Mathematisches Forschungsinstitut Oberwolfach

Report No. 11/2004

Computational Electromagnetism

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February 22nd – February 28th, 2004

Introduction by the Organisers

The field of computational electromagnetism is dedicated to the design and analysis of numerical methods for the approximate solution of electromagnetic field problems. Since the exploitation of electromagnetic phenomena is one of the foundations of modern technology, computational electromagnetics is of tremendous industrial relevance: in a sense, it is peer to computational solid and fluid mechanics and huge research efforts are spent on developing and enhancing simulation methods and software for electromagnetic field computations.

For a long time, computational electromagnetism remained a realm of engineering research with applied mathematics shunning the area. This was in stark contrast to elasticity and fluid mechanics, where mathematicians have been involved in the development of numerical methods from the very beginning. Maybe, the blame has to be laid on the incorrect belief of mathematicians who thought that the laws governing the behavior of electromagnetic fields basically boil down to well understood second-order elliptic problems.

Fortunately, the past fifteen years have seen a real surge of mathematical research activities in the area of computational electromagnetism. This resulted in insights that have begun to have a big impact on the numerical methods used in engineering and industrial environments. A prominent example is the explanation of so-called spurious solutions that can arise when using continuous "nodal" finite elements for the discretization of certain electromagnetic boundary value or eigenvalue problems, respectively. Another example is the appreciation of so-called edge finite elements and the construction of multilevel iterative solvers for the low-frequency setting.

Meanwhile, computational electromagnetism can claim to be a major area of numerical mathematics and scientific computing in its own right. This prompted us to ask the Mathematisches Forschungsinstitut Oberwolfach to host a one week workshop on computational electromagnetism, the first of its kind. Reflecting the growing importance of the subject, this workshop has been one of a series of events dedicated to mathematical issues in the computation of electromagnetic fields. We would like to mention, the NSF-CBMS Regional Conference in the Mathematical Sciences about "Numerical Methods in Forward and Inverse Electromagnetic Scattering", held in Golden, CO, June 3-7, 2002 (from which the book [2] arose), and the "LMS Durham Symposium on Computational methods for Wave Propagation in Direct Scattering", Durham, England, July 15-25, 2002 (see [1]).

This Oberwolfach workshop brought together some 50 experts in computational electromagnetism. The majority of the participants were applied mathematicians, but a sizable number of people with a background in engineering also attended, as appropriate for a field with close ties to engineering and the applied sciences. Nevertheless, the workshop had a clear mathematical focus, emphasizing rigorous theory, principles and ideas. Throughout, the presentations matched these expectations. A total of 29 presentations were given, of which ten were survey lectures offering broader treatment of a particular subject.

As is typical of an event that targets a specific area of application, it arose that a broad range of mathematical issues and techniques was addressed. Although it will certainly not do justice to many presentations, we will try categorize the talks as follows:

- **Mathematical modelling.** This subject did not play a central role, because most presentations took the model equations for granted. Modelling for practical engineering calculations was described by O. Bíró in his survey talk about *Practical Aspects of FEM in Electromagnetics*, p. 559, and by M. Clemens when speaking on *Formulations and Efficient Numerical Solution Techniques for Transient 3D Magneto-and Electro-Quasistatic Field Problems*, p. 572. Homogenization was addressed in the presentation by A. Bendali about *Two Scale Asymptotic Expansion for the Scattering of a TM-Electromagnetic Wave by a Rough Surface and Applications*, p. 556.
- **Spatial discretizations.** This turned out to be one of the core subjects of the workshop. The survey lectures of D. Boffi about *Theoretical Aspects of Edge Finite Elements*, p. 564 and I. Perugia on *Discontinuous Galerkin Methods for Maxwell's Equations*, p. 608, addressed the topic. Particular issues were discussed by S. Christiansen in his talk about the *Div-Curl Lemma for Edge Elements*, p. 571, and by J. Pasciak about *The Approximation of the Maxwell Eigenvalue Problem using a Least-Squares Method*, p. 606. M. Kaltenbacher gave an account of observations concerning finite element schemes in his presentation on *Nodal and Edge Finite Element Discretization of Maxwell's Equations*, p. 590.

Several presentations were devoted to higher order spatial discretization: the survey lecture of M. Ainsworth gave an account of the *Dispersive Properties of High Order Nédélec/Edge Elements for Maxwell's Equations*, p. 553, L. Demkowicz spoke about $H^1, \boldsymbol{H}(\mathrm{curl})$ *and* $\boldsymbol{H}(\mathrm{div})$*-Conforming Projection-Based Interpolation in Three Dimensions*, p. 582, and P. Ledger about *Computation of Maxwell Eigenvalues with Exponential Rates of Convergence*.

- **Timestepping.** There was only one contribution dealing with temporal discretization, namely the talk by T. Driscoll on *High-Order Time Stepping Methods for Electromagnetics*, p. 585.
- **Regularity of solutions.** Here one of the pioneers in the field, M. Dauge, gave a survey talk about *Singularities of Electromagnetic Fields in the Eddy Current Limit*, p. 574.
- **Integral equation methods**. Boundary element methods in the frequency domain were treated by S. Kurz in his talk on *A New View on Collocation*, p. 599. Conversely, time-domain integral equation methods were examined in the survey lecture by E. Michielssen on *Fast Time Domain Integral Equation Solvers*, p. 603, and P. Davies in her contribution on *Convergence of Collocation Methods for Time Domain Boundary Integral Equations*, p. 579. S. Börm talked about $\mathcal{H}^2$*-Matrices with Adaptive Cluster Bases Applied to an Eddy Current Problem*, p. 562, and presented a fast summation method for discrete frequency-domain integral equations.
- **Electromagnetic Scattering.** This topic was treated by R. Kress in his survey lecture on *Inverse Obstacle Scattering for Time-Harmonic Electromagnetic Waves*, p. 596. Also the talk by A. Bendali on *Two Scale Asymptotic Expansion for the Scattering of a TM-Electromagnetic Wave by a Rough Surface and Applications*, p. 556, addressed a particular scattering problem.
- **Absorbing boundary conditions.** A special incarnation of these was examined in the survey talk by F. Teixeira on *Perfectly Matched Layers*, p. 621. Details of a PML approach were studied by Z. Chen in his talk about *An Adaptive Perfectly Matched Layer Technique for Time-harmonic Scattering Problems*, p. 568. Other techniques were outlined by M. Grote (*Nonreflecting Boundary Conditions for Computational Electromagnetics*, p. 588) and F. Schmidt (*Pole Condition: A New Approach to Solve Scattering Problems*, p. 615).
- **Topological issues**. These were discussed in the talks of R. Kotiuga (*The Hurewicz Map Distinguishes Intuitive vs. Computable Topological Aspects of Computational Electromagnetics*, p. 593) and F. Rapetti (*Smith Normal Form as an Adequate Tool to Detect Mesh Defects as well as to Build Basis Fields for Domains with Loops and Holes*, p. 612).
- **Fast solvers.** Several speakers discussed fast algorithms for the solution of linear systems of equations arising from discretized field equations: it was the subject of J. Schöberl's survey lecture on *Preconditioning for Maxwell*

Equations, p. 617, and O. Sterz' talk on *Adaptive Multigrid-Methods for the Solution of Time-Harmonic Eddy-Current Problems*, p. 618. The use of multigrid methods was discussed in the contributions by M. Clemens on *Formulations and Efficient Numerical Solution Techniques for Transient 3D Magneto-and Electro-Quasistatic Field Problems*, p.'572, and by M. Kaltenbacher on *Nodal and Edge Finite Element Discretization of Maxwell's Equations*, p. 590. An enhancement for algebraic multigrid was proposed by P. Arbenz (*Treatment of Nullspace in Maxwell Problem*, p. 553). J. Zou dealt with domain decomposition methods in his contribution on *Some New Inexact Uzawa Methods and Non-overlapping DD Preconditioners for Solving Maxwell's Equations in Non-homogeneous Media*, p. 624.

- **Adaptive techniques.** Only one presentation, that of Z. Chen on *An Adaptive Perfectly Matched Layer Technique for Time-harmonic Scattering Problems*, p. 568, dealt with a special adaptive scheme.
- **Optimization.** This important subject reaches beyond the core of computational electromagnetism. An aspect was discussed in the talk by D. Lukáš on *Computational Shape and Topology Optimization with Applications to 3–Dimensional Magnetostatics* p. 601.

We would like to add our personal impression that two families of methods have been received with particular interest during the workshop:

- Time-domain integral equation methods,
- High-order spatial discretization schemes.

We are sure that the workshop will have made a substantial contribution to the progress of research in these and all other areas of computational electromagnetism.

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[2] P. Monk, *Finite Element Methods for Maxwell's Equations*, Clarendon Press, Oxford, UK, 2003.

Workshop on Computational Electromagnetism

Table of Contents

Abstracts

Dispersive Properties of High Order Nédélec/Edge Elements for Maxwell's Equations.

Mark Ainsworth

The dispersive behaviour of high order Nédélec element approximation of the time harmonic Maxwell equations at a prescribed temporal frequency ω on tensor product meshes of size h is analysed. A simple argument is presented showing that the discrete dispersion relation may be expressed in terms of the discrete dispersion relation for the approximation of the scalar Helmholtz equation in one dimension. An explicit form for the one dimensional dispersion relation is given, valid for arbitrary order of approximation. Explicit expressions for the leading term in the error in the regimes where (a) ωh is small, showing that the dispersion relation is accurate to order $2p$ for a p-th order method; and (b) in the high wave number limit where $1 \ll \omega h$, showing that in this case the error reduces at a super-exponential rate once the order of approximation exceeds a certain threshold which is given explicitly. Details have been published in the following work [1–3]

References

[1] ———, *Discrete dispersion relation for hp-version finite element approximation at high wave number*, SIAM J. Numer. Anal., (accepted for publication).

[2] ———, *Dispersive and dissipative behaviour of high order discontinuous Galerkin finite element methods*, J. Comput. Phys., (accepted for publication).

[3] ———, *Dispersive properties of high order Nédélec/edge element approximation of the time-harmonic Maxwell equations*, Phil. Trans. Roy. Soc. Series A, (accepted for publication).

Treatment of Nullspace in Maxwell Problems

Peter Arbenz

1. Introduction

The weak form of the magnetostatic equation reads: Find $\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega)$ such that

$$\begin{array}{lll} \text{(a)} & (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{\Psi}) = (\mathbf{r}, \mathbf{\Psi}), & \forall \mathbf{\Psi} \in H_0(\mathbf{curl}\,, \Omega), \\ \text{(b)} & (\mathbf{u}, \mathbf{grad}\, q) = 0, & \forall q \in H_0^1(\Omega), \end{array} \tag{1}$$

where $\Omega \in \mathbb{R}^3$ is a bounded domain with connected boundary $\partial\Omega$. We require that $(\mathbf{r}, \mathbf{grad}\, q) = 0$ for all $q \in H_0^1(\Omega)$ such that equation (1) is consistent.

The straightforward discretization of (1) by the finite element method yields the matrix equation

$$\begin{array}{lll} \text{(a)} & A\mathbf{x} = M\mathbf{r}, & C^T\mathbf{r} = \mathbf{0} \\ \text{(b)} & C^T\mathbf{x} = \mathbf{0} & \end{array} \tag{2}$$

where $a_{ij} = (\mathbf{curl}\,\mathbf{\Psi}_i, \mathbf{curl}\,\mathbf{\Psi}_j)$, $m_{ij} = (\mathbf{\Psi}_i, \mathbf{\Psi}_j)$, $c_{i\ell} = (\mathbf{\Psi}_i, \mathbf{grad}\,\varphi_\ell)$. Here, the $\mathbf{\Psi}_i$, $i = 1, \ldots, n$, form a basis of the space N_h of lowest order Nédélec edge elements and the φ_ℓ, $\ell = 1, \ldots, m$, form a basis of the lowest order Lagrange elements L_h, see [4].

A has a m-dimensional nullspace $\mathcal{N}(A)$ that satisfies [4, §III.5.3]

$$\mathcal{N}(A) = \{\mathbf{v}_h \in N_h \mid \mathbf{curl}\,\mathbf{v}_h = \mathbf{0}\} = \mathbf{grad}\,L_h. \tag{3}$$

Thus, the gradient of each φ_ℓ can be written as a linear combination of the edge basis functions $\mathbf{\Psi}_j$,

$$\mathbf{grad}\,\varphi_\ell = \sum_{j=1}^{n} y_{j\ell}\mathbf{\Psi}_j.$$

Let $Y \in \mathbb{R}^{n\times m}$ be the matrix with elements $y_{j\ell}$. Then, $AY = 0$ and $C = MY$. The columns of Y form a *sparse* null space basis of A, see e.g. [2]. Notice that Y can be constructed from geometric properties of the finite element mesh.

In [1] we have investigated the numerical solutions of consistent semi-definite equations of the form (2). The key idea is to employ the sparse null space basis to extract a positive definite submatrix of A of order $n-m$, the rank of A.

Reitzinger and Schöberl [5] introduced an algebraic multigrid method to solve (2) regularized by a term that is positive on $\mathcal{N}(A)$. Here we present a way how to extend the ideas of [1] to all levels of the Reitzinger-Schöberl AMG algorithm. In this way we get an AMG algorithm that works entirely on the largest subspace of N_h on which $\|\mathbf{curl}\,(\cdot)\|$ is a norm. Its dimension $n-m$ is considerably smaller than n.

2. Elimination of the nullspace

Let's assume that the last m rows of Y are linearly independent. Then [1]

$$W := \begin{bmatrix} I_{n-m} & Y_1 \\ O & Y_2 \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \qquad Y_2 \in \mathbb{R}^{m\times m}, \tag{4}$$

is nonsingular. We split A, C, $\mathbf{x}$, and $\mathbf{r}$ according to Y. Then (2) becomes

(5)
$$W^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} WW^{-1}\mathbf{x} = \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} W^{-1}\mathbf{x} = W^T M\mathbf{r} \quad \Longleftrightarrow \quad \begin{cases} A_{11}\mathbf{x}_1 = \mathbf{r}_1, \\ \mathbf{x}_2 = \mathbf{0}. \end{cases}$$

A_{11} is symmetric positive definite. The general solution of (2) has the form

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + Y\mathbf{a}.$$

To satisfy the constraint $C^T\mathbf{x} = \mathbf{0}$ we determine $\mathbf{a}$ by solving

$$H\mathbf{a} = -C_1^T\mathbf{x}_1. \tag{6}$$

Here, H is the symmetric positive definite matrix with elements

$h_{ij} = (\mathbf{grad}\,\varphi_i, \mathbf{grad}\,\varphi_j)$.

3. Application to the Reitzinger-Schöberl AMG algorithm

Reitzinger and Schöberl [5] introduced an Algebraic Multigrid (AMG) method for solving (1)-(2) that properly treats the solenoidal and curl-free portions of the vector fields. The authors start from an AMG method for solving the Poisson (or a similar elliptic) problem in L_h. Coarse grids are constructed from fine grids by aggregating nodes into 'virtual nodes'. Two aggregates are defined connected (through 'virtual edges') if they contain nodes that are connected in the fine grid [6]. The system matrices on the various levels are denoted by H_k, where $H_0 = H$ corresponds to the finest level. Because of the Galerkin principle, among two consecutive levels the relation

$$H_{k+1} = Q_k^T H_k Q_k$$

holds. Q_k prolongates (interpolates) from level k+1 to the finer level k. Reitzinger and Schöberl then construct a sequence of levels for the curl-curl matrix. The matrices on the various levels are denoted by A_k with $A_0 = A$ and

$$A_{k+1} = P_k^T A_k P_k$$

where the P_k now prolongates from coarse to fine edge space. Q_k and P_k are related via the compatibility condition

$$P_k Y_{k+1} = Y_k Q_k \tag{7}$$

such that coarse grid gradients are prolongated to fine grid gradients. Here, Y_k is a sparse nullspace basis of A_k.

To eliminate the nullspace on all levels of the Reitzinger-Schöberl AMG we arrange the matrices A_k such that the nullspace bases can be written in the form

$$Y_k = \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \end{bmatrix},$$

where $Y_{k,2}$ is a nonsingular submatrices of Y_k, cf. (4). With (5), we then get

$$\begin{aligned}
\begin{bmatrix} A_{k+1,11} & O \\ O & O \end{bmatrix} &= W_{k+1}^T \begin{bmatrix} A_{k+1,11} & A_{k+1,12} \\ A_{k+1,21} & A_{k+1,22} \end{bmatrix} W_{k+1} \\
&= W_{k+1}^T A_{k+1} W_{k+1} = W_{k+1}^T P_k^T A_k P_k W_{k+1} \\
&= W_{k+1}^T P_k^T \begin{bmatrix} A_{k,11} & A_{k,12} \\ A_{k,21} & A_{k,22} \end{bmatrix} P_k W_{k+1} \\
&= W_{k+1}^T P_k^T W_k^{-T} \begin{bmatrix} A_{k,11} & O \\ O & O \end{bmatrix} W_k^{-1} P_k W_{k+1}
\end{aligned}$$

and thus

$$W_k^{-1} P_k W_{k+1} = \begin{bmatrix} P_{k,11} - Y_{k,1} Y_{k,2}^{-1} P_{k,21} & O \\ Y_{k,2}^{-1} P_{k,21} & Q_k \end{bmatrix}.$$

So, the prolongator for the positive-definite portions of the systems is

$$\bar{P}_k := P_{k,11} - Y_{k,1} Y_{k,2}^{-1} P_{k,21}, \qquad A_{k+1,11} = \bar{P}_k^T A_{k,11} \bar{P}_k$$

These ideas can be incorporated in the Reitzinger-Schöberl AMG algorithm (or in a smoothed aggregation AMG algorithm like in [3]) as follows

(1) Build matrices the A_k and H_k. This implies that all the prolongators P_k and Q_k are available.
(2) Construct the nullspace bases Y_k on all levels.
(3) Reduce A_k to $A_{k,11}$
(4) Adapt the prolongators and smoothers.

A more memory-aware procedure works level by level starting with the finest.

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Two Scale Asymptotic Expansion for the Scattering of a TM-Electromagnetic Wave by a Rough Surface and Applications

A. Bendali

(joint work with P. Borderies, J.-R. Poirier)

In this study, mainly of methodological interest, we show how the two-scale asymptotic expansion method [7] can be used as a powerful tool in the study of the scattering of an electromagnetic wave by a highly oscillating perfectly conducting surface both from the theoretical and the practical standpoint. More specifically, we consider the following simple 2D model related to the scattering of an E-polarized incident time-harmonic electromagnetic wave $u^{\rm inc}$

$$(1) \qquad \begin{cases} \Delta u^\delta + k^2 u^\delta = 0 \ \text{ in } \ \Omega^\delta, \\ u^\delta = 0 \ \text{ on } \ \Gamma^\delta, \ x \to e^{-i\beta x} u^\delta(x,y) \text{ is periodic of period } L, \\ \text{Radiation Condition (RC) on } \ u^\delta - u^{\rm inc}\,. \end{cases}$$

The surface is considered as a periodic grating whose elementary cell is

$$\Omega^\delta := \big\{(x,y) \in \mathbb{R}^2 : 0 < x < L, \ \ y > \gamma_\delta(x)\big\}$$

in which $\Gamma^\delta := \big\{(x,y) \in \mathbb{R}^2 : 0 < x < L \mapsto \ \ y = \gamma_\delta(x)\big\}$ represents a sampling of the surface which is reproduced by periodicity. Data k and $\beta > 0$ are the wave number and the period respectively. The small parameter $\delta > 0$ characterizes

the rapid oscillations of the surface and their small amplitude in the following way $\gamma_\delta(x) = s(x, x/\delta)$ where $(x, \sigma) \mapsto s(x, \sigma)$ is a given function, assumed to be smooth for simplicity, doubly periodic of period L in x and d in σ. The radiation condition is expressed by means of the Floquet expansion of u^δ (e.g., [5]). The existence and uniqueness of a solution to (1) are ensured by the stability estimates needed also to establish error bounds on the asymptotic expansion. Even much more involved, the general case can be treated along the same lines [4].

We briefly describe how to obtain a two-scale asymptotic expansion for u^δ, to derive an homogenized boundary condition on a flat surface from this expansion and finally to establish bounds on the error resulting from replacing the rough boundary condition by the homogenized one. The full details can be found in [3]. Actually, the results are known and can be obtained by the method of correctors [1, 2] or by the matching asymptotic expansions [6]. However, the correctors technique, a step by step process, does not give a clear overall idea of the full asymptotic expansion. In the matching asymptotic expansions method, slow and rapid variables are mixed in the boundary layer resulting in intricate analytical calculations to separate them.

For the two-scale asymptotic expansion that is considered here, its determination is first done by means of a formal process. Proven error bounds give it a complete justification a posteriori.

The unknown u^δ is decomposed in the following form

$$u^\delta(x, y) = U^\delta(x, y) + \Pi^\delta(x, \sigma, \tau)|_{\sigma=x/\delta, \tau=y/\delta}.$$

The variable x will play the role of a parameter in the part $\Pi^\delta(x, \sigma, \tau)$ containing the fast variables. It is assumed next that both U^δ and Π^δ have the following asymptotic expansions

$$U^\delta(x, y) = u^0(x, y) + \delta u^1(x, y) + \cdots + \delta^n u^n(x, y) + \cdots,$$
$$\Pi^\delta(x, \sigma, \tau) = \Pi^0(x, \sigma, \tau) + \delta\Pi^1(x, \sigma, \tau) + \cdots + \delta^n\Pi^n(x, \sigma, \tau) + \cdots.$$

Inserting this expansion in the Helmholtz equation and equating to zero the coefficients of powers of δ gives the following system

$$\left(\Delta_{\sigma,\tau}\Pi^n + 2\partial_x\partial_\sigma\Pi^{n-1} + \left(\partial_x^2 + k^2\right)\Pi^{n-2}\right)(x, \sigma, \tau) + \left(\Delta u^n + k^2 u^n\right)(x, y) = 0,$$
$$n = 0, 1, \ldots$$

with 0 for any term involving a negative index. Now, assuming that every derivative of Π^n satisfies

$$\lim_{\tau\to+\infty} \partial^\alpha_{x,\sigma,\tau}\Pi^n(x, \sigma, \tau) = 0$$

makes possible a separation of the functions depending on the slow and the rapid variables

$$\left(\Delta_{\sigma,\tau}\Pi^n + 2\partial_x\partial_\sigma\Pi^{n-1} + \left(\partial_x^2 + k^2\right)\Pi^{n-2}\right)(x, \sigma, \tau) = 0, \quad \left(\Delta u^n + k^2 u^n\right)(x, y) = 0.$$

Since Π^δ is living in a boundary layer of the surface, the radiation condition is on the slow variables functions only

$$\text{RC on } u^0 - u^{\text{inc}}, \quad \text{RC on } u^n \text{ for } n \geq 1.$$

However, the decisive advantage of the two scale asymptotic expansion is its suitability to deal with the boundary condition

$$\sum_{n\geq 0} \delta^n \left(u^n(x,\delta s(x,\sigma)) + \Pi^n(x,\sigma,s(x,\sigma))\right) = 0.$$

A simple Taylor expansion for $u^n(x,\delta s(x,\sigma))$ yields

$$\Pi^n(x,\sigma,s(x,\sigma)) + u^n(x,0) + \sum_{k=1}^{n} \frac{s(x,\sigma)^k}{k!}\partial_y^k u^{n-k}(x,0) = 0.$$

In this way, all the equations needed to determine the asymptotic expansion have been obtained at once. The following theorem is the main tool to do this determination.

Theorem 1. *Let F be a given periodic function of period d in σ in $C^\infty(\overline{D})$ verifying*

$$\Delta_{\sigma,\tau}{}^{m-1} F = 0 \text{ in } D \text{ and } |F(\sigma,\tau)| \leq ce^{-\gamma\tau},$$

and $G \in C^\infty(\mathbb{R})$, periodic of period d. Then, the boundary-value problem

$$\Delta\Pi = F \text{ in } D, \quad \Pi(\sigma,s(\sigma)) = G(\sigma),\ 0 < \sigma < d.$$

admits one and only one solution satisfying $|\Pi(\sigma,\tau) - \Pi^\infty| \leq ce^{-\gamma\tau}$.

Proof. The proof is based on a variational formulation in a weighted Sobolev space and elliptic interior estimates and Fourier series expansion. □

The different terms of the asymptotic expansion are then determined recursively by solving boundary-value problems in the slow and the rapid variables. For the zero order terms, we have $\Pi^0 = 0$ and u^0 is the solution to the problem with a flat boundary

$$\begin{cases} \Delta u^0 + k^2 u^0 = 0 \text{ for } y > 0 \\ u^0 = 0 \text{ for } y = 0,\ x \to e^{-i\beta x}u^0(x,y) \text{ is periodic of period } L, \\ \text{Radiation Condition (RC) on } u^0 - u^{\text{inc}}. \end{cases}$$

Note that, contrary to the corrector method, the flat plane problem has been obtained only by calculations without passing to any limit. Solving the auxiliary problem

$$\Delta_{\sigma,\tau} H = 0, H(x,\sigma,s(x,\sigma)) = s(x,\sigma).$$

yields $h(x) = \lim_{\tau\to\infty} H(x,\sigma,\tau)$. The term u^1 is then the solution of the following boundary-value problem

$$\begin{cases} \Delta u^1 + k^2 u^1 = 0 \text{ for } y > 0, \\ u^1(x,0) + h(x)\partial_y u^0(x,0) = 0, \text{ RC on } u^1. \end{cases}$$

Proceeding in the same way, one can determine the asymptotic expansion at any order. The rigorous justification of the method is then obtained through the error bound given in the following theorem

Theorem 2. *For any given $y_0 > 0$, there exists a constant independent of δ such that*

$$\left\|(u^\delta - (u^0 + \delta u^1)|_{y_0<y}\right\| \leq c\delta^{3/2}.$$

Proof. Let be given a cut-off function $\chi \in \mathcal{D}(\mathbb{R})$ such that $\chi \equiv 1$ near 0 and $0 \leq \chi \leq 1$. The proof is obtained by means of an evaluation of the residuals of $u^\delta - (u^0 + \delta(u^1 + \chi(y)\Pi^1) + \delta^2\chi(y)\Pi^2)$ relatively to the equations of problem (1) and a suitable stability estimate for its solutions. □

The effective boundary condition at order 1 can then be written in terms of $h(x)$

$$u^{1,\delta}(x,0) + \delta h(x)\partial_y u^{1,\delta}(x,0) = 0. \tag{2}$$

and is used in place of the Dirichlet boundary condition in problem (1). The main result concerning the approximation by an effective boundary condition is stated in the following theorem.

Theorem 3. *As in the above theorem, the following bound holds*

$$\left\|(u^\delta - u^{1,\delta})|_{y_0<y}\right\| \leq c\delta^{3/2}.$$

Proof. The main step is to obtain an asymptotic expansion for the problem related to the effective boundary condition. One can readily verify that the first two terms of the expansion are exactly u^0 and u^1. The bound is then obtained through a stability estimate for the approximate problem. □

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Practical Aspects of FEM in Electromagnetics

Oszkár Bíró

The aim of this talk is to highlight two aspects of computational electromagnetism which concern practical low frequency applications. One of them is the question of taking account of the excitation through coils and the other is modelling magnetic nonlinearity.

In low frequency problems, the displacement current density can be neglected resulting in the so-called quasi-static approximation. If the electromagnetic field is generated by coils with known current density distribution, an *eddy current problem* is obtained. If, on the other hand, the current density of the coils is unknown, a *skin effect problem* is spoken of.

The boundary value problems are invariably formulated in terms of potentials. Scalar potentials are approximated on nodal elements and vector potentials on edge elements [6].

Magnetostatic problems in terms of a scalar potential. Magnetostatic fields are generated in nonconducting domains by coils with known current density. The magnetic field intensity can be described as the sum of the gradient of a scalar potential and of a vector function whose curl is the given current density (impressed vector potential). The scalar potential satisfies a Poisson equation with Dirichlet and Neumann boundary conditions. A great advantage of this approach is that the coils need not be modelled by the finite element mesh. It is shown, however, that if the smooth function representing the impressed vector potential is inserted into the finite element equations, wrong results are obtained. This is due to the fact that the impressed vector potential and the gradient of the scalar potential are in different function spaces. The remedy is representing the impressed vector potential in terms edge basis functions [2].

Magnetostatic problems in terms of a vector potential. An alternative to using the scalar potential is to describe the magnetic flux density as the curl of a magnetic vector potential. The current density of the coils appears then directly on the right hand side of the edge element equation system which is singular. Due to numerical integration errors the right hand side is not consistent and hence the equations cannot be solved by Krylov type iterative methods. Again, the remedy is to represent the current density by means of an impressed current vector potential and thus making the right hand side consistent. [4]

Using a reduced vector potential. A disadvantage of the vector potential approach is that the geometry of the coils has to be modelled by the finite element mesh. This can be avoided by writing the flux density as the sum of the curl of a known vector potential due to the coils in free space and of a reduced vector potential. [5].

Eddy current problems in terms of a current vector and a magnetic scalar potential or of a magnetic vector and an electric scalar potential. In case of eddy current problems, the excitation is represented by coils with given current density. Consequently, if their current is known, their treatment is similar to the approach followed in magnetostatic problems. In particular, if the eddy current field is represented by a current vector and a magnetic scalar potential, the coils are taken into account by means of an impressed current vector potential described by edge elements [6]. Similarly, if a magnetic vector and an electric scalar potential are

used, the alternatives to represent the excitation are by means of an impressed current vector potential or a reduced vector potential [6]. If, on the other hand, the voltage of the coils is given, the current can be treated as an additional unknown and a circuit equation added to the system [8, 11].

Skin effect problems in terms of a current vector and a magnetic scalar potential. In case of skin effect problems, the excitation is either the current or the voltage of conductors acting as coils with their current density distribution unknown. Current excitation can be incorporated into the finite element formulation by means of prescribing appropriate boundary conditions if a current vector potential and a magnetic scalar potential act as system variables. Conversely, it is the voltage driven case that can be treated through boundary conditions within the frame of the formulation using a magnetic vector and an electric scalar potential [7]. The voltage excitation can be taken into account in the formulation in terms of the current vector potential and the magnetic scalar potential by means of treating the current as unknown and writing circuit equations [8–10].

Treatment of nonlinearity. Due to the nonlinear relationship between the magnetic flux density and field intensity, the finite element method leads to nonlinear algebraic equations in case of magnetostatic problems and to nonlinear ordinary differential equations for time dependent problems. Their solution can be carried out by means of standard techniques [3]. Frequently, it is more advantageous to write the eddy current equations in the frequency domain instead of the time domain. This leads to the harmonic balance method, see e.g. [1].

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$\mathcal{H}^2$-Matrices with Adaptive Cluster Bases Applied to an Eddy Current Problem

Steffen Börm

$\mathcal{H}^2$-matrices [1, 5, 6, 16] can be used to find data-sparse representations of the densely populated matrices occurring, e.g., in boundary element methods.

The basic idea of hierarchical matrix techniques [3, 4, 10, 13–15] is to split the index set I into a hierarchy of subsets, the *cluster tree* $\mathcal{T}_I$, and to split the matrix into a hierarchy $\mathcal{T}_{I\times I}$ of subblocks $\tau \times \sigma$ corresponding to $\tau, \sigma \in \mathcal{T}_I$ that contains only small blocks and blocks that admit a separable approximation. The latter blocks are called *admissible*.

In a hierarchical matrix, an admissible block $\tau \times \sigma$ is approximated by a factorized rank-k-matrix $AB^\top$ ($A \in \mathbb{R}^{\tau\times k}$, $B \in \mathbb{R}^{\sigma\times k}$). The factorized form can be constructed by standard panel-clustering techniques [17], multipole expansion [11, 19] or interpolation [2].

In an $\mathcal{H}^2$-matrix, an admissible block $\tau \times \sigma$ is approximated by a special low-rank matrix of the form $V^\tau S^{\tau,\sigma} W^{\sigma\top}$ ($V^\tau \in \mathbb{R}^{\tau\times k}$, $W^\sigma \in \mathbb{R}^{\sigma\times k}$, $S^{\tau,\sigma} \in \mathbb{R}^{k\times k}$). By requiring the *row cluster bases* V^τ and the *column cluster bases* W^σ to be organized in a nested hierarchy (this is straightforward for polynomial approximation schemes [6, 9] and can also be achieved for multipole expansions [12]), we can reach algorithms with *linear* complexity in the number of degrees of freedom n.

While constructing an $\mathcal{H}^2$-matrix approximation of an integral operator by Lagrangian interpolation leads to a relatively general, simple and fast method, this approach also requires a large amount of storage, since polynomial bases are not adapted to the special characteristics of a given operator or a given geometry. This problem can be solved by combining the separable approximation scheme with an algebraic recompression algorithm that detects and eliminates redundant expansion functions by solving local symmetric eigenvalue problems [5, 7], which reduces the storage requirements significantly at the price of a moderate increase in computing time.

Since the recompression algorithm can be used without keeping the entire original $\mathcal{H}^2$-matrix approximation in memory, it is possible to treat boundary element problems with more than 100.000 degrees of freedom on standard PCs in less than ten minutes.

The combination of polynomial interpolation and algebraic recompression can not only be applied to standard Laplace problems, but also to more complicated

vector-valued eddy-current models for Maxwell's equation [8, 18]. An example is the vector-valued double layer potential

$$b(\mathbf{E},\phi) = \int_\Gamma \int_\Gamma \langle \mathbf{curl}_\Gamma\, \phi(y), \mathbf{E}(x)\rangle \langle \mathbf{grad}_x\, \Phi(x,y), \mathbf{n}(x)\rangle\, dy\, dx - \int_\Gamma \int_\Gamma \langle \mathbf{curl}_\Gamma\, \phi(y), \mathbf{n}(x)\rangle \langle \mathbf{grad}_x\, \Phi(x,y), \mathbf{E}(x)\rangle\, dy\, dx$$

for the fundamental solution $\Phi(x,y) := 1/(4\pi\|x-y\|)$. Even if Φ could be approximated by a single tensor product, the resulting matrix approximation would still have rank 3, since the variables x and y are coupled by a three-dimensional inner product. In practical approximation schemes, this implies that the rank required for the approximation of the vector-valued operator will be at least three times as high as in the case of scalar-valued operators.

Still, numerical experiments performed by applying the recompression algorithm to the vector-valued operator leads to storage requirements that are close to those of the scalar-valued operator. This result suggests that recompression is crucial for the efficient treatment of vector-valued problems, since the conventional fast approximation schemes like polynomial and multipole expansions seem to be incapable of taking advantage of their special structure.

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Theoretical Aspects of Edge Finite Elements

Daniele Boffi

Let us consider the time harmonic Maxwell system

$$\text{(TH)} \qquad \begin{cases} \operatorname{curl}(\mu^{-1}\operatorname{curl}\mathbf{u}) - \omega^2\varepsilon\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}(\varepsilon\mathbf{u}) = 0 & \text{in } \Omega \\ \mathbf{u}\times\mathbf{n} = 0 & \text{on } \Omega \end{cases}$$

where ω is the fixed frequency, $\operatorname{div}\mathbf{f} = 0$, and Ω is a polyhedral (or polygonal) domain with outward normal $\mathbf{n}$.

It is well known that problem (TH) is well posed if and only if ω^2 is not an interior Maxwell eigenvalue. A variational formulation of the problem under consideration is obtained, for instance, by imposing the divergence free condition in a weak sense in the spirit of Kikuchi [21] as follows.

Find $(\mathbf{u}, p) \in H_0(\operatorname{curl};\Omega)\times H_0^1(\Omega) = V\times Q$ such that

$$\text{(TH-V)} \qquad \begin{cases} (\mu^{-1}\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) - \omega^2(\varepsilon\mathbf{u},\mathbf{v}) + (\varepsilon\mathbf{v}, \operatorname{grad} p) = (\mathbf{f},\mathbf{v}) & \forall \mathbf{v}\in V \\ (\varepsilon\mathbf{u}, \operatorname{grad} q) = 0 & \forall q \in Q \end{cases}$$

A stability estimate of the solution of (TH-V) can be found, for instance, in [17].

$$\left(\|\mathbf{u}\|_{\operatorname{curl}}^2 + \|p\|_1^2\right)^{1/2} \le \sup_{i=1,2,\dots}\left(1+\omega^2, \frac{1+\lambda_i}{|\lambda_i-\omega^2|}\right)\|\mathbf{f}\|_0$$

where λ_i $(i = 1, 2, \dots)$ are the interior Maxwell eigenvalues. Given $V_h \subset V$ and $Q_h \subset Q$ we consider the discretization of problem (TH-V).

(TH-Vh)

Find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} (\mu^{-1}\operatorname{curl}\mathbf{u}_h, \operatorname{curl}\mathbf{v}) - \omega^2(\varepsilon\mathbf{u}_h,\mathbf{v}) + (\varepsilon\mathbf{v}, \operatorname{grad} p_h) = (\mathbf{f},\mathbf{v}) & \forall \mathbf{v}\in V_h \\ (\varepsilon\mathbf{u}_h, \operatorname{grad} q) = 0 & \forall q \in Q_h \end{cases}$$

Assuming the compatibility condition

(COMP) $$\operatorname{grad} Q_h \subset V_h$$

which guarantees a discrete inf-sup condition for problem (TH-Vh), we have the error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathrm{curl}} + \|p-p_h\|^2_1 \le \gamma^2 \inf_{(\mathbf{v}_h,q_h)\in V_h\times Q_h} \left(\|\mathbf{u}-\mathbf{v}_h\|^2_{\mathrm{curl}} + \|p-q_h\|^2_1\right)$$

with

$$\gamma \le 1 + \max_{i=1,2,\dots} \left(1+\omega^2, \frac{1+\lambda_{i,h}}{|\lambda_{i,h}-\omega^2|}\right)$$

where $\lambda_{i,h}$ are the discrete Maxwell eigenvalues. We explicitly notice that $\operatorname{div}\mathbf{f}=0$ implies $p=p_h=0$.

Several numerical experiments and theoretical results (see [8, 10], for instance) show that standard nodal elements do not approximate Maxwell eigenvalues in a correct way, even on special two dimensional meshes where the compatibility condition (COMP) is satisfied [8, 28]. On the other hand, edge finite elements have been proven to satisfy the discrete compactness property which guarantees the good approximation of the eigensolutions [6, 7, 14, 22–24] (see [20, 26] for a review on this topic).

In this talk we review some of the most important theoretical properties of edge finite elements, including discrete compactness, commuting diagram (de Rham complex), interpolation estimates. The commuting diagram property (see, for instance, [7, 12, 13, 18, 19], [16, 29] for possible extensions and [3] for a review) on a simply connected domain reads

$$\begin{array}{ccccccccc}
0 \to & Q & \xrightarrow{\mathrm{grad}} & V & \xrightarrow{\mathrm{curl}} & U & \xrightarrow{\mathrm{div}} & S/\mathbb{R} & \to 0 \\
& \downarrow \Pi_h^Q & & \downarrow \Pi_h^V & & \downarrow \Pi_h^U & & \downarrow \Pi_h^S & \\
0 \to & Q_h & \xrightarrow{\mathrm{grad}} & V_h & \xrightarrow{\mathrm{curl}} & U_h & \xrightarrow{\mathrm{div}} & S_h/\mathbb{R} & \to 0,
\end{array}$$

where $Q \subset H^1_0(\Omega)$, $V \subset H_0(\mathrm{curl})$, $U \subset H_0(\mathrm{div})$, and $S \subset L^2(\Omega)$ are suitable smooth function spaces, so that the corresponding interpolation operators can be defined and Q_h, V_h, U_h, and S_h are their discrete counterparts.

Standard interpolation estimates are (see, for instance, [1, 2, 15, 20, 25, 27])

$$\inf_{\mathbf{v}_h\in V_h} \|\mathbf{u}-\mathbf{v}_h\|_0 \le Ch^s(|\mathbf{u}|_s + \|\operatorname{curl}\mathbf{u}\|_s) \qquad 1/2 < s \le k+1$$
$$\inf_{\mathbf{v}_h\in V_h} \|\operatorname{curl}\mathbf{u} - \operatorname{curl}\mathbf{v}_h\|_0 \le Ch^s|\operatorname{curl}\mathbf{u}|_s \qquad 0 < s \le k+1$$

When $\operatorname{curl}\mathbf{u}$ is discrete, the improved estimate

$$\inf_{\mathbf{v}_h\in V_h} \|\mathbf{u}-\mathbf{v}_h\|_0 \le Ch^s|\mathbf{u}|_s \quad 1/2 < s \le k+1$$

has been used in [6], see also [20]. Recent results [9] show the improved estimate

$$\begin{aligned}
&\|\mathbf{u}-\Pi_h^V\mathbf{u}\|_{L^2} \le Ch^s\left(|\mathbf{u}|_{H^s}+\|\operatorname{curl}\mathbf{u}\|_{L^p}\right) && 1/2<s\le 1,\ p>2\\
&\|\mathbf{u}-\Pi_h^V\mathbf{u}\|_{L^2} \le Ch^s|\mathbf{u}|_{H^s} && 1<s\le k+1\\
&\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\Pi_h^V\mathbf{u}\|_{L^2} \le Ch^s|\operatorname{curl}\mathbf{u}|_{H^s} && 0<s\le k+1
\end{aligned}$$

These estimates, which do not require on $\operatorname{curl}\mathbf{u}$ more regularity than the one needed for the definition of the interpolant itself (see [2]), have been used in [9] for the analysis of the approximation of photonic crystals.

The last remark concerns the approximation properties achieved by edge finite elements on quadrilateral meshes. Recent results show that particular care has to be taken into account when dealing with general regular quadrilateral finite elements [4]. This issue is particularly significant for quadrilateral edge elements; the lowest order element does not achieve the convergence at all in the $H(\operatorname{curl})$ norm, the higher order elements are substantially suboptimal [5]. Some modifications of standard edge element, which provide a solution to this phenomenon, have been recently proposed [5, 11].

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An Adaptive Perfectly Matched Layer Technique for Time-harmonic Scattering Problems

Zhiming Chen
(joint work with Xueze Liu)

We propose and study an adaptive perfectly matched layer (PML) technique for solving the Helmholtz-type scattering problems with perfectly conducting boundary:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbf{R}^2 \backslash \bar{D}, \tag{1a}$$

$$\frac{\partial u}{\partial \mathbf{n}} = -g \quad \text{on } \Gamma_D, \tag{1b}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{1c}$$

Here $D \subset \mathbf{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ is determined by the incoming wave, and $\mathbf{n}$ is the unit outer normal to Γ_D. We assume the wave number $k \in \mathbb{R}$ is a constant. We remark that the results in this paper can be easily extended to solve the scattering problems with other boundary conditions such as Dirichlet or the impedance condition on Γ_D, or the acoustic wave propagation problems in inhomogeneous media which correspond to a variable wave number $k^2(x)$.

Since the work of Berenger [3] which proposed a PML layer for use with the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [17] for a review). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a finite thickness layer of the specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain. The PML equation for the time-harmonic scattering problem (1a) is derived in Collino and Monk [8] by a complex extension of the solution u in the exterior domain. It is proved in Lassas and Somersalo [11], Hohage, Schmidt and Zschiedrich [10] that the resultant PML solution converges exponentially to the solution of the original scattering problem as the PML layer thickness tends to infinite. We remark that in practical applications involving PML method, one cannot afford to use a very thick PML layer because it requires excessive grid points and hence more computer time and more storage. On the other hand, a thin PML layer requires a rapid variation of the artificial material property which deteriorates the accuracy if two corse mesh is used in the PML layer.

A posteriori error estimates are computable quantities in terms of the discrete solution and data that measure the actual discrete errors without the knowledge

of exact solutions. They are essential in designing algorithms for mesh modification which equi-distribute the computational effort and optimize the computation. Ever since the pioneering work of Babuška and Rheinboldt [2], the adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. The ability of error control and the asymptotically optimal approximation property (see e.g. Morin, Nochetto and Siebert [14], Chen and Dai [5]) make the adaptive finite element method attractive for complicated physical and industrial processes (cf. e.g. Chen and Dai [4], Chen, Nochetto and Schmidt [6]). For the efforts to solve scattering problems using adaptive methods based on a posterior error estimate, we refer to the recent work Monk [12], Monk and Süli [13].

It is proposed in Chen and Wu [7] for scattering problems by periodic structures, the grating problem, that one can use the a posteriori error estimate to determine the PML parameters. Moreover, the derived a posteriori error estimate in [7] has the nice feature of exponential decay in terms of the distance to the distance to the boundary of the fixed domain where the PML layer is placed. This property leads to coarse mesh size away from the fixed domain and thus makes the total computational cost insensitive to the thickness of the PML absorbing layer.

In this paper we extend the idea of using a posteriori error estimates to determine the PML parameters and propose an adaptive PML technique for solving the scattering problem (1a)-(1c). The first difficulty of the analysis is that in contrast to the grating problems in which there are only finite number of outgoing modes [7], now there are infinite number of outgoing modes expressed in terms of Hankel functions. We overcome this difficulty by using following uniform estimate for the Hankel functions H^1_ν, $\nu \in \mathbb{C}$,:

$$|H^{(1)}_\nu(z)| \le e^{-\mathrm{Im}\,(z)\left(1-\frac{\Theta^2}{|z|^2}\right)^{1/2}} |H^{(1)}_\nu(\Theta)|, \tag{2}$$

for any $z \in \mathbb{C}_{++}, \Theta \in \mathbf{R}$ such that $0 < \Theta \le |z|$, where $\mathbb{C}_{++} = \{z \in \mathbb{C} : \mathrm{Im}\,(z) \ge 0, \mathrm{Re}\,(z) \ge 0\}$. This sharp estimate, which seems first appeared in this paper, allows us to prove the exponentially decaying property of the PML solution without resorting to the integral equation technique in [11] or the representation formula in [10]. We remark that in [11], [10], it is required the fictitious absorbing coefficient must be linear after certain distance away from the bounary where the PML layer is placed.

The second difficulty is that the PML equation in the PML layer is not necessarily uniquely solvable for any wave number k^2. Let $\Omega^{\mathrm{PML}} = B_\rho \backslash \bar{B}_R$, where $0 < R < \rho$ and B_a denotes the circle of radius a for any $a > 0$. Let $\alpha = 1 + \mathbf{i}\sigma$ be the fictitious medium property. In practical applications, σ is usually taken as power functions:

$$\sigma = \sigma(r) = \sigma_0 \left(\frac{r-R}{\rho-R}\right)^m \quad \text{for some integer } m \ge 1, \tag{3}$$

where $\sigma_0 > 0$ is some constant. We prove that for any given R and ρ, the PML equation in the PML layer is uniquely solvable and its solution satisfies sharp

stability estimates if σ_0 is chosen sufficiently large. This allows us to complete the proof of the following key estimate between the Dirichlet-to-Neumann mapping for the original scattering problem $T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ and the PML problem $\hat{T}$, where $\Gamma_R = \partial B_R$,

$$\| T - \hat{T} \|_{L(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R))} \leq C(1 + k^2 R^2)|\alpha_0|^2 e^{-k\mathrm{Im}\,(\tilde{\rho})\left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}},$$

where $\alpha_0 = 1 + \mathbf{i}\sigma_0$, and $\tilde{\rho} = \int_0^\rho \alpha(t)dt$ is the complex radius corresponding to ρ.

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Div-Curl Lemma for Edge Elements

Snorre H. Christiansen

Given two sequences (u_h) and (u'_h) of vector fields converging weakly in L^2 on some open domain in $\mathbb{R}^3$ the div-curl lemma of Murat [5] and Tartar [7] gives sufficient conditions under which their scalar product converges in the weak-star sense of distributions to the right scalar field. Namely if the sequences $(\operatorname{div} u_h)$ and $(\operatorname{curl} u'_h)$ are relatively compact in H^{-1} then this convergence property holds. This lemma is useful in questions arizing in homogenization and certain non-linear PDEs and is an ingredient in the method of compensated compactness.

For the variational formulation of problems in electromagnetics on Nédélec's [6] edge element spaces X_h one can naturally obtain control over the L^2 norm of the involved fields. One can also expect to have sufficient control of the curl in H^{-1} (e.g. in the form of boundedness in L^2), due to energy considerations. However control of the divergence of a field $u_h \in X_h$ is obtained in the form of estimates on $\int u_h \cdot \operatorname{grad} p_h$ when p_h runs trough the maximal space Y_h of continuous piecewise polynomials which vanish on the boundary and such that the gradient operator maps Y_h into X_h. Since the space Y_h is smaller than H^1_0, the question arizes whether an L^2 bounded sequence of so-called discrete divergence free vector fields $u_h \in X_h$ has compact divergence in H^{-1}. This property is stronger than the discrete compactness property of Kikuchi which has come to play a central role in the numerical analysis of edge elements.

While we leave this question unanswered we prove in this talk the following div-curl lemma for edge elements on quasi-uniform meshes on bounded domains with smooth boundary[1]:

Lemma 1. *Suppose (u_h) and (u'_h) are sequences of vector fields $u_h, u'_h \in X_h$ converging weakly in L^2 to u and u'. Suppose furthermore that with the decomposition $u_h = v_h + \operatorname{grad} p_h$ with v_h in the L^2 orthogonal of $\operatorname{grad} Y_h$ in X_h, and $p_h \in Y_h$, (p_h) is relatively compact in H^1_0, and that $(\operatorname{curl} u'_h)$ is relatively compact in H^{-1}.*

Then $(u_h \cdot u'_h)$ converges to $u \cdot u'$ in the weak-star sense of distributions.

One of the main ingredients of the proof is a norm equivalence on a subspace of X_h which is uniform with respect to h and which strengthens the standard discrete compactness property (using a technique appearing in Lemma 4.1 in [4]). Another ingredient is a super-approximation property of the spaces Y_h. For the details of the proof I refer to the revised version of the preprint [2], which also contains bibliographical references in particular to the work by Boffi and Hiptmair on discrete compactness.

This work is related to a joint effort [3] to understand the variational formulation of constraints in the discretization of some non-linear PDEs, parts of which were presented in [1].

[1] The possibility of weaking these hypothesis was briefly discussed and is the object of current efforts.

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Formulations and Efficient Numerical Solution Techniques for Transient 3D Magneto-and Electro-Quasistatic Field Problems

Markus Clemens

(joint work with Galina Benderskaya, Herbert De Gersem, Stefan Feigh, Markus Wilke, Jing Yuan and Thomas Weiland)

The simulation of 3D quasistatic electric high-voltage fields and magnetic eddy currents field problems typically involves nonlinear material properties such as field dependent electric conductivities of insulator materials or saturation effects within ferromagnetic materials which may be even of hysteretic nature. In these cases and, more generally, for any non-periodical field excitation, time domain formulations of these problems are preferred. Using spatial discretization schemes such as the Whitney Finite Element method [3], the Cell Method [18]or the Finite Integration Technique [9, 19], for electro-quasistatic problems this will result in large systems of stiff ordinary differential equations of the form

$$\mathbf{G}^T\mathbf{M}_\varepsilon\mathbf{G}\frac{d}{dt}\mathbf{\Phi}(t) + \mathbf{G}^T\mathbf{M}_\kappa\big(\mathbf{\Phi}(t)\big)\mathbf{G}\mathbf{\Phi}(t) = 0. \tag{1}$$

where $\mathbf{G}^T$ and $\mathbf{G}$ are the discrete divergence and gradient matrices with the vector of electric grid voltages as $\widehat{\mathbf{e}} = -\mathbf{G}\mathbf{\Phi}$ and $\mathbf{M}_\varepsilon$ and $\mathbf{M}_\kappa = \mathbf{M}_\kappa(\mathbf{\Phi})$ are material matrices combining the permittivities and field dependent electrical conductivities with the metric information of the grid [7]. Magneto-quasistatic fields can be described with systems of differential-algebraic equations of index 1

$$\mathbf{M}_\kappa\frac{d}{dt}\widehat{\mathbf{a}}(t) + \mathbf{C}^T\mathbf{M}_\nu\big(\widehat{\mathbf{a}}(t)\big)\mathbf{C}\widehat{\mathbf{a}}(t) = \widehat{\widehat{\mathbf{j}}}_s(t), \tag{2}$$

where $\widehat{\mathbf{a}}$ is the vector of path integrated magnetic vector potentials, $\mathbf{C}$ is the incidence matrix discretizing the curl operator to yield the vector of magnetic fluxes

$\widehat{\widehat{\mathbf{b}}} = \mathbf{C}\widehat{\mathbf{a}}$, $\mathbf{M}_\nu = \mathbf{M}_\nu(\widehat{\mathbf{a}})$ is the matrix of flux dependent reluctivities and the $\widehat{\widehat{\mathbf{j}}}_s$ is the vector of current excitations [8, 12]. Today, efficient numerical techniques for the solution of the large systems of equations (1) and (2) involve time step adaptive higher order embedded time integration schemes such as singly diagonal implicit Runge-Kutta methods (SDIRK) or linear-implicit Rosenbrock-Wanner (ROW) methods [13, 15]. In these schemes the repeated solution of the algebraic systems of equations involves a combination of advanced numerical methods. Such methods are geometric or algebraic multigrid preconditioners specifically designed to interact with the above mentioned geometric discretization methods [4–6, 17], multiple-righthand side Lanczos-projection techniques and a subspace projection extrapolation scheme for the generation of optimal start vectors of the iterative solution methods [11, 14]. Specialized projection methods are used for the inclusion of floating potential areas and other complicated boundary conditions [16] and nonstandard time step-prediction schemes are developed for magnetodynamic field-circuit coupled formulations involving switching circuit elements [2]. Extensions of the magneto-quasistatic formulations also include models for motion-induced eddy currents as they occur e.g. in eddy current brakes using either Lagrangian or Eulerian coordinate descriptions [1, 10] and nonlinear iteration schemes adapted to hysteretic ferromagnetic material behavior described by Preisach or Jiles-Atherton hysteresis models [20–22].

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Singularities of Electromagnetic Fields in the Eddy Current Limit

Monique Dauge
(joint work with Martin Costabel and Serge Nicaise)

This talk discusses the notion of eddy current limit for a conductor surrounded by an exterior dielectric medium and presents results from [15–17] about the singularities of solutions when the conductor has corners and edges.

1. The eddy current limit

Let Ω_C be the conductor body. We assume that Ω_C is a three-dimensional polyhedron. To simplify the exposition we also assume that the boundary B of Ω_C has a single connected component. Let Ω be a ball, large enough to surround Ω_C. We consider the exterior domain $\Omega_E = \Omega \setminus \overline{\Omega}_C$. We denote by ε_C, μ_C and

σ_C the electric permittivity, the magnetic permeability and the conductivity of Ω_C, respectively, and by ε_E, μ_E and σ_E their values inside Ω_E. We assume that $\sigma_E = 0$. We consider the harmonic Maxwell equation at the given frequency ω:

$$(1)\qquad \begin{cases} (i) & \mathbf{curl}\ \mathbf{E} = -i\omega\,\mu\mathbf{H} & \text{in} & \Omega, \\ (ii) & \mathbf{curl}\ \mathbf{H} = (\sigma + i\omega\,\varepsilon)\mathbf{E} + \mathbf{j}_0 & \text{in} & \Omega, \\ (iii) & \mathbf{E}\times n = 0 \quad \text{and} \quad \mathbf{H}\cdot n = 0 & \text{on} & \partial\Omega. \end{cases}$$

Here $\mathbf{j}_0$ is a divergence free field (the source current density) with support inside Ω_C and σ denotes the piecewise constant equal to σ_C inside Ω_C and 0 inside Ω_E. Similar conventions hold for ε and μ. Taking the divergence of equation (1) *(ii)*, we obtain:

$$(2)\qquad \operatorname{div}(i\omega\varepsilon + \sigma)\mathbf{E} = 0 \quad \text{in} \quad \Omega.$$

The time-harmonic eddy current problem [2, 8, 11, 20] consists in neglecting $\omega\varepsilon$ in (1) in the case when $\sigma >> \omega\varepsilon$ and reads:

$$(3)\qquad \begin{cases} (i) & \mathbf{curl}\ \mathbf{E} = -i\omega\mu\mathbf{H} & \text{in } \Omega, \\ (ii) & \mathbf{curl}\ \mathbf{H} = \sigma\mathbf{E} + \mathbf{j}_0 & \text{in } \Omega, \\ (iii) & \mathbf{E}\times n = 0 \quad \text{and} \quad \mathbf{H}\cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us write $\mathbf{E}_C = \mathbf{E}|_{\Omega_C}$ and $\mathbf{E}_E = \mathbf{E}|_{\Omega_E}$. Taking the divergence of (3) *(ii)*, we only obtain $\operatorname{div}\mathbf{E}_C = 0$ in Ω_C and $\mathbf{E}_C \cdot n = 0$ on B, which *has to be completed by the gauge conditions*:

$$(4)\qquad \operatorname{div}\mathbf{E}_E = 0 \ \text{ in } \ \Omega_E \quad \text{and} \quad \int_B \mathbf{E}_E \cdot n \,\mathrm{d}S = 0.$$

Let us assume for simplicity that $\varepsilon_C \simeq \varepsilon_E$ and let us introduce our small parameter δ as

$$\delta = \frac{\varepsilon_C}{\sigma_C}\,.$$

Let us consider σ, μ and $\omega > 0$ as fixed and denote by $(\mathbf{E}^\delta, \mathbf{H}^\delta)$ the solution of (1) and by $(\mathbf{E}^0, \mathbf{H}^0)$ the solution of (3). We have proved in [16]

$$(5)\qquad \left\|\mathbf{E}^\delta - \mathbf{E}^0\right\|_{L^2(\Omega)} + \left\|\mathbf{H}^\delta - \mathbf{H}^0\right\|_{L^2(\Omega)} \le C\delta.$$

This notion of limit corresponds to that presented in [11, Ch.4], whereas it somewhat differs from the point of view adopted in [2] where a zero frequency limit is considered for both problems (1) and (3). However (5) does not answer completely the question of knowing whether the eddy current approximation is valid when we are given a set of parameters σ, μ, ε and ω. Let us set $\hat{\varepsilon} = \varepsilon/\delta$. The interior equations for the electric field $\mathbf{E}^\delta$ take the form:

$$(6)\qquad \begin{cases} (i) & \mathbf{curl}\ \mu^{-1}\,\mathbf{curl}\ \mathbf{E}^\delta + i\omega\sigma\mathbf{E}^\delta - \delta\omega^2\hat{\varepsilon}\,\mathbf{E}^\delta = -i\omega\mathbf{j}_0 & \text{in} & \Omega_C, \\ (ii) & \mathbf{curl}\ \mu^{-1}\,\mathbf{curl}\ \mathbf{E}^\delta - \delta\omega^2\hat{\varepsilon}\,\mathbf{E}^\delta = 0 & \text{in} & \Omega_E. \end{cases}$$

We can see that *(i)* tends to its eddy current counterpart as soon as $\delta\omega$ is small, whereas for equation *(ii)* approaching the eddy current limit requires that $\omega^2\varepsilon\mu$ also is small at the scale of Ω_E. Another asymptotic effect may occur when

$\omega\mu\sigma >> 1$: The skin effect produces a strong concentration of the electromagnetic field inside the conductor in a very narrow layer around its surface [9].

2. Singularities

The equations (6) combined with the zero divergence constraint inside $\Omega_C \cup \Omega_E$ and transmission conditions on B, produce an elliptic boundary value problem on Ω. Like any elliptic boundary value problem in a domain with corners and edges [18, 19, 22, 24], the electric or magnetic Maxwell problems have singular solutions (the "singularities") [13, 15]. In the present situation of a polyhedral conductor surrounded by a dielectric medium, the issue is the investigation [16, 17] of the singularities of the eddy current problem (3) together with the way in which the singularities of the transmission problem (1) transform as $\delta \to 0$ in the eddy current limit. Let us define $\alpha = (\alpha_C, \alpha_E)$ by

$$i\omega\alpha = i\omega\delta\hat{\varepsilon} + \sigma.$$

The "electric" singularities of problems (1) and (3) are those of the operator

$$(7) \qquad \begin{cases} (i) \ \ \mathbf{curl}\ \mu_C^{-1}\,\mathbf{curl}\ \mathsf{E} - \nabla \operatorname{div} \mathsf{E} & \text{in} \quad \Omega_C, \\ (ii) \ \ \mathbf{curl}\ \mu_E^{-1}\,\mathbf{curl}\ \mathsf{E} - \nabla \operatorname{div} \mathsf{E} & \text{in} \quad \Omega_E, \end{cases}$$

with the essential transmission conditions:

$$(8) \qquad [\mathsf{E} \times n] = 0 \quad \text{and} \quad [\alpha \mathsf{E} \cdot n] = 0 \qquad \text{on} \quad B,$$

which we complement by the Neumann type transmission conditions

$$(9) \qquad [\mu^{-1}\,\mathbf{curl}\ \mathsf{E} \times n] = 0 \quad \text{and} \quad [\operatorname{div} \alpha \mathsf{E}] = 0 \qquad \text{on} \quad B.$$

Problem (7)-(9) is the principal part of one of the regularized operators associated with problem (1).

According to the classification of [13, 15], problem (7)-(9) has mainly two types of singularities, Type 1 and Type 2, at each corner and each edge of Ω_C. To each corner or edge we associate two cones Γ_C and Γ_E together with their interface I. For a corner point $\mathbf{c}$, Γ_C and Γ_E coincide with Ω_C and Ω_E, respectively, in a neighborhood of $\mathbf{c}$. For an edge we have similar definitions where Γ_C and Γ_E are plane sectors such that the diehedra $\Gamma_C \times \mathbb{R}$ and $\Gamma_E \times \mathbb{R}$ coincide with Ω_C and Ω_E in a neighborhood of the edge. The singularities are homogeneous functions on $\Gamma_C \cup \Gamma_E$.

The singularities of Type 1 are the gradients $\nabla\Phi$ in $\Gamma_C \cup \Gamma_E$ of a potential function $\Phi = (\Phi_C, \Phi_E)$ which is itself a singularity of the scalar transmission problem, *cf* [25, 26]:

$$(10) \quad \Delta\Phi_C = 0 \ \text{ in } \ \Gamma_C, \quad \Delta\Phi_E = 0 \ \text{ in } \ \Gamma_E, \quad [\Phi] = 0 \ \text{ and } \ [\alpha\partial_n\Phi] = 0 \ \text{ on } \ I,$$

the last transmission condition becoming $\partial_n\Phi_C = 0$ on I in the eddy current limit $\delta = 0$: In the latter case, either $\Phi_C = 0$ and Φ_E is a Dirichlet singularity of the Laplace problem on Γ_E, or Φ_C is a Neumann singularity of the Laplace problem on Γ_C and Φ_E has the same Dirichlet traces as Φ_C (and the same degree of homogeneity).

The singularities of Type 2 are electric fields associated with magnetic fields of the form $\nabla\Psi$ where the scalar potential $\Psi = (\Psi_C, \Psi_E)$ is a singularity of the transmission problem

$$\Delta\Psi_C = 0 \quad \text{in } \Gamma_C, \quad \Delta\Psi_E = 0 \text{ in } \Gamma_E, \quad [\Psi] = 0 \text{ and } [\mu\partial_n\Psi] = 0 \text{ on } I. \tag{11}$$

If the permeability μ has no jump, solutions of (11) still exist, but they are polynomials and do not decrease the regularity of Maxwell solutions.

3. Regularity

Let β_α and β_μ be the limiting regularity Sobolev exponents for the transmission Laplace operators $\operatorname{div}\alpha\nabla$, *cf* singularities (10), and $\operatorname{div}\mu\nabla$ respectively, *cf* singularities (11). Then if the data $\mathbf{j}_0$ is regular enough, the solution $\mathbf{E}$ of (1) satisfies

$$\mathbf{E}_C \in H^s(\Omega_C) \quad \text{and} \quad \mathbf{E}_E \in H^s(\Omega_E), \quad \forall s < \min\{\beta_\alpha - 1, \beta_\mu\}.$$

Moreover we have a decomposition of [4, 5]'s type: $\mathbf{E}$ can be split into $\nabla\Phi + \mathbf{E}^{\mathsf{reg}}$ with

$$\mathbf{E}_C^{\mathsf{reg}} \in H^s(\Omega_C) \quad \text{and} \quad \mathbf{E}_E^{\mathsf{reg}} \in H^s(\Omega_E), \quad \forall s < \min\{\beta_\alpha\,, \beta_\mu\}.$$

Concerning the eddy current problem (3), we introduce the limiting regularity Sobolev exponents β_E^{Dir} and β_C^{Neu} for the Dirichlet problem on Ω_E and the Neumann problem on Ω_C, respectively. Let us assume for simplicity that μ has no jump (which amounts to setting $\beta_\mu = \infty$). Then the solution $\mathbf{E}$ of (3) satisfies

$$\mathbf{E}_C \in H^s(\Omega_C),\ \forall s < \beta_C^{\mathsf{Neu}} - 1, \quad \text{and} \quad \mathbf{E}_E \in H^s(\Omega_E),\ \forall s < \min\{\beta_C^{\mathsf{Neu}}, \beta_E^{\mathsf{Dir}}\} - 1.$$

Moreover, we may split $\mathbf{E}$ into $\nabla\Phi + \mathbf{E}^{\mathsf{reg}}$ with

$$\mathbf{E}_C^{\mathsf{reg}} \in H^s(\Omega_C),\ \forall s < \beta_C^{\mathsf{Neu}} \quad \text{and} \quad \mathbf{E}_E^{\mathsf{reg}} \in H^s(\Omega_E),\ \forall s < \min\{\beta_C^{\mathsf{Neu}}, \beta_E^{\mathsf{Dir}}\}.$$

Thus, if the conductor Ω_C is convex, it may happen that, in the eddy current limit, the solution inside the conductor is more regular than outside. This effect does not occur for $\delta \neq 0$. In fact, the conductor part Φ_C of certain singularities of (1) is vanishing as $\delta \to 0$.

4. Short conclusion about the numerical approximation

The resolution of the eddy current problem is made by eliminating either the electric field ($\mathbf{H}$-formulation or magnetic approach [1, 8, 11]) or the magnetic field ($\mathbf{E}$-formulation or electric approach [2, 3, 8, 11, 20]) or combining both [12]. The magnetic approach can be preferred because the magnetic field in Ω_E is irrotational. Thus a coupled FEM-BEM method can be used to compute $\mathbf{H}$ [10, 23]. Concerning the use of edge elements, see [6, 7].

We would like to end by the "usual" warning: In the presence of reentrant corners (i.e. any situation where Ω_C is a polyhedron in $\mathbf{E}$-formulation, and the case when Ω_C is a non-convex polyhedron in $\mathbf{H}$-formulation) certain methods lead to wrong results. This is the case for the *plain regularization* by a divergence term, used with nodal elements, or, even, certain edge elements which do not satisfy the discrete compactness property, see the review papers [14, 21].

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Convergence of Collocation Methods for Time Domain Boundary Integral Equations

Penny J Davies
(joint work with Dugald B Duncan)

The problem of interest is to calculate the current induced on a perfectly conducting surface Γ when it is subjected to a transient electromagnetic field. Time–stepping solution schemes for this problem are often numerically unstable (see e.g. [2, 7, 9]), and our aim is to develop stable collocation approximations. Here we concentrate on the more straightforward case of acoustic scattering, where the same stability issues arise. This problem is to find the solution u of

$$\int_\Gamma \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\Gamma = a(\boldsymbol{x}, t) \tag{1}$$

given $a(\boldsymbol{x}, t)$ on $\Gamma\times(0, T)$, and assuming causality, namely that $u \equiv 0$ and $a \equiv 0$ for all $t \leq 0$. Equation (1) is the single layer potential equation for acoustic scattering from the surface Γ, and we shall concentrate on the case in which Γ is a flat plate. Note that a can be calculated anywhere in space from (1) once $u|_\Gamma$ is known.

It follows from results of Ha-Duong [6, Thm. 3] and Lubich [8, §2.3] that for temporally smooth data $a(\cdot, t) \in H^{1/2}(\Gamma)$ which vanish near $t = 0$, equation (1) has a unique smooth solution $u(\cdot, t) \in H^{-1/2}(\Gamma)$.

Many authors have considered full Galerkin approximations (in time and space) for (1) and related boundary integral equations (see [7] for a description of the relevant theory and a survey of the literature). This approach is based on a sound theoretical framework, and stability is proved via an energy identity. However, the method is hard to implement (it involves evaluating integrals over complicated subregions of $\Gamma \times \Gamma \times (0, T)$), and collocation schemes are more frequently used in practice (the article [1] contains an overview of different solution methods for problems such as (1)).

In a collocation approximation we suppose that (1) holds at N_S points $\boldsymbol{x}_\beta \in \Gamma$ and at time $t^n = n\Delta t$ for $n = 1, 2, \ldots$

$$a(\boldsymbol{x}_\beta, t^n) = \int_\Gamma \frac{u(\boldsymbol{x}', t^n - |\boldsymbol{x}' - \boldsymbol{x}_\beta|)}{|\boldsymbol{x}' - \boldsymbol{x}_\beta|}\, d\Gamma\,. \tag{2}$$

The unknown u is then approximated (in time and space), and the integral is approximated or evaluated to give

$$\underline{a}^n = \sum_{m=0}^{n-1} Q_m\, \underline{U}^{n-m} \tag{3}$$

in terms of (very sparse) matrices $Q_m \in \mathbb{R}^{N_S \times N_S}$, where $\underline{U}^m = \left\{U_\beta^m\right\}_\beta$ and $U_\beta^m \approx u(\boldsymbol{x}_\beta, t^m)$. Rearranging gives the time–stepping algorithm

$$Q_0\, \underline{U}^n = \underline{a}^n - \sum_{m=1}^{n-1} Q_m\, \underline{U}^{n-m}\,.$$

The sparsity of Q_0 means that solving this equation for the unknown $\underline{U}^n$ is straightforward. However, numerical instability is often a problem for schemes of this type, with the computed solution typically exhibiting oscillating instabilities that grow exponentially in the time–step [2, 3, 9]. Insight can be obtained by comparing the continuous Fourier transform of (1) at spatial frequency $\boldsymbol{\omega}$ with the discrete Fourier transform of (3) at the same frequency, when Γ is assumed to be a flat infinite surface (i.e. $\Gamma = \mathbb{R}^2$), and the points $\boldsymbol{x}_\beta$ form a uniform square mesh. It can be shown [2, 3] that numerical stability in this case can be characterised by Fourier coefficients $p_n(\boldsymbol{\omega})$: if $|p_n(\boldsymbol{\omega})|$ remains bounded with n for all $\boldsymbol{\omega} \in S_h \equiv [-\pi/h, \pi/h]^2$, where h is the (spatial) grid size, then the scheme is stable. Unfortunately there appears to be no obvious way to verify this condition analytically, and we resort to testing it numerically for many individual frequencies $\boldsymbol{\omega} \in S_h$ to determine the stability of a collocation scheme for (1).

We have derived three new collocation schemes for (1), based on a piecewise linear approximation for u in space, and a piecewise linear or piecewise constant approximation for u in time. The resulting integrals are either evaluated exactly [5], or transformed to polar coordinates (R, θ) via the local change of variables $R = |\boldsymbol{x}' - \boldsymbol{x}_\beta|$ in (2), and then approximated using the trapezoidal rule in R and (nearly) exact integration in θ [4]. Numerical evaluation of the Fourier coefficients $p_n(\boldsymbol{\omega})$ for these three schemes indicate that they are all stable for any value of the mesh ratio $\Delta t/h$ [4, 5].

If a is assumed to be sufficiently smooth, then it can be shown that such stable schemes for (1) are second order convergent when Γ is an infinite flat plate. That is, there exists a constant C such that $\|u^n - U^n\|_h \le C\, h^2$ as $h \to 0$ for $t^n \le T$, where $\|\cdot\|_h$ denotes the discrete L^2-norm. The proof involves using estimates on (spatial) Fourier transforms [10], and (temporal) $Z-$transform techniques due to Lubich [8].

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$H^1, \boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$-Conforming Projection-Based Interpolation in Three Dimensions

L. Demkowicz
(joint work with A. Buffa)

The talk is concerned with optimal p- and hp-estimates for the *Projection Based Interpolation* operators [2, 3, 5].

Given a master tetrahedral element T, and a sequence of polynomial spaces reproducing the standard grad-curl-div exact sequence at the discrete level, we consider a family of projection-based interpolation operators [2, 3, 5], $\Pi, \Pi^{curl}, \Pi^{div}, P$ that make the de Rham diagram commute. The projection-based interpolation operators are defined through a sequence of projections done on edge, face, and element levels. A compact definition of the interpolation operators follows.

H^1-conforming::

$$\begin{cases} u_1(a) = u(a) \\ \|u - \Pi u\|_{\epsilon,e} \to \min \\ \|\boldsymbol{\nabla}_f (u - \Pi u)\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ \|\boldsymbol{\nabla}(u - \Pi u)\|_{0,K} \to \min \end{cases}$$

$\boldsymbol{H}$(curl)-conforming::

$$\begin{cases} \displaystyle\int_e \boldsymbol{E}_t - \Pi^{curl} \boldsymbol{E}_t = 0 \\ \| \displaystyle\int (\boldsymbol{E}_t - \Pi^{curl} \boldsymbol{E}_t)\|_{0,\epsilon} \to \min \\ \begin{cases} \|\mathrm{curl}_f (\boldsymbol{E}_t - \Pi^{curl} \boldsymbol{E}_t)\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ (\boldsymbol{E}_t - \Pi^{curl} \boldsymbol{E}_t, \boldsymbol{\nabla}_f \phi)_{-\frac{1}{2}+\epsilon,f} = 0, \quad \forall \phi \in P^{p_f+1}_{-1} \end{cases} \\ \begin{cases} \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \Pi^{curl} \boldsymbol{E})\|_{0,T} \to \min \\ (\boldsymbol{E} - \Pi^{curl} \boldsymbol{E}, \boldsymbol{\nabla} \phi)_{0,T} = 0, \quad \forall \phi \in P^{p+1}_{p_f+1,p_e+1} \end{cases} \end{cases}$$

$\boldsymbol{H}$(div)-conforming::

$$\begin{cases} \|\boldsymbol{F}_n - \Pi^{div} \boldsymbol{F}_n\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ \begin{cases} \|\boldsymbol{\nabla} \circ (\boldsymbol{F} - \Pi^{div} \boldsymbol{F})\|_{0,T} \to \min \\ (\boldsymbol{F} - \Pi^{div} \boldsymbol{F}, \boldsymbol{\nabla} \times \boldsymbol{\phi})_{0,T} = 0, \quad \forall \boldsymbol{\phi} \in P^{p+1}_{p_f+1} \end{cases} \end{cases}$$

The task is to develop optimal error estimates with respect to polynomial degree p. As the operators are polynomial preserving, this in turn, leads to optimal hp-estimates as well. A major difficulty in deriving such estimates in 3D is the "loss of traces" at vertices. The trace space of $H^{1+\epsilon}(\mathrm{T})$ for a face f is $H^{\frac{1}{2}+\epsilon}(f)$, and $H^{\epsilon}(e)$ for an edge e, but we have no trace at a vertex v. In other words, $H^{1+\epsilon}(T)$

is not continously embedded in the space of continuous functions. This lack of regularity prevents the use of the reasoning used in 2D [3].

The key idea in deriving the estimates, is to compare the interpolation errors with two families of commuting projections, defined on element T and face f levels, see the commuting diagrams below.

$$\begin{array}{ccccccc}
\mathbb{R}\rightarrow H^{\frac{3}{2}+\epsilon}(T) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}^{\epsilon}(\mathrm{curl},T)\cap\boldsymbol{H}^{\frac{1}{2}+\epsilon}(T) & \xrightarrow{\boldsymbol{\nabla}\times} & \boldsymbol{H}^{\epsilon}(\mathrm{div},T) & \xrightarrow{\boldsymbol{\nabla}\circ} & L^2 \\
\downarrow id \quad P^1\downarrow\Pi & & P^{curl}\downarrow\Pi^{\mathrm{curl}} & & P^{div}\downarrow\Pi^{\mathrm{div}} & & \downarrow P \\
\mathbb{R}\rightarrow P^{p+1}_{p_e+1,p_f+1} & \xrightarrow{\boldsymbol{\nabla}} & \mathbf{P}^{p}_{p_e,p_f} & \xrightarrow{\boldsymbol{\nabla}\times} & \mathbf{P}^{p-1}_{p_f-1,p_e} & \xrightarrow{\boldsymbol{\nabla}\circ} & P^{p-2}
\end{array}$$

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^{\frac{1}{2}+\epsilon}(f) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl},f) & \xrightarrow{\boldsymbol{\nabla}\times} & H^{-\frac{1}{2}+\epsilon}(f) & \longrightarrow & \mathbf{0} \\
\downarrow id & & P^{\frac{1}{2}+\epsilon}\downarrow\Pi & & \downarrow\Pi^{\mathrm{curl}} & & \downarrow P & & \\
\mathbb{R} & \longrightarrow & P^{p_f+1}_{p_e+1} & \xrightarrow{\boldsymbol{\nabla}} & \mathbf{P}^{p_f}_{p_e} & \xrightarrow{\boldsymbol{\nabla}\times} & P^{p_f-2} & \longrightarrow & \mathbf{0}
\end{array}$$

Besides the commuting projection operators, instrumental in deriving the estimates are

- the existence of polynomial preserving, extension operators defined for a tetrahedral face [1],

$$\begin{array}{ccc}
H^{\frac{1}{2}+\epsilon}(f) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl},f) \\
E^{grad}\uparrow\downarrow Tr^{grad} & & E^{curl}\uparrow\downarrow Tr^{curl} \\
H^{\epsilon}(\partial f) & \xrightarrow{\frac{\partial}{\partial s}} & H^{-1+\epsilon}(\partial f)
\end{array}$$

$$\begin{array}{ccc}
P^{p}_{p_e}(f) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{P}^{p-1}_{p_e-1}(f) \\
E^{grad}\uparrow\downarrow Tr^{grad} & & E^{curl}\uparrow\downarrow Tr^{curl} \\
P^{p_e}(\partial f) & \xrightarrow{\frac{\partial}{\partial s}} & P^{p_e-1}(\partial f)
\end{array}$$

and on the element lecel (*conjectured*),

$$\begin{array}{ccccc} H^1(T) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}(\mathrm{curl},T) & \xrightarrow{\boldsymbol{\nabla}\times} & \boldsymbol{H}(\mathrm{div},T) \\ E^{grad}\uparrow\downarrow Tr^{grad} & & E^{curl}\uparrow\downarrow Tr^{curl} & & E^{div}\uparrow\downarrow Tr^{div} \\ H^{\frac{1}{2}}(\partial T) & \xrightarrow{\boldsymbol{\nabla}_{\partial T}} & \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial T) & \xrightarrow{\boldsymbol{\nabla}_{\partial T}\times} & H^{-\frac{1}{2}}(\partial T) \end{array}$$

$$\begin{array}{ccccc} P^p_{p_f,p_e}(T) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{P}^{p-1}_{p_f-1,p_e-1}(T) & \xrightarrow{\boldsymbol{\nabla}\times} & \boldsymbol{P}^{p-2}_{p_f-2}(T) \\ E^{grad}\uparrow\downarrow Tr^{grad} & & E^{curl}\uparrow\downarrow Tr^{curl} & & E^{div}\uparrow\downarrow Tr^{div} \\ P^{p_f}_{p_e}(\partial T) & \xrightarrow{\boldsymbol{\nabla}_{\partial T}} & \boldsymbol{P}^{p_f-1}_{p_e-1}(\partial T) & \xrightarrow{\boldsymbol{\nabla}_{\partial T}\times} & P^{p_f-2}(\partial T) \end{array}$$

and,

- the existence of polynomial preserving, right inverses G, K, D of grad, curl, and div operators,

$$\begin{array}{ccccc} H^{\frac{1}{2}+\epsilon}(f) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl},f) & \xrightarrow{\boldsymbol{\nabla}\times} & H^{-\frac{1}{2}+\epsilon}(f) \\ P^{p_f+1}_{p_e+1} & \xleftarrow{G} & \mathbf{P}^{p_f}_{p_e} & \xleftarrow{K} & P^{p_f-1} \end{array}$$

that are instrumental in proving discrete versions of Friedrichs inequalities.

Under the conjecture on the existence of polynomial preserving, extension operators, we can prove the following interpolation error estimates.

$$\begin{aligned} \|u-\Pi u\|_{1,T} &\le Cp^{-(r-\epsilon)}\|u\|_{1+r,T} \\ \|\boldsymbol{E}-\Pi^{curl}\boldsymbol{E}\|_{0,curl,T} &\le Cp^{-(r-\epsilon)}\|\boldsymbol{E}\|_{r,curl,T} \\ \|\boldsymbol{F}-\Pi^{div}\boldsymbol{F}\|_{0,div,T} &\le Cp^{-(r-\epsilon)}\|\boldsymbol{F}\|_{r,div,T} \end{aligned}$$

The interpolation theory is not crucial for the convergence analysis but it forms the backbone on fully automatic *hp*-adaptive startegies that deliver exponential convergence for both elliptic and Maxwell problems [4].

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High-Order Time Stepping Methods for Electromagnetics

Tobin A. Driscoll

High-order and spectral methods for spatial discretization have significant advantages in accuracy and efficiency over first- and second-order schemes [6, 9]. Such discretizations are most naturally paired with high-order methods in time, which yield similar benefits.

There are two aspects of discretizing Maxwell's equations in particular that lead to consideration of special time-stepping methods: *staggering* and *linear stiffness*. Staggering in spacetime was suggested by Yee for his famous second-order scheme [12]; it improves equal-cost accuracy by a factor of four and stable time step size by a factor of two over the related collocated method. The benefits of staggering are significantly increased at higher orders of accuracy [5, 8]. Linear stiffness arises from perfectly matched absorbing layers [1] that decay signals rapidly. Such decay can severely constrain the allowable time step size of a standard method. However, since the stiff aspect of the problem is linear, there are several strategies for restoring large time step sizes in high-order methods.

1. Staggering

The nature of Maxwell's equations allows **E** and **H** field components to be interlaced in time, as Yee showed in [12]. (The staggering can be done in space as well; the choices of whether to stagger in space and time may be made independently.) Other pure propagation problems, such as elastic waves, follow the same pattern [8].

We represent the semidiscrete evolution of a system eligible for time staggering as

$$u_t = f(t, v), \qquad v_t = g(t, u). \tag{1}$$

For instance, the second-order leapfrog in time used by Yee can be expressed as

$$v_{n+1/2} - v_{n-1/2} = \Delta t\, g(t_n, u_n), \qquad u_{n+1} - u_n = \Delta t\, f(t_{n+1/2}, v_{n+1/2}).$$

This method has an error constant that is 1/4 of that for the same method on an integer-level-only grid, and a *stability ordinate* (extent of the stability region along the imaginary axis that represents propagation) twice as large. We can increase the order of accuracy of leapfrog by using either more past steps or more interior stages, as was shown in [8].

Multistep methods are created in two variants, corresponding to whether one uses past values of the solution or of its derivative. (As with nonstaggered classical methods, trying to use both simultaneously leads to unstable methods.) We

TABLE 1. Comparison of staggered to classical nonstaggered methods.

Error constants

order	AB	RK	ABS	BDS	RKS
2	0.417	0.667	0.042	0.042	0.042
3	0.375	1.125	0.042	0.042	0.646
4	0.349	2.133	0.039	0.037	0.133
7	0.304	–	0.031	unstable	?
8	0.295	–	0.029	unstable	?

Stability ordinates

order	AB	RK	ABS	BDS	RKS
2	0	0	2.00	2.00	2.00
3	0.72	0.58	1.71	1.67	1.04
4	0.43	0.71	1.33	1.00	1.43
7	0.06	–	0.37	unstable	?
8	0.03	–	0.21	unstable	?

call these variants *staggered backward differentiation* (BDS) and *staggered Adams–Bashforth* (ABS), respectively, by analogy with the classical methods.[2] For example, the fourth-order ABS formula is

$$v_{n+1/2} - v_{n-1/2} = \frac{\Delta t}{24}\big(26u_n - 5u_{n-1} + 4u_{n-2} - u_{n-3}\big)$$

The coefficients and stability regions of ABS and BDS methods are cataloged in [8]. Only BDS methods of orders 2–4 and ABS methods of orders 2 and 3,4,7,8,11,12,..., have nontrivial stability ordinates. Past second order, all these methods are dissipative.

Staggered multistage methods are constructed on a more ad-hoc basis. The best fourth-order method known is [8]

$$\begin{aligned} d_1 &= \Delta t\, f(t_{n+1/2}, v_{n+1/2}) \\ d_2 &= \Delta t\, g(t_n, u_n) \\ d_3 &= \Delta t\, f(t_{n+1/2} - \Delta t, v_{n+1/2} - d_2) \\ d_4 &= \Delta t\, g(t_n + \Delta t, u_n + d_1) \\ d_5 &= \Delta t\, f(t_{n+1/2} + \Delta t, v_{n+1/2} + d_4) \\ u_{n+1} &= u_n + \tfrac{11}{12} d_1 + \tfrac{1}{24} d_3 + \tfrac{1}{24} d_5, \end{aligned}$$

with a symmetric formula for advancing v. This method requires four full function evaluations per full step, and it has a stability ordinate (normalized by the number of stages) of 1.43, compared to $1/\sqrt{2}$ for classical fourth-order Runge–Kutta.

Staggered methods are compared to their classical counterparts in Table 1. The table clearly demonstrates that the accuracy benefits of staggering increase with

[2]Note, however, that they are all explicit, unlike classical BD formulas.

the order of accuracy, and the stability ordinates controlling stable time step sizes are better by a factor of two or more.

2. Linear stiffness

Suppose a state $u(t)$ evolves according to

$$u_t = f(u) - Su, \tag{2}$$

where S is a linear operator and $\rho(S) \gg \rho(f')$, where ρ is spectral radius. Such systems arise quite frequently: the nonlinear Schrödinger, Korteweg–de Vries, Kuramoto–Sivashinsky, Gray–Scott, and Navier–Stokes equations are a few examples. In these cases the large spectrum of S is due to the presence of high-order spatial derivatives. In the Maxwellian case S represents the (perhaps nonphysical) conductive losses due to a perfectly matched layer [1].

The large spectrum of S creates an unacceptably strict stability condition on the time step size of explicit methods, but in most cases fully implicit methods are infeasible. A number of strategies have been devised to circumvent this difficulty at high orders of accuracy. They all work best when S is diagonal, as is the case in Maxwell's equations and in nonlinear PDEs under Fourier discretization.

One of the simplest ideas is the *integrating factor*, which transforms (2) to

$$\tfrac{d}{dt}(e^{St}u) = e^{St}f(u). \tag{3}$$

The evolution of $e^{St}u$ encounters no stiffness. However, the presence of the rapidly-varying exponential creates an accuracy penalty for a classical method. A better approach is to discretize (3) using a specialized method that explicitly incorporates the exponential. Such methods go by the name of *exact linear part* or *exponential time differencing*, are available in both multistep and multistage forms, and perform well in practice [2, 3, 10].

Another approach is to generalize the well known second-order Strang splitting, in which (2) over $[0, \Delta t]$ is replaced by

$$u_t = 2f(u) \text{ on } [0, \tfrac{1}{4}\Delta t]; \qquad u_t = 2Su \text{ on } [\tfrac{1}{4}\Delta t, \tfrac{3}{4}\Delta t];$$
$$u_t = 2f(u) \text{ on } [\tfrac{3}{4}\Delta t, \Delta t].$$

Fractional time step sequences can be found to give split-step methods of any even order [11, 13]. For fourth and sixth order, 7 and 15 substeps per step are needed, respectively, and in each case some steps must be negative, which makes these methods problematic for diffusion. However, they can be designed to conserve energy and symplecticness.

A third approach is to use *linearly implicit* methods, which marry explicit methods for the nonlinear term f and an implicit method for the stiff, linear—and hopefully diagonal—S. These have been shown to be quite effective when used in a heterogeneous discretization [4, 7], in which nonstiff components (e.g., free space propagation in Maxwell) are propagated by classical methods.

The best methods in each approach have mild or no stability restrictions and are orders of magnitude more efficient than their second-order counterparts. The

composite method of [4], in particular, is easy to implement and appears to be at least as effective as any other of this type for a variety of test problems.

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Nonreflecting Boundary Conditions for Computational Electromagnetics

Marcus J. Grote

(joint work with Wolfgang Bangerth, Joseph B. Keller and Christoph Kirsch)

For the numerical solution of wave scattering problems in unbounded media, a well-known approach is to enclose all obstacles, inhomogeneities and nonlinearities with an artificial boundary B. A boundary condition is then imposed on B, which leads to a numerically solvable boundary-value problem in a finite computational domain Ω. The boundary condition should be chosen such that the solution of the problem in Ω coincides with the restriction to Ω of the solution in the original unbounded region. Otherwise spurious reflections will appear at B, which will travel back into the interior computational region and spoil the numerical solution throughout Ω.

If the scatterer consists of several obstacles, which are well separated from each other, the use of a single artificial boundary to enclose the entire scattering region,

becomes too expensive. Instead it is preferable to enclose every sub-scatterer by a separate artificial boundary B_j. Then we seek an exact boundary condition on $B = \bigcup B_j$, where each B_j surrounds a single computational sub-domain Ω_j. This boundary condition must not only let outgoing waves leave Ω_j without spurious reflection from B_j, but also propagate the outgoing wave from Ω_j to all other sub-domains Ω_ℓ, which it may reenter subsequently. To derive such an exact boundary condition, an analytic expression for the solution everywhere in the exterior region is needed. Neither absorbing boundary conditions [1, 2], nor perfectly matched layers [3] provide us with such a representation. Instead, we shall use exact Dirichlet-to-Neumann (DtN) conditions in the time-harmonic case, or nonreflecting boundary conditions (NBC) in the time dependent case, which are both based on a Fourier series representation of the solution in the exterior region.

In the time-harmonic case, Dirichlet-to-Neumann (DtN) maps yield exact nonreflecting conditions and thus avoid spurious reflections from B. They are explicitly known for various equations or geometries [4–8]. Once combined with a finite difference or finite element discretization inside Ω, they lead to a highly accurate and efficient numerical scheme. Here we extend the DtN approach to multiple scattering problems, where every scatterer is enclosed by a separate artificial boundary B_j [9]. Thus Ω consists of multiple disjoint components, Ω_j. We derive an exact DtN boundary condition on B, the disjoint union of all B_j, by combining multiple contributions from purely outgoing wave fields. We present theoretical results that show existence and uniqueness of the solution to the boundary value problem in Ω, as well as numerical results that demonstrate the accuracy and efficiency of our method.

In the time-dependent case, *exact nonreflecting boundary conditions* have been derived for the wave equation [10, 11] and Maxwell's equations [12]. These boundary conditions are *local in time* and involve only first derivatives of the solution. Therefore, they are easy to use with standard finite difference or finite element methods. As the accurate simulation of waves at high frequencies or the detailed representation of small scale geometric features requires the use of adaptive mesh strategies, explicit time integrators become prohibitively expensive because of the stringent CFL condition. Instead, implicit methods, such as Crank-Nicolson, are typically used, yet they require the solution of a large linear system of equations at every time step. Due to the nonreflecting boundary condition, this linear system is no longer symmetric, unlike the situation in bounded domains. However, it is possible to reformulate the discretized equations by decoupling the additional unknowns needed on the artificial boundary from the interior unknowns [13]. As a consequence the symmetry and positive definiteness of the linear system are restored, while the additional computational effort due to the nonreflecting boundary condition becomes negligible.

For time-dependent multiple scattering problems the use of a single artificial boundary surrounding all scatterers involved also becomes prohibitively expensive in memory requirement. Instead, it is judicious to enclose each scatterer within a single separate computational domain. Clearly waves that leave a certain domain,

Ω_1, will impinge upon a different domain, Ω_2, at later times; hence they are no longer purely outgoing waves. To transfer the time-retarded information from Ω_1 to Ω_2 an analytical representation of the solution in the unbounded medium becomes necessary. Again, such an analytical representation [14] is inherent to the exact nonreflecting boundary conditions described above.

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Nodal and Edge Finite Element Discretization of Maxwell's Equations

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(joint work with Barbara Kaltenbacher and Stefan Reitzinger)

The numerical computation of electromagnetic fields is performed for more then 20 years. For the domain discretization nodal as well as edge finite elements have been used successfully. Nevertheless, in the last years inaccurate results at material parameter interfaces in the magnetostatic as well as in the eddy current case,

and, spurious modes in Maxwell's eigenvalue problems have been reported. In this paper we will concentrate on the problems related with material parameter interfaces, where the magnetic reluctivity changes its value abruptly. We will describe a simple to implement method following the ideas reported in [3], which produces correct results. For the high frequency case we refer to [3].

The electromagnetic field is fully described by Maxwell's equations [9]. Restricting the problem class to the quasi-static (eddy current) case, we arrive at the following partial differential equation for the magnetic vector potential $\mathbf{A}$

$$\gamma \frac{\partial \mathbf{A}}{\partial t} + \nabla \times \nu \nabla \times \mathbf{A} = \mathbf{J}_{\mathrm{i}} \tag{1}$$

with boundary condition $\mathbf{n} \times \mathbf{A} = 0$ and $\mathbf{n}$ the unit outward normal vector. In (1) $\mathbf{J}_{\mathrm{i}}$ denotes the impressed current density, ν the magnetic reluctivity and γ the electric conductivity. Furtheron, the following interface conditions have to be fulfilled

$$[\mathbf{A} \times \mathbf{n}] = 0 \; ; \quad [\nu \; \mathbf{n} \times \nabla \times \mathbf{A}] = 0 \; ; \quad \left[\gamma \frac{\partial \mathbf{A}}{\partial t}\right] = 0 \tag{2}$$

with $[\mathbf{Z}] = \mathbf{Z}_{\mathrm{right}} - \mathbf{Z}_{\mathrm{left}}$. For further discussions let Ω be a bounded single connected convex domain with boundary $\partial\Omega = \Gamma$. Therewith, the variational formulation for (1) in the function space

$$\mathbf{H}_0^{\Sigma}(curl) = \{\mathbf{u} \in (L^2(\Omega))^3 \,|\nabla \times \; \mathbf{u} \in (L^2(\Omega))^3, \; \mathbf{u} \times \mathbf{n}|_{\Gamma} = 0, \; [\mathbf{n} \times \mathbf{u}]|_{\Sigma} = 0\} \tag{3}$$

reads as follows: Find $\mathbf{A} \in \mathbf{H}_0^{\Sigma}(curl)$ such that

$$\begin{aligned} \int_{\Omega} \gamma \mathbf{A}' \cdot \frac{\partial \mathbf{A}}{\partial t} \, d\Omega \quad &+ \quad \int_{\Omega} \nabla \times \mathbf{A}' \cdot \nu \, \nabla \times \mathbf{A} \; d\Omega \\ &= \quad \int_{\Omega} \mathbf{A}' \cdot \mathbf{J}_{\mathrm{i}} \, d\Omega \end{aligned} \tag{4}$$

for any $\mathbf{A}' \in \mathbf{H}_0^{\Sigma}(curl)$ is fulfilled.

It is well known, that an edge FE-discretization of (4) is $\mathbf{H}_0(curl)$-conform [6]. Nevertheless, the solution of the algebraic system requires special care in order to obtain an optimal multigrid solver (see e.g. [2], [5]). We suggest to add a fictive electric conductivity γ' to regions with zero electric conductivity to obtain a variational form, which is elliptic [8]. Of course, this fictive conductivity γ' has to be chosen small as compared to the reluctivity of the material. The proof of convergence even in the case of $\gamma' \to 0$ is given in [7].

For the application of nodal finite elements, we have to perform additional steps. According to [4] as well as [1] we decompose the magnetic vector potential $\mathbf{A}$ by

$$\mathbf{A} = \mathbf{w} + \nabla\phi \, , \quad \nabla \cdot \mathbf{w} = 0 \, , \tag{5}$$

with $(\mathbf{w}, \phi) \in \left((H_{\mathrm{T}}^1(\Omega))^3, H_0^1(\Omega)\right)$ and Ω being a convex domain. The same decomposition is done for the test function $\mathbf{A}' = \mathbf{v} + \nabla\psi$. Since we have to guarantee $\nabla \cdot \mathbf{w} = 0$, we do so by adding the penalty term $\int_{\Omega} \nu \; (\nabla \cdot \mathbf{v} \, \nabla \cdot \mathbf{w}) \; d\Omega$ to the

variational formulation. Therewith, the variational formulation can be stated as follows: Find $(\mathbf{w}, \phi) \in ((H^1_{\mathrm{T}}(\Omega))^3,\, H^1_0(\Omega))$ such that

$$\int_\Omega \nu \nabla \times \mathbf{v} \cdot \nabla \times \mathbf{w}\; d\Omega + \int_\Omega \nu\; \nabla \cdot \mathbf{v} \nabla \cdot \mathbf{w}\, d\Omega$$

$$+ \int_\Omega \gamma(\mathbf{v} + \nabla\psi) \cdot \frac{\partial}{\partial t}(\mathbf{w} + \nabla\phi)\, d\Omega = \int_\Omega \mathbf{J}_{\mathrm{i}} \cdot \mathbf{v}\, d\Omega\,. \tag{6}$$

for any $(\mathbf{v},\, \psi) \in ((H^1_{\mathrm{T}}(\Omega)^3,\, H^1_0(\Omega))$. Now, since for most practical eddy current problems the domain Ω is convex, the discretization of the above variational formulation with nodal finite elements will result in correct results. However, the question arises, if a domain Ω including subdomains of different material parameters (magnetic reluctivity or/and electric conductivity), is really convex? Let us consider the case of a ferromagnetic cube embedded in air (see Fig. 1). Assuming

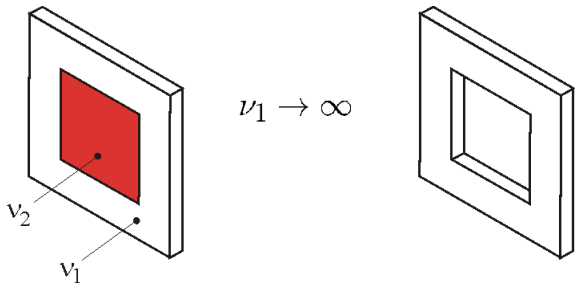

FIGURE 1. Ferromagnetic cube in air

the case $\nu_1 \to \infty$ (of course the limit of ν_1 is equal to $1/\mu_0$ with μ_0 being the permeability in vacuum), we arrive at a non-convex domain. Now according to [3], it is known, that for non-convex domains the discretization with nodal finite elements produces wrong solutions due to the non-density of smooth fields. In [3] the authors could proof, that by introducing a special weighting function inside the divergence integral, nodal finite elements can yet be used for the approximation. Therewith, the second term in the variational formulation (6) has to be changed to

$$\int_\Omega \nu\; \nabla \cdot \mathbf{v} \nabla \cdot \mathbf{w}\, d\Omega \to \int_\Omega \nu\; s\; \nabla \cdot \mathbf{v} \nabla \cdot \mathbf{w}\, d\Omega \tag{7}$$

with

$$s = \prod_{a \in \mathsf{Q}} r_a^\alpha\,. \tag{8}$$

In (8) Q denotes the set of all reentrant corners, r_a the distance to each reentrant corner, and α an exponent. We have implemented this idea in a simple way by setting the weighting function s to zero for finite elements near each interface of two subdomains with different material parameters.

The correctness of the weighted variational formulation have been demonstrated by numerical test cases (iron cube, thin iron plate) as well as industrial applications (electric power transformer, electromagnetic motor, magnetic resonance imaging scanner).

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The Hurewicz Map Distinguishes Intuitive vs. Computable Topological Aspects of Computational Electromagnetics

Robert Kotiuga

1. Abstract of talk

Answers to intuitive topological problems, such as checking if a space is contractible, are easily characterized in terms of homotopy groups. However, in four or more dimensions, such a characterization is provably computationally intractable. On the other hand, cohomology theory may not be intuitive, but it does provide a formal connection between Maxwell's equations and the lumped parameters occurring in Kirchhoff's laws. Furthermore, by exploiting sparse matrix algorithms and the Smith normal form, cohomological information is efficiently extracted from the data structures used in finite element analysis. A natural question then arises: Do engineers need to go beyond the linear algebra and sparse matrix techniques associated with homology calculations? It turns out that there are inverse problems involving near force-free magnetic fields where the conjectured characterization of the space of solutions, involves computationally intractable topological invariants such as the Thurston norm [4]. For this reason, it is imperative to investigate algebraic structures found in the data structures of finite element analysis, and

which yield topological insights not deducible from cohomological considerations alone.

The Hurewicz map takes representatives of generators of homotopy groups to their homology classes and is a well-defined map from homotopy groups to homology groups. In this sense, it provides a natural framework for comparing the intuitive but intractable aspects of homotopy theory with the computable but less intuitive aspects of homology theory. In particular, thorugh the use of the Hurewicz map, several important identifications can be made:

(1) The lower central series of the fundamental group is related to certain Massey products in the cohomology ring.
(2) The differential graded Lie algebras of rational homotopy theory are related to the minimal models of the cohomology ring.
(3) By Hopf's theorem, the cokernel of the second homology group under the Hurewicz map is characterized in terms of a presentation of the fundamental group.

The workshop talk concretely developed the relevance of these aspects of the Hurewicz map in the context of computational electromagnetics.

2. Putting my talk in the context of my previous work

Though originally developed as a natural outgrowth of multivariable calculus, algebraic topology and differential forms have become an essential tool used to formulate many basic laws of physics. Through my research this area of mathematics has found a natural application to many areas of electrical engineering and computational electromagnetics. A strong theme in my research is the identification of geometric and topological aspects, which shed light on dimensional dependence in the complexity of engineering problems and their algorithmic solution. This should be evident from the other publications I have selected to list below [7]-[15]. Much of my earlier work dealing with finite element analysis of electromagnetic fields and magnetic scalar potentials is summarized in the MSRI monograph coauthored with my Ph. D. student, Paul Gross [2].

If one were to seek a more mainstream characterization of my research interests, I could probably describe them in terms of the research interests listed on my resume:

- Electromagnetics;
- Numerical methods for 3-d vector fields;
- Whitney forms, the finite element method and the analysis of algorithms,
- Cuts for magnetic scalar potentials, formulation of eddy-current problems,
- Variational and symplectic techniques,
- Micromagnetics; nanoscale magnetics,
- Geometric inverse problems,
- Helicity functionals and near force-free magnetic fields; contact geometry,

My most recent research deals with how electromagnetic force constraints give rise to topological structures necessarily characterized by nonabelian algebraic

structures [1], [4]. This reseach is interesting both in terms of applications, and in defining the data structures which are useful for the finite element analysis of electromagnetic fields. The abstract of my workshop presentation above, is an attempt to get a handle on the latter aspects.

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Inverse Obstacle Scattering for Time-Harmonic Electromagnetic Waves

Rainer Kress

This presentation provides a survey on some recent developments in the theory and numerical solution of time-harmonic inverse scattering problems. Roughly speaking, one can distinguish two groups of inverse problems in this field, namely the inverse medium problem and the inverse obstacle problem. For time reasons, only inverse obstacle scattering will be covered. However, most of the ideas that are presented for inverse obstacle scattering have counter parts in inverse medium scattering. After formulating the inverse problem, the issue of uniqueness, that is, identifyability will be addressed. The uniqueness question is of its own mathematical interest and also interrelates with some of the more recently developed reconstruction algorithms. By considering one or two of its representatives the basic ideas of three groups of methods will be outlined, namely decomposition methods, iterative methods and sampling and probe methods. For illustration a couple of numerical examples will be included.

Consider the scattering of a time-harmonic electromagnetic plane wave E^i, H^i from an impenetrable scatterer described by a bounded domain D in $\Re^3$ either with a perfect conductor or an impedance boundary condition. The inverse obstacle scattering problem consists of finding the shape and location of D from the knowledge of the electric far field pattern E_∞ of the scattered wave E^s, H^s for one or several incident plane waves. The corresponding uniqueness result due to Kirsch and Kress [17] (see also [6]) confirms that the domain D and the boundary condition are uniquely determined by the far field pattern for infinitely many incident plane waves. The main idea of the proof is to exploit the fact that for scattering of electric dipole fields the scattered wave develops singularities when the source and observation points approach the boundary. Uniqueness for one incident plane wave remains a challenging open problem. Partial results were recently obtained for scattering from balls [20] and polyhedral scatterers [1].

Decomposition methods, in principle, separate the inverse problem into an ill-posed linear problem to reconstruct the scattered wave E^s, H^s from its far field pattern E_∞ and a nonlinear problem for the subsequent determination of the boundary ∂D of the scatterer from the boundary condition. These methods do not require the solution of the forward problem and some of them perform well without a priori information on the geometry of the obstacle. A typical representative of this approach is the potential method of Kirsch and Kress (see [9, 16]).

Iteration methods interpret the inverse obstacle scattering problem as a nonlinear ill-posed operator equation $A(\partial D) = E_\infty$ and apply iterative schemes such as regularized Newton type, Landweber or conjugate gradient methods for its solution. Here, A denotes the operator that, for a fixed incident field, maps the boundary ∂D of the scatterer onto the far field pattern of the scattered wave. The theoretical foundation for this approach requires to establish the differentiability of the operator A with respect to the boundary and to explicitly characterize

the derivative. For the perfect conductor boundary condition this was done by Potthast [22] via integral equation methods and by Kress [19] via a factorization formula. The latter method was recently extended to the impedance boundary condition by Haddar and Kress [10].

For details on the numerical implementation, among others, see [6, 8, 11, 14, 18]. The numerical examples provide amble evidence that iterative methods, in particular Newton iterations, yield very good reconstructions. However, they require the solution of the corresponding forward problem in each iteration step and a priori information on the geometry of the obstacle. Furthermore, although progress has been made through the work of Hohage [12] and Potthast [24], the convergence issue is not yet satisfactorily settled. A hybrid of Newton type iterations and decomposition methods was suggested in [21] and successfully tested for two-dimensional examples.

The main idea of the more recently developed so-called *sampling and probe methods* is to develop a criterium in terms of the behaviour of some ill-posed linear integral equation that decides on whether a point z lies inside or outside the scatterer D. Then the criterium is evaluated numerically for a grid of points to visualize the unknown scatterer. As opposed to the two previous types of methods that, in principle, only need the far field pattern for one incident direction, the sampling and probe methods need the far field pattern for all incident and observation directions and polarizations. However, as their main advantage they perform extremely well without any a priori information on the geometry. The linear sampling method as developed in acoustic scattering by Colton and Kirsch [5] has as its central piece the far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$ on the space of tangential L^2 fields on the unit sphere Ω. This operator is defined as an integral operator with the kernel given by the far field pattern $E_\infty(\hat{x}, d)$ for all observation directions $\hat{x} \in \Omega$ and all incident directions d. With the explicitly available far field pattern $E^i_{\infty,dip}(\cdot\,, z)p$ of the field of an electric dipole with polarization p located at the point z the linear sampling method is based on the ill-posed linear integral equation $Fg(\cdot\,, z) = E^i_{\infty,dip}(\cdot\,, z)p$. Although, this integral equation, in general, is not solvable, it can be approximately solved in the sense that for every $p \in \Re^3$, $\varepsilon > 0$, and $z \in D$ there exists $g(\cdot\,, z) \in L_t^2(\Omega)$ such that $\|Fg(\cdot\,, z) - E^i_{\infty,dip}(\cdot\,, z)p\|_{L^2(\Omega)} \leq \varepsilon$ and $\|g(\cdot\,, z)\|_{L^2(\Omega)} \to \infty$ as $z \to \partial D$. In the numerical implementation the far field integral equation is solved by Tikhonov regularization via Morozov's discrepancy principle and then ∂D is visualized through the points z where $\|g(\cdot\,, z)\|_{L^2(\Omega)}$ becomes large. For details on the theoretical foundation and numerical examples see [3, 4, 19].

A remaining gap in the theoretical foundation of the linear sampling method, namely, the question why the implementation via Tikhonov and Morozov actually picks the approximation g that is predicted by the above theoretical result was closed in acoustics through a recent contribution by Arens [2]. However, the gap remains open in electromagnetics, since Aren's analysis does not yet cover this case.

The factorization method may be considered as a variation of the linear sampling method in the sense that it replaces F in the far field equation by $(F^*F)^{1/4}$, that is, it is based on the equation $(F^*F)^{1/4}g(\cdot\,,z) = E^i_{\infty,dip}(\cdot\,,z)p$. This equation is more satisfying since it is to be expected that it is solvable if and only if $z \in D$. The corresponding result in acoustics is valid as shown in a pioneering paper by Kirsch [15]. However it is open for electromagnetics. The numerical implementation of the factorization is similar to that of the linear sampling method. The procedure is known as factorization method, since it relies on a factorization of the far field operator.

The linear sampling method and the factorization method may be viewed as dual to the uniqueness proof of Kirsch and Kress, since, in principle, their foundation is based on letting source points of electric dipole fields approach the boundary from inside of D whereas in the uniqueness proof the sourec points approch the boundary from outside of D. The latter idea is mimiced in the point source and singular source methods of Potthast [7, 23] and the probe method of Ikehata [13].

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A New View on Collocation

S. Kurz
(joint work with O. Rain, V. Rischmüller, S. Rjasanow)

In recent years, a remarkable amount of papers has been published that treat continuous and discrete electromagnetics in terms of differential forms (DFs). For a good account on this topic, see, e.g., [2] and [7]. However, most of these papers focus on (generalised) finite difference and finite element methods. There are only rare papers that deal with the boundary element method [1, 3, 6, 11].

The aim of this talk is to show how the integral equations of electromagnetics can be expressed in the language of DFs. The integral kernels become double forms [5]. These are DFs in one space with coefficients that are DFs in another space, or DF-valued DFs [12]. We restrict ourselves to the static case. Similar schemes can be derived for time dependent problems. The formulation in terms of DFs enables a uniform treatment of electrostatics (Kirchhoff representation formula) and magnetostatics (Stratton-Chu representation formula).

Since DFs possess discrete counterparts, known as Whitney forms, such schemes lend themselves naturally to discretisation. As an example, a boundary integral equation for the double curl operator is considered. This equation has been investigated in a variational setting in [8]. A detailed discussion of the Sobolev spaces being involved can be found in

[4, 10].

In the present contribution we wish to highlight an alternative approach. The proposed discretisation scheme generalises the well-known collocation technique by using de Rham maps on dual grid systems [6, 11]. Depending on the integral

operator to be discretised, the 1-form valued residual is forced to be zero either over the 1-chains of the primal or the dual grid. The viability of the method will be demonstrated by means of a numerical example, where a sphere is immersed in the field of a circular current loop.

For an extended version of this contribution see [9].

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Computation of Maxwell Eigenvalues with Exponential Rates of Convergence

Paul Ledger

Our interest in this work lies in the accurate calculation of Maxwell eigenvalues for closed cavities. The results are important for many applications such as the design of microwave devices and charged particle accelerators. The solution of such problems remains far from trivial due to the fact that realistic cavities often contain multi–materials, have small scale feature and contain many sharp corners, which all give rise to highly singular eigenfunctions.

Using a nodal finite element basis for each component of the electric field is known to be inappropriate, as the resulting solution is polluted by spurious modes. Instead, we choose to solve these problems using the $\vec{H}(\mathrm{curl})$ conforming finite elements that were first introduced by Nédélec [1]. Using such elements is known to overcome the problems of spurious modes and allow the easy incorporation of material interfaces and boundary conditions.

We follow a finite element approach which allows for arbitrary increases in polynomial order p. In particular we use the recent hierarchic basis of Ainsworth and Coyle [2, 3] with both p and h (mesh) refinements. Indeed, when the h and p refinements are correctly combined, we are able to observe the theoretically predicted exponential rates of convergence for the Maxwell eigenvalues. Numerical examples show that the exponential rates of convergence can be obtained in practice for a series of benchmark problems discretised with tetrahedral meshes in three–dimensions [4, 5].

Recent extensions include the application of *hp* finite elements to axisymmetric problems with rotational symmetry [6]. For such cases it is possible to reduce a three-dimensional problem to a sequence of two–dimensional problems. Again, exponential rates of convergence have been observed for the computed eigenvalues of closed cavities.

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Computational Shape and Topology Optimization with Applications to 3–Dimensional Magnetostatics

D. Lukáš

(joint work with U. Langer, E. Lindner, R. Stainko, J. Pištora)

In the talk we mainly discussed computational aspects of shape and topology optimization governed with 3–dimensional linear and nonlinear magnetostatics,

respectively. This is covered in the speaker's thesis [2] and in [3]. The acknowledgment is due to the Special Research Initiative SFB F013 "Numerical and symbolic scientific computing", subproject "Multilevel solvers for large scale discretized optimization problems" at the University of Linz, Austria. The speaker especially thanks to Dr. Joachim Schöberl for his kind software support during the week in Oberwolfach.

The presentation started with a motivation from physics. We described electromagnets that are used for measurements of magnetooptic effects on thin layers. We aim at designing their optimal topology and shape so that in the area where the measurements take place the magnetic field is as constant as possible and above a prescribed magnitude. Throughout the presentation we instantiate the ideas for this application.

Next, we recalled an abstract optimal shape design problem, its finite element approximation and we discussed the existence and convergence issues following the theory in [1], which is based on the compactness and continuity arguments. We optimize the interface between the air and ferromagnetics, rather than the boundary of the computational domain as usual in mechanics. We pointed out a drawback that on fine discretizations the non–design grid nodes cannot follow large perturbations of the design shape. The mapping from the shape to the grid nodes is carried over an artificial linear elasticity problem with the prescribed displacements along the design shape interface. Then, we presented the algebraic approach to the shape sensitivity analysis and its efficient software implementation, see [5]. The user is only supposed to dessignate the shape and to code the objective in terms of the state solution. The underlying finite element code provides the sensitivity of element contributions to the bilinear form with respect to the grid displacements. The optimization package is now to be included into the NgSolve, see [7].

Further, we presented numerical results for both 2– and 3–dimensional shape optimization problems. After the 2d optimized design the electromagnets were manufactured and the measurements of the magnetic field showed the 4.5–times improvements in terms of the objective functional, compared to the initial design.

We presented a multilevel optimization approach. Here, hierarchies of discretizations of both the state and design space are considered. We begin with the optimization on a coarse discretization for only two design parameters. The multilevel algorithm then proceeds such that the optimized shape is used on a finer level as the initial guess. Moreover, we prolonged the 2d coarse optimized shape to the third dimension and used that as the initial guess in the multilevel 3d optimization. In the 2d case for 7 design and 12.000 state unknowns we achieved the speedup 4.5. In the 3d case for 12 design and 30.000 state unknowns the speedup was more than 10–times.

Finally, we formulated a corresponding topology optimization problem governed by nonlinear magnetostatics. In the 2d case we solved for 3.920 design variables with 4.832 state ones and the computation typically proceeded within 8 steepest descent iterations and 8 nested nonlinear state Newton iterations. Just during the week in Oberwolfach we managed to run 3d topology optimization governed by

linear magnetostatics and we were able to solve problems of up to 1 million design unknowns in hours. The optimal design is close to a sphere around the area where the constant magnetic field is required. The talk was ended with the outlook concerned on using nonlinear multigrid techniques, all–at–once optimization approach and preconditioning techniques for the arising KKT–systems and adaptivity with respect to the cost functional.

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Fast Time Domain Integral Equation Solvers

Eric Michielssen

(joint work with Mingyu Lu and Balasubramaniam Shanke)

Efficient schemes for analyzing transient electromagnetic wave scattering and short-pulse radiation phenomena are important in disciplines ranging from electromagnetics to acoustics, geophysics, and elastodynamics. The analysis of transient scattering from perfectly conducting as well as potentially inhomogeneous penetrable bodies often is effected using marching on in time (MOT) based time domain integral equation (TDIE) methods.

A typical TDIE solver for analyzing transient electromagnetic scattering from perfect electrically conducting (PEC) surfaces residing in unbounded 3D lossless environments operates as follows. The extinction theorem states that the electromagnetic field anywhere in space can be evaluated upon specification of the incident field and the total magnetic field, or, equivalently, the current, on the scatterer's surface. By enforcing the tangential component of the total electric field along the surface to vanish, the surface current can be related to the incident field through an electric field TDIE. To solve this TDIE by MOT methods, the surface current is represented in terms of N_s spatial basis functions with unknown

amplitudes at N_t time steps. Then, the instantaneous total electric field is expressed as a superposition of the incident and scattered fields. The evaluation of the latter requires the computation of a retarded time boundary integral over the basis functions representing the field. This procedure leads to a system of equations that can be solved for the coefficients of the basis functions representing the surface field at a given time step. Depending on the choice of the time step size, the basis functions, and the testing procedure, the matrix to be inverted may be diagonal or sparse, yielding explicit or implicit time stepping schemes, respectively. It has been empirically shown that implicitness and accurate evaluation of retarded time boundary integrals contribute to the stability of a MOT scheme. Unfortunately, the overall computational cost of this procedure scales as $O(N_t N_s^2)$, which prevents the application of classical MOT-based TDIE solvers to the study of practical, real-world problems. It is noted that the above cost estimate is linear in only because the 3D lossless medium Green propagator is local in time. When the above procedure is applied to the study of scattering from 2D objects, or surfaces embedded in dissipative or structured (e.g., layered) environments, then the computational complexity would scale as $O(N_t^2 N_s^2)$, as Green propagators in such media all have a wake.

The recently introduced plane wave time-domain (PWTD) algorithm permits the efficient evaluation of transient wave fields generated by temporally bandlimited sources. The original PWTD scheme targeted sources residing in 3D homogeneous and lossless backgrounds [1]. This PWTD scheme constitutes the extension of the frequency domain (Helmholtz equation) fast multipole method [2, 3] to the time domain (wave equation) and, when coupled to the above described MOT-based TDIE solvers, reduces their computational complexity to $O(N_t N_s \log^2 N_s)$. To date, this PWTD scheme has been successfully used to construct (i) fast marching schemes for solving time domain integral equations [4] and (ii) fast boundary kernels for augmenting finite difference time domain simulators [5]. It even has been extended to 2D [6], layered [7], and dissipative environments [8] with only minor changes in the resulting computational complexity estimates. All PWTD schemes express wave fields as a superposition of plane waves. The evolution of these plane waves is either known analytically, or governed by one-dimensional wave equations. In 2D and in layered environments, a Hilbert transform acts on the plane wave superposition for it to yield the correct transient field. At present, spectral schemes have been developed that control the accuracy of each and every step in these various PWTD schemes; as a result, they can be hybridized with classical MOT-based TDIE solvers, thereby greatly improving their computational complexity and memory requirements, without affecting their accuracy. At present, PWTD-accelerated MOT-based TDIE solvers have been applied to the analysis of scattering and radiation from conducting [4, 9], resistive and impedance boundary condition surfaces [10], penetrable lossless [11], lossy [12], and dispersive volumes [13], and the analysis of hybrid lumped-distributed circuits [14, 15] involving up to hundreds of thousands of spatial unknowns, all this for thousands of time steps.

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The Approximation of the Maxwell Eigenvalue Problem using a Least-Squares Method

Joseph E. Pasciak

(joint work with James H. Bramble and Tsanio V. Kolev)

In this talk, I consider the eigenvalue problem problem associated with Maxwell's equations. These equations can, for example, be used to determine the frequencies which will propagate through a medium such as a waveguide or photonic crystal [6, 11, 16].

Although two dimensional versions of Maxwell's eigenvalue problem often result in eigenvalue problems involving the Laplacian, three dimensional problems are significantly more complicated as they result in an eigenvalue problem involving curl-curl, an operator which is not elliptic. Accordingly, the inverse is no longer compact leading to a much more complicated analysis. However, as we shall see, a compact "pseudo" inverse can be constructed which has the same nonzero eigenvectors.

One of the more popular approaches for approximating Maxwell's eigenvalue is based on using curl-conforming spaces such as those developed by Nedelec (cf. [18, 19]). In such a method one looks for solutions to the problem in $\boldsymbol{H}(\mathbf{curl})$, the space of vector function which, along with their curls, are in $\boldsymbol{L}^2(\Omega)$. Analysis of the eigenvalue problem using these spaces either involves proving collective compactness [14, 17] or proving convergence in norm [1, 2].

Early engineering approximations to these equations were often attempted using conforming finite element spaces [3]. These were known to have problems due to low regularity solutions and multiple valued potentials [10, 12, 15]. Recently, new methods for dealing with these problems have been proposed [7, 8, 20]. The methods of [8] depend on weighted functional with weights depending on the strength of the singularities at corners and edges. In [20], discontinuous Galerkin methods are proposed.

The approach which we take in this talk is to first relate the problem to a block system involving the solution of div-curl systems. These div-curl systems are formulated as variational problems following [5] where the solution is posed in $\boldsymbol{L}^2(\Omega)$ and the (components of the) test functions are in various subspaces of the Sobolev space $H^1(\Omega)$. This results in a very weak formulation of the div-curl problem where the data can reside in a negative norm space, e.g., in the dual of the test spaces. That the test functions are in $H^1(\Omega)$ is a critical attribute of the method which we take advantage of in our subsequent analysis of the Maxwell eigenvalue problem. Indeed, this leads to solution operators for the div-curl problem which are bounded from $H^{-1}(\Omega)$ into $L^2(\Omega)$ in the continuous as well as the discrete case. Since the approximation is based in $\boldsymbol{L}^2(\Omega)$, our approximation subspaces can be very simple, for example, we can use discontinuous functions at the material interfaces where the solutions jump while using C^0 elements in the interior where the solution is smooth.

In this talk, I show how this variational form of the div-curl system can be used to develop a stable approximation to the Maxwell's eigenvalue problem. The eigenfunctions with non-zero eigenvalues are also eigenfunctions of a block compact skew-Hermitian problem where the blocks correspond to div-curl problems. We use the div-curl approximation to derive a sequence of approximation operators which converge in norm to the above mentioned compact operator.

Actual three dimensional applications necessarily contain large numbers of unknowns (on the order of millions). Such a large number of unknowns result from complicated device geometry and the mesh refinement necessary for resolving singular behavior in the solutions. Since the systems are too large for conventional direct eigensolvers, the eigenvalues must be computed iteratively. To obtain a system which is more amenable to iterative computation, we show that the original eigenpairs can be computed from those of a compact symmetric real operator. This system can be approximated in norm by the discrete operator for one div-curl system and its adjoint and results in a symmetric discrete eigenvalue problem. The development of effective iterative techniques for computing the eigenvalues of large symmetric problems has been the subject of intensive research in the past two decades, e.g., [4, 9, 13]. These methods are more efficient and robust than those developed for non-symmetric and/or indefinite systems. Thus, the reformulation of the problem as a symmetric real system represents a significant computational advantage.

Theorems on the rate of convergence of the discrete eigenvalues are given and supported by computational experiments.

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Discontinuous Galerkin Methods for Maxwell's Equations

Ilaria Perugia

In recent years, there has been considerable interest in nonconforming finite element methods that are based on discontinuous piecewise polynomial approximation spaces and on local (element-by-element) variational formulations. Such approaches are referred to as discontinuous Galerkin (DG) methods. The main advantages of these methods lie in their ability to treat a wide range of problems within the same unified framework, and their great flexibility in the mesh-design. Indeed, DG methods can naturally handle non-matching grids and non-uniform, even anisotropic, polynomial approximation degrees; for this reason, DG methods are particularly suited within *hp*-adaptive procedures and for dealing with multi-material problems. In the following, a short survey on DG methods for the approximation of Maxwell's equation is presented.

The original DG method was introduced in [15] for the neutron transport equation. It is constructed by multiplying the equation by smooth test functions, integrating by parts element-by-element on a given mesh, replacing trial and test functions by discontinuous piecewise polynomial functions, and replacing interelement traces by *numerical fluxes*. Development of DG techniques in the context of conservation laws lead to the introduction of the Runge-Kutta (RK) DG method in [3], a high-order method based on a spatial approximation by means of discontinuous polynomials of order k with upwind *numerical fluxes*, and a special $(k+1)$-stage RK method for time-stepping, in combination with *slope limiters* in the case of nonlinear problems (see [4] for a review).

In the context of Maxwell's equations, RKDG-type methods have been applied to the problem in time-domain

$$\varepsilon_r \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}, \qquad \mu_r \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E},$$

written in conservation form:

$$Q(x)\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{q}) = \mathbf{S}$$

with $\mathbf{q} = [\mathbf{E}, \mathbf{H}]^T$, $\mathbf{F}_i(\mathbf{q}) = [-\mathbf{e}_i \times \mathbf{H}, \mathbf{e}_i \times \mathbf{E}]^T$, and $Q = \mathrm{diag}(\varepsilon_r, \varepsilon_r, \varepsilon_r, \mu_r, \mu_r, \mu_r)$. The use of DG methods in this context is motivated by the possibility of using unstructured, even non-matching, meshes for dealing with complex geometries, by the simplicity of incorporating spatially varying coefficients, and by the possibility of constructing high order methods by simply choosing basis functions; moreover, the mass matrices are diagonal (or block diagonal), which is advantageous for time-stepping.

Schemes based on a DG discretization in space with upwind numerical fluxes and RK time-stepping have been presented several papers: in [14], in combination with a mortar method for treating nonmatching grids; in [6], together with stability analysis and *hp*-error bounds of the proposed scheme (a divergence-free variant of which can be found in [7]); and in [16], where a unified DG method is constructed within the computational and the PML regions. A method using centered numerical fluxes and leap-frog time-stepping in order to reduce dissipation has been introduced in [20].

Finally, a DG space-time approach has been adopted in [5] and in [17], in order to obtain schemes with only *local* CFL control of the time-step for stability, allowing for larger time-steps in larger space elements. These methods use space-time DG methods on meshes generated by advancing front techniques. In particular, in [5], for the case of smooth coefficients, an *explicit* mesh is constructed, allowing for an ordering of the elements with respect to domain of dependence, and therefore, for an explicit element-by-element advancing front solution. In [17], in order to deal with inhomogeneous media, the constraints on the meshes are weakened, allowing for meshes aligned with the discontinuities of the coefficients, and a *semi-implicit* method, based on an ordering of the mesh by macroelements, is constructed.

For the Maxwell's equations in frequency-domain, consider, to fix the ideas, the following electric field-based formulation:

$$\begin{aligned} \nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} + i\omega\sigma\mathbf{E} &= -i\omega \mathbf{J}_s \quad \text{in } \Omega \\ \mathbf{n} \times \mathbf{E} &= \mathbf{0} \qquad \text{on } \partial\Omega. \end{aligned}$$

The term $\omega^2 \varepsilon \mathbf{E}$ is neglected in the low-frequency case. The solutions of this problem are typically highly oscillatory or strongly singular. DG methods are particularly suited for capturing such solutions, since they allow for an easy implementation of high-order elements and *hp*-adaptive procedures. The main ingredient for the construction of DG schemes, in this context, is the DG approximation of the

second order *curl-curl* operator. For a unified presentation of various DG methods for elliptic problems and their theoretical analysis, see [2].

For the low-frequency case, in the simple case of conductivity $\sigma \neq 0$, the problem is elliptic and optimal error estimates can be easily obtained (the case of irregular meshes and only piecewise smooth material coefficients is studied in [18], where *hp*-error bounds are derived). For the high-frequency case, optimal error estimated have been obtained in [9] in the case of smooth coefficients. Mixed methods for imposing the divergence-free constraint on the electric field in the regions where $\sigma = 0$, in the low-frequency case, and for providing control on the divergence of the electric field, in the high-frequency case, have been presented and analyzed in [11] and [19], respectively. An energy-norm *a posteriori* estimator for the mixed method in the low-frequency case has been studied in [10].

Finally, the Maxwell eigenvalue problem has been addressed in [8], where a nonstabilized local discontinuous Galerkin method is used. Numerical results have shown that, in the two-dimensional case, the method correctly captures the eigenmodes, and no spurious mode pollutes the spectrum. In the three-dimensional case, small spurious modes appear, which can be eliminated by adding a suitable stabilization to the scheme.

We conclude with some remarks. For the Maxwell equations in frequency-domain, the eddy-current and the stationary problems, extensive and comparative studies still need to be performed. The same for coupled field-based and potential-based formulations. Up to now, a rigorous analysis of the Maxwell eigenvalue problem has not been performed, as well as a theoretical analysis of the high-frequency problem in a framework which allows for treating discontinuous material coefficients. We finally mention that, in addition to some numerical studies (see [13], [21], [12]), a complete theoretical analysis of dispersion and dissipation errors for DG methods has been carried out in [1].

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Smith Normal Form as an Adequate Tool to Detect Mesh Defects as well as to Build Basis Fields for Domains with Loops and Holes

Francesca Rapetti
(joint work with Alain Bossavit (L.G.E.P.) and François Dubois (C.N.A.M.))

A precise description of industrial geometries relies on the use of computer assisted design (C.A.D.) tools. Submeshes are generally created when complex domains with millions of element volumes are concerned. Accidental errors (due to human mistakes, to roundings, to bugs, ...) when gluing together separately created parts will result in spurious holes and/or loops. *How can we perform an automatic mesh defect detection ?*

The Hodge decomposition for a vector $\mathbf{u} \in L^2(\Omega)^3$ consists in its representation as the sum of three orthogonal components $\mathbf{u} = \operatorname{grad}\phi + \operatorname{curl} w + \theta$, the third component θ depending on the domain topology. *How can we build a basis for θ ?*

Algebraic topology and linear algebra help giving an answer to these or other questions.

Let $A : \mathcal{X} \to \mathcal{Y}$ be a linear operator between vector spaces of dimension m and n respectively. If bases are selected in both spaces, A is represented by a $(n \times m)$-matrix $\overline{A}$. One can choose bases in such a way that

$$\overline{A} = \begin{bmatrix} 0_{k,m-k} & \mathrm{Id}_{k,k} \\ 0_{n-k,m-k} & 0_{n-k,k} \end{bmatrix}.$$

This is the *Smith normal form* of A [6]. The normal form clearly exhibits the rank k, the null space (spanned by the first $m-k$ basis vectors in $\mathcal{X}$) and the range (spanned by the last $n-k$ basis vectors in $\mathcal{Y}$) of A (see Figure 1).

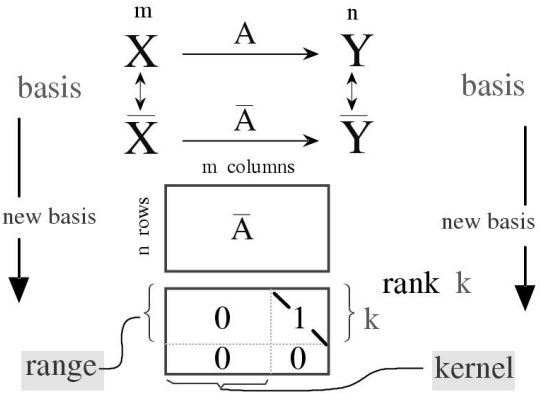

FIGURE 1. Smith normal form for the matrix $\overline{A}$.

Suppose now that one has a *complex* of linear maps $\partial_p : \mathcal{X}_p \to \mathcal{X}_{p-1}$, such that $\partial_p \partial_{p+1} = 0$. By a suitable choice of bases, one can put them all in Smith form,

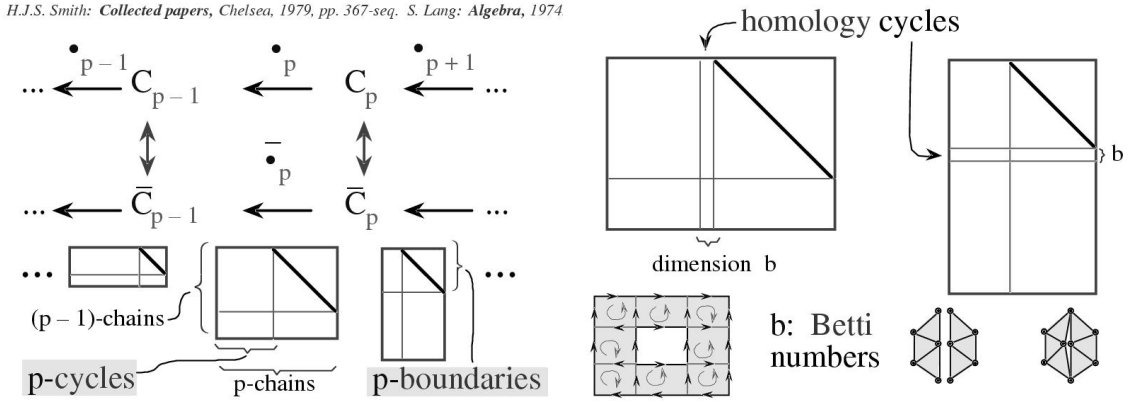

FIGURE 2. Computational configuration and analytical solution on the interface.

thus obtaining a complex of matrices on which one can spot the successive ranges and kernels, and most importantly the quotients $H_p = \ker(\partial_p)/\mathrm{ran}(\partial_{p+1})$.

A case in which this is valuable is when the ∂_ps are the boundary operators acting on chains based on p-cells of some discretization mesh (see Figure 2 (left) where the boundary operators are denoted by black dots carrying the dimension of the cells they act on). The original ∂_p are then the incidence matrices of this mesh, and take the above form when suitable bases are chosen in the chain spaces $\mathcal{X}_p$. One can then easily identify the *cycles* (chains with empty boundary), the *boundaries* (p-chains which bound a $(p+1)$-one), the *homology spaces* H_p and their dimensions b_p, the so-called *Betti numbers*, which are topological invariants (characteristics of the computational domain, not of the particular mesh), telling about the numbers of "holes" and "loops" in the meshed region (see Figure 2 (right)).

Such information is useful as a way to check whether the mesh has been consistently built. For istance, the mesh defects occurring when merging submeshes will result in spurious holes and/or loops, and thus can be detected this way [4]. Hence the interest for *algorithms* to reduce incidence matrices to normal form, with a competitive computational cost. They fall in two classes, depending on whether one works on the primal or the dual mesh.

In [4], we have proposed an algorithm working in $O(s^2)$ where $s = \max(n, m)$ for the considered $(n \times m)$-matrices. The results of the proposed algorithm applied to the incidence matrices of a simplicial discretization of a torus surface are shown in Figure 3. In this case, we are not looking for mesh defects but to an automatic way to compute the generators of H_p, $p = 1, 2$. We work with (incidence) matrices whose entries are *integers*, in particular 0, -1, +1. The Smith normal form of a (n, m)-matrix $\overline{A}$ is computed with *unimodular* transformations, represented by integer matrices with integer matrix inverses and determinants are ± 1. Elementary row operations

- exchange row i with row j
- multiply row i by -1

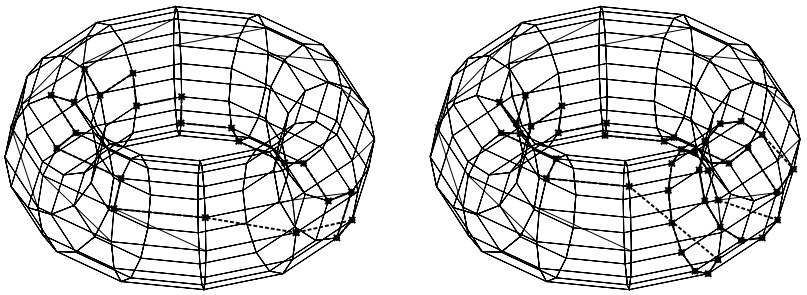

FIGURE 3. Wireframe representation of two loops, generators of the first homology group H_1 of the torus surface, see [5].

- replace row i by (row i) + α (row j), where α is an integer and $k \neq j$

Each of these operations corresponds to a change of basis in Y and similar column operations correspond to a change of basis in X.

These successive changes are stocked in two unimodular matrices, a (n, n)-matrix $\overline{Q}$ and a (m, m)-matrix $\overline{P}$. So, we look for $\overline{Q}$ and $\overline{P}$ such that $\overline{QAP}$ is in Smith form. Then, $\ker(\overline{A})$ is spanned by the first $m - k$ column vectors of $\overline{P}$ and $\mathrm{imag}(\overline{A})$ is spanned by the last $n - k$ row vectors of $\overline{Q}$ multiplied by the leading elements.

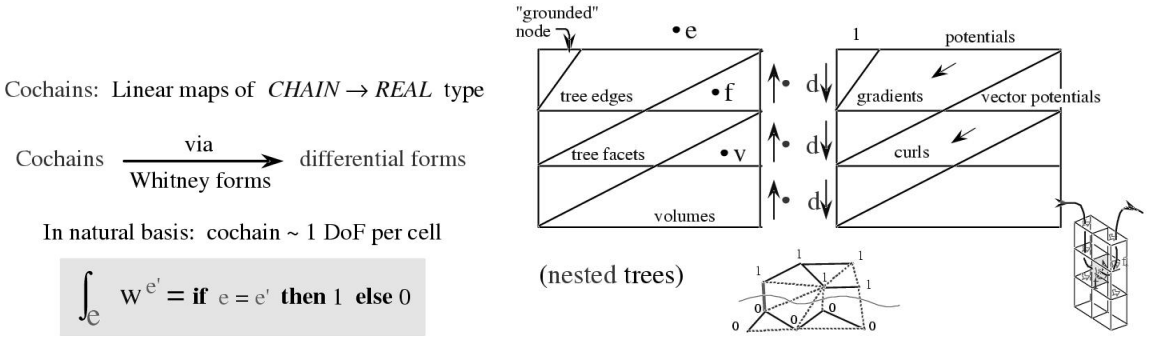

FIGURE 4. The dual side: cohomology.

There is more: by duality, a change of basis for chains induces one on cochains, which are the discrete representation of electromagnetic fields (see Figure 4). In particular, when loops or holes are present, there is a need [1] to construct "non-local" basis fields associated with them, which complete the basis of cell-related Whitney forms, as considered in [5]. Such fields can be read off from the Smith normal form, thanks to the geometric interpretation of the coefficients of the passage matrices. A classification of all possible ways to build representatives of the cohomology classes ("collars", "thick cuts", "tunnels", etc., as found in the work of Kotiuga [3], Kettunen [2], etc.) is thus obtained.

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Pole Condition: A new Approach to Solve Scattering Problems

F. Schmidt

(joint work with T. Hohage and L. Zschiedrich)

The pole condition concept is an approach to investigate certain classes of wave propagation problems on unbounded domains, including the time-dependent Schrödinger equation, the Helmholtz equation and time-harmonic Maxwell equations. The basic idea has been developed originally to solve the 1D time-dependent Schrödinger equation with *non-constant* exterior potentials [8, 9]. The convenient handling of *heterogeneous* exterior domains in 1D situations obtained there was the motivation to extend this concept to higher space dimensions as well as to time-harmonic problems. It turned out that the desired generalization can be done a very natural way.

We discuss the pole condition concept for solving time-harmonic scattering problems modeled by Helmholtz and Maxwell's equations on unbounded domains. The essential aspects are the following. First, the entire space is decomposed into an interior domain containing the scatterer and an exterior domain. The exterior domain may have a heterogeneous structure. Among the admissible types of inhomogeneous exterior domains are waveguide-like inhomogeneities which play an important role in applications. For the special case of 1D problems it was shown [7] that even exterior domains with periodic permittivities can be treated.

The basic idea of the pole condition approach is to consider the Laplace transform of the field in the exterior domain in radial direction. Here, radial direction denotes the distance-like direction in the exterior when covered by a prismatoidal coordinate system [11]. If we fix the angular-like coordinate of the exterior system and let the distance-like coordinate tend to infinity, we move on a ray from the boundary of the interior domain towards infinity. We characterize the exterior fields by the poles of their Laplace transforms along all possible rays and say that a field satisfies the pole condition if its Laplace transform has no pole in the lower half of the complex plane. Fields which satisfy the pole condition are outgoing fields.

A formulation of a scattering problem based on the pole condition consists of three parts: the interior problem, the coupling to the exterior problem, and the exterior problem in its Laplace transformed version. Additionally we have to ensure that the solution of the Laplace part contains only functions that satisfy the pole condition. The latter can be achieved in a number of different ways. One way is to use an extra condition in form of an integral condition [7], another to restrict the possible space of Laplace transformed functions by construction. For the continuous form of the pole condition based formulation of the Helmholtz scattering problem we obtained a number of results. First, the pole condition is equivalent to Sommerfeld's radiation condition in case of homogeneous exterior domains [5], second, the pole condition yields a new representation formula for for the exterior solution, third, parts of theorems concerning the series expansions of exterior fields (theorems of Karp and Wilcox) could be extended. A further surprising result states that the pole condition and the famous PML method are very closely related to each other [6].

The different continuous formulations of the pole condition leads to different numerical algorithms. Until now we investigated mainly two realizations: the cut function approach [5, 7] which is also the basis if the theoretical analysis and the real axis method [3, 7]. Whereas the first one allows directly to compute the exterior fields from the obtained data, the second one yields only the interior solution but has a simpler structure and can easily be extended to solve, e. g., eigenproblems on unbounded domains. A first numerical comparison between the pole condition approach and the PML methods [3] shows that both cause roughly the same numerical costs with a slight favour for PML. However, PML is not able to reproduce the exterior solution.

There are a number of new theoretical results offering new application areas of the pole condition. In [1] it has been shown by Arens and Hohage that the pole condition and the *upward propagating radiation condition* are equivalent. This enables a new approach in solving scattering problems involving unbounded obstacles. In a recent paper [4] Hohage and Stratis proved the equivalence of the pole condition and the Silver-Müller condition for electromagnetic scattering problems. The discrete electromagnetic scattering problem in 2D has been considered in [2]. Another application area of the pole condition concept is the computation of eigensolutions and resonances of open systems. In [10] we develop a convergence theory for the 1D Schrödinger case which allows a safe determination of converged resonances. The complete algorithm and main parts of the theory apply to higher dimensions as well. In [11] Zschiedrich gives a review on the current state of results related to the pole condition concept, a number of new results for time-dependent equations and 2D and 3D applications of our code solving time-harmonic electromagnetic scattering problems.

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Preconditioning for Maxwell Equations

Joachim Schöberl

In this talk, we discuss the construction and analysis of multigrid preconditioners for $H(curl)$ elliptic variational problems. We explain the smoothers of Hiptmair, and Arnold-Falk-Winther. These smoothers take care of components in the discrete kernel of the *curl*-operator, what is the gradient of the H^1 finite element space.

We sketch a new technique for the analysis of multi-level preconditioners in $H(curl)$. It is based on a multi-level decomposition by recently introduced commuting quasi-interpolation operators [1].

The second main topics in the talk is the discussion of algebraic multigrid methods in $H(curl)$. The idea is to define a coarsening algorithm for all finite element spaces in H^1, $H(curl)$, $H(div)$, and L_2, which maintains the complete sequence property on each multigrid level [2]. Thus, the the same smoothers work as in the geometric multigrid.

The last topics are new high order finite elements for all the spaces H^1, $H(curl)$, $H(div)$, and L_2. The high order H^1 elements have lowest order vertex functions, high order edge-, face-, and element-based shape functions. The $H(curl)$ elements have lowest order Nédélec (edge) shape functions, and high order edge-, face-, and element-based shape functions. Next, the $H(div)$ has lowest order Raviart-Thomas (face) shape functions, and high order face- and element-based ones. Finally, the

L_2 element has the constants, and high-order element functions. We stress the advantages of the new elements satisfying localized complete sequence properties for the lowerst order, edge-based, face-base, and element-based shape functions:

$$
\begin{array}{llllll}
W^V_{h,\,p+1=1} & \xrightarrow{\nabla} V^{\mathcal{N}_0}_h & \xrightarrow{\text{curl}} & Q^{\mathcal{RT}_0}_h & \xrightarrow{\text{div}} & S_{h,\,0} \\
W^E_{p_E+1} & \xrightarrow{\nabla} V^E_{p_E} & & & & \\
W^F_{p_F+1} & \xrightarrow{\nabla} V^F_{p_F} & \xrightarrow{\text{curl}} & Q^F_{p_F-1} & & \\
W^I_{p_I+1} & \xrightarrow{\nabla} V^I_{p_I} & \xrightarrow{\text{curl}} & Q^I_{p_I-1} & \xrightarrow{\text{div}} & S^I_{p_I-2}
\end{array}
$$

For the linear system of equations obtained by these basis function, simple block-diagonal preconditioners (the blocks contain unknows associated with edges, faces, and elements) in connection with an good coarse grid solver is efficient for $H(curl)$-elliptic problems in the following sense: The condition number is independent of the relative scaling of the L_2-part and the $curl$-semi-norm in the quadratic form, as well as independent of the mesh size. The dependency of p depends on the choice of the basis functions, and is currently a major point in research.

An other advantage of these basis function is that the order of the gradient functions and rotational functions can be chosen independently. In the limit case of a magnetostatic problem, the gradient functions can be totally skipped, which improves computation time about by a factor of 4. These new high order basis functions are explained in the upcoming paper [3].

All results are available from `http://www.hpfem.jku.at`

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Adaptive Multigrid-Methods for the Solution of Time-Harmonic Eddy-Current Problems

O. Sterz

An important class of electromagnetic problems are low frequency applications where the magnetic energy dominates the electric energy. Examples are devices from power engineering like motors, generators, transformers and switch gears as well as medical hyperthermia applications in cancer therapy. Here, the eddy-current approximation of the full Maxwell equations can be employed to describe the electromagnetic fields.

An upper bound for the modeling error of the eddy-current approximation of the full Maxwell-equations at a fixed angular frequency ω, as well as an asymptotic analysis for $\omega \to 0$, is given in [10], further details will be presented in [8]. Concerning the justification of the eddy current model by an asymptotic analysis, we also want to mention the pioneering works [1] and [2].

Assuming perfect conductor conditions $\mathbf{n} \times \mathbf{E} = 0$ at the boundary of the domain Ω, a variational formulation based on the electric field reads: Find $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$, such that $\forall \mathbf{E}' \in \mathbf{H}_0(\mathbf{curl}; \Omega)$

$$(\mu^{-1} \, \mathbf{curl}\, \mathbf{E}, \mathbf{curl}\, \mathbf{E}')_{\mathbf{L}^2(\Omega)} + i\omega(\sigma \mathbf{E}, \mathbf{E}')_{\mathbf{L}^2(\Omega_C)} = -i\omega(\mathbf{J_G}, \mathbf{E}')_{\mathbf{L}^2(\Omega)} \, .$$

¿From this formulation we do not get a unique electric field $\mathbf{E}$ in the insulating sub-domain, since we do not control the divergence of $\mathbf{E}$ and the total charges of the conductors. However, the magnetic field $\mathbf{H} = -(i\omega\mu)^{-1} \, \mathbf{curl}\, \mathbf{E}$, which is the interesting quantity in most cases of eddy current modeling, is unique. The discretization is done by edge elements on simplex grids (Whitney-1-forms) as the most natural choice.

To resolve local phenomena like singular behavior of the fields at edges and corners as well as small penetration depths (skin effect), we rely on an adaptive algorithm. With the help of an residual error estimator, see [5], the elements with the largest estimated error contribution are marked (maximum strategy) and refined (red/green-refinement). This results in a hierarchy of consistent grids.

The computation of real-world problems needs a large number of unknowns, up to several millions on a single processor machine are possible. Thus, for the solution of the linear systems of equations, as the most time consuming task, a fast method is essential. Therefore, multigrid methods are applied since they offer optimal complexity. For the smoothing in the multigrid cycles a standard algorithm like Gauß-Seidel is used in the insulating part of the domain ($\sigma = 0$), whereas the smoothing in the conductive part ($\sigma > 0$) needs a modification: Here, we may apply the idea proposed in [6], which is based on a Helmholtz decomposition and results in an additional smoothing step in the space of scalar potentials. Another possibility is the application of an overlapping block smoother, see [3].

In case of locally adapted grids the overall complexity may not be optimal unless the smoothing is restricted to the refined region. This leads to the implementation of *local* multigrid methods, which can be realized by grids that do not cover the whole computational domain at each level, see Fig. 5.

We finally mention, that the singularity of the arising linear system of equations is not a problem, as long as we take care of two things:

(1) The right hand side is in the range space of the matrix to guarantee solvability.
(2) During the iterative solution process the kernel components of the solution will not grow (or will grow slowly enough) to prevent cancellation errors.

The first condition can be satisfied by an adequate computation of the discrete excitation currents. To comply with the latter condition, we apply some approximate projections onto the orthogonal complement of the kernel of the **curl**-operator.

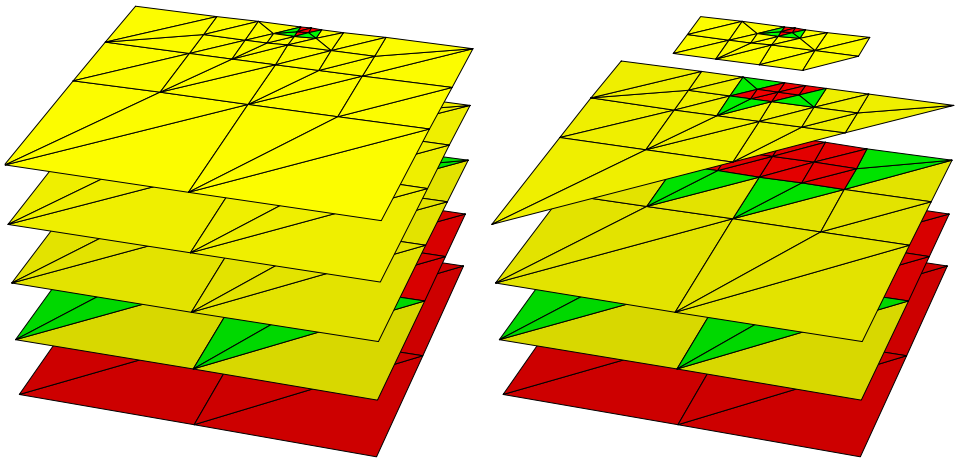

FIGURE 5. Example of a locally refined unit square: global grids (left) and local grids (right) of the multigrid hierarchy.

This can be realized with low costs by additional multigrid-sweeps on a Poisson-problem, see [10, 11].

All these concepts has been implemented in the adaptive finite element software $\mathsf{EM}_{\mathcal{UG}}$ (*electromagnetics on unstructured grids*), which is based on the simulation toolbox $\mathcal{UG}$, see [4, 10]. $\mathsf{EM}_{\mathcal{UG}}$ has been successfully applied to benchmark problems as well as realistic problems. A parallel prototype of the electromagnetic simulation tool is currently being developed.

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Perfectly Matched Layers

Fernando L. Teixeira

The simulation of electromagnetic problems in unbounded regions with partial differential equation (PDE) based methods, such as finite element (FE) and finite difference (FD) methods, necessitates the use of an absorbing boundary condition (ABC) to emulate the radiation condition at infinity. Perfectly matched layers (PML) are absorption layers used toward his purpose. The PML achieves a reflectionless absorption of electromagnetic waves in the continuum limit as the mesh discretization size goes to zero. The absorption inside the PML operates through conductive losses, so that an exponential decay for the fields inside the PML is obtained. Therefore, when the computational domain is surrounded by a PML region, spurious reflections from the grid boundaries can be made exponentially smaller. Being a *local* ABC, the PML retains the nearest-neighbor interaction characteristic of PDE-based methods, and therefore it is particularly suited for PDE-bases simulations on parallel computers. Also because of this property, the PML retains the inherent sparsity and (low) computational complexity of PDE-based methods.

When first introduced in the literature [1], the PML relied upon the use of matched artificial electric and magnetic conductivities and the splitting of the electromagnetic field components into two subcomponents each (split-field formulation). Because of this, the resulting fields inside the PML layer were rendered nonphysical (non-Maxwellian). The PML was later shown to be equivalent to a complex coordinate stretching of the coordinate space [2] or a complex coordinate transformation (analytic continuation of the coordinate space) [3],[4],[5]. Via such transformation, the (real) spatial coordinates are mapped as

$$\zeta \to \tilde{\zeta} = \int_0^{\zeta} s_\zeta(\zeta')d\zeta'$$

where s_ζ, with $\zeta = x, y, z$, are the so-called *complex stretching variables*, given by

$$s_\zeta(\zeta, \omega) = a_\zeta(\zeta) + i\frac{\Omega_\zeta(\zeta)}{\omega}$$

with $a_\zeta \geq 1$ and $\Omega_\zeta \geq 0$ (profile functions). The first inequality ensures that evanescent waves will have a faster exponential decay in the PML region, and the second inequality ensures that propagating waves will also decay exponentially along the respective coordinates in the PML. The ordinary Maxwell's equations are recovered from the above when $s_\zeta = 1$. Therefore, the complex stretching variables can be seen as added degrees of freedom to Maxwell's equations.

The PML has also found an interesting dual formulation (Maxwellian PML) with a more clear physical interpretation whereby the PML is represented by

frequency dependent material (constitutive) tensors $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ [6],[7]. These tensors also produce reflectionless absorption in the continuum limit. In addition to a more direct physical interpretation, the Maxwellian PML yields an easier interfacing with FE codes and a strongly well-posed formulation, as opposed to a weakly well-posed formulation in the original split-field PML [8].

The PML was first developed for planar grid terminations (Cartesian coordinates) [1],[2]. In order to be used with more general grid terminations, the PML later extended to curvilinear coordinates [3],[4],[5]. Although the first of such extensions have dealt with non-Maxwellian formulations only, it was later shown that Maxwellian PMLs could also be obtained in curvilinear geometries [9],[10]. In its most general form (for doubly curved surfaces), the curvilinear PML correspond to a *conformal* layer of anisotropic material tensors with inhomogeneous constitutive properties that depend on the local geometry (principal curvatures) of the mesh termination surface S [10],[11]. These PML constitutive parameters are given by $\overline{\overline{\mu}} = \mu\overline{\overline{\Lambda}}$ and $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{\Lambda}}$, with [10]

$$\overline{\overline{\Lambda}} = \hat{t}_1\hat{t}_1\left(\frac{sh_1\tilde{h}_2}{\tilde{h}_1h_2}\right) + \hat{t}_2\hat{t}_2\left(\frac{s\tilde{h}_1h_2}{h_1\tilde{h}_2}\right) + \hat{n}\hat{n}\left(\frac{\tilde{h}_1\tilde{h}_2}{sh_1h_2}\right).$$

Here s is the complex tretching coordinate along the normal coordinate ξ_3 at a point P in the mesh termination surface S, and h_i and $\tilde{h}_i$, $i = 1, 2$ are the nonstretched and stretched, respectively, (local) metric coefficients [10]. The unit vectors $\hat{t}_i$, $i = 1, 2$ are tangent to S at P along the principal lines of curvature that define tangential orthogonal coordinates ξ_1 and ξ_2, and $\hat{n}$ is the unit normal vector at that point (outward). The metric coefficients are given by $h_i = r_i/r_{0i}$, where r_{0i} are the principal radii of curvature at P and $r_i = r_{0i} + \xi_3$, $i = 1, 2$. The conformal PML is hence constructed over *parallel* surfaces to S. A basic limitation that exist in this general case, however, is that both radii of curvature should be non-negative (i.e., the PML can only be defined over planar or concave termination surfaces as viewed from inside the computational domain). Otherwise, dynamical instabilities ensue [11]. We note that the Cartesian, cylindrical, and spherical PMLs are special cases of this general curvilinear case, followed (possibly) by a successive application of the analytic continuation in orthogonal directions, if needed to achieve absorption in corner regions.

It is also possible to generalize the PML to terminate problems in more complex media, such as linear interior media exhibiting frequency dispersion and/or (bi) anisotropy [12]. This is in contrast to other local ABC, where an exact extension is often not possible in such cases. This extension is particularly important, for example, in electromagnetic simulations involving subsurface problems or complex materials [13]. For example, given an arbitrary dispersive and/or (bi)anisotropic linear interior media in a Cartesian domain with constitutive tensors $\overline{\overline{\epsilon}}(\omega)$, $\overline{\overline{\xi}}(\omega)$, $\overline{\overline{\zeta}}(\omega)$, $\overline{\overline{\mu}}(\omega)$, the corresponding Maxwellian PML bianisotropic constitutive parameters are given as [12]

$$\overline{\overline{\lambda}}_{PML}(\omega) = (\det\overline{\overline{S}})^{-1}\left(\overline{\overline{S}}\cdot\overline{\overline{\lambda}}(\omega)\cdot\overline{\overline{S}}\right),$$

where the symbol $\overline{\overline{\lambda}}$ stands for any of the above four constitutive tensors, and

$$\overline{\overline{S}}(\omega) = diag\{s_x^{-1}, s_y^{-1}, s_z^{-1}\}$$

Finally, we note that PML concept also admits a *geometric* interpretation as a complexification of the *metric* of space in the Fourier domain [14]. By exploring this interpretation, it can be shown that the differential forms language [15],[16] provides an elegant framework to unify the various PML formulations and obtain further generalizations. This is because the metric invariance of Maxwell's equations (in the sense of [17],[18]) is explicitely manifest in such language. A modification in the metric (diffeomorphism) corresponds to a modification on the Hodge operator, which fully incorporates the constitutive relations. The existence of Maxwellian PMLs can be seen as a simple consequence of the metric invariance of Maxwell's equations. The various PML formulations in the vector calculus language arise from the different choices on how to map differential forms to vector fields. This map fixes an isomorphism between differential forms and vectors and it depends on a metric. If the real metric is chosen to define such map, then the Maxwellian PML is receovered. Alternatively, if the complex (stretched) metric is chosen, then the non-Maxwellian PML is recovered. This also reveals that if other consistent metrics are chosen to fix the form-vector isomorphism (e.g., hybridizations of the previous ones), other (indeed, infinitely many) PML formulations are possible, albeit more cumbersome for practical implementation in numerical algorithms [14]. In such context, the existing PML formulations are particular cases of these choices.

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Some New Inexact Uzawa Methods and Non-overlapping DD Preconditioners for Solving Maxwell's Equations in Non-homogeneous Media

Jun Zou
(joint work with Qiya Hu)

This talk will review some new preconditioned Uzawa iterative methods for solving saddle-point systems, and a non-overlapping domain decomposition preconditioner for solving three-dimensional Maxwell's equations in non-homogeneous media.

Iterative methods for saddle-point system. Consider the system

$$Ax + By = f\,, \quad B^t x = g \tag{1}$$

where A is a symmetric and positive definite $n \times n$ matrix, and B is an $n \times m$ matrix with $m \leq n$. The system (1) is assumed to be nonsingular, so the Schur complement matrix $C = B^t A^{-1} B$ is positive definite. Linear systems such as (1) arise often from finite element discretizations of Maxwell equations and Navier-Stokes equations. Solving the saddle-point system (1) is usually much more difficult than solving the SPD system $Ax = b$. Recently the following inexact preconditioned Uzawa-type algorithm:

$$x_{i+1} = x_i + \hat{A}^{-1}[f - (Ax_i + By_i)]\,, \qquad y_{i+1} = y_i + \hat{C}^{-1}(B^t x_{i+1} - g) \tag{2}$$

has been widely used and studied (cf. [1] [2] [3]) for solving (1). Here $\hat{A}$ and $\hat{C}$ are preconditioners for A and C. The existing convergence results indicate that these algorithms converges assuming some good knowledge of the spectrum of the preconditioned matrices $\hat{A}^{-1}A$ and $\hat{C}^{-1}C$ or under some proper scalings of the

preconditioners $\hat{A}$ and $\hat{C}$. This "preprocessing" may not be easy and convenient to achieve in some applications.

Is it possible to introduce some relaxation parameters in (2) so that the resulting algorithm always converges for any SPD preconditioners $\hat{A}$ and $\hat{C}$, and converges with good rate when good preconditioners are available ? The following algorithm was proposed for this purpose (cf.[4]):

$$x_{i+1} = x_i + \omega_i \hat{A}^{-1}[f - (Ax_i + By_i)]\,, \qquad y_{i+1} = y_i + \tau_i \hat{C}^{-1}(B^t x_{i+1} - g) \tag{3}$$

where two parameters ω_i and τ_i can be updated using only the actions of $\hat{A}^{-1}$ and $\hat{C}^{-1}$.

The detailed convergence and convergence rate of algorithm (3) were given in terms of the condition numbers $\kappa(\hat{A}^{-1}A)$ and $\kappa(\hat{C}^{-1}C)$, without any conditions on preconditioner $\hat{C}$, see [4]. Unfortunately our proofs hold only with the condition that $\hat{A}$ is properly scaled so that the eigenvalues of $A^{-1}\hat{A}$ are bounded by one, although numerical experiments still demonstrated convergence when this condition is violated.

When a good preconditioner $\hat{A}$ is not available, one may replace the preconditioning part of $\hat{A}$ in (3) by some nonlinear iteration. This leads to the following algorithm (cf. [5]):

$$x_{i+1} = x_i + \Psi(f - (Ax_i + By_i))\,, \qquad y_{i+1} = y_i + \tau_i \hat{C}^{-1}(B^t x_{i+1} - g) \tag{4}$$

where Ψ is a nonlinear map in R^n such that for any $\phi \in R^n$, $\Psi(\phi)$ approximates the solution ξ of $A\xi = \phi$. And the parameter τ_i can be updated using only the actions of $\hat{C}^{-1}$ and Ψ.

The detailed convergence and convergence rate of the algorithm (4) can be given in terms of the condition number $\kappa(\hat{C}^{-1}C)$ and the tolerance parameter used for Ψ, and no any conditions on the preconditioner $\hat{C}$ are needed.

The algorithm (4) may not work well when the conditioning of the preconditioned Schur complement $\hat{C}^{-1}C$ is much worse than the conditioning of system $\hat{A}^{-1}A$. In this case, we may use a few PCG iterations with preconditioner $\hat{C}$ to improve the conditioning of $\hat{C}^{-1}C$, then apply the algorithm (4). This suggests the following algorithm (cf. [7]):

$$x_{i+1} = x_i + \Psi(f - (Ax_i + By_i))\,, \qquad y_{i+1} = y_i + \tau_i \Psi_H(B^t x_{i+1} - g), \tag{5}$$

where $\Psi_H(g_i)$ for any g_i is the iterate generated by the PCG method with preconditioner $\hat{C}$ for solving $H\psi = g_i$ with $H = B^T \hat{A}^{-1} B$ such that for some $\delta \in (0, 1)$,

$$\|\Psi_H(g_i) - H^{-1}g_i\|_H \le \delta \,\|H^{-1}g_i\|_H\,.$$

The actual effect of $\Psi_H(g_i)$ amounts to generating a new preconditioner $\hat{Q}_i$ such that the conditioning of $\hat{Q}_i^{-1}C$ is much improved than the one of $\hat{C}^{-1}C$ and $\kappa(\hat{Q}_i^{-1}C)$ is about the same as $\kappa(\hat{A}^{-1}A)$ (cf. [7]). The convergence and convergence rate of this algorithm was given in [7] and also applied to solving nonlinear saddle-point system like

$$F(x) + By = f, \quad B^t x = g\,.$$

Non-overlapping domain decomposition methods. Consider the Maxwell system:

$$\begin{cases} \nabla \times (\alpha \nabla \times \mathbf{u}) + \gamma_0 \beta \mathbf{u} = \mathbf{f} & \text{in} \quad \Omega \\ \nabla \cdot (\beta \mathbf{u}) = g & \text{in} \quad \Omega \end{cases} \tag{6}$$

where Ω is a Lipschitz polyhedral domain in $\mathbf{R}^3$, not necessarily convex. $\alpha(x)$ and $\beta(x)$ are positive but may be discontinuous in Ω. The perfect boundary condition $\mathbf{u} \times \mathbf{n} = 0$ is assumed on $\partial\Omega$. The constant γ_0 is non-negative, and it is allowed to be identically zero. It is this extreme case that causes the most troublesome technical difficulty in the analysis.

The variational saddle-point problem associated with system (6) is formulated as follows:

Find $(\mathbf{u}, p) \in H_0(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$\begin{cases} (\alpha \nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \gamma_0 (\beta \mathbf{u}, \mathbf{v}) + (\nabla p, \beta \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \\ (\beta \mathbf{u}, \nabla q) = (g, q), & \forall q \in H_0^1(\Omega). \end{cases} \tag{7}$$

Domain decompositions and edge elements. Decompose Ω into N non-overlapping tetrahedral subdomains $\{\Omega_i\}_i^N$, with each Ω_i of size d. The common face of subdomains Ω_i and Ω_j is denoted by Γ_{ij}, and set $\Gamma = \cup \Gamma_{ij}$, and $\Gamma_i = \Gamma \cap \partial\Omega_i$. Then we divide each Ω_i into smaller tetrahedral elements of size h so that elements from two neighboring subdomains match with each other on the interface Γ. Let T_h be the resulting triangulation of the domain Ω. We shall approximate the field $\mathbf{u}$ and multiplier p by the N*é*d*é*lec edge element space of lowest order and the piecewise linear nodal element space of $H_0^1(\Omega)$, denoted by $V_h(\Omega)$ and $Z_h(\Omega)$. Then the edge element approximation of system (7) is to find $(\mathbf{u}_h, p_h) \in V_h(\Omega) \times Z_h(\Omega)$ such that

$$\begin{cases} (\alpha \nabla \times \mathbf{u}_h, \nabla \times \mathbf{v}_h) + \gamma_0 (\beta \mathbf{u}_h, \mathbf{v}_h) + (\nabla p_h, \beta \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h(\Omega) \\ (\beta \mathbf{u}_h, \nabla q_h) = (g, q_h), & \forall q_h \in Z_h(\Omega). \end{cases} \tag{8}$$

For any face F of Ω_i, F_b denotes the union of all T_h-induced (closed) triangles on F, which have either one single vertex or one edge lying on ∂F, and F_∂ denotes the open set $\mathrm{F} \backslash \mathrm{F}_b$. For any subdomain Ω_i, define $\Delta_i = \cup_{\mathrm{F} \subset \Gamma_i} \mathrm{F}_b$. With each Ω_i, we define a local operator A_i on $V_h(\Omega_i)$, a standard restriction space of $V_h(\Omega)$ on Ω_i, by

$$(A_i \mathbf{u}, \mathbf{v}) = (\alpha \nabla \times \mathbf{u}, \nabla \times \mathbf{v})_{\Omega_i} + (\alpha \mathbf{u}, \mathbf{v})_{\Omega_i}, \quad \forall \mathbf{u},\ \mathbf{v} \in V_h(\Omega_i).$$

And $\tilde{A}$ is defined similarly to A_i but on the global space $V_h(\Omega)$. For any $\Phi \in V_h(\Gamma_i)$, we define its discrete A_i-extension $\mathbf{R}_h^i \Phi$ in $V_h(\Gamma_i)$: $\mathbf{R}_h^i \Phi \times \mathbf{n} = \Phi$ on Γ_i and solves

$$(A_i \mathbf{R}_h^i \Phi, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h^0(\Omega_i).$$

We can write system (8) as the operator form:

$$(\bar{A} + \gamma_0 \, \beta \, I) \mathbf{u}_h + B p_h = \bar{\mathbf{f}}_h, \quad B^t \mathbf{u}_h = g_h. \tag{9}$$

Noting that the operator $\bar{A}$ is singular in $V_h(\Omega)$, we can not apply the existing Uzawa-type iterative solvers for solving the saddle-point system when $\gamma_0 = 0$. To avoid the difficulty, we rewrite (9) into the following equivalent system

$$A \, \mathbf{u}_h + B p_h = \mathbf{f}_h, \quad B^t \mathbf{u}_h = g_h \tag{10}$$

where $A = \bar{A} + \gamma_0 \, \beta \, I$ for $\gamma_0 \neq 0$ and $A = \bar{A} + r_0 B \hat{C}^{-1} B^t$ if $\gamma_0 = 0$. Now one can apply, for example, the inexact Uzawa algorithm (3) for (10). It is important to note that the action of $\hat{C}^{-1}$ needs only once in each iteration, and the convergence rate of this algorithm is determined by $\kappa(\hat{A}^{-1} A)$ and $\kappa(\hat{C}^{-1} B^t \hat{A}^{-1} B)$.

Construction of preconditioners for A and $B^tA^{-1}B$.

One can show (cf. [6]) that if $\hat{C}$ is a preconditioner for $B^tA^{-1}B$ such that $(\beta\nabla\phi,\nabla\phi) \lesssim (\hat{C}\phi,\phi) \lesssim G(d/h)(\beta\nabla\phi,\nabla\phi)$ for all $\phi \in Z_h(\Omega)$,
then we have $G(d/h)^{-1}(\tilde{A}\mathbf{v}_h,\mathbf{v}_h) \lesssim (A\mathbf{v}_h,\mathbf{v}_h) \lesssim (\tilde{A}\mathbf{v}_h,\mathbf{v}_h)$, for all $\mathbf{v}_h \in V_h(\Omega)$. So it suffices to construct a preconditioner for $\tilde{A}$, instead of A.

Let $\lambda_e(\mathbf{v})$ be the moment of $\mathbf{v}$ on any edge e, $V^H(\Omega) \subset V_h(\Omega)$, consisting of all discrete A_i-extensions in each Ω_i, and

$$V^p(\Omega) = \prod_{k=1}^{N} V_h^0(\Omega_k), \quad V^{ij}(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \mathrm{supp}(\mathbf{v}) \subset \Omega_i \cup \Omega_j \cup \Gamma_{ij}\Big\},$$

$$V^0(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \lambda_e(\mathbf{v}) = 0 \text{ for each } e \in \mathrm{F}_\partial \text{ with } \mathrm{F} \subset \Gamma\Big\}$$

while $\hat{A}_p$ and $\hat{A}_{ij}$ are operators on $V^p(\Omega)$ and $V^{ij}(\Omega)$, and $\hat{A}_0$ the coarse solver in $V^0(\Omega)$:

$$(\hat{A}_p\mathbf{v},\mathbf{v}) \eqsim \sum_{k=1}^{N}(A_k\mathbf{v},\mathbf{v})_{\Omega_k}\ \forall\mathbf{v} \in V^p(\Omega);$$

$$(\hat{A}_{ij}\mathbf{v},\mathbf{v}) \eqsim (A_i\mathbf{v}_i,\mathbf{v}_i)_{\Omega_i} + (A_j\mathbf{v}_j,\mathbf{v}_j)_{\Omega_j}\ \forall\mathbf{v} \in V^{ij}(\Omega)$$

$$(\hat{A}_0\mathbf{v},\mathbf{w}) = h[1+\log(d/h)]\sum_{i=1}^{N}\alpha_i\Big\{\langle \mathrm{div}_\tau(\mathbf{v}\times\mathbf{n})|_{\Gamma_i}, \mathrm{div}_\tau(\mathbf{w}\times\mathbf{n})|_{\Gamma_i}\rangle_{\Delta_i} + \langle \mathbf{v}\times\mathbf{n}, \mathbf{w}\times\mathbf{n}\rangle_{\Delta_i}\Big\}$$

Then the additive preconditioner $\hat{A}$ formed by $\hat{A}_0$, $\hat{A}_p$ and $\hat{A}_{ij}$ is nearly optimal, i.e. $\kappa(\hat{A}^{-1}A) \lesssim G(d/h)[1+\log(d/h)]^2$, also independent of jumps of material coefficients (cf.[6]).

Acknowledgments. Jun Zou was supported by HKRGC Grants (No. CUHK4048/02P and 403403); Qiya Hu was supported by Special Funds for Major State Basic Research Projects of China G1999032804.

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