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# Arithmetic Algebraic Geometry

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## Introduction by the Organisers

At this workshop various aspects of Arithmetic Algebraic Geometry have been discussed. It was organized by G. Faltings, G. Harder (Max-Planck-Institute for Mathematics Bonn) and N. Katz (Princeton university)

The main goal of this field is, to obtain information on the solution of diophantine problems by applying the tools provided by algebraic geometry.

The workshop was attended by 42 participants and we had the total number of 18 talks.

A very interesting diophantine problem, which has a very geometric flavour, is the investigation of the structure of the Brauer group of a scheme or more in geometric terms of a complex algebraic variety. This aspect has been discussed in the talks by Gabber, who reported on some recent progress in direction of the purity conjecture. It has also been discussed in the talks of Lieblich and de Jong, in which some more geometric questions have been discussed.

Another tool to obtain information on diophantine problems is provided by $p$-adic methods. Here certain analogies between classical analytic theory over $\mathbf{C}$ and $p$-adic analytic theory have to be developed. We have to understand the meaning of local systems. This topic was discussed in the talks of Deninger and Ramero. Minhyong Kim outlined a program how to used a $p$-adic unipotent Albanese map to prove finiteness in diophantine geometry, for instance the classical theorem of Siegel.

Shimura varieties are certainly interesting objects in Arithmetic Algebraic Geometry, they provide interesting examples of algebraic varieties. Ben Moonen's talk was on the borderline between $p$-adic methods and Shimura varieties. Laumon reported on the fundamental Lemma for $U(n)$. Various talks discussed the Galois representations attached to automorphic forms and abelan varieties (Bültel, Edixhoven,Harris, Diamond). In the talks of Pink and Ullmo Wildeshaus, Rapoport, M. Kings and Ullmo some other aspects of this field were discussed.

Goncharov discussed some interesting aspects of higher Teichmüller theory.

## Workshop: Arithmetic Algebraic Geometry

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# Abstracts

## On purity for the Brauer group

Ofer Gabber

For a scheme $X$ let $\mathrm{Br}'(X)$ be the torsion subgroup of $H^2(X_{et}, \mathbb{G}_m)$ (an element is torsion iff it is killed by a locally constant function $X \to \mathbb{Z}_{>0}$). We have an injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ which is an isomorphism if $X$ is affine ([3] II) and if $X$ admits an ample line bundle (proved differently by the author and by de Jong). For a local ring $R$ with maximal ideal $m$ let $U_R$ denote the punctured spectrum $\mathrm{Spec}(R) - \{m\}$. The purity conjecture for the Brauer group (Auslander-Goldman [1] 7.4, Grothendieck [6] III §6) asserts that if $X$ is a regular (locally nœtherian) scheme and $Z \subset X$ a closed subset of codimension $\geq 2$ then $\mathrm{Br}'(X) \to \mathrm{Br}'(X - Z)$ is an isomorphism. This is equivalent to

**Conjecture 1.** Let $R$ be a strictly henselian regular local ring. Then

$$(*) \qquad (\dim(R) \geq 2) \Rightarrow \mathrm{Br}'(U_R) = 0\,.$$

Conjecture 1 holds if $\dim(R) = 2$ (cited references) and if $\dim(R) = 3$ ([3] I). The theorem announced on [3] p. 204 implies that if $X$ is a regular scheme and $Y \subset X$ a nowhere dense regular subscheme then $\mathrm{Br}'(X) \xrightarrow{\sim} \mathrm{Br}'(B\ell_Y X)$. This together with Br-purity in dimension $\leq 3$ and purity of the branch locus can be used to show the following.

**Theorem 1.** *Let $X$ be a regular scheme and $\pi : X' \to X$ a birational blowing-up with regular center. If $x \in X$ and for every $y \in \pi^{-1}(x)$ the strict henselization $\mathcal{O}^{sh}_{X',y}$ satisfies $(*)$, then $\mathcal{O}^{sh}_{X,x}$ satisfies $(*)$.*

Conjecture 1 holds for $n$-torsion when $1/n \in R$ by [2], when $R$ is equal characteristic, and when $R$ is formally smooth over a discrete valuation ring ([4] 2.10). We note that $(*)$ for $R$ is equivalent to $(*)$ for the completion $\hat{R}$. We have the following fact (used below in a non nœtherian case).

**Theorem 2.** *Let $(A, fA)$ be an henselian pair with $f$ s.t. $Ann_A(f^n) = Ann_A(f^{n+1})$ for some $n$, $\hat{A}$ the $f$-adic completion of $A$, $U \subset \mathrm{Spec}(A)$ a quasi-compact open subset containing $D(f)$, $\hat{U}$ the inverse image of $U$ in $\mathrm{Spec}(\hat{A})$. Then*

$$\mathrm{Br}'(U) \xrightarrow{\sim} \mathrm{Br}'(\hat{U})\,.$$

One can generalize Conjecture 1.

**Conjecture 2.** Let $R$ be a strictly henselian complete intersection nœtherian local ring of dimension $\geq 4$. Then $\mathrm{Br}'(U_R) = 0$.

**Conjecture 3.** Let $R$ be a complete intersection nœtherian local ring of dimension 3. Then $\mathrm{Pic}(U_R)$ is torsion free.

Conjecture 2 implies Conjecture 3 (apply Conjecture 2 to $R^{sh}[[x,y]]/(xy)$). One can restate Conjecture 3 as vanishing of local flat cohomology: $H^2_{\{m\}}(\mathrm{Spec}(R),\mu_n) = 0$ for every $n > 0$. This holds for constant coefficients since for $R = R^{sh}$, $U_R$ is simply connected ([5] X 3.4).

We recall ([5] XI 3.13) that for $R$ as in Conjecture 2, $\mathrm{Pic}(U_R) = 0$. Thus Conjecture 2 is equivalent to the vanishing of $H^2(U_R,\mu_n)$ for every $n > 0$. One can show that Conjecture 2 implies a similar vanishing with coefficients in any finite flat commutative group scheme over $R$.

Conjecture 2 holds for $n$-torsion when $1/n \in R$. More generally:

**Theorem 3.** *Let $R = R^{sh}$ be a complete intersection nœtherian local ring of dimension $d$. Then for all $i < d$ and $n > 0$ an integer invertible in $R$,*

$$H^i_{\{m\}}(\mathrm{Spec}(R),\mathbb{Z}/n\mathbb{Z}) = 0\,.$$

This can be reduced as in [2] to the case where $R$ is essentially of finite type over a discrete valuation ring, which holds by ([7] 2.6).

**Theorem 4.** *If $R \to R'$ is a finite flat relative complete intersection extension of local rings as in Conjecture 2 then $\mathrm{Br}'(U_R) \to \mathrm{Br}'(U_{R'})$ is injective.*

Consider $\pi : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ and the $R$ group scheme $G = \pi_* \mathbb{G}_m/\mathbb{G}_m$. The statement of the theorem is equivalent to $H^1(U_R, G) = 0$. We may assume $R$ is complete, write $R = R_0/I$ where $R_0$ is a formal power series ring over a Cohen ring and $I$ is generated by a regular sequence, and find a finite flat extension $R_0 \to R'_0$ s.t. $R' \simeq R'_0 \underset{R_0}{\otimes} R$. Let $\mathfrak{X}$ be the formal completion of $U_{R_0}$ along $U_R$. One checks that a $G$-torsor on $U_R$ lifts to a $G_0$-torsor on $\mathfrak{X}$, and using $\mathrm{Leff}(U_{R_0}, U_R)$ ([5] X 2) the torsor extends from $\mathfrak{X}$ to an open subscheme $V \subset U_{R_0}$, and $H^1(V, G_0) = 0$ using above mentioned results.

**Theorem 5.**

(1) *Conjecture 2 holds if $R \supset \mathbb{F}_p$.*
(2) *Conjecture 2 holds if $\dim(R) \geq 5$.*

It suffices to consider Conjecture 2 for $p$-torsion elements where $p > 0$ is the residue characteristic of $R$. Part (1) is shown by applying Theorem 4 to extensions of the form $R[f_1^{1/p},\dots,f_n^{1/p}]$. In part (2) one can apply Theorem 2 to reduce to the case that $R$ is complete and $p$ is a non-zero-divisor in $R$. Let $\mathfrak{X}$ be the $p$-adic formal completion of $\mathrm{Spec}(R)$. Using part (1) one shows that if $A$ is an Azumaya algebra on $U_R$ then $A|\mathfrak{X}$ is $\mathcal{E}nd$ of a vector bundle. By Leff, $A|V$ is $\mathcal{E}nd$ of a vector bundle for some open subscheme $V$ of $U_R$ containing $U_{R/pR}$. By parafactoriality this extends to $U_R$.

**Theorem 6.** *Conjecture 1 holds for $p$-torsion when $R$ is of dimension 4, of mixed characteristic $(0,p)$, and contains $\mu_p$.*

**Corollary.** *Conjecture 1 holds when $R$ is of dimension 4 and of mixed characteristic $(0,p)$ and there is a sequence of birational blowing-ups with regular centers*

$$X_n \to \ldots \to X_0 = \mathrm{Spec}(\hat{R})$$

*s.t. the components of multiplicity not divisible by $p-1$ of the divisor of $p$ on $X_n$ constitute a normal crossings divisor.*

## References


[1] M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367-409.

[2] K. Fujiwara, A proof of the Absolute Purity Conjecture (after Gabber), pp. 153-183 *in* Algebraic Geometry 2000, Azumino, *Advanced Studies in Pure Mathematics* **36**, 2002.

[3] O. Gabber, Some Theorems on Azumaya Algebras, pp. 129-209 *in* Groupe de Brauer, LNM 844, Springer, 1981.

[4] O. Gabber, An injectivity property for étale cohomology, *Compositio Math.* **86** (1993), 1-14.

[5] A. Grothendieck, Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux (SGA2), North-Holland, 1968.

[6] A. Grothendieck, Le groupe de Brauer I, II, III, *in* Dix Exposés sur la Cohomologie des Schémas (J. Giraud et al), North-Holland, 1968.

[7] L. Illusie, Perversité et variation, *manuscripta math.* **112** (2003), 271-295.


## Moduli of Azumaya Algebras

Max Lieblich

Let $X$ be a smooth, geometrically connected, projective variety over a field $k$. Given a reductive algebraic group $G$, it is natural to wonder about the structure of the stack of $G$-torsors on $X$. When $G = \mathrm{PGL}_n$, this is essentially the same as the stack of Azumaya algebras on $X$ of degree $n$. Thus, one might hope that there is an interaction between the Brauer group of $X$ and the geometry of the moduli space of $\mathrm{PGL}_n$-bundles. This suspicion indeed bears fruit; in particular, one can use the structure of the moduli space when $X$ is a smooth, geometrically connected curve or surface over a finite, local, or algebraically closed field to deduce information about the "period-index problem" for unramified Brauer classes.

Using Giraud's non-abelian cohomology [3], one can associate to any $\mathrm{PGL}_n$-torsor $T$ an algebraic stack $\mathscr{X}$ which is a $\boldsymbol{\mu}_n$-gerbe over $X$ and a locally free sheaf $\mathscr{V}$ of rank $n$ on $\mathscr{X}$ such that the torsor associated to $\mathscr{E}nd(\mathscr{V})$ is naturally isomorphic to $T$. Furthermore, the isomorphism class of $\mathscr{X}$ as an $X$-stack depends only on the class of the torsor $T$ in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ arising from the presentation $1 \to \boldsymbol{\mu}_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$. Thus, one relates moduli of $\mathrm{PGL}_n$-torsors on $X$ with a fixed cohomology class to moduli of certain sheaves on a fixed algebraic stack $\mathscr{X}$. The subclass of sheaves arising in this way are the *twisted sheaves*, and they are characterized by the way in which the inertia stack of $\mathscr{X}$ acts on them. From this (sheaf-theoretic) point of view, one immediately gets a candidate for a compactification of the space of $\mathrm{PGL}_n$-torsors; namely, allow torsion free twisted sheaves $\mathscr{F}$ on $\mathscr{X}$. Instead of taking the endomorphism algebra $\mathscr{E}nd(\mathscr{F})$, one must take the "derived endomorphism algebra" $\mathbf{R}\mathscr{E}nd(\mathscr{F})$, yielding a *generalized Azumaya algebra*. There is a sense in which these objects form an Artin stack, giving a compactification of the stack of Azumaya algebras. One can also define a natural notion of stability, which agrees with classical definitions of stability for $\mathrm{PGL}_n$-torsors when the characteristic is 0 but possibly diverges in positive characteristic. (Details may be found in [6].)

Using these ideas, one can prove the following theorems, among others. First, suppose $C$ is a smooth curve over an algebraically closed field $k$ and $\mathscr{C} \to C$ is a $\boldsymbol{\mu}_n$-gerbe. By Tsen's theorem, $\mathscr{C}$ is isomorphic to the gerbe of $n$th roots of an invertible sheaf $\mathscr{L}$. Given an invertible sheaf $\mathscr{M}$, let $\mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{M})$ denote the stack of semistable $\mathscr{C}$-twisted sheaves of rank $n$ and determinant $\mathscr{M}$, and let $\mathbf{Sh}^{ss}_{C/k}(n, \mathscr{M})$ denote the stack of semistable sheaves on $X$ of rank $n$ and determinant $\mathscr{M}$.

**Theorem.** *There is an isomorphism*

$$\mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{M}) \xrightarrow{\sim} \mathbf{Sh}^{ss}_{C/k}(n, \mathscr{M} \otimes \mathscr{L}^\vee)$$

*which preserves the stable loci.*

Now suppose $X$ is a surface and let $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, c)$ denote the Artin stack of semistable twisted sheaves of rank $n$, trivial determinant, and second Chern class $c$. Let $\mathbf{GAz}^{ss}_{X/k}(n, [\mathscr{X}], 2nc)$ denote the associated stack of semistable generalized Azumaya algebras on $X$ of rank $n^2$, cohomology class $[\mathscr{X}]$, and second Chern class $2nc$.

**Theorem.** *For sufficiently large c, the stack* $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, c)$ *is non-empty, l.c.i., generically smooth and geometrically irreducible. The same is true of the stack* $\mathbf{GAz}^{ss}_{X/k}(n, [\mathscr{X}], 2nc)$.

(For the geometrically minded, this theorem is a very weak algebraic analogue of results of Taubes on the stable topology of the space of self-dual connections on a $G$-bundle [9].) The proof proceeds along lines very similar to O'Grady's proof of the corresponding result for untwisted sheaves [4, 8].

Recall that to any Brauer class over a field $\alpha \in \mathrm{Br}(K)$, one can associate two natural invariants: the *period* of $\alpha$ is the order of $\alpha$ in $\mathrm{Br}(K)$, while the *index* of $\alpha$ is the square root of the rank of a central division $K$-algebra representing $\alpha$. It is well-known that $\mathrm{per}(\alpha)$ divides $\mathrm{ind}(\alpha)$ and that $\mathrm{ind}(\alpha)$ divides some power of $\mathrm{per}(\alpha)$. If $K$ is the function field of an $n$-dimensional variety over an algebraically closed field, one is naturally led to conjecture that for any $\alpha \in \mathrm{Br}(K)$ one has $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{n-1}$. When $n = 1$, this follows trivially from Tsen's theorem. When $n = 2$, this was recently proven by de Jong (for $\mathrm{per}(\alpha)$ prime to $\mathrm{char}(k)$) [1]. One consequence of the above theorems is the following.

**Corollary.** *Let $X$ be a smooth geometrically connected surface over a field $k$, and let $\alpha \in \mathrm{Br}(X)$ be a Brauer class of period prime to* $\mathrm{char}(k)$.

(*i*) *If $k$ is finite or algebraically closed, then* $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.
(*ii*) *If $k$ is local and $X$ has smooth reduction, then* $\mathrm{per}(\alpha) | \mathrm{ind}(\alpha)^2$

*(iii) If in (ii) one has that $\alpha$ is unramified on a smooth model of $X$ over the integers of $k$, then* $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.

From part *(i)*, one recovers de Jong's result in the unramified case (and one can in fact recover the ramified case using the same techniques of proof). The proof in this case uses de Jong and Starr's generalization of the Graber-Harris-Starr theorem [2], applied to a space of twisted sheaves on a curve over $k(t)$. The proof for finite and local $k$ ultimately follows from an application of the Lang-Weil estimates [5] to the spaces of twisted sheaves on a surface. In both cases, the goal is to find a rational point on the relevant moduli space. Further details may be found in [6] or [7].

## References


[1] de Jong, A. J., *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. 123 (2004), 71–94.

[2] de Jong, A. J. and Starr, J., *Every rationally connected variety over the function field of a curve has a rational point*, Amer. J. Math. 125 (2003), 567–580.

[3] Giraud, Jean, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften, Band 179: Springer-Verlag, 1971.

[4] Huybrechts, Daniel and Lehn, Manfred, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31: Friedr. Vieweg & Sohn, 1997.

[5] *Number of points of varieties in finite fields*, Amer. J. Math. 76 (1954), 819–827.

[6] Lieblich, Max, *Moduli of twisted sheaves and generalized Azumaya algebras*, PhD Thesis, Massachusetts Institute of Technology, 2004.

[7] Lieblich, Max, *Twisted sheaves and the period-index problem*, in preparation (2004).

[8] O'Grady, Kieran G., *Moduli of vector bundles on projective surfaces: some basic results*, Invent. Math. 123 (1996), 141–207.

[9] Taubes, Clifford Henry, *The stable topology of self-dual moduli spaces*, J. Differential Geom. 29 (1989), 163–230.


## The unipotent Albanese map and Diophantine geometry

Minhyong Kim

Given a compact smooth hyperbolic curve $X$ over a number field $F$, we can consider its points $X(F_v)$ in some non-archimedean completion $F_v$ of $F$. Depending on some mild hypotheses one can define a $p-$adic logarithmic Albanese map

$$j_1 : X(F_v) \to H_1(X_v)/F^0$$

where the $H_1$ refers to De Rham homology and we are taking the quotient by the Hodge filtration. This map can be used (Chabauty) to prove Faltings' theorem on finiteness of $X(F)$ in certain circumstances, for example, if the Mordell-Weil rank of the Jacobian is strictly less than the genus of $X$. One proves this by showing that the pull-back via $j_1$ of some non-trivial linear function on $H_1/F^0$ has to vanish on the global points. On the other hand, such a pull-back lies inside the ring of Coleman functions, and hence, has only finitely many zeros on $X(F_v)$.

Our program is to generalize this technique by lifting $j_1$ to

$$j_n : X(F_v) \to U_n/F^0$$

where $U = \pi_1(X_v, x)$ is the pro-unipotent De Rham fundamental group of $X_v$ and $U_n$ is the quotient via the $n+1$-th level of the descending central series. This lift is constructed by using the crystalline structure on the De Rham fundamental group. The idea then is that the coordinate ring of $U_n/F^0$ provides many more functions ($p-$adic multiple polylogarithms) with which one attempts to annihilate the global points. At the moment, this program is not realized because of a lack of control in global Galois cohomology. However, there are interesting relations to vanishing conjectures of Jannsen, and the finiteness theorem of Siegel on $\mathbf{P}^1\setminus\{0,1,\infty\}$ can be proved using this idea and Soulé's vanishing theorem for Tate twists. Eventually, a careful analysis of the function spaces involved should give some effectivity (on numbers of points) and results in higher-dimensions as well.

## Local systems on the $p$-adic punctured disc

Lorenzo Ramero

Let us fix the following notation :

- $K$ is an algebraically closed field of characteristic zero, complete for a non-archimedean norm $|\cdot| : K \to \mathbb{R}$, and of residue characteristic $p > 0$.
- $\ell \neq p$ is a prime number, and $\Lambda$ is the algebraic closure of the finite field $\mathbb{F}_\ell$.
- $\Delta$ is the abelian group $\mathbb{Q} \times \mathbb{R}$, which we endow with the lexicographic ordering (relative to the standard ordering on the factors $\mathbb{Q}$ and $\mathbb{R}$). We let also $\Delta^+ := \{\delta \in \Delta \mid \delta \geq 0\}$, and for any $\delta := (q, r) \in \Delta$, we set $\delta^\natural := q$.

For any real number $r > 0$, the *punctured disc over $K$ of radius $r$* is the open subset $\mathbb{D}(r)^*$ of the analytic affine line $\mathbb{A}^1_K$, whose set of $K$-rational points is :

$$\mathbb{D}(r)^*(K) = \{x \in K \mid 0 < |x| \leq r\}.$$

Depending on one's favorite viewpoint, $\mathbb{D}(r)^*$ can be viewed as a rigid analytic variety ([3]), or as an analytic adic space ([4]), or alternatively as a non-archimedean analytic space à la Berkovich ([2]). Then one may endow $\mathbb{D}(r)^*$ with an étale topology, and a $\Lambda$-*local system* on the punctured disc is a locally constant sheaf of finite dimensional $\Lambda$-vector spaces on the resulting étale site $\mathbb{D}(r)^*_{\text{ét}}$. (The category of such $\Lambda$-local systems is independent of the chosen foundational viewpoint.)

In my talk I explained the following result.

**Theorem 1.** ([5, Th.4.2.42]) *Let $\mathscr{F}$ be a $\Lambda$-local system on the punctured disc $\mathbb{D}(1)^*$, and suppose that $H^1(\mathbb{D}(1)^*_{\text{ét}}, \mathscr{F})$ is a finite dimensional $\Lambda$-vector space (in which case we say that $\mathscr{F}$ has* bounded ramification*). Then there exist :*

(a) *a connected subdomain $U \subset \mathbb{D}(1)^*$ such that $U \cap \mathbb{D}(r)^* \neq \varnothing$ for every $r > 0$;*

(b) *a* break decomposition *of the restriction to $U$ of $\mathscr{F}$ :*

$$\mathscr{F}_{|U} \simeq \bigoplus_{\delta \in \Delta^+} \mathscr{F}(\delta)$$

*consisting of local systems $\mathscr{F}(\delta)$ (on the étale site of $U$).* $\square$

These break decompositions enjoy the following properties :

- (Functoriality). Denote by $\mathsf{C}$ the category consisting of all pairs $(\mathscr{G}, U)$, where $U \subset \mathbb{D}(1)^*$ is an open subset fulfilling condition (a) of theorem 1, and $\mathscr{G}$ is a $\Lambda$-local system on $U_{\text{ét}}$; a morphism $(\mathscr{G}, U) \to (\mathscr{G}', U')$ in $\mathsf{C}$ is a map of $\Lambda$-modules $\mathscr{G}_{|U\cap U'} \to \mathscr{G}'_{|U\cap U'}$. Then, for every $\delta \in \Delta^+$, the rule $\mathscr{F} \mapsto (\mathscr{F}(\delta), U)$ defines a functor from the category of $\Lambda$-local systems on $\mathbb{D}(1)^*$ with bounded ramification, to the category $\mathsf{C}$.
- (Tannakian condition). Moreover, suppose that $\mathscr{F}$ and $\mathscr{G}$ are two $\Lambda$-local systems on $\mathbb{D}(1)^*$ with bounded ramification. Then, for every $\delta, \delta' \in \Delta^+$ we have :
$$\mathscr{F}(\delta) \otimes_\Lambda \mathscr{G}(\delta') \subset \begin{cases} (\mathscr{F} \otimes_\Lambda \mathscr{G})(\max(\delta, \delta')) & \text{if } \delta \neq \delta' \\ \oplus_{\rho \leq \delta} (\mathscr{F} \otimes_\Lambda \mathscr{G})(\rho) & \text{otherwise} \end{cases}$$
and an analogous compatibility condition with the break decomposition of $\mathscr{H}om_\Lambda(\mathscr{F}, \mathscr{G})$.
- (Hasse-Arf theorem). We have the integrality condition :
$$\delta^\natural \cdot \mathrm{rk}_\Lambda \mathscr{F}(\delta) \in \mathbb{N} \qquad \text{for every } \delta \in \Delta^+.$$

**Remarks.** (i) More generally, the theorem holds when $\Lambda$ is an artinian local ring in which $p$ is invertible, and such that $\Lambda$ is the filtered union of its finite subrings. (Then, one says that a $\Lambda$-local system has bounded ramification if its cohomology is a $\Lambda$-module of finite type.)

(ii) The boundedness condition on $\mathscr{F}$ is independent of the radius of the punctured disc. That is, $\mathscr{F}$ has bounded ramification if and only if there exists $r \in (0, 1]$ such that $H^1(\mathbb{D}(r)^*_{\text{ét}}, \mathscr{F})$ is of finite type over $\Lambda$, if and only if the same holds for all $r \in (0, 1]$. This condition really means that the Swan conductor of $\mathscr{F}$ *"at the missing origin"* of the punctured disc, is finite.

(iii) I expect that the subset $U$ appearing in theorem 1 can always be chosen to be a punctured disc (of some possibly smaller radius) centered at the origin. In this case, one could rephrase the theorem by saying that the Tannakian category of *germs of $\Lambda$-local systems* around the origin admits a *Hasse-Arf filtration*, defined as in the recent paper by Y.André [1]. The methods of [1] would then allow to derive very strong structural properties for such local systems. I hope to address this question in a future work.

## References


[1] Y.André, Filtrations de type Hasse-Arf et monodromie $p$-adique. *Invent.Math.* 148 (2002), pp.285–317.

[2] V.Berkovich, Spectral theory and analytic geometry over non-archimedean fields. *AMS Math. Surveys and Monographs* 33 (1990).

[3] J.Fresnel, M.van der Put, Rigid Geometry and Its Applications. *Birkhäuser Progress in Math.* 218 (2004).

[4] R.Huber, Étale cohomology of rigid analytic varieties and adic spaces. *Vieweg Aspects of Math.* 30 (1996).

[5] L.Ramero, Local monodromy in non-archimedean analytic geometry. *Preprint* (2004).

## Fundamental Lemma and Hitchin Fibration for Unitary Groups

Gerard Laumon
(joint work with Ngô Bao Châu)

Let $G$ be an unramified reductive group over a non archimedean local field $F$. The *Langlands Fundamental Lemma* is a family of conjectural identities between orbital integrals for $G(F)$ and orbital integrals for endoscopic groups of $G$ which have been precisely formulated by Langlands and Shelstad. The Langlands Fundamental Lemma is a key tool for proving many cases of Langlands functoriality and for computing the zeta functions of Shimura varieties.

Our main result is that the Langlands Fundamental Lemma holds in the particular case where $F$ is a finite extension of $\mathbf{F}_p((t))$, $G = \mathrm{U}(n)$ is a unitary group and $p > n$. Hales and Waldspurger have shown that this particular case implies the Langlands fundamental lemma for unitary groups of rank $< p$ when $F$ is any finite extension of $\mathbf{Q}_p$.

We follow in part a strategy initiated by Goresky, Kottwitz and MacPherson. Our main new tool is a deformation of orbital integrals which is constructed with the help of the Hitchin fibration for unitary groups over projective curves.

More precisely both sides of the "Fundamental Lemma" are linear combinations of orbital integrals. Those orbital integrals count selfdual $\mathcal{O}_F$-lattices in finite dimensional hermitian $F$-vector spaces $V$ which are stable under certain hermitian endomorphisms of $V$. In particular they are the numbers of rational points of varieties over the residue field of $F$, the *affine Springer fibers*.

The affine Springer fibers do not behave well in families. However, up to homeomorphisms the affine Springer fibers are isomorphic to étale coverings of compactified Picard schemes of projective curves, which behave well in families. This is our first key observation.

Our second key observation is that the Hitchin fibrations give us natural group theoretical families of compactified Jacobians of curves. In fact we deduce the Langlands Fundamental Lemma from a global result which relates the $\ell$-adic cohomology of some Hitchin fiber for $\mathrm{U}(n)$ and the $\ell$-adic cohomology of a corresponding Hitchin fiber for the endoscopic group $\mathrm{U}(n_1) \times \mathrm{U}(n_2)$, $n = n_1 + n_2$. In addition to the Hitchin fibration, our main tools are the Atiyah-Borel-Segal localization theorem in equivariant cohomology, Deligne's purity theorem in $\ell$-adic cohomology and a Bertini theorem of Poonen.

### References

[A-I-K] A. ALTMAN, A. IARROBINO, S. KLEIMAN – Irreducibility of the Compactified Jacobian, dans "Real and complex singularities. Proceedings, Oslo 1976, P. Holm (ed.)", Sijthoff & Nordhoff, (1977), 1-12

[G-K-M] M. GORESKY, R. KOTTWITZ, R. MACPHERSON – Homology of affine Springer fibers in the unramified case, http://arxiv.org/abs/math.RT/0305144, (2003)

[Hit] N. HITCHIN – Stable bundles and integrable systems *Duke Math. J.* **54**, (1987), 91-114

[Ka-Lu] D. KAZHDAN, G. LUSZTIG – Fixed Point Varieties on Affine Flag Manifolds *Israel J. of Math.* **62**, (1988), (129-168)
[La-Sh] R.P. LANGLANDS, D. SHELSTAD – On the definition of the transfer factors *Math. Ann.* **278**, (1987), (219-271)
[La-Ng] G. LAUMON, B. C. NGÔ – Le lemme fondamental pour les groupes unitaires, math.AG/0404454,(2004)
[Ngo] B. C. NGÔ – Fibration de Hitchin et endoscopie, math.AG/0406599, (2004)
[Poo] B. POONEN – Bertini theorems over finite fields, math.AG/0204002, (2002)
[Wal] J.-L. WALDSPURGER – Endoscopie et changement de caractéristiques, prépublication, (2004)

## Construction of abelian varieties with given monodromy

Oliver Bültel

### 1. Description of the theorem

The following result is a consequence of the main theorem of [2]:

**Theorem 1.** *Let $G/\mathbb{Q}_\ell$ be a connected semisimple group, and write $\gamma$ for the order of the automorphism group of its root datum (over $\mathbb{Q}_\ell^{ac}$). Let $\rho : G \to \mathrm{GL}(C/\mathbb{Q}_\ell)$ be a faithful, finite dimensional, linear representation, and $p \notin \{2, \ell\}$ be a prime. Then there exists a polarized abelian scheme $(Y, \lambda)$ of dimension smaller than or equal to*

$$\gamma^2((\dim_{\mathbb{Q}_\ell} C)^3 - (\dim_{\mathbb{Q}_\ell} C)^2)$$
$$+\gamma((1 + \dim_{\mathbb{Q}_\ell} G)(\dim_{\mathbb{Q}_\ell} C)^2 - \dim_{\mathbb{Q}_\ell} G \dim_{\mathbb{Q}_\ell} C)$$

*over a projective and smooth pointed $\mathbb{F}_p^{ac}$-curve $S \stackrel{\xi}{\leftarrow} \operatorname{Spec} \mathbb{F}_p^{ac}$, together with an embedding $f : C \hookrightarrow V_\ell Y_\xi$, such that:*

(i) *the image of $f$ is $\pi_1(S, \xi)$-invariant and totally isotropic with respect to the $\ell$-adic Weil pairing,*
(ii) *the image of the $\pi_1(S, \xi)$-operation on $C$ (pulled back by means of $f$) is a compact open subgroup of $\rho(G(\mathbb{Q}_\ell))$.*

A key ingredient in the proof is a careful study of certain maps between PEL type Shimura varieties. This gives one the possibility to apply the theorems of Tate, Zahrin, and Serre-Tate to more than one abelian variety.
More specifically let $n$ be a positive integer, let $L$ be a CM field and let $Y^{(0)}$ and $Y^{(1)}$ be two complex abelian varieties with $\mathcal{O}_L$-operation such that for all $x \in \mathcal{O}_L$:

$$\mathrm{tr}(x|_{\mathrm{Lie}\, Y^{(k)}}) = \begin{cases} \Phi(x) & k = 0 \\ (n-1)\Phi(x) + \Phi'(x) & k = 1 \end{cases},$$

where $\Phi, \Phi' : L \to \mathbb{C}$ are two CM types for $L$. Then for every $k \in \{0, \dots, n\}$ there is one and only one complex abelian variety $Y^{(k)}$ whose period lattice is canonically isomorphic to:

$$H_1(Y^{(0)}(\mathbb{C}),\mathbb{Z})^{\otimes_{\mathcal{O}_L} 1-k} \otimes_{\mathcal{O}_L} \bigwedge_{\mathcal{O}_L}^{k} H_1(Y^{(1)}(\mathbb{C}),\mathbb{Z}), \tag{1}$$

moreover $Y^{(k)}$ has an operation of $\mathcal{O}_L$ and the formula

$$\operatorname{tr}(x|_{\operatorname{Lie} Y^{(k)}}) = \binom{n-1}{k}\Phi(x) + \binom{n-1}{k-1}\Phi'(x)$$

holds. In order to utilize our $Y^{(k)}$ in characteristic $p$ we need to know the following:

**Lemma 2.** *There exists a canonical algebra morphism*

$$op^{(k)} : \operatorname{sym}_L^k \operatorname{End}_L^0(Y^{(1)}) \to \operatorname{End}_L^0(Y^{(k)}),$$

*which in the $\ell$-adic and crystalline homologies of $Y^{(1)}$ and $Y^{(k)}$ looks like*

$$f^{\otimes_L k} \mapsto (x_0^{1-k} x_1 \wedge \cdots \wedge x_k \mapsto f(x_0)^{1-k} f(x_1) \wedge \cdots \wedge f(x_k)).$$

*Remark* 3. The formula for $f$ given above is only meaningful if one has, analogs of (1) in the $\ell$-adic and crystalline settings. However, these are obtained from the main result of [1].

The proof of the above lemma in characteristic 0 is easy, in characteristic $p$ it is more roundabout. There are several steps:

**Step 1.** *Use the theorem of Tate to prove that the operator defined by the formula in lemma 2 sends $\operatorname{sym}_L^k \operatorname{End}_L^0(Y^{(1)}) \otimes_L L_\lambda$ into $\operatorname{End}_L^0(Y^{(k)}) \otimes_L L_\lambda$ for all primes $\lambda$ (including those over $p$).*

**Step 2.** *Prove that the formation of $Y^{(k)}$ preserves isogeny classes. This is done by factoring an isogeny into several factors of degree $p$, and by checking that degree $p$ isogenies can be lifted into characteristic 0. In particular this shows that $op^{(k)}$ has the requested property when restricted to the set $\{f^{\otimes_L k} | f \in \operatorname{End}_L^0(Y^{(1)})^\times, ff^* = 1\}$.*

**Step 3.** *Use the steps 1 and 2 and a density argument to show that $op^{(k)}$ behaves well on elements of the form $f^{\otimes_L k}$. However these elements generate the $L$-vector space $\operatorname{sym}_L^k \operatorname{End}_L^0(Y^{(1)})$.*

*Example* 4. Let us call $Y^{(1)}$ "supersingular" if and only if $\operatorname{End}_L^0(Y^{(1)})$ is a central simple algebra of rank $n^2$ over $L$. This may differ from the usual definition, as the Newton cocharacter may not be a scalar but is merely contained in the center of the unitary group which defines the Shimura variety. For example, $Y^{(1)}$ is supersingular if there exists an isogeny to a product of $n$ copies of an abelian $\frac{1}{2}[L : \mathbb{Q}]$-fold $\overline{X}$, assumed to have a $\mathcal{O}_L$-operation. In this case $op^{(k)}$ is surjective so that lemma 2 gives an explicit description of the algebra $\operatorname{End}_L^0(Y^{(k)})$ as $\operatorname{Mat}(\binom{n}{k}, L)$.

The proof of theorem 1 is now a deformation argument: We start with a supersingular $Y^{(1)}$ and pick a deformation $\tilde{Y}^{(1)}$ over a powerseries ring such that $\operatorname{End}_L^0(\tilde{Y}^{(k)})$ is a prescribed subalgebra of $\operatorname{End}_L^0(Y^{(k)})$. Using Zarhin's theorem and some bookkeeping of tensors in $\operatorname{End}_L(\bigwedge_L^2 \ldots)$, one gets the result.

### 2. Further questions

The Mumford-Tate conjecture implies a strong link between monodromy groups of abelian varieties in characteristic 0 and the theory of Shimura varieties of Hodge type, i.e. subvarieties $X \subset \mathcal{A}_g \times \mathbb{Q}$, that can be described as a quotient of a symmetric-Hermitian domain by a congruence subgroup.
However, the curious occurence of all groups in theorem 1, including say $G_2$ (in this case there is also a construction of Nick Katz, [3]) cries for a generalization of the concept of a Shimura variety in characteristic $p$. In particular this envisaged generalization should explain the structure of the corresponding deformation spaces of crystals with additional structure.
A related problem is the study of moduli spaces of abelian varieties with certain additional endomorphisms, cf. [4] for example: It is not known whether there exists a simple abelian variety $Y$ over an algebraically closed field, such that $\frac{2\dim Y}{\dim \mathrm{End}^0(Y)} = \frac{3}{2}$. It seems likely that one can use the methods of [5] to compute the dimension of moduli spaces of such abelian varieties, but this does not settle whether or not the endomorphism algebra changes at the generic point.

### References

[1] D. Blasius, *A p-adic property of Hodge classes on Abelian Varieties*, Proc. Symp. Pure Math. 55(1994) part 2, p.293-308
[2] O. Bültel, *Construction of abelian varieties with given monodromy http : //www.rzuser.uni − heidelberg.de/˜t91/bueltel/*
[3] N. Katz, *Notes on $G_2$, determinants, and equidistribution*
[4] F. Oort, *"Question 2A"*, in: *Some questions in algebraic geometry*, 1995
[5] M. Rapoport, T. Zink, *Period Spaces for p-divisible Groups*, Annals of Mathematics Studies 141, Princeton University Press 1996

## Computation of mod $l$ Galois representations associated to modular forms

Bas Edixhoven

For simplicity, I just concentrated on the modular form

$$\Delta = q \prod_{n\geq 1} (1-q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n$$

of level one and weight 12. For $l$ prime, let $V_l$ be the 2-dimensional mod $l$ Galois representation associated to $\Delta$. We assume that the image of the Galois group contains the special linear group of $V_l$ (this excludes only finitely many $l$).

The project is then to show that one can compute $V_l$ in time polynomial in $l$, where "computing $V_l$" means: give the minimum polynomial over the rationals of a generator of the finite Galois extension $K_l$ whose group is the image of the representation on $V_l$. The space $V_l$ occurs in the $l$-torsion of the jacobian of the modular curve $X_1(l)$. The problem is that the genus of this curve grows quadratically with $l$, so that one cannot find $V_l$ using computer algebra. The idea

(Couveignes) is to use numerical computations and a height estimate of a suitably constructed generator of $K_l$. In December 2000 (see streaming video at the MSRI) I described such a construction of a generator, and how to use Arakelov theory to get the height estimate.

Since 2003 this is joint work with Couveignes and Robin de Jong. Recent results of Franz Merkl, and of Jorgenson-Kramer, on Green's functions allow us to finish the height estimate, and to show that the numerical work can be done $p$-adically for $p$ a suitable small prime. The results are being written up now.

## Rational Curves on Varieties

A.J. de Jong
(joint work with Jason Starr)

### Introduction

In this note we work over an uncountable algebraically closed field $k$ of characteristic zero. First we recall some definitions and results. The term "rationally 0-connected" defined below is usually referred to as "rationally connected" in the literature.

**Definition.** (a) A variety $X$ is called rationally 0-connected if for two general points $x_1, x_2$ in $X$ there exists a morphism $f : \mathbf{P}^1 \to X$ whose image contains both $x_1$ and $x_2$.
(b) A smooth projective variety $X$ is called Fano if the canonical divisor $K_X$ is anti-ample, i.e., $-K_X$ is ample.

**Theorem.** (Campana, Kollár-Miyaoka-Mori) A Fano variety is rationally 0-connected.

**Theorem.** (Graber-Harris-Starr, see [B]) Suppose that $f : Y \to C$ is a nonconstant projective morphism of varieties and $B$ is a curve. Assume that $Y_{\overline{k(C)}}$ is rationally 0-connected. Then $f$ has a rational section, i.e., $X(k(C)) \neq \emptyset$.

In the talk we described briefly some thoughts on higher dimensional versions of these two theorems. Higher dimensional in the sense that 0-connectedness gets replaced by 1-connectedness and "curve" is replaced by "surface".

### Speculation

We would like to propose the following guiding principle even though it is quite possibly false as stated:

**(GP)** Suppose $f : Y \to S$ is a dominant projective morphism of varieties such that $S$ is a surface and $Y_{\overline{k(S)}}$ is rationally 1-connected. Then the only obstruction to $f$ having a rational section should be an element $\alpha \in Br(k(S))$. More precisely: for $S' \to S$ a dominant morphism of surfaces, we have $Y(k(S')) \neq \emptyset$ if and only if $\alpha|_{k(S')} = 0$.

**Remarks.** (a) It may be necessary to assume that $S$ is projective and to add conditions on the singular fibres of $f$ as well as the discriminant divisor.
(b) We do not have a good definition of the term "rationally 1-connected" except in the special case described below.
(c) It is possible that one should replace the Brauer class $\alpha$ with another type of cohomology class, but we insist that it should be in a target group that can be defined independently of $f : Y \to S$. Having a class is necessary as is shown by the existence of Brauer-Severi varieties.

**Example Definition.** Suppose that $X$ is smooth and projective and that $Pic(X) = \mathbf{Z}\mathcal{O}_X(1)$. In this case we say that $X$ is rationally 1-connected if and only if $X$ is rationally 0-connected and for all $e >> 0$ and general points $x_1, x_2$ of $X$ the space

$$\mathcal{M}_{x_1,x_2}(X, e)$$

is an open subvariety of a rationally 0-connected variety. Here $\mathcal{M}_{x_1,x_2}(X, e)$ is the space of maps $\phi : (C, c_1, c_2) \to (X, x_1, x_2)$, where $C$ is a smooth projective genus zero curve, $\deg \phi^*\mathcal{O}_X(1) = e$, and $\phi(c_i) = x_i$.

### Higher order Fano conditions

In this subsection we use some terminology and notation from the theory of Kontsevich mapping spaces and we assume that $k = \mathbf{C}$. Let $X$ be a smooth projective variety over $k$. Let $\beta \in H_2(X, Z)$ be a homology class. Let $\bar{\mathcal{M}}$ be the Kontsevich mapping space

$$\bar{\mathcal{M}} = \bar{\mathcal{M}}_{0,0}(X, \beta).$$

(Note that this is actually a stack.) There is a universal curve $p : \mathcal{C} \to \bar{\mathcal{M}}$ and a universal map $f : \mathcal{C} \to X$. The expected dimension of $\bar{\mathcal{M}}$ is $\langle -K_X, \beta\rangle + \dim X - 3$, and every component has at least this dimension. Assume that $\bar{\mathcal{M}}$ actually has this dimension. If this is so then one can show that the canonical class $K_{\bar{\mathcal{M}}}$ of $\bar{\mathcal{M}}$ is equal to the following expression

$$p_* f^* \big(c_2(\Omega^1_X) - \frac{1}{2}K^2_X\big) - \frac{1}{2\langle -K_X, \beta\rangle} p_* f^* K^2_X$$
$$+ \sum_{\beta=\beta'+\beta''} \frac{\langle -K_X, \beta'\rangle\langle -K_X, \beta''\rangle - 4\langle -K_X, \beta\rangle}{2\langle -K_X, \beta\rangle} \Delta_{\beta',\beta''}$$

This seems to suggest that the combination $c_2(\Omega^1_X) - \frac{1}{2}K^2_X$ plays an important role in determining the Kodaira dimension of the moduli space of rational curves on $X$. The coefficients in front of the boundary components $\Delta_{\beta',\beta''}$ are "usually" positive, but somehow these contributions are of a lower order of importance. Anyhow, it occurred to us that this suggests the following question.

**Question.** Suppose that $X$ is Fano and that $c_2(\Omega^1_X) - \frac{1}{2}K^2_X$ is negative (eg. its intersection product with every surface on $X$ is negative). Does it follow that $X$ is rationally 1-connected?

Of course for the moment this question only makes sense if $X$ has Picard group $\mathbf{Z}$. We can compute what this condition means when $X$ is a smooth complete intersection of type $(d_1, \dots, d_c)$ in $\mathbf{P}^n$:

$$X \text{ Fano} \Leftrightarrow \sum d_i \leq n,$$

$$c_2(\Omega^1_X) - \frac{1}{2}K_X^2 < 0 \Leftrightarrow \sum d_i^2 \leq n$$

This is suggestive since the second condition also comes up in Tsen's theorem.

### General hypersurfaces

**Theorem.** (Starr following work by Harris-Starr) A general hypersurface of degree $d$ in $\mathbf{P}^n$ is rationally 1-connected if $d^2 \leq n$.

Work in progress. The theorem remains true for $d^2 \leq n+1$, $d \geq 3$. We can probably also deal with the case of general complete intersections of type $(d_1, \dots, d_c)$ in $\mathbf{P}^n$ where $\sum d_i^2 \leq n+1$.

Note that this is for the general hypersurface or complete intersection, and not for any given hypersurface. In particular we don't know whether a Fermat hypersurface of degree $d$ in $\mathbf{P}^n$, with $d^2 \leq n+1$ is rationally 1-connected (except when $d = 1, 2$ of course). This is interesting because the standard example of a family of smooth hypersurfaces over a surface which doesn't have a rational point is the family

$$\Sigma_{0 \leq i,j \leq d-1} s^i t^j X^d_{(i,j)} = 0$$

over $S$ with $k(S) = k(s,t)$. This lies in a projective space with homogenous coordinates $X_{(i,j)}$, i.e., in $\mathbf{P}^{d^2-1}$. We do not think there is a class $\alpha \in Br(k(S))$ as in (GP) above (allthough we haven't proved this). Hence, if the Fermat hypersurface of degree $d \geq 3$ in $\mathbf{P}^{d^2-1}$ is rationally 1-connected then the family above is a counter examople to (GP).

### How to prove (GP)?

A very sketchy outline of a possible proof of the guiding principle is the following. As a first step one chooses a Lefschetz fibration $S \to \mathbf{P}^1$. Then for every $t \in \mathbf{P}^1$ one considers the restriction of $f : Y \to S$ to the fibre over $t$, i.e., this gives $f_t : Y_t \to S_t$. For each integer $e >> 0$ one considers the space of sections $\Sigma^e_t$ of $f_t$ of degree $e$ (for example take the degree with respect to some auxilliary ample invertible sheaf on $Y$). Putting together the spaces $\Sigma^e_t$ gives a space $\Sigma^e \to \mathbf{P}^1$. To show that $f : Y \to S$ has a rational section is equivalent to proving that for some $e$ the map $\Sigma^e \to \mathbf{P}^1$ has a rational section. By the Graber-Harris-Starr theorem of the first section it suffices to show that $\Sigma^e_t$ (or a suitable irreducible component of it) is rationally 0-connected. This is where the hypothesis on the rational 1-connectedness of the general fibre of $Y \to S$ is supposed to come in, but for the moment we can only succeed in proving this under additional hypotheses.

Without going into further detail on the precise form of the theorem, let us mention a consequence which is of independent interest.

**Theorem.** Let $k$ be any algebraically closed field. Fix integers $1 \leq \ell < \frac{1}{2}n$, and let $G(\ell, n)$ denote the Grassmanian of $\ell$-planes in an $n$-dimensional vector space. Suppose we have $Y \to S$, $\mathcal{L} \in Pic(Y)$ such that
(a) $S$ is a smooth and projective surface over $k$,
(b) for all $s \in S(k)$, we have $Y_s \cong G(\ell, n)$, and
(c) $\mathcal{L}|_{Y_s}$ is the ample generator of $Pic$.
Then $Y(k(S)) \neq \emptyset$.

This is the theorem that allows us to improve the "period equals index" result of [dJ] to all possible brauer classes on surfaces in any characteristic.

**References**

[B] T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, pp. 57–67.

[dJ] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface,* Duke Math. J. **123** (2004), no. 1, pp. 71–94.

## The Conjectures of Mordell-Lang and André-Oort

Richard Pink

The Mordell-Lang conjecture, proved by the combination of work of Faltings, Raynaud, Vojta, and Hindry, is the following statement:

**Conjecture 1** (Mordell-Lang)**.** *Let $A$ be an abelian variety over $\mathbb{C}$ and*

$$\Lambda := \{a \in A \mid \exists n \in \mathbb{Z}^{>0} : na \in \Lambda_0\}$$

*the division group of a finitely generated subgroup $\Lambda_0 \subset A$. Let $Z \subset A$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in $Z$. Then $Z$ is a translate of an abelian subvariety of $A$.*

The Manin-Mumford conjecture is the special case $\Lambda_0 = 0$, where $\Lambda$ is the group of torsion points on $A$.

Next, an irreducible component of a Shimura subvariety of a Shimura variety $S$, or of its image under a Hecke operator, is called a *special subvariety of $S$*. A *special point $s \in S$* is one for which $\{s\}$ is a special subvariety of dimension zero. The André-Oort conjecture, which to date is only partially known by work of André, Edixhoven, Moonen, and Yafaev, states:

**Conjecture 2** (André-Oort)**.** *Let $S$ be a Shimura variety over $\mathbb{C}$, and let $\Lambda \subset S$ denote the set of all its special points. Let $Z \subset S$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in $Z$. Then $Z$ is a special subvariety of $S$.*

The two conjectures are related not only by formal analogy, but also by the fact that special points on Shimura varieties are intimately connected with torsion points of abelian varieties with complex multiplication. In fact, the direct analogue of the André-Oort conjecture for mixed Shimura varieties includes the Manin-Mumford conjecture.

Like André earlier, I was motivated by this to try to encompass the two kinds of conjectures into a single natural general conjecture. So far I have succeeded only partially. The natural framework seems to be that of *mixed Shimura varieties*, and one must generalize the notion of special subvarieties. A *Shimura morphism* is a morphism between two mixed Shimura varieties that is induced by a homomorphism between the underlying algebraic groups. Consider two Shimura morphisms $i : T \to S$ and $\varphi : T \to T'$ and a point $t' \in T'$. Then an irreducible component of $i\big(\varphi^{-1}(t')\big)$, or of its image under a Hecke operator, is called a *weakly special subvariety of* $S$. The proposed conjecture is this:

**Conjecture 3.** *Let $S$ be a mixed Shimura variety over $\mathbb{C}$ and $\Lambda \subset S$ the generalized Hecke orbit of a point $s \in S$. Let $Z \subset S$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in $Z$. Then $Z$ is a weakly special subvariety of $S$.*

There are three kinds of evidence for this, each of which centers on a different extremal case. The first concerns special points and easily reduces to the André-Oort conjecture:

**Remark 4.** *For a pure Shimura variety and the generalized Hecke orbit of a special point, Conjecture 3 becomes a particular case of the André-Oort conjecture. If in addition $Z$ is a curve, it is thus proved by Edixhoven and Yafaev.*

The second case concerns a family of abelian varieties $\pi : A \to S$, where $A$ is a mixed Shimura variety, $S$ is a pure Shimura variety, and $\pi$ is a Shimura morphism:

**Theorem 5.** *For subvarieties of $A$ contained in a fiber of $\pi$, for all $\pi : A \to S$ as above, Conjecture 3 is equivalent to the Mordell-Lang conjecture, and hence known.*

The third case concerns the opposite extreme of special points. It results from work of Serre on the Mumford-Tate conjecture and from results of Clozel and Ullmo on equidistribution of Hecke operators:

**Theorem 6.** *Let $S$ be a Siegel moduli space of abelian varieties of dimension $g$, where $g$ is odd or $2$ or $6$. Let $s \in S$ be a point that does not lie in any proper special subvariety of $S$. Then any infinite subset of the generalized Hecke orbit of $s$ is Zariski dense in $S$. In particular, Conjecture 3 is true for $s \in S$ and any $Z$.*

In each case the reduction is relatively simple; the real hard work is done in the cited literature. One may hope to prove the conjecture eventually by a combination of the individual approaches.

For full details and references see [1].

## References

[1] Pink, R.: *A Combination of the Conjectures by Mordell-Lang and André-Oort.* Preprint August 2004, 27p. See http://www.math.ethz.ch/~pink.

## Boundary motive of a Shimura variety

Jörg Wildeshaus

In recent work [W], I introduced the notion of the *boundary motive* $\partial M_{gm}(X)$ of a scheme $X$ over a perfect field. By definition, it measures the difference between the motive $M_{gm}(X)$ and the motive with compact support $M^c_{gm}(X)$, as defined and studied in the book [VSF]. I developed a number of tools to compute the boundary motive in terms of the geometry of a compactification of $X$: *co-localization*, *invariance under abstract blow-up*, and *analytical invariance.*

The purpose of the talk was to sketch what is necessary from this theory to obtain a proof of the motivic version of the theorems on degeneration of mixed sheaves in the Baily–Borel compactification of a Shimura variety ([P] for étale and $\ell$-adic sheaves, [BW] for Hodge modules).

We use the usual notation from Shimura data and varieties (see e.g. [BW, Sect. 1]). Let $S^K = S^K(G, \mathcal{H})$ be a pure Shimura vaiety, and $S^K_1$ a boundary stratum. It is (up to an error due to the free action of a finite group) itself a pure Shimura variety $S^{\pi(K_1)} = S^{\pi(K_1)}(G_1, \mathcal{H}_1)$. The group $G_1$ is the maximal reductive quotient of a certain subgroup $P_1$ of $G$. Denote by $i$ the closed immersion of $S^K_1$, and by $j$ the open immersion of $S^K$ in the Baily–Borel compactification $(S^K)^*$. In the terminology introduced in [W, Sect. 3], the theorem on degeneration is about the identification of $M_{gm}(S^K_1, i^! j_! \mathbb{Z})$, the *motive of $S^K_1$ with coefficients in $i^! j_! \mathbb{Z}$.*

The geometric approach is of course the one from [P] and [BW]; that the main reduction steps of [loc. cit.] are possible in the motivic setting results from the tools developed in [W]. Invariance under abstract blow-up [W, Thm. 4.1] replaces the usual application of proper base change, in order to show that one can perform the identification of $M_{gm}(S^K_1, i^! j_! \mathbb{Z})$ in a toroidal compactification $S^K(\mathfrak{S})$ of $S^K$. Denote the pre-image $S^K_1$ in $S^K(\mathfrak{S})$ by $S'$. It is then true that the formal completion of $S^K(\mathfrak{S})$ along $S'$ is isomorphic to the quotient by the free action of a certain arithmetic group $\Delta_1$ of the formal completion of a relative torus embedding along a certain union $Z$ of strata. The generic stratum is the (mixed) Shimura variety $S^{K_1} = S^{K_1}(P_1, \mathcal{X}_1)$ associated to the group $P_1$, and the combinatorics of $Z$ is contractible. $S^{K_1}$ is a torus torsor of relative dimension $u_1$ over an Abelian scheme over $S^{\pi(K_1)}$. One uses a $\Delta_1$-equivariant version of analytical invariance [W, Thm. 5.1] in order to identify $M_{gm}(S^K_1, i^! j_! \mathbb{Z})$ with $R\Gamma(\Delta_1, M_{gm}(Z, i^! j_! \mathbb{Z}))$. Hence the problem is reduced to compute the motive with coefficients in $i^! j_! \mathbb{Z}$ of a union of strata in a torus embedding. This can be done in some generality. Since the combinatorics of $Z$ is contractible, the result has a particularly easy shape: one has

$$M_{gm}(Z, i^! j_! \mathbb{Z}) = M_{gm}(S^{K_1})[u_1] \ .$$

Putting everything together, we get the analogue of Pink's Theorem:

**Theorem.** There is a canonical isomorphism

$$M_{gm}(S_1^K, i^! j_! \mathbb{Z}) \xrightarrow{\sim} R\Gamma(\Delta_1, M_{gm}(S^{K_1}))[u_1] \ .$$

## References


[BW] J.I. Burgos, J. Wildeshaus, *Hodge modules on Shimura varieties and their higher direct images in the Baily–Borel compactification*, Ann. Scient. Éc. Norm. Sup. **37** (2004), 363–413.

[P] R. Pink, *On $\ell$-adic sheaves on Shimura varieties and their higher direct images in the Baily–Borel compactification*, Math. Ann. **292** (1992), 197–240.

[VSF] V. Voevodsky, A. Suslin, E.M. Friedlander, *Cycles, Transfers, and Motivic Homology Theories*, Ann. of Math. Studies **143**, Princeton Univ. Press 2000.

[W] J. Wildeshaus, *The boundary motive: definition and basic properties*, Prépublications du Laboratoire d'Analyse, Géométrie et Applications de l'Université Paris 13, no. **2004-20**, July 2004, 29 pages, available under `http://www-math.univ-paris13.fr/prepub/pp2004/`


## Mod $p$ period domains


Ben Moonen

(joint work with Torsten Wedhorn)


The goal of my talk was to report on joint work with Torsten Wedhorn, about certain objects that we call *$F$-zips*; see [5]. If $S$ is a base scheme of characteristic $p > 0$ then by an $F$-zip over $S$ we mean a vector bundle $M$ equipped with a descending filtration $C^\bullet$, an ascending filtration $D_\bullet$, and with a collection of $\mathcal{O}_S$-linear isomorphisms $\phi_i \colon \left(\mathrm{gr}^i_C\right)^{(p)} \xrightarrow{\sim} \mathrm{gr}^D_i$.

Our motivation for looking at such objects comes from the following geometric example. Let $f \colon X \to S$ be a smooth proper morphism of schemes in characteristic $p > 0$. We assume that the sheaves $R^b f_* \Omega^a_{X/S}$ are locally free and that the Hodge-de Rham spectral sequence degenerates at $E_1$-level. The de Rham cohomology sheaves $M = H^m_{\mathrm{dR}}(X/S)$ are then locally free $\mathcal{O}_S$-modules that come naturally equipped with a structure of an $F$-zip, taking $C^\bullet$ to be the Hodge filtration, $D_\bullet$ the conjugate filtration, and $\phi_i$ the isomorphism given by the (inverse) Cartier operator.

In our work we obtain a complete classification of $F$-zips over an algebraically closed field. It turns out that $F$-zips over $k = \bar{k}$ are essentially combinatorical objects. In order to state our result, let us first define the *type* of an $F$-zip $\underline{M}$ over a connected basis $S$ as the function $\tau \colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $\tau(i) = \mathrm{rank}_{\mathcal{O}_S}\left(\mathrm{gr}^i_C\right)$. In the geometric example considered above, the type is given by the Hodge numbers $h^{i,m-i}$ of the fibres of $f$.

**Theorem 1.** *Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support $i_1 < \cdots < i_r$. Let $n_j := \tau(i_j)$, write $J = (n_r, \ldots, n_1)$, and let $n := n_1 + \cdots + n_r$. Then there is a bijection*

$$\left\{\begin{matrix}\text{isomorphism classes of } F\text{-zips} \\ \text{of type } \tau \text{ over } k\end{matrix}\right\} \longleftrightarrow (S_{n_r} \times \cdots \times S_{n_1})\backslash S_n =: {}^J W\,.$$

*More precisely, to each $u \in {}^J W$ we associate a "standard $F$-zip" $\underline{M}^u_\tau$ over $\mathbb{F}_p$ such that any $F$-zip $\underline{M}$ over $k$ is isomorphic to $\underline{M}^u_\tau \otimes_{\mathbb{F}_p} k$ for some uniquely determined $u \in {}^J W$.*

In the case of an abelian variety $X$ over a perfect field $k$, the $F$-zip structure on $H^1_{\mathrm{dR}}(X/k)$ gives the Dieudonné module of the $p$-kernel group scheme $X[p]$. In this special case, our classification theorem was proven (up to differences in terminology) by Kraft in [1]. It was realized by Ekedahl and Oort that this can be used to define a stratification of the moduli space $\mathcal{A}_g$ of abelian varieties in characteristic $p$. This Ekedahl-Oort stratification is a very useful tool in the study of $\mathcal{A}_g$; see Oort, [6] and [7].

Our theory of $F$-zips enables us to extend these ideas to arbitrary families $f\colon X \to S$ satisfying certain conditions, and to de Rham cohomology in arbitrary degree. We define a generalized Ekedahl-Oort stratification of the base scheme $S$. In fact, our theory gives a natural scheme-theoretic definition of these strata, which is new even in the case of abelian varieties. The result can be stated as follows.

**Theorem 2.** *Let $\tau$ and ${}^J W$ be as in Theorem 1. Let $\underline{M} = (M, C^\bullet, D_\bullet, \phi_\bullet)$ be an $F$-zip of type $\tau$ over a scheme $S$ of characteristic $p$. For $u \in {}^J W$ we define a subfunctor $S^u_{\underline{M}}$ of $S$ by the condition that a morphism $g\colon T \to S$ factors through $S^u_{\underline{M}}$ if and only if $g^*\underline{M}$ is fppf-locally isomorphic to $\underline{M}^u_\tau \otimes_{\mathbb{F}_p} \mathcal{O}_T$. Then $S^u_{\underline{M}} \subset S$ is representable by a locally closed subscheme of $S$, and the map*

$$\coprod_{u \in {}^J W} S^u_{\underline{M}} \lhook\joinrel\longrightarrow S$$

*is a bijective monomorphism (i.e., a partition of $S$).*

For the proof of Theorems 1 and 2 we study the $\mathbb{F}_p$-scheme $X_\tau$ whose $S$-valued points are the triples $(C^\bullet, D_\bullet, \phi_\bullet)$ such that $(\mathcal{O}^n_S, C^\bullet, D_\bullet, \phi_\bullet)$ is an $F$-zip of type $\tau$ over $S$. The algebraic group $\mathrm{GL}_{n,\mathbb{F}_p}$ naturally acts on $X_\tau$. Theorem 1 amounts to a classification of the $\mathrm{GL}_n$-orbits in $X_\tau$.

We think of $X_\tau$ as a "mod $p$ analogue" of a compactified period domain. Indeed, if we let ${}^\# S \to S$ be the $\mathrm{GL}_n$-torsor of trivialisations of the underlying vector bundle $M$ then we get a natural "mod $p$ period map" ${}^\# S \to X_\tau$, analogous to the period maps arising in Hodge theory. It turns out that there is a unique open $\mathrm{GL}_n$-orbit $X^{\mathrm{ord}}_\tau \subset X_\tau$, the "ordinary locus", which is to be thought of as the interior of the period domain. In this picture, the other strata correspond to degenerations of the data that constitute an $F$-zip.

In order to study the $\mathrm{GL}_n$-orbits in $X_\tau$, we express the latter in more group-theoretical terms. We introduce varieties $Z_J$ that are semi-linear variants of the varieties studied by Lusztig in [2]. We consider these varieties in the general context of a (not necessarily connected) reductive group $G$ over a finite field. Write $(W, I)$ for the Weyl group of $G$ with its set of simple reflections. As further input for the definition of $Z_J$ we need two subsets $J$, $K \subseteq I$, and a Weyl group element $x \in W$ satisfying certain assumptions. Write $U_P$ for the unipotent radical of a parabolic $P \subset G$. Then $Z_J$ is the Zariski sheafification of the functor that classifies triples $\big(P, Q, [g]\big)$ with $P$ and $Q$ parabolic subgroups of types $J$ and $K$, respectively, and with $[g]$ a double coset in $U_Q\backslash G/F(U_P)$ such that $Q$ and ${}^gF(P)$ are in relative position $x$. We prove that $Z_J$ is a smooth variety of dimension equal to $\dim(G)$. The group $G$ naturally acts on $Z_J$.

The connection with the theory of $F$-zips is as follows. Let $\tau$ and $J$ be as in Theorem 1, and take $G = \mathrm{GL}_{n,\mathbb{F}_p}$. We identify $W = S_n$. The ordered partition $J = (n_r, \ldots, n_1)$ corresponds to a subset of the set $I$ of simple reflections. For $K \subseteq I$ we take the subset corresponding to the opposite partition $(n_1, \ldots, n_r)$, and for $x$ we take the element of minimal length in the double coset $W_K w_0 W_J$, where $W_J$ and $W_K \subset W$ are the subgroups generated by $J$ and $K$, respectively, and where $w_0 \in W$ is the longest element. We show that with these choices, there is a $\mathrm{GL}_n$-equivariant isomorphism between $X_\tau$ and the variety $Z_J$. Theorem 1 is then a consequence of the following general result about the varieties $Z_J$.

**Theorem 3.** *There is a bijection between the set of $G$-orbits in $Z_J$ and the set ${}^JW \subset W$ of elements $w \in W$ that are of minimal length in their coset $W_J w$. (So ${}^JW$ is in bijection with $W_J\backslash W$.)*

The idea for the proof of this theorem is the following. Let $\big(P, Q, [g]\big)$ be a point of $Z_J$. We define a new pair of parabolics $(P_1, Q_1)$ by

$$P_1 := (P \cap Q)U_P\,, \qquad Q_1 := (Q \cap gF(P_1)g^{-1})U_Q\,.$$

In a sense that can be made precise, the pair $(P_1, Q_1)$ is a refinement of the pair $(P, Q)$. Repeating this process, we get a sequence of pairs $(P_n, Q_n)$ that stabilizes. Then the bijection in Theorem 3 is obtained by sending the point $\big(P, Q, [g]\big)$ to the element of $W$ that measures the relative position of $P_n$ and $Q_n$ for $n \gg 0$.

The same ideas as sketched here can be applied to study $F$-zips with certain additional structures, such as a bilinear form or an action of a semi-simple algebra. We apply this to abelian varieties, K3-surfaces, and to good reductions of PEL-Shimura varieties. In this last case, we give a new proof of the dimension formula for Ekedahl-Oort strata that was obtained in [4] using the results of [9]. In fact, this is a consequence of the following general result on the dimensions of the $G$-orbits in $Z_J$.

**Theorem 4.** *For $u \in {}^JW$, let $O^u \subset Z_J$ be the corresponding $G$-orbit under the bijection of Theorem 3. Then*

$$\operatorname{codim}(O^u, Z_J) = \dim(\mathrm{Par}_J) - \ell(u)\,,$$

*where $\ell(u)$ is the length of $u$ in the Coxeter group $W$, and where* $\mathrm{Par}_J$ *is the variety of parabolics of type $J$.*

For further results on the ordinary locus in good reductions of PEL moduli spaces we refer to [3] and [8].

## References


[1] H. Kraft: *Kommutative algebraische p-Gruppen (mit Anwendungen auf p-divisible Gruppen und abelsche Varietäten)*, manuscript, Univ. Bonn, Sept. 1975, 86 pp. (Unpublished)

[2] G. Lusztig: *Parabolic character sheaves I*, Moscow Math. J. **4** (2004), 153–179 and —, *II*, preprint math.RT/0302317.

[3] B.J.J. Moonen: *Serre-Tate theory for moduli spaces of PEL type*, Ann. scient. Éc. Norm. Sup. (4), **37** (2004), 223–269.

[4] B.J.J. Moonen: *A dimension formula for Ekedahl-Oort strata*, Ann. Inst. Fourier (Grenoble) **54** (2004), 666–698.

[5] B.J.J. Moonen and T. Wedhorn: *Discrete invariants of varieties in positive characteristic*, to appear in the IMRN.

[6] F. Oort: *A stratification of a moduli space of polarized abelian varieties in positive characteristic*, in: C. Faber and E. Looijenga (eds.), Moduli of curves and abelian varieties, Aspects Math. **E33**, Vieweg, Braunschweig, 1999, pp. 47–64.

[7] F. Oort: *A stratification of a moduli space of abelian varieties*, in: Moduli of abelian varieties, C. Faber, G. van der Geer, F. Oort (eds.), Progr. in Math. **195**, Birkhäuser Verlag, Basel, 2001, pp. 435–416.

[8] T. Wedhorn: *Ordinariness in good reductions of Shimura varieties of PEL-type*, Ann. scient. Éc. Norm. Sup. (4), **32**, (1999), 575–618.

[9] T. Wedhorn: *The dimension of Oort strata of Shimura varieties of PEL-type*, in: Moduli of abelian varieties, C. Faber, G. van der Geer, F. Oort (eds.), Progr. in Math. **195**, Birkhäuser Verlag, Basel, 2001, pp. 441–471.


## Modular forms from Shimura curves


Michael Rapoport

(joint work with S. Kudla and T. Yang)


This is a report on joint work with S. Kudla and T. Yang on generating series arising from special cycles on Shimura curves.

Let $B$ be an indefinite quaternion division algebra over $\mathbb{Q}$. Let $\Gamma = O_B^\times$, where $O_B$ is a fixed maximal order in $B$. The "Shimura curve" associated to $B$ is the orbifold

$$[\Gamma\backslash\mathcal{H}_\pm].$$

It is related to the following moduli problem $\mathcal{M}$. To a scheme $S$, it associates the category of pairs $(A, \iota)$ consisting of an abelian scheme $A$ over $S$ and an action $\iota : O_B \to \mathrm{End}(A)$ satisfying the *determinant condition.* If $S$ is a scheme of characteristic zero, this last condition simply says that $A$ is of relative dimension 2. Then

$$\mathcal{M}(\mathbb{C}) = [\Gamma\backslash\mathcal{H}_\pm].$$

Let $t \in \mathbb{Z}_{>0}$. Consider the moduli problem $\mathcal{Z}_t$ over $\mathcal{M}$. It associates to a scheme $S$ the category of triples $(A, \iota, x)$ where $x \in \mathrm{End}(A, \iota)$ satisfies $\mathrm{tr}(x) = 0$ and $\mathrm{Nm}(x) = t \cdot \mathrm{id}_A$. We form the generating series

$$\phi_1 = -\operatorname{vol}(\mathcal{M}(\mathbb{C})) + \sum_{t>0} \deg(Z_t) q^t \in \mathbb{C}[\![q]\!].$$

**Proposition 1.1.** *$\phi_1$ is the $q$-expansion of a holomorphic modular form of weight $\frac{3}{2}$.*

This seems to be a classical result. It is proved in [4] by identifying $\phi_1$ with an Eisenstein series which is the analogue for $B$ of Zagier's Eisenstein series [6] for $M_2(\mathbb{Q})$.

In the remainder of the talk I sketched arithmetic analogues of the above statement. The moduli problem $\mathcal{M}$ is represented by a Deligne-Mumford-stack which is proper and flat over Spec $\mathbb{Z}$. It is smooth outside the ramification primes of $B$ and semistable everywhere. Similarly, for every $t \in \mathbb{Z}_{>0}$, the moduli problem $\mathcal{Z}_t$ is representable by a DM-stack which is finite and unramified over $\mathcal{M}$. It is finite over Spec $\mathbb{Z}$ outside the ramification primes of $B$, but can contain irreducible components of the special fiber of $\mathcal{M}$ at primes of bad reduction. Set $\mathcal{Z}_t = \emptyset$ for $t < 0$. Then for any $t \in \mathbb{Z} \setminus \{0\}$, Kudla [1] has constructed a Green's function $g_t(v)$ for $\mathcal{Z}_t(\mathbb{C})$ which depends on a parameter $v \in \mathbb{R}_{>0}$. The pair $(\mathcal{Z}_t, g_t(v))$ defines an element $\hat{\mathcal{Z}}_t(v)$ in the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M})$. We form

$$\hat{\phi}_1 = \sum \hat{\mathcal{Z}}_t(v) q^t.$$

Here $\hat{\mathcal{Z}}_0(v) = -[\hat{\omega}] - (0, \log v + c)$, where $[\hat{\omega}]$ is the class of the metrized Hodge line bundle under the natural identification $\widehat{\mathrm{Pic}}(\mathcal{M}) = \widehat{\mathrm{CH}}^1(\mathcal{M})$ and where $c$ is an explicit constant. This is a Laurent series in $q$ with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{M})$ depending on $v$.

**Theorem 1.2.** *$\hat{\phi}_1$ is a (non-holomorphic) modular form of weight $\frac{3}{2}$ and of level $\Gamma_0(4D(B))$ with values in $\widehat{CH}^1(\mathcal{M})$.*

Here $D(B)$ is the product of the ramification primes of $B$.

To explain the meaning of the statement of the theorem, recall that the arithmetic Chow group splits canonically into a direct sum of a finite-dimensional $\mathbb{C}$-vector space $\widehat{\mathrm{CH}}^1(\mathcal{M})^0$ and the vector space $C^\infty(\mathcal{M}(\mathbb{C}))_0$ of smooth functions on $\mathcal{M}(\mathbb{C})$ with total volume 0. Correspondingly, the series $\hat{\phi}_1$ is the sum of a series $\hat{\phi}_1^0$ in $q$ with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{M})^0$ and a series $\hat{\phi}_1^\infty$ in $q$ with coefficients in $C^\infty(\mathcal{M}(\mathbb{C}))_0$. The statement of the theorem is that there is a smooth function on $\mathcal{H}_+$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{M})^0$ which satisfies the usual transformation law for a modular form of weight $\frac{3}{2}$ and of level $\Gamma_0(4D(B))$ whose $q$-expansion is equal to $\hat{\phi}_1^0$, and a smooth function on $\mathcal{H}_+ \times \mathcal{M}(\mathbb{C})$ which satisfies the usual transformation law for a modular form of weight $\frac{3}{2}$ and of level $\Gamma_0(4D(B))$ in the first variable and whose $q$-expansion in the first variable is equal to $\hat{\phi}_1^\infty$. Obviously, the series $\hat{\phi}_1^0$ satisfies the above condition if for any linear form $\ell : \widehat{\mathrm{CH}}^1(\mathcal{M})^0 \to \mathbb{C}$ the series $\ell(\hat{\phi}_1)$ with coefficients in $\mathbb{C}$ is a non-holomorphic modular form of weight $\frac{3}{2}$ and level $\Gamma_0(4D(B))$ in the usual sense.

An important ingredient of the proof of the above theorem is the Borcherds construction of special divisors [3]. For some linear forms $\ell$ one can explicitly identify the modular form $\ell(\hat{\phi}_1)$. For instance, the first proposition can be viewed as giving an explicit expression for $\ell(\hat{\phi}_1)$, where $\ell$ is the degree map

$$\deg : \widehat{\mathrm{CH}}^1(\mathcal{M}) \longrightarrow \mathrm{CH}^1(\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{C}) \longrightarrow \mathbb{Z}.$$

Similarly, consider the Gillet-Soulé pairing

$$\langle\ ,\ \rangle : \widehat{\mathrm{CH}}^1(\mathcal{M}) \times \widehat{\mathrm{CH}}^1(\mathcal{M}) \longrightarrow \widehat{\mathrm{CH}}^2_{\mathbb{R}}(\mathcal{M}) = \mathbb{R}.$$

The last identification comes from the fact that $\mathcal{M}$ is geometrically connected. If $\ell_\omega(\hat{\phi}_1) = \langle [\hat{\omega}], \hat{\phi}_1 \rangle$, then $\ell_\omega(\hat{\phi}_1)$ can be identified with the derivative of a specific Eisenstein series [4].

One can form a similar generating series for 0-cycles on $\mathcal{M}$ instead of divisors on $\mathcal{M}$. It has the form

$$\hat{\phi}_2 = \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})^\vee} \hat{\mathcal{Z}}_T(v) q^T.$$

The coefficients $\hat{\mathcal{Z}}_T(v) \in \mathbb{R}$ depend on $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$. Let $T$ be positive-definite, and consider the DM-stack $\mathcal{Z}_T$ over $\mathcal{M}$ which to $S$ associates the category of triples $(A, \iota, \underline{x})$, where $\underline{x} = (x_1, x_2) \in \mathrm{End}(A, \iota)^2$ satisfies $\mathrm{tr}(x_1) = \mathrm{tr}(x_2) = 0$ and $\frac{1}{2}(\underline{x}, \underline{x}) = T$. Then if $\mathcal{Z}_T$ has support disjoint from the ramification primes of $B$, it has finite support and we put

$$\hat{\mathcal{Z}}_T(v) = \log |\mathcal{Z}_T| \qquad \text{(independent of } v).$$

The extension of this definition to the remaining $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ is somewhat arbitrary, and is in part dictated by the desire to make the following theorem hold true.

**Theorem 1.3.** *$\hat{\phi}_2$ is (the q-expansion of) a non-holomorphic Siegel modular form of genus 2, of weight $\frac{3}{2}$ and level $\Gamma_0(4D(B))$.*

This modular form can be identified with the derivative of an explicit Siegel Eisenstein series.

The previous two theorems are related by the following inner product formula. Let

$$d : \mathcal{H}_+ \times \mathcal{H}_+ \longrightarrow \mathcal{H}_+^{(2)}$$

be the "diagonal" embedding into the Siegel upper half space of genus 2, with $d(\tau_1, \tau_2) = \mathrm{diag}(\tau_1, \tau_2) = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$.

**Theorem 1.4.** *We have*

$$\langle \hat{\phi}_1(\tau_1), \hat{\phi}_1(\tau_2)\rangle = d^*(\hat{\phi}_2(\tau_1,\tau_2)).$$

Explicitly, for any $t_1, t_2 \in \mathbb{Z}$ and $v_1, v_2 \in \mathbb{R}_{>0}$, we have

$$\langle \hat{\mathcal{Z}}_{t_1}(v_1), \hat{\mathcal{Z}}_{t_2}(v_2)\rangle = \sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Z})^\vee \\ diag(T)=(t_1,t_2)}} \hat{\mathcal{Z}}_T(\mathrm{diag}(v_1,v_2)).$$

As a consequence of these theorems one obtains a formula for the self-intersection number of the metrized Hodge bundle, which is reminiscent of the corresponding formula of Bost and Kühn [5] in the case of the modular curve.

**Corollary 1.5.**

$$\langle \hat{\omega}_0, \hat{\omega}_0\rangle = 4\cdot \zeta^{D(B)}(-1)\cdot \left[\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} - \frac{1}{4}\sum_{p\mid D(B)} \frac{p+1}{p-1}\log p\right]$$

Here the notation $\hat{\omega}_0$ indicates that we have taken Bost's normalization of the metric on $\omega$.

## References


[1] S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math., **146** (1997), 545–646.

[2] S. Kudla, *Modular forms and arithmetic geometry*, Current Developments in Math., Nov. 2002, to appear.

[3] S. Kudla, *Integrals of Borcherds forms*, Compositio Math., **137** (2003), 293–349.

[4] S. Kudla, M. Rapoport, T. Yang, *Derivatives of Eisenstein series and Faltings heights*, Compositio Math., **140** (2004), 887–951.

[5] U. Kühn, *Generalized arithmetic intersection numbers*, J. reine angew. Math., **534** (2001), 209–236.

[6] D. Zagier, *Nombres de classes et formes modulaires de poids 3/2*, C. R. Acad. Sc. Paris, **281** (1975), 883–886.


## Vector bundles on $p$-adic curves and parallel transport


Christopher Deninger

(joint work with Annette Werner)


On a compact Riemann surface every finite dimensional complex representation of the fundamental group gives rise to a flat vector bundle and hence to a holomorphic vector bundle. By a theorem of Weil, one obtains precisely the holomorphic bundles whose indecomposable components have degree zero [W]. It was proved by Narasimhan and Seshadri [Na-Se] that unitary representations give rise to semistable bundles of degree zero. Moreover, every stable bundle of degree zero comes from an irreducible unitary representation.

We have established a partial $p$-adic analogue of this theory, generalized to representations of the fundamental groupoid [De-We1]. The following is one of our main results. Recall that a vector bundle on a smooth projective curve over a field of characteristic $p$ is called strongly semistable if $\mathrm{Fr}_p^{\nu *}E$ is semistable for all $\nu \geq 0$. Here $\mathrm{Fr}_p$ is the absolute Frobenius morphism.

Let $X$ be a smooth projective curve over $\overline{\mathbb{Q}}_p$ and assume for simplicity that $X$ has a smooth model $\mathfrak{X}$ over $\overline{\mathbb{Z}}_p$. The special fibre $\mathfrak{X}_k$ is then a smooth projective curve over $k = \overline{\mathbb{F}}_p$.

**Theorem.** *Let $E$ be a vector bundle of degree zero on $X$ which extends to a bundle on $\mathfrak{X}$ with strongly semistable reduction. Then there are functorial isomorphisms of "parallel transport" along étale paths between the fibres of $E_{\mathbb{C}_p}$ on $X_{\mathbb{C}_p}$. In particular one obtains a representation $\rho_{E,x}$ of $\pi_1(X,x)$ on $E_x$ for every point $x$ in $X(\mathbb{C}_p)$. The parallel transport is compatible with tensor products, duals, internal homs, pullbacks and Galois conjugation.*

In fact, the theorem holds more generally if the reduction of the model $\mathfrak{X}$ is a semistable irreducible curve.

The theorem applies in particular to line bundles of degree zero on $X$. In this case the $p$-part of the corresponding character of $\pi_1(X,x)$ was already constructed by Tate using Cartier duality for the $p$-divisible group of the abelian scheme $\mathrm{Pic}^0_{\mathfrak{X}/\overline{\mathbb{Z}}_p}$ cf. [Ta] § 4 and [De-We2]. His method does not extend to bundles of higher rank.

Let us now discuss our theory in more detail. After this we can sketch the proof of the theorem.

We first investigate the category $\mathcal{S}_{\mathfrak{X},D}$ consisting of finitely presented proper $\overline{\mathbb{Z}}_p$-morphisms $\pi : \mathcal{Y} \to \mathfrak{X}$ whose generic fibre is a finite covering of $Y$ which is étale outside of a divisor $D$ on $X$. The important point is that for given $\pi$ in $\mathcal{S}_{\mathfrak{X},D}$ there is an object $\pi' : \mathcal{Y}' \to \mathfrak{X}$ in $\mathcal{S}_{\mathfrak{X},D}$ lying over $\pi$ with better properties, e.g. cohomologically flat of dimension zero or even semistable. We also construct certain coverings $\pi$ using the theory of the Picard functor which are used several times.

Then we define and investigate categories $\mathfrak{B}_{X_{\mathbb{C}_p},D}$ and $\mathfrak{B}^\sharp_{X_{\mathbb{C}_p},D}$ involving a divisor $D$ on $X$ and also an analogous category $\mathfrak{B}_{\mathfrak{X}_\mathfrak{o},D}$ for a fixed model $\mathfrak{X}$ of $X$. Here $\mathfrak{o}$ is the ring of integers in $\mathbb{C}_p$. These are defined as follows. The category $\mathfrak{B}_{\mathfrak{X}_\mathfrak{o},D}$ consists of all vector bundles $E$ on $\mathfrak{X}_\mathfrak{o}$ such that for all $n \geq 1$ there is a covering $\pi$ in $\mathcal{S}_{\mathfrak{X},D}$ with $\pi^*E$ trivial modulo $p^n$. We prove that this condition has to be checked for $n = 1$ only. If $E$ is already defined on $\mathfrak{X}$ then it even suffices that $\pi_k^* E_k$ is trivial where $\pi_k$ is the special fibre of some $\pi$.

Next, $\mathfrak{B}_{X_{\mathbb{C}_p},D}$ consists of all bundles which are isomorphic to the generic fibre of a bundle $E$ in $\mathfrak{B}_{\mathfrak{X}_\mathfrak{o},D}$ for some model $\mathfrak{X}$ of $X$. These categories are additive and stable under extensions. We prove that for every bundle $E$ on $X_{\mathbb{C}_p}$ there is a finite étale covering $\alpha : Y \to X$ such that $\alpha^*E$ extends to a vector bundle on a model of $Y$. This is the motivation for defining $\mathfrak{B}^\sharp_{X_{\mathbb{C}_p},D}$ as the category of vector bundles on $X_{\mathbb{C}_p}$ whose pullback along $\alpha$ lies in $\mathfrak{B}_{Y_{\mathbb{C}_p},\alpha^*D}$ for some finite $\alpha$ which is étale over $X \setminus D$. We obtain an additive category which is closed under extensions and contains all line bundles of degree zero. All vector bundles in $\mathfrak{B}^\sharp$ are semistable of degree zero.

We then define and study certain isomorphisms of parallel transport along étale paths in $U = X \setminus D$ for the bundles in the category $\mathfrak{B}^\sharp_{X_{\mathbb{C}_p},D}$. In more technical terms, we construct an exact $\otimes$-functor $\rho$ from $\mathfrak{B}^\sharp_{X_{\mathbb{C}_p},D}$ to the category of continuous representations of the étale fundamental groupoid $\Pi_1(U)$ on $\mathbb{C}_p$-vector spaces. The basic idea is this: Consider a bundle $E$ in $\mathfrak{B}_{\mathfrak{X}_\mathfrak{o},D}$ and for a given $n \geq 1$ let $\pi : \mathcal{Y} \to \mathfrak{X}$ be an object of $\mathcal{S}_{\mathfrak{X},D}$ such that $\pi_n^* E_n$ is a trivial bundle on $\mathcal{Y}_n$. Here the index $n$ denotes reduction modulo $p^n$. Consider points $x$ and $x'$ in $X(\mathbb{C}_p) = \mathfrak{X}(\mathfrak{o})$ and choose a point $y$ in $Y = \mathcal{Y}_{\mathbb{C}_p}$ above $x$. For an étale path $\gamma$ from $x$ to $x'$ i.e. an isomorphism of fibre functors, let $\gamma y$ be the corresponding point above $x'$. For a "good" cover $\pi$ we have isomorphisms

$$E_{x_n} \xleftarrow[\sim]{y_n^*} \Gamma(\mathcal{Y}_n, \pi_n^* E_n) \xrightarrow[\sim]{(\gamma y)_n^*} E_{x'_n} \ .$$

We define the parallel transport $\rho_E(\gamma) : E_x \xrightarrow{\sim} E_{x'}$ as the projective limit of the maps $\rho_{E,n}(\gamma) = (\gamma y)_n^* \circ (y_n^*)^{-1}$. This parallel transport is then extended to $\mathfrak{B}_{X_{\mathbb{C}_p},D}$ and $\mathfrak{B}^\sharp_{X_{\mathbb{C}_p},D}$. We also prove that the functor mapping a bundle $E$ in $\mathfrak{B}^\sharp_{X_{\mathbb{C}_p},D}$ to its fibre in a point $x \in U(\mathbb{C}_p)$ is faithful.

Using a Seifert–van Kampen theorem for étale groupoids we show that for a bundle $E$ which is in $\mathfrak{B}^\sharp$ for two disjoint divisors, one actually obtains a parallel transport along all étale paths in $X$.

The proof of the theorem above starts with a characterization due to Lange and Stuhler [LS] of the strongly semistable bundles on a smooth projective curve over a finite field: These are exactly the bundles whose pullback by a finite surjective morphism becomes trivial. Hence we have to lift finite covers in characteristic $p$ to characteristic zero. The main point here is to construct a morphism of models whose reduction factors over a given power of Frobenius. Somewhat surprisingly this is possible. In fact our method allows us to construct two coverings $\pi$ in $\mathcal{S}_{\mathfrak{X},D}$ and $\pi'$ in $\mathcal{S}_{\mathfrak{X},D'}$ for two disjoint divisors $D$ and $D'$ such that $\pi_k^* E_k$ and $\pi_k'^* E_k$ are both trivial. By the above theory, one gets the parallel transport on all of $X_{\mathbb{C}_p}$.

Recently Faltings has announced a $p$-adic version of non-abelian Hodge theory [Fa]. He proves an equivalence of categories between vector bundles on $X_{\mathbb{C}_p}$ endowed with a $p$-adic Higgs field and a certain category of "generalized representations" which contains the representations of $\pi_1(X, x)$ as a full subcategory. His methods are different from ours. In particular Faltings uses his theory of almost étale extensions. It follows from his results that in the theorem above the functor sending $E$ to the representation $\rho_{E,x} : \pi_1(X, x) \to \mathrm{GL}(E_x)$ is not only faithful but also full. Note that the category of continuous representations of the fundamental groupoid $\Pi_1(X)$ is a full subcategory of the category of continuous representations of $\pi_1(X, x)$ via the fibre functor in $x$.

## References


[De-We1] C. Deninger, A. Werner: Vector bundles on $p$-adic curves and parallel transport. ArXiv, math.AG/0403516, March 2004

[De-We2] C. Deninger, A. Werner: Line bundles and $p$-adic characters. Preprint 2004

[Fa] G. Faltings: A $p$-adic Simpson correspondence. Notes 2003

[LS] H. Lange, U. Stuhler: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe. Math. Z. **156** (1977), 73–83

[Na-Se] M.S. Narasimhan, C.S. Seshadri: Stable and unitary vector bundles on a compact Riemann surface. Ann. Math. **82** (1965), 540–567

[Ta] J. Tate: $p$-divisible groups. Proceedings of a Conference on local fields. Driebergen 1966, 158–183

[W] A. Weil: Généralisation des fonctions abéliennes. J. de Math. P. et App. (IX) 17 (1938) 47-87


## Moduli spaces of local systems, positivity and higher Teichmüller theory


Alexander Goncharov

(joint work with V. V. Fock)


I. Let $S$ be an oriented surface with boundary: $S = \bar{S} - D_1 \cup \ldots \cup D_n, n > 0$. Assume that $S$ is hyperbolic. Usually $\chi(S) < 0$. The *Teichmüller space* for $S$ is defined as

$$\begin{aligned}\mathcal{T}(S) :&= \{\text{complex structures on } S\}/\operatorname{Diff}_0(S)\\ &\cong \{\text{faithful representations}\\ &\qquad \rho : \pi_1(s) \to PSL_2(\mathbb{R})\}/PSL_2(\mathbb{R}) - \text{conjug.}\end{aligned}$$

The mapping class group $\Gamma_S = \operatorname{Diff}(S)/\operatorname{Diff}_0(S)$ acts on $\mathcal{T}(S)$. Let $\mathcal{T}''(S) \subset \mathcal{T}(S)$ be the subset of representations with unipotent monodromy around each boundary component. The $\mathcal{T}(S)$ has a boundary with corners with the deepest stratum $\mathcal{T}''(S)$. Define

$$\begin{aligned}\widetilde{\mathcal{T}}(S) := \{p \in \mathcal{T}(S),\ &\text{plus choice of an eigenvalue}\\ &\text{for the monodromy around each } \partial D_i\}.\end{aligned}$$

In the case when $S$ is compact, N. Hitchin defined in 1992 a component in a space of representations $\pi_1(S) \to G(\mathbb{R})$ where $G$ is a simple Lie group with trivial center. He proved that given a complex structure on $S$, this component is isomorphic to $\mathbb{C}^N$.

We are looking for an algebraic geometric avatar of the Teichmüller-Thurston theory, which can be generalized to any $G$ (split, semi-simple algebraic group over $\mathbb{Q}$ with trivial center).

II. The *moduli space* $\mathcal{X}_{G,\hat{S}}$. Let $\hat{S}$ be a pair $(S, \{x_1, \ldots, x_m\})$, where the second element of the pair is a collection of marked points on the boundary $\partial S$.

**Definition 1.** *A* ***framed G-local system*** *on $\hat{S}$ is a pair $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a $G$-local system on $S$ and $\beta$ is a flat section of the restriction of $\mathcal{L} \times_G \mathcal{B}$ to the punctured boundary $\partial S - \{x_1, \ldots, x_m\}$, where $\mathcal{B}$ is the flag variety for $G$.*

**Definition 2.** *$\mathcal{X}_{G,\hat{S}}$ is the moduli space of framed $G$-local systems on $\hat{S}$.*

**Example 3.** *If $\hat{S}$ is a disc $\hat{D}_n$ with $n$ marked points on the boundary then*

$$\mathcal{X}_{G,n} := \mathcal{X}_{G,\hat{D}_n} \cong G\backslash\mathcal{B}^n.$$

Let $\mathscr{T}$ be an ideal triangulation of a surface $S'$ with $n$ punctures ($S' \underset{\text{h.e.}}{\sim} S$). Restricting $(\mathcal{L}, \beta)$ to triangles and rectangles of the triangulation we get a rational map

$$\Pi_{\mathscr{T}} : \mathcal{X}_{G,S} \longrightarrow \prod_{\text{triangles of } \mathscr{T}} \mathcal{X}_{G,3} \times H^{\{\text{edges of } \mathscr{T}\}}.$$

**Theorem 4.** *This map is a birational isomorphism.*

Using different ideal triangulations $\mathscr{T}$ we get a $\Gamma_S$-equivariant atlas on $\mathcal{X}_{G,S}$. We prove that the transition functions for this atlas are *subtraction-free*. Thus for any semifield $K$, e.g. $K = \mathbb{R}_{>0}$, we can define the set of $K$-valued points of $\mathcal{X}_{G,\hat{S}}$.

**Definition 5.** *The* ***higher Teichmüller space*** *for $G$ is $\mathcal{X}_{G,\hat{S}}(\mathbb{R}_{>0})$.*

**Definition 6.** *The* ***lamination space*** *for $G$ is $\mathcal{X}_{G,\hat{S}}(\mathbb{R}^t)$, where $\mathbb{R}^t$ is the tropical semifield.*

**Theorem 7.** *For $G = PSL_2$ we get the classical Teichmüller space $\widetilde{\mathcal{T}}(S)$ and the space of Thurston's measured laminations on $S$.*

Our main conjecture relates $\mathcal{X}_{G,S}$ to get another moduli space $\mathcal{A}_{^LG,S}$ related to the Langland's dual $^LG$, which also has a canonical positive atlas on it.

## On Serre's conjecture over totally real fields

Fred Diamond

Serre conjectured that all continuous, irreducible, odd representations $\rho : G_{\mathbf{Q}} \to GL_2(\bar{\mathbf{F}}_p)$ arise from modular forms. If $\rho$ is modular, then proven refinements provide recipes for the possible weights and levels of the forms giving rise to it. A natural generalization to the context of a totally real field $F$ predicts that any continuous, irreducible, totally odd $\rho : G_F \to \mathrm{GL}_2(\bar{\mathbf{F}}_p)$ arises from Hilbert modular forms. As in the case of $F = \mathbf{Q}$, one expects that the minimal level at which $\rho$ arises is the Artin conductor of $\rho$, up to powers of primes over $p$. This is known to hold in most cases by work of Jarvis, Fujiwara and Rajaei, assuming $\rho$ is modular.

A conjectural recipe for the possible weights has been formulated (jointly with Buzzard and Jarvis) in the case where $p$ is unramified in $F$. First one defines a *weight* $\sigma$ to be an irreducible $\bar{\mathbf{F}}_p$ representation of $\mathrm{GL}_2(\mathcal{O}_F/p)$. There is then a natural notion of $\rho$ being modular of weight $\sigma$ from which one can recover the classical weights and $p$-level structures of characteristic zero Hilbert modular eigenforms giving rise to $\rho$. The weight part of Serre's conjecture is then a recipe giving the set of weights for which $\rho$ is modular in terms of the local representations $\rho_v = \rho|G_{F_v}$. More precisely, for each prime $v$ over $p$, the restriction $\rho_v$ determines a set of representations $\Sigma(\rho_v)$ of $\mathrm{GL}_2(\mathcal{O}_F/v)$, and the conjectural set of weights for $\rho$ consists of their tensor products.

When $\rho_v$ is semsimple, it turns out that $\Sigma(\rho_v)$ typically has $2^{f_v}$ and consists precisely of the Jordan-Holder constituents of the reduction of an irreducible *characteristic zero* representation of $\mathrm{GL}_2(\mathcal{O}_F/v)$. Moreover this observation establishes a curious bijection between {2-dimensional semisimple $\bar{F}_p$-representations of $G_{F_v}$}/inertial equivalence and {irreducible $\bar{\mathbf{Q}}_p$-representations of $\mathrm{GL}_2(\mathcal{O}_F/v)$ not factoring through det}, with a rather different flavor from the local Langlands correspondence.

Suppose now that $\rho_v$ is not semisimple and write $\rho_v \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix}$. Then $\Sigma(\rho_v)$ is a subset of $\Sigma(\rho_v^{\mathrm{ss}})$ determined by the corresponding class $c_{\rho_v} \in V = H^1(F_v, \xi_1\xi_2^{-1})$. More precisely, suppose for simplicity that $\xi_1\xi_2^{-1}$ is not trivial or cyclotomic. Then for each subset $S$ of the set of embeddings of $\mathcal{O}_F/v$ in $\bar{\mathbf{F}}_p$, there is an $|S|$-dimensional subspace $V_S \subset V$ and a weight $\sigma_S(\rho_v) \in \Sigma(\rho_v^{\mathrm{ss}})$ such that

$$\sigma_S(\rho_v) \in \Sigma(\rho_v) \quad \Leftrightarrow \quad c_{\rho_v} \in V_S.$$

Joint work with Dembélé and Roberts provides some numerical evidence that the set of weights is as conjectured, and work of Jarvis and Gee provides some theoretical evidence, but on the whole, the problem of proving it for $F \neq \mathbf{Q}$ is very much open.

## Deformations of automorphic Galois representations and applications

Michael Harris

(joint work with Richard Taylor and in part with Laurent Clozel)

This is a report on joint work in progress with Richard Taylor, and in part with Laurent Clozel. Any errors or omissions in the present text are my sole responsibility.

The original objective of my project with Taylor, begun in 1996, was to extend the theorem of Taylor-Wiles to automorphic forms in higher dimension, where the Galois representations are of dimension $n > 2$ in general. I remind you that the Taylor-Wiles theorem, which has been generalized in various directions by Diamond and Fujiwara, asserts that, in favorable circumstances, if a mod $l$ representation $\bar{\rho}$ of $Gal(\bar{\mathbb{Q}}/E)$ lifts to an $l$-adic representation which comes from automorphic forms on $GL(n)$ of a certain type, where $E$ is totally real or CM field, then any lifting with minimal additional ramification also comes from automorphic forms. By "favorable circumstances" I mean, for example, that $l > n$, so we can apply Fontaine-Laffaille (nearly ordinary would also work) and $\bar{\rho}$ should be absolutely irreducible and with image not too small. The work of Wiles on $GL(2)$, and later Skinner-Wiles and Breuil-Conrad-Diamond-Taylor, showed that the lifting theorem remained true under more general ramification conditions. This is the problem of "level raising" that involves a different range of techniques.

Work of Taylor's student Russ Mann, together with base change arguments, showed that the minimality condition could be dropped, assuming a specific generalization to $GL(n)$ of a step in the level-raising theory called Ihara's Lemma, which I will formulate later. I will assume an appropriate version of Ihara's Lemma, which we have so far been unable to prove, and will state our main theorem under this assumption.

Let $F$ (resp. $E$) be a number field, which we will assume totally real (resp. CM). For the moment we stick with $F$. Let $S_l$ denote the set of primes of $F$ dividing $l$. Following Fontaine and Mazur, we say an $n$-dimensional $l$-adic representation $\rho$ of $Gal(\bar{F}/F)$ is *of geometric type* if it is unramified outside the finite set $S \coprod S_l$ of primes of $F$, where $S \cap S_l = \emptyset$; and if at every $v \in S_l$ it has Fontaine's de Rham property, which means in particular that it allows us to associate a set of Hodge-Tate numbers $h^p(\rho)$ to $\rho$, with $n = \sum h^p(\rho)$ (varying with $v$ in general).

Following Clozel, we define an automorphic representation $\pi$ of $GL(n, \mathbb{Q})$ if $\pi_\infty$ has integral infinitesimal character – which means it can be associated to a Hodge structure.

**Conjecture.** *(a) Let $\rho$ be an irreducible $n$-dimensional $l$-adic representation of $G_F$ of geometric type. Then there is a cuspidal automorphic representation of $GL(n, \mathbb{Q})$ $\pi_\rho$ of algebraic type associated to $\rho$, in the sense that $L(s, \pi_\rho) = L(s, \rho)$ where the former is the $L$-function associated by automorphic theory. In particular, $L(s, \rho)$ has an analytic continuation to an entire function satisfying the usual sort of functional equation.*

*(b) Conversely, if $\pi$ is an automorphic representation of $GL(n, \mathbb{Q})$ of algebraic type, then there exists an $l$-adic representation $\rho_\pi$ of geometric type associated to $\pi$.*

A theorem of Taylor-Wiles type is the following:

**Prototype of generalized Taylor-Wiles theorem.** *Suppose $\rho$ is as in (a), and suppose $\pi_\rho$ exists. Let $\rho'$ be a second $n$-dimensional $\mathfrak{p}$-adic representation, with $\rho' \equiv \rho \pmod{\mathfrak{p}}$. Suppose moreover that*

(i) *$\rho'$ satisfies a minimality condition, typically that $\rho'$ is no more ramified than $\bar{\rho} := \rho \pmod{l}$ at primes in $S$ or otherwise, apart from primes in $S_l$.*
(ii) *Some more precise condition on the restriction of $\rho'$ to $Gal(K_{\rho,q}/\mathbb{Q}_q)$, for $q \in S - S_l$ e.g. that $\rho'$ and $\rho$ have isomorphic restrictions to $Gal(K_{\rho',q}/\mathbb{Q}_q)$.*
(iii) *A specific Fontaine-type condition on the restriction of $\rho'$ to $Gal(K_{\rho',v}/F_v)$ for $v \in S_l$; in practice, $\rho$ and $\rho'$ are assumed crystalline and have the same Hodge-Tate numbers).*

(iv) *Additional conditions, e.g.*
   (a) $\rho \pmod{l}$ *is irreducible, even after restriction to* $F(\zeta_l)^+$ *and has big image,*
   (b) $l > n$ *and is unramified in* $F$.

*Then* $\rho'$ *also satisfies the Fontaine-Mazur conjecture (a).*

Here is an unenlightening definition of "big image". Let $\mathcal{O} = \mathcal{O}_K$. The polarization condition implies that $\rho$ extends to a homomorphism

$$\tilde{\rho} : Gal(\overline{\mathbb{Q}}/F) \to \mathcal{G}_n(\mathcal{O}) := [GL(n, \mathcal{O}) \times GL(1, \mathcal{O})] \ltimes \{1, c\}$$

where $c(g, \mu) = (\mu^t g^{-1}, \mu)$ (more than one extension is in principle possible...) Let $\bar{\tilde{\rho}} := \tilde{\rho} \pmod{l}$, $H = Im(\bar{\tilde{\rho}}) \cap Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l)^+)$. We say $H$ is "big" if

(i) $H^0(H, ad(\bar{\rho})) = H^1(H, ad(\bar{\rho})) = 0$;

(ii) For any irreducible submodule $W \subset \mathbf{A}(\bar{\rho})$ there exists $h \in H \cap GL(n, k)$ and $\alpha \in k$ such that the $\alpha$-generalized eigenspace $V_{h,\alpha}$ of $h$ is of dimension 1 and $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq 0$ where $\pi_{h,\alpha}$ and $i_{h,\alpha}$ are respectively the $h$-equivariant projection on $V_{h,\alpha}$ and the inclusion.

The original Taylor-Wiles theorem applies to $n = 2$ and for specific conditions (ii-iv). The generalization due to Wiles, and Taylor and his collaborators more generally,in the case $n = 2$ removes condition (i). We always assume that all primes in $S$ split completely in $E/F$.

**Theorem A. (MH-Taylor).** *(F totally real.) Let* $K$ *be a finite extension of* $\mathbb{Q}_\ell$, *with residue field* $k$. *Suppose* $V_l$ *is an* $n$*-dimensional* $K$*-vector space and* $\rho : Gal(\overline{\mathbb{Q}}/F) \to GL(n, K)$ *is as in (a), and suppose* $\pi_\rho$ *exists and is cohomological at infinity (this will be automatic). Suppose moreover that*

*(1) Polarized of weight* $n-1$*: there is a Galois-invariant bilinear form*

$$V_l \otimes V_l \to \mathbb{Q}(1-n),$$

*of parity* $(-1)^{n-1}$;

*(2) Regular:* $h^p(\rho) \leq 1$ *for all (p), and* $h^p = h^{n-1-p}$ *(in practice* $\rho$ *will be pure of weight* $n-1$ *by construction)*

*(3) At some finite place* $v_0$ *of* $F$ $\pi_\rho$ *is either supercuspidal or Steinberg. In the supercuspidal case this implies that* $\rho\mid_{G_{v_0}}$ *is irreducible, and we assume this is still true of* $\bar{\rho}$. *In the Steinberg case there is a way around this.*

*Suppose* $\rho'$ *is as above and satisfies conditions (i)-(iv) and is polarized ((2) and (3) are automatic). Then* $\pi_{\rho'}$ *also exists.*

For $E$ CM rather than totally real, we replace condition (1) by the hypothesis that $V_l^\vee \xrightarrow{\sim} V_l^c(1-n)$. The weight condition can be relaxed somewhat. In fact, we only consider CM fields of the form $E = F \cdot \mathcal{K}$ where $\mathcal{K}$ is imaginary quadratic, though it should be possible to treat more general cases by a descent argument.

The theorem is proved by working with automorphic forms on definite unitary groups $U(B)$ attached to division algebras $U(B)$ over $E$ (= $F \cdot \mathcal{K}$ if necessary). These are like the groups that occur in my book with Taylor, except they are positive-definite everywhere. This allows us to construct Taylor-Wiles systems without difficulty, because the Hecke algebra modules in question are in 0-dimensional cohomology. Conditions (1) and (2) are unavoidable for cohomological representations. There are reasons to believe "most" representations, in some sense, are of this type. On the other hand, (3) is a symptom of our dependence on the current state of the stable trace formula, and is likely to be unnecessary in the near future. Note that (1) and (2) are conditions on $\rho'$ whereas (3) is a condition on $\pi_{\rho'}$, but by the results of my book with Taylor (3) is in fact a condition on $\rho'$.

The above theorem was proved by 1998, under versions of condition (ii) that have been gradually relaxed as we learned more about the compatibility of local and global correspondences; some minor restrictions remain. We are presently in the process of writing an article (possibly with Clozel) with the same statement, except that the end is as follows. From now on I will work with $E$ CM rather than $F$.

**Theorem B (almost completely verified).** *Suppose $\rho$ and $\pi_\rho$ are as in the previous theorem and satisfy (1)-(3). Suppose $\rho'$ is as in the Fontaine-Mazur conjecture, is polarized, and satisfies conditions (ii)-(iv) (i) is optional). Assume an appropriate version of Ihara's Lemma, e.g. Conjecture 1, below. Then $\pi_{\rho'}$ also exists.*

Recall that $k$ is the residue field of $\mathcal{O}$.

**Conjecture 1 (Ihara's Lemma, version 1).** *Let $M_k$ denote the module of $k$-valued modular forms on $U(B)$ which are fixed by the Iwahori subgroup at $v \in S'$, and let $M_k^v$ denote the $k[GL(n, F_v)]$-module generated by $M_k$. Localize at a non-Eisenstein prime of $\mathbb{T}$. Then every irreducible $GL(n, F_v)$-submodule of $M_k^v$ is generic (has a Whittaker model over $k$).*

## Degeneration of polylogarithms and special values of L-functions of totally real fields

Guido Kings

In this talk we explained the degeneration of the polylogarithm on Hilbert modular varieties $S$ for a totally real field $F$ and its connection with special values of partial zeta functions for the field $F$. This generalizes a previous result of Huber and the author [HK] and is connected to work of Sczech [S] and Nori [N] through the formulation given by Beilinson and Levin [BL] (unpublished).

To be more precise we need some notation: Let $\pi : \mathcal{A} \to S$ be the universal abelian variety and $\mathcal{H}$ the dual of $R^1\pi_*\mathbb{Q}$. For any $x \in \mathcal{A}(S)_{tors} \setminus 0$ we get *Eisensteinclasses* by specializing the polylogarithm on $\mathcal{A}$ (as defined by Beilinson, Levin and Wildeshaus)

$$x^*pol^k \in Ext^{2g-1}_{MHM(S)}(\mathbb{Q}, Sym^k\mathcal{H}(g)).$$

Let $\partial S$ be the boundary of the Baily-Borel compactification $\bar{S}$ of $S$ and $i : \partial S \hookrightarrow \bar{S}$ resp. $j : S \hookrightarrow \bar{S}$ the associated immersions. We get

$$(1)\quad Ext^{2g-1}_{MHM(S)}(\mathbb{Q}, Sym^k\mathcal{H}(g)) \to Ext^{2g-1}_{MHM(\partial S)}(\mathbb{Q}, i^*Rj_*Sym^k\mathcal{H}(g)) \to Hom_{MHM(\partial S)}(\mathbb{Q}, i^*Rj_*Sym^k\mathcal{H}(g)) \cong Hom_{MHM(\partial S)}(\mathbb{Q}, \mathbb{Q}).$$

The aim is to compute the image of $x^*pol^k$ under this composition, which we call the residue map. A cusp of $S$ is given by a quotient $p : \mathcal{O}_F^{\oplus 2} \to \mathfrak{b}$, which is projective $\mathcal{O}_F$-module of rank 1. Let $\mathfrak{f} \subset \mathcal{O}_F$ be a fractional ideal, $\mathcal{O}_\mathfrak{f}^* := \{a \in \mathcal{O}_F^* | a \text{ totally positive}, a \cong 1 \bmod \mathfrak{f}\}$ and $\epsilon : (F \otimes \mathbb{R})^* \to \{\pm 1\}$ a sign character, $\delta$ the different of $F$. Define

$$L(\mathfrak{b}, x, \mathfrak{f}, \epsilon, s) := \sum_{\lambda \in (\mathfrak{b}\delta)^{-1}/\mathcal{O}_\mathfrak{f}^*} \frac{\epsilon(\lambda) e^{2\pi i Tr x\lambda}}{|N(\lambda)|^s}$$

for $Res > 1$. For $x \in (\mathfrak{f}^{-1})^{\oplus 2}$ we get $p(x) \in \mathfrak{b}\mathfrak{f}^{-1}$. The main result is that the residue of $x^*pol^k$ is ($g = [F : \mathbb{Q}]$)

$$res(x^*pol^k) = |N(\delta)|^{-\frac{1}{2}} N(\mathfrak{b}) \frac{2^g(-1)^{g-1}((n-1)!)^g}{(2\pi i)^k} L(\mathfrak{b}, p(x), \mathfrak{f}, \epsilon, n)$$

if $k = gn$ and $\epsilon$ is trivial if $n$ is even, or $\epsilon$ is the product of $g$ non-trivial characters if $n$ is odd. In all other cases the residue is zero. The proof uses purely topological methods and the result of Sczech, Nori in the form of Beilinson-Levin. David Blottière considers in his Ph.D. thesis this question from the point of view of Levin's explicit computation of the polylogarithmic currents in [L].

## References


[BL] A. Beilinson, A. Levin: unpublished notes.

[HK] A. Huber, G. Kings: Degeneration of $l$-adic Eisenstein classes and of the elliptic polylog, Invent. math. **135** (1999), 545–594 .

[L] A. Levin: Polylogarithmic Current on Abelian Varieties, in: A. Reznikov, N. Schappacher (eds.): Regulators in Analysis, Geometry and Number theory, Birkhäuser 1999, 207-229.

[N] M. Nori: Some Eisenstein classes for the integral unimodular group, ICM Zürich 1994, 690-694.

[S] R. Sczech: Eisenstein group cocycles for $Gl_n$ and values of L-functions, Invent. math. **113** (1993), 581-616.

[W] J. Wildeshaus: Realizations of polylogarithms, LNM 1650 Springer 1997.


## Equidistribution of special subvarieties of Shimura varieties


Emmanuel Ullmo

(joint work with L. Clozel)


This is a report on a joint work with L. Clozel [1].

Let $S$ be a Shimura variety over $\mathbb{C}$. One can define a set of special points of $S$ (points with complex multiplication) and a set of special subvarieties (subvarieties of Hodge type). The André-Oort conjecture says that a component of the Zariski closure of a set of special points is a special subvariety. An other formulation is:

**Conjecture 0.1.** *(**André-Oort**) Let $X$ be a subvariety of $S$. There exists a finite set of special subvarieties $(Z_1, \dots, Z_r)$ of $S$ contained in $X$ such that any special subvariety of $S$ contained in $X$ is contained in $\cup_{i=1}^r Z_i$.*

A Shimura variety is associated to a Shimura datum $(G_{\mathbb{Q}}, X)$ where $G_{\mathbb{Q}}$ is a reductive $\mathbb{Q}$-algebraic group and $X$ is the $G(\mathbb{R})$-conjugacy class of a morphism $\alpha:\ \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$. A special subvariety $Z$ is associated to a sub-Shimura datum $(H_{\mathbb{Q}}, X_H)$. Special points are associated to a sub-Shimura datum $(T_{\mathbb{Q}}, x_T)$ where $T$ is a torus (and $x_T$ is a zero-dimentional variety).

**Definition 0.2.** *A "strongly special" subvariety of $S$ is a special subvariety of $S$ associated to a sub-Shimura datum $(H_{\mathbb{Q}}, X_H)$ such that*
*(i) $H_{\mathbb{Q}}$ is semi-simple.*
*(ii) $H_{\mathbb{Q}}$ is not contained in a proper $\mathbb{Q}$-parabolic subgroup of $G_{\mathbb{Q}}$.*

The dimension of a strongly special subvariety is $> 0$.

**Theorem 0.3.** *Let $X$ be a subvariety of $S$. There exists a finite set of strongly special subvarieties $(Z_1, \dots, Z_r)$ of $S$ contained in $X$ such that any strongly special subvariety of $S$ contained in $X$ is contained in $\cup_{i=1}^r Z_i$.*

A special subvariety $Z$ of $S$ is endowed with a canonical probability measure $\mu_Z$ on $S$ with support $Z$. The theorem is a consequence of the folowing "ergodic" result:

**Theorem 0.4.** *Let $Z_n$ be a sequence of strongly special subvarieties of $S$ and $\mu_n$ the associated sequence of probability measures. There exists a strongly special subvariety $Z$ and a subsequence $Z_{n_k}$ such that $\mu_{n_k}$ converges weakly to $\mu_Z$. Moreover for $k$ big enough $Z_{n_k} \subset Z$.*

The proof uses results from ergodic theory (Ratner's theory [4] and some results by Mozes-Shah [3]) and the theory of Shimura varieties as explained by Deligne [2].

**Example** Let $S$ be the moduli space of principally polarized abelian varieties of dimension $g$. For all totally real number field $F$ of degree $g$ we can define a familly of strongly special subvarieties parmetrizing abelian varieties with multiplication by $F$ (Hilbert modular varieties). If $Z_n$ is a "strict" sequence of Hilbert modular varieties of $S$, then the associated sequence of'probability measure $\mu_n$ weakly converges to the total measure $\mu_S$. A sequence of special subvarieties $Z_n$ is said to be strict if for all special subvarieties $Z \neq S$ of $S$ the set $\{n \in \mathbb{N} \mid Z_n \subset Z\}$ is finite.

## References

[1] L. Clozel, E. Ullmo *Equidistribution de sous-variétés spéciales* à paraître dans Annals of Maths.

[2] P. Deligne . *Variétés de Shimura: interprétation modulaire et techniques de construction de modèles canoniques*, dans *Automorphic Forms, Representations, and L-functions* part. **2**; Editeurs: A. Borel et W Casselman; Proc. of Symp. in Pure Math. **33**, American Mathematical Society, 1979, p. 247–290.

[3] S. Mozes, N. Shah *On the space of ergodic invariant measures of unipotent flows*, Ergod. Th. and Dynam. Sys. **15**, (1995), p. 149-159.

[4] M. Ratner *Interaction between Ergodic Theory Lie Groups and Number Theory.* Proceedings of ICM 1994 (Zurich), vol **1**, Birkhauser (1995), p. 157–182.

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