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Discrepancy Theory and Its Applications

Organised by
Bernard Chazelle (Princeton)
William Chen (Sydney)
Anand Srivastav (Kiel)

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Introduction by the Organisers

The meeting was organized by Bernard Chazelle (Princeton), William Chen (Sydney) and Anand Srivastav (Kiel), and was attended by some twenty participants from over ten countries and three continents.

The purpose of the meeting was to encourage and enhance dialogue and collaboration between the theoretical and practical aspects of discrepancy theory. The topics covered included:

- (1) Classical discrepancy theory, including low discrepancy sequences, geometric discrepancy and number theoretical aspects.
- (2) Combinatorial discrepancy theory, including coloring of hypergraphs and arithmetic structures.
- (3) Algorithms and complexity, including relations of discrepancy theory to derandomization of probabilistic algorithms and pseudorandomness, complexity classes, data structures in computational geometry and applications in combinatorial optimization.
- (4) Numerical integration in high dimension and its complexity.

Nineteen talks were presented, including a few of a survey nature as well as others that concentrated on specific recent results. These talks demonstrated the diversity on all four areas and their inter-relationships, as well as the vitality of these areas of research.

The organizers and participants would like to take this opportunity to thank again the “Mathematisches Forschungsinstitut Oberwolfach” for having provided a comfortable and inspiring environment for the meeting and the scientific work.

The pleasant atmosphere and superb facilities contributed to the overall success of the meeting.

We include the abstracts of all the talks in alphabetical order of the speakers.

Workshop on Discrepancy Theory and Its Applications

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Abstracts

Balanced partitions of vector sequences

Imre Bárány

Let $d, N \in \mathbb{N}$. Let $\|\cdot\|$ be any norm on \mathbb{R}^d and $B = \{v \in \mathbb{R}^d : \|v\| \leq 1\}$ its unit ball. Some time ago I proved the following result [2]: Let $v_1, v_2, \dots, v_N \subseteq B$ be a finite sequence of vectors. Then there are signs $\varepsilon_i \in \{-1, 1\}$ such that

$$\left\| \sum_{i \in [n]} \varepsilon_i v_i \right\| \leq 2d$$

for all $n \in [N] = \{1, 2, \dots, N\}$. In other words, there is a partition $[N] = I_1 \cup I_2$ such that

$$\left\| \sum_{i \in I_j \cap [n]} v_i - \frac{1}{2} \sum_{i \in [n]} v_i \right\| \leq d$$

for all $n \in [N]$ and $j \in [2]$.

This partitioning version of the theorem was extended to partitions into $r > 2$ classes with error bound $(r-1)d$ in [3]. In my talk, I explained how the factor $(r-1)$ can be replaced by a constant. To state this result precisely, we introduce some convenient notation: Let V be the given sequence of vectors v_1, v_2, \dots, v_N . We use the (non-standard) notation

$$\sum_k V = \sum_{i=1}^k v_i.$$

Further, for a subsequence X of V , we define

$$\sum_k X = \sum_{i \leq k, v_i \in X} v_i.$$

Theorem 1. *For every sequence $V \subset B$, and for every integer $r \geq 2$, there is a partition of V into r subsequences X_1, \dots, X_r such that for all k and j ,*

$$\sum_k X_j \in \frac{1}{r} \sum_k V + C(r)dB,$$

where $C(r)$ is a constant depending only on r . In particular, $C(2) \leq 1$, $C(3) \leq 1.5$, and $C(r) \leq 2.005$ always.

It is worth mentioning here that the result holds for all norms in \mathbb{R}^d . This is due to the fact that proofs use linear dependence among some vectors, with the norm playing very little role. But most likely, much better bounds are valid for particular norms. For instance, it is conjectured that for the $r = 2$ and Euclidean norm case the best bound is of order \sqrt{d} .

Moreover, for some norms the bound given above is tight, apart from the precise value of $C(r)$. An example showing this is the ℓ_1 norm, when the sequence is just e_1, e_2, \dots, e_d (the standard basis) and r is much smaller than d .

The proof of the theorem, with further results of this type, is to appear in [1].

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Limitations to regularity

József Beck

In 1981 I [1] proved the following “irregularity” result. For every n , there is an n -element point set in the unit square which does not have a balanced 2-coloring in the following quantitative sense: Whatever way one 2-colors the n points red and blue, there is always an axis-parallel rectangle in which the number of red points differs from the number of blue points by at least $(\log n)/100$.

My argument was nonconstructive. I could not provide an explicit example of such an n -set. Note that the usual grid is not good. A chessboard type alternating 2-coloring is so balanced that the two color classes differ by at most one.

Roth [4] later gave the following explicit example. Consider the tilted $\sqrt{n} \times \sqrt{n}$ grid, where the slope is a quadratic irrational like $\sqrt{2}$. An equivalent reformulation of Roth’s theorem goes as follows. Given any 2-coloring of the $n \times n$ square lattice (“grid”), there is always a tilted rectangle of slope $\sqrt{2}$, say, such that the number of red points differs from the number of blue points by at least $c \log n$, where $c > 0$ is an absolute constant. Note, however, that the size of the “unbalanced” rectangle cannot be specified in advance.

The following questions arise naturally:

- (1) What happens in the case of circles?
- (2) Can one specify the radius of the circle in advance?
- (3) How about one-sided discrepancy for circles?
- (4) Is there any other “natural geometric shape” for which translated copies alone give “unbounded irregularity”?

The first question was basically solved by Schmidt [5] many years earlier. His integral equation method, developed in the late 1960’s, can be easily adapted to show that the 2-coloring discrepancy of circles is as large as a power of n , rather than $\log n$. Unfortunately Schmidt’s method does not work for circles of fixed radius, and cannot handle one-sided discrepancy.

In the early 1990’s I could answer Question 4. My natural shape was a “hyperbola segment”. Consider the region between the two curves $y = 1/x$ and $y = -1/x$ where $1 \leq x \leq n$; I call it the hyperbolic needle of length n . It has area $2 \log n$,

and it has the following remarkable extra large irregularity property: Given any 2-coloring of the $n \times n$ square lattice, there is always a translated copy of the hyperbolic needle of length n with slope $\sqrt{2}$, say, in which the number of red points differs from the number of blue points by at least $c \log n$, where $c > 0$ is an absolute constant. Since the area of the hyperbolic needle is $2 \log n$, it means that the irregularity is proportional to the area. This explains the name “extra large irregularity”.

I could even prove a one-sided version [3] as follows. Assume that n is even, and the $n \times n$ square lattice has a globally balanced 2-coloring, in the sense that there are $n^2/2$ red points and $n^2/2$ blue points. Assume also that the $n \times n$ square lattice is a torus, so that we can “wrap around” the hyperbolic needles. Then there is always a translated copy of the hyperbolic needle of length n with slope $\sqrt{2}$, say, on the $n \times n$ torus in which the number of red points is more than the number of blue points by at least $c \log n$, where $c > 0$ is an absolute constant.

Note that the theorem holds for hyperbolic needle of any length $\ell < n$. Then the corresponding irregularity is constant times $\log \ell$ instead of $\log n$.

Recently I could give an affirmative answer to Questions 2 and 3. The main result, which answers both questions at the same time, goes as follows: Again assume that n is even, and that the $n \times n$ square lattice has a globally balanced 2-coloring in the sense that there are $n^2/2$ red points and $n^2/2$ blue points. Let R be an arbitrary real number with $2 < R < n/2$. Also assume that the $n \times n$ square lattice is a torus, so that we can “wrap around” the circles of radius R . Then there is always a circle of radius R on the $n \times n$ torus in which the number of red points is more than the number of blue points by at least $c\sqrt{\log R}$, where $c > 0$ is an absolute constant. In the case of varying radius, a weaker result, see [2].

The order $\sqrt{\log R}$ is almost certainly very far from optimal. I conjecture that the truth is a power of R rather than a power of $\log R$, but I do not have the slightest idea how to prove it. But I am not complaining – I was more than happy to prove anything “tending to infinity”.

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Classical discrepancy

William Chen

(joint work with Maxim Skriganov)

Let \mathcal{P} be a distribution of N points in the unit square $[0, 1]^2$. For every $\mathbf{x} = (x_1, x_2)$ in $[0, 1]^2$, let $Z[\mathcal{P}; B(\mathbf{x})] = |\mathcal{P} \cap B(\mathbf{x})|$ denote the number of points of the distribution \mathcal{P} that fall into the rectangle $B(\mathbf{x}) = [0, x_1) \times [0, x_2)$, and consider the corresponding discrepancy function $D[\mathcal{P}; B(\mathbf{x})] = Z[\mathcal{P}; B(\mathbf{x})] - Nx_1x_2$.

Theorem 1.

- (i) *There exists a positive absolute constant c such that for every positive integer N and every distribution \mathcal{P} of N points in the unit square $[0, 1]^2$, we have*

$$\int_{[0,1]^2} |D[\mathcal{P}; B(\mathbf{x})]|^2 d\mathbf{x} > c \log N.$$

- (ii) *There exists a positive absolute constant C such that for every integer $N \geq 2$, there exists a distribution \mathcal{P} of N points in the unit square $[0, 1]^2$ such that*

$$\int_{[0,1]^2} |D[\mathcal{P}; B(\mathbf{x})]|^2 d\mathbf{x} < C \log N.$$

The lower bound was obtained by Roth [7] in 1954, while the upper bound was obtained by Davenport [5] in 1956. Indeed, the lower bound can be extended to point distributions in the k -dimensional unit cube for arbitrary $k \geq 2$ without any extra difficulty, as shown in Roth [7] with lower bound $c(k)(\log N)^{k-1}$. However, ideas different from those of Davenport are necessary to extend the upper bound to the k -dimensional unit cube for arbitrary $k \geq 2$.

Much work in connection with the upper bound involves the van der Corput point sets and their generalizations. The van der Corput point set of 2^h points in $[0, 1]^2$ is given by

$$(1) \quad \mathcal{P}(2^h) = \{(0.a_1 \dots a_h, 0.a_h \dots a_1) : a_1, \dots, a_h \in \{0, 1\}\},$$

where we have used digit expansion base 2 on the right hand side. However,

$$(2) \quad \int_{[0,1]^2} |D[\mathcal{P}(2^h); B(\mathbf{x})]|^2 d\mathbf{x} = 2^{-6}h^2 + O(h),$$

as shown by Halton and Zaremba [6], and so this does not give a proof of the upper bound.

This difficulty was studied in detail by Chen and Skriganov [2], using classical Fourier analysis, since the van der Corput point sets have nice periodicity properties. Recall that $\mathbf{x} = (x_1, x_2)$ denotes the top right vertex of the rectangle $B(\mathbf{x})$. Suppose that $x_1 \neq 1$. Then it can be shown that there exists a finite set $\mathcal{I}(x_1) \subseteq \{1, \dots, h\}$ such that

$$D[\mathcal{P}(2^h); B(\mathbf{x})] = \sum_{s \in \mathcal{I}(x_1)} \left(c_s - \psi \left(\frac{x_2 + z_s}{2^{s-h}} \right) \right) + O(1).$$

One therefore needs to study sums of the form

$$\sum_{s' \in \mathcal{I}(x_1)} \sum_{s'' \in \mathcal{I}(x_1)} \left(c_{s'} - \psi \left(\frac{x_2 + z_{s'}}{2^{s'-h}} \right) \right) \left(c_{s''} - \psi \left(\frac{x_2 + z_{s''}}{2^{s''-h}} \right) \right).$$

Using Fourier analysis and integrating with respect to the variable x_2 over the interval $[0, 1]$, one can show that each of the summands above gives rise to an integral

$$\int_0^1 \left(c_{s'} - \psi \left(\frac{x_2 + z_{s'}}{2^{s'-h}} \right) \right) \left(c_{s''} - \psi \left(\frac{x_2 + z_{s''}}{2^{s''-h}} \right) \right) dx_2 = c_{s'} c_{s''} + O \left(\frac{2^{2 \min\{s', s''\}}}{2^{s'+s''}} \right).$$

Unfortunately, the sum

$$\sum_{s' \in \mathcal{I}(x_1)} \sum_{s'' \in \mathcal{I}(x_1)} c_{s'} c_{s''}$$

leads to the term $2^{-6}h^2$ in (2).

There are various ways of overcoming this difficulty. In Roth [8], one uses a translation variable t and translates the point set $\mathcal{P}(2^h)$ vertically modulo 1 to obtain the point set $\mathcal{P}(2^h; t)$ and a corresponding discrepancy function

$$D[\mathcal{P}(2^h; t); B(\mathbf{x})] = \sum_{s \in \mathcal{I}(x_1)} \left(\psi \left(\frac{z_s + t}{2^{s-h}} \right) - \psi \left(\frac{w_s + t}{2^{s-h}} \right) \right) + O(1),$$

where z_2 and w_2 are constants that depend on x_2 . Squaring and integrating with respect to the variable t over the interval $[0, 1]$, we now handle integrals of the form

$$\int_0^1 \psi \left(\frac{z_{s'} + t}{2^{s'-h}} \right) \psi \left(\frac{z_{s''} + t}{2^{s''-h}} \right) dt = O \left(\frac{2^{2 \min\{s', s''\}}}{2^{s'+s''}} \right).$$

In Chen [1], one uses digit translations to modify the point set $\mathcal{P}(2^h)$ horizontally to obtain the point set $\mathcal{P}(2^h; \chi)$ and a corresponding discrepancy function

$$D[\mathcal{P}(2^h; \chi); B(\mathbf{x})] = \sum_{s \in \mathcal{I}(x_1)} \left(c_s(\chi) + \psi \left(\frac{x_2 + z_s(\chi)}{2^{s-h}} \right) \right) + O(1).$$

Squaring and integrating with respect to the variable x_2 over the interval $[0, 1]$ and being economical with the truth, we now essentially handle integrals of the form

$$\begin{aligned} & \int_0^1 \left(c_{s'}(\chi) + \psi \left(\frac{x_2 + z_{s'}(\chi)}{2^{s'-h}} \right) \right) \left(c_{s''}(\chi) + \psi \left(\frac{x_2 + z_{s''}(\chi)}{2^{s''-h}} \right) \right) dx_2 \\ &= c_{s'}(\chi) c_{s''}(\chi) + O \left(\frac{2^{2 \min\{s', s''\}}}{2^{s'+s''}} \right). \end{aligned}$$

Furthermore, over a large collection of digit translations χ , the sum

$$\sum_{s' \in \mathcal{I}(x_1)} \sum_{s'' \in \mathcal{I}(x_1)} c_{s'}(\chi) c_{s''}(\chi)$$

has a small average. However, both of these involve probabilistic variables, and so no explicit point sets \mathcal{P} satisfying the conclusion of Theorem 1(ii) are obtained.

The van der Corput point sets (1) also possess nice group structure. Clearly $\mathcal{P}(2^h)$ forms a group under coordinatewise and digitwise addition modulo 2, and is isomorphic to the direct product \mathbb{Z}_2^h . This observation immediately invites the use of Fourier-Walsh functions and series. The discussion can be conducted in general in base p , where p is a fixed prime. In other words, we consider the generalization of the classical van der Corput point sets $\mathcal{P}(2^h)$ to sets of the form

$$\mathcal{P}(p^h) = \{(0.a_1 \dots a_h, 0.a_h \dots a_1) : a_1, \dots, a_h \in \{0, 1, \dots, p-1\}\},$$

where we now use digit expansion base p on the right hand side. Clearly $\mathcal{P}(p^h)$ forms a group of p^h elements under coordinatewise and digitwise addition modulo p , and is isomorphic to the direct product \mathbb{Z}_p^h . This suggests the use of Fourier-Walsh functions and series base p . Using the abbreviation \mathcal{P} for the point set $\mathcal{P}(p^h)$, one can show that an approximation $D_h[\mathcal{P}; B(\mathbf{x})]$ of the discrepancy function $D[\mathcal{P}; B(\mathbf{x})]$ satisfies

$$D_h[\mathcal{P}; B(\mathbf{x})] = \sum_{\substack{\ell_1=0 \\ (\ell_1, \ell_2) \neq (0,0)}}^{p^h-1} \sum_{\ell_2=0}^{p^h-1} \left(\sum_{\mathbf{p} \in \mathcal{P}} \overline{w_{\ell_1}(p_1)w_{\ell_2}(p_2)} \right) \tilde{\chi}_{\ell_1}(x_1)\tilde{\chi}_{\ell_2}(x_2).$$

Here w_ℓ , where $\ell \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, denotes the ℓ -th base p Walsh function, and $\tilde{\chi}_\ell(x)$ denotes the ℓ -th coefficient of Fourier-Walsh series of the characteristic function of the interval $[0, x)$. Since the Walsh functions are characters of the group \mathcal{P} , the orthogonality relationship

$$\sum_{\mathbf{p} \in \mathcal{P}} w_{\ell_1}(p_1)w_{\ell_2}(p_2) = \begin{cases} p^h & \text{if } (\ell_1, \ell_2) \in \mathcal{P}^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{P}^\perp \subseteq \mathbb{N}_0^2$ is the orthogonal dual to the group \mathcal{P} , gives

$$D_h[\mathcal{P}; B(\mathbf{x})] = p^h \sum_{(\ell_1, \ell_2) \in \mathcal{P}^\perp \setminus \{(0,0)\}} \tilde{\chi}_{\ell_1}(x_1)\tilde{\chi}_{\ell_2}(x_2).$$

One would like to square this expression and then integrate with respect to $\mathbf{x} = (x_1, x_2)$ over the unit square $[0, 1]^2$. Unfortunately, the Fourier-Walsh coefficients

$$(3) \quad \tilde{\chi}_{\ell_1}(x_1)\tilde{\chi}_{\ell_2}(x_2), \quad (\ell_1, \ell_2) \in \mathcal{P}^\perp \setminus \{(0,0)\},$$

are not orthogonal in $L^2([0, 1]^2)$ in general. In Chen and Skrikanov [3], it is shown that as long as the prime p is chosen large enough, there exist groups \mathcal{P} of p^h elements in the square $[0, 1]^2$, in the spirit of van der Corput, such that the Fourier-Walsh coefficients (3) are quasi-orthonormal in $L^2([0, 1]^2)$. Indeed, they are able to establish Theorem 1(ii) for arbitrary dimensions with explicitly constructed point sets. More recently, Chen and Skrikanov [4] have shown that in fact, as long as the prime p is chosen large enough, there exist groups \mathcal{P} of p^h elements in the square $[0, 1]^2$, in the spirit of van der Corput, such that the Fourier-Walsh coefficients (3) are orthogonal in $L^2([0, 1]^2)$, so that

$$\int_{[0,1]^2} |D_h[\mathcal{P}; B(\mathbf{x})]|^2 d\mathbf{x} = p^{2h} \sum_{(\ell_1, \ell_2) \in \mathcal{P}^\perp \setminus \{(0,0)\}} \int_{[0,1]^2} |\tilde{\chi}_{\ell_1}(x_1)\tilde{\chi}_{\ell_2}(x_2)|^2 d\mathbf{x}.$$

Furthermore, they have shown that, corresponding to the group \mathcal{P} of p^h elements in the square $[0, 1]^2$, there is a group \mathcal{G} of order p^{2h} of digit shifts such that

$$\frac{1}{|\mathcal{G}|} \sum_{\mathbf{t} \in \mathcal{G}} \int_{[0,1]^2} |D_h[\mathcal{P} \oplus \mathbf{t}; B(\mathbf{x})]|^2 d\mathbf{x} = p^{2h} \sum_{(\ell_1, \ell_2) \in \mathcal{P}^\perp \setminus \{(0,0)\}} \int_{[0,1]^2} |\tilde{\chi}_{\ell_1}(x_1) \tilde{\chi}_{\ell_2}(x_2)|^2 d\mathbf{x}.$$

This is a consequence of the orthogonality relationship

$$\sum_{\mathbf{t} \in \mathcal{G}} \overline{w_{\ell'_1}(t_1) w_{\ell'_2}(t_2)} w_{\ell''_1}(t_1) w_{\ell''_2}(t_2) = \begin{cases} p^{2h} & \text{if } (\ell'_1, \ell'_2) = (\ell''_1, \ell''_2), \\ 0 & \text{otherwise.} \end{cases}$$

We therefore now have a better understanding of the probabilistic argument of Chen [1].

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Multi-color discrepancies

Benjamin Doerr

(joint work with Anand Srivastav)

We extend the notion of combinatorial discrepancy of hypergraphs to arbitrary numbers of colors. Unless otherwise stated, the following results appeared in [5]. Let $\mathcal{H} = (X, \mathcal{E})$ denote a finite hypergraph, *i.e.*, X is a finite set and \mathcal{E} is a family of subsets of X . Put $n = |X|$ and $m = |\mathcal{E}|$. A c -coloring of \mathcal{H} is a mapping $\chi : X \rightarrow M$, where M is any set of cardinality c . Usually, we take $M = [c] := \{1, \dots, c\}$. The basic idea of measuring the deviation from perfect balance motivates these definitions of the discrepancy of \mathcal{H} with respect to χ and the discrepancy of \mathcal{H} in c colors:

$$\text{disc}(\mathcal{H}, \chi, c) := \max_{i \in M, E \in \mathcal{E}} \left| |\chi^{-1}(i) \cap E| - \frac{|E|}{c} \right|,$$

$$\text{disc}(\mathcal{H}, c) := \min_{\chi: X \rightarrow [c]} \text{disc}(\mathcal{H}, \chi, c).$$

Let us start with an example which shows that a hypergraph may have very different discrepancies in different numbers of colors. Let $k \in \mathbb{N}$ and $n = 4k$. Set

$$\mathcal{H}_n = ([n], \{X \subseteq [n] : |X \cap [n/2]| = |X \setminus [n/2]|\}).$$

Obviously, \mathcal{H}_n has 2-color discrepancy zero, but $\text{disc}(\mathcal{H}_n, 4) = \frac{1}{8}n$.

In fact, such examples exist for nearly any two numbers of colors. Unless c_1 divides c_2 , there are hypergraphs \mathcal{H}_n on n vertices having discrepancy $\Theta(n)$ in c_1 colors and zero discrepancy in c_2 colors. This has been investigated in [2].

For some 2-color discrepancy results, the proofs seem to rely heavily on the fact that only two colors are used. This applies in particular to those where the partial coloring method introduced by Beck [1] is used. A key step there is to construct a low discrepancy partial coloring $\chi := \frac{1}{2}(\chi_1 - \chi_2)$ from two colorings χ_1, χ_2 with $\chi_1(E) \approx \chi_2(E)$ for all $E \in \mathcal{E}$. It is not clear to us how this idea can be extended to c colors.

The idea of recursive coloring is to successively enlarge the number of partition classes. We start with a suitable 2-coloring of X with color classes X_1, X_2 and then iterate this process on the subhypergraphs induced by X_1 and X_2 . If the weighted 2-color discrepancies of the induced subhypergraphs are bounded, such a recursive approach can be analyzed, even if c is not a power of 2. For $p \in [0, 1]$, denote the discrepancy of \mathcal{H} with respect to the weight $(p, 1 - p)$ by

$$\text{disc}(\mathcal{H}, (p, 1 - p)) = \min_{\chi: X \rightarrow [2]} \max_{E \in \mathcal{E}} \left| |E \cap \chi^{-1}(1)| - p|E| \right|.$$

Theorem 1. *Let $\text{disc}(\mathcal{H}_0, (p, 1 - p)) \leq K$ for all induced subgraphs \mathcal{H}_0 of \mathcal{H} and all $p \in [0, 1]$. Then the inequality $\text{disc}(\mathcal{H}, c) \leq 2.0005K$ holds for all numbers c of colors.*

For many classical results, a refinement of the above ideas yields even stronger bounds that decrease for larger numbers of colors. For reasons of space we are not able to state the general result precisely. Roughly speaking, we have that if induced subhypergraphs on n_0 vertices have 2-color discrepancy at most $O(n_0^\alpha)$ for some $\alpha \in]0, 1[$, then $\text{disc}(\mathcal{H}, c) = O((n/c)^\alpha)$. This gives, among many others, the following bounds, where in all cases, the implicit constants do not depend on c :

- General bound: $\text{disc}(\mathcal{H}, c) \leq 45\sqrt{(n/c) \log(4m)} + 1$.
- Spencer's six standard deviations [7]: For all hypergraphs \mathcal{H} having $n = m$ vertices and hyperedges, $\text{disc}(\mathcal{H}, c) = O(\sqrt{(n/c) \log c})$.
- Arithmetic progressions: The hypergraph \mathcal{A}_n of arithmetic progressions in $[n]$ satisfies $\text{disc}(\mathcal{A}_n, c) = O(c^{-0.16}n^{0.25})$ for $c \leq n^{0.25}$. This extends the bound of Matoušek and Spencer [6].

A second general approach is to mimic the proofs of two-color results. Since the choice of the colors ± 1 for two colors allows several powerful arguments, the key problem is to choose a suitable set of colors for the general case. The colors we use are vectors in \mathbb{R}^c . We obtain a multi-color analogue of the Beck-Fiala theorem showing that $\text{disc}(\mathcal{H}, c) \leq 2\Delta(\mathcal{H})$ and one of the Bárány-Grunberg theorems. The

latter was improved by Bárány in his talk by reducing the multiplicative dependence on the number of colors to a constant.

An analogue of an eigenvalue bound attributed to Lovász and Sós shows that

$$\text{disc}(\mathcal{H}, c) \geq \sqrt{\frac{n(c-1)}{mc^2}} \lambda_{\min}(A^\top A),$$

where A^\top is an incidence matrix of A . This can be used to show a lower bound of $0.04c^{-1/2}n^{1/4}$ for the c -color discrepancy of the arithmetic progressions in $[n]$.

For hypergraph having $n = m$ vertices and edges, using a random construction we recently showed that our upper bound in Spencer's six standard deviations is sharp apart from constant factors [4].

Theorem 2. *For all $c \in \mathbb{N}_{\geq 2}$ and $n \geq c \log c$, there is a hypergraph having n vertices, n hyperedges and c -color discrepancy at least $\frac{1}{40} \sqrt{(n/c) \log c}$.*

In contrast to the (ordinary) c -color discrepancy, there is a strong correlation between the hereditary discrepancies of a hypergraph in different numbers of colors.

Theorem 3. *For any two numbers of colors $c_1, c_2 \in \mathbb{N}_{\geq 2}$ and all hypergraphs \mathcal{H} , we have*

$$\text{herdisc}(\mathcal{H}, c_2) \leq 3c_1^2 \text{herdisc}(\mathcal{H}, c_1).$$

Hence $\text{herdisc}(\cdot, c_2) = \Theta_{c_1, c_2}(\text{herdisc}(\cdot, c_1))$. The proof given in [3] actually solves a more general problem, namely it reduces the color rounding problem in c_2 colors to the hereditary discrepancy problem in c_1 colors. We currently have no purely combinatorial proof.

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Digital expansions and uniformly distributed sequences modulo 1

Michael Drmota

Let $s_q(n)$ denote the sum-of-digits function of the q -ary digital expansion of the non-negative integer n . Then it is well known that the sequence $(s_q(n)\alpha)$ is uniformly distributed modulo 1 if and only if α is irrational. The purpose of this talk is to present a survey of recent results of this kind and also to present the methods that are used. We will deal with the following topics:

- (1) Uniform distribution of $(f(n)\alpha)$ for additive functions related to various digital expansions.
- (2) Discrepancy bounds for $(s_q(n)\alpha)$ in terms of the continued fraction expansion of α .
- (3) Uniform distribution of $(s_{q_1}(n)\alpha_1, \dots, s_{q_d}(n)\alpha_d)$ for coprime bases q_1, \dots, q_d and irrational $\alpha_1, \dots, \alpha_d$.
- (4) Uniform distribution of $(s_q(n)\alpha)_{n \in S}$ for certain subsequences S ; for example, squares.

Let $f(n)$ be a q -additive function, *e.g.*, the q -ary sum-of-digits function $s_q(n)$. Then it is worth considering the generating function

$$F_N(x) = \sum_{n < N} x^{f(n)}.$$

Owing to the recursive structure of q -ary digital expansion, one directly gets recurrences for $F_N(x)$ that (usually) lead to (more or less) explicit (or asymptotic) expressions for $F_N(x)$. For example, for the binary sum-of-digits function one has

$$\sum_{n < 2^k} x^{s_2(n)} = (1+x)^2.$$

The advantage of these kinds of representation is that they directly imply results on

- the distribution $\#\{n < N : f(n) \leq x\}$ as $N \rightarrow \infty$ (Gaussian limiting distributions), and
- uniform distribution and discrepancy estimates of the sequence $(f(n)\alpha)$ for irrational numbers α .

For example, in order to treat uniform distribution of $(f(n)\alpha)$, one has to evaluate $F_N(x)$ for $x = e^{2\pi i h \alpha}$.

Of course, with the help of this method one obtains upper bounds for exponential sums and for the discrepancy, however, usually not optimal ones. Nevertheless, it is possible to get more precise discrepancy estimates by using the continued fraction expansion of α (see [3]). For example, if α has bounded continued fraction expansion, then one gets

$$\frac{1}{\sqrt{\log N}} \ll D_N(s_q(n)\alpha) \ll \frac{\log \log \log N}{\sqrt{\log N}}.$$

It is also an interesting problem to consider d -dimensional sequences

$$(s_{q_1}(n)\alpha_1, \dots, s_{q_d}(n)\alpha_d)$$

for coprime bases q_1, \dots, q_d and irrational $\alpha_1, \dots, \alpha_d$. With the help of exponential sum estimates (see [3]), it follows that these kinds of sequences are uniformly distributed modulo 1 for all irrational numbers $\alpha_1, \dots, \alpha_d$.

In other words, this is a mathematical formulation of the well accepted fact that q -ary digital expansions are independent if the bases q_1, \dots, q_d are coprime. Interestingly one can be even more precise. With the help of methods of Bassily and Katai [1] and by proper use of Baker's theorem on linear forms in logarithms,

it follows that the joint distribution of $(s_{q_1}(n), s_{q_2}(n))$ is asymptotically Gaussian and independent if q_1 and q_2 are coprime (see [2]). It is even possible to derive asymptotic expansions for the numbers $\#\{n < N : s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\}$.

Finally we consider the (binary) sum-of-digits function $s(n^2)$ of squares. There are no precise results on the distribution of squares. For example, it is an unsolved problem of Gelfond [5] whether the asymptotic frequency of $s(n^2)$ being even is $1/2$ or not. Equivalently we can ask whether

$$\sum_{n < N} (-1)^{s(n^2)} = o(N)?$$

We could not answer this question. However, in joint work with Rivat [4], the sum of binary digits $s(n^2)$ is split into two parts $s_{<k}(n^2) + s_{\geq k}(n^2)$, where $s_{<k}(n^2) = s(n^2 \bmod 2^k)$ collects the first k digits and $s_{\geq k}(n^2) = s(\lfloor n^2/2^k \rfloor)$ collects the remaining digits. With the help of the generating function approach mentioned above, we derive very precise results on the distribution on $s_{<k}(n^2)$ and $s_{\geq k}(n^2)$. We provide asymptotic formulas for the numbers $\#\{n < 2^k : s_{<k}(n^2) = m\}$ and $\#\{n < 2^k : s_{\geq k}(n^2) = m\}$ and show that the sequences $(s_{<k}(n^2)\alpha)_{n < 2^k}$ and $(s_{\geq k}(n^2)\alpha)_{n < 2^k}$ are very well distributed modulo 1.

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Geometric discrepancies and δ -covers

Michael Gnewuch

(joint work with Benjamin Doerr)

It is of interest to derive bounds for geometric discrepancies, *e.g.*, the $*$ - or the unanchored discrepancy, with good behaviour in the parameter of dimension d . An upper bound for the $*$ -discrepancy with a nearly optimal behaviour in d and explicitly known constants was proved; see Theorem 1 in [1]. Here we introduce the notion of δ -covers on the d -dimensional unit cube $[0, 1]^d$ and give bounds for their minimal cardinality. From these estimates we obtain upper bounds for the $*$ -discrepancy and its inverse, which improve the results of [1]. We achieve similar results for the unanchored discrepancy.

For $x, y \in [0, 1]^d$, we write $[x, y[= \prod_{i \in [d]} [x_i, y_i[$. Let $\delta > 0$. We say that some finite subset Γ is a δ -cover of $[0, 1]^d$ if for all $y \in [0, 1]^d$, there are $x, z \in \Gamma \cup \{0\}$

such that $x_i \leq y_i \leq z_i$ for all i and $\text{vol}([0, z]) - \text{vol}([0, x]) \leq \delta$. We denote the minimal cardinality of all δ -covers by $N(d, \delta)$.

We get a first bound on $N(d, \delta)$ by considering an equidistant grid Γ_m with mesh size $1/m$, where $m = \lceil d/\delta \rceil$. Obviously Γ_m is a δ -cover of $[0, 1]^d$ with cardinality $(m + 1)^d$. We then derive a better bound by calculating the coordinates of a non-equidistant grid $\Gamma = \{x_0, \dots, x_{\kappa(\delta, d)}\}^d$ with the following recursive procedure:

$$\begin{aligned}
 & x_0 := 1 \\
 & x_1 := (1 - \delta)^{1/d} \\
 (1) \quad & \text{for } i \geq 1 \text{ do} \\
 & \quad x_{i+1} := (x_i - \delta)x_1^{1-d} \\
 & \quad \text{if } x_{i+1} \leq \delta, \text{ then } \kappa(\delta, d) := i + 1 \text{ and stop}
 \end{aligned}$$

The sequence x_0, x_1, \dots is finite and strictly decreasing. Γ is a δ -cover of $[0, 1]^d$, which establishes the following bound on $N(d, \delta)$.

Theorem 1. *Let $d \geq 2$ and $0 \leq \delta < 1$. Then $N(d, \delta) \leq (\kappa(\delta, d) + 1)^d$, where*

$$(2) \quad \kappa(\delta, d) = \left\lceil \frac{d}{d-1} \frac{\log(1 - (1 - \delta)^{1/d}) - \log(\delta)}{\log(1 - \delta)} \right\rceil.$$

The estimate

$$\kappa(\delta, d) \leq \left\lceil \frac{d}{d-1} \frac{\log d}{\delta} \right\rceil$$

holds, and the quotient of the left and the right hand sides of the inequality converges to 1 as $\delta \rightarrow 0$.

Another recursive construction gives us a bound with better asymptotic behaviour in d . The construction in dimension d uses the $(d - 1)$ -dimensional construction and a scaling property and leads to the next theorem. Note that all O -notation refer to the variable δ^{-1} only.

Theorem 2. *Let $d \geq 2$ and $0 < \delta < 1$. Then, with a constant $C \leq 2e$,*

$$(3) \quad N(d, \delta) \leq 2^d \frac{d^d}{d!} \left(\delta^{-1} + \frac{d+1}{4} - \frac{1}{2d} \right)^d \leq C^d \delta^{-d} + O(\delta^{-d+1}).$$

A lower bound for the cardinality of each δ -cover is stated in the next theorem.

Theorem 3. *Let $\delta \in]0, 1]$. Then, with a constant $c \geq e^{-1}$,*

$$N(d, \delta) \geq \frac{2}{5} \frac{d!}{d^d} \delta^{-d} - \frac{2}{5} d! \sum_{k=0}^{d-1} \frac{d^k (\log(d\delta))^{-1k}}{k!} \geq c^d \delta^{-d} + O((\log \delta^{-1})^{d-1}).$$

We now discuss applications to $*$ -discrepancy. The L^∞ - $*$ -discrepancy is given by

$$d_\infty^*(n, d) = \inf_{t_1, \dots, t_n \in [0, 1]^d} d_\infty^*(t_1, \dots, t_n),$$

where

$$d_{\infty}^*(t_1, \dots, t_n) = \sup_{x \in [0,1]^d} \left| \text{vol}([0, x[) - \frac{1}{n} \sum_{k=1}^n 1_{[0,x[}(t_k) \right|.$$

The inverse of the *-discrepancy is defined by

$$n_{\infty}^*(\varepsilon, d) = \min\{n \in \mathbb{N} : d_{\infty}^*(n, d) \leq \varepsilon\}$$

for given $\varepsilon > 0$. For any δ -cover Γ of $[0, 1]^d$, the following approximation property holds: For every $t_1, \dots, t_n \in [0, 1]^d$, we have

$$d_{\infty}^*(t_1, \dots, t_n) \leq \max_{x \in \Gamma} \left| \text{vol}([0, x[) - \frac{1}{n} \sum_{i=1}^n 1_{[0,x[}(t_i) \right| + \delta.$$

Using this and our results on δ -covers, we obtain the following result.

Theorem 4. *Let $d \geq 2$ and $\varepsilon > 0$. If $\varepsilon \leq 8/(d + 1)$, then there exists a constant $C \leq 8e$, independent of ε and d , with*

$$n_{\infty}^*(\varepsilon, d) \leq \left\lceil 2\varepsilon^{-2} \left(d \log \left(\frac{C}{\varepsilon} \right) + \log 2 \right) \right\rceil.$$

For all $0 < \varepsilon \leq 1$, we have

$$(4) \quad n_{\infty}^*(\varepsilon, d) \leq \left\lceil 2\varepsilon^{-2} \left(d \log \left(\kappa \left(\frac{\varepsilon}{2}, d \right) + 1 \right) + \log 2 \right) \right\rceil,$$

where $\kappa(\varepsilon/2, d)$ is defined as in (2). If

$$n \geq 2 \left(d \log \left(\left\lceil \frac{2d}{d-1} \log d \right\rceil + 1 \right) + \log 2 \right),$$

then, with $\rho = 2\sqrt{2 \log 2/5} < 1.0532$,

$$d_{\infty}^*(n, d) \leq \sqrt{2} n^{-1/2} (d \log(\lceil \rho n^{1/2} \rceil + 1) + \log 2)^{1/2}.$$

We verified this theorem by adapting the proof idea from Theorem 1 of [1]. The proof considers n equally distributed independent random variables representing the possible point configurations, and in this situation our approximation property above allows us to make use of Hoeffding’s inequality. Note that the same proof technique was also employed in [2] and [3].

Using (4), we can give explicit bounds for the inverse of the *-discrepancy. Corresponding to the same values of d and ε as in Section 2 of [1], we have the following bounds:

| | |
|------------------------------------|------------------------------------|
| $n_{\infty}^*(0.45, 5) \leq 116$ | $n_{\infty}^*(0.1, 5) \leq 3828$ |
| $n_{\infty}^*(0.45, 10) \leq 244$ | $n_{\infty}^*(0.1, 10) \leq 8003$ |
| $n_{\infty}^*(0.45, 20) \leq 514$ | $n_{\infty}^*(0.1, 20) \leq 16648$ |
| $n_{\infty}^*(0.45, 40) \leq 1103$ | $n_{\infty}^*(0.1, 40) \leq 34679$ |
| $n_{\infty}^*(0.45, 60) \leq 1686$ | $n_{\infty}^*(0.1, 60) \leq 53020$ |
| $n_{\infty}^*(0.45, 80) \leq 2291$ | $n_{\infty}^*(0.1, 80) \leq 71777$ |

The bounds in [1] were achieved by using the same technique that we adapted in the proof of Theorem 4 and by analysing the average behaviour of the L^p -*-discrepancy for even integers p . Our bounds are smaller by factors between 5 and 8.1 than the bounds in [1] that make use of Hoeffding's inequality, and they are still smaller than the bounds resulting from the average L^p -*-discrepancy analysis – roughly by a factor 3 for $\varepsilon = 0.45$ and 1.6 for $\varepsilon = 0.1$.

We conclude by making some remarks on unanchored discrepancy. Instead of δ -covers of $[0, 1]^d$ we can define δ -covers for characteristic functions of axis-parallel boxes in $[0, 1]^d$. This definition is a special case of the more general notion of one-sided (μ, δ) -covers in [3]. We use our results about $N(d, \delta)$ to get bounds for the minimal cardinality of these modified δ -covers, which lead to upper bounds for the unanchored discrepancy. Those bounds are similar to the ones for the *-discrepancy in Theorem 4 – more or less, we just have to substitute d by $2d$ in each estimate.

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Discrepancy and declustering

Nils Hebbinghaus

(joint work with Benjamin Doerr and Sören Werth)

The declustering problem is to assign data blocks from a multi-dimensional grid system to one of M storage devices in a balanced manner. More precisely, we consider a grid $V = [n_1] \times \dots \times [n_d]$ for some positive integers n_1, \dots, n_d . Here we use the notations $[n] := \{1, 2, \dots, n\}$ and $[n..m] := \{n, n+1, \dots, m\}$ for $n, m \in \mathbb{N}$ with $n \leq m$.

A query Q requests the data assigned to a sub-grid $[x_1..y_1] \times \dots \times [x_d..y_d]$ for some integers $1 \leq x_i \leq y_i \leq n_i$. We assume that the time to process such a query is proportional to the maximum number of requested data blocks that are stored in a single device. If we represent the assignment of the data blocks to the devices by a mapping $\chi : V \rightarrow [M]$, then the query time of the query above is

$$\max_{i \in [M]} |\chi^{-1}(i) \cap Q|,$$

where we identify the query Q with its associated sub-grid. Clearly, no declustering scheme can do better than $|Q|/M$. Hence a natural performance measure is the additive deviation from this lower bound.

Thus the problem turns out to be a combinatorial discrepancy problem in M colors. Denote by \mathcal{E} the set of all sub-grids in V . Then $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph.

For a coloring $\chi : V \rightarrow [M]$, the positive discrepancy of \mathcal{H} with respect to χ and the positive discrepancy of \mathcal{H} in M colors are respectively

$$\begin{aligned} \text{disc}^+(\mathcal{H}, \chi) &:= \max_{i \in [M], E \in \mathcal{E}} \left(|\chi^{-1}(i) \cap E| - \frac{1}{M}|E| \right), \\ \text{disc}^+(\mathcal{H}, M) &:= \min_{\chi: V \rightarrow [M]} \text{disc}^+(\mathcal{H}, \chi). \end{aligned}$$

A similar definition was introduced by Srivastav and the first author in [4]. The only difference is that we regard positive instead of absolute deviations. Independently, Anstee, Demetrovics, Katona and Sali [1] and Sinha, Bhatia and Chen [8] proved a lower bound of $\Omega(\log M)$ for the additive error of any declustering scheme in dimension 2. Sinha *et al.* [8] also gave the bound $\Omega(\log^{(d-1)/2} M)$ for arbitrary $d \geq 3$, but their proof contains a crucial error.

The current best upper bounds in arbitrary dimension for the declustering schemes are proposed by C.-M. Chen and C. Cheng [3]. They present two schemes for d -dimensional problems with an additive error $O(\log^{d-1} M)$. The first one works if $M = p^k$ for some $k \in \mathbb{N}$ and p a prime such that $p \geq d$, whereas the second works for arbitrary M , but the error increases with N .

For the upper bounds, we present an improved scheme that yields an additive error of $O(\log^{d-1} M)$ for a broader range of values of M , which is independent of the data size. Our requirement on M is that if $M = q_1 \dots q_k$, where $q_1 < \dots < q_k$, is the canonical factorization of M into prime powers, we require $d \leq q_1 + 1$. Thus, in particular, our schemes work for M being a power of 2 (such that $M \geq d - 1$) and without any restriction on M in dimensions 2 and 3, which is very useful from the viewpoint of application. We also show that the latin hypercube construction used by Chen *et al.* [3] is much better than claimed. Where they show that the latin hypercube coloring extended to the whole grid has an error of at most 2^d times the one of the latin hypercube, we show that both errors are the same.

For the lower bound, we present the first correct proof of the $\Omega(\log^{(d-1)/2} M)$ bound. Again, a more careful analysis shows that the positive discrepancy is at least $1/2d$ times the normal discrepancy instead of 3^{-d} as claimed in [8]. Note that in typical applications with M between 16 and 1024, these 2^d and 3^d factors are at least as important as finding the right exponent of the $\log M$ term.

Since a central result of our investigation is on discrepancy bounds that are independent of the size of the grid, we usually work with the hypergraph $\mathcal{H}_N^d = ([N]^d, \mathcal{E}_N^d)$, where

$$\mathcal{E}_N^d = \left\{ \prod_{i=1}^d [x_i..y_i] : 1 \leq x_i \leq y_i \leq N \right\}$$

for some sufficiently large integer N . We have the following result.

Theorem 1. *Let $M, d \geq 2$ be positive integers and q_1 the smallest prime power in the canonical factorization of M into prime powers. We have*

- (i) $\text{disc}^+(\mathcal{H}_N^d, M) = O(\log^{d-1} M)$ for $d \leq q_1 + 1$, independently of $N \in \mathbb{N}$;
and

(ii) $\text{disc}^+(\mathcal{H}_N^d, M) = \Omega(\log^{(d-1)/2} M)$ for $N \geq M$.

The combinatorial discrepancy results are shown via strong results from geometric discrepancy theory. The problem of geometric discrepancy in the unit cube $[0, 1]^d$ is to distribute $n \in \mathbb{N}$ points evenly with respect to axis-parallel boxes: In every box R there should be approximately $n \text{vol}(R)$ points, where $\text{vol}(R)$ denotes the volume of R . Again, discrepancy quantifies the distance to a perfect distribution. The discrepancy of a point set \mathcal{P} with respect to a box $R \subseteq [0, 1]^d$ and the set of all axis-parallel boxes \mathcal{R}_d are defined by

$$D(\mathcal{P}, R) = |\mathcal{P} \cap R| - n \text{vol}(R),$$

$$D(\mathcal{P}, \mathcal{R}_d) = \sup_{R \in \mathcal{R}_d} |D(\mathcal{P}, R)|.$$

The general idea in the proofs of the lower bound in Sinha *et al.* [8] and Anstee *et al.* [1] is the same, described here in two dimensions.

Starting with an arbitrary M -coloring of $[M]^2$, there is a monochromatic set \hat{P} with M vertices. Based on this set, an M -point set \mathcal{P} in $[0, 1]^2$ is constructed. By discrepancy theory [7], there is a rectangle R such that $D(\mathcal{P}, R) = \Omega(\log M)$. Rounding R to the $[M]^2$ grid, they construct a hyperedge \hat{R} that has almost the volume as R . Additionally \hat{R} contains as many vertices of \hat{P} as R points of \mathcal{P} . With the help of \hat{R} and a short calculation the lower bound of the additive error $\Omega(\log M)$ is shown.

The small, but crucial, mistake in the proof of Sinha *et al.* [8] lies in the transfer from the geometric discrepancy setting back to the combinatorial one. Recall that the authors started with a color class of exactly M^{d-1} points. They scaled it down by a factor of M to a set in the unit cube (note that this is a subset of $\{0, 1/M, 2/M, \dots, (M-1)/M\}^d$). Then their geometric discrepancy argument yields a rectangle of polylogarithmic discrepancy. However, the rectangle $[0, (M-1)/M]^d$ has a much larger discrepancy: It contains all M^{d-1} points, but has a volume of $((M-1)/M)^d$ only. This immediately yields a discrepancy of $M^{d-1}(1 - ((M-1)/M)^d) = \Omega(M^{d-2})$. For dimension $d \geq 3$, this is larger than the upper bound, also indicating an error in the proof of Sinha *et al.* [8]. The last argument also shows that rounding an arbitrary box to a box in the grid can cause a roundoff error, which is of magnitude larger than the discrepancy. For this reason, a direct generalization using the lower bound of Roth [6] is not possible. A more careful analysis is needed. In particular, we have to ensure the existence of a small box having large discrepancy. Using ideas of Beck [2], we show the following.

Theorem 2. *For any n -point set \mathcal{P} in the unit cube $[0, 1]^d$, there is an axis-parallel cube Q with side at most $n^{-(2d-3)d/(d-1)^2(2d+1)}$ fully contained in $[0, 1]^d$ with*

$$D(\mathcal{P}, Q) = \Omega(\log^{(d-1)/2} n).$$

Now Theorem 1(ii) follows from Theorem 2 using the roundoff reduction of Anstee *et al.* [1] and Sinha *et al.* [8].

For the proof of our upper bound, we use geometric discrepancies to construct a declustering scheme. The notation of Niederreiter [5] is used in the following.

For an integer $b \geq 2$, an elementary interval in base b is an interval of the form

$$E = \prod_{i=1}^d [a_i b^{-d_i}, (a_i + 1) b^{-d_i}[,$$

with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq d$. For integers t, m such that $0 \leq t \leq m$, a (t, m, d) -net in base b is a point set of b^m points in $[0, 1]^d$ such that all elementary intervals with volume b^{t-m} contain exactly b^t points. Note that any elementary interval with volume b^{t-m} has discrepancy zero in a (t, m, d) -net. Since any subset of an elementary interval of volume b^{t-m} has discrepancy at most b^t and any box can be packed with elementary intervals in a way that the uncovered part can be covered by $O(\log^{d-1} n)$ elementary intervals of volume b^{t-m} , the following is immediate.

Theorem 3. *A (t, m, d) -net \mathcal{P} has discrepancy $D(\mathcal{P}, \mathcal{R}_d) = O(\log^{d-1} n)$.*

The central argument in our proof of the upper bound is the following result of Niederreiter [5] on the existence of $(0, m, d)$ -nets. From the viewpoint of application it is important that his proof is constructive.

Theorem 4. *Let $b \geq 2$ be an arbitrary base and $b = q_1 q_2 \dots q_u$ be the canonical factorization of b into prime powers such that $q_1 < \dots < q_u$. Then for any $m \geq 0$ and $d \leq q_1 + 1$, there exists a $(0, m, d)$ -net in base b .*

We construct colorings of \mathcal{H}_N^d from $(0, m, d)$ -nets with small discrepancy. We start with colorings for \mathcal{H}_M^d .

Theorem 5. *Let \mathcal{P}_{net} be a $(0, d - 1, d)$ -net in base M in $[0, 1]^d$. Then there is an M -coloring χ_M of $\mathcal{H}_M^d = ([M]^d, \mathcal{E}_M^d)$ such that all rows of $[M]^d$ contain every color exactly once and $\text{disc}(\mathcal{H}_M^d, \chi_M) \leq D(\mathcal{P}_{\text{net}}, \mathcal{R}_d)$.*

In Theorem 6 below, we show that it is sufficient to consider the discrepancy of \mathcal{H}_M^d with respect to these colorings for determining the upper bound of the discrepancy of \mathcal{H}_N^d . Theorem 6 is a reasonable improvement of Theorem 4.2 in [3], where

$$\text{disc}(\mathcal{H}_N^d, \chi) \leq 2^d \text{disc}(\mathcal{H}_M^d, \chi_M)$$

is shown. Note that this reduces the implicit constant in the upper bound by factor of 2^d .

Theorem 6. *Let χ_M be an M -coloring of \mathcal{H}_M^d such that all rows of $[M]^d$ contain every color exactly once and χ a coloring of \mathcal{H}_N^d defined by $\chi(x_1, \dots, x_d) = \chi_M(y_1, \dots, y_d)$ such that $x_i \equiv y_i \pmod{M}$ for $i \in [d]$, $x_i \in [N]$ and $y_i \in [M]$. Then*

$$\text{disc}(\mathcal{H}_N^d, \chi) = \text{disc}(\mathcal{H}_M^d, \chi_M).$$

The upper bound in Theorem 1 follows from the above.

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Quantum algorithms for numerical integration**Stefan Heinrich**

One of the most challenging questions of today, in the overlap of computer science, mathematics, and physics, is the exploration of potential capabilities of quantum computers. Milestones which intensified and enlarged research considerably were the algorithm of Shor [6], who showed that quantum computers could factor large integers efficiently (which is widely believed to be infeasible on classical computers) and the quantum search algorithm of Grover [1], which provides a quadratic speedup over deterministic and randomized classical algorithms of searching a database.

So far research was mainly concentrated on discrete problems like the above and many others one encounters in computer science. Much less is known about computational problems of analysis, including such typical field of application as high dimensional integration. We seek to understand how well these problems can be solved in the quantum model of computation (that is, on a – hypothetical – quantum computer) and how the outcome compares to the efficiency of deterministic or Monte Carlo algorithms on a classical (*i.e.* non-quantum) computer.

First steps were made by Novak [5], who considered integration of functions from Hölder spaces. This line of research was continued by the author [2], where quantum algorithms for the integration of L_p -functions and, as a key prerequisite, for the computation of the mean of p -summable sequences were constructed. In [2], a rigorous model of quantum computation for numerical problems was developed, as well. The case of integration of functions from Sobolev spaces is considered in [3], and more on the computation of the mean was presented in [4]. These papers also established matching lower bounds.

Combining these results with previous ones of information-based complexity theory about the best possible ways of solving the respective problems deterministically or by Monte Carlo on classical computers, we are now in a position to fairly well answer the question where quantum computation can provide a speedup in high dimensional integration and where not. We know cases among the above where quantum algorithms yield an exponential speedup over deterministic algorithms and a quadratic speedup over randomized ones (on classical computers). The talk gives an overview about the state of the art in this field.

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Geometric transversal problems

Jiří Matoušek

A fairly general formulation of the basic problem in discrepancy theory is this: We are given a ground set X (often \mathbb{R}^d), a system \mathcal{F} of subsets of X (such as all axis-parallel boxes), a probability measure μ on X such that all sets of \mathcal{F} are measurable, and a parameter $\varepsilon > 0$, and we want to find a probability measure ν supported on n points of X , with $n = n(\mathcal{F}, \mu, \varepsilon)$ as small as possible, such that $|\mu(F) - \nu(F)| \leq \varepsilon$ for all $F \in \mathcal{F}$. A related problem discussed in this talk is that of finding a small transversal for all large sets in \mathcal{F} , that is, a set $N \subseteq X$ with $N \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ with $\mu(F) \geq \varepsilon$. Such an N is called a weak ε -net for \mathcal{F} with respect to μ .

Well known results of Vapnik and Chervonenkis and of Haussler and Welzl show that if the VC-dimension of \mathcal{F} is finite, then there is ν as above supported on $O(\varepsilon^{-2} \log \varepsilon^{-1})$ points and N of size $O(\varepsilon^{-1} \log \varepsilon^{-1})$. Moreover, finite VC-dimension is necessary if we want ν or N of size bounded in terms of ε and \mathcal{F} hereditarily (also for \mathcal{F} restricted to any subset $Y \subseteq X$).

Interestingly, if we do not consider restrictions of \mathcal{F} to subsets of X , then weak ε -nets of bounded size exist for some set systems of infinite VC-dimension too. A prime example is the system of all convex sets in \mathbb{R}^d . Let us denote by

$f(d, \varepsilon)$ the smallest number such that every probability measure μ in \mathbb{R}^d admits a weak ε -net for convex sets with respect to μ . It is nontrivial to prove that $f(d, \varepsilon) < \infty$ for all d and $\varepsilon > 0$ (this was first done by Alon, Bárány, Füredi, and Kleitman). The best known upper bound is $f(d, \varepsilon) = O(\varepsilon^{-d}(\log \varepsilon^{-1})^{c(d)})$ for every fixed d with a suitable constant $c(d)$. The only known nontrivial lower bound is $f(d, 0.01) = e^{\Omega(\sqrt{d})}$ as $d \rightarrow \infty$. It would be very interesting to improve the quadratic upper bound on $f(2, \varepsilon)$, say, or to provide a superlinear lower bound. A nice (and perhaps hard) problem in high-dimensional convex geometry is to improve bounds on the minimum size of a weak ε -net for convex sets in \mathbb{R}^d with respect to μ , the uniform measure on S^{d-1} (this is the example used for the $e^{\Omega(\sqrt{d})}$ lower bound mentioned above).

For general set systems \mathcal{F} , the existence of weak ε -nets of bounded size seems closely related to the fractional Helly property, which is weaker than finite VC-dimension, but no satisfactory characterization is known.

Most of the material of this talk is covered in detail, for example, in the book [2], where detailed references are also provided. Some more recent results from [3], [1], and [4] are also reported.

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New bounds for the star discrepancy

Erich Novak

(joint work with Aicke Hinrichs)

Can we compute, up to some error $\varepsilon > 0$, the integral

$$I_d(f) = \int_{[0,1]^d} f(x) dx$$

for $f : [0, 1]^d \rightarrow \mathbb{R}$ from F_d in polynomial time, *i.e.*,

$$\text{cost}(\varepsilon, F_d) \leq C\varepsilon^{-\gamma}d^\beta?$$

In some applications the dimension d is (very) large. The answer depends on the classes F_d , see the survey [4]. For certain F_d , we have to study the star-discrepancy.

Let $M_n = \{t_1, \dots, t_n\} \subset [0, 1]^d$. The star-discrepancy $\text{disc}_\infty(M_n)$ is defined by

$$\text{disc}_\infty(M_n) = \sup_{x \in [0,1]^d} \left| x_1 \dots x_d - \frac{1}{n} \sum_{i=1}^n 1_{[0,x)}(t_i) \right|.$$

Low discrepancy sequences are quite often used in numerical analysis for the so called quasi-Monte Carlo methods. One obtains

$$\text{disc}_\infty(M_n) \leq C_d n^{-1}(\log n)^{d-1},$$

or similar upper bounds. It is not known whether these known M_n have a small discrepancy if d is large (say $d > 10$) and n is moderate (say $n \approx 10d^2$). In this direction we present and comment on the following main results, established respectively in [1] and [2].

Theorem 1. *There exists $c > 0$ such that for any $n, d \in \mathbb{N}$, there exists M_n with*

$$(1) \quad \text{disc}_\infty(M_n) \leq c\sqrt{\frac{d}{n}}.$$

Theorem 2. *There exists $k > 0$ such that*

$$\text{disc}_\infty(M_n) \geq k \min\left(\frac{d}{n}, 1\right)$$

for all M_n and all $n, d \in \mathbb{N}$.

Both results are proved using the Vapnik-Červonenkis dimension. The proof of the upper bound is probabilistic. It is not known how we can construct points $M_n \subset [0, 1]^d$ in polynomial (in n and d) time such that (1), or a slightly weaker estimate, holds.

Can we prove results with “less randomness”? Can we find a “small” subset of $[0, 1]^{dn}$ containing a low discrepancy set M_n ? We now discuss how the p -discrepancy might be of some help for these. The p -discrepancy of M_n is defined by

$$\text{disc}_p(M_n) = \left(\int_{[0,1]^d} \left| x_1 \dots x_d - \frac{1}{n} \sum_{i=1}^n 1_{[0,x)}(t_i) \right|^p dx \right)^{1/p}.$$

The discrepancy function is not “too peaked”, one can obtain upper bounds for $\text{disc}_\infty(M_n)$ from upper bounds of $\text{disc}_p(M_n)$. The idea is to compute the expectation $\mathbb{E}(\text{disc}_p^p(M_n))$ for even p with different distributions on $[0, 1]^{nd}$. We consider the Lebesgue measure λ and another measure. For even p , we obtain

$$\text{disc}_p^p(M_n) = \sum_{j=0}^p \binom{p}{j} (-n)^{-j} \sum_{(u_1, \dots, u_j) \in \{1, \dots, n\}^j} (p-j+1)^{-d} \prod_{m=1}^d \min_{k=1, \dots, j} (1 - t_{u_k, m}^{p-j+1}),$$

and so

$$\begin{aligned} & \mathbb{E}(\text{disc}_p^p(M_n)) \\ &= \sum_{j=0}^p \binom{p}{j} (-n)^{-j} \sum_{(u_1, \dots, u_j) \in \{1, \dots, n\}^j} (p-j+1)^{-d} \mathbb{E} \left(\prod_{m=1}^d \min_{k=1, \dots, j} (1 - t_{u_k, m}^{p-j+1}) \right). \end{aligned}$$

We consider first the case of Lebesgue measure. We obtain

$$\mathbb{E}_\lambda \left(\prod_{m=1}^d \min_{k=1, \dots, j} (1 - t_{k, m}^\alpha) \right) = \left(\frac{\alpha}{\alpha + j} \right)^d.$$

Let $\#(j, k, n)$ be the number of tuples $(u_1, \dots, u_j) \in \{1, \dots, n\}^j$ such that k different elements occur. Then

$$\mathbb{E}_\lambda(\text{disc}_p^p(M_n)) = \sum_{j=0}^p \binom{p}{j} (-n)^{-j} \sum_{k=0}^j (k + p - j + 1)^{-d} \#(j, k, n).$$

The numbers $\#(j, k, n)$ can be written with the Stirling numbers of first and second type. Using the fact that

$$\sum_{k=0}^{p-r+j} \binom{p}{r+k-j} (-1)^k s(k, k-j) S(k-j+r, k) = 0$$

for $p = 2m$ even, $r = 0, \dots, m - 1$ and $j = 0, \dots, r$, we obtain

$$(\mathbb{E}_\lambda(\text{disc}_p^p(M_n)))^{1/p} \leq 4p(p+2)^{1/p} 2^{-d/p} n^{-1/2}.$$

Compared with Theorem 1, one obtains a slightly weaker upper bound, see [1].

An improvement is possible using symmetrization. Let $X_i(M_n)(x) = 1_{[0,x)}(t_i)$. Then X_1, \dots, X_n are i.i.d. random variables with values in L_p and one gets, see [3],

$$\mathbb{E}_\lambda \left(\left\| \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}X_i) \right\|^p \right) \leq \mathbb{E}_{\lambda,\varepsilon} \left(2 \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i X_i \right\|^p \right).$$

Similar computation as above yields

$$\mathbb{E}_{\lambda,\varepsilon} \left(2 \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i X_i \right\|^p \right) = 2^p n^{-p} \sum_{k=0}^{p/2} (k+1)^{-d} \#(p/2, k, n).$$

Observe that now there is no cancellation of positive and negative terms, and one gets

$$(\mathbb{E}_\lambda(\text{disc}_p^p(M_n)))^{1/p} \leq 2\sqrt{p}(p+2)^{1/p} 2^{-d/p} n^{-1/2}.$$

The upper bound (n proportional to d) follows.

Next, we consider generalized lattices with shift. Now

$$M_n^{z,\Delta} = \{t_j = jz + \Delta \pmod{1} : j = 0, \dots, n-1\}$$

with $z, \Delta \in [0, 1]^d$. Consider

$$\mathbb{E}_{z,\Delta}(\text{disc}_p^p(M_n^{z,\Delta})) = \int_{[0,1]^{2d}} \text{disc}_p^p(M_n^{z,\Delta}) dz d\Delta.$$

Is it true that

$$\mathbb{E}_{z,\Delta}(\text{disc}_p^p(M_n^{z,\Delta})) \leq \mathbb{E}_\lambda(\text{disc}_p^p(M_n))?$$

One would need the numbers

$$\mathbb{E}_{z,\Delta} \left(\prod_{m=1}^d \min_{k=1, \dots, j} (1 - t_{u_k, m}^\alpha) \right),$$

i.e., the two-dimensional integrals

$$\int_0^1 \int_0^1 \max((j_1 z + \Delta)_1^\alpha, (j_2 z + \Delta)_1^\alpha, \dots, (j_k z + \Delta)_1^\alpha) dz d\Delta,$$

with j_i different natural numbers and $(j_i z + \Delta)_1$ are modulo 1, *i.e.*, $x = [x] + (x)_1$, $\alpha \in \{1, \dots, p+1\}$ and $k \in \{0, 1, \dots, p\}$.

An open problem is to prove an upper bound, such as (1), for lattices M_n .

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Discrepancy of $(0, 1)$ -sequences Friedrich Pillichshammer (joint work with Gerhard Larcher)

For a sequence x_0, x_1, \dots of points in the 1-dimensional unit interval $[0, 1)$, the discrepancy function Δ_N , where $N \geq 1$, is defined as $\Delta_N(\alpha) := A_N([0, \alpha])/N - \alpha$, for $0 \leq \alpha \leq 1$, where $A_N([0, \alpha))$ denotes the number of indices i satisfying $0 \leq i \leq N-1$ and $x_i \in [0, \alpha)$. Now the L_p -discrepancy $L_{p,N}$, for $p \geq 1$, of the sequence is defined as the L_p -norm of the discrepancy function Δ_N , *i.e.*, for $1 \leq p < \infty$, we set

$$L_{p,N} = L_{p,N}(x_0, x_1, \dots) := \left(\int_0^1 |\Delta_N(\alpha)|^p d\alpha \right)^{1/p}.$$

For $p = \infty$, we get the usual star discrepancy

$$D_N^* = D_N^*(x_0, x_1, \dots) := \sup_{0 \leq \alpha \leq 1} |\Delta_N(\alpha)|$$

of the sequence.

We consider the discrepancy of a special class of sequences in $[0, 1)$, namely the class of the so-called digital $(0, 1)$ -sequences. Digital $(0, 1)$ -sequences or, more generally, digital (t, s) -sequences were introduced by Niederreiter [3, 4], and they provide at the moment the most efficient method for generating sequences with small discrepancy.

We consider the discrepancy of digital $(0, 1)$ -sequences over \mathbb{Z}_2 . Choose an $\mathbb{N} \times \mathbb{N}$ matrix C over \mathbb{Z}_2 such that every left upper $m \times m$ matrix $C(m)$ has full rank over \mathbb{Z}_2 . For $n \geq 0$, let $n = n_0 + n_1 2 + n_2 2^2 + \dots$ be the base 2 representation of n . Then multiply the vector $\vec{n} = (n_0, n_1, \dots)^T$ with the matrix C to obtain

$$C\vec{n} =: (y_1(n), y_2(n), \dots)^T \in \mathbb{Z}_2^\infty,$$

and set

$$x_n := \frac{y_1(n)}{2} + \frac{y_2(n)}{2^2} + \dots$$

Every sequence constructed in this way is called digital $(0, 1)$ -sequence over \mathbb{Z}_2 .

The most famous digital $(0, 1)$ -sequence over \mathbb{Z}_2 is the well known van der Corput sequence which is generated by the $\mathbb{N} \times \mathbb{N}$ identity matrix.

Niederreiter [3, 4] proved that for any digital $(0, 1)$ -sequence over \mathbb{Z}_2 , we have

$$ND_N^* \leq \frac{\log N}{2 \log 2} + O(1)$$

for any $N \in \mathbb{N}$. There is also a well known lower bound due to Schmidt [8] which tells us that for any sequence in $[0, 1)$, for the star discrepancy D_N^* , we have

$$ND_N^* \geq \frac{\log N}{66 \log 4}$$

for infinitely many values of $N \in \mathbb{N}$. Hence the star discrepancy of any digital $(0, 1)$ -sequence over \mathbb{Z}_2 is of best possible order in N .

Our first result [5] is the following improvement of Niederreiter's result.

Theorem 1. *Let \tilde{D}_N^* denote the star discrepancy of any digital $(0, 1)$ -sequence over \mathbb{Z}_2 . For every $N \geq 1$, we have*

$$N\tilde{D}_N^* \leq ND_N^* \leq \frac{\log N}{3 \log 2} + 1,$$

where D_N^* denotes the star discrepancy of the van der Corput sequence.

Hence the van der Corput sequence is the worst distributed digital $(0, 1)$ -sequence over \mathbb{Z}_2 with respect to star discrepancy. For the star discrepancy of the van der Corput sequence we can say even more.

Theorem 2. *Let D_n^* denote the star discrepancy of the first n elements of the van der Corput sequence. For every $\varepsilon > 0$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ n \leq N : \frac{\log 2}{4} - \varepsilon \leq \frac{nD_n^*}{\log n} \leq \frac{\log 2}{4} + \varepsilon \right\} \right| = 1.$$

Finally we consider the L_2 -discrepancy of digital $(0, 1)$ -sequences. We can prove the following.

Theorem 3. *For the L_2 -discrepancy of any digital $(0, 1)$ -sequence over \mathbb{Z}_2 generated by a non-singular upper triangular (NUT) matrix, we have*

$$(NL_{2,N})^2 \leq \left(\frac{\log N}{6 \log 2} \right)^2 + O(\log N).$$

This is a generalization of a result of Faure [2] who proved this bound for the L_2 -discrepancy of the van der Corput sequence. Further, we know from [1, 5, 6] that

$$\limsup_{N \rightarrow \infty} \frac{NL_{2,N}}{\log N} = \frac{1}{6 \log 2}$$

for the L_2 -discrepancy of the van der Corput sequence. Hence we have the following consequence.

Theorem 4. *We have*

$$\limsup_{N \rightarrow \infty} \sup \frac{NL_{2,N}}{\log N} = \frac{1}{6 \log 2},$$

where the sup is extended over all digital $(0, 1)$ -sequences generated by an NUT matrix. In other words, the van der Corput sequence is essentially the worst distributed digital $(0, 1)$ -sequence over \mathbb{Z}_2 which is generated by an NUT matrix.

We compare this result with the lower bound of Roth [7] which tells us that there exists a constant $c > 0$ such that for the L_2 -discrepancy, for any sequence in $[0, 1)$, we have

$$L_{2,N} \geq c \frac{\sqrt{\log N}}{N}$$

for infinitely many values of $N \in \mathbb{N}$. So our upper bound is not best possible in the sense of Roth's lower bound. The following question arises: Is there a digital $(0, 1)$ -sequence over \mathbb{Z}_2 generated by an NUT matrix C such that for the L_2 -discrepancy of this sequence, we have

$$L_{2,N} \leq c_1 \frac{\sqrt{\log N}}{N}$$

for any $N \geq 2$, where $c_1 > 0$?

Until now no such sequence is known. Motivated by results from [5], we consider the digital $(0, 1)$ -sequence generated by the matrix

$$(1) \quad C = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 5. *For the L_2 -discrepancy of the digital $(0, 1)$ -sequence generated by the matrix C from (1), we have, for any $\varepsilon > 0$,*

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \left| \left\{ n \leq N : L_{2,n} \leq c \frac{(\log n)^{1/2+\varepsilon}}{n} \right\} \right| = 1.$$

Theorem 6. *For the digital $(0, 1)$ -sequence from Theorem 5, we have*

$$L_{2,N} > c \frac{\log N}{N}$$

for infinitely many $N \in \mathbb{N}$, where $c > 0$.

To summarize, it is well known that the star discrepancy of any digital $(0, 1)$ -sequence over \mathbb{Z}_2 is best possible in the order of magnitude in N . On the other hand, the question on whether there exist digital $(0, 1)$ -sequences with best possible order of L_2 -discrepancy (in the sense of Roth) or not seems to be a very difficult open problem.

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Combinatorial complexity of convex sequences and some other hard Erdős problems
Mischa Rudnev

I am not a 100% aware whether there is a precise definition of what constitutes a “hard Erdős problem”. However, there is a sort of general agreement about some of those great many questions posed by Erdős. Take for example the “distance conjecture”: Let $P_N \subset \mathbb{R}^d$ be a point set of N elements, where N is large. Let

$$\Delta(P_N) = \{t = \|x - y\| : x, y \in P_N\}$$

be the Euclidean distance set of P_N . Prove that its cardinality

$$(1) \quad |\Delta(P_N)| = \Omega_\varepsilon(N^{2/d}).$$

Above and below, the symbols Ω , \gtrsim or Ω_ε , \gtrsim_ε (O , \lesssim or O_ε , \lesssim_ε) are used to indicate lower (upper) bounds in the usual way. The symbol \approx stands for equality up to a constant (depending on d).

The distance conjecture has been mostly approached by methods of combinatorial geometry. See for example the book of Matoušek [10] for the state-of-the-art. The best result so far, specifically in $d = 2$, is ε slightly below $1/7$, due to Solymosi and Toth [12], improved a bit by Tardos [14].

The case when P_N is well-distributed (*i.e.*, when there exists a cube Q containing P_N and a pair of constants (c, C) such that a ball of radius c centered at any $p \in P_N$ contains no other points of P_N , while any ball of radius C , centered anywhere in Q does contain some $p \in P_N$) is of special interest. For example, if $P_N = \mathbb{Z}^2 \cap [0, \sqrt{N}]^2$, then we have $|\Delta(P_N)| \approx N/\sqrt{\log N}$, so for $d = 2$, the bound (1) is best possible.

It is expected [6] that in the well-distributed case, the distance conjecture should be true with the bound $|\Delta(P_N)| = \Omega(N^{2/d}/\log^2 N)$, using the methods of Fourier analysis. Turn each $p \in P_N$ into a small ball, endow the resulting set with a natural probability measure μ , and then study the distance measure ν_μ , generated by μ .

The theory was neatly set up by Mattila [11], in connection with the question of Falconer [3], whether any Borel set of Hausdorff dimension $s > d/2$ has a distance set of positive Lebesgue measure. An example with integer lattice points shows that $d/2$ would be the best possible, see [3]. The present status of Falconer's conjecture is $s = d(d+1)/2(d+1)$ due to Wolff [15] and Erdogan [2].

Application of Fourier analysis methods to the well-distributed set case prompts one to try to appeal to the methods developed for mean square discrepancy of the lattice point distribution, see, *e.g.*, [8]. However, in the latter case, there is the Poisson summation formula, which results in a curious fact that the corresponding distance measure ν_μ are commensurable point-wise.

The results obtained via Fourier analysis are easily extendible to non-isotropic distances, determined by a symmetric strictly convex body K , with a smooth boundary (K is a Euclidean ball for the Euclidean distance $\|\cdot\|$), as long as there is a lower bound for Gaussian curvature on ∂K . The effect of curvature is crucial and displays itself in a variety questions, one of which is discussed below. The motivation for it comes from estimating the L^p -norms of some trigonometric polynomials and a theorem of Konyagin [9]. See below.

The rest exposes the results of our recent work [7]. Let $B = \{1, 2, \dots, N\}$ be a "base" set. Let $S = \{s_j\}_{j=1}^N$ be a strictly convex sequence, *i.e.*, the differences $s_{j+1} - s_j$ are strictly monotone in j . One can assume that $s_j = f(j)$, $j \in B$, for some strictly convex function f . There is no bound on $D^2 f$ from below, except for $D^2 f > 0$.

Consider the equation

$$(2) \quad s_{j_1} + \dots + s_{j_d} = s_{j_{d+1}} + \dots + s_{j_{2d}}.$$

Let \mathcal{C}_d be the number of solutions of (2), with all j 's in B . It appears reasonable to conjecture that without any algebraic assumptions on f , one has

$$(3) \quad \mathcal{C}_d = O_\varepsilon(N^{2d-2}).$$

Traditionally, problems like this one have been studied by algebraic methods, see the survey [5]. For example, (3) follows easily if $f(x) = x^m$, where $m = 2, 3, \dots$. But Konyagin used combinatorics, namely the Szemerédi-Trotter (henceforth ST theorem – see [13] for a proof - bounding the number of incidences I for an arrangement $(\mathcal{L}, \mathcal{P})$ of lines (curves) and points in \mathbb{R}^2 as $I \lesssim |\mathcal{L}| + |\mathcal{P}| + (|\mathcal{L}||\mathcal{P}|)^{2/3}$) to get a robust bound

$$\mathcal{C}_2 = O(N^{5/2}),$$

no matter what S , as long as it is strictly convex. A paper by Elekes *et al.* [1] falls short of proving this result, instead giving the lower bound $N^{3/2}$ for the number of the elements of the sumset $2S = S + S$. Konyagin's result was repeated by Garaev [4], who removed ST as the (only) prerequisite for the proof.

The following theorem of Iosevich, Ten and the author generalizes Konyagin's theorem for $d \geq 2$.

Theorem 1. *For $d \geq 2$, let $\alpha = 2(1 - 2^{-d})$. Then $|\mathcal{C}_d| = O(N^{2d-\alpha})$.*

At the moment, the author is confident that conjecture (3) can be vindicated for a wide class of f 's by using the Fourier transform. The approach was roughly outlined in the final section of [7]. Our recent proof [6] of the Erdős distance conjecture for well-distributed sets uses the same ideas. However the constant in estimate (3) may end up being dependent on the lower bound for $D^2 f$.

If the sequence $\{s_i\}_{i \in B}$ is integer-valued, we deduce an estimate for the L^{2d} -norm of the Dirichlet kernel associated with S .

Theorem 2. *If $S \subset \mathbb{Z}$, for $\theta \in \mathbb{T}^1$, let*

$$F_N(\theta) = \sum_{j=1}^N e^{2\pi i b_j \theta}.$$

Then

$$\|f_N\|_{2d} = O\left(N^{1-\frac{1-2^{-d}}{d}}\right).$$

Theorem 1 was proved by induction in dimension, starting off $d = 2$. However, the higher-dimensional set-up is not amenable to the standard ST, unless one adds weights to it. Reduction to weighted ST is not obvious, as one is tempted to turn towards higher-dimensional versions of ST, which are nothing as good as the case $d = 2$. Though Fourier analysis is much more robust, as far as the dimension is concerned, see [6].

In fact, [1] proves a very similar estimate $\Omega(N^{2-2^{-d+1}})$ for cardinality of the sumset dS . However, for the latter estimate, no weighted ST turns out to be necessary; it also arises as a by-product in [7]. From the harmonic analysis point of view the two estimates end up being equivalent, see [6].

Weighted (in some sense) versions of ST have been around for a while, see [13]. But it is in connection with the weights where the main difficulty arises. On the inductive step $d \rightarrow d + 1$ of our proof, the lines involved in the incidences have weights, which they have inherited from the previous step. These weights are equal to the multiplicity of $c \in dS$, available from the previous step via a certain majorant, bounding the distribution function $\nu(t)$ of multiplicities (weights) over the elements of the sumset dS . Complexity \mathcal{C}_d is just the square of the L^2 -norm of ν .

If one writes down the incidence bound in the weighted set-up, it incorporates the L^∞ -norm of ν , which is too large. So the most non-trivial part of the proof of Theorem 1 is a lemma, which states that one can partition the weighted set of curves \mathcal{L} into some $\log \log N$ pieces, so that eventually one can use the L^1 -norm of ν in the estimate for the total number of incidences. This enables one to get an exponentially small error 2^{-d+1} , with respect to the conjectured bound.

Both works [7] and [6] are in essence based on the same simple principle: the L^2 -norm of the function ν in the former case and the distance measure ν_μ in the latter case should not be too large in comparison with the L^1 -norm, at most $O_\varepsilon(N)$ times greater. In both cases, this prevents the quantity in question from being supported on a thin set, yielding the desired result.

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Extremal additive intersective sets**Tomasz Schoen**

For a set $S = \{s_1, s_2, \dots\} \subseteq \mathbb{N}$, denote its counting function by $S(n) = |S \cap [n]|$, where $[n] = \{1, 2, \dots, n\}$. As usual, let $A+B$ be the set of all numbers represented in the form $a+b$, where $a \in A$ and $b \in B$. Let

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n \rightarrow \infty} \frac{A(n)}{n}.$$

Define

$$\underline{\text{int}}(S) = \sup_{(A+A) \cap S = \emptyset} \underline{d}(A).$$

We say that a set S has no intersective property if there is a set A such that

$$(A+A) \cap S = \emptyset \quad \text{and} \quad \underline{d}(A) = \frac{1}{2}.$$

Consider the following question of Erdős. Put $S(d, r, n) = |\{s \in S \cap [n] : s \equiv r \pmod{d}\}|$ and suppose that S satisfies the two conditions

$$(1) \quad \frac{S(d, r, n)}{S(n)} \rightarrow \frac{1}{d} \quad \text{as } n \rightarrow \infty \text{ for all } d, r \in \mathbb{N},$$

and

$$(2) \quad \frac{s_n}{s_{n+1}} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Is it true that for every set $A \subseteq \mathbb{N}$ with $(A + A) \cap S = \emptyset$, we have $\underline{d}(A) = \frac{1}{2}$?

We prove the following result which solves the problem of Erdős in the negative.

Theorem 1. *Let $\omega(n)$ be any increasing function tending to infinity as $n \rightarrow \infty$. Then there is a set $S \subseteq \mathbb{N}$ satisfying (1) and (2), having no intersective property and such that $S(n) \geq n/\omega(n)$ for every $n \in \mathbb{N}$.*

We also show that every sufficiently sparse set has no intersective property.

Theorem 2. *For every $\varepsilon > 0$, there is a set $S \subseteq \mathbb{N}$ such that $S(n) \leq \varepsilon \log n$ for every $n \in \mathbb{N}$, and*

$$\underline{\text{int}}(S) \leq \frac{1}{2} - \frac{1}{4 \cdot 2^{300/\varepsilon}}.$$

Theorem 3. *Let $S \subseteq \mathbb{N}$ be any set with $S(n) = o(\log n)$. Then S has no intersective property, so that there exists a set $A \subseteq \mathbb{N}$ with $\underline{d}(A) = \frac{1}{2}$ such that $(A + A) \cap S = \emptyset$.*

Theorem 4. *Let $\frac{1}{10} > \varepsilon > 0$, and let $S \subseteq \mathbb{N}$ be an arbitrary set with $S(n) \leq \varepsilon \log n$ for all sufficiently large n . Then*

$$\underline{\text{int}}(S) \geq \frac{1}{2} - \frac{4}{2^{1/\varepsilon}}.$$

A set S is called sum-intersective if for every set A with $\overline{d}(A) > 0$, we have $(A + A) \cap S \neq \emptyset$ (or $\overline{\text{int}}(S) = 0$). We know from Erdős and Sárközy [1] that if S is sum-intersective, then $S(n) = o(\log^2 n)$ is impossible. We also know from Ruzsa [2] that if $\omega(n) \rightarrow \infty$, then there is a sum-intersective set S with $S(n) = O(\omega(n) \log^2 n)$.

Our next result shows that Ruzsa's theorem is sharp.

Theorem 5. *If there is a constant C such that the inequality $S(n) \leq C \log^2 n$ has infinitely many solutions, then*

$$\overline{\text{int}}(S) \geq \frac{1}{2^{20C}}.$$

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Variation of the number of lattice points in large balls
Maxim Skriganov
(joint work with Alexander Sobolev)

Let $\Gamma \subset \mathbb{R}^d$, $d \geq 2$, be a lattice in the d -dimensional Euclidean space. For any bounded set $\mathcal{C} \subset \mathbb{R}^d$, we denote by $\mathcal{N}[\mathcal{C}]$ the number of lattice points in \mathcal{C} , that is

$$\mathcal{N}[\mathcal{C}] = \#\{\gamma \in \Gamma : \gamma \in \mathcal{C}\}.$$

Denote by

$$B(r; \mathbf{k}) = \{\boldsymbol{\xi} : |\boldsymbol{\xi} - \mathbf{k}| < r\}$$

the open ball of radius $r > 0$ centered at the point $\mathbf{k} \in \mathbb{R}^d$. The function $\mathcal{N}[B(r; \mathbf{k})]$ is a periodic function of the variable \mathbf{k} with the period lattice Γ , and hence it is bounded. We are interested in the variation of the quantity $\mathcal{N}[B(r; \mathbf{k})]$ as a function of \mathbf{k} . Define for all $r > 0$

$$\mathcal{N}^+(r) = \max_{\mathbf{k}} \mathcal{N}[B(r; \mathbf{k})], \quad \mathcal{N}^-(r) = \min_{\mathbf{k}} \mathcal{N}[B(r; \mathbf{k})],$$

and introduce the δ -variation of the counting function by writing

$$V(\lambda, \delta) = \mathcal{N}^+(\sqrt{\lambda - \delta}) - \mathcal{N}^-(\sqrt{\lambda + \delta})$$

for $\lambda \geq 0$ and $\delta \in [0, \lambda]$. Our objective is to find out when the δ -variation is non-negative and to obtain lower bounds for $V(\lambda, \delta)$ for small δ and large λ under the assumption that the lattice Γ is rational.

A lattice $\Gamma \subset \mathbb{R}^d$ is said to be rational if for any two vectors $\gamma_1, \gamma_2 \in \Gamma$, the inner product satisfies the relation

$$\langle \gamma_1, \gamma_2 \rangle = \beta_\Gamma r_{12},$$

where $\beta_\Gamma \neq 0$ is a real-valued constant independent of γ_1 and γ_2 , and where $r_{12} = r_{21}$ is an integer. Otherwise the lattice is called irrational.

For the cases $d = 2, 3$ quite precise lower bounds for $V(\lambda, \delta)$ are known to hold without any assumptions on the arithmetic properties of Γ . However, in higher dimensions these become important. Our main results are contained in Theorems 1, 2 and 3.

Theorem 1. *Let $\Gamma \subset \mathbb{R}^d$ be a rational lattice and let $d \geq 5$. Then there are three positive constants $\delta_0 = \delta_0(\Gamma)$, $\lambda_0 = \lambda_0(\Gamma)$ and c_Γ such that for all $\delta \in [0, \delta_0]$ and all $\lambda \geq \lambda_0$, we have*

$$(1) \quad V(\lambda, \delta) \geq c_\Gamma \lambda^{(d-2)/2}.$$

The bound (1) is sharp.

Theorem 2. *Let $\Gamma \subset \mathbb{R}^4$ be a rational lattice. Then there are three positive constants $\delta_0 = \delta_0(\Gamma)$, $\lambda_0 = \lambda_0(\Gamma)$ and c_Γ such that for all $\delta \in [0, \delta_0]$ and all $\lambda \geq \lambda_0$, we have*

$$(2) \quad V(\lambda, \delta) \geq c_\Gamma \lambda (\log \log \lambda)^{-1}.$$

It is not yet clear whether one can get rid of the log log-factor in (2) for general rational lattices. However, for the case of a cubic lattice Γ , this can be done.

Theorem 3. *Let $\Gamma = \mathbb{Z}^4$. Then for each $\delta \in [0, 10^{-4}]$, all sufficiently large $\lambda \geq \lambda_0 > 0$ and some $c > 0$, one has the bound*

$$V(\lambda, \delta) > c\lambda.$$

The proofs of Theorems 1, 2 and 3 are based on the classical results on representation of integers by the integer quadratic forms and some arguments from the geometry of numbers.

One-sided discrepancy of hyperplanes in \mathbb{F}_q^r

Anand Srivastav

(joint work with Nils Hebbinghaus and Tomasz Schoen)

We study the one-sided discrepancy and discrepancy of the hypergraph $\mathcal{H}_{q,r} = (\mathbb{F}_q^r, \mathcal{E}_{q,r,1})$ of linear hyperplanes in \mathbb{F}_q^r , where \mathbb{F}_q^r is the r -dimensional vector space over \mathbb{F}_q and $\mathcal{E}_{q,r,1}$ is the set of all its linear hyperplanes, *i.e.*, the subspaces of codimension one. Let $n := q^r$.

The bounds on the discrepancy can be derived with standard methods (lower bound with the eigenvalue technique and the upper bound via the VC-dimension) and are given by

$$\frac{\sqrt{z(1-z)}}{q} \sqrt{n} - \frac{c-1}{c} \geq \text{disc}(\mathcal{H}_{q,r,1}, c) \geq \alpha \sqrt{\frac{n}{qc}} c^{1/2(r-1)}.$$

Since the one-sided discrepancy satisfies $\text{disc}^+(\cdot) \leq \text{disc}(\cdot)$, we have the same upper bound. Our main result is the proof of the lower bound for the one-sided discrepancy $\text{disc}^+(\mathcal{H}_{q,r,1}, c)$, given by

$$\text{disc}^+(\mathcal{H}_{q,r,1}, c) \geq \frac{\sqrt{z(1-z)}}{4q(q-1)\sqrt{c}} \sqrt{n} - \frac{q-1}{q}.$$

This is accomplished by Fourier analysis on the additive group \mathbb{F}_q^r . Note that for $q = O(1)$ and $c = O(1)$, the bounds are tight and give a new example for Spencer's six-standard-deviation theorem [6].

Finally, we generalise our main result for the one-sided discrepancy to the hypergraph $\mathcal{H}_{q,r,m} = (\mathbb{F}_q^r, \mathcal{E}_{q,r,m})$, where $\mathcal{E}_{q,r,m}$ is the set of all subspaces of \mathbb{F}_q^r of codimension m , where $m \leq r-3$.

Let V be a finite set and \mathcal{E} a subset of 2^V . Then $\mathcal{H} := (V, \mathcal{E})$ is called a hypergraph. A c -coloring of \mathcal{H} is a function $\chi : V \rightarrow M_c$, where M_c is any set of cardinality c . For convenience we take $M_c = \{1, 2, \dots, c\} =: [c]$, but in applications a different choice of M_c can be helpful (see [1]).

Let $A_i := \chi^{-1}(i)$ be the color-class of color $i \in [c]$ in V . The c -color discrepancy of \mathcal{H} with respect to χ is defined by

$$\text{disc}(\mathcal{H}, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left| |A_i \cap E| - \frac{|E|}{c} \right|,$$

and the c -color discrepancy of \mathcal{H} is

$$\text{disc}(\mathcal{H}, c) = \min_{\chi: V \rightarrow [c]} \text{disc}(\mathcal{H}, \chi, c).$$

For $c = 2$, the c -color discrepancy is exactly half of the common two-color discrepancy where the two colors are represented by 1 and -1 . For further information on discrepancies, we refer to Beck and Sós [2] and Matoušek [4]. For our purposes, a related discrepancy notion will be relevant. The one-sided c -color discrepancy of \mathcal{H} with respect to χ is

$$\text{disc}^+(\mathcal{H}, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left(|A_i \cap E| - \frac{|E|}{c} \right),$$

and the one-sided c -color discrepancy of \mathcal{H} is

$$\text{disc}^+(\mathcal{H}, c) = \min_{\chi: V \rightarrow [c]} \text{disc}^+(\mathcal{H}, \chi, c).$$

Trivially we have $\text{disc}^+(\mathcal{H}, c) \leq \text{disc}(\mathcal{H}, c)$, where equality holds for $c = 2$.

Let \mathbb{F}_q be the field of q elements, where $q = p^k$ is a power of a prime p , $V := \mathbb{F}_q^r$ the r -dimensional vector space over \mathbb{F}_q , and let $\mathcal{E}_{q,r,m}$ be the set of all subspaces of V of codimension m . Put $n := |V| = q^r$. For a set $S \subseteq \mathbb{F}_q^r$ define $S^\# := S \setminus \{0\}$. We investigate the discrepancy of the hypergraph $\mathcal{H}_{q,r,m} = (V, \mathcal{E}_{q,r,m})$. Note that $\mathcal{H}_{q,r} = (V, \mathcal{E}_{q,r,1})$ is an (n/q) -uniform hypergraph on n vertices with $|\mathcal{E}_{q,r,1}| = (n-1)/(q-1)$ hyperedges.

We define a new hypergraph $\mathcal{H}' := (V', \mathcal{E}')$ with $V' := V \setminus \{0\}$ and

$$\mathcal{E}' := \{E \cap V' : E \in \mathcal{E}_{q,r,1}\}.$$

For $q = 2$, this hypergraph has constant pair-degree, *i.e.*, there exists a $\lambda \in \mathbb{N}$ with

$$|\{E \in \mathcal{E} : i, j \in E\}| = \lambda,$$

for all $i, j \in V, i \neq j$. For such hypergraphs $\mathcal{H} = (V, \mathcal{E})$, we can extend the “trace”-lower bound of Beck and Sós [2] to c -colors and obtain

$$\text{disc}(\mathcal{H}, c) \geq \left(\frac{1}{c^2 |\mathcal{E}|} \sum_{v \in V} (d_v - \lambda) \right)^{1/2},$$

with d_v as degree of v . In fact, this lower bound can be extended to cover also the incidence matrix of $\mathcal{H}_{q,r,1}$ for $q > 2$, where we do not have constant pair-degree, but the pair-degree cannot vary too much. This yields the bound of the following theorem. The upper bound is obtained by a c -color generalisation of a theorem of Matoušek [4] for hypergraphs with bounded VC-dimension by Doerr and Srivastav [3].

Theorem 1. Let $z := \frac{(q-1) \bmod c}{c}$. Then there is a constant $\alpha > 0$ with

$$\frac{\sqrt{z(1-z)}}{q} \sqrt{n} - \frac{c-1}{c} \leq \text{disc}(\mathcal{H}_{q,r,1}, c) \leq \alpha \sqrt{\frac{n}{qc}} c^{1/2(r-1)}.$$

For one-sided discrepancy, we invoke the Fourier transform on \mathbb{F}_q^r in the following way. For simplicity, we take here $q = 2$. A subspace $E \subseteq \mathbb{F}_q^r$ of codimension 1 is uniquely determined by a vector $z \in \mathbb{F}_q^r$, where $E^\perp = \langle z \rangle$. Thus, for $A \subseteq \mathbb{F}_q^r$, the function

$$E \rightarrow |A \cap E| - \frac{|E|}{c}$$

is a function of z , denoted by $f(z)$, and we may build $\widehat{f}(z)$. A sophisticated interplay between the growth of Fourier coefficients and the size of color classes leads to the following main result.

Theorem 2. Let $z := \frac{(q-1) \bmod c}{c}$ and $q^{r-1} \geq q^{r/2} + 6q^2$. There exists a constant $\alpha > 0$ such that for every $c \geq 2$, we have

$$\frac{\sqrt{z(1-z)}}{4q(q-1)\sqrt{c}} \sqrt{n} - \frac{q-1}{q} \leq \text{disc}(\mathcal{H}_{q,r}, c) \leq \alpha \sqrt{\frac{n}{qc}} c^{1/2(r-1)}.$$

Note that the lower bounds for discrepancy and one-sided discrepancy differ by a factor of about $4(q-1)\sqrt{c}$.

Using our theorems, we can extend the result from linear hyperplanes to subspaces of codimensions $m \leq r-3$.

Theorem 3. Let $z := \frac{(q-1) \bmod c}{c}$. If $q^{r-m} \geq q^{(r-m+1)/2} + 6q^2$, there is a constant $\alpha > 0$ such that for $m \leq r-3$, we have

$$\frac{\sqrt{z(1-z)}}{4(q-1)\sqrt{c}} \sqrt{\frac{n}{q^{m+1}}} - \frac{q-1}{q} \leq \text{disc}(\mathcal{H}_{q,r,m}, c) \leq \alpha \sqrt{\frac{n}{qc}} c^{1/2(r-m)}.$$

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Metric discrepancy theory

Robert Tichy

In the first part of the lecture a survey on normal numbers and metric theory of uniform distribution is given. In the second part metric theorems for distribution measures of pseudorandom sequences are discussed – joint work with W. Philipp.

Let $\chi(x) = 2\mathbf{1}_{[0, \frac{1}{2})}(\{x\}) - 1$, where $\{x\}$ denotes the fractional part of x and $\mathbf{1}_A$ the indicator function of the set A . Throughout this abstract (n_k) denotes an increasing sequence of positive integers and $\omega \in [0, 1)$. For $k \geq 1$, we define

$$(1) \quad e_k := \chi(n_k\omega).$$

The well-distribution measure of stage N of the sequence (1) is defined as

$$(2) \quad W_N := \max_{a,b,t} \left| \sum_{j \leq t} e_{a+bj} \right|, \quad N \geq 1,$$

where the maximum is extended over all $a \in \mathbb{Z}$ and $b, t \in \mathbb{N}$ such that $1 \leq a + b \leq a + bt \leq N$. This measure of pseudorandomness was first introduced by Mauduit and Sárközy [1]. As was already noted by them, there is nothing special about the interval $[0, \frac{1}{2})$ since W_N can be bounded by the discrepancy D_t of the defining sequence $(n_k\omega, k \geq 1)$ in the form

$$(3) \quad W_N \leq \max_{a,b,t} t D_t(\{n_{a+bj}\omega\}).$$

Here, for a fixed sequence (x_j) with $0 \leq x_j < 1$,

$$D_t(x_j) := \sup \left(\left| \frac{1}{t} \sum_{j \leq t} \left(\mathbf{1}_{[\alpha, \beta)}(x_j) - (\beta - \alpha) \right) \right| : 0 \leq \alpha < \beta \leq 1 \right)$$

denotes the discrepancy in the sense of uniform distribution modulo 1. In view of relation (3) we will formulate our results in terms of discrepancies.

Among other things, Mauduit and Sárközy [2, 3] prove metric results for sequences $n_k = k^d$, where $d \in \mathbb{N}$. Our first result can handle arbitrary increasing sequences (n_k) and for $d \geq 3$ it yields a sharper error term.

Theorem 1. *Let $(n_k, k \geq 1)$ be an increasing sequence of positive integers. Then for almost all ω and arbitrary $\varepsilon > 0$,*

$$(4) \quad \max \left(t D_t(\{n_{a+bj}\omega\}) : a \in \mathbb{Z}, b, t \in \mathbb{N}, 1 \leq a + b \leq a + bt \leq N \right) \ll N^{2/3} (\log N)^{1+\varepsilon}.$$

The third part is devoted to the analysis of pair correlations as studied by Rudnick, Sarnak and Zaharescu. Here some joint results of I. Berkes, W. Philipp and R. Tichy are presented.

We prove a Glivenko-Cantelli type strong law of large numbers for the pair correlation of independent random variables. Except for a few powers of logarithms the results obtained are sharp. Similar estimates hold for the pair correlation of lacunary sequences $\{n_k\omega\}$ modulo 1.

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Average decay of Fourier transforms, geometry of planar convex bodies, and discrepancy theory
Giancarlo Travaglini

A number of facts in discrepancy theory depends on estimates for the decay of the Fourier transform (see [2, 8, 10, 14]). Our first example is given by the following result, which extends a theorem of D. Kendall (see [9, 7]): Let $B \subset \mathbb{R}^d$ be a convex body. For $\rho \geq 2$, $\sigma \in SO(d)$ and $t \in \mathbb{T}^d$, consider the discrepancy

$$D_{\sigma(B)-t}(\rho) = \text{card}((\rho\sigma(B) - t) \cap \mathbb{Z}^d) - \rho^d|B|.$$

Then

$$(1) \quad \int_{\mathbb{T}^d} \int_{SO(d)} |D_{\sigma(B)-t}(\rho)|^2 d\sigma dt \leq c\rho^{d-1}.$$

The inequality (1) depends on the inequality

$$\int_{\Sigma_{d-1}} |\widehat{\chi}_B(\rho\gamma)|^2 d\gamma \leq c\rho^{-d-1}$$

(see [11, 5]). L^2 results such as (1) do not depend on the shape of B , which instead plays a role when we replace L^2 with L^p , $p < 2$. We consider L^1 and we state our second example, which depends on upper and lower estimates for

$$\int_{\Sigma_1} |\widehat{\chi}_B(\rho\gamma)| d\gamma$$

(see [12, 13, 3, 4, 7]).

Theorem 1. *Let P be a convex polygon and let K be a planar convex body with piecewise smooth boundary, different from a polygon. Then*

$$(2) \quad c_1 \log \rho \leq \int_{\mathbb{T}^2} \int_{SO(2)} |D_{\sigma^{-1}(P)-t}(\rho)| d\sigma dt \leq c_2 \log^2 \rho,$$

$$(3) \quad c_1 \rho^{1/2} \leq \int_{\mathbb{T}^2} \int_{SO(2)} |D_{\sigma^{-1}(K)-t}(\rho)| d\sigma dt \leq c_2 \rho^{1/2}.$$

Here we wish to show nearly best possible results (see [6]) for intermediate cases between (2) and (3). In order to do this, we scale between discs and polygons in two different, although related, ways. The first one consists of approximating the convex body B with certain polygons, especially tailored for the Fourier transform, and then counting the number of sides of these polygons. The second one consists

of a fractal measure of the image of the Gauss map on ∂B . In both cases we need estimates of the Fourier transform of the characteristic function of a polygon with many sides. These estimates depend partially on a development of the following remark.

Remark. Let T be a triangle and let $\widehat{\chi}_T(\rho\Theta)$ be the Fourier transform of its characteristic function, written in polar coordinates $\rho \geq 2$ and $\Theta = (\cos \theta, \sin \theta)$. Then we have $|\widehat{\chi}_T(\rho\Theta)| \leq c\theta\rho^{-2}$ when Θ is not orthogonal to a side of T , while we only have $|\widehat{\chi}_T(\rho\Theta)| \leq c\rho^{-1}$ in the three remaining directions. Then (see [4, 7]) one can prove that

$$\int_0^{2\pi} |\widehat{\chi}_T(\rho\Theta)| \, d\theta \leq c\rho^{-2} \log \rho.$$

Now let $P = P_N$ be a polygon with N sides, of lengths not greater than 1. By splitting P into triangles, we obviously get

$$\int_0^{2\pi} |\widehat{\chi}_P(\rho\Theta)| \, d\theta \leq cN\rho^{-2} \log \rho,$$

with c independent of N . It turns out that this last “trivial” inequality is nearly sharp, since for any $\varepsilon > 0$, we cannot replace N in the right hand side above by $N^{1-\varepsilon}$ (see [14]).

Estimates for the decay of the Fourier transform can be also used to prove lower bounds for irregularities of distribution. As an example we consider the following theorem, which is a basic result in the theory and it has been independently proved in [1] and [10].

Theorem 2. *Let B be a convex body in \mathbb{T}^2 . For every finite set $\{u(j)\}_{j=1}^N \subset \mathbb{T}^2$, we have*

$$(4) \quad \int_0^1 \int_{SO(2)} \int_{\mathbb{T}^2} \left| -Ns^2|B| + \sum_{j=1}^N \chi_{s\sigma^{-1}(B)-t}(u(j)) \right|^2 \, dt d\sigma ds \geq cN^{1/2}.$$

It is possible to prove that for certain choices of B the inequality (4) holds and it is best possible even without averaging over dilations.

Theorem 3. *Let T be a triangle in \mathbb{T}^2 . For every finite set $\{u(j)\}_{j=1}^N \subset \mathbb{T}^2$, we have*

$$\int_{SO(2)} \int_{\mathbb{T}^2} \left| -N|T| + \sum_{j=1}^N \chi_{\sigma^{-1}(T)-t}(u(j)) \right|^2 \, dt d\sigma \geq cN^{1/2}.$$

The lower bound depends on the argument in [10] and on estimates in [4], while the upper bound runs as in (1).

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Polynomial-time algorithms for multivariate linear problems with finite-order weights

Grzegorz Wasilkowski

(joint work with Henryk Woźniakowski)

There is a host of practical problems that deal with functions of very many variables. In many cases, the required error tolerance for such problems is not too small. Then the classical estimates are asymptotic for n going to ∞ and for fixed the number d of variables, and they are usually of no practical value if n is fixed and d is very large. For instance, the classical discrepancy bounds are of the form $n^{-1}(\log n)^{d-1}$ and become meaningful only when the number n of function evaluations significantly exceeds e^d . This is why, since its introduction in 1994, see [12], there has been an increasing interest in the study of tractability of multivariate problems. Recall that a problem is tractable if it is possible to reduce the initial error ε -times by using a polynomial number of evaluations in ε^{-1} and d ; and it is strongly tractable if this number is independent of d . We stress that the upper bound on the number of evaluations should hold for all $\varepsilon \in (0, 1)$ and all $d = 1, 2, \dots$, including the case of huge d and relatively large ε , say $\varepsilon = 10^{-1}$. Algorithms that compute an ε -approximation and use a polynomial number of evaluations in ε^{-1} and d are called polynomial-time algorithms; and if this number does not depend on d they are called strongly polynomial-time algorithms.

There are many results on the tractability of multivariate problems. However, quite a few of them are not constructive, see the survey paper [4] and many papers cited there. Most of the results are obtained for problems defined over general tensor product spaces, including Banach spaces, see, *e.g.*, [3]. As observed in a number of papers, see, *e.g.*, [1, 5, 7, 8], there are important problems, including problems in mathematical finance and physics, that deal with functions which only depend on groups of few variables. That is, the functions depend on all d variables; however, they are sums of terms each of which depends only on few, say q^* , variables. For some applications, the number q^* is fairly small, *e.g.*, $q^* = 1$ or 2. An example of such functions with $q^* = 2m$ is provided by the Coulomb potential function where

$$f(\mathbf{x}) = \sum_{i \neq j, i, j=1}^d (\|x_i - x_j\|_2 + \alpha)^{-1}$$

for vectors $x_j \in \mathbb{R}^m$ and a positive α . That is, f only depends on groups of two variables each being an m -dimensional vector.

Functions of d variables can be written as the sum of functions of groups x_u of variables with u varying through all subsets of the index set $\{1, 2, \dots, d\}$. That is, for $\mathbf{x} = [x_1, x_2, \dots, x_d]$, we have

$$f(\mathbf{x}) = \sum_{u \subset \{1, 2, \dots, d\}} \gamma_{d,u} f_u(\mathbf{x})$$

for some functions f_u depending only on x_j for $j \in u$, and non-negative weights $\gamma_{d,u}$. The essence of the example with the Coulomb potential function is that $\gamma_{d,u} = 0$ for all u with cardinality greater than $2m$. If such a special structure of functions is present in a specific problem, it is said that the problem has finite-order weights; see [2, 5] where the concept of finite-order weights has been introduced.

When such a structure is properly used we might be able to obtain efficient algorithms that are polynomial-time or even strongly polynomial time algorithms. Indeed, it has recently been shown in [2, 5] that this is the case for approximating integrals

$$\int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x}$$

for Sobolev and Korobov spaces of functions equipped with finite-order weights. In this case, the quasi-Monte Carlo algorithms based on such classical low discrepancy points as Niederreiter, Halton, Sobol, lattice rules and shifted lattice rules are polynomial or even strongly polynomial-time algorithms.

More general problems, including the weighted L_2 -approximation problem, have been studied in a recent paper [10]. It was shown there that, under a special assumption (1), these problems are tractable or even strongly tractable for reproducing kernel Hilbert spaces equipped with finite-order weights. More specifically, an upper bound on the number of evaluations needed to compute an ε -approximation was shown to be independent on d and of order ε^{-2} or ε^{-4} ; the former dependence

for algorithms that use properly chosen linear functional evaluations, and the latter for algorithms that use only function evaluations at properly chosen points. For some problems these bounds are not sharp; however, in full generality, the bound of order ε^{-2} cannot be improved. The bound ε^{-4} is probably not sharp, and the proof of it is non-constructive.

The research presented here may be viewed as a continuation of [10]. Indeed, under slightly different assumptions and using different proof techniques, we provide constructions of polynomial-time algorithms that use only function evaluations for linear problems over reproducing kernel Hilbert spaces equipped with finite-order weights. These algorithms are derived for arbitrary $d \geq 2$ in terms of tensor products of algorithms for $d = 1$ in a way similar to weighted Smolyak algorithms studied in [9], see also [6]. Upper bounds on the number of evaluations needed to compute an ε -approximation for general d are practically the same as for $d = 1$ as far the dependence on ε^{-1} is concerned. Hence, these upper bounds are sharp in ε^{-1} if we use optimal algorithms for $d = 1$. The dependence on d is polynomial and the degree of this polynomial depends on the order of the weights, *i.e.*, on the largest cardinality of u for which $\gamma_{d,u}$ is still non-zero.

We explain our results in more technical terms for the following simplified version of weighted approximation problem, where one wants to recover f with the error measured in a weighted L_2 -norm

$$\sqrt{\int_{D_d} |f(\mathbf{x}) - (Af)(\mathbf{x})|^2 \rho_d(\mathbf{x}) \, d\mathbf{x}}.$$

Here $D_d = D \times \dots \times D$ with $D \subset \mathbb{R}$, $\rho_d = \prod_{k=1}^d \rho(x_k)$ is a probability density function on D_d , and Af is an approximation given by an algorithm A . We assume that functions f belong to a reproducing kernel Hilbert space F_d whose formal definition was presented during the talk, see also [11]. The condition (1) from [10] relates the kernel K defining the space and the probability density ρ by assuming that

$$(1) \quad \int_D K(x, x) \rho(x) \, dx < \infty.$$

Let $A_{d,\varepsilon}$ be one of the proposed algorithms that computes an ε -approximation for the d -dimensional case. Letting $\text{card}(A_{d,\varepsilon})$ denote the corresponding number of function evaluations used by the algorithm $A_{d,\varepsilon}$, we show that for any positive δ , there exists a positive number a_δ such that

$$(2) \quad \text{card}(A_{d,\varepsilon}) \leq a_\delta \varepsilon^{-p(1+\delta)} d^{q^*} \quad \forall d, \varepsilon.$$

Here p can be chosen as the smallest exponent for the case $d = 1$, and q^* is the order of the weights, *i.e.*, $\gamma_{d,u} = 0$ for all $u \in \{1, 2, \dots, d\}$ with the cardinality $|u| > q^*$. In particular, this means that, modulo $(1 + \delta)$, the exponent of ε^{-1} is as small as possible. Since we do not assume that the condition (1) is satisfied, the exponent p can be arbitrarily large. As in [10], $p < 4$ if (1) holds. For smooth problems, however, p is much smaller than 4.

We now comment on the results concerning algorithms that may use arbitrary functional evaluations. As already mentioned, general results with constructive proofs have been obtained in [10] with the exponent $p = 2$. Under an additional assumption and using different proof techniques, we construct optimal algorithms with $\text{card}(A_{d,\varepsilon})$ bounded as in (2). Hence, we may have the exponent p much smaller than 2. We also show that this bound is sharp in both ε^{-1} and d .

Under yet an additional assumption that the weights $\gamma_{d,u}$ depend on u only via $|u|$, we show a necessary and sufficient condition for the approximation problem to be strongly tractable and we present strongly polynomial-time algorithms. We also show that sometimes there is a tradeoff between the minimal exponents of ε^{-1} and d . Indeed, for strongly tractable problems we have a sharp bound of the form

$$\text{card}(A_{d,\varepsilon}) \leq c\varepsilon^{-p'} \quad \forall d, \varepsilon.$$

Furthermore, (2) also holds; however, the exponent p' is in general larger than p in (2). This means that by increasing the exponent of ε^{-1} we can obtain the bound independent of d .

The results discussed here will be published in [11].

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Integration, tractability, discrepancy

Henryk Woźniakowski

In this talk we discuss recent progress on solving multivariate integration when the number d of integrand variables is in hundreds or thousands. Such high dimensional integrals occur in many applications including financial mathematics and computational physics. We want to approximate

$$I_d(f) = \int_{D_d} \rho_d(t) f(t) dt,$$

where $D_d \subset \mathbb{R}^d$, the function ρ_d is non-negative and its integral over D_d is one, and real f belongs to a normed class F_d of integrable functions.

We restrict our attention to the worst case setting although different settings such as average, randomized and quantum are also studied. We approximate $I_d(f)$ by a quadrature rule

$$Q_{n,d} = \sum_{j=1}^n a_j f(t_j).$$

Here, t_j are sample points from the domain of f , and a_j are real numbers. For QMC (quasi-Monte Carlo) rules we have $a_j = 1/n$. The number n denotes the total number of function values used by $Q_{n,d}$.

The worst case error of $Q_{n,d}$ is defined as its worst performance for approximating integrals for the unit ball of F_d ,

$$e(Q_{n,d}) = \sup_{f \in F_d, \|f\| \leq 1} |I_d(f) - Q_{n,d}(f)|.$$

Clearly, the cost of using $Q_{n,d}$ is proportional to n , and therefore we would like to use n as small as possible with the worst case error below a given threshold. For $n = 0$, we formally set $Q_{0,d} = 0$, and then the worst case error is called the initial error which is the norm $\|I_d\|$ of the integration in the space F_d .

We consider two error criteria. The first one is the absolute error criterion in which we want to guarantee that the worst case error is at most ε , *i.e.*, $e(Q_{n,d}) \leq \varepsilon$. The second one is the normalized error criterion in which we want to guarantee that the worst case reduces the initial error by a factor of ε , *i.e.*, $e(Q_{n,d}) \leq \varepsilon \|I_d\|$. Here, the error parameter $\varepsilon \in (0, 1)$.

Define $n(\varepsilon, F_d)$ as the minimal number of function values needed to satisfy the absolute or normalized error criterion. If $n(\varepsilon, F_d)$ can be bounded by a polynomial in ε^{-1} and d , then multivariate integration in F_d is called tractable, *i.e.*, there exist non-negative numbers C, p and q such that

$$n(\varepsilon, F_d) \leq C \varepsilon^{-p} d^q \quad \forall \varepsilon \in (0, 1), d = 1, 2, \dots$$

If $q = 0$ in the bound above, then $n(\varepsilon, F_d)$ is bounded by a polynomial in ε^{-1} independently of d , and then multivariate integration in F_d is called strongly tractable.

The study of tractability, not only for multivariate integration, has recently become a popular research subject. The main point is to identify classes F_d for

which strong tractability or tractability hold. A survey of current results and approaches may be found in [4].

For some spaces F_d , the worst case error is the same as the L_2 or star discrepancy. In this case, tractability is equivalent to finding discrepancy bounds of n sample points with polynomial dependence on d and converging to zero as a positive power of n^{-1} . For instance, consider the Sobolev space of functions defined over $[0, 1]^d$ which are one time differentiable with respect to all variables and satisfying the boundary condition $f(t) = 0$ if at least one of the components of t is zero. The norm of f in this space is defined by the L_2 norm of $\partial^d f / \partial t_1 \dots \partial t_d$. Then the worst case of $Q_{n,d}$ is exactly the L_2 discrepancy of t_j . It turns out that for the absolute error criterion, multivariate integration is strongly tractable. On the other hand, for the normalized error criterion, multivariate integration is intractable.

If we remove the boundary condition and redefine the norm in the L_p sense by taking projections of f as in the Zaremba and Koksma-Hlawka (in)equalities, the situation changes. For the L_2 case, multivariate integration is intractable for the two error criteria. Surprisingly enough, if we switch to the L_1 case, then the worst case error is the same as the star discrepancy. In this case, the two error criteria are the same since the initial error is one. It turns out that we now have tractability, but not strong tractability, as proven in [2].

It was observed in many papers that integrands of practical importance have additional properties which are not properly modeled by classical spaces. Namely, in many cases, integrands are sums of functions that depend only on groups of a few variables, or that they depend on the successive variables in the diminishing sense. This additional structure of integrands may be modeled by weighted spaces of functions in which each group of variables has a weight moderating its importance. Tractability for weighted spaces has been initiated in [5]. For some spaces we know necessary and sufficient conditions on the weights to obtain strong tractability or tractability. For instance, take the Sobolev space without the boundary condition with the L_2 norm as above, and equip the space with the weight γ_j for each variable. This means that the norm of the space is redefined and $\|f\| \leq 1$ with small γ_j means that f weakly depends on the j th variable. Then, in particular, strong tractability of multivariate integration holds iff $\sum_{j=1}^{\infty} \gamma_j < \infty$ as proven in [5, 3].

What seems especially promising is the idea of finite-order weights as introduced in [1] and [6]. The weights are finite-order if they are zero for all groups of variables of cardinality greater than, say, k . Here k is independent of d and usually relatively small. For instance, for some financial problems, $k = 1$ or $k = 2$, and for some problems in computational physics, $k = 6$. It turns out that for finite-order weights multivariate integration is tractable and often strongly tractable. However, the error bounds are exponential in k . This, in turn, is not dangerous as long as k is not large. Furthermore, classical sample points such as Halton, Niederreiter or Sobol lead to tractable error bounds.

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Participants

Prof. Dr. Imre Barany

barany@renyi.hu
barany@math.ucl.ac.uk
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Prof. Dr. Jozsef Beck

jbeck@math.rutgers.edu
Dept. of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08903 – USA

Prof. Dr. William W.L. Chen

wchen@maths.mq.edu.au
Department of Mathematics
Macquarie University
Sydney NSW 2109 – Australia

Dr. Benjamin Doerr

bed@numerik.uni-kiel.de
Mathematisches Seminar
Bereich 2
Christian-Albrechts-Universität
Christian-Albrechts-Platz 4
D-24098 Kiel

Dr. Michael Drmota

michael.drmota@tuwien.ac.at
Michael.drmota@dmg.tuwien.ac.at
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedener Hauptstr. 8 - 10
A-1040 Wien

Dr. Michael Gnewuch

mig@numerik.uni-kiel.de
Mathematisches Seminar
Christian-Albrechts-Universität
Kiel
D-24098 Kiel

Nils Hebbinghaus

nhe@numerik.uni-kiel.de
Mathematisches Seminar
Bereich 2
Christian-Albrechts-Universität
Christian-Albrechts-Platz 4
D-24098 Kiel

Prof. Dr. Stefan Heinrich

heinrich@informatik.uni-kl.de
Fachbereich Informatik
Universität Kaiserslautern
D-67653 Kaiserslautern

Prof. Dr. Jiri Matousek

matousek@kam.mff.cuni.cz
Department of Applied Mathematics
Charles University
Malostranske nam. 25
118 00 Praha 1 – Czech Republic

Prof. Dr. Erich Novak

novak@minet.uni-jena.de
novak@mathematik.uni-jena.de
Fakultät für Mathematik
und Informatik
Friedrich-Schiller-Universität
Ernst-Abbe-Platz 2
D-07743 Jena

Prof. Dr. Friedrich Pillichshammer

friedrich.pillichshammer@jku.at
Institut für Analysis
Universität Linz
Altenbergstr. 69
A-4040 Linz

Prof. Dr. Michael Rudnev

M.Rudnev@bris.ac.uk
School of Mathematics
University of Bristol
University Walk
GB-Bristol BS8 1TW

Dr. Tomasz Schoen

schoen@amu.edu.pl
Zakad Matematyki Dyskretnej
Adam Mickiewicz University
pokoj nr B3-31
ul. Umultowska 87
61-614 Poznan – Poland

Prof. Dr. Maxim Skriganov

skrig@pdmi.ras.ru
St. Petersburg Branch of
Mathematical Institute of
Russian Academy of Science
Fontanka 27
191023 St. Petersburg – Russia

Prof. Dr. Anand Srivastav

asr@numerik.uni-kiel.de
Mathematisches Seminar
Christian-Albrechts-Universität
Kiel
Ludewig-Meyn-Str. 4
D-24118 Kiel

Prof. Dr. Robert F. Tichy

tichy@weyl.math.tu-graz.ac.at
tichy@tugraz.at
Institut für Mathematik
Technische Universität Graz
Steyrergasse 30
A-8010 Graz

Prof. Dr. Giancarlo Travaglini

travaglini@matapp.unimib.it
Dipt. Matematica e Applicazioni
Universita' Milano Bicocca
Via Bicocca degli Arcimboldi 8
I-20126 Milano

Prof. Dr. Grzegorz W. Wasilkowski

greg@cs.engr.uky.edu
Computer Science Department
University of Kentucky
773 Anderson Hall
Lexington, KY 40506-0046 – USA

Prof. Dr. Emo Welzl

emo@inf.ethz.ch
Theoretische Informatik
ETH-Zürich
ETH-Zentrum
CH-8092 Zürich

Prof. Dr. Henryk Wozniakowski

henryk@cs.columbia.edu
Department of Computer Science
Columbia University
450 Computer Science Building
1214 Amsterdam Avenue
New York, NY 10027-7003 – USA