + \Theta_{\mathbb{Y}} g_0) = \Theta_{\mathbb{Y}}(g_1 - g_0)$. Notice that this yields a vector $f_1 = \Theta_{\mathbb{Y}}(g_1 - g_0)$ in the range of $\Theta_{\mathbb{Y}}$. Encode the plaintext a third time, with output e_2, and compute $f_2 = e_2 - e_0$. Compute $f_3, \ldots, f_m$ until the collection $\{f_1, \ldots, f_m\}$ contains a linearly independent subset of size N. This then determines the subspace $Z := \Theta_{\mathbb{Y}}(\mathbb{R}^N) \subset \mathbb{R}^M$.

Step 2. Determine the range $T := \Theta_{\mathbb{X}}(\mathbb{R}^N) \subset \mathbb{R}^M$. Choose any (non-zero) plaintext m_1 of size N; encode the plaintext, with output e_1; then project e_1 onto the orthogonal complement of Z. This yields a vector x_1 in T. Choose another plaintext m_2 and repeat, yielding vector $x_2 \in T$. Repeat until the collection $\{x_1, \dots, x_q\}$ contains a linearly independent subset of size N. This set determines T.

Step 3. Determine the matrix $\Theta_{\mathbb{X}}$. Suppose in Step 2, $\{m_1, \dots, m_N\}$ is such that $\{x_1, \dots, x_N\}$ is linearly independent. We then have

$$\Theta_{\mathbb{X}} m_k = x_k \text{ for } k = 1, \dots N.$$

Given this system of equations, now solve for $\Theta_{\mathbb{X}}$.

Step 4. Unencode cipher texts. Given any ciphertext c, the adversary computes the following:

$$\begin{aligned} \Theta_{\mathbb{X}}^* c &= \Theta_{\mathbb{X}}^*(\Theta_{\mathbb{X}} m + \Theta_{\mathbb{Y}} g) \\ &= \Theta_{\mathbb{X}}^* \Theta_{\mathbb{X}} m \\ &= m \end{aligned}$$

since $\mathbb{X}$ was a Parseval frame.

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Open Problems

How close can an L^1-Function be to a Convolution Idempotent?

Hans G. Feichtinger

On one hand, it is well known that $(L^1(\mathbb{R}^d), *)$ is a convolution algebra which does not contain unit, in the sense that it does not exists any function $e \in L^1(\mathbb{R}^d)$ such that $g * e = g$, for all $g \in L^1(\mathbb{R}^d)$. On the other hand, it is always possible to construct a sequence $(e_n)_{n\in\mathbb{N}}$ of functions in $L^1(\mathbb{R}^d)$ such that $\|g * e_n - g\|_1 \to 0$, for $n \to +\infty$.

The open problem suggested by Feichtinger is the following (It has been first stated at an Oberwolfach conference in 1980):

What is the infimum for expression of the form $\|g * g - g\|_1$, where g is a symmetric function $g \in L^1(\mathbb{R})$ with $\|g\|_1 = 1$. Is there a function which minimizes $\|g * g - g\|$? (of course it cannot be uniquely determined, since the problem is invariant under L^1-normalized dilations, but maybe this is the only form of ambiguity).

Nowadays the problem appears again as very interesting because it not obvious how to attack it numerically, because it cannot be formulated in a non-trivial way over discrete groups.

Let us show that this problem well posed and, in particular, that

$$\inf_{g\in L^1(\mathbb{R}),\|g\|_1=1} \|g * g - g\|_1 \geq \frac{1}{4}. \tag{1}$$

Because of the assumptions on g, its Fourier transform $\mathcal{F}g$ has the following properties: $\mathcal{F}g$ is real valued, continuous, vanishing at infinity, and $\mathcal{F}g(0) = 1$. The norm $\|g * g - g\|_1$ can be estimated from below by

$$\|g * g - g\|_1 \geq \|(\mathcal{F}g)^2 - \mathcal{F}g\|_\infty.$$

In particular, one has $\|(\mathcal{F}g)^2 - \mathcal{F}g\|_\infty \geq \left\| |\mathcal{F}g|^2 - |\mathcal{F}g| \right\|_\infty$. Since $\mathcal{F}g$ is continuous, vanishing at infinity, and $\mathcal{F}g(0) = 1$, there exists $\omega_0 \in \mathbb{R}^d$ such that $|\mathcal{F}g(\omega_0)| = \frac{1}{2}$. Therefore

$$\sup_{\omega\in\mathbb{R}^d} \left| |\mathcal{F}g(\omega)|^2 - |\mathcal{F}g(\omega)| \right| \geq \frac{1}{4}.$$

This immediately implies (1). Numerical experiments indicate that (naturally) the Gaussian (up to dilation) is a “strong candidate” with a value around $0.31 < 1/3$, so one may expect that the “true value of the infimum” is in the interval $[1/4, 1/3]$.

Approximation of Frames by Normalized Tight Ones

Michael Frank

Let $\{x_i\}_i$ be a frame of a Hilbert subspace $K \subseteq H$ of a given (separable) Hilbert space H with upper and lower frame bounds B and A. The resulting frame transform is the map $\theta : K \to l_2$, $\theta(x) = \{\langle x, g_i \rangle\}_i$, and its adjoint operator is $\theta^* : l_2 \to K$, $\theta^*(e_i) = g_i$, $i \in \mathbb{N}$, for the standard orthonormal basis $\{e_i\}_i$ of l_2. Let $S = (\theta^*\theta)^{-1}$ be the frame operator defined on K. It is positive and invertible. There exists an orthogonal projection $P : l_2 \to \theta\theta^*(l_2) \subseteq l_2$ onto the range of the frame transform.

Problem:

Are there distance measures on the set of frames of all Hilbert subspaces L of H with respect to which a multiple of the normalized tight frame $\{S^{1/2}(x_i)\}_i$ is the closest normalized tight frame to the given frame $\{x_i\}_i$ of the Hilbert subspace $K \subseteq H$, or at least one of the closest normalized tight frames?

If there are other closest normalized tight frames with respect to the selected distance measures, do they span the same Hilbert subspaces of H? If not, how are the positions of the subspaces with respect to $K \subseteq H$?

To obtain at least partial results authors usually have applied some additional restrictions to the set of frames to be considered: (i) resort to similar frames, (ii) resort to the case $K = L = H$, (iii) resort to special classes of frames like Gabor (Weyl-Heisenberg) or wavelet frames, and others. So one goal might be to lessen the restrictions in the suppositions.

We would like to list some existing results from [1], [3] and [4] to give a flavor of the existing successful approaches and to outline the wide open field of research to be filled. From recent correspondences with R. Balan we know about new findings of him and Z. Landau to be published in the near future ([2]).

First recall the major results by R. Balan ([1]): The frame $\{x_i\}_i$ of the Hilbert space H is said to be *quadratically close* to the frame $\{y_i\}_i$ of H if there exists a non-negative number C such that the inequality

$$\left\| \sum\nolimits_i c_i (x_i - y_i) \right\| \leq C \cdot \left\| \sum\nolimits_i c_i y_i \right\|$$

is satisfied. The infimum of all such constants C is denoted by $c(y,x)$. In general, if $C \geq c(y,x)$ then $C(1-C)^{-1} \geq c(x,y)$, however this distance measure is not reflexive. Two frames $\{x_i\}_i$ and $\{y_i\}_i$ of a Hilbert space H are said to be *near* if $d(x,y) = \log(\max(c(x,y), c(y,x)) + 1) < \infty$. They are near if and only if they are similar, [1, Th. 2.4]. The distance measure $d(x,y)$ is an equivalence relation and fulfills the triangle inequality.

Theorem 1([1]) For a given frame $\{x_i\}_i$ of H the distance measures admit their infima at

$$\min c(y,x) = \min c(x,y) = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}\,, \ \min d(x,y) = \frac{1}{4}(\log(B)-\log(A))\,.$$

These values are achieved by the tight frames

$$\left\{\frac{\sqrt{A}+\sqrt{B}}{2}S^{1/2}(x_i)\right\}_i\,, \left\{\frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}}S^{1/2}(x_i)\right\}_i\,, \{\sqrt[4]{AB}S^{1/2}(x_i)\}_i\,,$$

in the same order as the three measures are listed above. The solution may not be unique, in general, however any tight frame $\{y_i\}_i$ of H that achieves the minimum of one of the three distance measures $c(y,x)$, $c(x,y)$ and $d(x,y)$ is unitarily equivalent to the corresponding solutions listed above. The difference of the connecting unitary operator and the product of minimal distance times either $S^{1/2}$ or $S^{-1/2}$ fulfills a measure-specific operator norm equality.

A second class of examples has been treated by T. R. Tiballi, V. I. Paulsen and the author in 1998 ([3]). The foundations were laid by T. R. Tiballi in his Master Thesis in 1991 ([6]). Therein he was dealing with the symmetric orthogonalization of orthonormal bases of Hilbert spaces in a way that did not make use of the linear independence of the elements. So his techniques have been extendable to the situation of frames giving rise to the symmetric approximation of frames by normalized tight ones.

Theorem 2([3]) The operator $(P-|\theta^*|)$ is Hilbert-Schmidt if and only if the sum $\sum_{j=1}^{\infty}\|\mu_j - x_j\|^2$ is finite for at least one normalized tight frame $\{\mu_i\}_i$ of a Hilbert subspace L of H that is similar to $\{x_i\}_i$. In this situation the estimate

$$\sum\nolimits_{j=1}^{\infty}\|\mu_j - x_j\|^2 \geq \sum\nolimits_{j=1}^{\infty}\|S^{1/2}(x_j)-x_j\|^2 = \|(P-|\theta^*|)\|_{c_2}^2$$

is valid for every normalized tight frame $\{\mu_i\}_i$ of any Hilbert subspace L of H that is similar to $\{x_i\}_i$. (The left sum can be infinite for some choices of subspaces L and normalized tight frames $\{\mu_i\}_i$ for them.)

Equality appears if and only if $\mu_i = S^{1/2}(x_i)$ for any $i \in \mathbb{N}$. Consequently, *the symmetric approximation of a frame* $\{x_i\}_i$ *in a Hilbert space* $K \subseteq H$ is the normalized tight frame $\{S^{1/2}(x_i)\}_i$ spanning the same Hilbert subspace $L \equiv K$ of H and being similar to $\{x_i\}_i$ via the invertible operator $S^{-1/2}$.

Remark: (see [5]) If $\{x_i\}_i$ is a Riesz basis, then $\{S^{1/2}(x_i)\}_i$ is *the symmetric orthogonalization* of this basis. This is why the denotation 'symmetric approximation' has been selected.

A third approach has been developed by Deguang Han investigating approximation of Gabor (Weyl-Heisenberg) and wavelet frames. His starting point are countable *unitary systems* $\mathcal{U}$ on separable Hilbert spaces that contain the identity operator. In particular, $\mathcal{U}$ is supposed to be *group-like*, i.e. *group*$(\mathcal{U}) \subseteq \mathbb{T}\mathcal{U} = \{\lambda U : \lambda \in \mathbb{T}, U \in \mathcal{U}\}$. A vector $\phi \in H$ is a *complete frame vector* (resp., a *normalized tight frame vector*) for $\mathcal{U}$ if the set $\mathcal{U}\phi := \{U(\phi) : U \in \mathcal{U}\}$ is a frame (resp., a

normalized tight frame) of H. Two *frame vectors* $\phi, \psi \in H$ are said to be *similar* if the two frames $\mathcal{U}\phi$ and $\mathcal{U}\psi$ are similar frames in H. Let $\mathcal{T}(\mathcal{U})$ denote the set of all normalized tight frame vectors of H with respect to the action of $\mathcal{U}$.
As a matter of fact the distance measure used in [3] gives $\sum_{U\in\mathcal{U}} \|U(\xi)-U(\eta)\|^2 = \infty$ if $\mathcal{U}$ is an infinite set and $\xi \neq \eta$. Also, $\mathcal{U}\xi$ and $\mathcal{U}\eta$ are not similar, in general, cf. [1]. So define a vector $\psi \in \mathcal{T}(\mathcal{U})$ to be a *best normalized tight frame (NTF) approximation* for a given complete frame vector $\phi \in H$ of $\mathcal{U}$ if

$$\|\psi - \phi\| := \mathrm{dist}(\phi, \mathcal{T}(\mathcal{U})) := \inf\{\|\eta - \phi\| \,:\, \eta \in \mathcal{T}(\mathcal{U})\}\,.$$

Theorem 3([4]) Let $\mathcal{U}$ be a group-like unitary system acting on a Hilbert space H. Let $\phi \in H$ be a complete frame vector for $\mathcal{U}$. Then the vector $S^{1/2}(\phi)$ is the unique best NTF approximation for ϕ, where $S = (\theta^*\theta)^{-1}$ is the frame operator for ϕ.

Theorem 4([4]) Let $\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$ be a full-rank lattice and g be a Gabor frame generator associated with Λ. Then the vector $S^{1/2}(g)$ is the unique best NTF approximation for g, where S is the frame operator for g. ($S^{1/2}(g)$ is a Gabor frame generator, again.)

Considering the wavelet situation where the generating unitary systems sometimes are not group-like some obstacles are encountered. For example, D. Han found that for an orthonormal wavelet $\mathcal{U}_{D,T}(g)$ the vector $\phi = 1/4 \cdot g$ possesses better NTF approximations than $S^{1/2}(\phi)$. In ongoing discussions of D. Han with I. Daubechies, J. Wexler and M. Bownik examples of wavelet frames have been found for which there does not exist any wavelet-type dual frame. It is unknown whether $\{S^{1/2}(x_i)\}_i$ has always wavelet structure for wavelet frames $\{x_i\}_i$, or not.

Theorem 5([4]) Suppose, ϕ is the generator of a semi-orthogonal wavelet frame, i.e. $\phi_{m,k} \perp \phi_{n,l}$ for $\phi_{m,k} := |\det(M)|^{m/2}\phi(M^m x - k)$ and for any $k, l \in \mathbb{Z}^d$, all $m, n \in \mathbb{Z}$ with $m \neq n$. Denote by $\mathcal{U}_{D,T}$ the unitary system generating the initial wavelet frame. Then there exists a unique normalized tight wavelet frame generated by ψ such that the equality

$$\|\phi - \psi\| = \min\{\|h - \phi\| \,:\, h \in \mathcal{T}(\mathcal{U}_{D.T}),\, h \sim \phi\}$$

holds. Moreover, $\psi = S^{1/2}(\phi)$.

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A Reproducing Kernel without (?) Discretization

Hartmut Führ

For any measurable subset $B \subset \mathbb{R}$, let $\mathcal{H}_B$ denote the space of functions in $\mathrm{L}^2(\mathbb{R})$ whose Fourier transforms are supported in B. It is a translation-invariant closed subspace of $\mathrm{L}^2(\mathbb{R})$.

Now pick an open, dense subset $A \subset \mathbb{R}$ of finite measure, and let $\mathcal{H} = \mathcal{H}_A$. Then the fact that the characteristic function of A is in L^2 implies for all f that $f = f * g$, where g is the inverse Fourier transform of said characteristic function. Hence convolution with g acts as a reproducing kernel on $\mathcal{H}$. (Put differently: The function g is a coherent state.)

Question: Does there exist a subset $\Gamma \subset \mathbb{R}$ and the function $\eta \in \mathcal{H}$ such that the Γ-shifts of η are a frame? (A tight frame even?). The reason for choosing this particular set A is the following simple observation:

Proposition. *There exists $g \in \mathcal{H}_B$ such that the $\alpha\mathbb{Z}$-shifts of g are total in $\mathcal{H}_B$ iff $|\frac{k}{\alpha} + B \cap B| = 0$ for all nonzero $k \in \mathbb{Z}$, where $|\cdot|$ denotes Lebesgue measure.*

Hence, by choice of A, we have for all $\Gamma = \alpha\mathbb{Z}$, that the Γ-shifts of an arbitrary function are not total in $\mathcal{H}$. This fact suggests that there cannot exist a frame consisting of shifts of a single function, but I have not been able to prove it.

Understanding the problem could help to clarify the role of the integrability conditions which appear in the sampling theorems of Feichtinger and Gröchenig (e.g., [2] and related papers). The problem can also be phrased as follows: Does there exist a frame of exponentials for $\mathrm{L}^2(A)$? This formulation is reminiscent of spectral sets and the Fuglede conjecture [3]. Finally, the fact that regularly spaced sampling sets do not work, no matter how small the step-size, suggests using perturbation techniques (see e.g. [1]). Hence an understanding of this problem would also shed some light on the scope of these techniques.

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Density for Gabor Schauder Bases

Christopher Heil

Let H be a Hilbert space. A sequence $\{f_i\}_{i\in\mathbf{N}}$ of vectors in H is a *Schauder basis* for H if for each $f \in H$ there exist unique scalars $c_i(f)$ such that $f = \sum_{i=1}^{\infty} c_i(f) f_i$.

In this case there exists a *dual basis* $\{\tilde{f}_i\}_{i\in\mathbf{N}}$ such that $f = \sum_{i=1}^{\infty}\langle f, \tilde{f}_i\rangle\, f_i$. However, in general this series may converge only conditionally, i.e., it might not converge if a different ordering of the series is used.

A sequence $\{f_i\}_{i\in\mathbf{N}}$ is a *frame* for H if there exist constants A, $B > 0$ such that $A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2$ for every $f \in H$. In this case there exists a *dual frame* $\{\tilde{f}_i\}_{i\in\mathbf{N}}$ such that $f = \sum_{i=1}^{\infty}\langle f, \tilde{f}_i\rangle\, f_i$. Moreover, this series converges unconditionally, i.e., every reordering converges. However, the coefficients $\langle f, \tilde{f}_i\rangle$ in the series need not be unique, i.e., there may exist some other coefficients c_i such that $f = \sum_{i=1}^{\infty} c_i f_i$.

A frame is a Schauder basis if and only if it is a Riesz basis, i.e., the image of an orthonormal basis under a continuous bijection of H onto itself. For more information on frames, Schauder bases, and Riesz bases, see [Hei97] or [Chr03].

Let $T_a f(x) = f(x-a)$ denote the operation of translation. In [Zal78], [Zal80], Zalik gave some necessary and some sufficient conditions on $g \in L^2(\mathbf{R})$ and countable subsets $\Gamma \subset \mathbf{R}$ such that $\{T_a g\}_{a\in\Gamma}$ is complete in $L^2(\mathbf{R})$. Olson and Zalik proved in [OZ92] that no such system of pure translations can be a Riesz basis for $L^2(\mathbf{R})$, and conjectured that no such system can be a Schauder basis. This conjecture is still open.

In [CDH99], it was observed that no such system of pure translations can form a frame for $L^2(\mathbf{R})$. This is a corollary of the following general result due to Ramanathan and Steger [RS95].

Theorem 1. Let $g \in L^2(\mathbf{R})$ and let $\Lambda \subset \mathbf{R}^2$ be given. Then the *Gabor system* $\mathcal{G}(g,\Lambda) = \{e^{2\pi i b x} g(x-a)\}_{(a,b)\in\Lambda}$ has the following properties.

(a) If $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbf{R})$, then $1 \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.

(b) If $\mathcal{G}(g,\Lambda)$ is a Riesz basis for $L^2(\mathbf{R})$, then $D^-(\Lambda) = D^+(\Lambda) = 1$.

(c) If $D^-(\Lambda) < 1$ then $\mathcal{G}(g,\Lambda)$ is not a frame for $L^2(\mathbf{R})$.

In this result, $D^{\pm}(\Lambda)$ denote the Beurling densities of Λ, which provide in some sense upper and lower limits to the average number of points of Λ inside unit squares. More precisely, to compute Beurling density we count the average number of points inside squares of larger and larger radii and take the limit, yielding the definitions

$$D^-(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbf{R}^2} \frac{|\Lambda \cap Q_r(z)|}{r^2},$$
$$D^+(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbf{R}^2} \frac{|\Lambda \cap Q_r(z)|}{r^2},$$

for the lower and upper Beurling densities of Λ. Here $Q_r(z)$ is the square in $\mathbf{R}^2$ centered at z with side lengths r and $|E|$ denotes the cardinality of a set E. In particular, the Beurling density of a rectangular lattice is $D^-(\alpha\mathbf{Z}\times\beta\mathbf{Z}) = D^+(\alpha\mathbf{Z}\times\beta\mathbf{Z}) = \frac{1}{\alpha\beta}$.

Some corrections and extensions to Ramanathan and Steger's result are given in [CDH99], and a suite of new results on redundancy of frames partly inspired by their proof are given in [BCHL03a], [BCHL03b], [BCHL04].

Since $\{T_a g\}_{a\in\Gamma} = \mathcal{G}(g, \Gamma\times\{0\})$ and $D^-(\Gamma\times\{0\}) = 0$, it follows from Theorem 1 that such a system can never be a frame for $L^2(\mathbf{R})$.

Little is known about Gabor systems that are Schauder bases but not Riesz bases for $L^2(\mathbf{R})$. One example of such a system is $\mathcal{G}(g, \mathbf{Z}^2)$ where

$$g(x) = |x|^\alpha \, \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), \ 0 < \alpha < \frac{1}{2}.$$

It was conjectured in [DH00] that Gabor Schauder bases follow the same Nyquist-type rules as Gabor Riesz bases, i.e., if $\mathcal{G}(g, \Lambda)$ is a Gabor Schauder basis then $D^-(\Lambda) = D^+(\Lambda) = 1$. Some partial results were obtained in [DH00], but the conjecture remains open. If this conjecture is proved, then the Olson/Zalik conjecture follows as a corollary.

Another open question is whether there is an analogue of the Balian–Low Theorem for Gabor Schauder bases. Qualitatively, the Balian–Low Theorem states that any Gabor Riesz basis will be generated by a function which is either not smooth, or has very poor decay at infinity. For a survey of the Balian–Low Theorem, see [BHW95].

Finally, for recent wavelet versions of density theorems, see [HK03]. There are interesting differences between the density-type theorems for Gabor and wavelet frames, most notably that there is no Nyquist-like cutoff in the possible densities for wavelets. An open general problem is to derive more powerful necessary or sufficient conditions for the existence of Gabor or wavelet frames.

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Two Problems on Frames and Decomposition of Operators

David R. Larson

In the *Abstracts of Talks* section we showed a relation between frame theory and projection (and other) decompositions of positive operators on a Hilbert space H. If S is a positive invertible operator in $B(H)$, for H a real or complex separable Hilbert space with infinite dimension, and if $\|S\|_{ess} > 1$, then S can be written $S = \sum_n P_n$, where $\{P_n\}$ is a sequence of self-adjoint (i.e.*orthogonal*) projections. This is equivalent to the property that S is the frame operator for a frame (for all of H) consisting of unit vectors. More generally, it was shown that if T is a positive operator (not necessarily invertible) which has essential norm strictly greater than 1, then T admits such a projection decomposition. If T has closed range, then writing $T = \sum_n x_n \otimes x_n$,where the x_n are unit vectors, yields that $\{x_n\}$ is a frame of unit vectors for the range of T. If T does not have close range, then $\{x_n\}$ is a sequence of unit vectors which does not constitute a frame for its closed span (i.e. the closed range of T), but can be filled out in many ways with unit vectors to give a tight frame for its closed span. (Just choose a positive operator R of norm > 1 such that $T + R$ is a scalar multiple of P, where P is the orthogonal projection onto the closure of range(T).) Since projection decompositions of positive operators seem to be useful when they exist, this suggests some problems in single operator theory.

PROBLEM A: When does a positive operator $T \in B(H)$ have a *finite* projection decomposition? That is, when can it be written as a *finite* sum of orthogonal projections?

Suppose, in fact, we assume that T has an *infinite* projection decomposition. Is it a common occurrence for T to also admit a *finite* projection decomposition? Or does this rarely happen?

In the context of the above problem, it is clear that if T is an invertible operator which has a finite projection decomposition, then it is the frame operator for a frame which is the union of *finitely many orthonormal sets* of vectors.

Also in the context of Problem A, we mention that it is easy to show (it is a lemma in the second paper) that if a positive operator of norm exactly 1 has a projection decomposition, then in fact it *must be* a projection. So it has a finite projection decomposition consisting of one projection. On the other hand, if a positive operator has essential norm strictly greater than 1, then we know it has an infinite projection decomposition (by a theorem in the extended abstract), but does it also have a finite projection decomposition?

We now discuss a problem concerning more general *targeted* decompositions of positive operators. *Targeted* means that we are asking for a decomposition as a (strongly convergent) series of simpler positive operators (such as projections, or rank-one operators satisfying specified norm or other properties). In paper 2 that was discussed in the abstract, which was joint work with K. Kornelson, we found techniques to deal with problems of specified norm targeted decompositions. In this Workshop, in response to a short talk on targeted decompositions presented by D. Larson in a problem session, H. Feichtinger and K. Grochenig pointed out that similar techniques just may lead to progress on a certain problem in modulation space theory. Subsequently, Larson and C. Heil discussed this matter, and there are plans to follow up on this lead. The following problem *seems* to be pointing in the right direction. At the least, it seems to be a *representative problem* on the concept of *targeted decompositions*, which is mathematically interesting (at least to this investigator) as a problem in Hilbert space operator theory, and which was motivated by Workshop discussions. We present it in this spirit. It concerns targeted decompositions of trace-class operators, hence is a problem in a different direction from the results in both papers discussed in the *Abstract*.

PROBLEM B: Let H be an infinite dimensional separable Hilbert space. As usual, denote the Hilbert space norm on H by $\|\cdot\|$. If x and y are vectors in H, then $x \otimes y$ will denote the operator of rank one defined by $(x \otimes y)z = (z, y)x$. The operator norm of $x \otimes y$ is then just the product of $\|x\|$ and $\|y\|$.

Fix an orthonormal basis $\{e_n\}_n$ for H. For each vector v in H, define

$$|||v||| = \sum_n |(v, e_n)|$$

This may be $+\infty$.

Let L be the set of all vectors v in H for which $|||v|||$ is finite. Then L is a dense linear subspace of H, and is a Banach space in the *triple norm*. It is of course isomorphic to ℓ^1

Let T be any positive trace-class operator in $B(H)$. The usual eigenvector decomposition for T expresses T as a strongly convergent series of operators $h_n \otimes h_n$, where $\{h_n\}$ is an orthogonal sequence of eigenvectors of T. That is,

$$T = \sum_n h_n \otimes h_n$$

In this representation the eigenvalue corresponding to the eigenvector h_n is the square of the norm: $\|h_n\|^2$. The trace of T is then

$$\sum_n \|h_n\|^2$$

and since T is positive this is also the trace-class norm of T.

Let us say that T is of *typeA* with respect to the orthonormal basis $\{e_n\}$ if, for the eigenvectors $\{h_n\}$ as above, we have that $\sum_n |||h_n|||^2$ is finite. Note that this is just the (somewhat unusual) formula displayed above for the trace of T with the triple norm used in place of the usual Hilbert space norm of the vectors $\{h_n\}$. (So,in particular, such operators T must be of trace class.)

And let us say that T is of *typeB* with respect to the orthonormal basis $\{e_n\}$ if there is *some* sequence of vectors $\{v_n\}$ in H with $\sum_n |||v_n|||^2$ *finite* such that

$$T = \sum_n v_n \otimes v_n$$

where the convergence of this series is in the strong operator topology. (Of course, *typeA* wrt a basis implies *typeB* wrt that basis. It is the *converse* direction that we want to consider.)

The problem we wish to isolate is the following: Let $\{e_n\}$ be an orthonormal basis for H. Find a characterization of all positive trace class operators T that are of *typeB* with respect to $\{e_n\}$. In particular, is the class of *typeB* operators with respect to a fixed orthonormal basis for H much larger than the class of *typeA* operators (with respect to that basis)?

Quantitative Behaviour of Wavelet Bases

Karsten Urban

We demonstrate a typical wavelet discretization of an elliptic problem and give some examples of condition numbers indicating the corresponding research problems.

Wavelet Representation of Differential Operators. For simplicity, let us consider the periodic 1D problem finding a function $u : [0,1] \to \mathbb{R}$ such that

$$\text{(1)} \qquad -u''(x) + u(x) = f(x), \quad x \in (0,1), \qquad u(0) = u(1).$$

Let $\psi \in H^1(\mathbb{R})$ be a sufficiently smooth wavelet, we consider the periodic wavelet basis

$$\text{(2)} \qquad \psi_{j,k}(x) := 2^{j/2} \sum_{\ell \in \mathbb{Z}} \psi(2^j(x+\ell) - k)|_{[0,1]}$$

and the corresponding wavelet spaces $W_j := \operatorname{span}(\Psi_j)$ for $\Psi_j := \{\psi_{j,k} : k = 0, \ldots, 2^j - 1\}$, for $j \geq 0$ and $S_0 := \{c : c \in \mathbb{R}\} = \operatorname{span}(\Phi_0)$, $\Phi_0 = \{\chi_{(0,1)}\}$, so that

$$H^1_{\text{per}}(\mathbb{R}) = S_0 \oplus \bigoplus_{j \in \mathbb{N}} W_j,$$

and $\Psi := \Phi_0 \cup \bigcup_{j \in \mathbb{N}} \Psi_j$ is a Riesz basis for $L_2(0,1)$. Let us describe the structure of the wavelet representation $a(\Psi, \Psi)$ of the differential operator in (1), where here we have $a(u,v) := (u', v')_0 + (u,v)_0$. In order to give an impression of the entries let us consider index pairs (j,k) and (ℓ, m) such that the corresponding shifted translates are completely located inside the interval $(0,1)$, i.e.,

$$\psi_{j,k}(x) = 2^{j/2}\, \psi(2^j x - k), \qquad \psi_{\ell,m}(x) = 2^{\ell/2}\, \psi(2^\ell x - m), \qquad x \in [0,1].$$

Then, we obtain by a simple change of variables $y = 2^j x - k$

$$\begin{aligned}(\psi_{j,k}, \psi_{\ell,m})_0 &= 2^{(j+\ell)/2} \int_{\mathbb{R}} \psi(2^j x - k)\, \psi(2^\ell x - m)\, dx \\ &= 2^{(\ell-j)/2} \int_{\mathbb{R}} \psi(y)\, \psi(2^\ell(2^{-j}(y+k) - m)\, dy \\ &= 2^{(\ell-j)/2} (\psi, \psi_{l-j, m-2^{l-j}k})_0,\end{aligned}$$

i.e., the inner product only depends on the *level difference* and the relative location in space. In particular, there is only a fixed number of non-zero values per level difference $\ell - j$.

For the second-order term in $a(\cdot,\cdot)$ we use the fact that for every (sufficiently smooth) wavelet ψ there exists a second wavelet ψ^* such that $\psi'(x) = 4\,\psi^*(x)$, [1]. Then, a similar calculation as above gives

$$(\psi'_{j,k}, \psi'_{\ell,m})_0 = 2^{j+\ell+4}\, 2^{(\ell-j)/2} (\psi^*, \psi^*_{l-j, m-2^{l-j}k})_0.$$

We observe the same behaviour as above, i.e., the values depend only on the level difference and there is only a fixed number of non-zero entries per level. This shows the finger structure of the matrix which is ordered level-wise. This block-structure is also shown in the figure.

Preconditioning. Recalling the norm equivalence

$$\left\| \sum_{j,k} d_{j,k}\, \psi_{j,k} \right\|_s^2 \sim \sum_{j,k} 2^{2js} |d_{j,k}|^2, \qquad s \in (-\tilde{\gamma}, \gamma),$$

where $\gamma, \tilde{\gamma} > 1$ depend on the wavelet ψ and its dual $\tilde{\psi}$, we obtain the following preconditioning for $u = \boldsymbol{d}^T \Psi = \sum_{j,k} d_{j,k} \psi_{j,k}$

$$a(u,u) = \|u'\|_0^2 + \|u\|_0^2 = \|u\|_1^2 \sim \sum_{j,k} 2^{2j} |d_{j,k}|^2,$$

i.e., we obtain the preconditioner $\boldsymbol{C}_1 = \mathrm{diag}(2^{|\lambda|})$. As an alternative, the norm equivalence also leads to the preconditioner $\boldsymbol{C}_2 = \mathrm{diag}(\sqrt{2^{|\lambda|+1}})$. Both preconditioners are asymptotically optimale, i.e.,

$$\mathrm{cond}(\boldsymbol{C}_i^{-1} a(\Psi,\Psi) \boldsymbol{C}_i^{-1}) < \infty, \quad i = 1, 2.$$

It is not clear a priori which one is better in a practical application.

From a practical point of view, one can also try to use the diagonal of $a(\Psi,\Psi)$ as a preconditioner which is easily accessible and already contains information of the matrix. In Table 1, we have listed condition numbers of slices of the matrices corresponding to the level j.

j	cond($a(\Psi,\Psi)$)	$\boldsymbol{C}_1$	$\|\boldsymbol{A}\|$	diag	$\boldsymbol{C}_2$
2	87.47	17.74	15.07	4.07	14.58
3	352.81	21.40	18.73	5.94	20.15
4	1.43e+3	25.19	20.43	6.67	24.52
5	5.75e+3	27.52	21.05	7.47	28.53
6	2.30e+4	29.12	21.26	8.45	30.50
7	9.20e+4	29.68	21.41	9.47	31.34
8	3.67e+5	29.93	21.46	9.62	32.06
9	1.47e+6	30.13	21.47	10.06	32.35
10	5.88e+6	30.20	21.48	10.76	32.57
11	2.35e+7	30.27	21.48	9.82	32.68

Table 1. Condition numbers.

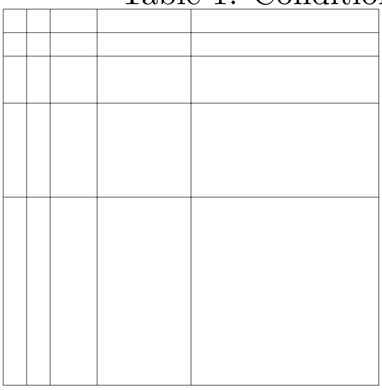

Figure 1. Block structure of $\boldsymbol{A}$.

Even though these numbers give a quite impressive rate of reduction, one should keep several facts in mind:

- The periodic case is the most simple one and of academic character only. When introducing Dirichlet boundary conditions even on an interval, the numbers increase significantly.
- When considering problems in 2D or 3D, even on the unit square or unit cube using tensor products, one has to square or cubic the condition numbers.
- When dealing with complex domains on needs several unit cubes and also certain combinations.

Adaptive Richardson Iteration. When using the adaptive wavelet method within the Richardson iteration directly, the condition number of $\boldsymbol{A}$ influences the error reduction factor ρ directly. In fact, it has been observed that $\rho \approx 1$ in many situations. Several attempts have been made in order to improve the condition numbers as listed in Table 1. However, none of them was really successful, which might also be explained by the low numbers for the diagonal preconditioners.

An alternative to the 'standard' adaptive Richardson iteration would be to combine the Richardson method using the adaptive approximate operator application APPLY from [2] as an outer iteration with an inner loop solving the Galerkin problem on a fixed set of unknowns. It has in fact been observed that such an iteration is quantitatively better in several situations:

- $[\boldsymbol{v}, \Lambda] = APPLY(\boldsymbol{u}^{(i)}, \varepsilon_i)$;
- solve $A_\Lambda u_\Lambda = f_\Lambda$ and call the numerical approximation $\boldsymbol{u}^{(i+1)}$.

So far there is no theoretical backup for the behaviour of this algorithm. It could be possible to explain the quantitative improvement with the aid of the *finite section method* known in frame theory.

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Two Problems on the Generation of Wavelet and Random Frames

Eric Weber

Problem 1. If $g \in L^2(\mathbb{R})$ is the Gaussian, and $\Lambda := \{(a_z, b_z) : z \in \mathbb{Z}^2\} \subset \mathbb{R}^2$ is any set such that

$$1 < D^-(\Lambda) \leq D^+(\Lambda) < \infty,$$

then the Gabor system

$$G(g, \Lambda) = \{e^{-2\pi i a_z x} g(x - b_z) : z \in \mathbb{Z}^2\}$$

is a frame for $L^2(\mathbb{R})$, where $D^-(\Lambda)$, $D^+(\Lambda)$ are the lower and upper Beurling densities, respectively, of Λ.

Is there a corresponding statement regarding wavelet frames? More specifically, is there a function $\psi \in L^2(\mathbb{R})$ such that for any set $\Gamma := \{(a_z, b_z) : z \in \mathbb{Z}^2\} \subset \mathbb{R}^2$ with

$$1 < D^-(\Gamma) \leq D^+(\Gamma) < \infty,$$

then the wavelet system

$$W(\psi, \Gamma) = \{|a_z|^{1/2} g(a_z x - b_z) : z \in \mathbb{Z}^2\}$$

is a frame for $L^2(\mathbb{R})$, where $D^-(\Gamma)$, $D^+(\Gamma)$ are the lower and upper affine densities, respectively, of Γ (see [Heil et al. 2003]).

Problem 2. Let H be a finite dimensional Hilbert space.

i) What is a reasonable definition of "random frame"?
ii) How does one construct a "random frame"?

We make the following remarks:

a) The idea of a random orthonormal basis has a reasonably good definition. Fix any orthonormal basis of H; each orthonormal basis of H then corresponds to a unitary operator from the fixed basis to the new one. The group of unitary operators on H is a compact group, hence possesses a finite Haar measure, which can be normalized to give a probability measure. This probability measure would correspond to a uniform density, since the Haar measure is invariant under multiplication.
b) There are several ways of constructing random orthonormal bases for $\mathbb{R}^d$. Randomly choose $d(d+1)/2$ numbers and place in the upper triangle of a matrix B; fill in the remaining entries such that $B^T = -B$. The spectrum then is purely imaginary, whence e^B has spectrum on the unit circle and hence is unitary. (The matrix e^B can only be approximated). A second method is due to Stewart [Stewart 1980].
c) A frame with N elements for a Hilbert space H of dimension d can be obtained by choosing any basis of a Hilbert space K of dimension N and projecting the basis onto any subspace of dimension d. Thus, it is possible to construct a "random" frame from a random (orthonormal) basis.

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