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Multiplier Ideal Sheaves in Algebraic and Complex Geometry

Organised by
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Introduction by the Organisers

The conference was organized by Joseph Kohn (Princeton), Georg Schumacher (Marburg), and Yum-Tong Siu (Harvard), and was attended by 44 participants. Its aim was to put together a group from both complex analysis and algebraic geometry, reflecting recent developments, where the title of the conference stands for phenomena and methods, closely related to both of these areas.

The original approach involving the theory of partial differential equations and subelliptic estimates was addressed in several contributions, including estimates for the $\overline{\partial}$-Neumann problem, subelliptic PDE's and sub-Riemannian Geometry, and subelliptic estimates from an algebraic-geometric point of view.

Main areas were also applications to the abundance conjecture, pseudoeffective bundles, and the use of the twisted Nakano identity to investigate the cohomology of multiplier ideals. Critical points of sections of holomorphic vector bundles were discussed from a probabilistic viewpoint.

Concerning the hyperbolicity of complex manifolds, for entire analytic curves the multiplicity of the associated current with respect to subsets was studied, and holomorphic curves in semi-abelian varieties.

Another main topic was recent results on invariants arising from multiplier ideal sheaves, and critical exponents of analytic functions related to jumping coefficients, applications to local analytic geometry, and also results concerning the Fujita conjecture. Finally recent progress on transcendental Morse inequalities was presented.

Workshop on Multiplier Ideal Sheaves in Algebraic and Complex Geometry

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Abstracts

Estimates for the $\overline{\partial}$-Neumann problem

David Catlin

For a given smoothly bounded pseudoconvex domain one can prove that subelliptic estimates hold for the $\overline{\partial}$-Neumann problem, provided that it is possible to construct plurisubharmonic functions φ_δ in Ω so that $|\varphi_\delta| \le 1$ and so that the Hessian

$$H(\varphi_\delta)(z) \ge \delta^{-2\varepsilon}|z|^2, \text{ if } -\delta \le \Lambda(z) \le 0.$$

In order to construct φ_δ one can define for each point $z \in \partial\Omega$ the multi-type $m(z) = (m_1, \cdots, m_n)$ which measures the "flatness" of $\partial\Omega$ at z. If the type $T(z_0) < \infty$, then the function φ_δ can be constructed for some $\varepsilon > 0$.

Geometric stability of the cotangent bundle and the universal cover of a projective manifold

Thomas Peternell

Let X_n be a complex projective n-dimensional manifold and $\tilde{X}$ its universal cover. The Shafarevitch conjecture asserts that $\tilde{X}$ is holomorphically convex, i.e. admits a proper holomorphic map onto a Stein space. There are two extremal cases, namely that this map is constant, i.e. $\tilde{X}$ is compact. This means that $\pi_1(X)$ is finite and not much can be said further. Or the map is a modification, i.e through the general point of $\tilde{X}$ there is no positive-dimensional compact subvariety, e.g. $\tilde{X}$ is Stein. This happens in particular for X an Abelian variety or a quotient of a bounded domain. It is conjectured (see [Ko93], and [CZ04] for the Kähler case) that X should then have a holomorphic submersion onto a variety of general type with Abelian varieties as fibres, after a suitable finite étale cover and birational modification. This follows up to dimension 3 from the solutions of the conjectures of the Minimal Model Program. We prove here a weaker statement in every dimension:

Theorem 1. *Let X_n be a projective manifold and suppose that its universal cover is not covered by its positive-dimensional compact subvarieties. Then either X is of general type, or $\chi(\mathcal{O}_X) = 0$. In particular, if $\tilde{X}$ is Stein (or does not contain a compact subvariety of positive dimension), then either K_X is ample, or: K_X is nef, $K_X^n = 0$, and $\chi(\mathcal{O}_X) = 0$.*

This theorem is deduced (via the comparison theorem [Ca95], which relates the geometric positivity of subsheaves in the cotangent bundle to the geometry of $\tilde{X}$) from the following more general:

Theorem 2. *Let X_n be a projective manifold. Suppose that Ω_X^p contains for some p a subsheaf whose determinant is big, i.e. has maximal Kodaira dimension n. Then K_X is big, i.e. $\kappa(X) = n$.*

This in turn follows from a slight generalization of Miyaoka's theorem that the cotangent bundle of a projective manifold is "generically nef" unless the manifold is uniruled, and from a characterization of pseudo-effective line bundles by moving curves [BDPP04]. We show indeed that quotients of Ω^p_X have a pseudo-effective determinant if X is not uniruled:

Theorem 3. *Let X be a projective manifold and suppose that X is not uniruled. Let Q be a torsion free quotient of $(\Omega^1_X)^{\otimes m}$ for some $m > 0$. Then* $\det Q$ *is pseudo-effective; i.e. it's class is contained in the closure of the effective cone.*

The uniruledness criterion has also other applications, e.g. one can prove that a variety admitting a section in a tensor power of the tangent bundle with a zero, must be uniruled.
Theorem 2 is actually a piece in a larger framework. To explain this, we consider subsheaves $\mathcal{F} \subset \Omega^p_X$ for some $p > 0$. Then one can form $\kappa(\det \mathcal{F})$ and take the supremum over all $\mathcal{F}$. This gives a modified Kodaira dimension $\kappa^+(X)$, introduced in [Ca95]. Conjecturally

$$\kappa^+(X) = \kappa(X)$$

unless X is uniruled. Theorem 2 is nothing than this conjecture in case $\kappa^+(X) = \dim X$. We shall prove the conjecture in several cases, e.g. in dimension 4. It is very much related to (actually a consequence of) the following:

Conjecture: *Suppose X is a projective manifold, and suppose a decomposition*

$$NK_X = A + B$$

with some positive integer N and a pseudo-effective line bundle B (one may assume A spanned). Then

$$\kappa(X) \geq \kappa(A).$$

The special case $A = \mathcal{O}_X$ implies that $\kappa(X) \geq 0$ if X is not uniruled, using the preceding result, and the pseudo-effectiveness of K_X when X is not uniruled ([BDPP04]). In another direction we obtain the special case in which B is numerically trivial:

Theorem 4. *Let X be a projective complex manifold, and $L \in \mathrm{Pic}(X)$ be numerically trivial. Then:*

1. $\kappa(X, K_X + L) \leq \kappa(X)$.
2. *If $\kappa(X) = 0$, and if $\kappa(X, K_X + L) = \kappa(X)$, then L is a torsion element in the group* $\mathrm{Pic}^0(X)$.

In particular, if mK_X is numerically equivalent to an effective divisor, then $\kappa(X) \geq 0$. The proof uses an important result of Simpson [Si93].

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The Hodge theory of algebraic maps

Luca Migliorini

(joint work with Mark A. de Cataldo)

In the talk given at the Oberwolfach seminar I reported on joint work with M.A.de Cataldo, (SUNY at Stony Brook). Most of the results described are contained in the preprint AG0306030, [dCM2]. Let X be a projective variety. The main theme is to investigate the failure of the Hard Lefschetz theorem when we consider the cup product with the first Chern class of a line bundle L on X which is not ample but only NEF, whose class is therefore at the boundary of the ample cone. Unfortunately it seems that the problem in general is quite hard. The problem admits nevertheless a complete solution if L is in addition semiample (i.e. a power of the bundle is globally generated). Our previous paper [dCM]characterizes those semiample bundle for which the Hard Lefschetz theorem continues to hold . The condition is that the map that such a bundle defines is semismall, i.e. the map $f : X \to Y$ defined by some multiple of L verifies the condition:
If $V \subseteq X$ is a subvariety, $2\dim V - \dim f(V) \leq \dim X$
The fact that this condition is precisely the condition which ensures that the direct image of the constant sheaf $\mathbb{Q}_X[n]$ is perverse suggests a relation between the problem and the decomposition theorem due to Beilinson, Bernstein, Deligne and Gabber [BBD]. To give a measure of the failure of the Hard Lefschetz theorem we consider the "weight filtration" introduced already in [D]. It is a filtration on $\oplus H^k(X)$ related to the Jordan form of the nilpotent operator given by the cup-product with $c_1(L)$. By its very construction this filtration is given by Hodge substructures. On the other hand, if a multiple of L defines the map f, there is a different filtration on the same vector space arising from the theory of t-structures. We consider a stratification $Y = \coprod Y_l$ for f, i.e. an algebraic stratification for Y such that the map

$$f : f^{-1}(Y_l) \to Y_l$$

is a topologically trivial fibration, and the self-dual (middle-perversity) t-structure on Y. this gives a series of truncations of $f_*\mathbb{Q}_X[n]$ which originate a filtration on $\oplus H^k(X)$ after taking hypercohomology. Observe that this filtration is defined locally by purely topological constructions. The main theorem is that these two apriori unrelated filtrations coincide up to some shift. This relation of a topological invariant of the map (the perverse filtration) with a Hodge theoretic global invariant can be thought of as "relative Hodge theory". The result gives several strong gemetric consequences especially non-trivial contraction criteria, and a proof of a refined (polarized) version of the Decomposition Theorem [BBD]for complex varieties. Related results are also contained in the series of papers by Saito [Sai1], [Sai2], [Sai3]which present a D-module theoretic approach and in the recent paper by Sabbah [Sab]which suggests a vast generalization of the theory.

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Holomorphic Curves in Semi-Abelian Varieties and Applications

Junjiro Noguchi

(joint work with J. Winkelmann and K. Yamanoi)

1. Introduction

Let $f : \mathbb{C} \to V$ be a holomorphic curve into a complex projective manifold V with Zariski dense image and let D be an effective reduced divisor on V. For a real (1,1)-form ω on V we define the order function $T_f(r, \omega)$ by

$$T_f(r, \omega) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\omega,$$

where $\Delta(t)$ is the disk of radius t with center at the origin. When ω is positive definite, we abbreviate $T_f(r) = T_f(r, \omega)$. If ω belongs to the Chern class $c_1(L(D))$ of the line bundle $L(D)$ given by D, we write $T_f(r, L(D)) = T_f(r, \omega)$.
If $f(\mathbb{C}) \not\subset D$ we have the pull-back divisor $f^*D = \sum_{a \in f^{-1}D} \mathrm{ord}_a f^*D \cdot a$. We define the truncated counting functions to level $l \leq \infty$ by

$$n_l(t, f^*D) = \sum_{a \in f^{-1}(D) \cap \Delta(t)} \min\{\mathrm{ord}_a f^*D, l\}, \quad N_l(r f^*D) = \int_1^r \frac{n_l(t, f^*D)}{t} dt.$$

As one of the most elementary facts derived from the First Main Theorem, we have Nevanlinna's inequality

$$N_\infty(r, f^*D) < T_f(r, L(D)) + O(1), \tag{1.1}$$

where the term $O(1)$ is independent from D belonging to the linear system.
The next and more interesting problem is to establish the so-called Second Main Theorem which provides an opposite inequality to (1.1). It was proved in [3] and [4] that if the space $H^0(V, \Omega^1_V(\log D))$ of logarithmic 1-forms along D is "ample" in a sense, then we have the following inequalities of the second main theorem type,

$$\begin{aligned} \kappa T_f(r) &\le N_\infty(r; f^*D) + O(\log r) + O(\log T_f(r))||, \\ \kappa' T_f(r) &\le N_1(r; f^*D) + O(\log r) + O(\log T_f(r))||, \end{aligned}$$

where "$||$" stands for the inequality to hold for every $r > 1$ outside a Borel set of finite Lebesgue measure. It is an interesting and fundamental problem to determine the constant κ or κ'. In the case where V is the compactification of a semi-abelian variety A this problem is related to what kind of compactification V of A we take.
The value distribution theory for semi-abelian varieties is important since it unifies the classical Borel Theorem and Bloch-Ochiai's Theorem (log-Bloch-Ochiai's theorem; [1], [11], [3], [4], [6]) The other related motivations are as follows:
(i) Applications to Kobayashi hyperbolicity questions.
(ii) Analogue to *abc*-Conjecture over semi-abelian variety.
(iii) Analogous result over algebraic function fields by [8].
(iv) Analysis of holomorphic jet flows.
In our former paper [9] we proved that for a holomorphic curve $f : \mathbb{C} \to A$ into a semi-abelian variety A of dimension n and an algebraic divisor D on A, there is a number l_0 such that

$$T_f(r; L(\bar{D})) \le N_{l_0}(r; f^*D) + O(\log r) + O(\log T_f(r; L(\bar{D})))||. \tag{1.2}$$

Here we used a compactification $\bar{A}$ of A such that the maximal affine subgroup $(\mathbb{C}^*)^t$ of A was compactified by $(\mathbb{P}^1(\mathbb{C}))^t$, and we assumed a boundary condition (Condition 4.11 in [9]) for the closure $\bar{D}$ of D in $\bar{A}$; this roughly meant the divisor $\bar{D} + (\bar{A} \setminus A)$ to be in general position.
Let $J_l(\bar{A}; \log \partial A) \cong \bar{A} \times \mathbb{C}^{nl}$ be the l-th logarithmic jet space and let $J_l(f) : \mathbb{C} \to J_l(\bar{A}; \log \partial A)$ be the l-th jet lift of f. Let $X_l(f)$ be the Zariski closure of $J_l(f)(\mathbb{C})$ in $J_l(\bar{A}; \log \partial A)$ and let $\pi_2 : J_l(\bar{A}; \log \partial A) \to \mathbb{C}^{nl}$ be the projection.
The key idea of the proof of (1.2) was a lemma:

Lemma 1. *Let the notation be as above. Then there is a number $l_0 = l_0(f, D)$ such that*

$$\pi_2(J_l(\bar{D}; \log \partial A) \cap X_l(f)) \ne \pi_2(X_l(f)), \quad l \ge l_0.$$

Bloch's idea ([1]) was to find an ample part of jet-differentials restricted to $X_l(f)$, but to this Lemma 1 the ampleness of jet-differentials restricted to $X_l(f)$ is totally irrelevant.

For abelian A Lemma 1 was also used in Siu-Yeung [12] (Claim 1) and they improved the dependence of l_0 so that $l_0 = l_0(c_1(L(D)))$ depends only on the Chern number of $L(D)$. Lately K. Yamanoi [14] proved that for abelian A

$$T_f(r; L(D)) \leq N_1(r; f^*D) + \epsilon T_f(r; L(D)))||_\epsilon. \tag{1.3}$$

Our problem here is
(i) to remove the technical boundary Condition 4.11 in [9];
(ii) to extend (1.3) to the case of semi-abelian varieties.

2. Main results

We keep the notation in §1.

Main Theorem ([10]). *Let A be a semi-abelian variety. Let $f : \mathbb{C} \to A$ be a holomorphic curve with Zariski dense image. Let D be an effective reduced Cartier divisor on $X_k(f)$ ($k \geq 0$). Then there exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that*

$$T(r; \omega_{\bar{D}, J_k(f)}) \leq N_1(r; J_k(f)^*D) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0, \tag{2.1}$$

where $\bar{D}$ is the closure of D in $\bar{X}_k(f)$.
In the case of $k = 0$ the compactification $\bar{A}$ of A can be chosen as smooth, equivariant with respect to the A-action, and independent of f; furthermore, (2.1)takes the form

$$T_f(r; L(\bar{D})) \leq N_1(r; f^*D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad \forall \epsilon > 0. \tag{2.2}$$

Note that in the above estimate (2.1) or (2.2) the term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log r) + O(\log T_f(r))$" (see [9] Example (5.36)). It is interesting to observe that the small error term being "$O(\log r) + O(\log T_f(r; L(\bar{D})))||$", the truncation level l in (1.2) has to depend on D, but the small error term being allowed to be "$\epsilon T_f(r; L(\bar{D})))||_\epsilon$", l can be one, the smallest possible.
To deal with semi-abelian varieties the main difficulties are caused by the following two points:
(i) Semi-abelian varieties are not compact and need some good compactifications.
(ii) There is no Poincaré reducibility theorem for semi-abelian varieties.
It is also noted that some part of the proof of the Main Theorem for abelian varieties in [14] does not hold for semi-abelian varieties ([14] §3 Claim).
The following are important steps in the proof of the Main Theorem:

Proposition 2. *Let B be the stabilizer of $X = X_k(f)$. (Then* $\dim B > 0$ *([5], [7]).) Assume that $B \cap \mathrm{St}(D) = \{0\}$. Then there is a compactification $\bar{X}$ of X and a B-equivariant blow-up $\psi : X^\dagger \to \bar{X}$ such that $X^\dagger$ has a stratification by B-invariant strata $X^\dagger = \cup_\lambda \Gamma_\lambda$ satisfying the following properties:*

- (i) *$\Gamma_\lambda \cong X/B_x$ $(x \in \Gamma_\lambda)$ where $B_x = \{b \in B : b \cdot x = x\}$ is the isotropy group at x.*
- (ii) *The closure of D in $X^\dagger$ contains none of the strata Γ_λ.*

(iii) *The open subset X of $X^\dagger$ coincides with one of the strata Γ_λ.*

Theorem 3. *Let $f : \mathbb{C} \to A$ be a holomorphic curve and let $Z \subset X_k(f)$ be a subvariety of $\operatorname{codim}_{X_k(f)} Z \geq 2$. Then*

$$N_1(r; J_k(f)^* Z) \leq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

3. Applications

(a) In [2] M. Green discussed the algebraic degeneracy of a holomorphic curve $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ omitting an effective reduced divisor D on $\mathbb{P}^n(\mathbb{C})$ with normal crossings and of degree $\geq n+2$. He proved the following theorem and conjectured that it would hold without the condition of finite order for f:

Theorem 4. (M. Green [2]) *Let $f : \mathbb{C} \longrightarrow \mathbb{P}^2(\mathbb{C})$ be a holomorphic curve of finite order and let $[x_0, x_1, x_2]$ be the homogeneous coordinate system of $\mathbb{P}^2(\mathbb{C})$. Assume that f omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then the image $g(\mathbb{C})$ lies in a line or a conic.*

Here we answer his conjecture in more general form:

Theorem 5. *Let $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let $[x_0, \ldots, x_n]$ be the homogeneous coordinate system of $\mathbb{P}^n(\mathbb{C})$. Assume that f omits hyperplanes given by*

$$x_i = 0, \qquad 1 \leq i \leq n, \tag{3.1}$$

and a hypersurface defined by

$$x_0^q + \cdots + x_n^q = 0, \qquad q \geq 2.$$

Then f is algebraically degenerate.

(b) Let A be a semi-abelian variety as above and let $X \subset J_k(A)$ be an irreducible algebraic subvariety. We consider the existence problem of an algebraically nondegenerate entire holomorphic curve $f : \mathbb{C} \to A$ such that $J_k(f)(\mathbb{C}) \subset X$ and $J_k(f)(\mathbb{C})$ is Zariski dense in X. This is a problem of a system of algebraic differential equations described by the equations defining the subvariety X.
The first necessary condition for the existence of such solution f is that $\operatorname{St}(X) \neq \{0\}$ (cf. [5], [7]). Now we assume the existence of such f. Then we take a big line bundle $L \to X$ and a section $\sigma \in H0(X, L)$ which defines a reduced divisor on X. We arbitrarily fix a trivialization

$$J_k(f)^* L \cong \mathbb{C} \times \mathbb{C}, \tag{3.2}$$

and regard $J_k(f)^*\sigma$ as an entire function.

Theorem 6. *Let the notation be as above. Then there is no entire function $\psi(z)$ such that every zero of $\psi(z)$ has degree ≥ 2 and*

$$J_k(f)^*\sigma(z) = \psi(z), \qquad z \in \mathbb{C}. \tag{3.3}$$

In particular, there is no entire function $\psi(z)$ satisfying

$$J_k(f)^*\sigma(z) = (\psi(z))^q, \qquad z \in \mathbb{C}, \tag{3.4}$$

where $q \geq 2$ is an integer.

The property given by (3.3) or (3.4) is independent of the choice of the trivialization (3.2).

(c) The truncation level one in the Second Main theorem (2.2) allows the following immediate improvement of Theorem 6.1. in [9].

Theorem 7. *Let A be a compact complex torus and D a divisor which contains no positive-dimensional translate of a subtorus of A. Let $\pi : X \to A$ be a finite ramified covering which is ramified at all points in $\pi^{-1}(D)$. Then X is Kobayashi hyperbolic.*

Further applications to Kobayashi hyperbolicity questions will be discussed in a future article.

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Numerically Trivial Foliations

Thomas Eckl

Let L be a nef line bundle on a smooth projective variety $/\mathbb{C}$. Then the numerical dimension $\nu(X, L)$ of L is $\geq$ the Kodaira dimension $\kappa(X, L)$. The (generalized) abundance conjecture predicts that for $L = K_X$ we even have

$$\nu(X, K_X) = \kappa(X, K_X).$$

AIM: To give a geometric reason for strict inequality

$$\nu(X, L) > \kappa(X, L).$$

A partial answer to this request is the construction of **numerically trivial foliations** [Eck03a].
This concept was developed to answer a slightly different question:

Theorem 1 (Nef-reduction, [BCE+00])**.** *Let X, L be as above. Then there exists an almost holomorphic map $f : X \dashrightarrow Y$ with connected fibers such that*

(i) *$L_{|F} \equiv 0$ on fibers F of f, and*
(ii) *for general $x \in X$, for every irreducible curve $C \ni x$ with $\dim f(C) > 0$ we have $L.C > 0$.*

The dimension of Y is called the **nef dimension** $n(X, L)$ of L, and it is easy to show that $\nu(X, L) \leq n(X, L)$.

QUESTION: Are there geometric reasons for **strict** inequality ?

Now, in [Eck03a] the following is proven:

Theorem 2. *There always exists a foliation (with singularities) and a sequence (h_k) of metrics on L such that*

(i) *the curvature form $T_k = i\Theta_{h_k} \geq -\epsilon_k \omega$ for some (fixed) Kähler form ω, and $\epsilon_k \to 0$,*
(ii) *for all $U \cong \Delta$ open on a leave:*

$$\lim_{k\to\infty} \int_U T_k = 0,$$

and the foliation contains all foliations satisfying (i) and (ii).

Furthermore it is shown that the nef fibration from above is just the biggest fibration contained in this numerically trivial foliation [Eck03a, Prop. 3.12]. Hence a geometric reason for $\nu(X, L) < n(X, L)$ is that the numerically trivial foliation is not a fibration.
Unfortunately this reason is not necessary: A missing transversality statement like

$$\forall U \cong \Delta \text{ open in a transversal curve} : \inf_k \int_U T_k > 0$$

allows counter examples (on families of $\mathbb{P}^2$ blown up in 9 points where 8 points are fixed and the 9th varies, [Eck03a, 4.3]) on which

$$\nu(X, L) < \text{codimension of the leaves}$$

of the numerically trivial foliation.
If the numerically trivial foliation satisfies the transversality statement from above we would have a necessary and sufficient geometric reason for $\nu(X, L) > \kappa(X, L)$:

Theorem 3 (Geometric Abundance)**.** *The numerically trivial foliation for a nef line bundle L on a smooth projective variety X is a fibration with base dimension $= \nu(X, L)$ iff $\nu(X, L) = \kappa(X, L)$.*

So finally, the whole picture calls for another

AIM: Define something more general than a foliation — e.g. a lamination where only the leaves are holomorphic but not their (local) parametrization — whose leaves are numerically trivial and have codimension $= \nu(X, L)$, and generalize the Geometric Abundance Theorem from above.

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Recent applications of the twisted Nakano identity

Takeo Ohsawa

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1. Let M be a pseudoconvex manifold of dimension n, let $E \to M$ be a holomorphic vector bundle, and let $H^{p,q}(M, E)$ be the E-valued $\overline{\partial}$-cohomology group of M, of type (p, q). It is a basic fact in several complex variables that $H^{p,q}(M, E)$ vanishes under some curvature condition on E. Recall that E is said to be Nakano positive (resp. Nakano semipositive) if there exists a C^∞ fiber metric h of E such that its curvature form

$$\Theta := \overline{\partial} \circ h^{-1} \circ \partial \circ h + h^{-1} \circ \partial \circ h \circ \overline{\partial}$$

as a $\mathrm{Hom(E, E)}$-valued $(1, 1)$-form is positive (resp. semipositive) in the sense that the Hermitian form on the fibers of $T_M^{1,0} \otimes E$ induced by $h\Theta$ is positive. Here $T_M^{1,0}$ denotes the holomorphic tangent bundle of M. Generalizing the Kodaira vanishing theorem and Cartan's theorem B, vanishing theorems for the Nakano positive and Nakano semipositive bundles have been obtained (cf. [N],[A-N],[K],[G-R],[T-1]).

2. Skoda [S] was the first to analyse the cohomology of (possibly nowhere positive) Nakano semipositive bundles. He established a division theorem with L^2 conditions by generalizing the works of Hörmander [H-1,2]. This amounts to giving a curvature condition for the injectivity of the homomorphism

$$H^{n,q}(M,S) \to H^{n,q}(M,E)$$

for an inclusion homomorphism $S \hookrightarrow E$.

3. Skoda's work inspired the author and Takegoshi [O-T] to prove an extension theorem with L^2 conditions. Recently it turned out that the L^2 extension theorem, in a suitably generalized form (cf. [O-1]), implies an L^2 division theorem that improves Skoda's L^2 division theorem in some case (cf. [O-2,3]). It turned out also, more recently, that the curvature condition for the generalized L^2 extension theorem looks very similar to that for the cohomology injectivity theorems due to Tankeev [Tn], Kollár [Kl], Enoki [E] and Takegoshi [T-2]. Accordingly it is reasonable to look for a viewpoint which will give a unified understanding of these results. For that, the main question is whether or not one can remove the smoothness assumption for the submanifold in the L^2 extension theorem without strengthening the curvature condition. Refining Takegoshi's method of combining Enoki's idea with the twisted Nakano identity, we have obtained the following.

Theorem 1 (cf.[O-4, Theorem 2]). *Let M be a weakly 1-complete (i.e. C^∞-pseudoconvex) Kähler manifold, let (E,h) be a Hermitian holomorphic vector bundle over M, and let $\mathcal{I}$ be a coherent ideal sheaf of the structure sheaf of M. Suppose that there exists a C^2 function $\sigma : M \to [0,\infty)$ with $\sigma^{-1}(0) = \mathrm{supp}\mathcal{O}_\mathrm{M}/\mathcal{I}$ such that*

$$\Theta + (1+t)\mathrm{Id}_\mathrm{E} \otimes \partial\bar{\partial}\log\sigma \geq 0$$

holds on $M \backslash \sigma^{-1}(0)$ for $0 \leq t < \varepsilon$ for some $\varepsilon > 0$, and that

$$\mathcal{I}_x = \left\{ f_x \,\middle|\, \begin{array}{l} f \text{ is holomorphic on some coordinate neighborhood } U \ni x \\ \text{and satisfies } \int_U |f|^2/\sigma < \infty \end{array} \right\}$$

Then the kernel of the homomorphism

$$H^q(M, K_M \otimes E \otimes \mathcal{I}) \to H^q(M, K_M \otimes E)$$

is contained in the closure of zero for any $q \geq 0$. Here K_M denotes the canonical bundle of M.

Remark: The homomorphism $H^q(M, K_M \otimes E \otimes \mathcal{I}) \to H^q(M, K_M \otimes E)$ becomes injective if M is moreover holomophically convex, in virtue of Grauert's direct image theorem.

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Critical points of holomorphic sections and supersymmetric vacua

Bernard Shiffman

(joint work with Mike Douglas and Steve Zelditch)

The number of critical points of a section s of a holomorphic line bundle L depends on the Hermitian metric h on L. The critical points of s are the points where $\nabla_h s = 0$, where ∇_h is the Hermitian connection, or equivalently, where $d \log \|s\|_h = 0$. Even the average number of critical points, as s varies in the space $H^0(M, L)$, depends on the curvature of h. When we speak of the average, we use the natural Gaussian probability measure on $H^0(M, L)$ adapted to the geometry of the manifold and the line bundle.

We shall discuss the average density and average total number of critical points and how it depends on the line bundle L and the metric h. In particular, for a fixed line bundle, is there a metric for which the number is minimal? Is this metric unique?

For example, let (L, h) be a positively curved Hermitian line bundle on a compact complex curve M, and let $s \in H^0(M, L^N)$ be a random holomorphic section of a power L^N of the line bundle L. We show that the expected number of critical points of s of index -1 (i.e, the number of local maxima of $\|s\|_{h^N}$) is given by the asymptotic formula

$$\mathcal{N}^{\max}(N) = \frac{1}{3}\, c_1(L)\, N - \frac{1}{9}\,(2g-2) + \left(\frac{1}{27\pi} \int_C \rho^2 \omega_h\right) N^{-1} + O(N^{-2})\ ,$$

where ρ is the Gaussian curvature (i.e., scalar curvature) of h. Thus the average number of local maxima of the length of a section s is asymptotic to 1/3 the number of its zeros (and hence by Morse theory, the average total number of critical points is asymptotic to 5/3 times the number of its zeros).

For the higher dimensions, we have the following result:

Theorem 1. *For any positive Hermitian line bundle $(L, h) \to (M, \omega)$ over any compact m-dimensional manifold with Kähler form $\omega_h = -\frac{i}{2}\partial\bar\partial \log h$, the expected number of critical points on M is of the form*

$$\begin{aligned} \mathcal{N}^{crit}_{N,h} \;\sim\; & [\beta_0\, c_1(L)^m]\, N^m + \left[\beta_1\, c_1(M) \cdot c_1(L)^{m-1}\right] N^{m-1} \\ + \;& \left[k_0 \int_M \rho^2 dVol_h + \beta_2\, c_1(M)^2 \cdot c_1(L)^{m-2} + \beta_3\, c_2(M) \cdot c_1(L)^{m-2}\right] N^{m-2} \\ + \;& \cdots , \end{aligned}$$

where $k_0, \beta_0, \ldots, \beta_3$ are constants depending only on the dimension m.

This formula implies that the average number of critical points for global sections of a sufficiently high power of a positively curved line bundle (L, h) is minimal over all possible metrics if h minimizes Calabi's functional $\int_M \rho_h^2\, \omega_h^m$, where ρ_h is the scalar curvature of ω_h. In particular, the average number of critical points is asymptotically minimal if ω_h is Kähler-Einstein.

The motivation for this work comes from string theory, where a supersymmetric vacuum ("universe") is associated with a critical point Y of a "superpotential" W, i.e., a holomorphic section of the natural line bundle L over the moduli space $\mathcal{M}$ of complex structures on a Calabi-Yau 3-fold. (Here $\mathcal{M}$ is noncompact and L is negative, and we apply our results to a special family of sections.) Since (W, Y) is unknown and there are myriad possibilities, an important physical problem is to determine how many vacua there are and how they are distributed.

This report is on joint work with Mike Douglas and Steve Zelditch [DSZ1, DSZ2].

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Invariants from multiplier ideals

Lawrence Ein

Let X be a nonsingular complex variety and $Y \subset X$ be a closed subscheme. Denote by $\mathcal{I}(I_Y^{\lambda})$ the multiplier ideal of Y with weight λ. λ is said to be a jumping number of Y if

$$\mathcal{I}(I_Y^{\lambda}) \subsetneq \mathcal{I}(I_Y^{\lambda-\varepsilon})$$

for $\varepsilon > 0$. The log-canonical threshold of (X, Y) is defined as $c = lc(X, Y) =$ smallest jumping number. The following theorems describe some properties of jumping numbers

Theorem 1 (Ein, Lazarsfeld, Smith, Varolin).
Let $c = lc(X, Y)$ and λ be a jumping number of Y. Then
1) $\lambda \in \mathbb{Q}$
2) For $\lambda > \dim X - 1$, λ is a jumping number, iff $\lambda + 1$ is a jumping number.
3) For $a \geq 0$ there is a jumping number in $(a, a + c]$.
4) Suppose $X = \mathbb{C}^n$ and $I_Y = < f >$. If λ is a jumping number of f in $[0, 1]$, then $b_f(-\lambda) = 0$ where $b_f(t)$ is the Bernstein-Saito polynomial.

Theorem 2 (de Fernex, Ein, Mustata).
Suppose $p \in X$. Let $\mathcal{R} = \mathcal{O}_{X,p}$ and $\mathcal{A} = I_{Y,p}$. Assume $\sqrt{\mathcal{A}} = \mathcal{M}_p$. Then

$$\ell(\mathcal{R}/\mathcal{A}) \geq \frac{n^n}{n! lc(\mathcal{A})^n} \text{ where } n = \dim \mathcal{R}.$$

$$e(\mathcal{A}) = \text{ Samuel multiplicity } \geq \frac{n^n}{lc(\mathcal{A})^n}$$

As an application of the above theorem we prove

Theorem 3 (de Fernex, Ein, Mustata).
Let X be a smooth projective hypersurface of degree n in $\mathbb{P}^n$. Assume that $4 \leq n \leq 12$. Then $Aut_{\mathbb{C}}(\mathbb{C}(X)) \cong Bir_{\mathbb{C}}(X)$ is isomorphic to $Aut_{\mathbb{C}}(X)$ which is a finite group.

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Subelliptic PDE's and SubRiemannian Geometry

Peter Greiner

I would like to propose geometrically invariant formulas for the fundamental solutions, heat kernels, and wave kernels of PDE's of the form

$$\Delta = \frac{1}{2} \sum_{j=1}^{m} X_j^2 + \cdots , \tag{1}$$

where $X_1, \dots, X_m$ are linearly independent vector fields on a manifold M_n of dimension n.

- a) $m = n \Rightarrow \Delta$ is elliptic,
- b) $m < n$ and the brackets of the X_j's yield all of $TM_n \Rightarrow \Delta$ is subelliptic. If one bracket suffices we call it step 2, otherwise higher step.

To illustrate the proposed structure, I shall discuss an example for which "explicit" fundamental solutions given in geometric terms are available. We are in 3 dimensions, $(x_0, x_1, x_2) = (x_0, x')$, with 2 vector fields,

$$X_1 = \frac{\partial}{\partial x_1} + 2kx_2|x'|^{2k-2}\frac{\partial}{\partial x_0}, \qquad X_2 = \frac{\partial}{\partial x_2} - 2kx_1|x'|^{2k-2}\frac{\partial}{\partial x_0}, \tag{2}$$

and the differential operator one wants to invert is

$$\Delta_\lambda = \frac{1}{2}(X_1^2 + X_2^2) - \frac{1}{2}i\lambda[X_1, X_2]. \tag{3}$$

Δ_λ is step 2 at points $|x'|^2 \neq 0$, and step $2k$ otherwise. The fundamental solution $K_\lambda(x, x^0)$ of Δ_λ is the distribution solution of

$$\Delta_{\lambda,x} K_\lambda(x, x^{(0)}) = \delta(x - x^{(0)}). \tag{4}$$

We shall look for K_λ in the form

$$K_\lambda(x, x^{(0)}) = \int_{\mathbb{R}} \frac{E(x, x^{(0)}, \tau) v_\lambda(x, x^{(0)}, \tau) d\tau}{g(x, x^{(0)}, \tau)}, \tag{5}$$

where the function g is a solution of the Hamilton-Jacobi equation

$$\frac{\partial g}{\partial \tau} + \frac{1}{2}(X_1 g)^2 + \frac{1}{2}(X_2 g)^2 = 0, \tag{6}$$

given by the modified action integral of a complex Hamiltonian problem

$$\begin{aligned} g &= -i(x_0(0) - x_0^{(0)}) + \int_0^\tau \big[\xi(s)\cdot\dot{x}(s) - H\big(x(s), \xi(s)\big)\big] ds \\ &= -i(x^{(0)} - x_0^{(0)}) + \Big(1 - \frac{1}{k}\Big)E \\ &\quad + \frac{1}{2k}\operatorname{sgn}\tau\big[\big(2E|x'|^2 + W(|x'|^2)^2\big)^{\frac{1}{2}} - \big(2E|x'|^2 + W\big(|x^{(0)\prime}|^2\big)^2\big)^{\frac{1}{2}}\big], \end{aligned} \tag{7}$$

where one uses the principal branch of the square roots. Let

$$H(x, \xi) = \frac{1}{2}\big(\xi_1 + 2kx_2|x'|^{2k-2}\xi_0\big)^2 + \frac{1}{2}\big(\xi_2 - 2kx_1|x'|^{2k-2}\xi_0\big)^2 \tag{8}$$

denote the Hamiltonian with dual variables $\xi = (\xi_0, \xi_1, \xi_2) = (\xi_0, \xi')$. The complex bicharacteristics are solutions of the usual Hamiltonian system of differential operators

$$\dot{x}_j = H_{\xi_j}, \quad \dot{\xi}_j = -H_{x_j}, \quad j = 0, 1, 2, \tag{9}$$

with the unusual boundary conditions

$$\begin{aligned} &\xi_0(0) = -i, \\ &x_1(0) = x_1^{(0)},\ x_2(0) = x_2^{(0)}, \\ x_0(\tau) = x_0, \quad &x_1(\tau) = x_1, \quad x_2(\tau) = x_2, \end{aligned} \tag{10}$$

in time τ. $E = -\partial g/\partial\tau = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2$ is the energy, the first invariant of motion. $\Omega = \Omega(x^{(0)}, x, \tau)$ is the angular momentum, and we set $W(u) = 2ku^k - \Omega$. The volume element v_λ is the solution of the following second order transport equation:

$$\Delta_\lambda(Ev_\lambda) + \frac{\partial}{\partial\tau}\big[Tv_\lambda + (\Delta_\lambda g)v_\lambda\big] = 0, \tag{11}$$

where

$$T = \frac{\partial}{\partial\tau} + \sum_{j=1}^{2}(X_j g)X_j \tag{12}$$

is differentiation along the bicharacteristic.

Theorem 1. *The fundamental solution $K_\lambda(x^{(0)}, x)$ of Δ_λ has the following invariant representation:*

$$K_\lambda = \int_{\mathbb{R}} \frac{Ev_\lambda d\tau}{g}, \tag{13}$$

where the second order transport equation for v_λ may be reduced to an Euler-Poisson-Darboux equation and solved explicitly as a function of E and Ω. Namely,

$$v_\lambda = \frac{c_\lambda}{k}(A_+ - g)^{-\frac{1-\lambda}{2}}(A_- + g)^{-\frac{1+\lambda}{2}}F_\lambda(q_+, q_-), \tag{14}$$

with,

$$c_\lambda = -e^{i\pi\frac{1-\lambda}{2}}4\pi^2\Gamma\Big(\frac{1-\lambda}{2}\Big)\Gamma\Big(\frac{1+\lambda}{2}\Big), \quad A_\pm = \frac{1}{k}\Omega_\pm + g_\pm, \quad \Omega_\pm = \lim_{\tau\to\pm\infty}\Omega, \tag{15}$$

$$g_\pm = \lim_{\tau\to\pm\infty} g,$$

$$q_\pm = \frac{2^{1/k}(x_1 \pm ix_2)(x_1^{(0)} \mp ix_2^{(0)})}{(A_\pm \mp g)^{1/k}}, \tag{16}$$

and $F_\lambda(q_+,q_-)$ *is the following hypergeometric function of 2 variables,*

$$F_\lambda(q_+,q_-) = \frac{1}{\Gamma\left(\frac{1-\lambda}{2}\right)\Gamma\left(\frac{1+\lambda}{2}\right)}\int_0^1\int_0^1 \frac{ds_+ds_-}{s_+s_-}\left\{\left(\frac{s_+}{1-s_+}\right)^{\frac{1-\lambda}{2}}\left(\frac{s_-}{1-s_-}\right)^{\frac{1+\lambda}{2}}\cdot\right. \tag{17}$$

$$\left.\cdot\,\frac{1-q_+q_-(s_+s_-)^{1/k}}{(1-q_+s_+^{1/k})(1-q_-s_-^{1/k})\big(1-(q_+q_-)^k s_+s_-\big)}\right\}.$$

Remark. Subelliptic differential operators of the form (1) should be understood as precisely as the Laplace-Beltrami operator is understood. To this end we note that formula (5) is fully invariant and makes sense for arbitrary operators of the form (1). In the present example it may be looked upon as a complex distance to the correct negative power being summed over the characteristic variety with density v_λ, and thus includes the well known Newton potential of elliptic operators.

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On finite type and subelliptic estimates

Gordon Heier

In the first half of the talk, real real-analytic hypersurfaces in $\mathbb{C}^n$ of finite type are discussed from an algebraic-geometric point of view, without regard for subelliptic estimates. Basic references here are [D'A82] and [D'A93].

By means of a well-known polarization argument, the study of the type of a real real-analytic hypersurface in $\mathbb{C}^n$ at a given point can be reduced to the study of the type of ideals of germs of holomorphic functions in $\mathbb{C}^n$ (see [D'A82]).

Definition 1. *Let* $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ *be an ideal. Define the type of* I *to be*

$$T(I) := \sup_{\gamma\in\Gamma}\inf_{f\in I}\frac{\mathrm{ord}_0 f\circ\gamma}{\mathrm{ord}_0\gamma},$$

where Γ *is the set of all germs of holomorphic curves* $\gamma : \Delta \to \mathbb{C}^n$ *with* $\gamma(0) = 0$. *If* $T(I)$ *is finite, we say that* I *has finite type.*

If I is generated by $f_1,\ldots,f_N$, then one clearly has

$$T(I) = \sup_{\gamma\in\Gamma}\inf_{f\in I}\frac{\mathrm{ord}_0 f\circ\gamma}{\mathrm{ord}_0\gamma} = \sup_{\gamma\in\Gamma}\min\left\{\frac{\mathrm{ord}_0 f_1\circ\gamma}{\mathrm{ord}_0\gamma},\ldots,\frac{\mathrm{ord}_0 f_N\circ\gamma}{\mathrm{ord}_0\gamma}\right\}.$$

In the case of two variables, we sketch an elementary self-contained proof of the following theorem. The strategy of the proof is to blow up points until a certain "model situation" is achieved, at which point the theorem can be verified.

Theorem 1. *Let $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^2,0}$ such that $V(I) = \{0\}$ for $I = (f_1, f_2)$, which is the ideal generated by f_1, f_2. Moreover, assume that the zero-sets of f_1 and f_2 each are irreducible germs of curves. (The schemes defined by the principal ideals (f_1) and (f_2) don't have to be reduced.) Then*

$$T(I) = \frac{\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/I}{\min\{\mathrm{ord}_0 f_1, \mathrm{ord}_0 f_2\}}.$$

In [D'A82, Theorem 2.7], it is shown that $T(I) \leq \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/I$. Therefore, our result can be understood as an improvement on D'Angelo's inequality in this case. The case of I having more than two generators and of the generators not being irreducible is only notationally more difficult. A perhaps surprising corollary is

Corollary 2. *Among the three pairwise intersection numbers of three irreducible local curves at the origin in $\mathbb{C}^2$ with the same multiplicity at the origin, at least two coincide.*

In higher dimensions, we conjecture the following formula. Future work in this direction may be done jointly with J. McNeal and A. Némethi, the first of whom initially suggested the formula contained in Theorem 1. In this workshop, J. McNeal will also discuss this problem, using, in the two dimensional case, the Puiseux characteristics of the curves involved and other methods from singularity theory.

Conjecture 3. *Let $f_1, \ldots, f_N \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $V(I) = \{0\}$ for $I = (f_1, \ldots, f_N)$. Let P be the set of all subsets S of $\{1, \ldots, N\}$ such that $V((f_i)_{i \in S})$ has components of dimension one. For $S \in P$, let $\bigcup_{\nu=1,\ldots,k_S} C_{\nu,S}$ be a decomposition of the one dimensional part of $V((f_i)_{i\in S})$ into irreducible curves. Then*

$$T(I) = \max\left\{ \min_{i\in\{1,\ldots,N\}} \left\{ \frac{\mathrm{ord}_0 f_i|_{C_{\nu,S}}}{\mathrm{mult}_0 C_{\nu,S}} \right\} \right\},$$

where the maximum is taken over $S \in P$ and $\nu \in 1, \ldots, k_S$.

In the second half of the talk, effective subelliptic estimates for weakly pseudoconvex domains Ω in $\mathbb{C}^n$ with non-singular real-analytic boundary of finite type are discussed. This is joint work with A. Nicoara and I. Coskun. For precise definitions of the terms appearing in the sequel and for further background information, we refer to the lucid survey article [DK99].

In his seminal paper [Koh79], Kohn introduced the method of multiplier ideals. For a point p in the boundary of Ω (Ω not necessarily of finite type at p) the stalk of Kohn's multiplier ideal sheaf is defined to be the ideal of germs of real-valued C^∞ functions f such that subelliptic estimates hold on $(0,1)$-forms for some Sobolev order ε *after* multiplication by f. With this terminology, it is clear that subelliptic estimates hold at p if and only if the constant function 1 is a multiplier. The central new idea in [Koh79] is to introduce a non-decreasing sequence of ideals of multipliers (ie all contained in the stalk of the multiplier ideal sheaf at p) which terminates with the unit ideal if and only if p is a point of finite type, ie there are no local complex curves through p contained in the boundary of Ω. In other words, subelliptic estimates on $(0,1)$-forms hold at p for some Sobolev order ε if and only if p is a point of finite type. We remark that in order to obtain this particular formulation of his result, Kohn used results of Diederich and Fornaess in [DF78].

The starting point of our investigation is the observation that Kohn's theorem is not effective in the sense that the method makes no statement about the Sobolev order ε for which subelliptic estimates hold. It is known that at strongly pseudoconvex boundary points, subelliptic estimates always hold for $\varepsilon = \frac{1}{2}$. Moreover, Catlin ([Cat83]) observed that ε can never be larger than the reciprocal of the type of Ω at the given boundary point. The true nature of the dependence of the maximal possible value of ε on the type is unknown, although Catlin showed in [Cat87] by different methods that for a point of type t, subelliptic estimates hold at least for $\varepsilon = t^{-n^2 t^{n^2}}$. The aim of our work is to provide – by a close analysis of Kohn's method of multiplier ideals – a better lower bound for the maximal value by exhibiting an ε, written as the reciprocal of a polynomial in t whose degree is a function of n only, for which subelliptic estimates hold.

Our approach to this problem essentially is to take an entirely algebraic-geometric point of view on the moves in Kohn's algorithm which are used to get new multipliers from old ones. From the point of view of effectivity, Kohn's use of real radicals for which there is no order bound is particularly problematic. The methods that we have developed so far give the following theorem.

Theorem 4. *Let $h_1(z_1,\ldots,z_{n-1}),\ldots,h_k(z_1,\ldots,z_{n-1})$ ($k \geq n-1$) be holomorphic functions on an open neighborhood of the origin in $\mathbb{C}^{n-1} \subset \mathbb{C}^{n-1} \times \mathbb{C}$ whose common zero set is precisely the origin in $\mathbb{C}^{n-1}$. Let*

$$r(z_1,\ldots,z_n) = \mathrm{Re}(z_n) + \sum_{j=1}^{k} |h_j(z_1,\ldots,z_{n-1})|^2$$

and

$$\Omega = \{r < 0\}.$$

Then subelliptic estimates on $(0,1)$-forms of Sobolev order at least

$$\frac{1}{2^{n+2}(2t)^{n!}}$$

hold at the origin in $\mathbb{C}^n$.

It is work in progress to extend this result to the case of general real-analytic boundaries via the polarization mentioned in the first half of the talk, thereby obtaining an effective version of Kohn's method of multiplier ideals.

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The order of contact of a holomorphic ideal in $\mathbf{C}^2$

Jeffery D. McNeal

(joint work with András Némethi)

Let $I \subset \mathcal{O}(\mathbb{C}^n, 0)$ be an ideal of (germs of) holomorphic functions near 0. Let Γ denote the set of holomorphic mappings $\gamma : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0)$ with $\gamma(0) = 0$. For a function $h : \mathbb{C} \longrightarrow \mathbb{C}$ such that $h(0) = 0$, let $\mathrm{ord}(h)$ denote the order of vanishing of h at 0, and for vector-valued mappings such as γ, let $\mathrm{ord}(\gamma) = \min_k \{\mathrm{ord}(\gamma_k)\}$ (see section 2 for exact definitions). The purpose of this paper is to establish an alternate way to compute the invariant

$$T(I) = \sup_{\gamma \in \Gamma} \left(\inf_{f \in I} \frac{\mathrm{ord}(f \circ \gamma)}{\mathrm{ord}(\gamma)} \right), \tag{1.1}$$

called the *order of contact of* I, when $n = 2$.

This invariant was introduced by D'Angelo. Similar notions, however, also were considered earlier in algebraic geometry, see for example the book of Brieskorn-Knörner, page 484. The invariant plays an important role in complex function theory on domains in $\mathbb{C}^n$, especially for L^2 subelliptic estimates for the Cauchy-Riemann equations. Given a domain $\Omega \subset \mathbb{C}^n$ with smooth boundary $b\Omega$, consider a defining function, $r : \mathbb{C}^n \longrightarrow \mathbb{R}$, for $b\Omega$: $b\Omega = \{r = 0\}$ and $dr \neq 0$ when $r = 0$. It was observed by Kohn that, for certain domains Ω, if some $\gamma \in \Gamma$ lies in $b\Omega$, then the Cauchy-Riemann system $\bar{\partial}$ fails to be hypoelliptic, i.e. there are $\bar{\partial}$-closed $(0,1)$-forms α such that every solution u, to the system of equations $\bar{\partial} u = \alpha$, has the property that singsupp $u \not\subseteq$ singsupp α. Catlin extended this result to any smoothly bounded domain Ω. Moreover, Catlin later showed that holomorphic curves with high order of tangency to $b\Omega$ prevent certain analytic estimates on $\bar{\partial}$, which are quantified versions of hypoellipticity, from holding. He showed that if $p \in b\Omega$ and there exists an analytic curve γ passing through p whose order of tangency with $b\Omega$ is M, then the subelliptic estimate

$$||u||_\epsilon^2 \leq C\left(||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2\right), \qquad u \in \text{Dom }(\bar{\partial}^*) \tag{1.2}$$

cannot hold in a neighborhood of p if $\epsilon > \frac{1}{M}$. Here $||\cdot||_\epsilon$ denotes the Sobolev norm of order ϵ and $||\cdot||$ denotes the L^2 norm. To clarify the connection between (1.2) and the contact order of curves γ with $b\Omega$, D'Angelo defined the invariant $\Delta_1(b\Omega, p)$, the *type of* p, by

$$\Delta_1(b\Omega, p) = \sup_{\gamma \in \Gamma} \left(\frac{\text{ord}(r \circ \gamma)}{\text{ord}(\gamma - \gamma(p))} \right), \tag{1.3}$$

and established the invariant's basic properties. Catlin then proved the remarkable result that (1.2) holds in a neighborhood of p (for some $\epsilon > 0$) if and only if $\Delta_1(b\Omega, p) < \infty$.

In order to establish the basic properties of $\Delta_1(b\Omega, p)$, D'Angelo associated to the pair $(r; p)$ – the defining function r and a point $p \in b\Omega$ – a family of ideals of holomorphic functions I_β. The exact association of I_β to $(r; p)$ involves a truncation and a polarization of the defining function r; these points are not relevant to our discussion in this paper, so we refer to the book of D'Angelo for the definition of I_β. However we point out that D'Angelo showed that $\Delta_1(b\Omega, p)$ is bounded, from above and below, by simple functions of $\max_\beta T(I_\beta)$. Thus, in principle, the computation of $\Delta_1(b\Omega, p)$ is reduced to computing the type of the associated ideals I_β.

The difficulty in computing $T(I)$ is the supremum, over the infinite dimensional space of curves Γ, on the right hand side of (1.1). There is also a counter-intuitive property of $T(I)$ which stems from the fact that Γ is infinite dimensional: $T(I)$ is not an upper semicontinuous function of parameters on I. That is, suppose that for each $s \in (-1, 1)$, I_s is an ideal of holomorphic functions which varies continuously in s; it can happen that $T(I_0) < T(I_s)$ for s arbitrarily near 0. It is, therefore, tempting to discard $T(I)$ and replace it with another, better behaved invariant, for example one which is easier to compute (e.g. replace Γ by $\mathbb{C}P^1$ in (1.1)) and/or upper semicontinuous with respect to parameters (e.g. the intersection multiplicity of I). However, results such as Catlin's show that each curve counts and may influence the function theory: *any* complex analytic curve γ contained in $b\Omega$ causes $\bar{\partial}$ to be non-hypoelliptic, and so we cannot, apriori, ignore the order with any curve.

We want to point out that the *finiteness* of $T(I)$ maybe be shown by comparison with other invariants and, indeed, the statement $T(I) < \infty$ may be verified without directly knowing the order of contact of any curves, e.g. by computing the intersection number of the ideal I. The *value* of $T(I)$, however, is a different matter; the order of contact of I with all complex curves must be measured and this means grappling with the large space of curves Γ.

The following theorem shows, nevertheless, that $T(I)$ may always be computed using only a finite number of testing curves.

Theorem 1. *Let $I \subset \mathcal{O}\left(\mathbb{C}^2, 0\right)$ be an ideal and fix any set of generators $f_1, \ldots, f_k$ of I. For each $1 \leq a \leq k$, let*

$$f_a = \prod_b (f_{ab})^{p_{ab}}$$

be the irreducible decomposition of f_a in the ring $\mathcal{O}\left(\mathbb{C}^2, 0\right)$. Then

$$T(I) = \max_{a,b} \left(\min_r \frac{ord\,(f_r \circ \gamma_{ab})}{ord(\gamma_{ab})} \right),$$

where γ_{ab} is a (minimal) parameterization of $C_{ab} = \{f_{ab} = 0\}$.

In other words, in order to compute $T(I)$, we have to only consider that finite set of irreducible (parameterized) curves which appear as irreducible components in at least one of the generators $f_1, \ldots, f_k$. Of course, as the choice of generators $f_1, \ldots, f_k$ is not unique, different choices of generators may provide a different set of testing curves C_{ab}.
Theorem **1** is, thus, a kind of "selection theorem" and a particularly simple one: if I is generated by k irreducible functions, then we must only test against k curves. For higher dimensional ideals, $I \subset \mathcal{O}_n$, $n > 2$, we can also show that $T(I)$ can be computed by testing against only a finite number of curves (rather than considering all curves in Γ). However, our proof of this fact for $n > 2$ is much more involved than the transparent 2-dimensional argument below. Also, the size of the set of testing curves we obtain grows as n increases and is not obviously the minimal set we obtain when $n = 2$. For these reasons we discuss only Theorem **1** in this talk and defer the discussion of the case $n > 2$ to another occasion.

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Geometry of varieties of general type

Hajime Tsuji

1. Rigidity of automorphisms of projective varieties

Let X be a smooth projective variety defined over complex numbers. Let $\mathrm{Aut}(X)$ denote the (holomorphic) automorphism of X. It is known that $\mathrm{Aut}(X)$ is a complex Lie group. Let $\mathrm{Aut}^0(X)$ denote the identity component of $\mathrm{Aut}(X)$. Let

$$\lambda : \mathrm{Aut}(X) \longrightarrow \pi_0\mathrm{Diff}(X)$$

be the homomorphism which sends an element of $\mathrm{Aut}(X)$ to it isotopy class. Then it is clear that $\mathrm{Aut}^0(X)$ is contained in the kernel of λ. Hence we have the natural homomorphism

$$\bar\lambda : \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \longrightarrow \pi_0\mathrm{Diff}(X).$$

Theorem 1. *Let X be a smooth projective variety defined over complex numbers. Then the natural homomorphism*

$$\bar\lambda : Aut(X)/Aut^0(X) \longrightarrow \pi_0 Diff(X)$$

is injective.

Remark 2. *Under the same assumption as in Theorem 1, it would be expected that the homotopic automorphisms are identical up to the action of $Aut^0(X)$.*

We note that every smooth projective variety of general type, the automorphism group is finite.

Corollary 3. *Let X be a smooth projective variety of general type. Then the natural homomorphism*

$$Aut(X) \longrightarrow Iso(X)$$

is injective.

The key idea is to look at the maps from the space of smooth curves on X into a quotient Teichmüller space. In this case the target has negative sectonal curvature with respect to the Weil-Petersson Kähler metric although it is not complete.

The method depends on the uniqueness theorem for harmonic mappings into a quotient of Teichmüller spaces ([3, p.1206,Lemma 3.1]) based on the argument in [5]. The key ingredient here is the convexity of Teichmüller spaces. This ensures that homotopic harmonic maps with finite energy from a complete Riemannian manifolds with finite volume into a quotient of Teichmüller space can be joined by a family of geodesics with constant length and the energy density is constant along the geodesics. Then the uniqueness comes from the second variation formula.

2. Minimal volume of orbifolds of general type

The second topic is the minimal volume of orbifolds of general type.

Theorem 4. *([8]) There exists a positive constant c_n depending only on n such that for every orbifold of general type (X, Δ) of dimension n*

$$\mu(X, K_X + \Delta) \geqq c_n$$

holds, the the volume $\mu(X, K_X)$ with respect to K_X is defined by

$$\mu(X, K_X) := n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(\lceil m(K_X + \Delta)\rceil)).$$

This is equivalent to the following theorem.

Theorem 5. *([8]) There exists a positive constant ν_n depending only on n such that for every orbifold of general type (X, Δ) of dimension n and $m \geqq \nu_n$, $|\lfloor m(K_X + \Delta)\rfloor|$ gives a birational embedding of X into a projective space.*

These are the extension of the results in [6] to the case of orbifolds. As in [6], the subadjunction theorems of [4] and [7] are essential for the proof. These are used for the comparison of the canonical divisor of the minimal center of logcanonical singularities and the canonical divisor of the ambient space. In relation to this theorem, Dr. Hisamatsu proved the following theorems.

Theorem 6. *([2]) There exists a constant C_2 such that for every surface (X, D) of log general type.*

$$\mu(X, K_X + D) \geqq C_2$$

holds.

Theorem 7. *([2]) There exists a positive constant d_2 such that such that for every orbifold of general type (X, Δ) of dimension n and $m \geqq d_2$, $\mid m(K_X + \Delta) \mid$ gives a birational embedding of X into a projective space.*

It is an very interesting problem to estimate d_2.

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Currents associated to transcendental entire curves on compact Kähler manifolds

Mihai Paun

In this talk, we give an overview of our paper [17]. We are concerned with the asymptotic behavior of the Zariski dense holomorphic curves $\varphi : \mathbb{C} \mapsto X$, where X is a compact Kähler manifold of dimension n. We recall that M. McQuillan ([13]) showed that one can associate to a transcendental entire curve φ a closed positive current, of $(n-1, n-1)$ type, denoted by $T[\varphi]$ (which morally speaking, should reflect the behavior of the curve at infinity).
In the first place, we would like to present here some of the numerical properties of the current $T[\varphi]$. For instance, we show that if the curve φ acquires a singularity at some point of the manifold, then the cohomology class of the current $T[\varphi]$ is big (this will be a consequence of more general results stated below).
The main results of the paper [17] are stated in the section two. It turns out that for currents associated to entire curves, given an analytic set $Y \subset X$, there are two ways of measuring the singularity of $T[\varphi]$ along Y: the first one is the classical generic Lelong number, and the second one is the multiplicity. Then we have

Theorem 1. *Let X be a compact Kähler manifold, and let $\varphi : \mathbb{C} \mapsto X$ be a Zariski dense holomorphic curve. If* $\mathrm{mult(T[\varphi], Y)} > 0$ *for some irreducible analytic set $Y \subset X$ of codimension equal to p, then for any nef class $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ such that $p + \nu(\alpha) > n$ we have $(T[\varphi], \alpha) > 0$.*
(we denote by $\nu(\alpha)$ the numerical dimension of the class $\{\alpha\}$).

Now let us give an overview of the proof of the Theorem 1. The main step in the proof is to show that if we consider the Kähler approximations $\{\alpha + \delta\omega\}$ of our class, then we can get a positive current $T_\delta \in \{\alpha + \delta\omega\}$ such that the generic Lelong number $\nu_Y(T_\delta) \geq C\delta^{\frac{n-\nu(\alpha)}{p}}$ (this is done by a careful analysis of a family of Monge–Ampère equations). Remark that if the class $\{\alpha\}$ is the first Chern class of some holomorphic line bundle, this statement is obtained by the usual techniques (i.e. the Riemann–Roch theorem).
It is now clear why the hypothesis $p + \nu(\alpha) > n$ is needed: we want to insure that the singularity of our current T_δ is greater than the distance between the class $\{\alpha + \delta\omega\}$ and the class $\{\alpha\}$, i.e. $\delta^{\frac{n-\nu(\alpha)}{p}} \gg \delta$, as $\delta \mapsto 0$. In the algebraic context, this approach was already used by McQuillan, see [15]; however, the techniques used here are quite different. Another important technical tool will be the regularization theorem of closed positive currents given by Demailly in [5].
The Theorem 1 has a certain number of consequences, see [17].

Now given an analytic subset $Y \subset X$, we define and study the generalized Lelong number of the current $T[\varphi]$ with respect to Y, and we establish some very basic

properties of this number (compare with [4]). The motivation for introducing and studying such numbers is the following. Consider a surface X, endowed with a holomorphic foliation $\mathcal{F}$, and a Zariski dense entire curve φ tangent to $\mathcal{F}$. In the paper [13], McQuillan show that the Lelong numbers of $T[\varphi]$ at the singular points of $\mathcal{F}$ appear as an obstruction to the positivity of the intersection of $T[\varphi]$ with the normal bundle of the foliation. Next he is able to show that the numerical properties of $T[\varphi]$ are substantially improved by the occurance of a Lelong number at some point (page ??). Our guess is that in higher dimensions, it is very likely that the obstructions one has to deal with are the generalized Lelong numbers.
In the third section, we obtain some results concerning the Siu decomposition of the current $T[\varphi]$. One of them is the following

Theorem 2. *Let (X, V) be a projective directed manifold, and let $\varphi : \mathbb{C} \mapsto X$ be an entire curve tangent to V. We denote by $\varphi_{[1]} : \mathbb{C} \mapsto \mathbb{P}(V)$ the derivative of φ. Then we have $(\chi_{C_1} T[\varphi_{[1]}], \pi^*\omega) = (\chi_C T[\varphi], \omega)$, where C is an algebraic curve of X tangent to V, and C_1 is the canonical lifting of C to the first jet space $X_1 := \mathbb{P}(V)$.*

An easy consequence of the Theorem 2 is the following observation.

Remark 3. *Let X be a surface of general type, and let $\varphi : \mathbb{C} \mapsto X$ be an entire curve, such that $T[\varphi] = [C]$, where C is a smooth, algebraic curve. Then the image of φ contained in C.*

It is very likely that the techniques presented here can be used to prove similar statements, for currents such that $R[\varphi] = 0$.
Now the natural expectation about the curves which occur in the algebraic part of a current $T[\varphi]$ should be the following:

Question. *Assume that for some positive λ we have $T[\varphi] \geq \lambda[C]$. Then the geometric genus of the curve C is strictly less than 2.*
We will address in [17] only some particular cases of this question. In the case of foliated surfaces, the question above is completely solved, by the work of M. Brunella and M. McQuillan. Very recent and encouraging progress for the general case was made by J. Duval.

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Multiplier Ideals and Basepoint Freeness

Stefan Helmke

Let X be a smooth projective variety, L a nef divisor on X and Σ a zero-dimensional subscheme of X. If Σ fails to impose independent conditions on global sections of the line bundle $\omega_X(L)$, then one expects that the geometric reason for this failure is a flag of subvarieties $X = Z_0 \supset Z_1 \supset \cdots \supset Z_{r+1} \subset \Sigma$ with relatively small degree. More precisely, examples as well as techniques suggest that the degrees of those subvarieties Z_i should be bounded by

$$(*) \qquad \left(\frac{e_1}{e_0}\right)^{\frac{1}{\delta_0}} < \cdots < \left(\frac{e_{r+1}}{e_r}\right)^{\frac{1}{\delta_r}} \quad \text{and} \quad \delta_0 \left(\frac{e_1}{e_0}\right)^{\frac{1}{\delta_0}} + \cdots + \delta_r \left(\frac{e_{r+1}}{e_r}\right)^{\frac{1}{\delta_r}} \geq 1,$$

where $d_i = \dim Z_i$, $\delta_i = d_i - d_{i+1}$ and $e_i = L^{d_i} Z_i/\text{length}(Z_i \cap \Sigma)$.
The existence of such a flag of subvarieties would imply the following well known (cf. e.g. [TV00]).

Conjecture. Let H be an ample divisor on a smooth projective variety X of dimensionn. Then, $\omega_X(mH)$ is k-very ample if $m > k + n$ or if $m = k + n$ and $H^n > 1$.
In particular, for $k = 0$ and $k = 1$ this is Fujita's basepoint free and very ample-

ness conjecture [Fu87] respectively. The existence of subvarieties Z_i satisfying the strong bound($*$) is known for surfaces [La00]. For arbitrary dimensional varieties, only some essentially weaker bounds are proved in [AS95] and [He97], but for low dimensions, the bounds are not so bad. For instance, the first part of the previous conjecture for $k = 0$ was already proved in [EL93] for threefolds and in [Ka97] for fourfolds. The second part for $n = 3$ and $k = 0$ is proved in [Fu93].
The method used by A. Langer to obtain the optimal bounds for surfaces [La00] and log-surfaces [La01] is a generalization of Reider's argument [Re88], the so-called rank-2 vector bundle technique. Whereas this technique is certainly very elegant and implies the strongest results, it seems not to generalize to higher dimensions. In contrast, L.Ein and R.Lazarsfeld [EL93] used a technique based on a generalization of Kodaira's vanishing theorem (cf. [Ka82] and [Vi82]), which also works in higher dimensions, but gives weaker results. In order to get the second part of the conjecture for threefolds in the case $k = 0$, T.Fujita [Fu93] considerably improved the Riemann-Roch argument. His argument was then finally optimized in [He99] for a hypersurface $Z_1 \subset X$, but the higher codimensional case is still unclear. However, since this is the only case one has to consider for projective surfaces, we can now use this method to prove the optimal bounds even for log-surfaces, at least if the embedding dimension of Σ is at most 1. Without the improvements from [He99], the results are essentially weaker (compare [Ma99]).

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Pseudo Hermitian Manifolds

Song-Ying Li
(joint work with H.-S. Luk)

This talk is based on a joint work with H.-S. Luk in [11], providing an explicit formula for Webster pseudo Ricci curvature and using pseudo scalar curvature to characterize a ball in $\mathbb{C}^{n+1}$.

Let M be a $(2n+1)$-dimensional CR manifold with CR dimension n. We say that (M, θ) is a strictly pseudoconvex pseudohermitian manifold in the sense of Webster [12], if there is a real one-form θ (contact form) on M and at each point of M there is a neighbourhood with a local basis $\theta^1, \cdots, \theta^n$ for the holomorphic cotangent space $T^{1,0}$ so that

$$d\theta = i\, h_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}, \tag{1}$$

where $[h_{\alpha\overline{\beta}}]$ is a positive definite $n \times n$ matrix, determined by the Levi-form L_θ on M. Here

$$L_\theta(w, \overline{v}) = -i\, d\theta(w, \overline{v}), \quad w, v \in T_{1,0}(M). \tag{2}$$

Let $R_{\alpha\overline{\beta}}$ be the Webster pseudo Ricci curvature and $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ be the pseudo scalar curvature. It is known that the contact form is neither unique nor CR invariant, but lies in a conformal class ($\theta_f = e^f\theta$ for some smooth function f). By earlier works of Greenleaf [6] and Li and Luk [10] on the sharp estimates of the first positive eigenvalue of sub-Laplace $\Delta_{sb} = \Re(\Box_b)$ on M and many others indicated that it is very important to find good contact form θ so that we can have nice formula for Webster pseudo Ricci curvature, this is obtained in [11]. In particular, when $\theta = (\partial\rho - \overline{\partial}\rho)/(2i)$ when $M = \partial D$ and $D = \{\rho(z) < 0\}$ is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ and ρ is potential function of Fefferman metric, i.e. $\rho \in C^3(\overline{D})$ so that

$$J(\rho) = -\det \begin{bmatrix} \rho & \rho_{\overline{j}} \\ \rho_i & \rho_{i\overline{j}} \end{bmatrix} \equiv 1 \text{ in } D, \quad \rho = 0 \text{ on } \partial D, \tag{3}$$

we have

$$\mathrm{Ric}(w, \overline{v}) = (n+1) \det H(\rho) L_\theta(w, \overline{v}) \text{ for any } w, v \in T_{1,0}(M). \tag{4}$$

The CR Yamabe problem: Find a contact form θ_f so that the Webster pseudo scalar curvature R_f with respect to the θ_f is a given constant. In other words, the variational equation:

$$\lambda(M) = \inf_\theta \lambda(\theta) = \inf_\theta \frac{\int_M R\,\theta \wedge (d\theta)^n}{(\int_M \theta \wedge (d\theta)^n)^{\frac{n}{n+1}}}, \tag{5}$$

has a minimum. The problem was solved by D. Jerison and J. Lee in [8], and N. Gamara and R. Yacoub in [5] and [4]. In [8], Jerison and Lee proved that $\lambda(M) \le \lambda(\mathbb{S}^{2n+1})$. In particular, it was proved in [8] that, if $\lambda(M) = \lambda(\mathbb{S}^{2n+1})$, then M is locally sphere. If $M = \partial D$ where D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$. In addition, if D is simply connected then D, it

was proved by Chern and Ji in [2] that locally spherical implies global spherical, or D biholomorphically equivalent to the ball in $\mathbb{C}^{n+1}$. It was proved by Huang and Ji in [7] that, if $M = \partial D$ is locally sperical, then D is biholomorphically equivalent to the unit ball when M is algebraic; and a counterexample was constructed by Burns and Shnieder [1] that the algebraic condition can not be replaced by real analytic. Combining (4) and main theorem in [9] on characterization for D being a ball in $\mathbb{C}^{n+1}$, the following theorem is proved in [11]: Let $M = \partial D$ and $\theta = (-i/2)(\partial\rho - \overline{\partial}\rho)$. Let $\theta_0 = \frac{1}{2i}(\partial\rho_0 - \overline{\partial}\rho_0)$ with $\rho_0(z) = |z|^2$. If $J(\rho) \equiv 1$ on D and the Webster pseudo scalar curvature $R \geq n(n+1)(\int_M \theta \wedge (d\theta)^n)/(\int_M \theta_0 \wedge (d\theta_0)^n)$ then D must be biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$ having a constant Jacobian biholomorphic map.
Note: When M is finte type, it is hopeful, one can define and calculate pseudo Ricci curvature by using multiplier ideal sheaves.

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Geometric and arithmetic properties of complex two ball quotients

Sai-Kee Yeung

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The purpose of this short note is to summarize our recent results on three different arithmetic and geometric problems on complex two ball quotients.The first one is on the criterion for integrality and arithmeticity of a lattice in $PU(2,1)$. The

second one is on the virtual positivity of the first Betti number on an arbitrary complex two ball quotient. The last one is on the finiteness of rational points on a complex two ball quotient which is defined over a number field. We put the three problems together because of the similarity of a step in the methods of proof. Let G be a semi-simple Lie group of non-compact type and K be a maximal compact subgroup of G. G/K is then a symmetric space of non-compact type. Let Γ be a cocompact lattice of G. According to the Arithmeticity Theorem of Margulis and the results of Corlette and Gromov-Schoen, any lattice of $G \neq PO(n,1), PU(n,1)$ has to be an arithmetic lattice (cf.[Ma]). For real hyperbolic spaces there exist a lot of non-arithmetic lattices in each dimension as constructed by Gromov and Piatetski-Shapiro. For complex hyperbolic spaces, there are examples of non-arithmetic lattice in dimension 2 and 3 as explained by Deligne-Mostow (cf. [Ma]). It is then an interesting problem to find criteria for the arithmeticity of lattices in $PU(n,1)$. In this aspect, there is the conjecture of Rogawski that a torsion-free cocompact lattice Γ of $PU(2,1)$ satisfying hypothesis
(a) first Betti number $b_1(M) = 0$, and
(b) the Neron-Severi group $H^{1,1}(M) \cap H^2(M,\mathbb{Q}) = \mathbb{Q}$,
where $M = \Gamma\backslash PU(2,1)/P(U(2)\times U(1))$, is arithmetic.

Theorem 1.
A. Let Γ be a torsion-free cocompact lattice of $PU(2,1)$ so that the quotient $M = \Gamma\backslash PU(2,1)/P(U(2)\times U(1))$ satisfies hypothesis (a) and (b). Then Γ is integral.
B. Assume that Γ satisfies hypothesis (a), (b) and that the canonical line bundle K_M is three times the generator of the Neron-Severi group $H^{1,1}(M)\cap H^2(M,\mathbb{Z}) = \mathbb{Z}$ modulo torsion. Then Γ is arithmetic.
C. Assume that Γ satisfies the hypothesis (a), (b) and furthermore that the corresponding ball quotient M contains no immersed totally geodesic curve. Then either Γ is arithmetic of second type, or Γ is a non-arithmetic lattice of $PU(2,1)$ and is a subgroup of infinite index in an arithmetic lattice of second type of some $PU(2,1)^p$ with $p > 1$.

Here we say that a lattice is integral in $G = PU(n,1)$ if there is a form G_F of G over a number field F such that a subgroup of finite index is contained in $G(\mathcal{O}_F)$. Arithmetic lattices of second type of $PU(2,1)$ are the ones defined over a central divison algebra of degree $d \geq 2$ over a quadratic extension of a number field k. Examples satisfying Theorem 1B include fake projective plane which are complex two ball quotients different from $P^2_{\mathbb{C}}$ but have the same rational cohomology ring as the projective plane and hence the canonical line bundle is three times the generator of the Neron-Severi group. The first such examples are constructed by Mumford [M]. As a corollary, lattices associated with fake projective planes are arithmetic. This result is proved independently by Klingler in [Kl], and for the original example of Mumford by Kato [Ka], who actually describes the number field involved in defining the lattice. More recently, we are able to generalize the arguments in Theorem 1 to the following result in non-Archimedean superrigidity.

Theorem 2 (Y4)**.** *Let M be a projective algebraic manifold of complex dimension n satisfying $H^{1,1}(M) \cap H^2(M, \mathbb{Q}) = \mathbb{Q}$ and $b_1(M) = 0$. Let $G(k_p)$ be an absolutely almost simple algebraic group defined over a p-adic field k_p with $\mathrm{rank}_{k_p}(G) \leq n$. Then any homomorphism of Γ in $G(k_p)$ is compact.*

As a consequence, most of the statements in Theorem 1 are generalized to arbitrary dimension $n > 2$ in [Y4]. However, we do not know of any example of fake projective space of higher dimenson, in the sense that the rational cohomology ring is isomorphic to the corresponding one of the projective space of the same dimension. In fact, it is proved in [Y4] that such examples do not exist for any odd dimension n.
We now move to our second topic. It is a conjecture of Thurston that the first Betti number of a compact smooth real hyperbolic space form, in particular, a compact three dimensional real hyperbolic space form $\Gamma\backslash SO(3,1)/S(O(3)\times O(1))$, is virtually positive. The question is generalized by Borel to a complex ball quotient in [Bo]. Most of the general techniques known for this type of problems for complex ball quotients are applicable only to arithmetic lattices. Our strategy towards the conjecture of Borel-Thurston is solve the problem for non-integral lattices, and then try to generalize known techniques for arithmetic lattices to integral cases as well.

Theorem 3. *Let Γ be a co-compact torsion-free lattice of $PU(2,1)$. Let M be the associated compact complex two ball quotient $\Gamma\backslash PU(2,1)/P(U(2) \times U(1))$. Consider the realization of Γ as a subgroup of G_F for an algebraic group G defined over a real number field F with $G \otimes_F \mathbb{R} \cong PU(2,1)$. Assume that Γ is non-integral in G_F. Then the first Betti number of M is virtually positive.*

For integral lattices, we made progress recently in a joined work with S.H.Tang [TY]. In particular, the conjecture of Borel-Thurston is not known only for the case of integral lattices that we mention in Theorem 1C, corresponding to a subgroup of an arithmetic lattice of second type in $PU(2,1)$.
The last topic to be discussed is about finiteness of rational points on a smooth complex varieity defined over a number field whose underlying complex manifold is biholomorphic to a complex two ball quotient. We recall that it is a conjecture of Lang that an algebraic variety over a number field k which is complex hyperbolic has at most a finite number of rational points over k. Metrically speaking, complex ball quotients are the simpliest type of hyperbolic manifolds since the Bergman metric involved has constant negative holomorphic sectional curvature. Mordell Conjecture, which is solved by Faltings [F1], corresponds to $B^1_{\mathbb{C}}/\Gamma$, complex one ball quotients. We begin with the following general observation.

Proposition 4 (Y3)**.** *Let M be a smooth algebraic surface over a number field k. Assume that there exists a finite unramified covering $p : M' \to M$ with $h^{1,0}(M') \geq 3$ and that ρ is rigid. Assume that M contains no genus 0, 1 curves. Then the cardinality of $M(k)$ is finite.*

Then we consider the following generalization of the results in Theorem 3 to the following setting.

Theorem 5 (Y3). *Let Γ be cocompact torsion-free lattice of $PU(2,1)$. $M = B^2_{\mathbb{C}}/\Gamma$. Then either*
(a): Γ is non-integral and virtual $b_1(M) \geq 5$, or
(b): Γ is integral and parametrizes a holomorphic family of principally polarized abelian varieties.

Combining Proposition 4 and Theorem 5 with the help of arguments related to rigidity (cf. [Siu]) we obtain the following result confirming the conjecture of Lang for arbitrary complex two ball quotients.

Theorem 6 (Y3). *Let M be an algebraic variety over a number field k such that the underlying complex variety is a smooth complex two ball quotient. Then the cardinality of $M(k)$ is finite.*

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Periods for irregular connections on complex curves

Helene Esnault
(joint work with A. Beilinson, S. Bloch, P. Deligne)

Given an affine curve X and a connection (E, ∇) defined over a subfield $k \subset \mathbb{C}$ of the complex numbers, one defines the global epsilon de Rham k-line $\epsilon_{DR}(\nabla)$ as the determinant of de Rham cohomology of ∇ on X, and the global epsilon Betti

F-line $\epsilon_B(\nabla)^\vee$ as the determinant of the rapid decay homology of $\nabla^\vee$, where the subfield $F \subset \mathbb{C}$ of the complex numbers contains the monodromy field and is such that all the local Stokes structures at the singularities $\bar{X} \setminus X$ of ∇ are defined over F. The Riemann-Hilbert correspondence of Deligne-Malgrange-Sibuya etc. for irregular connections says that integration establishes an isomorphism

$$\theta(\nabla) : \epsilon_{DR}(\nabla) \otimes_k \mathbb{C} \cong \epsilon_B(\nabla) \otimes_F \mathbb{C}. \tag{0.5}$$

To the choice of $0 \neq \omega \in \Omega^1_{k(X)}$, we construct functorially for all closed points $x \in \bar{X}$ de Rham and Betti lines $\epsilon_{DR}(x, \nabla, \omega), \epsilon_B(x, \nabla, \omega)$ which are k and F-lines respectively, together with an isomorphism

$$\theta(x, \nabla, \omega) : \epsilon_{DR}(x, \nabla, \omega) \otimes_k \mathbb{C} \cong \epsilon_B(x, \nabla, \omega)) \otimes_F \mathbb{C} \tag{0.6}$$

so that (0.1) fulfills a reciprocity formula

$$\begin{array}{ccc} \epsilon_{DR}(\nabla) & \xrightarrow{\theta} & \epsilon_B(\nabla) \\ \cong\downarrow & & \downarrow\cong \\ \otimes_{x\in\bar{X}}\epsilon(x,\nabla,\omega) & \xrightarrow[\otimes_{x\in\bar{X}}\theta(x,\nabla,\omega)]{} & \otimes_{x\in\bar{X}}\epsilon_B(x,\nabla,\omega) \end{array} \tag{0.7}$$

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Volumes of complex hyperbolic manifolds with cusps

Jun-Muk Hwang

We discuss the problem of bounding the number of cusps of a complex hyperbolic manifold in terms of its volume. This problem has been studied by topologists, as a generalization of the corresponding problem for real hyperbolic manifolds ([HP], [Pa]).

Applying algebro-geometric method, we get a bound which is considerably better than those obtained previously by methods of geometric topology. Our main result is the following. Let

$$P(\ell) := \frac{(n\ell + n + \ell)!}{n!(n\ell + \ell)!}.$$

Theorem *Let X be an n-dimensional complex hyperbolic manifold of finite volume. Let k be the number of cusps of X and let $Vol(X)$ be the volume of X with respect to the Bergmann metric with holomorphic sectional curvature -1. Then*

$$Vol(X) \geq \frac{(4\pi)^n}{n!(P(4) - P(2))}(k-1).$$

This is obtained by examining the dimensions of the spaces of certain cusp forms. The proof depends essentially on the existence of a toroidal compactification of X and its metric property which was established by Mumford [Mu] for X defined by an arithmetic group and generalized to arbitrary X by N. Mok and W.-K. To [To]. Excepting these results, we only use standard methods in algebraic geometry for the proof.

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Critical exponents of analytic functions

Jacob Sturm
(joint work with D.H.Phong)

Let $f(z_1, ..., z_n)$ be an analytic function defined on a neighbourhood of the origin in $\mathbb{C}^n$. Then $\delta(f)$, the critical exponent of f, is defined as follows:

$$\delta(f) = \sup\{c \in \mathbb{R} : \int_B |f|^{-c} dV < \infty \text{ for some ball } B \text{ centered at } 0 \in \mathbb{C}^n\}.$$

The critical exponent $\delta(f)$ is well-known in singularity theory and in harmonic analysis, where it is a measure of the singularity of $f(z)$ at 0 (see e.g. [AGV], [DK], [M], [P]). It is also the first pole of the zeta function $\zeta(s) = \int_B |f|^{-s} dV$ where B is a sufficently small ball. More generally, let the multiplier ideal sheaf $\mathcal{I}_c(f)$ be defined by

$$\mathcal{I}_c(f) = \{g \text{ holomorphic germ } : \int_B |g||f|^{-c} dV < \infty\}$$

for some ball B centered at $0 \in \mathbb{C}^n$. Then the "jumping coefficients" defined by Ein, Lazarsfeld, Mustata et al. are the numbers δ_i where $\mathcal{I}_{c_i}(f) \neq \mathcal{I}_{c_i - \epsilon}(f)$ for $\epsilon > 0$ small (see [ELSV] and references therein). The lowest jumping coefficient is also called the log canonical threshold in algebraic geometry (see e.g. [K]). It coincides with $\delta(f)$ and hence with the first pole of $\zeta(s)$. One can show that the higher jumping coefficients correspond to the higher poles of $\zeta(s)$. Multiplier ideal sheaves and critical exponents have emerged as powerful tools in complex geometry (see e.g. [AS], [S], [DK], [W1]). Global versions of $\delta(f)$ play a major role in the search for canonical metrics in Kähler geometry ([Y], [T1-2], [N], [So], [W2]). Finally, analogues of $\delta(f)$ defined for kernels are central to the study of decay rates of oscillatory integral and sublevel set operators ([PS], [PSS2]).

One approach to the study of critical exponents, and, more generally, to integrals of the type in $\delta(f)$, is that of algebraic estimates, which we now describe:

• Let $P(z)$, $Q(z)$ be polynomials with complex coefficients of degrees M and N respectively, with $Q(z)$ monic. Let $S = \{\alpha : Q(\alpha) = 0\}$ denote the set of roots of $Q(z)$, counted with multiplicity (so S is a set with N elements).

• If $A \subset \mathbb{C}$, define the diameter $d(A)$ of A by $d(A) = \sup_{\alpha,\beta \in A} |\alpha - \beta|$.

• For $0 \leq k \leq N-1$ and $\alpha \in S$ define $L_k(\alpha) = \inf\{d(S_{N-k}(\alpha))\}$ where the infimum is taken over all subsets $S_{N-k}(\alpha) \subset S$ such that $|S_{N-k}(\alpha)| = N-k$ and $\alpha \in S_{N-k}(\alpha)$.

We shall make the following two assumptions:

• $S \subset B_{\Lambda/2}$.

• $\nu\epsilon + 2 - (N-k)\delta \neq 0$, for all integers k, ν with $0 \leq k \leq N$, $0 \leq \nu \leq M$.

For each $\nu \geq 0$, we define k_ν by $k_\nu = -1$ if $\nu\epsilon + 2 > N\delta$, and otherwise as the integer between 0 and $N-1$ satisfying

$$(N - k_\nu - 1)\delta < \nu\epsilon + 2 < (N - k_\nu)\delta.$$

Evidently, the integral $\int_{B_\Lambda} \frac{|P(z)|^\epsilon}{|Q(z)|^\delta} dV$ is finite if and only if $L_{k_\nu}(\alpha) > 0$ for any root α of $Q(\alpha) = 0$, where ν is the order of vanishing of $P(z)$ at α, and k_ν is defined as above. Since the cluster scales $L_k(\alpha)$ are decreasing in k, this condition is actually equivalent to the seemingly more restrictive condition that for all ν with $P^{(\nu)}(\alpha) \neq 0$, we have $L_{k_\nu}(\alpha) > 0$. The following theorem gives a precise, quantitative version of this statement.

Theorem 1 (PS1). *Under the preceding assumptions, the integral* $\int_{B_\Lambda} \frac{|P(z)|^\epsilon}{|Q(z)|^\delta} dV$ *is of size*

$$\int_{B_\Lambda} \frac{|P(z)|^\epsilon}{|Q(z)|^\delta} dV \sim \sum_{\{\alpha : Q(\alpha)=0\}} \sum_{\{\nu : P^{(\nu)}(\alpha) \neq 0\}} \frac{|P^{(\nu)}(\alpha)|^\epsilon}{\Phi_{\nu,k_\nu}(\alpha)}$$

where $\Phi_{\nu,k}(\alpha)$ *is defined by*

$$\Phi_{\nu,k}(\alpha) = \begin{cases} L_k(\alpha)^{(N-k)\delta-(\nu\epsilon+2)} \prod_{0 \le i < k} L_i(\alpha)^\delta, & \text{if } k \ge 0 \\ \Lambda^{N\delta-(\nu\epsilon+2)}, & \text{if } k < 0 \end{cases}$$

Moreover, if we let $\mathcal{B}$ denote the set of pairs (P, Q) (which we view, via the coefficients, as an open subset of a finite dimensional vector space), then there is a family of closed varieties $\emptyset = V_r \subset V_{r-1} \subset \cdots \subset V_0 = \mathcal{B}$, and a sequence $T_\lambda(b)$ of absolute rational functions (i.e. ratios of absolute polynomials, that is, functions of the form $\frac{(\sum_i |P_i|^2)^{\epsilon'}}{(\sum_j |Q_j|^2)^{\delta'}}$, with P_i, Q_j a finite collection of polynomials and ϵ', δ' positive real numbers) with the property

$$\int_{B_\Lambda} \frac{|P(z)|^\epsilon}{|Q(z)|^\delta} dV \sim T_\lambda(b) \text{ for all } b \in V_\lambda \backslash V_{\lambda-1}.$$

Here the equivalence $\sim$ means that each side is bounded by positive constant multiples of the other side, with constants which depend only on ϵ, δ, and the degrees M and N of $P(z)$ and $Q(z)$. The constants are independent of the choice of P and Q.
This theorem, and its real analytic analogues, provides a tool for studying critical exponents. For example, the following results may be obtained using these methods:

1. The real analytic setting: Let f be real analytic, and $B \subset \mathbb{R}^n$ a ball centered at the origin. Let $B' \subset B$ be a ball whose radius is smaller than that of B.

Theorem 2 (PSS1). *If* $n = 2$, *the map* $f \mapsto \int_{B'} |f|^{-\delta}$ *is continuous (with respect to the sup norm on* f *on* B*). If* $n = 3$, *then the map is continuous provided* $\delta < 2/N$, *where* N *is the multiplicity of* f *at the origin.*

This last result is sharp since Varchenko's example shows that continuity fails for $\delta > 2/N$.

2. Strong semi-continuity and the ascending chain condition: Demailly and Kollar make the following conjectures in [DK]:
A: Let f be analytic on a ball $B \subset \mathbb{C}^n$. Then there exists $\alpha > 0$ such that if g is analytic on B and $||g - f||_{C^0} < \alpha \Rightarrow \delta(g) \ge \delta(f)$.
B: Let $\mathcal{C}(n) = \{\delta \in \mathbb{R} : \delta = \delta(f) \text{ for some } f\}$. Then $\mathcal{C}(n)$ satisfies the ascending chain condition: If $x_k \in \mathcal{C}(n)$ and $x_k \le x_{k+1}$ then $x_k = x_{k+1}$ for k sufficiently

large.
In [DK], it was shown that conjecture **B** implies conjecture **A**, and that the following weaker version of Conjecture A holds: Fix f as above. Then for any $\epsilon > 0$, there exists $\alpha(\epsilon) > 0$ so that $||g - f||_{C^0} < \alpha(\epsilon) \Rightarrow \delta(g) \geq \delta(f) - \epsilon$. Special cases of this weaker version had been obtained in [T2], [AS] and [PS1]. Using algebraic estimates, we can show:

Theorem 3 (PS2). *Conjectures A and B are true if $n = 2$ and f is either complex analytic or real analytic.*

Theorem 4. *Conjecture A is true if $n = 3$, f is complex analytic and $\delta(f) < \frac{4}{N}$, where N is the multiplicity of f at the origin.*

Very recently, Hou [H] has established Conjecture **A** under some additional hypotheses, such as the origin being an isolated singularity.

3. The global setting:

Theorem 5 (PS3). *Let P_i and Q_j be a finite collection of polynomials on $\mathbb{C}^n$, and let ϵ, δ be positive real numbers. Then the integral*

$$\int_{\mathbb{C}^n} \frac{(\sum_i |P_i|^2)^\epsilon}{(\sum_j |Q_j|^2)^\delta}$$

when viewed as a function of the coefficients of the P_i and Q_j, is continuous on a non-empty Zariski open set.

4. The non-archimedean setting: Thus far we have restricted our attention to polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$, but in applications it is useful to allow other rings of coefficients: Let p be a prime number a let $Q \in \mathbb{Z}_p[X]$ be a monic polynomial of degree N. Let K be a finite extension of $\mathbb{Q}_p$ containing all the roots of Q, and assume that the roots lie in $B_{1/2}$, the ball of radius $\frac{1}{2}$ centered at $0 \in K$. Let $Q = \prod_{\mu=1}^r Q_\mu$ the factorization of Q in $\mathbb{Z}_p[Z]$ into irreducible factors. Let $N_\mu = deg(Q_\mu)$.

Theorem 6. *Let $\delta > 0$ be a rational number such that $\frac{1}{\delta} \notin \mathbb{Z}$. There exist absolute polynomials $F_1, ..., F_t$, depending only on k and N_μ and δ, such that*

$$\int_{B_1} \frac{1}{|Q(z)|^\delta} \sim \sum_{j=1}^t \frac{1}{F_j(b)}$$

where $b \in \mathcal{B}$, the space of coefficients of the Q_μ.

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Complexification of group actions

Stefano Trapani

The geometric invariant theory of Mumford and its interpretation in terms of moment maps of hamiltonian actions, is dealing essentially with the holomorphic (algebraic) action of a reductive complex Lie group on a complex space see [9]. A reductive complex group is the complexification of a compact real Lie group see [6]. Suppose that a complex space X is endowed with an action of a real Lie group G by holomorphic tranformations. In order to study this in the framework of geometric invariant theory it is convenient to embed X as an open G invariant subset of a good complex space X^* with an holomorphic action of the complexified group $G^{\mathbb{C}}$. More precisely we say that a complex space X^* with an holomorphic $G^{\mathbb{C}}$ action is a universal complexification of X, if there exists a G equivariant open embedding $i : X \to X^*$ such that $G^{\mathbb{C}} i(X) = X^*$, and for every pair (Y, ϕ) where Y is a complex space with an holomorphic $G^{\mathbb{C}}$ action and ϕ is an holomorphic G equivariant map from X to Y, there exists a unique holomorphic $G^{\mathbb{C}}$ equivariant map $\tilde{\phi} : X^* \to Y$ such that $\tilde{\phi} \circ i = \phi$. Clearly if a universal complexification exists, then it is unique.

By the results of Palais [10] and Heinzner-Iannuzzi [5], in a broad variety of situations a canonical space X^* as above can be constructed as a non necessarily Hausdorff complex space. However Heinzner proved in [3] that, if G is compact and X is Stein, then X^* exists, is Hausdorff and in fact it is a Stein space. Later Heinzer and Iannuzzi proved that, if G is compact and X is holomorphically convex, then X^* exists is Hausdorff and holomorphically convex. Moreover in [12] the following statement is proved: Assume that G is a compact torus and that X is weakly pseudoconvex or C^k weakly pseudoconvex for some $k \geq 0$, then the universal complexification X^* exists, it is Hausdorff and it has the same pseudoconvexity property of the original space X. Here by pseudoconvex, respectively C^k pseudoconvex space we mean a complex space having a plurisubharmonic, respectively C^k plurisubharmonic, exhaustion function.

Let us now discuss the results in [1], [7] and [8] concerning the action of some non compact connected real Lie groups on a Stein manifold. Let G be a connected real Lie group with Lie algebra g. We say that G is pseudocompact, if it satisfies the following assumptions:

i) There exists a discrete cocompact subgroup Γ of G such that $G^{\mathbb{C}}/\Gamma$ is Stein
ii) The map $(g, \xi) \to g\ exp\ (i\xi)$ from $G \times g$ to $G^{\mathbb{C}}$ is a diffeomorphism.

For example if G is a connected Lie group which is compact, or simply connected and abelian, or more generally simply connected and solvable with rational structure constants, then G is pseudo compact. Assume now that X is a Stein manifold with the holomorphic action of a pseudo compact Lie group G. In [1] we proved that the universal complexification X^* exists and it is a Stein manifold if and only if the complex manifold $G^{\mathbb{C}} \times X/\Gamma$ is Stein. Here the action of Γ on $G^{\mathbb{C}}$ is by left translations, and the action on $G^{\mathbb{C}} \times X$ is diagonal. This theorem establishes a bridge between the problem of constructing the universal complexification with the old Serre problem about fiber bundles with Stein base and Stein fiber. Moreover by using a result of Forstnerich [2] we see that if there exists no non constant bounded plurisubharmonic function on X, then $X = X^*$, so that X^* exists and it is Stein. On the other extreme, by [11], if X is a bounded domain in $\mathbb{C}^n$, then X^* exists and it is Stein. If we do not insist on the complexification to be Stein, we have the following partial results:

In [7] it is proved that, if X is a taut manifold, then the universal complexification X^* exists, i.e. it is a Hausdorff complex manifold. On the other hand in [8] it is proved, that if X admits a Bergman metric, in other words if the space of

global L^2 sections of the canonical line bundle generate first jets at each point of X, then the universal complexification X^* exists. Actually the same method of proof would give us the following statement: Let H be a holomorphic line bundle over X together with a holomorphic lifting of the G action. Assume that there exists a Banach subspace B of the space of global holomorphic sections of H such that the elements of B generate first jets at each point of X, then the universal complexification X^* exists.
As a corollary one can derive that, if X admits either a smooth bounded strictly plurisubharmonic function, or a G invariant smooth strictly plurisubharmonic function, then the universal complexification X^* exists.
We now want to outline the construction of the "Palais globalization X^*":
For simplicity let us assume X to be smooth. We consider of $G^{\mathbb{C}} \times X$ the diagonal action of G corresponding to the right action of G on $G^{\mathbb{C}}$. Let $\xi_1, \cdots, \xi_n$ be a base of the Lie algebra g and let $\hat{\xi_1}, \cdots \hat{\xi_n}$ be the corresponding vector fields on $G^{\mathbb{C}} \times X$ induced by the action. Let J be the complex structure on $G^{\mathbb{C}} \times X$. Then the distribution $Span\{\hat{\xi_1}, \ldots \hat{\xi_n}\} + JSpan\{\hat{\xi_1}, \cdots \hat{\xi_n}\}$ is Frobenius integrable. Let X^* be the quotient space of $G^{\mathbb{C}} \times X$ modulo the leaves of the foliation induced by the above distribution. Let us consider the $G^{\mathbb{C}}$ action given by the left action of $G^{\mathbb{C}}$ on the first factor of $G^{\mathbb{C}} \times X$. This induces an action on X^*. Let $i : X \to X^*$ be the map induced by the embedding $x \to (e, x)$, where $e \in G^{\mathbb{C}}$ is the identity element. We say that the action is univalent, if the map i is injective. If the action is univalent, then we can endow X^* with a structure of a smooth locally euclidean complex space for which the map i becomes an open G equivariant holomorphic embedding. Moreover it is easy to see that X^* becomes a universal complexification. However for a manifold X which is not Stein the Palais complexification X^* is in general not Hausdorff. For example let us take $X = \mathbb{C}^2 \setminus \{(z, w) \in \mathbb{C}^2 : z = 0\ |w| = 1\}$ with the $\mathbb{S}^1$ action $e^{i\theta}(z, w) = (e^{i\theta}z, e^{i\theta}w)$. Then the action is univalent but the Palais complexification X^* can be seen to be non Hausdorff.

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Linear coverings of complex manifolds

Philippe Eyssidieux

A classical theorem of Riemann asserts that a simply connected Riemann surface is biholomorphic to $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$. Uniformization in several complex variables is well-known to be more difficult than in one variable. The structure of the universal covering space of a projective manifold of dimension greater than 2 is known to constrain strongly the geometry of the manifold [Cam95][Kol93].
The most reasonable analogue of this theorem so-called Shafarevich conjecture of holomorphic convexity. It predicts that the universal covering space of a complex projective manifold should be holomorphically convex. A complex manifold is holomorphically convex if it has a proper map to a Stein space, or equivalently, for every sequence of points going to infinity there is an holomorphic function whose values on the sequence tends to infinity.
Progress on that conjecture has been accomplished by [Cam94] [Kol93]. But these developpements do not adress the construction of holomorphic functions on the universal covering space.
Due to our difficulties in understanding the part of the fundamental group which is not seen by finite dimensionnal representations (actually groups violating Kazhdan's property (T) are also somewhat better), it is difficult to make progress on this conjecture without assuming existence of finite dimensionnal representations of infinite image.
On the other, using the powerful techniques of [Sim1988] and [GroSch92] regarding holomorphic convexity of spaces of the form, it is possible to try and prove holomorphic convexity of $\widetilde{X}/H_S$ where S is some set of finite dimensionnal representations of the fundamental group.
It is well known, in the abelian case, that one needs Hodge-theoretic conditions on S to ensure holomorphic convexity. In the talk, I reported on two recent results in the non abelian case.

Theorem 1. *(*[Eys2004]*) Let X be a connected projective algebraic complex manifold. Let M be an absolute constructible (in the sense of* [Sim1993]*) quasi compact set of conjugacy classes of linear reductive representations of $\pi_1(X)$ in some reductive algebraic group G over $\mathbb{Q}$.*
The Galois covering space of X, $\widetilde{X}_M = \widetilde{X^{univ}} / \cap_{\rho \in M(\overline{\mathbb{Q}})} \ker(\rho)$ corresponding to the intersection of the kernels of elements of M is holomorphically convex.

Rather than giving full details on absolute constructible sets, we will be content of recalling the following facts:

- The full moduli space $M(X,G)$ of representations of $\pi_1(X)$ in G defined is absolute constructible and quasi compact (acqc).
- Whenever ρ is an isolated point in $M(X,G)$, $\{\rho\}$ is acqc.
- Acqc sets are invariant under various operations; given $f : Y \to X$ a morphism of smooth connected projective varieties, by taking image and reciprocal image under $f^* : M(X,G) \to M(Y,G)$; given $\mu : G \to G'$ a morphism, by taking image and reciprocal image under $\mu_* : M(X,G) \to M(X,G')$.
- Given $f : Y \to X$ a dominant morphism and $i \in \mathbb{N}$ the set $M^i_f(X, GL_n)$ of local systems V on Y such that $R^i f_* V$ is a local system is ac and taking image and reciprocal image under $R^i f_* : M^i_f(X, GL_n) \to M(Y, GL_{n'})$ preserve acqc sets.
- The complex points of a closed acqc set M are stable under the $\mathbb{C}^*$ actions defined by [Sim1988] in terms of Higgs bundles, whose fixed point set M^{VHS} consists in polarizable complex Variations of Hodge structure (VHS, in short). M is then the smallest closed acqc set containing M^{VHS}.

To state the second result, which is a joint work in progress with L. Katzarkov, T. Pantev and M. Ramachandran, we have to give some notations. Let $\mathcal{T}^{VHS}_M$ be the full Tannakian subcategory of the Tannakian category of local systems generated by the elements of M^{VHS}. Its objects may be described as follows. They are the subquotients of $\alpha_1(\rho_1) \otimes \ldots \otimes \alpha_s(\rho_s)$, where $\alpha_1, \ldots, \alpha_s$ is a finite collection of linear rational representations of G and $\rho_1, \ldots, \rho_s$ are elements of M^{VHS}. The objects of $\mathcal{T}^{VHS}_M$ underly polarizable complex VHS. Let T^{VMHS}_M be the thick Tannakian category of Variations of Mixed Hodge Structures whose graded constituents are objects of $\mathcal{T}^{VHS}_M$. Let Γ^∞_M be the quotient of $\pi_1(X)$ by the intersection of the kernels of the objects of T^{VMHS}_M and of the objects of M. The group Γ^∞_M has quotients Γ^k_M corresponding to taking intersection of the kernels of the VMHS in T^{VMHS}_M with a weight filtration of length $k+1$. In particular $\Gamma^0_M = \Gamma_M$.
Notation: H^k_M will denote the kernel of $\pi_1(X) \to \Gamma^k_M$.

Theorem 2. [EysKatPanRam2004] *Let X be a connected projective algebraic complex manifold. Let M be an absolute constructible quasi compact set, defined over $\mathbb{Q}$ of conjugacy classes of linear reductive representations of $\pi_1(X)$ in some reductive algebraic group G over $\mathbb{Q}$.*
For any subgroup H of $\pi_1(X)$ such that $H^\infty_M \subset H \subset H^1_M$, the Galois covering space of X, $\widetilde{X^{univ}}/H$ is holomorphically convex.

In particular the Shafarevich conjecture is true when the fundamental group is almost linear.

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Integration of meromorphic cohomology classes and applications

Daniel Barlet
(joint work with Jon Magnusson)

In this talk we explain the results of our recent paper in collaboration with Jon Magnusson (Reykjavik, Island) [B-Mg.03] to appear in the Asian Journal of Mathematics 2003. We begin by recalling the setting and results of our previous papers [B-Mg.98] and [B-Mg.99]. They describe how positivity conditions on the normal bundle of a compact complex submanifold Y of codimension $n+1$ in a complex manifold Z can be transformed into positivity conditions for a Cartier divisor in a space parametrizing $n-$cycles in Z. Using a more general setting where the pole Y is a generically finite pole for the given analytic family of n-cycles in Z and results obtained in [B-K.03] , we are able to precise a filtered version of integration of meromorphic classes which is (locally) optimal and to deduce from it a better (and more usefull) theorem of transfert of positivity form the normal bundle to Y to the line bundle associated to the corresponding incidence divisor in S. Of course this is a way to produce in a geometric way big line bundles on the Cycle' space of a projective manifold.

As an application of our results we prove that the following problem has a positive answer in many cases :

Let Z be a compact connected complex manifold of dimension $n+p$. Let $Y \subset Z$ a submanifold of Z of dimension $p-1$ whose normal bundle $N_{Y|Z}$ is (Griffiths) positive. We assume that there exists a covering analytic family $(X_s)_{s\in S}$ of compact $n-$cycles in Z parametrized by a compact normal complex space S.

Is the algebraic dimension of $Z \geq p$?

A.M.S. Classification : 32 F 12 , 32 J 25 , 14 C 20 , 32 C 30 , 32 C 36 .

Keywords : Integration of cohomology classes , Incidence divisor, Ample normal bundle, conormal filtration, local cohomology , family of cycles .

Main References :

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Effective behaviour of multiple linear systems on an algebraic surface

Sheng-Li Tan

Multiplier ideal sheaves have been applied successfully to obtain some effective results in algebraic geometry, for example, effective bounds on Matsusaka's big theorem and Fujita's conjecture (see the references in [7]). The purpose of this talk is to give optimal effective bounds for some other well-known theorems above multiple linear systems on an algebraic surface X, including the linear systems in the classical sense, i.e., possessing given types of singularities. By Zariski decomposition, we only need to consider the most general linear system $|nA+L|$, where A is nef and big, and L is arbitrary. Let

$$\mathcal{B}(A,L) = \frac{((K_X-L)A+2)^2}{4A^2} - \frac{(K_X-L)^2}{4},$$
$$\beta(A,L) = [\mathcal{B}(A,L)] \quad \text{(integral part)}.$$

Note that $n > \mathcal{B}(A,L) \iff n \geq 1+\beta(A,L)$.

Definition 1. *Let $k \geq 0$ be an integer.*

(1) *$|D|$ is called (-1)-**very ample** if $H^1(D)=0$.*
(2) *$|D|$ is called k-**very ample** if any zero-dimensional subscheme $\Delta \subset X$ with $\deg\Delta \leq k+1$ gives $\deg\Delta$ independent conditions on $|D|$.*

All of the desired optimal effective bounds follow from the following main theorem.

Theorem 2. *Let $k \geq 0$. Assume that either $n > k+\mathcal{B}(A,L)$, or $n = \mathcal{B}(A,L)$ when $k=0$ and $L \sim K_X+\lambda A$ for some $\lambda \in \mathbb{Q}$. Suppose $|nA+L|$ is not $(k-1)$-very ample, i.e., there is a zero dimensional subscheme Δ on X with minimal degree $\deg\Delta \leq k$ such that it does not give independent conditions on $|nA+L|$. Then there is an effective divisor $D \neq 0$ containing Δ such that*

$$\begin{cases} LD - D^2 - K_XD \leq k, \\ DA = 0. \end{cases}$$

For example, if $A = H$ is ample, then we get an effective version of Serre's theorem.

Corollary 3. (Effective Serre Theorem) *If $n > k + \mathcal{B}(H, L)$, then $|nH + L|$ is $(k-1)$-very ample. Equivalently, $|nH + L|$ is $(n - \beta - 2)$-very ample when $n \geq \beta + 1$.*

The following example shows that the bounds are optimal.

Example 4. *$X = \mathbb{P}^2$, $H = \mathcal{O}_{\mathbb{P}^2}(1)$ and L is arbitrary.*

If $L = K_X$, then Corollary 3 implies Fujita's conjecture. If $k = 0$, then we obtain a solution to the classical problem on effective postulation (see [6], p.444).

Corollary 5. (Effective Postulation) *Assume that either $n > \mathcal{B}(H, L)$ or $n = \mathcal{B}(H, L)$ when $L \sim K_X + \lambda H$ for some $\lambda \in \mathbb{Q}$. Then*

$$\dim |nH + L| = \chi(\mathcal{O}_X) - 1 + \frac{1}{2}(nH + L)(nH + L - K_X).$$

If $L = 0$ and $k = 2$, then Corollary 3 is an effective Matsusaka's big theorem.

Corollary 6. (Effective Matsusaka's Big Theorem) *If*

$$n > 2 + \frac{(K_X H + 2)^2}{4H^2} - \frac{K_X^2}{4},$$

then nH is very ample.

The bound is optimal, see

Example 7. *Any smooth double cover $\pi : X \to \mathbb{P}^2$, $H = \pi^*(\mathcal{O}(1))$.*

In general, we set

$$\alpha(A, L) = \begin{cases} \min_D \{ LD - K_X D - D^2 \}, & \text{if } A \text{ is not ample,} \\ +\infty, & \text{if } A \text{ is ample,} \end{cases}$$

where D runs over all effective divisors $D \neq 0$ such that $DA = 0$. Then we have

Corollary 8. *Assume that $\alpha = \alpha(A, L) \geq 1$ and $n \geq \beta + 1 = \beta(A, L) + 1$. Then $|nA + L|$ is $\min\{ \alpha - 2,\ n - \beta - 2 \}$-very ample.*

Some well-known conditions on linear systems are those satisfying $\alpha \geq 1$. For example, $\alpha(A, K_X) = \min_D \{ -D^2 \} \geq 1$ (Fujita's condition). $\alpha(A, 0) = 2$ if and only if $p_a(D) \leq 0$ for any D (Artin's condition). Laufer-Ramanujan's condition is that $LD \geq K_X D$ for any D, which implies also that $\alpha \geq 1$.

Corollary 9. (Effective Artin Theorem [1]**)** *Assume that A satisfies Artin's condition, i.e., $\alpha(A, 0) = 2$. Then we have*

1. *If $n \geq 1 + \beta(A, 0)$, or $n = \mathcal{B}(A, 0)$, $A \sim \lambda K_X$, then $H^1(nA) = H^2(nA) = 0$.*
2. *If $n \geq 2 + \beta(A, 0)$, then $|nA|$ is base point free.*
3. *If $n \geq 3 + \beta(A, 0)$, then Φ_{nA} is the contraction map with connected fibers.*

The canonical divisor $A = K_X$ of a minimal surface of general type satisfies Artin's condition. In this case, the Effective Artin Theorem gives the vanishing theorem of Mumford and theorem of Bombieri [3]. So the bounds are optimal.

Corollary 10. (Effective Zariski Theorem [8]**)** *Assume that $|A|$ has no fixed part.*

(1) *If $n \geq 2 + \beta(A, 0)$, then $|nA|$ is base point free.*
(2) *If $n \geq 3 + \beta(A, 0)$, then Φ_{nA} is a contraction map.*

Now we consider the linear systems in the classical sense. Let S be a curve singularity, and let $m_1, \cdots, m_r$ be its multiplicity sequence.

$$\beta(S) := \sum_{i=1}^{r} \frac{(m_i + 1)^2}{4}, \qquad \ell(S) := \sum_{i=1}^{r} \frac{m_i(m_i + 1)}{2}.$$

Denote by $|C - S_1p_1 - \cdots - S_kp_k|$ the linear subsystem of $|C|$ whose curves admit a singularity of type S_i or "worse" at p_i for each i (see [9]).

Corollary 11. (1) *If $n > \mathcal{B}(H, L) + \sum_{i=1}^{k} \beta(S_i)$, then*

$$\dim |nH + L - S_1p_1 - \cdots - S_kp_k| = \dim |nH + L| - \sum_{i=1}^{k} \ell(S_i)$$

(2) *If $n > \mathcal{B}(H, L) + \sum_{i=1}^{k} \beta(S_i) + 1$, then generic curves in $|nH + L - S_1p_1 - \cdots - S_kp_k|$ are irreducible, and have exactly curve singularity of type S_i at p_i.*

This is a problem studied by Cayley, Noether, Hilbert, Zariski, $\cdots$, (see [9], p.51). Severi Variety $V = V_{|nH+L|}(S_1, \cdots, S_k)$ consists of irreducible curves in $|nH + L|$ having exactly singularities of *topological types* $S_1, \cdots, S_k$.

Corollary 12. *If $n > \mathcal{B}(H, L) + \sum_{i=1}^{k} \beta(S_i) + 1$, then*

$$\dim V = \dim |nH + L| - \sum_{i=1}^{k} c(S_i),$$

where $c(S_i)$ is the number of conditions given by S_i. V is a Zariski open subset of a variety $\overline{V} = \overline{V}_{|nH+L|}(S_1, \cdots, S_k)$ which is a $\mathbb{P}^N$-fibration.

Using Zariski's decomposition and our main theorem, we obtain also effective solutions to Riemann-Roch problem ([8]) and some related theorems ([2, 4, 5]). We find also the effective behavior of the rational maps $\Phi_{|nD|}$ and $\Phi_{|nA+L|}$, and effective bounds on the degrees of generators of the section ring $R(D) = \oplus_{n\geq 0} H^0(X, \mathcal{O}(nD))$ for some D.

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The Hitchin-Kobayashi correspondence

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The purpose of this note is to explain our result on stability for compact extremal Kähler manifolds with emphasis on its historical backgrounds. Our work is a generalization of a recent result of Donaldson on stability for a projective algebraic manifold polarized by a Kähler metric of constant scalar curvature.

[I] Let M be a compact connected Kähler manifold. Historically, the following conjecture of Calabi is well-known.

Conjecture. (a) *If* $c_1(M)_{\mathbb{R}} < 0$*, then* M *admits a unique Kähler-Einstein metric* ω *such that* $\mathrm{Ric}(\omega) = -\omega$.
(b) *If* $c_1(M)_{\mathbb{R}} = 0$*, then each Kähler class on* M *admits a unique Kähler-Einstein metric* ω *such that* $\mathrm{Ric}(\omega) = 0$.
(c) *For* $c_1(M)_{\mathbb{R}} < 0$*, find a suitable condition for* M *to admit a Kähler-Einstein metric* ω *such that* $\mathrm{Ric}(\omega) = \omega$.

For (a) and (b), after some pioneering works of Calabi and Aubin, a complete affirmative answer was given by Yau by solving systematically certain complex Monge-Ampère equations. As to (c), only partial results are known by Siu, Nadel, Tian and Yau, while very nice by-products such as Nadel's vanishing theorem was obtained in application of the theory of multiplier ideal sheaves to the existence problem of Kähler-Einstein metrics.

[II] On the other hand, for the study of moduli problems, stability has been a very important subject in algebraic geometry. Recall, for instance, the following famous result:

Fact (Mumford): *If* L *is an ample line bundle of degree* $d \geq 2g+1$ *over a nonsingular projective algebraic curve* C *of genus* $g \geq 1$*, then* (M, L) *is Chow-stable.*

For the pluri-canonical bundles $K_M^{\otimes m}$, $m \gg 1$, using the stability in the sense of Hilbert schemes, Gieseker generalized this result to the case where M is a surface of general type. For higher dimensional cases, there is a stability result by Viehweg in the case where the canonical bundle K_M is semipositive. However, both for the results of Gieseker and Viehweg, their proofs are fairly complicated.

[III] It is quite natural for us to suspect the relationship between the existence of Kähler-Einstein metrics (or more generally, that of extremal Kähler metrics) and the stability of the underlying manifolds. This suspicion may become a belief in view of the following Hitchin-Kobayashi correspondence for vector bundles:

Fact: *For an indecomposable holomorphic vector bundle E over a compact connected Kähler manifold, E is stable in the sense of Takemoto-Mumford if and only if E admits a Hermitian-Einstein metric.*

This fact was established in 1980's by Kobayashi, Lübke, Donaldson, Uhlenbeck and Yau. As a manifold analogue of this conjecture, we can naturally ask whether the following conjecture (known as Yau's conjecture) on the Hitchin-Kobayashi correspondence for manifolds is true:

Conjecture. *For a polarized algebraic manifold (M, L), the polarization class admits a Kähler metric of constant scalar curvature (or more generally an extremal Kähler metric) if and only if (M, L) is asymptotically stable in a certain sense of GIT.*

Actually, after a first breakthrough by Tian, there was a very remarkable work of Donaldson which establishes the stability of a polarized algebraic Kähler manifold M of constant scalar curvature essentially when the connected linear algebraic part $G(M)$ of the group of holomorphic automorphisms of M is semisimple.

[IV] By generalizing the concept of stability to the case where $G(M)$ is not semisimple, we extend Donaldson's result to extremal Kähler cases even when $G(M)$ is not semisimple. Namely, we obtained the following:

Theorem. *A polarized projective algebraic manifold with an extremal Kähler metric in the polarization class is always stable in this generalized sense.*

This in particular implies that an extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold M is unique, if any, modulo the action of $G(M)$.

Transcendental holomorphic Morse inequalities

Jean-Pierre Demailly

(joint work with S. Boucksom, M. Paun, Th. Peternell)

Classical holomorphic Morse inequalities [De85] (see also [Siu84, 90]) give estimates of the growth of cohomology groups of a hermitian line bundle (L, h) on a compact complex n-dimensional manifold X in terms of the $(1,1)$-curvature form $\Theta_h(L) = -\frac{i}{2\pi}\partial\overline{\partial}\log h$ of the line bundle. Especially, they provide lower bounds for the volume

$$\mathrm{Vol(L)} := \limsup_{\mathrm{k}\to+\infty} \frac{\mathrm{n!}}{\mathrm{k^n}} \mathrm{h^0(X, L^{\otimes k})}, \tag{1}$$

which is a measure of the asymptotic growth of sections. When L is ample or nef, the Riemann-Roch formula simply gives $\mathrm{Vol(L)} = \mathrm{c_1(L)^n}$. Our ultimate goal would be to extend Morse inequalities to arbitrary transcendental classes $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$. In the non Kähler case, one has to use here the Bott-Chern cohomology groups, namely d-closed (p,q)-forms modulo $\partial\overline{\partial}$-exact forms. We say that the $\partial\overline{\partial}$-cohomology class $\{\alpha\}$ is *pseudo-effective* ([De90, DPS94, DPS00]) if it contains a closed positive $(1,1)$-current, i.e. if there exists a locally L^1 potential φ such that $T = \alpha + i\partial\overline{\partial}\varphi \geq 0$. Then, assuming $\{\alpha\}$ pseudo-effective, one defines

$$\mathrm{Vol}(\{\alpha\}) = \sup_{\mathrm{T}\in\{\alpha\},\mathrm{T}\geq 0} \int_{\mathrm{X}} \mathrm{T}_{\mathrm{ac}}^{\mathrm{n}}, \tag{2}$$

where T_{ac} is the absolutely continuous part in the Lebesgue decomposition of the current T. S. Boucksom [Bou02] showed that this coincides with (1) in case $\alpha = c_1(L)$. The following conjecture would extend to "transcendental" $(1,1)$-classes results which are known to hold in the case of integral cohomology classes ([De85]).

Conjecture 1. *Let X be a compact complex manifold, and $n = \dim X$.*
(i) *Let α be a closed, $(1,1)$-form on X. We denote by $X(\alpha,\leq 1)$ the set of points $x \in X$ such that α_x has at most one negative eigenvalue. If $\int_{X(\alpha,\leq 1)} \alpha^n > 0$, the class $\{\alpha\}$ contains a Kähler current and*

$$\mathrm{Vol}(\alpha) \geq \int_{\mathrm{X}(\alpha,\leq 1)} \alpha^{\mathrm{n}}.$$

(ii) *Let $\{\alpha\}$ and $\{\beta\}$ be nef cohomology classes of type $(1,1)$ on X satisfying the inequality $\alpha^n - n\alpha^{n-1}\cdot\beta > 0$. Then $\{\alpha-\beta\}$ contains a Kähler current and*

$$\mathrm{Vol}(\alpha-\beta) \geq \alpha^{\mathrm{n}} - \mathrm{n}\alpha^{\mathrm{n}-1}\cdot\beta.$$

A Kähler current is a current T such that $T \geq \varepsilon\omega$ for some $\varepsilon > 0$ and some smooth positive form ω on X. It is known [DPa03] that X possesses a Kähler current if and only if X belongs to the Fujiki class $\mathcal{C}$ (of manifolds bimeromorphic to compact Kähler manifolds). Especially, the existence of α as in (i) should imply $X \in \mathcal{C}$.
In general, (ii) is a consequence of (i). In fact, if α and β are smooth positive definite $(1,1)$-forms and

$$\lambda_1 \geq \ldots \geq \lambda_n > 0$$

are the eigenvalues of β with respect to α, then $X(\alpha-\beta,\leq 1) = \{x\in X, \quad \lambda_2(x) < 1\}$ and

$$\mathbf{1}_{X(\alpha-\beta,\leq 1)}(\alpha-\beta)^n = \mathbf{1}_{X(\alpha-\beta,\leq 1)}(1-\lambda_1)\ldots(1-\lambda_n) \geq 1-(\lambda_1+\ldots+\lambda_n)$$

everywhere on X. This is proved by an easy induction on n. An integration on X then yields inequality (ii). The special cases of the conjecture that we can prove at the moment are summarized in the following two theorems.

Theorem 1. *The conjecture holds true for compact Kähler manifolds X which are limits by deformation of projective algebraic manifolds X_t with maximal Picard number $\rho(X_t) = h^{1,1}(X_t)$. In particular, the conjecture holds true for compact hyperkähler manifolds.*

Theorem 2. *Let X be a projective manifold of dimension n. Then there exists a constant c_n depending only on dimension (actually one can take $c_n = (n+1)^2/4$), such that the inequality*

$$\mathrm{Vol}(\omega - \mathrm{A}) \geq \int_{\mathrm{X}} \omega^{\mathrm{n}} - \mathrm{c_n} \int_{\mathrm{X}} \omega^{\mathrm{n}-1} \wedge \mathrm{c}_1(\mathrm{A})$$

holds for every Kähler metric ω and every ample line bundle A on X.

Theorem 1 is derived from the algebraic case via a rather standard deformation argument. We use the fact that, thanks to the assumption on X, arbitrary transcendental classes on X are limits of rational classes on the X_t's. On the other hand, the proof of Theorem 2 is deeper and is based on a concentration of mass technique using Monge-Ampère equations and the Calabi-Yau theorem [Yau78], along the lines of [DPa03]. The conjecture (with the optimal constant $c_n = n$) would be badly needed in view of the following observation.

Theorem 3. *Assume that the conjecture holds true for all compact Kähler manifolds bimeromorphic to a given compact Kähler n-fold X. Then the cone $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of pseudo-effective classes is dual to the cone $\mathcal{M} \subset H^{n-1,n-1}(X, \mathbb{R})$ of "movable" current classes, namely classes of $(n-1, n-1)$-currents of the form*

$$\mu_*(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$$

where $\mu : \widetilde{X} \to X$ is an arbitrary modification and $\widetilde{\omega}_j$, $1 \leq j \leq n-1$ are Kähler forms on $\widetilde{X}$.

The proof relies on the following *orthogonality estimate* for approximate Zariski decompositions $\mu^* T = [E] + \beta$ of a given Kähler current T with analytic singularities ([DPS00, BDPP04]), where $\mu : \widetilde{X} \to X$ a log-resolution of the singularities, E an effective $\mathbb{Q}$-divisor on $\widetilde{X}$ and β a smooth closed semi-positive $(1,1)$-form on X:

$$(E \cdot \beta^{n-1})^2 \leq C(\mathrm{Vol}(\{\mathrm{T}\}) - \beta^{\mathrm{n}}) \tag{3}$$

where C is a constant depending only on the numerical class $\{T\} \in H^{1,1}(X, \mathbb{R})$. Inequality (3) is itself derived from a variational argument for the volume, which requires inequality (ii) of the conjecture. Since the conjecture is known to hold for integral classes on projective manifolds, Theorem 3 is actually verified (unconditionally) in that case; then the Kähler forms $\widetilde{\omega}_j$ can be taken to be themselves integral, i.e. of the form $c_1(A_j, h_j)$ for suitable ample line bundles A_j.

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