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String-Theorie und Geometrie

Organised by
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Abstract. The conference brought together mathematicians working in algebraic and differential geometry (14J32, 37J35, 53C15, 53C80) and physicists working on conformal quantum field theory and string theory (81T30, 81T40). In particular, new developments in mirror symmetry, topological quantum field theory and generalised complex geometry were discussed.

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Introduction by the Organisers

Of the three organizers of the conference N. Hitchin is a mathematician, A. Kapustin and W. Nahm are physicists. Ideas from physics, in particular from string theory and related areas had a profound influence on algebraic and differential geometry. Conversely, many developments in string theory use very recent mathematical results. There were 19 talks at the workshop, 12 by mathematicians and 7 by physicists, but in many cases an outside observer would have had difficulties to sort them out.

Quantum field theory often allows continuous interpolations between geometrical structures of topologically different manifolds. A particularly important example is mirror symmetry, where the complexified Kähler cone of one manifold corresponds to the moduli space of complex structures of the mirror manifold. Several talks explored geometrical and number theoretical aspects of this connection. Mirror symmetry can be understood in terms of torus fibrations and an isomorphism between the quantum field theories given by maps to a torus and to its dual (T-duality).

The B-field of string theory can be described in terms of non-commutative geometry on the tori, and its effect on T-duality has been investigated. The quantum field theories which provide these dualities are supersymmetric conformally invariant quantum field theories in two dimensions. In mathematics, large parts of their structures have been axiomatised in the language of vertex operator algebras, but much remains to be done. A promising mathematical approach is provided by the chiral de Rham complex. This cohomology of this complex seems to be invariant under mirror symmetry, but its physical meaning is not yet clear. This led to many discussions between mathematicians and physicists. Indeed, the free time for discussions was at least as important as the lectures for learning the language of researchers with a different background and for making use of their ideas.

An area in which the common understanding is well advance is topological field theory. Here one has good axioms and many of the analytic problems of conformally invariant quantum field theories can be ignored. One can solve rather complex problems and one can make contact with new geometrical ideas, in particular generalised complex structures. The latter allow to interpolate between complex and symplectic geometry, which should become important for both physics and mathematics. To understand all of conformal field theory at a similar mathematical depth will take more time, but already now relations to mixed Hodge structures have given much insight. Eventually many so far intractable problems should be solvable by the use of quantum field theoretical ideas, for example the construction of the non-singular Ricci flat metrics on K3.

Perturbative string theory can be described in terms of conformal field theory, but the complete theory has more excitations than strings. Among them the best understood are the D-branes, many features of which can be understood in terms of boundary states in conformal field theory. Some of these branes are related to complex submanifolds, others to certain Lagrangian submanifolds, and their description involves twisted K-theory and derived categories.

The development of a common language between mathematicians and physicists is well under way, and the workshop has made a contribution which all participants found very stimulating. In Europe such meetings between mathematicians and string theorists are rare, so the workshop was particularly useful. Much of the time was reserved for informal discussions. On one evening there was a discussion on the relation between classical and quantum geometry. Weather predictions for the free afternoon had been bad, but finally the weather was splendid and the supply of Black Forest cake matched the demand.

Workshop: String-Theorie und Geometrie

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Abstracts

Generalized geometry and the Hodge decomposition

Marco Gualtieri

In this talk, we review some of the concepts of generalized geometry, as introduced by Hitchin and developed in the speaker's thesis. We also prove a Hodge decomposition for the twisted cohomology of a compact twisted generalized Kähler manifold.

I. Geometry of $T \oplus T^*$. The sum $T \oplus T^*$ of the tangent and cotangent bundle of an n-dimensional manifold has a natural $O(n,n)$ structure, and

$$\mathfrak{so}(n,n) = \wedge^2 T \oplus \mathrm{End}(T) \oplus \wedge^2 T^*.$$

Hence we may view 2-forms B and bivectors β as infinitesimal symmetries of $T \oplus T^*$. We may also form the Clifford algebra $CL(T \oplus T^*)$, which has a spin representation on the Clifford module $\wedge^\bullet T^*$ as described by Cartan:

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho,$$

for $X + \xi \in T \oplus T^*$ and $\rho \in \wedge^\bullet T^*$. This means that we may view differential forms as spinors[1] for $T \oplus T^*$. From the general theory of spinors, this implies that there is a $Spin_o(n,n)$-invariant bilinear form

$$\langle , \rangle : \wedge^\bullet T^* \times \wedge^\bullet T^* \longrightarrow \det T^*,$$

given by $\langle \alpha, \beta \rangle = (\alpha \wedge \sigma(\beta))_n$, where σ is the operator which reverses the order of any product.

Another structure emerging from the interpretation of forms as spinors is the Courant bracket $[,]_H$ on sections of $T \oplus T^*$, obtained as the derived bracket of the natural differential operator $d + H \wedge \cdot$ acting on differential forms, where d is the exterior derivative and $H \in \Omega^3_{cl}(M)$. When $H = 0$, we have the following

Proposition. *The group of automorphisms of the Courant bracket for $H = 0$ is a semidirect product of $\mathrm{Diff}(M)$ and $\Omega^2_{cl}(M)$, where $B \in \Omega^2_{cl}(M)$ acts as the shear $\exp(B)$ on $T \oplus T^*$.*

[1] Actually, the bundle of spinors differs from $\wedge^\bullet T^*$ by tensoring with a line bundle, which can be taken to be trivializable; we assume a trivialization has been chosen – this is called a *dilaton* by physicists.

II. Generalized complex geometry. A generalized complex structure is an integrable reduction of $T \oplus T^*$ from $O(2n, 2n)$ to $U(n, n)$ (only possible when $\dim_{\mathbb{R}} M = 2n$), which is equivalent to the choice of an orthogonal complex structure

$$\mathcal{J} \in O(T \oplus T^*), \ \mathcal{J}^2 = -1.$$

The integrability condition is that the $+i$-eigenbundle of $\mathcal{J}$,

$$E < (T \oplus T^*) \otimes \mathcal{C},$$

must be closed under the Courant bracket. If H is nonzero we call this a twisted generalized complex structure. The Courant bracket is a Lie bracket when restricted to E and therefore we may form the differential graded algebra

$$\mathcal{E} = (\wedge^\bullet E^*, d_E).$$

Theorem. *The dga $\mathcal{E}$ is an elliptic complex and it gives rise to a Kuranishi deformation space for any generalized complex structure. The tangent space to the deformation space, in the unobstructed case, is $H^2(\mathcal{E})$.*

For example, let $J \in \mathrm{End}(T)$ be a usual complex structure, and form the generalized complex structure

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

Then, $E = T_{0,1} \oplus T^*_{1,0}$, so that $\mathcal{E}$ is simply the Dolbeault complex of the holomorphic multivectors. Consequently

$$H^2(\mathcal{E}) = H^0(M, \wedge^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O}).$$

For a complex surface, a holomorphic bivector β always integrates to an actual deformation, and so for $\mathcal{C}P^2$, for example, we obtain a new generalized complex structure which is complex along an anticanonical divisor (the vanishing locus of β) and the B-field transform of a symplectic structure outside the cubic.

This provides an alternative interpretation of the extended deformation parameter β, which is normally viewed as a noncommutative deformation of the algebra defining $\mathcal{C}P^2$. The usual translation parameter along the commutative elliptic curve can be obtained by differentiating β along its vanishing set.

The previous example indicates that the algebraic type of a generalized complex structure may jump along loci in the manifold. Indeed a generalized complex structure on a $2n$-manifold may have type $0, \cdots, n$, with 0 denoting the (generic) symplectic type and n denoting the complex type. Type may jump up, but only by an even number.

Theorem (Generalized Darboux theorem)**.** *Away from type jumping loci, a generalized complex manifold of type k is locally isomorphic, via a diffeomorphism and a B-field transform, to $\mathcal{C}^k \times \mathbb{R}^{2n-2k}_{\omega_0}$, where ω_0 is the usual Darboux symplectic form.*

Generalized complex manifolds also have natural sub-objects, called generalized complex submanifolds. These sub-objects correspond exactly with the physicists' notion of topological D-branes; in particular, one recovers in the symplectic case the co-isotropic A-branes of Kapustin and Orlov. There is also a natural notion of generalized holomorphic bundle supported on a generalized complex submanifold, a concept which seems to correspond to D-branes of higher rank. One can even see how such a brane could deform into several branes of lower rank. We will not discuss these processes at this time.

III. Generalized Riemannian geometry. A generalized Riemannian metric is a reduction of $T \oplus T^*$ from $O(n,n)$ to $O(n) \times O(n)$. This is equivalent to specifying a maximal positive-definite subbundle $C_+ < T \oplus T^*$, which can be described as the graph of $b + g$, where g is a usual Riemannian metric and b is a 2-form.

Let $* = a_1 \cdots a_n$ be the product in $CL(C_+) < CL(T \oplus T^*)$ of an oriented orthonormal basis for C_+. This volume element acts on the differential forms via the spin representation, and is related to the Hodge star operator $\star$: if $b = 0$ then

$$\star\rho = \sigma(\sigma(*) \cdot \rho).$$

For any generalized Riemannian structure, we may define the following positive-definite Hermitian inner product on differential forms

$$h(\alpha, \beta) = \int_M \langle \alpha, \sigma(*)\bar{\beta} \rangle,$$

which we call the Born-Infeld inner product, to coincide with the physics terminology. In particular, we have the expression

$$\langle \alpha, \sigma(*)\bar{\beta} \rangle = G(\alpha, \beta)\langle 1, \sigma(*)1 \rangle = G(\alpha, \beta)\frac{\det(g + b)}{\det^{1/2} g},$$

where $G(\alpha, \beta)$ is a positive-definite Hermitian metric on forms depending on g and b satisfying $(1, 1) = 1$. The Born-Infeld inner product is a direct generalization of the Hodge inner product of Riemannian geometry.

On an even-dimensional manifold, the adjoint of the twisted exterior derivative d_H is simply

$$d_H^* = * \cdot d_H \cdot \sigma(*) = * \cdot d_H \cdot *^{-1}.$$

As in the Riemannian case, $d_H + d_H^*$ is an elliptic operator and so, therefore, is the Laplacian $\Delta_{d_H} = d_H d_H^* + d_H^* d_H$.

Proceeding in the usual way, we may conclude that on a compact generalized Riemannian manifold, every H-twisted de Rham cohomology class has a unique Δ_{d_H}-harmonic representative. There is a gauge freedom here, in the sense that given any 2-form b', the automorphism $e^{b'}$ takes harmonic representatives for (g, b, H) to those for $(g, b + b', H - db')$.

IV. Generalized Kähler geometry and Hodge decomposition. A generalized Kähler structure is given by two commuting generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ such that $-\mathcal{J}_1\mathcal{J}_2 = G$ defines a generalized Riemannian metric on $T \oplus T^*$, i.e. define C_+ as the $+1$-eigenbundle of G. Both $\mathcal{J}_1, \mathcal{J}_2$ are in $\mathfrak{so}(T \oplus T^*)$, and via the Spin representation they act on differential forms. $\mathcal{J}_1$ induces a decomposition of forms into its eigenspaces

$$\wedge^\bullet T^* \otimes \mathcal{C} = U_{-n} \oplus \cdots \oplus U_0 \oplus \cdots \oplus U_n,$$

Where U_k is the ik-eigenspace of $\mathcal{J}_1$. Furthermore, the exterior derivative d_H acting on U_k decomposes as the sum of the two projections $\overline{\partial}_1, \partial_1$ to U_{k+1}, U_{k-1} respectively, i.e.

$$C^\infty(U_k) \underset{\partial_1}{\overset{\overline{\partial}_1}{\rightleftarrows}} C^\infty(U_{k+1}) \; .$$

The commuting endomorphism $\mathcal{J}_2$ engenders a further decomposition of the U_k:

$$U_k = U_{k,|k|-n} \oplus U_{k,|k|-n+2} \oplus \cdots \oplus U_{k,n-|k|}.$$

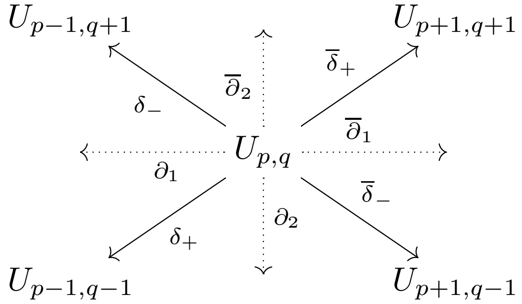
In this way we obtain an orthogonal (p,q) decomposition of the differential forms into the following diamond

$$\begin{array}{ccccccc} & & & U_{0,n} & & & \\ & & \cdots & & \cdots & & \\ & U_{-n+1,1} & & & & U_{n-1,1} & \\ U_{-n,0} & & & \cdots & & & U_{n,0} \\ & U_{-n+1,-1} & & & & U_{n-1,-1} & \\ & & \cdots & & \cdots & & \\ & & & U_{0,-n} & & & \end{array}$$

The exterior derivative then breaks into four components

$$d_H = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-,$$

according to the (p,q) decomposition, as follows:



where we have, for definiteness, $\overline{\partial}_1 = \overline{\delta}_+ + \overline{\delta}_-$ and $\overline{\partial}_2 = \overline{\delta}_+ + \delta_-$. These operators all fit into elliptic complexes, and the crucial calculation is that

$$\overline{\delta}_+^* = -\delta_+ \;\text{ and }\; \overline{\delta}_-^* = \delta_-.$$

These simple generalized Kähler identities imply the equality of all Laplacians in sight:

$$\Delta_{d_H} = 2\Delta_{\overline{\partial}_{1/2}} = 2\Delta_{\partial_{1/2}} = 4\Delta_{\overline{\delta}_\pm} = 4\Delta_{\delta_\pm},$$

and so we obtain a (p, q) decomposition for the twisted cohomology of any compact generalized Kähler manifold. Note that in the usual Kähler case, this (p, q) decomposition is *not* the Dolbeault decomposition: it was called the Clifford decomposition by Michelsohn, and there is an orthogonal transformation called the Hodge automorphism taking it to the usual Dolbeault decomposition.

D-branes and π-stability

Tom Bridgeland

An N=2 superconformal field theory of the sort arising from the nonlinear sigma model on a Calabi-Yau manifold has an associated topologically twisted theory which in turn determines a triangulated category $\mathcal{D}$ whose objects are the branes or boundary conditions of the theory. In fact there are two topological twistings, usually called the A- and B-model, and these are exchanged by mirror symmetry. The original SCFT contains more data than the topologically twisted theory, and Douglas has shown that one manifestation of this extra data is the existence of certain subcategories $\mathcal{P} \subset \mathcal{D}$ whose objects are the so-called BPS branes.

The notion of a stability condition on a triangulated category was introduced in order to axiomatise the properties of these subcategories of BPS branes. The space of all stability conditions $\mathrm{Stab}(\mathcal{D})$ on a fixed category triangulated category $\mathcal{D}$ provides a mathematical analogue of certain moduli spaces of SCFTs arising in physics.

In this talk I gave the definition of a stability condition and defined the natural topology arising on the space of all stability conditions $\mathrm{Stab}(\mathcal{D})$. I also related stability conditions to the more familiar notion of a t-structure.

In the second half of the talk I discussed an example where $\mathcal{D}$ is the derived category of coherent sheaves of the non-compact Calabi-Yau threefold X which is the total space of the canonical line bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$. In this case the space $\mathrm{Stab}(\mathcal{D})$ has a natural action of the annular braid group $\mathbb{C}B_n$ which is the fundamental group of the configuration space of three unordered points in $\mathbb{C}^*$. The fundamental domains for this action are labelled by quivers with relations, and the module category of each such quiver is derived equivalent to the category of coherent sheaves on X.

tt^* geometry and mixed Hodge structures

Claus Hertling

tt^* geometry is a generalization of the notion of variation of Hodge structures.

Cecotti and Vafa [2] [3] established and studied it in 1991 and later in the context of $N = 2$ supersymmetric field theories, especially in the case of Landau-Ginzburg models. Here the central object is an algebraic function $f : Y \to \mathbb{C}$ on an affine algebraic manifold Y such that f has only isolated singularities and such that f is tame at infinity.

A mathematically safe way to this tt^* geometry uses oscillating integrals. Lefschetz thimbles and oscillating integrals are discussed in [11] [12] [5, ch. 6]. The closely related case of germs of holomorphic functions is treated in [7, ch. 8]; there also the way to tt^* geometry is given in detail. General results on the Gauss-Manin system, its Fourier transform and the Brieskorn lattice of $f : Y \to \mathbb{C}$ are provided in [14]. The corresponding structures for unfoldings of $f : Y \to \mathbb{C}$ are established in [4].

One starts with the flat bundle of homology classes of Lefschetz thimbles in $\mathbb{C}^*$ and the dual bundle $H' \to \mathbb{C}^*$. It contains a flat lattice bundle $H'_{\mathbb{Z}}$ and a flat real bundle $H'_{\mathbb{R}}$. The intersection form for Lefschetz thimbles induces a flat pairing $P : H'_z \times H'_{-z} \to \mathbb{C}$. These are the topological data. Via oscillating integrals one obtains a natural extension of $H' \to \mathbb{C}^*$ to a holomorphic vector bundle $H \to \mathbb{C}$. Then the flat connection ∇ on H' has a pole of order ≤ 2 at 0. The flat pairing evaluated on germs of holomorphic sections in $H \to \mathbb{C}$ takes values in $z^n\mathcal{O}_{\mathbb{C},0}$ (here $\dim Y = n$). The coefficients of the different powers of z are essentially K. Saito's higher residue pairings [12].

The data $(H \to \mathbb{C}, H' \to \mathbb{C}^*, H'_{\mathbb{R}}, \nabla, P)$ with their properties are called a (TERP)-structure in [7] [8]. If they satisfy certain conditions, they give a generalization of a polarized Hodge structure and then they are called a (pos. def. tr. TERP)-structure. This generalization has to be understood as follows (cf. [8] [7]).

$H' \to \mathbb{C}^*$ and $H'_{\mathbb{R}} \to \mathbb{C}^*$ are the analoga of a vector space and a real subspace. The extension of $H' \to \mathbb{C}^*$ to $H \to \mathbb{C}$ generalizes a Hodge filtration. Using $H'_{\mathbb{R}}$ and the flat structure, one can construct from H an extension to a holomorphic vector bundle $\hat{H} \to \mathbb{P}^1$ with a pole of order ≤ 2 at ∞. This extension to ∞ generalizes the complex conjugate filtration of a Hodge filtration. In a Hodge structure the Hodge filtration and the complex conjugate filtration are opposite. The analogon of this condition is that $\hat{H} \to \mathbb{P}^1$ should be the trivial bundle. This is a nontrivial condition. If it holds one can construct on the fiber H_0 a $\mathbb{C}$-antilinear involution and, using the pairing P, a hermitian metric h. The second nontrivial condition is that this metric h should be positive definite. Then one would have a (pos.def.tr.TERP)-structure, generalizing a polarized Hodge structure.

Theorem/Conjecture. *The (TERP)-structure for $f : Y \to \mathbb{C}$ as above is a (pos.def.tr.TERP)-structure.*

For physicists this should be a theorem, because it should follow from properties of the Landau-Ginzburg model; the metric h is the ground state metric. For mathematicians this is a very interesting conjecture.

If true, it would be comparable to the fact that the cohomology of compact Kähler manifolds carries Hodge structures. It would give a new viewpoint on and a new understanding of the mixed Hodge structure of Sabbah [13] [14] for $f : Y \to \mathbb{C}$ and the mixed Hodge structure of Steenbrink for germs of holomorphic functions with isolated singularities.

Using the mixed Hodge structures of Sabbah and Steenbrink as well as [1, Cor. 3.13] and a generalization of an argument of Dubrovin [6, Prop 2.2], the following result, which is weaker than the conjecture above, can be proved.

Theorem. *For $r > 0$ close to 0 and for $r \gg 0$, the (TERP)-structure of $r \cdot f$ is a (pos.def.tr.TERP)-structure.*

The rescaling of f to $r \cdot f$ is the physicists renormalization. Denoting $\pi_r : \mathbb{C} \to \mathbb{C}$, $z \mapsto \frac{1}{r}z$, one finds $(TERP)(r \cdot f) = \pi_r^*(TERP)(f)$. Therefore, if one (TERP)-structure $(TERP)$ is given, one should study simultaneously the whole family $\pi_r^*(TERP)$, $r > 0$.

This leads to two generalizations for (TERP)-structures of a fundamental 1–1 correspondence between polarized mixed Hodge structures and nilpotent orbits of pure Hodge structures ([16, Thm. 6.16] and [1, Cor. 3.13]). The first generalization concerns small $r > 0$, the second large $r > 0$.

The first generalization says that for $r > 0$ small $\pi_r^*(TERP)$ is a (pos. def. tr. TERP)-structure if and only if a certain filtration $F^\bullet_{Sabbah}$, constructed in [14, math.AG], is part of a polarized mixed Hodge structure. This can be proved using [16] [1].

Together with Sabbah's result [13] and [14, math.AG] that in the case of $(TERP)(f)$ his filtration is part of a mixed Hodge structure, it shows the first half of theorem 2. (In [13] [14] he does not discuss the polarization, but this can be done.)

The second generalization gives conjecturally the following correspondence: for $r \gg 0$ $\pi_r^*(TERP)$ is a (pos.def.tr.TERP)-structure if and only if $(TERP)$ is a (mixed.TERP)-structure.

The notion of a (mixed.TERP)-structure is defined in [8]; roughly it says that the germ at 0 of the (TERP)-structure is formally isomorphic to a sum of tensor products of rank one irregular connections and some regular singular (TERP)-structures, that each of these regular singular (TERP)-structures gives rise to a polarized mixed Hodge structure, and that the Stokes structure is compatible with the real structure.

The direction $\Leftarrow$ in this correspondence can be proved using [1] and a generalization of an argument of Dubrovin [6, Prop 2.2]. The direction $\Rightarrow$ can be proved in the case when the pole part of the order 2 pole at 0 is nilpotent using [16] and in the rank 2 case using properties of Painlevé III, which were established in [10] [9].

$(TERP)(f)$ is a (mixed.TERP)-structure. Together with the direction $\Leftarrow$ in the last correspondence it shows the second half of theorem 2.

I hope that the (TERP)-structures and tt^*-geometry will have many applications to moduli of singularities, the distribution of their spectral numbers, the relation to quantum cohomology and to mirror symmetry.

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Homological mirror symmetry for Fano surfaces

Denis Auroux

The goal of the work presented here (joint with L. Katzarkov and D. Orlov) is to explicitly verify Kontsevich's homological mirror symmetry conjecture for some simple examples of Fano varieties, following ideas of Hori, Vafa, Kontsevich, Seidel, ...

The homological mirror symmetry conjecture treats mirror symmetry as an equivalence between two categories naturally attached to a mirror pair of Calabi-Yau manifolds X, Y: the (bounded) derived category $D^bCoh(X)$ of coherent sheaves on X is equivalent to the (derived) *Fukaya category* $DF(Y)$ of Y, and vice-versa.

We focus on a different setting, that of Fano manifolds (i.e., X with $c_1(TX) > 0$), which mirror symmetry puts in correspondence with *Landau-Ginzburg models*, i.e. pairs (Y, w) where Y is a noncompact manifold and $w : Y \to \mathbf{C}$ is a holomorphic function. The derived category of coherent sheaves of X is then expected to be equivalent to a derived category of Lagrangian vanishing cycles associated to the singularities of w, that we denote by $DLag_{vc}(w)$. This category, which should include not only compact Lagrangian submanifolds of Y, but also some noncompact objects which outside of a compact subset fiber in a special way above real half-lines, is not rigorously defined yet in general. However, in the special case where w is a complex Morse function, i.e. the critical points of w are isolated and non-degenerate, a precise definition of the category of Lagrangian vanishing cycles has been given by Seidel, which makes the explicit verification of homological mirror symmetry possible on various examples.

More precisely, in this case $DLag_{vc}(w)$ can be realized as the (split-closed) derived category of a finite directed A_∞ category $Lag_{vc}(w, \{\gamma_i\})$ associated to an ordered collection of arcs $(\gamma_i)_{1\le i\le r}$ joining a reference point $\lambda_0 \in \mathbf{C}$ to the various critical values $\lambda_1, \dots, \lambda_r$ of w. The arcs γ_i determine vanishing cycles $L_1, \dots, L_r$, which are Lagrangian spheres inside the reference fiber $\Sigma_0 = w^{-1}(\lambda_0)$. The objects of $Lag_{vc}(w, \{\gamma_i\})$ are $L_1, \dots, L_r$, with morphisms given by

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} \mathbf{C}^{|L_i \cap L_j|} & \text{if } i < j, \\ \mathbf{C} \cdot Id & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Morphism spaces are graded by Maslov index, and differentials and compositions are defined using Floer theory in Σ_0, i.e. by counting pseudoholomorphic discs $u : (D^2, \partial D^2) \to (\Sigma_0, \bigcup L_i)$ with corners at specified intersection points, with weights $\pm \exp(-\int_{D^2} u^*\omega)$. It is a result of Seidel that, while $Lag_{vc}(w, \{\gamma_i\})$ depends very much on the choice of $\{\gamma_i\}$, the derived category does not (in a sense, different systems of arcs correspond to different “presentations” of the same derived category).

The main examples that we consider are the weighted projective planes $X = \mathbf{CP}^2(a,b,c) = (\mathbf{C}^3 - \{0\})/\mathbf{C}^*$, where a, b, c are positive integers and $\mathbf{C}^*$ acts by $t \cdot (x, y, z) = (t^a x, t^b y, t^c z)$. X is a Fano orbifold, and its derived category is generated by the exceptional collection $\langle O, O(1), \ldots, O(a + b + c - 1)\rangle$, with $\mathrm{Hom}(O(i), O(j))$ isomorphic to the degree $(j - i)$ part of the symmetric algebra $\mathbf{C}[x, y, z]$ where $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$.

The mirror of X is the affine hypersurface $Y = \{x^a y^b z^c = 1\} \subset (\mathbf{C}^*)^3$ equipped with the superpotential $w = x+y+z$ and an exact symplectic form. The $a+b+c$ critical points of w are nondegenerate, and by explicitly determining the vanishing cycles one can describe completely the category $Lag_{vc}(w, \{\gamma_i\})$ for a suitable choice of $\{\gamma_i\}$ (morphisms, gradings, composition formulas). Putting these vanishing cycles in relation with the coherent sheaves generating $D^bCoh(X)$, one can prove that $D^bCoh(X) \simeq DLag_{vc}(w)$, i.e. homological mirror symmetry is verified for weighted projective planes. Similar results have been obtained for Hirzebruch surfaces or for certain Del Pezzo surfaces; in addition, partial results have been obtained for higher-dimensional weighted projective spaces.

Perhaps the most exciting development coming out of these calculations is a relationship between non-exact symplectic deformations of Y and certain noncommutative deformations of X. Namely, in the case of weighted projective planes $Y \simeq (\mathbf{C}^*)^2$, so $H^2(Y, \mathbf{C}) = \mathbf{C}$, and hence Y carries non-exact symplectic forms. If we equip Y with a symplectic form and a B-field such that $\int_T [B + i\omega] = \tau \in \mathbf{C}$, where $T = Y \cap \{|x| = |y| = |z| = 1\}$ is the generator of $H_2(Y, \mathbf{Z})$, this deforms $Lag_{vc}(w)$ by modifying the coefficients in the composition formulas (the areas of the pseudoholomorphic discs in $(\Sigma_0, \bigcup L_i)$ are modified). On the complex side, this corresponds to a noncommutative deformation of $\mathbf{CP}^2(a, b, c)$, where the underlying weighted symmetric algebra $\mathbf{C}[x, y, z]$ is deformed by setting $yz = \lambda zy$, $zx = \mu xz$, $xy = \nu yx$, where $\lambda^a \mu^b \nu^c = \exp(i\tau)$. Moreover, in the case of the usual $\mathbf{CP}^2$ additional noncommutative deformations can be obtained by considering a fiberwise compactification of Y, whose second cohomology has rank 2.

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Moduli spaces of local systems, positivity and higher Teichmüller theory

Alexander Goncharov
(joint work with V. V. Fock)

Goncharov, Alexander

I. Let S be an oriented surface with boundary: $S = \bar{S} - D_1 \cup \ldots \cup D_n, n > 0$. Assume that S is hyperbolic. Usually $\chi(S) < 0$. The *Teichmüller space* for S is defined as

$$\begin{aligned}\mathcal{T}(S) :&= \{\text{complex structures on } S\}/\operatorname{Diff}_0(S)\\ &\cong \{\text{faithful representations}\\ &\qquad \rho : \pi_1(s) \to PSL_2(\mathbb{R})\}/PSL_2(\mathbb{R}) - \text{conjug.}\end{aligned}$$

The mapping class group $\Gamma_S = \operatorname{Diff}(S)/\operatorname{Diff}_0(S)$ acts on $\mathcal{T}(S)$. Let $\mathcal{T}''(S) \subset \mathcal{T}(S)$ be the subset of representations with unipotent monodromy around each boundary component. The $\mathcal{T}(S)$ has a boundary with corners with the deepest stratum $\mathcal{T}''(S)$. Define

$$\begin{aligned}\widetilde{\mathcal{T}}(S) := \{p \in \mathcal{T}(S),\ &\text{plus choice of an eigenvalue}\\ &\text{for the monodromy around each } \partial D_i\}.\end{aligned}$$

In the case when S is compact, N. Hitchin defined in 1992 a component in a space of representations $\pi_1(S) \to G(\mathbb{R})$ where G is a simple Lie group with trivial center. He proved that given a complex structure on S, this component is isomorphic to $\mathbb{C}^N$.

We are looking for an algebraic geometric avatar of the Teichmüller-Thurston theory, which can be generalized to any G (split, semi-simple algebraic group over $\mathbb{Q}$ with trivial center).

II. The *moduli space* $\mathcal{X}_{G,\hat{S}}$. Let $\hat{S}$ be a pair $(S, \{x_1, \ldots, x_m\})$, where the second element of the pair is a collection of marked points on the boundary ∂S.

Definition. *A* ***framed*** ***G-local system*** *on $\hat{S}$ is a pair $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a G-local system on S and β is a flat section of the restriction of $\mathcal{L} \times_G \mathcal{B}$ to the punctured boundary $\partial S - \{x_1, \ldots, x_m\}$, where $\mathcal{B}$ is the flag variety for G.*

Definition. *$\mathcal{X}_{G,\hat{S}}$ is the moduli space of framed G-local systems on $\hat{S}$.*

Example. *If $\hat{S}$ is a disc $\hat{D}_n$ with n marked points on the boundary then*

$$\mathcal{X}_{G,n} := \mathcal{X}_{G,\hat{D}_n} \cong G\backslash\mathcal{B}^n.$$

Let $\mathscr{T}$ be an ideal triangulation of a surface S' with n punctures ($S' \underset{\text{h.e.}}{\sim} S$). Restricting $(\mathcal{L}, \beta)$ to triangles and rectangles of the triangulation we get a rational map

$$\Pi_{\mathscr{T}} : \mathcal{X}_{G,S} \longrightarrow \prod_{\text{triangles of } \mathscr{T}} \mathcal{X}_{G,3} \times H^{\{\text{edges of } \mathscr{T}\}}.$$

Theorem. *This map is a birational isomorphism.*

Using different ideal triangulations $\mathscr{T}$ we get a Γ_S-equivariant atlas on $\mathcal{X}_{G,S}$. We prove that the transition functions for this atlas are *subtraction-free.* Thus for any semifield K, e.g. $K = \mathbb{R}_{>0}$, we can define the set of K-valued points of $\mathcal{X}_{G,\hat{S}}$.

Definition. *The* ***higher Teichumüller space*** *for* G *is* $\mathcal{X}_{G,\hat{S}}(\mathbb{R}_{>0})$.

Definition. *The* ***lamination space*** *for* G *is* $\mathcal{X}_{G,\hat{S}}(\mathbb{R}^t)$*, where* $\mathbb{R}^t$ *is the tropical semifield.*

Theorem. *For* $G = PSL_2$ *we get the classical Teichumüller space* $\widetilde{\mathcal{T}}(S)$ *and the space of Thurston's measured laminations on* S*.*

Our main conjecture relates $\mathcal{X}_{G,S}$ to get another moduli space $\mathcal{A}_{{}^L G,S}$ related to the Langland's dual ${}^L G$, which also has a canonical positive atlas on it.

Introduction to the chiral de Rham complex

Fyodor Malikov

This is a brief and elementary review of the work done jointly with A. Vaintrob, V. Schechtman, and V. Gorbounov on sheaves of vertex algebras over smooth manifolds. Let U be étale over $\mathbb{C}$ with an étale coordinate system $x = \{x_i\}$, $\partial = \{\partial_{x_i}\}$, $1 \leq i \leq N$. To the pair (U, x) one attaches two vertex algebras:

(1) $\mathcal{O}^{ch}_{U,x}$, generated by a pair of even fields, $x(z)$, $\partial(z)$, s.t. $[\partial(z), x(w)] = \delta(z-w)$, and

(2) $\Omega^{ch}_{U,x}$, generated by $x(z)$, $\partial(z)$ as above and a pair of odd fields , $dx(z)$, $\partial_{dx}(z)$, s.t. $[\partial_{dx}(z), dx(w)] = \delta(z-w)$.

Next one considers an arbitrary smooth manifold X along with a covering by (U, x) as above and attempts to sheafify the just introduced vertex algebras by glueing over the intersections. This can be accomplished in the case of Ω^{ch}_N.

Lemma. *Let* $Aut(U)$ *be the automorphism group of* U *and* $\widehat{Aut}(U)$ *the automorphism group of* $\Omega^{ch}_{U,x}$*. There is an injection*

$$Aut(U) \to \widehat{Aut}(U).$$

This lemma allows one to define for any X a sheaf of vertex algebras Ω^{ch}_X to be called a *chiral de Rham complex.* The correspondence

$$X \to \Omega^{ch}_X$$

is natural w.r.t. to étale morphisms $X \to Y$.

One property of Ω_X^{ch} is that it is actually a vertex algebra with a square zero differential. The usual de Rham complex, Ω_X, embeds into it so that $\Omega_X \hookrightarrow \Omega_X^{ch}$ is a quasi-isomorphism.

Another property of Ω_X^{ch} was discovered by Borisov and Libgober.

Theorem (Borisov-Libgober)**.** *Let $Ell_X(q,y)$ be the elliptic genus of X. Then $Ell_X(q,y)$ equals the Euler character of Ω_X^{ch} w.r.t. to an appropriate double grading of Ω_X^{ch}.*

Here are two applications of the chiral de Rham complex.

- The following character formula is valid
$$2ch\, H^0(\mathbb{P}^{2N}, \Omega^{ch}_{\mathbb{P}^{2N}}) = Ell_{\mathbb{P}^{2N}}(q,1) + 1.$$
It follows that the elliptic genus has positive coefficients. One can argue, therefore, that the vertex algebra $H^0(\mathbb{P}^{2N}, \Omega^{ch}_{\mathbb{P}^{2N}})$ provides a realization of the elliptic genus just as the Monster vertex algebra provided a realization of the j-function.
- Let $\Sigma \subset \mathbb{P}^{N-1}$ be a Calabi-Yau hypersurface. There is a spectral sequence, (E_r, d_r), s.t.
  (a) $(E_r, d_r) \Rightarrow H^*(\Sigma, \Omega^{ch}_\Sigma)$,
  (b) the 1st term, E_1, is almost isomorphic to Witten's chiral algebra of the corresponding Landau-Ginzburg orbifold.

This implies the following explicit orbifold formula for the elliptic genus of Σ:
$$Ell_\Sigma(q,y) = \frac{1}{N}\sum_{j,l=0}^{N-1}(-1)^{N(j+l)+sj} e^{-\pi i(N-2)(2s+l-j^2)} \left(\frac{\theta_1((1-1/N)s+\frac{1}{N}(j\tau+l),\tau)}{\theta_1(\frac{1}{N}(s-j\tau-l),\tau)}\right),$$
where $q = e^{2\pi i\tau}$, $y = e^{2\pi i s}$.

Let us now turn to the problem of sheafification of the purely even vertex algebra $\mathcal{O}^{ch}_{U,x}$. In this case there is no analogue of the lemma, and there is no universal such sheaf, but one defines a natural gerbe of such sheaves, to be called the gerbe of chiral differential operators. The following holds true (for the sake of simplicity we keep to the case of a complex analytic X):

- a sheaf of chiral differential operators exists over X iff $c_2 - \frac{1}{2}c_1^2 = 0$;
- if non-empty, the set of isomorphism classes of sheaves of chiral differential operators over X is a torsor over $H^1(X, \Omega_X^{2,cl})$;
- the automorphism group of any such sheaf is isomorphic to $H^0(X, \Omega_X^{2,cl})$.

An interesting example of this construction is provided by an algebraic semi-simple complex Lie group G, in which case one establishes a 1-1 correspondence between algebras of chiral differential operators over G and complex numbers – very much in spirit of the true physicists' WZW model. (This assertion can be made much more precise.) The BRST reduction of these gives a classification of chiral differential operators over the corresponding flag manifold recovering thereby the Wakimoto modules.

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Minimal Models and K-Theory

Volker Braun
(joint work with Sakura Schäfer-Nameki)

There are only certain values of the central charge which can occur in a $N = 2$ unitary CFT, they must be $c \in \{\frac{3k}{k+2}, k \in \mathbb{Z}_{\geq}\} \cup [3, \infty)$. The discrete values $c < 3$ are rational with respect to the $N = 2$ chiral algebra, the so-called minimal models[1]. There are 3 widely used constructions for these CFTs.

- The coset $\dfrac{\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2}{\mathfrak{u}(1)_{k+2}}$ with diagonal modular invariant.
- MM_k, the $W = x^{k+2} + y^2$ LG model.
- $MM_k/\mathbb{Z}_2$, the $W = x^{k+2}$ LG model.

The D-brane charges for the coset model are well understood. They can either be determined by renormalization group flow, or as equivariant K-groups for twisted complex K-theory [1, 5, 6], where the twist class is related to the level k.

$$ {}^tK^0_{U(1)}\Big(SU(2)\Big)^{\mathbb{Z}_2} = 0\,, \quad {}^tK^1_{U(1)}\Big(SU(2)\Big)^{\mathbb{Z}_2} = \mathbb{Z}^{k+1}\,. $$

Especially, they are torsion free abelian groups. Now recently Hori [2] argued that the $W = x^{k+2}$ Landau-Ginsburg (LG) model allows for non-vanishing charges for the B type D-branes. We can verify that by an honest K-theory computation. In summary, there are the following D-brane charge groups.

Model	coset	MM_k	$MM_k/\mathbb{Z}_2$
K(A-branes)	$\mathbb{Z}^{k+1}$	$\mathbb{Z}^{k+1}$	$\mathbb{Z}^{k+1}$
K(B-branes)	0	0	$\mathbb{Z}_{k+2}$

Obviously, the MM_k and $MM_k/\mathbb{Z}_2$ are different CFTs, and it is well known that one can be obtained from the other as $\mathbb{Z}_2$ orbifold.

The LG B-branes can be understood as follows. If one were to use the usual LG action on worldsheets with boundary, then the supersymmetry variation is not zero but a boundary term. To cancel this and obtain an $\mathcal{N} = 2$ SCFT one must add a boundary action. A popular ansatz [3] contains a choice of matrix factorization

$$ \phi_0, \phi_1 \in \mathrm{Mat}(n, R) : \quad W \cdot \mathbf{1}_n = \phi_0\phi_1 = \phi_1\phi_0\,, $$

where $R \stackrel{\text{def}}{=} \mathbb{C}[\bar{z}]$ is the polynomial ring in the LG fields. Different factorizations yield different boundary theories, and hence describe different D-branes. This can be formalized to a category **MF** whose objects are 2-periodic complexes

$$ \cdots \xrightarrow{\phi_0} (R/W)^n \xrightarrow{\phi_1} (R/W)^n \xrightarrow{\phi_0} \cdots\,, $$

and maps are ordinary chain maps. In analogy with the usual B-model on a Calabi-Yau manifold X, **MF** plays the role of $\mathrm{Coh}(X)$. This category needs to be

[1] In the following, I will only consider the A type minimal models for simplicity. The D and E type can be treated similarly.

extended to a triangulated category, and Kontsevich proposed the category **DB**. It can be obtained as the homotopy category of **MF**, or as the stable category associated to **MF**. This category then plays the role analogous to the derived category $D(\mathrm{Coh}\, X)$. Orlov [4] showed that $\mathbf{DB} \simeq D_{\mathrm{sg}}\big(\{W=0\}\big)$, which gives a nice geometrical interpretation as sheaves on the singularity.

One can [7] identify **MF** with the category of Cohen-Macaulay modules over R/W, which gives a computationally useful way to understand the matrix factorizations. Here one must mod out the trivial matrix factorizations and the trivial module R/W, but we will ignore this subtlety in the following. The Auslander-Reiten (AR) quivers of the module categories are known. Especially, the AR quiver for $W = x^n + y^2$ is the $\mathbb{Z}_2$ orbifolds of the $W = x^n$ AR quiver. The $\mathbb{Z}_2$ action fixes one of the modules if n is even, and acts freely if n is odd. In the former case one has to add an extra module, corresponding to a twisted sector. In any case, one can easily compute the Grothendieck group of the module category, which I already listed in the beginning.

The true importance of the minimal models is that they serve as building blocks for string theory compactifications. For this, one has to construct a suitable $c = 9$ SCFT and then impose the GSO projection. This can be archived by tensoring minimal models, a construction is known as Gepner models. For example, the $(k = 3)^5$ Gepner model corresponds to the Fermat quintic. We can check that

$$ {}^{t}K^{i}_{U(1)^5\times\mathbb{Z}_5}\Big(SU(2)^5\Big)^{(\mathbb{Z}_2)^5} \otimes_{\mathbb{Z}} \mathbb{C} = K^i(\mathrm{Quintic}) \otimes_{\mathbb{Z}} \mathbb{C} = \begin{cases} \mathbb{C}^{204} & i=1 \\ \mathbb{C}^{4} & i=0\,. \end{cases} $$

Of course, such an identity should be lifted to an equivalence of derived categories.

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Lagrangian Tori on the quartic surface

Paul Seidel

Let Y_0 be a $K3$ surface. More concretely, we want to think of this as a quartic surface in $\mathbb{C}P^3$, and equip it with the Kähler form obtained from the Fubini-Study form; since we are taking the point of view of symplectic topology, any other Kähler form in the same cohomology class (in particular the unique Ricci-flat one) will do just as well, due to Moser's Lemma.

An oriented Lagrangian surface $L_0 \subset Y_0$ has two basic topological invariants: its homology class $[L_0]$ and its Maslov class $\mu_{L_0} \in H^1(L_0)$. For instance, the tori which are fibres of the SYZ fibrations have zero Maslov class but nonzero homology class, and so do all other special Lagrangian surfaces; while the small Clifford tori lying in coordinate charts have zero homology class, but nonzero Maslov class.

Conjecture. *If $\mu_{L_0} = 0$, then necessarily $[L_0] \neq 0$.*

Note that this concerns Lagrangian tori only, because all other oriented Lagrangian surfaces automatically have nontrivial homology classes ($L_0 \cdot L_0 = -\chi(L_0)$ being nonzero). Questions concerning Lagrangian tori have a certain importance in symplectic topology, because Luttinger surgery along such tori can be used to produce new symplectic four-manifolds (the condition $\mu_{L_0} = 0$ ensures that the manifold obtained from the $K3$ surface by such a surgery still has $c_1 = 0$).

There is some geometric evidence for the conjecture: first of all, similar results are true for T^4 (by passing to the universal cover, and using the fact that Lagrangian tori in $\mathbb{R}^4$ must have nonzero Maslov class, which is due to Polterovich) and for resolutions of ADE surface singularities (by a more complicated geometric argument, still using Floer cohomology). Secondly, one can imagine a "proof" based on the Thomas-Yau conjecture, together with the fact that after a generic deformation of the Calabi-Yau structure inside its moduli space, there are no more special Lagrangian surfaces (just as after a generic deformation of the complex structure, there are no more divisors).

The aim of this talk is to explain another approach to the conjecture: we retain only the idea of generic deformation, and translate that to pure algebraic geometry using Kontsevich's homological mirror symmetry. Since homological mirror symmetry is known to hold in the case of the quartic surface, one can expect to actually obtain a proof of our conjecture in this way. However, several details still remain to be checked.

Let X_0 be any smooth projective variety, and A_0 the category of coherent sheaves on it. We will consider deformations A of A_0 over $\mathbb{C}[[q]]/q^{N+1}$ ($N = \infty$ is also allowed, with suitable caution). One can think of A in terms of the abstract deformation theory of abelian categories, or more geometrically in terms of generalized deformations of X_0 (including twisted sheaves and deformation quantization). Up to first order, such deformations are classified by an extended Kodaira-Spencer class

$$\kappa \in H^0(\Lambda^2 TX_0) \oplus H^1(TX_0) \oplus H^2(\mathcal{O}_{X_0}).$$

Any object E of A comes with a filtration $q^k E$ whose quotients $q^k E/q^{k+1}E$ are objects of A_0. Multiplication by q yields an epimorphism $q^k E/q^{k+1}E \to q^{k+1}E/q^{k+2}E$. Suppose now that $N = \infty$; because of the noetherian nature of A_0, $q^k E/q^{k+1}E$ must then stabilize for $k \gg 0$. If it stabilizes to 0, which means that $q^k E = 0$ for some k, we say that E is q-torsion. There is an obstruction theory associated to this problem; in the case of $K3$s, one can use it to show that *if κ is generic, all objects E of A are q-torsion.*

We now pass to the derived category, or rather a differential graded version of that. Namely, let C_0 be the dg category of bounded complexes of injective quasicoherent modules on X_0. A generalized deformation of X_0 induces a deformation (in a much more obvious sense as before) of C_0 to a dg category C linear over $\mathbb{C}[[q]]/q^{N+1}$. If F is an object of C then its cohomologies $H^k(F)$ are objects of A. Together with the Eilenberg-Moore spectral sequence

$$Ext^*(H^*(F), H^*(F)) \Longrightarrow H^*(hom_C(F, F)),$$

this implies that *if κ is generic, the endomorphism ring $H(hom_C(F, F))$ of any object F of C is a torsion q-module.*

Finally let's return to the quartic surface Y_0. A Lagrangian torus with $\mu_{L_0} = 0$ defines an object of the Fukaya category fuk_0 of Y_0. The deformation theory of that category is governed by the Hochschild cohomology group $HH^2(fuk_0)$. A particular kind of deformations is obtained by changing the symplectic class; infinitesimally, this is expressed by a map $H^2(Y_0) \to HH^2(fuk_0)$. For the quartic surface, one can prove that this map is an isomorphism, so these are in fact all deformations. If our Lagrangian torus satisfies $[L_0] = 0$, it gives rise to an object L of any deformed Fukaya category fuk, whose endomorphism ring is free over $\mathbb{C}[[q]]/q^{N+1}$:

$$H(hom_{fuk}(L, L)) = H^*(T^2) \otimes \mathbb{C}[[q]]/q^{N+1}.$$

The (cohomologically) full and faithful embedding $fuk_0 \to C_0$ which is a weak form of homologically mirror symmetry extends to an embedding $fuk \to C$. For a generic choice of deformation C, the image of L under this would be an object F which violates the previously stated property. In this way, a proof of the conjecture would be obtained.

Twisted generalised complex structures and topological field theory

Anton Kapustin

It was pointed out by E. Witten in 1988 that one can construct interesting examples of topological field theories by "twisting" supersymmetric field theories. [1] This observation turned out to be very important for quantum field theory and string theory, since observables in topologically twisted theories are effectively computable on one hand and can be interpreted in terms of the untwisted theory on the other. In other words, supersymmetric field theories tend to have large integrable sectors.

From the string theory viewpoint, the most important class of supersymmetric field theories admitting a topological twist are sigma-models with $(2,2)$ supersymmetry. Usually one considers the case when the B-field is a closed 2-form, in which case $(2,2)$ supersymmetry requires the target M to be a Kähler manifold. In this case the theory admits two different twists, which give rise to two different topological field theories, known as the A and B-models. (More precisely, the B-model makes sense on the quantum level if and only if M is a Calabi-Yau manifold. For the A-model, the Calabi-Yau condition is unnecessary.) In any topological field theory, observables form a supercommutative ring. For the A-model, this ring turns out to be a deformation of the complex de Rham cohomology ring of M, which is known as the quantum cohomology ring. It depends on the symplectic (Kähler) form on M, but not on its complex structure. For the B-model, the ring of observables turns out to be isomorphic to

$$\oplus_{p,q} H^p(\Lambda^q T^{1,0} M),$$

which obviously depends only on the complex structure of M. Furthermore, it turns out that all correlators in the A-model are symplectic invariants of M, while all correlators in the B-model are invariants of the complex structure on M [2].

In the paper [3] we analyzed more general topological sigma-models for which $H = dB$ is not necessarily zero. It is well-known that for $H \neq 0$ $(2,2)$ supersymmetry requires the target manifold M to be "Kähler with torsion" [4]. This means that we have two different complex structures $I_\pm$ for right-movers and left-movers, such that the Riemannian metric g is Hermitian with respect to either one of them, and I_+ and I_- are parallel with respect to two different connections with torsion. The torsion is proportional to $\pm H$. The presence of torsion implies that the geometry is not Kähler (the forms $\omega_\pm = gI_\pm$ are not closed). Upon topological twisting, one obtains a topological field theory, and one would like to describe its correlators in terms of geometric data on M. As in the Kähler case, there are two different twists (A and B), and by analogy one expects that the correlators of either model depend only on "half" of the available geometric data. Furthermore, it is plausible that there exist pairs of $(2,2)$ sigma-models with H-flux for which the A-model are B-model are "exchanged." This would provide an interesting generalization of Mirror Symmetry to non-Kähler manifolds.

The main result of Ref. [3] is that topological observables can be described in terms of a (twisted) generalized complex structure on M. This notion was introduced by N. Hitchin [5] and studied in detail by M. Gualtieri [6]; we review it below. One can show that the geometric data H, g, I_+, I_- can be repackaged as a pair of commuting twisted generalized complex structures on M [6]. We show in Ref. [3] that on the classical level the ring of topological observables and the topological metric on this ring depend only on one of the two twisted generalized complex structures. This strongly suggests that all the correlators of either A or B-models (encoded by an appropriate Frobenius manifold) are invariants of only one twisted generalized complex structure. Therefore, if M and M' are related by Mirror Symmetry (i.e. if the A-model of M is isomorphic to the B-model of M' and vice versa), then the appropriate moduli spaces of twisted generalized complex structures on M and M' will be isomorphic.

To state our results more precisely, we need to recall the definition of the (twisted) generalized complex structure (TGC-structure for short). Let M be a smooth even-dimensional manifold, and let H be a closed 3-form on M. The bundle $TM \oplus TM^*$ has a binary operation, called the twisted Dorfman bracket. It is defined, for arbitrary $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(TM^*)$, as

$$(X \oplus \xi) \circ (Y \oplus \eta) = [X, Y] \oplus (\mathcal{L}_X \eta - i_Y d\xi + \iota_Y \iota_X H)\,.$$

It is not skew-symmetric, but satisfies a kind of Jacobi identity. Its skew - symmetrization is called the twisted Courant bracket. The bundle $TM \oplus TM^*$ also has an obvious pseudo-Euclidean metric q of signature (n, n).

A TGC-structure on M is a bundle map $\mathcal{J}$ from $TM \oplus TM^*$ to itself which satisfies the following three requirements:

- $\mathcal{J}^2 = -1$.
- $\mathcal{J}$ preserves q, i.e. $q(\mathcal{J}u, \mathcal{J}v) = q(u, v)$ for any $u, v \in TM \oplus TM^*$.
- The eigenbundle of $\mathcal{J}$ with eigenvalue i is closed with respect to the twisted Dorfman bracket. (One may replace the Dorfman bracket with the Courant bracket without any harm).

To any TGC-structure on M one can canonically associate a complex Lie algebroid E (which is, roughly, a complex vector bundle with a Lie bracket which has properties similar to that of a complexification of the tangent bundle of M). From any complex Lie algebroid E one can construct a differential complex whose underlying vector space is the space of sections of $\Lambda^\bullet(E^*)$ (which generalizes the complexified de Rham complex of M). In Ref. [3] we showed that the space of topological observables is isomorphic to the cohomology of this differential complex. We also derived a formula for the metric on the cohomology, which makes it into a supercommutative Frobenius algebra. Both the ring structure and the topological metric depend only on one of the two TGC-structures available.

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Gerbe-modules and WZW branes

Krzysztof Gawędzki

(Bundle) gerbes [9] and gerbe-modules [1, 7], both equipped with a hermitian connection, find natural application in analysis of sigma models in topologically non-trivial situations. In (bosonic) sigma models, one considers maps

$$\phi : \Sigma \longrightarrow M$$

from a Riemann surface Σ to the target manifold M. The latter is equipped with a metric γ, Kalb-Ramond 2-form B and a (possibly non-abelian) Chan-Paton gauge field A. One attempts to compute Feynman path integrals

$$\int \cdots \ \mathrm{e}^{-S(\phi)} \ D\phi$$

with $\cdots$ standing for various insertions and the (abusively denoted) "classical amplitudes"

$$\mathrm{e}^{-S(\phi)} \ = \ \exp\Big[-\, \|d\phi\|^2_{L^2} + \mathsf{i}\int_\Sigma \phi^* B \Big] \ \prod_i \mathrm{tr}\, \mathcal{P}\, \mathrm{e}^{\mathsf{i}\int_{\ell_i} \phi^* A} \,.$$

The last product is over the traces of holonomy of the gauge field A along the ϕ-images of the boundary loops $\ell_i \subset \partial\Sigma$. We ignore eventual contributions from the dylatonic or tachyonic potential.

In the WZW model [11], M is a Lie group G (assumed compact and simple here) with the left-right invariant metric defined by the bilinear form $\frac{\mathsf{k}}{4\pi}\,\mathrm{tr}\, XY$ on the Lie algebra $\mathbf{g}$. Only the exterior derivative $dB = H$ is given: $H = \frac{\mathsf{k}}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3$. Since H is a closed but not exact 3-form on G, the B field exists only locally and is determined up to closed 2-forms. In the presence of such topologically non-trivial 2-form field, the definition of the classical amplitude (1) needs a refinement.

Let $\mathcal{G}$ be a (bundle) gerbe over M with (hermitian connection of) curvature H [9]. Such gerbes exist if and only if the periods of H are in $2\pi\mathbf{Z}$. In the latter case, non(-stably)-isomorphic [10] choices of $\mathcal{G}$ are labeled by elements of $H^2(M, U(1))$. In particular, the isomorphism class of $\phi^*\mathcal{G}$ belongs to $H^2(\Sigma, U(1))$. For $\partial\Sigma = \emptyset$ the latter group is canonically isomorphic to $U(1)$ and one may take the corresponding number $Hol_{\mathcal{G}}(\phi)$ (the "holonomy" of ϕ w.r.t. $\mathcal{G}$) as the definition of $\exp[\mathsf{i}\int_\Sigma \phi^* B]$ in the closed string sector.

For simply connected group G, the gerbe $\mathcal{G}$ with curvature H exists if and only if k is an integer (in the normalization where $\mathrm{tr}\,\alpha^2 = 2$ for long roots). It is then unique up to isomorphism. An explicit construction of $\mathcal{G}$ has been given in [8]. For non-simply connected groups $G' = G/Z$, where Z is a subgroup of the center of G, there is an obstruction $[U] \in H^3(Z, U(1))$ to the existence of the corresponding gerbe $\mathcal{G}'$ that pulls back to $\mathcal{G}$ [6]. Let the map $Z \ni z \mapsto w_z \in N(T)$, with $N(T) \subset G$ the normalizer of the Cartan subgroup T, be defined by the relation

$$z\,\mathrm{e}^{2\pi\mathrm{i}\,\tau} \;=\; w_z^{-1}\mathrm{e}^{2\pi\mathrm{i}\,(z\tau)}w_z$$

for τ and $z\tau$ belonging to the positive Weyl alcove $\mathcal{A}_W$ in the Lie algebra $\mathbf{t}$ of T. Let $b_{z,z'} \in \mathbf{t}$ be such that $w_z w'_z w_{zz'}^{-1} = \mathrm{e}^{\mathrm{i}b_{z,z'}}$. Then one may take [5,6]

$$U_{z,z',z''} \;=\; \mathrm{e}^{\mathrm{i}\mathsf{k}\,\mathrm{tr}\,\lambda_z b_{z',z''}}\,,$$

where λ_z is the simple weight for which $z^{-1} = \mathrm{e}^{2\pi\mathrm{i}\lambda_z}$. Triviality of the cohomology class $[U]$ selects the values of k for which $\mathcal{G}'$ exists. They coincide with the values of k found in [2] by demanding that the 3-form H' on G' pulling back to H has periods in $2\pi\mathbf{Z}$. For $[U] = 1$, the non-isomorphic gerbes $\mathcal{G}'$ are labeled by the elements of $H^2(Z, U(1))$. The latter group is trivial for cyclic Z. For simple compact groups only $Spin(4n)$ has non-cyclic center $\mathbf{Z}_2 \times \mathbf{Z}_2$. Since $H^2(\mathbf{Z}_2 \times \mathbf{Z}_2, U(1)) = \mathbf{Z}_2$ there are two non-isomorphic gerbes $\mathcal{G}'_\pm$ on $Spin(4n)/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ and two WZW theories, as already noted in [2]. The ambiguity is an example of Vafa's discrete torsion.

The $\mathcal{G}$-module $\mathcal{E}$ of rank N [1] is a version of an N-dimensional vector bundle (with a hermitian connection) over M twisted by gerbe $\mathcal{G}$. The twist renders the holonomy $Hol_{\mathcal{E}}(\varphi)$ of loops $\varphi : S^1 \to M$ ambiguous but in such a way that for $\partial\Sigma = \bigsqcup_i \ell_i$,

$$Hol_{\mathcal{G}}(\phi)\,\prod_i \mathrm{tr}\,Hol_{\mathcal{E}_i}(\phi|_{\ell_i})$$

is defined unambiguously. Given $\mathcal{G}$ and the $\mathcal{G}$-modules $\mathcal{E}_i$, the latter combination may serve as the definition of the non-metric contributions to the open string amplitude (1). This works however only when H is an exact form since existence of $\mathcal{G}$-modules implies the latter property.

To escape that limitation, let us define a $\mathcal{G}$-brane $\mathcal{D}$ as a pair $(D, \mathcal{E})$ where D is a submanifold of M and $\mathcal{E}$ is a $\mathcal{G}|_D$-module (whose existence requires only that $H|_D$ be exact). Then the expression (1) may be still used to define the non-metric contributions to (1) if $\mathcal{D}_i = (D_i, \mathcal{E}_i)$ are $\mathcal{G}$-branes and the classical field satisfies the boundary conditions

$$\phi(\ell_i) \;\in\; D_i\,,$$

i.e. take the boundary values in the $\mathcal{G}$-brane supports.

One is specially interested in boundary conditions that preserve the large part of the bulk symmetries of the sigma models. The symmetry algebra of the closed string sector of the WZW theory is $\widehat{\mathbf{g}}\oplus\widehat{\mathbf{g}}$, the double affine algebra. The $\mathcal{G}$-branes $(D,\mathcal{E})$ that do not break the diagonal affine subalgebra are called "symmetric". For symmetric branes, D must be a conjugacy class in G and the curvature 2-form of the twisted gauge field of $\mathcal{E}$ must be equal to the scalar 2-form

$$F_{\mathcal{E}} \;=\; \frac{1}{8\pi}\,\mathrm{tr}\,(g^{-1}dg)\,\frac{1+Ad_g}{1-Ad_g}(g^{-1}dg)\,.$$

Let, for τ in the positive Weyl alcove $\mathcal{A}_W$, $\mathcal{C}_\tau$ denote the conjugacy class containing $\mathrm{e}^{2\pi\mathrm{i}\tau}$. The rank 1 symmetric $\mathcal{G}$-branes in G are of the form $(\mathcal{C}_{\lambda/\mathsf{k}},\mathcal{E}_1)$ for λ a weight and $\mathcal{E}_1$ a rank 1 $\mathcal{G}|_{\mathcal{C}_{\lambda/\mathsf{k}}}$-module (unique up to isomorphism). Since weights λ such that λ/k is in $\mathcal{A}_W$ also label the unitary representations of the symmetry algebra $\widehat{\mathbf{g}}$ at fixed k, there is a one-to-one correspondence between the latter and the rank 1 symmetric branes (so called Cardy's case). The general symmetric branes $(\mathcal{C}_{\lambda/\mathsf{k}},\mathcal{E})$ in G have

$$\mathcal{E}\;\cong\;\mathcal{E}_1\oplus\,\ldots\,\oplus\mathcal{E}_1$$

i.e. are isomorphic to "stacks" of rank 1 branes in physicists' terminology.

In the non-simply connected group $G'=G/Z$, the conjugacy classes are labelled by Z-orbits $[\tau]$ in $\mathcal{A}_W$, with the class $\mathcal{C}'_{[\tau]}$ pulling back to the union of $\mathcal{C}_\tau\subset G$ with $\tau\in[\tau]$. Only the conjugacy classes $\mathcal{C}'_{[\lambda/\mathsf{k}]}$ may support symmetric branes. There is an obstruction

$$[\mathcal{V}]\in H^2(Z,U(1)^{[\lambda/\mathsf{k}]})\cong H^2(Z_{[\lambda/\mathsf{k}]},U(1))$$

to the existence of a rank 1 symmetric brane $(\mathcal{C}'_{[\lambda/\mathsf{k}]},\mathcal{E}'_1)$, where $Z_{[\tau]}$ denotes the stabilizer subgroup $\subset Z$ of $\tau\in[\tau]$ and $U(1)^{[\tau]}$ the Z-module of $U(1)$-valued functions on $[\tau]$. Explicitly, one may take [5]

$$\mathcal{V}_{\lambda/\mathsf{k};\,z,z'}\;=\;\mathrm{e}^{\mathrm{i}\,\mathrm{tr}\,\lambda b_{z,z'}}V_{z,z'}\,,$$

where $\delta V=U$. If $[\mathcal{V}]$ is trivial (e.g. if $Z_{[\lambda/\mathsf{k}]}$ is cyclic) then non-isomorphic choices of $\mathcal{E}'_1$ are labeled by elements of $H^1(Z,U(1)^{[\lambda/\mathsf{k}]})\cong H^1(Z_{[\lambda/\mathsf{k}]},U(1))$, i.e. by characters of $Z_{\lambda/\mathsf{k}]}$ so that we have $|Z_{[\lambda/\mathsf{k}]}|$ non-isomorphic rank 1 $\mathcal{G}'$-modules $\mathcal{E}'_1(1),\ldots\mathcal{E}'_1(|Z_{[\lambda/\mathsf{k}]}|)$. The general rank N symmetric $\mathcal{G}'$-brane $(\mathcal{C}'_{[\lambda/\mathsf{k}]},\mathcal{E}')$ are then isomorphic to "stacks" of rank 1 branes with

$$\mathcal{E}'\;\cong\;\mathcal{E}'_1(i_1)\oplus\cdots\oplus\mathcal{E}'_1(i_N)\,.$$

The only cases with non vanishing obstruction $[\mathcal{V}]$ are the ones for the group $G'=Spin(4n)/(\mathbf{Z}_2\times\mathbf{Z}_2)$ with gerbe $\mathcal{G}'_-$ and the conjugacy classes $\mathcal{C}'_{\{\lambda/\mathsf{k}\}}$ corresponding to 1-point orbits of λ/k. They support only even rank symmetric branes $(\mathcal{C}'_{\{\lambda/\mathsf{k}\}},\mathcal{E}')$ isomorphic to "stacks" of rank 2 branes, i.e. with

$$\mathcal{E}'\;\cong\;\mathcal{E}'_2\oplus\cdots\oplus\mathcal{E}'_2\,,$$

where $\mathcal{E}'_2$ is a unique (up to isomorphism) rank 2 $\mathcal{G}'_-|_{\mathcal{C}'_{\{\lambda/\mathsf{k}\}}}$-module giving rise to a symmetric brane. In particular, there are no rank 1 (abelian) $\mathcal{G}'_-$-branes supported by conjugacy classes $\mathcal{C}'_{\{\lambda/\mathsf{k}\}}$. This is the phenomenon of spontaneous gauge symmetry enhancement observed in [4] in the context of the Gepner models.

The above geometric constructions, entering the consistent definition of the probability amplitudes of the classical field configurations, permit to extract specific information about the quantized theory. This is so because the WZW model may be easily (geometrically) quantized by transgression. In particular, a gerbe on $\mathcal{G}$ over group G induces canonically a line bundle $\mathcal{L}_{\mathcal{G}}$ (with a hermitian connection) over the loop group LG and the space of the states of the theory in the closed string sector may be taken as the space $\Gamma(\mathcal{L}_{\mathcal{G}})$ of sections of $\mathcal{L}_{\mathcal{G}}$ with a geometric action of the double affine algebra $\widehat{\mathbf{g}}\otimes\widehat{\mathbf{g}}$. To find the bulk spectrum of the theory it is then enough to identify the highest weight sections in $\Gamma(\mathcal{L}_{\mathcal{G}})$. This was the route followed in [2] to obtain the toroidal partition functions of the WZW models with non-simply connected target groups given by modular invariant sesqui-linear combinations of affine characters. Similarly, a gerbe $\mathcal{G}$ over G and a pair of symmetric $\mathcal{G}$-branes $(\mathcal{D}_0, \mathcal{D}_1)$ induce canonically a vector bundle $\mathcal{E}^{\mathcal{D}_1}_{\mathcal{D}_0}$ (again with a hermitian connection) over the space $I^{D_1}_{D_0}$ of maps of an interval into G mapping the ends into the brane supports. $\Gamma(\mathcal{E}^{\mathcal{D}_1}_{\mathcal{D}_0})$ provides then the spaces of boundary states carrying a geometric action of $\widehat{\mathbf{g}}$. The spectrum of the theory in the open string sector may be again determined by identification of the highest weight sections. This allowed [5] to obtain a simple explicit relation between the spaces of boundary states in the group G and G/Z theories, shed new light on the formulae for the annulus partition functions of the WZW models with G/Z targets previously postulated in [3] and to obtain similar formulae for the operator product of the boundary operators.

The geometric approach to WZW theories sketched above provides a uniform treatment of classical and quantum theories and should extend to other classes of conformal boundary conditions, other target groups as well as to coset conformal field theories.

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A-model and Chern-Simons Theory

A. Schwarz

E. Witten [1] has shown that open strings in A-model are related to Chern-Simons theory in the case when the tangent space is a Calabi-Yau threefold. Replacing Chern-Simons theory with its multidimensional analogy constructed in [2] (AKSZ) one can generalise this statemsnt to an arbitrary symplectic manifold M.

It seems that one can start with modification of original Witten's arfuments relating open strings ending in Lagrangian manifold $L \subset M$ with Chern-Simons theory on L. To take into account instanton contributions one can combine the results of [3](Fukaya) and [4](Cattaneo-Frohlich-Pedrini).

Let us take the map (iterated integra) $a \to \int h$ constructed in [4]. This map transforms chains in string space $S(L)$ into preobservables of Chern-Simons theory on L and obeys the relation (9) (see page 4 of [4]). (This relation is stated in [4] only for cycles in $S(L)$ when $\int h$ is an observable, but it seems that the proof goes through also for chains). From other side, Fukaya [3] constructed a chain α in string space $S(L)$ that obeys

$$\partial\alpha + \frac{1}{2}\{\alpha, \alpha\} = 0$$

starting with pseudoholomorphic disks in M with boundary L. Applying the iterated integral reconstruction to α we obtain a preobservable $\tilde{\alpha} = \int h$ of Chern-Simons theory on L. The preobservable $\tilde{\alpha}$ obeys $\pm\delta\tilde{\alpha} + \frac{1}{2}\{\tilde{\alpha}, \tilde{\alpha}\} = 0$ this follows from (39) of [4]. Adding $\tilde{\alpha}$ to the Chern-Simons action of S we obtain an instanton corrected action obeying equation $\{S+\tilde{\alpha}, S+\tilde{\alpha}\} = 0$. Corresponding differential on the space of preobservables that can be considered as completed free algebra generated by forms $C \in \Omega^*(L)$ specifies an A_∞-algebra structure. This A_∞-algebra is equipped with inner product coming from (even or odd) symplectic structure. It coincides with A_∞-algebra of Fukaya.

It is necessary to stress that the considerations based on results of [3], [4] give only instanton corrections to Chern-Simons action. One of the possible ways to obtain Chern-Simons action from A-model is based on the results of [5], [6]. In these papers and A-model with several Lagrangian manifolds was analysed in the limit when the distance between Lagrangian manifolds tends to zero. (This limit corresponds to N coinciding D-branes in physics). Less rigorous way is based on generalization of original Witten's arguments.

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Arithmetic of Calabi-Yau Manifolds

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(joint work with X. de la Ossa and F. Rodriguez-Villegas)

Calabi-Yau manifolds owe many remarkable properties to the special role that they play in relation to supersymmetry and to string theory. It is a fundamental fact that these manifolds depend holomorphically on parameters that determine the complex structure and Kähler-class. The variation of the complex structure of a Calabi-Yau manifold is naturally studied through it periods. These periods are very 'physical' in that they enter prominently into the calculation of physical quantities as, for example, when one calculates couplings that govern the low energy effective theory that results from the compactification of string theory on a manifold. It comes, perhaps, as a surprise to a mathematical physicist that the periods have also an arithmetic significance, a fact that is known to number theorists.

Our first main result expresses the number of rational points of the quintic threefold over $\mathbb{F}_p$, the field with p elements, in terms of the periods. To write out the expression we have to recall some facts. The periods satisfy a system of differential equations, the Picard-Fuchs equations, with respect to the parameters. Specifically, for the family of quintic threefolds corresponding to the polynomial

$$P(x,\,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5$$

there are 204 periods, but among them there are four periods which we shall denote by ϖ_0, ϖ_1, ϖ_2 and ϖ_3, which coincide with the periods of the mirror manifold and satisfy the equation

$$\mathcal{L}\,\varpi_j \;=\; 0 \text{ with } \mathcal{L} \;=\; \vartheta^4 - 5\lambda \prod_{i=1}^{4}(5\vartheta + i)\ ,$$

where here and in the following it is convenient to take the parameter to be $\lambda = 1/(5\psi)^5$, and ϑ denotes the logarithmic derivative $\lambda\frac{d}{d\lambda}$. Consider now the behaviour of the periods in the neighborhood of $\lambda = 0$. The Picard-Fuchs equation has all four of its indices equal to zero. Thus the solutions are asymptotically like 1, $\log\lambda$, $\log^2\lambda$ and $\log^3\lambda$. We denote by f_0 the solution that is analytic at $\lambda = 0$; specifically this is the series

$$f_0(\lambda) \;=\; \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5}\, \lambda^m \;.$$

We can choose the four periods to be of the form

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda) \\
\varpi_1(\lambda) &= f_0(\lambda)\,\log\lambda + f_1(\lambda) \\
\varpi_2(\lambda) &= f_0(\lambda)\,\log^2\lambda + 2f_1(\lambda)\,\log\lambda + f_2(\lambda) \\
\varpi_3(\lambda) &= f_0(\lambda)\,\log^3\lambda + 3f_1(\lambda)\,\log^2\lambda + 3f_2(\lambda)\,\log\lambda + f_3(\lambda)
\end{aligned}$$

where the $f_j(\lambda)$ are power series in λ.

For the case of Calabi-Yau manifolds that can be realised as hypersurfaces in toric varieties, which is a wide class with the quintic as the simplest example, the manifold can be associated with the Newton polyhedron, Δ, of the monomials that appear in the polynomial that defines the hypersurface. In these cases, given Δ, there is a purely combinatoric way of finding a differential system that the periods satisfy; this yields the GKZ system. A fact which is not fully understood is that the GKZ system is often of higher order than the Picard-Fuchs system. So while it is true that the periods satisfy the differential system that one deduces from Δ there are also often additional solutions of this system that are not periods. These extra solutions are called semiperiods and their appearance is somewhat mysterious. It turns out that for the quintic there is a semiperiod and it plays a role in our expressions.

For the quintic the GKZ operator, $\mathcal{L}^\Delta$, is related to the Picard-Fuchs operator by

$$\mathcal{L}^\Delta \;=\; \vartheta\,\mathcal{L} \;=\; \vartheta^5 - \lambda \prod_{i=1}^{5}(5\vartheta + i)\;.$$

The first equality shows that the periods $\varpi_j(\lambda)$ satisfy the new equation and the second equality shows that the new operator has all five of the indices corresponding to $\lambda = 0$ equal to zero. The semiperiod is thus of the form

$$\varpi_4(\lambda) \;=\; f_0(\lambda)\,\log^4\lambda + 4\,f_1(\lambda)\,\log^3\lambda + 6\,f_2(\lambda)\,\log^2\lambda + 4\,f_3(\lambda)\,\log\lambda + f_4(\lambda)\;,$$

with the power series f_0, f_1, f_2, f_3 as before. For $\psi \in \mathbb{F}_p$, $p \neq 5$, we denote by $\nu(\psi)$ the number of rational points of the manifold

$$\nu(\psi) \;=\; \#\{x \in \mathbb{F}_p^5 \mid P(x,\psi) = 0\,\}\;.$$

We denote also by ${}^n f_j$ the truncation of the series f_j to n terms. Thus for example

$${}^n f_0(\lambda) \;=\; \sum_{m=0}^{n-1} \frac{(5m)!}{(m!)^5}\, \lambda^m \;.$$

With these conventions we can state our result most simply for the case that 5 does not divide $p-1$

$$\nu(\psi) = {}^p f_0(\lambda^{p^4}) + \left(\frac{p}{1-p}\right)^p f_1'(\lambda^{p^4}) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2 {}^p f_2''(\lambda^{p^4}) + \\ + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3 {}^p f_3'''(\lambda^{p^4}) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4 {}^p f_4''''(\lambda^{p^4}) \pmod{p^5}$$

where the coefficients $\frac{1}{j!}\left(\frac{p}{1-p}\right)^j$ are understood to be expanded p-adically. We extend this result in three directions: by writing an exact p-adic expression, by extending the result to finer fields with $q = p^r$ elements and by extending the result to cover the interesting case that $5|q-1$.

The expressions we obtain for the numbers of $\mathbb{F}_p$-rational points are computable in a practical sense and this allows us to make some observations on the structure of the ζ-function for these varieties based on numerical experiment. The ζ-function is defined in terms of the N_r, the number of $\mathbb{F}_p$-rational points of the projective variety, by the expression

$$\zeta(t,\psi) \;=\; \exp\left(\sum_{r=1}^{\infty} N_r(\psi)\,\frac{t^r}{r}\right) \; .$$

and has an interesting structure which we discuss.

Our principal result in this direction is that for general ψ (that is $\psi^5 \neq 0, 1, \infty$) the ζ-function has the form

$$\zeta_{\mathcal{M}}(t,\psi) = \frac{R_{\mathbf{1}}(t,\psi)R_{\mathcal{A}}(p^\rho t^\rho,\psi)^{\frac{20}{\rho}} R_{\mathcal{B}}(p^\rho t^\rho,\psi)^{\frac{30}{\rho}}}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}$$

In this expression the R's are quartic polynomials in their first argument and the quantity ρ $(=1,2$ or $4)$ is the least integer such that $p^\rho - 1$ is divisible by 5. Thus $R_{\mathbf{1}}$, for example, has the structure

$$R_{\mathbf{1}}(t,\psi) \;=\; 1 + a_{\mathbf{1}}(\psi)\,t + b_{\mathbf{1}}(\psi)\,pt^2 + a_{\mathbf{1}}(\psi)\,p^3t^3 + p^6t^4$$

with $a_{\mathbf{1}}$ and $b_{\mathbf{1}}$ integers that vary with ψ. The other factors $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$ have a similar structure. It is intriguing that these factors are related to certain genus 4 Riemann curves.

Much is known about the structure of the ζ-function in virtue of the the Weil conjectures (since proved). In particular we know even before performing specific computation that the ζ-function for a Calabi-Yau threefold is a rational funcion of t which has the structure

$$\zeta(t) \;=\; \frac{\text{Numerator of degree } 2h^{21}+2 \text{ depending on the complex structure of } \mathcal{M}}{\text{Denominator of degree } 2h^{11}+2} \; .$$

It is immediately apparent that the ζ-function does not treat the complex structure and Kähler parameters symmetrically since the numerator depends nontrivially on the complex structure parameters, for our family of quintics the numerator varies nontrivially with ψ, while on the other hand the denominator depends only on the number of Kähler parameters.

We would perhaps like to introduce a modified or quantum ζ-function, ζ^Q, that respects mirror symmetry and which would have the property

$$\zeta^Q_{\mathcal{M}}(t) \;=\; \frac{1}{\zeta^Q_{\mathcal{W}}(t)}$$

however such a function cannot be given by the classical definition since this would immediately contradict the positivity of the N_r by giving $N_{\mathcal{M},r} = -N_{\mathcal{W},r}$. It is of course possible to define

$$\zeta^Q_{\mathcal{M}} \;=\; \frac{\text{numerator of } \zeta_{\mathcal{M}}}{\text{numerator of } \zeta_{\mathcal{W}}}$$

which will satisfy the desired relation; however without a more intrinsic definition this does not seem to be very fruitful. While we do not propose here such an intrinsic definition nor do we know that such a definition exists nevertheless we find it interesting to point out insights from mirror symmetry that may be pertinent.

An analog of the large complex structure limit seems to follow from the 5-adic expansion of the ζ-function; specifically we find

$$\zeta_{\mathcal{M}}(t,\psi) \;=\; \frac{1}{\zeta_{\mathcal{W}}(t,\psi)} + \mathcal{O}(5^2)$$

moreover this relation holds independent of ψ. Thus we close with the strange suggestion that the higher terms in the 5-adic expansion should be understood as 'quantum corrections'.

Geometry of the Quantum Hall Effect in QED in 2+1 dimensions

Marianne Leitner

Introduction. When, in a two dimensional device, an electric field is turned on, a transversal current is induced. By the Ohm-Hall law, particle flow and electric force are coupled by e^{-2} times the conductivity matrix, $\vec{J}/e = \sigma/e^2(e\vec{E})$, and $\sigma_H := \sigma_{21}/e^2$. In experiments, one additionally applies a constant magnetic field perpendicularly to the plane, of strength F^{12}, in order to obtain nonzero Hall conductivity σ_H. Here, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ for $\vec{A} \in \Gamma(T\mathbb{R}^2)$, a gauge field. Provided $F^{\mu\nu}$ is sufficiently large and the Fermi energy E_F lies in a gap, at zero temperature, $2\pi\sigma_H$ turns out to integral (Integer Quantum Hall Effect). A very similar effect occurs in QED_{2+1}, but without magnetic field and with half integral

$2\pi\sigma_H$. A comparison with the methods of solid state physics allows a better understanding of this quantum field theory [1].

Integer Quantum Hall Effect. Soon after its discovery in 1980, this effect was revealed to be of topological nature ([2], [3]). For a single particle of charge e in a potential A, the Hamiltonian H^A on $C^\infty(\mathbb{R}^2)$ is polynomial in $(\vec{p}-e\vec{A})$. To describe particles in finite area, we introduce a lattice $\Lambda \subset \mathbb{R}^2$ w.r.t. which the potential function is periodic. In particular, $[(\vec{p}-e\vec{A}), T^{\vec{A}}] = 0$, where $T^{\vec{A}} := e^{i\tau^{\vec{A}}}T$ is a twisted translation operator. For given $\vec{\lambda} \in \Lambda$, T is the usual translation defined by $T_{\vec{\lambda}} f(\vec{x}) := f(\vec{x}+\vec{\lambda})$, and $\tau^{\vec{A}} : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$ is given by $\tau^{\vec{A}}_{\vec{\lambda}}(\vec{x}) \equiv e\vec{A}(\vec{\lambda})\vec{x} + const.$ Assuming the magnetic flux per unit cell F of Λ to be an even integer so that $T^{\vec{A}}$ defines a representation of Λ on $C^\infty(\mathbb{R}^2)$, we restrict the domain of H^A to F by imposing quasi-periodic boundary conditions $\vec{k} \in T^* := \mathbb{R}^2/\Lambda^*$. To be precise, H^A acts on functions $\psi \equiv \psi_{\vec{k}}$ decomposing as $\psi(\vec{x}) = e^{i\vec{k}\vec{x}}u(\vec{x})$ with $T^{\vec{A}}u = u$. Here, u is a section of the vector bundle over $T = \mathbb{R}^2/\Lambda$ given by the magnetic phase solely, i.e. $u \equiv u_{\vec{k}} \in L^2_{\vec{A}}(T)$. Conjugation by $e^{-i\vec{k}\vec{x}}$ yields a Hamiltonian $H^A(\vec{k})$ with covariant derivative $(\vec{p}-i\vec{k}-e\vec{A})$. We consider the eigenvalue problem $H^A(\vec{k})u^{(n)}_{\vec{k}} = E^n(\vec{k})u^{(n)}_{\vec{k}}$. If $E^n(\vec{k})$ is nondegenerate for all $\vec{k}$, the corresponding smooth family of spectral projectors $P^{(n)}_{\vec{k}}$ defines a complex rank-one eigenspace bundle $\mathcal{E}^{(n)}$ equipped with the adiabatic connection $P^{(n)}_{\vec{k}} \circ \nabla_{\vec{k}}$ which is Hermitian and has curvature equal to

$$\text{(1)} \qquad \mathrm{Tr}_{\mathcal{H}}\{P^{(n)}_{\vec{k}}[\partial_{k_1}P^{(n)}_{\vec{k}}, \partial_{k_2}P^{(n)}_{\vec{k}}]\} = i\sigma^{(n)}_H(\vec{k});$$

on the other hand, $\sigma^{(n)}_H(\vec{k})$ the n-th energy contribution to the Hall conductivity associated to $H^A(\vec{k})$. It follows that in the average over the boundary conditions,

$$\text{(2)} \qquad \sigma^{(n)}_H = \frac{1}{(2\pi)^2}\int_{T^*} \sigma^{(n)}_H(\vec{k})\, d^2k = \frac{1}{2\pi}c_1[\mathcal{E}^{(n)}]$$

i.e. $2\pi\sigma^{(n)}_H$ is the Chern number of $\mathcal{E}^{(n)}$.

This procedure can be generalized to multi-fermion systems, letting $P^{(0)}_{\vec{k}}$ be the projector onto the multi-fermion ground state. If the particles don't interact, the total Hall conductivity σ_H equals the sum over all $\sigma^{(n)}_H$ for $u^{(n)}_{\vec{k}}$ in the Fermi sea. In many situations, this yields a line bundle $\mathcal{E}^{(0)}_{\mathrm{multi}}$ of multi-fermion ground states.

Zero Field Hall Effect. In case H^A is the Schrödinger operator, the Chern number vanishes if $\vec{A} \equiv \vec{0}$. Actually, a time reversal symmetry breaking term in the Hamiltonian might suffice to produce nonzero Hall conductivity [4]. Let

$$H := \vec{\sigma}\,\vec{p} + m\sigma_3, \quad \vec{\sigma} := (\sigma_j)^2_{j=1}, \quad m \in \mathbb{R}\setminus\{0\},$$

be the (nonmagnetic) massive Dirac operator in 2+1 dimensions. Then, provided $E_F \in (-|m|, |m|)$, the Hall conductivity at zero temperature is a Dirac sea quantity, $\sigma_H^{(T=0)} \equiv \sigma_H^{(-)}$. We choose the Hilbert space $\mathcal{H} = L^2(T, \mathbb{C}^2)$ for $T := \mathbb{R}^2/\mathbb{Z}^2$. Then, under Fourier transformation, for each $\vec{k} \in T^*$, $H(\vec{k})$ gets mapped to $\oplus_{\vec{K}\in(2\pi\mathbb{Z})^2}((\vec{k}+\vec{K})\,\vec{\sigma} + m\sigma_3)$, and (1) becomes

$$(3) \qquad i\sigma_H^{(-)}(\vec{k}) = \sum_{\vec{K}\in(2\pi\mathbb{Z})^2} \mathrm{Tr}_{\mathbb{C}^2}\{\widehat{P}^{(-)}_{\vec{k}+\vec{K}}[\partial_{k_1}\widehat{P}^{(-)}_{\vec{k}+\vec{K}}, \partial_{k_2}\widehat{P}^{(-)}_{\vec{k}+\vec{K}}]\}$$

$$(4) \qquad = -\frac{i}{2}\sum_{\vec{K}\in(2\pi\mathbb{Z})^2} \frac{m}{\{(\vec{k}+\vec{K})^2 + m^2\}^{3/2}}$$

$$= -\frac{i}{4\pi}\mathrm{sgn}(m)\sum_{\vec{R}\in\mathbb{Z}^2} e^{-|m||\vec{R}|-i\vec{k}\vec{R}}$$

by Poisson summation. In the average (2), the correction terms ($|\vec{R}| \neq 0$) vanish, i.e.

$$(5) \qquad 2\pi\sigma_H^{(-)} = -\frac{1}{2}\mathrm{sgn}(m).$$

Since degeneracies occur over T^*, there is no need for $2\pi\sigma_H^{(-)}$ to be an integer.
In the multi-fermion description, (3) equals the trace (1) applied to the projector $\widehat{P}^{(-)}_{\vec{k}}$ onto the multi-fermion ground state in Hilbert space $\wedge_{\vec{K}\in(2\pi\mathbb{Z})^2}\mathbb{C}^2$. Going around the dual torus interchanges the wedge factors. Since there are infinitely many interchanges, the corresponding total phase cannot be determined, and the ground states fail to define a line bundle over T^*.

Zero Field Hall Effect geometrically. Variation of $\vec{k}$ over T^* amounts to considering the family $\widehat{H}(\vec{k}) = \vec{\sigma}\vec{k} + m\sigma_3$ parametrized over $\mathbb{R}^2$ resp. replacing the sum in (4) by an integral over $\mathbb{R}^2$. We extend to $\mathbb{R}^3$ by writing $\widehat{H}(\underline{k}) = \underline{k\sigma}$ for $\underline{\sigma} := (\sigma_j)_{j=1}^3$ and $\underline{k} \in F_m := \{\underline{k} \in \mathbb{R}^3 | k_3 = m\} \cong \mathbb{R}^2$. By (4), the two form $\frac{1}{2\pi}\sigma_H^{(-)}(\vec{k})dk_1 \wedge dk_2$ generalizes naturally to

$$\eta^{(-)} := -\frac{1}{8\pi}\varepsilon^{\alpha\beta\gamma}\frac{k_\alpha dk_\beta \wedge dk_\gamma}{|\underline{k}|^3}.$$

Namely, $\eta^{(-)}$ is rotationally invariant. It is also homogeneous with $\int_{S^2}\eta^{(-)} = -1$. Thus

$$\int_{F_m\subset\mathbb{R}^3}\eta^{(-)} = \int_{\tilde{F}_m\subset S^2}\eta^{(-)} = -\frac{1}{2\pi}\mathrm{sgn}(m),$$

where $\tilde{F}_m \subset S^2$ denote the open upper (in case $m > 0$) resp. lower ($m < 0$) half-sphere onto which F_m projects homeomorphically. This proves (5) geometrically, describing $\sigma_H^{(-)}$ as a solid angle.

The Kubo formula in QM. The Kubo formula for the response of the current J^μ to an electric perturbation of strength E^ν has the form

$$(6) \qquad \langle J^\mu\rangle(t) = \int_{-\infty}^{t} \sigma^{\mu\nu}(t-t')\, E_\nu\, dt' + O(\vec{E}^{\,2}).$$

The Hall conductivity is obtained in the zero frequency limit. With eigenbasis $|u_{\vec{k}}^{(m)}\rangle$ to the Hamiltonian $H(\vec{k})$ as introduced above, standard first order perturbation theory yields

$$(7) \quad \sigma_H^{(0)}(\vec{k}) = \frac{\delta\langle u_{\vec{k}}^{(0)}|J^2(\vec{k})|u_{\vec{k}}^{(0)}\rangle}{e^2\delta E_1} = -2\,\mathrm{Im}\sum_{m\neq 0}\langle u_{\vec{k}}^{(0)}|v^2\frac{|u_{\vec{k}}^{(m)}\rangle\langle u_{\vec{k}}^{(m)}|}{(E_m(\vec{k})-E_0(\vec{k}))^2}v^1|u_{\vec{k}}^{(0)}\rangle$$

$$= -i\mathrm{Tr}_{\mathcal{H}}\{P_{\vec{k}}^{(0)}[\partial_{k_1}P_{\vec{k}}^{(0)}, \partial_{k_2}P_{\vec{k}}^{(0)}]\}.$$

More generally, if, for some index set $I \subseteq \mathbb{N}$, $\{u_{\vec{k}}^i\}_{i\in I} \subseteq \mathcal{H}$ is an orthonormal basis of the eigenspace to the ground energy of $H(\vec{k})$, then $U_{\vec{k}} := \wedge_{i\in I}\, u_{\vec{k}}^i$ defines a multi-fermion ground state in the Hilbert space $\mathcal{H}_I := \wedge_{i\in I}\mathcal{H}$, and

$$\mathrm{Tr}_{\mathcal{H}_I}\{P_U[\partial_{k_1}P_U, \partial_{k_2}P_U]\} = \sum_{i\in I}\mathrm{Tr}_{\mathcal{H}}\{P_{u^i}[\partial_{k_1}P_{u^i}, \partial_{k_2}P_{u^i}]\},$$

where P_U and P_{u^i} denote the projector on $U \equiv U_{\vec{k}}$ in $\mathcal{H}_I$ and $u^i \equiv u_{\vec{k}}^i$ in $\mathcal{H}$ respectively. An example has been discussed for the constant Dirac operator in $L^2(T, \mathbb{C}^2)$.

The Kubo formula in (QED)$_3$. The ground state current of the Euclidean Dirac operator $H_{\mathrm{eucl.}}^{\underline{A}}$ with positive mass term in 3 dimensional QED in a background field $(F^{\nu\eta})_{\nu,\eta=0}^2$ is ([5], [6])

$$(8) \qquad \langle 0|J_\mu|0\rangle = -\frac{1}{8\pi}\varepsilon_{\mu\nu\eta}(eF^{\nu\eta})sgn(m), \quad \mu = 0,1,2.$$

Here, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ for $\underline{A} = (A^j)_{j=0}^2 \in \Gamma(T\mathbb{R}^3)$. It has been remarked [7] that, for $\mu, \nu \in \{1,2\}$, (8) is just the Ohm-Hall law. In particular, the value of σ_H can be read off, giving (5), as well as its independence of the field strength F^{12}. (8) is deduced as follows:

$$(9) \qquad \frac{\delta S_{\mathrm{eff}}[\underline{A}]}{\delta A_\mu(\underline{x})} =: e\langle 0|J_\mu(\underline{x})|0\rangle = -i\int_{\mathbb{R}^3}\mathcal{K}^{\mu\nu}(\underline{x}-\underline{x}')\, eA_\nu(\underline{x}')\, d^3x' + O(\underline{A}^2),$$

with

$$\mathcal{K}^{\mu\nu}(\underline{x}-\underline{x}') := \mathrm{Tr}_{\mathrm{spin}}\{\gamma_{\mathrm{eucl.}}^\mu G_{\mathrm{eucl.}}(\underline{x}-\underline{x}')\gamma_{\mathrm{eucl.}}^\nu G_{\mathrm{eucl.}}(\underline{x}'-\underline{x})\}.$$

Here the Euclidean Dirac matrices $\gamma_{\mathrm{eucl.}}^\mu$ stand for the velocity operators and $G_{\mathrm{eucl.}}$ denotes the Green's function for the nonmagnetic ($\underline{A} \equiv \underline{0}$) operator $H_{\mathrm{eucl.}}$. (9) is the relativistically invariant version of the quantum mechanical Kubo formula (6) and its consequence (7).

Gauge invariant regularization reduces the effective action to second order in $\underline{A}$ to the Chern Simons action

$$S_{CS}[\underline{A}] = \kappa_{CS} \frac{1}{8\pi^2} \varepsilon^{\mu\nu\eta} \int_{\mathbb{R}^3} (\partial_\mu A_\nu) A_\eta \, d^3x$$

for $\kappa_{CS} = -2\pi \,\mathrm{sgn}(m)$, thus giving (8). In particular, σ_H given by (5) is the Hall conductivity for the regularized theory.

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Topological String Amplitudes for Regular K3 Fibrations

Emanuel Scheidegger

(joint work with Albrecht Klemm, Maximilian Kreuzer and Erwin Riegler)

I present work done in collaboration with Albrecht Klemm, Maximilian Kreuzer and Erwin Riegler [1]. The talk is a direct continuation of the talk by Albrecht Klemm on general aspects of topological string theory and its various incarnations and relations to mathematics. Therefore, we first recall his table

Topological model	Open string	Closed string	Heterotic string
A–model	Chern–Simons theory	Gromov–Witten theory Gopakumar–Vafa theory Donaldson–Thomas theory	Elliptic genus
B–model	Holomorphic Chern–Simons theory	Kodaira–Spencer theory of gravity	

The two rows are related by mirror symmetry, the open and closed string columns are related by large N transitions, and the closed and heterotic string columns are related by a kind of S–duality. The aim is to compute amplitudes of the topological string because, from a mathematical point of view, they contain information about the enumerative geometry of the target space, and, from a physical point of view, they compute certain F–terms in the low–energy effective action in four dimensions. The direct computation of the Gromov–Witten, Gopakumar–Vafa, or Donaldson–Thomas invariants is in most cases not known, presently. We will therefore present two indirect methods, the holomorphic anomaly equation in the Kodaira–Spencer theory of gravity together with mirror symmetry, and heterotic–type II string duality applied to the elliptic genus, and show that they allow for a computation of these invariants in cases that are out of reach by the direct ways.

Given a holomorphic map $\phi : \Sigma_g \to X$, from a Riemann surface Σ_g of genus $g > 1$ into a Calabi–Yau threefold X, the basic quantity of interest is the topological string amplitude

$$\mathcal{F}^{(g)}(t,\bar{t}) = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} \mathrm{d}\tau_i \mathrm{d}\bar{\tau}_i \prod_{k=1}^{3g-3} \left\langle |G \cdot \mu_k|^2 \right\rangle$$

The integral is over the moduli space $\mathcal{M}_g$ of genus g stable curves. The brackets $\langle \ldots \rangle$ denote the path integral $\int \mathcal{D}\phi \ldots \exp(-S[\phi] - \sum_i t_i \phi_i^{(2)})$ where $S[\phi]$ is the action of a twisted non-linear σ–model associated to a $N = (2,2)$ superconformal algebra. This algebra has four generators, the energy-momentum tensor T, the BRST current Q, satisfying $Q^2 = 0$, another odd current G, and the $\mathrm{U}(1)_R$ current J. The most important relation of this algebra is $T = \{Q, G\}$ which says that the energy–momentum is BRST exact, i.e. the theory is topological. The current G is folded into the Beltrami differentials μ_k of Σ_g via $G \cdot \mu_k = \int_{\Sigma_g} \mathrm{d}^2 z G_{zz} {\mu_k}^z{}_{\bar{z}}$, and similarly for the right–movers. Given a basis ω_i generating the Kähler cone in $H^2(X, \mathbb{Z})$, we expand the complexified Kähler class as $\omega = \sum_i t_i \omega_i$, and finally, we define $\phi_i^{(2)} = \{Q, [Q, \phi]\}$, an exactly marginal operator which is represents a deformation of the Kähler structure in the A–model, or a deformation of the complex structure in the B–model. In other words, the t_i are coordinates on the moduli space $\mathcal{M}$ of Kähler structures in the A–model, which by mirror symmetry yield coordinates $z_i = z_i(t)$ of the moduli space of complex structures on the mirror X^* in the B–model.

From these definitions one would naively expect that $\mathcal{F}^{(g)}$ is holomorphic in t. It turns out, however, that the contributions to the integral from the boundary of $\mathcal{M}_g$ are not BRST trivial, and lead to a dependence in $\bar{t}$ which is captured by the holomorphic anomaly equation [2]

$$\bar{\partial}_{\bar{k}} \mathcal{F}^{(g)} = \frac{1}{2} \bar{C}_{\bar{k}}^{ij} \left(D_j D_k \mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_j \mathcal{F}^{(r)} D_k \mathcal{F}^{(g-r)} \right) \tag{1}$$

where $\mathcal{D}_i$ is the covariant derivative on $\mathcal{M}$ with respect to both the Levi–Civita connection for the Weil–Petersson metric $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ and the line bundle $\mathcal{L} \to \mathcal{M}$

with $c_1(\mathcal{L}) = \omega$. From the three–point function C_{ijk} on the sphere, we obtain $\bar{C}^{ij}_{\bar{k}} = e^{2K} G^{i\bar{\imath}} G^{j\bar{\jmath}} \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}}$ is related to the antiholomorphic coupling $\bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}}$, which is symmetric in its indices and fulfils an integrability condition, known as the WDVV equations, $D_{\bar{\imath}}\bar{C}_{\bar{\jmath}\bar{k}\bar{l}} = D_{\bar{\jmath}}\bar{C}_{\bar{\imath}\bar{k}\bar{l}}$. It can therefore be derived from a section S of $\mathcal{L}^{-2}$ as $\bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} = e^{-2K} D_{\bar{\imath}} D_{\bar{\jmath}} D_{\bar{k}} S$. One of the main prerequisites to solve (1) recursively is the construction of S. It is convenient to define as intermediate steps of the above integration $S^i = G^{i\bar{\jmath}}\bar{\partial}_{\bar{\jmath}} S$, $S^{ik} = G^{k\bar{k}}\bar{\partial}_{\bar{k}} S^i$, such that $\bar{C}^{ij}_{\bar{k}} = \partial_{\bar{k}} S^{ij}$. Integrating these differential equations leads to an overdetermined system of equations, which can be solved by using ansatz which turns out be a rational function in $z = z(t)$.

The solution to the holomorphic anomaly equations can be written as [2] schematically as

$$\mathcal{F}^{(g)}(t, \bar{t}) = f(C_{ijk}, C^{(1)}_i, S, S^i, S^{ij}, \chi) + f_g(t) \tag{2}$$

where $f_2(t)$ is the holomorphic ambiguity coming from the integration of (1). We should stress that the solutions for $\mathcal{F}^{(g)}$ in together with the Kähler potential K and the Weil–Petersson metric $G_{i\bar{\jmath}}$ are sufficient to obtain the full $(t, \bar{t})$ dependence of $\mathcal{F}^{(g)}$. However, the determination of $f_g(t)$ is the main difficulty in finding an explicit solution for a given X. The general form of $f_g(t), g > 0$ is expected to be, written in terms of the complex structure moduli $z = z(t)$,

$$f_g(z) = \sum_{i=1}^{D} \sum_{k=0}^{2g-2} \frac{p^{(k)}_i(z)}{{\Delta_i}^k} \tag{3}$$

where D is the number of components Δ_i of the discriminant, and $p^{(k)}_i(z)$ are polynomials of degree k. At present, it is not known how to determine the $p^{(k)}_i(z)$ a priori. Their structure is presumably related to the compactification of the complex structure moduli space $\mathcal{M}$, i.e. it is encoded in the boundary of $\mathcal{M}$.

The only currently known way to derive the free energy of the topological A-model on a Calabi–Yau manifold X

$$\mathcal{F}(\lambda, t, \bar{t}) = \sum_{\lambda=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(t, \bar{t})$$

is to use mirror symmetry, i.e. invert the mirror map $t = t(z)$, and the solutions to the holomorphic anomaly equation just described. $\mathcal{F}^{\mathrm{hol}}(\lambda, t) = \lim_{\bar{t}\to 0} \mathcal{F}^{(g)}(\lambda, t, \bar{t})$ can be expanded in terms of BPS state sums by performing a Schwinger loop calculation [6], [3], [4] in the effective supergravity theory in four dimensions as

$$\mathcal{F}^{\mathrm{hol}}(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(t) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g=0}^{\infty} \sum_{Q,m=1} n^{(g)}_Q \frac{1}{m} \left(2 \sin \frac{m\lambda}{2}\right)^{2g-2} q^{Qm},$$

with $q^Q = \exp(i\langle Q, t\rangle)$. The cubic term $c(t)$ is the classical part of the prepotential $\mathcal{F}^{(0)}$ without the constant term, and $l(t) = \sum_{i=1}^{h^{1,1}} \frac{t_i}{24} \int_X c_2\, \omega_i$ is the classical part of $\mathcal{F}^{(1)}$. If we expand $\mathcal{F}^{\text{hol}}(\lambda,t)$, we find for the genus 2 contribution

$$\mathcal{F}^{(2)}(t)=\frac{\chi}{5760}+\sum_Q\left(\frac{1}{240}n_Q^{(0)}+n_Q^{(2)}\right)\mathrm{Li}_{-1}(q^Q)$$

We can compare this expression with the $\bar{t}\to 0$ limit of (2). However, in order to find the genus 2 instanton numbers $n_Q^{(2)}$ we need in general additional information. We have seen previously that the system of equations determining $\mathcal{F}^{(g)}$ is overdetermined, and we have solved it using an ansatz. Furthermore, we made an ansatz for the holomorphic ambiguity $f_g(t)$ in (3). We therefore need additional consistency checks in order to fix all the ambiguities. We can obtain them from the following six sources. For low degrees Q, some of the $n_Q^{(g)}$ can be computed using classical algebraic geometry [5]. Next, we can use the expected behaviour of the $\mathcal{F}^{(g)}$ near the conifold locus in $\mathcal{M}$, described by the vanishing of Δ_{con}. It is given by an asymptotic expansion of the $c=1$ string at the selfdual radius. Furthermore, at other singularities in the moduli space, a Calabi–Yau manifold X can degenerate to another Calabi–Yau manifold X', having less moduli in general. This degeneration is typically described via a birational map $f: X \dashrightarrow X'$. The instanton numbers in this case are related by $n_{Q'}^{(g)}(X')=\sum_{Q:f(Q)=Q'}n_Q^{(g)}(X)$. Fourth, if X admits an elliptic fibration, we can take the local limit of a large elliptic fiber, and obtain a non-compact Calabi–Yau manifold Y. Therefore, we can use the results for local Calabi–Yau manifolds to check that for $Q\in H_2(Y)\subset H_2(X)$ we have $n_Q^{(g)}(X)=n_Q^{(g)}(Y)$. On the other hand, if X admits a K3 fibration $\mathcal{F}^{\text{hol}}$ has been evaluated using the heterotic-type II duality in the limit where the base of the fibration is large by [6] and [8]. We extend their argument in several directions, and discuss in the remainder of the talk.

For C a curve in the class $[C]$ in the K3 with $C^2=2g-2$ a formula for the topological free energy was given in [5]. It is based on a specific model of the moduli space of M2 branes, which leads to the Hilbert scheme of points on K3 [9]

$$\mathcal{H}^{\text{hol}}(\lambda,t)=\left(\frac{1}{2}\sin(\tfrac{\lambda}{2})\right)^2\prod_{n\geq 1}\frac{1}{(1-e^{i\lambda}q^n)^2(1-q^n)^{20}(1-e^{-i\lambda}q^n)^2}\ .$$

Extending the argument in [5] for $T^2\times$K3 we argue that for regular K3 fibrations the higher genus invariants in the fiber classes are given by

$$\mathcal{F}^{\text{hol}}(\lambda,t)=\frac{\Theta(q)}{q}\mathcal{H}^{\text{hol}}(\lambda,t) \tag{4}$$

where $\Theta(q)$ is determined from the lattice embedding of $i:\mathrm{Pic}(\mathrm{K3})\hookrightarrow H^2(X,\mathbb{Z})$. As such, it depends on global properties of the fibration and in particular not only on Pic(K3). This formula is clearly inspired by the results of heterotic-type II duality [7], [8]. $\Theta(q)$ is related to an automorphic form of the classical duality group $\mathrm{SO}(2,h^{1,1}(X)-1,\mathbb{Z})$ by the Borcherds lifting. One of our main results is that we can construct $\Theta(q)$ purely from the geometric data of the fibration and in particular in cases where the heterotic dual is not known.

For two–parameter K3 fibrations with $\chi = -4(4+7k)$, $k = 1, \dots, 4$, and Pic(K3) = $\langle 2 \rangle$ we find

$$\Theta^{(k)}(q) = 2^{-8} W E_4 \left(4\, W^{12} - (76+7k)\, W^8 X^4 - (180-6k)\, W^4 X^8 - (4-k)\, X^{12}\right)$$

are modular forms for $\Gamma^0(4)$ with $W = \theta_3(\frac{\tau}{2})$, $X = \theta_4(\frac{\tau}{2})$ and $E_4(\tau)$ being an Eisenstein series. This allows us to completely fix $f_2(t)$, e.g. for $k=1$,

$$\begin{aligned} f_2(z_1, z_2) &= \left(\frac{2051}{103680} + \frac{965}{5184} z_2 - \frac{615}{4} z_1 (1 - 4 z_2)\right) \frac{1}{\Delta_s} \\ &\quad + \frac{-2 + 6757 z_1 - 5643648 z_1^2}{60 \Delta_{con}} + \frac{(1 - 1728 z_1)^3}{120 \Delta_{con}^2} \end{aligned}$$

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Formal T-duality for holomorphic non-commutative tori

Tony Pantev

It is commonly believed [3, 7, 8] that the most general deformations of a complex algebraic space X are captured by the deformations of the category $D^b(X)$ of coherent sheaves on X. We investigate the deformations of such derived categories and the equivalences between them in the case when X is a complex analytic torus.

A particular family of infinitesimal deformations of $D^b(X)$ comes from deforming the identity functor on $D^b(X)$. This family is naturally parameterized by the second Hochschild cohomology $HH^2(X)$ of X [6]. By definition $HH^i(X)$ is the cohomology of $R\operatorname{Hom}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$. If X is a manifold, the geometric version of the Hochschild-Kostant-Rosenberg theorem [4,8] identifies $HH^i(X)$ with the coherent cohomology of the holomorphic polyvector fields on X. In particular

$$HH^2(X) \cong H^0(X, \wedge^2 T_X) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X). \tag{1}$$

Viewing $HH^2(X)$ as infinitesimal deformations of $D^b(X)$ we can interpret the pieces in (1) as follows. Elements in $H^1(X, T_X)$ correspond to deformations of X as a complex manifold. Elements in $H^0(X, \wedge^2 T_X)$ correspond to deforming the multiplication on $\mathcal{O}_X$ as an associative $\star$-product. Finally, elements in $H^2(X, \mathcal{O}_X)$ correspond to deforming the trivial $\mathcal{O}^\times$-gerbe on X.

Given two complex manifolds X and Y and an equivalence $\varphi : D^b(X) \to D^b(Y)$ one obtains a natural isomorphism $\tilde{\varphi} : HH^2(X)\tilde{\to}HH^2(Y)$. In particular to every deformation direction $\xi \in HH^2(X)$ for $D^b(X)$ we can associate a deformation direction $\tilde{\varphi}(\xi) \in HH^2(Y)$ for $D^b(Y)$. The question we would like to investigate in general is whether the equivalence $\tilde{\varphi}$ deforms along with $D^b(X)$ and $D^b(Y)$ in the directions ξ and $\tilde{\varphi}(\xi)$.

An important special case is when X is a complex torus, $Y = X^\vee$ is the dual torus, and φ is the classical Fourier-Mukai equivalence. An interesting feature of this case is that $\tilde{\varphi}$ exchanges the non-commutative deformations of X with the gerby deformations of Y and vise versa. Thus the corresponding deformation of φ, if it exists, will have to exchange sheaves of different geometric origin. We carry out this program to show that φ deforms to an equivalence of the derived category of a formal non-commutative deformation of X and the derived category of a formal gerby deformation of $X^\vee$.

More precisely, let $X = V/\Lambda$ be a complex torus equipped with a holomorphic Poisson structure $\mathbf{\Pi}$. Since the holomorphic tangent bundle of a complex torus is trivial, the bitensor $\mathbf{\Pi} \in H^0(X, \wedge^2 T_X)$ will necessarily be translation invariant and hence will be of constant rank on X. The formal $\star$-quantizations of a complex manifold equipped with a Poisson structure of constant rank are known to be parameterized [2, 9, 10] by an affine space. In the case of a Poisson complex torus $(X, \mathbf{\Pi})$ the picture simplifies since one can use the Moyal product to construct a canonical point $X_{\mathbf{\Pi}}$ in the moduli space of quantizations of $(X, \mathbf{\Pi})$. This is the *the Moyal quantization* of $(X, \mathbf{\Pi})$. It is a formal space fibering $X_{\mathbf{\Pi}} \to \mathbb{D}$ over the one dimensional formal disk $\mathbb{D}$.

Next note that for a torus X and the dual torus $X^\vee$ we have a canonical isomorphism $H^0(\wedge^2 T_X) \cong H^2(\mathcal{O}_{X^\vee})$. Let $\boldsymbol{B} := \tilde{\varphi}(\mathbf{\Pi}) \in H^2(\mathcal{O}_{X^\vee})$ be the element corresponding to $\mathbf{\Pi} \in H^0(\wedge^2 T_X)$. An analysis of the $N = 2$ sigma model with target X lead Kapustin [5] to predict that the classical Fourier-Mukai equivalence φ will deform to an equivalence between the derived category of sheaves on a quantum deformation of X corresponding to $\mathbf{\Pi}$ and the derived category of $\boldsymbol{B}$-twisted sheaves on $X^\vee$. (Similar conjecture was made before by D.Orlov.)

In a joint work with O.Ben-Bassat and J.Block [1] we prove a formal version of this conjecture. More presicely, we show that the cohomology class $\exp(\hbar \boldsymbol{B}) \in H^2(\mathcal{O}_{X^\vee}[[\hbar]]^\times)$ classifies an $\mathcal{O}^\times$ gerbe ${}_{\boldsymbol{B}}X^\vee$ on the formal space $X^\vee \times \mathbb{D}$. First we prove the following geometric statement:

Theorem. *Let X be a complex analytic torus, $X^\vee$ be the dual torus and $\boldsymbol{\Pi} \in H^0(\wedge^2 T_X)$ and $\boldsymbol{B} \in H^2(X^\vee, \mathcal{O}_\vee)$ be two matching infinitesimal deformations of the categories $D^b(X)$ and $D^b(X^\vee)$ respectively. Then*

(1) *The gerbe ${}_{\boldsymbol{B}}X^\vee \to \mathbb{D}$ admits a natural group structure. With this structure ${}_{\boldsymbol{B}}X^\vee$ is a group stack (in general a non-commutative one) over $\mathbb{D}$, wich is an extension of the commutative group space $X^\vee \times \mathbb{D} \to \mathbb{D}$ by the commutative group stack $B\mathcal{O}_{\mathbb{D}}^\times$.*

(2) *The relative Picard stack $\mathcal{P}ic^0(X_{\boldsymbol{\Pi}}/\mathbb{D})$, parameterizing line bundles on the formal non-commutative space, is a formal geometric stack (in the sense of Artin) and is canonically isomorphic to ${}_{\boldsymbol{B}}X^\vee$ as a group stack over $\mathbb{D}$.*

(3) *There is a universal line bundle $\mathcal{P}$ on the product $X_{\boldsymbol{\Pi}} \times_{\mathbb{D}} {}_{\boldsymbol{B}}X^\vee$ which over the closed point $0 \in \mathbb{D}$ restricts to the standard Poincare bundle on $X \times X^\vee$.*

Finally, using the existence of $\mathcal{P}$ we prove a formal variant of the Kapustin-Orlov conjecture:

Theorem. *The Poincare sheaf $\mathcal{P}$ defines an integral transform between the derived category of coherent sheaves on the non-commutative space $X_{\boldsymbol{\Pi}}$ and the derived category of weight one coherent sheaves on the gerbe ${}_{\boldsymbol{B}}X^\vee$. This equivalence is a formal deformation of the standard Fourier-Mukai transform.*

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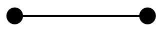
Quivers

Alistair King

 $\leftrightsquigarrow$ family of algebras $A_{n,m} = \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}$

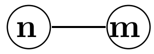
Morita equivalent to basic $A = \underbrace{\mathbb{C}e_1 \oplus \mathbb{C}e_2}_{S_1 \oplus S_2} \oplus \underbrace{\mathbb{C}x}_{X_{12}}$

structure:

$$\begin{array}{ccc} {e_1}^2 = e_1 & {e_2}^2 = e_2 & e_1e_2 = 0 = e_2e_1 \\ e_1x = x & xe_2 = x & x^2 = 0 \\ e_2x = 0 & xe_1 = 0 & \end{array}$$

$S = S_1 \oplus S_2$ semisimple, $X = X_{12}$ S, S-bimodule, $X \otimes_s X = 0$

e.g.

for general A_{nm}, $S = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}$, $X = \begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix}$, one might write 

For general Q,

$$S = \bigoplus_i S_i \qquad \textit{and} \qquad X = \bigoplus_{i,j} X_{ij}$$

$$\begin{aligned} A &= S\langle X \rangle \qquad \text{tensor algebra of } X \text{ over } S \\ &= S \oplus X \oplus (X \otimes_s X) \oplus \ldots \end{aligned}$$

Representations. V is a right A-module $\leftrightsquigarrow$ $V = Ve_1 \oplus Ve_2$ and $\phi_x : Ve_1 \to Ve_2$.

$$ve_1x = vx \qquad vxe_2 = vx$$

Similarly left A-module $\leftrightsquigarrow$ $\psi x : e_2V \to e_1V$

More invariantly

$V_A \leftrightsquigarrow V_S$ and right S-map $V \otimes_S X \to V$

${}_AV \leftrightsquigarrow {}_SV$ and left S-map $X \otimes_S V \to V$

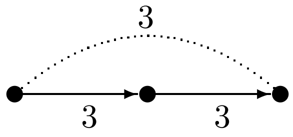
Theorem. *For any finite-dimensional algebra A over C,*

$\{\{A$-modules$\}\}$ equivalent to $\{\{$ representations of Q with relarions$\}\}$

e.g. $X_{ij} = Ext^1(S_i, S_j)^*$

Quivers with Relations.

Example

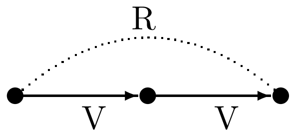


Arrows: x_1 x_2 / y_1 y_2 / z_1 z_2

Relations:

$$y_1z_2 - z_1y_2 = 0$$
$$z_1x_2 - x_1z_2 = 0$$
$$x_1y_2 - y_1x_2 = 0$$

invariantly



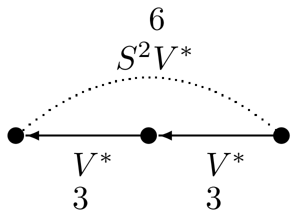
$R = \Lambda^2 V \subseteq V \otimes V$

$$\begin{aligned} S &= C^{\{1,2,3\}} \\ X_{12} &= X_{23} = V \\ A &= S\langle X \rangle/(R) \end{aligned}$$

Theorem. *(Beilinson)* $D(A) \cong D(P^2)$

Koszul Duality.

B:



$D(B) \cong D(A)$

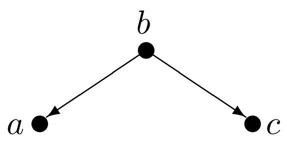
$(0 \to S^2V^* \to V^* \otimes V^* \to R^* \to 0)$ mutation [Bondal...]

R comes from A_∞-structure on $\mathrm{Ext}^i(S,S)$ (Feynman diagram in string theory)

$$\bigoplus_{n\geq 2} Ext^1(S,S)^{\otimes n} \longrightarrow Ext^2(S,S)$$

Anslander-Reiten Quiver.

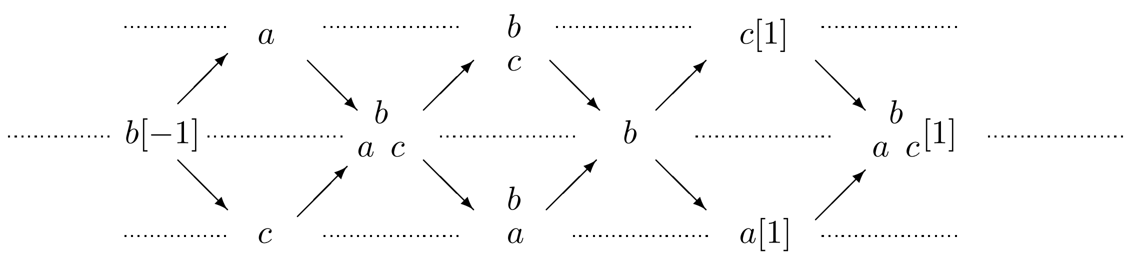
e.g. A



$$\begin{array}{llll} \text{Projectives:} & P_a = a & P_b = \begin{smallmatrix} & b & \\ a & & c \end{smallmatrix} & P_c = c \\ \text{Simples:} & S_a = a & S_b = b & S_c = c \end{array}$$

$$Hom(P_i, P_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Start from projectives and knit



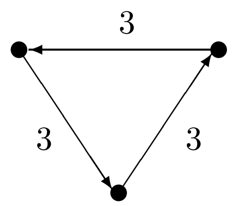
by short exact sequences, e.g.

$$0 \to a \to \begin{smallmatrix} & b & \\ a & & c \end{smallmatrix} \to \begin{smallmatrix} b \\ c \end{smallmatrix} \to 0$$

$$0 \to \begin{smallmatrix} & b & \\ a & & c \end{smallmatrix} \to \begin{smallmatrix} b \\ c \end{smallmatrix} \oplus \begin{smallmatrix} b \\ a \end{smallmatrix} \to b \to 0$$

Quivers in String Theory (With $N = 1$ super symmetry).

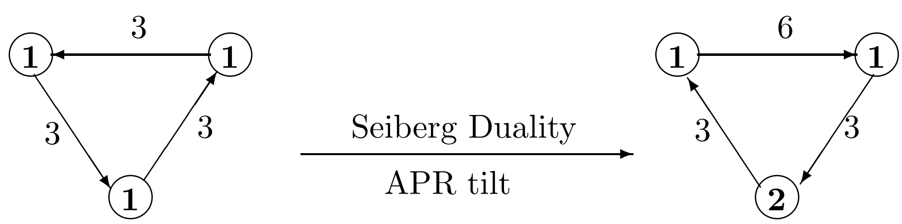
e.g. C^3/Z_3 orbifold



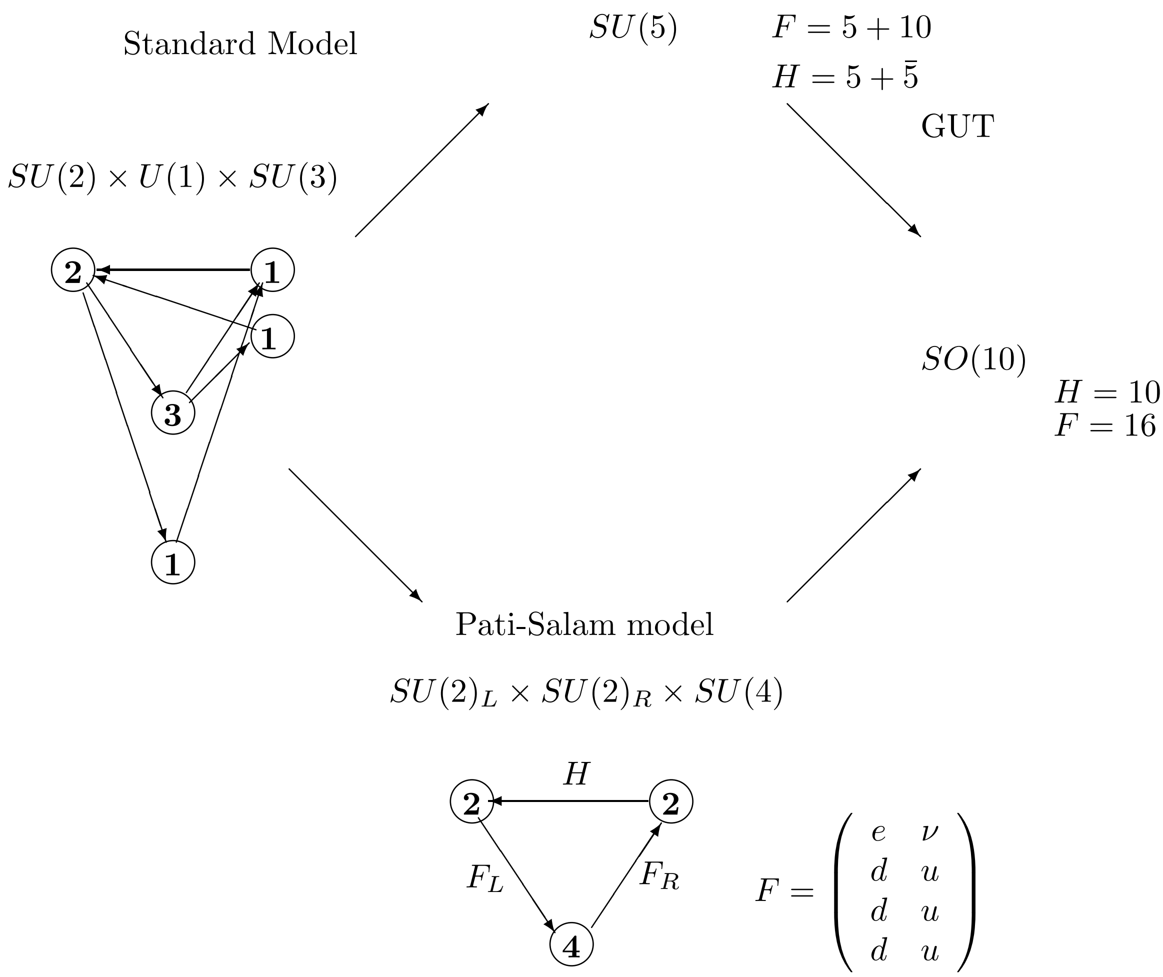
relations are implicit from $CY3$ and dual to the arrows. (Serre duality in dim 3)

$Ext^2(S_i, S_j) \cong Ext^1(S_j, S_i)^*$ and $D\left(\frac{L\mathbb{C}^3}{Z_3}\right) \cong D(Y)$

Theorem. *(McKay)*
Crepant resolution of $\mathbb{C}/\mathbb{Z}_3$ *is* $Y = \text{total space of} - \omega_{\mathbb{P}^2}$



Quivers in Particle Physics.



Mirror Symmetry and Non-Archimedean Analytic Spaces

Yan Soibelman
(joint work with Maxim Kontsevich)

1. An integral affine structure on a manifold of dimension n is given by a torsion-free flat connection with the monodromy reduced to $GL(n, \mathbf{Z})$.

There are two basic situations in which integral affine structures occur naturally. One is the case of classical integrable systems . Most interesting for us is a class of examples arising from analytic manifolds over non-archimedean fields. It is motivated by the approach to Mirror Symmetry suggested by us few years ago. From our point of view manifolds with integral affine structure appear in Mirror Symmetry in two ways. One considers the Gromov-Hausdorff collapse of degenerating families of Calabi-Yau manifolds. The limiting space can be interpreted either as a contraction of an analytic manifold over a non-archimedean field of Laurent series $\mathbf{C}((t))$, or as a base of a fibration of a Calabi-Yau manifold by Lagrangian tori (with respect to the symplectic Kähler 2-form). On a dense open subset of the limiting space one gets two integral affine structures associated with two interpretations, the non-archimedean one and the symplectic one. Mirror dual family of degenerating Calabi-Yau manifolds should have metrically the same Gromov-Hausdorff limit, with the roles of two integral affine structures interchanged.

Very interesting question arises: how to reconstruct these families of Calabi-Yau manifolds from the corresponding manifolds with integral affine structures? This question was one of the main motivations for our project.

2. Our approach to the reconstruction of analytic Calabi-Yau manifolds from real manifolds with integral affine structure can be illustrated in the following toy-model example. Let $S^1 = \mathbf{R}/\mathbf{Z}$ be a circle equipped with the induced from $\mathbf{R}$ affine structure. We equip S^1 with the canonical sheaf $\mathcal{O}_{S^1}^{can}$ of Noetherian $\mathbf{C}((q))$-algebras. By definition, for an open interval $U \subset S^1$ algebra $\mathcal{O}_{S^1}^{can}(U)$ consists of formal series $f = \sum_{m,n \in \mathbf{Z}} a_{m,n} q^m z^n,\ a_{m,n} \in \mathbf{C}$ such that $\inf_{a_{m,n} \neq 0}(m + nx) > -\infty$. Here $x \in \mathbf{R}$ is any point in a connected component of the pre-image of U in $\mathbf{R}$, the choice of a different component $x \to x + k,\ k \in \mathbf{Z}$ corresponds to the substitution $z \mapsto q^k z$. The corresponding analytic space is the Tate elliptic curve $(E, \mathcal{O}_E)$, and there is a continuous map $\pi : E \to S^1$ such that $\pi_*(\mathcal{O}_E) = \mathcal{O}_{S^1}^{can}$.

In the case of K3 surfaces one starts with S^2. The corresponding integral affine structure is well-defined on the set $S^2 \setminus \{x_1, ..., x_{24}\} \subset S^2$, where $x_1, ..., x_{24}$ are distinct points. Similarly to the above toy-model example one can construct the canonical sheaf $\mathcal{O}_{S^2 \setminus \{x_1,...,x_{24}\}}^{can}$ of algebras, an open 2-dimensional smooth analytic surface X' with the trivial canonical bundle (Calabi-Yau manifold), and a continuous projection $\pi' : X' \to S^2 \setminus \{x_1, ..., x_{24}\}$ such that $\pi'_*(\mathcal{O}_{X'}) = \mathcal{O}_{S^2 \setminus \{x_1,...,x_{24}\}}^{can}$. The problem is to find a sheaf $\mathcal{O}_{S^2}$ whose restriction to $S^2 \setminus \{x_1, ..., x_{24}\}$ is locally isomorphic to $\mathcal{O}_{S^2 \setminus \{x_1,...,x_{24}\}}^{can}$, an analytic compact K3 surface X, and a continuous projection $\pi : X \to S^2$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_{S^2}$. We call this problem (in general case) the Lifting Problem. At this time we do not know the conditions one should impose on singularities of the affine structure, so that the Lifting Problem would have a solution. We have solved it in the case of K3 surfaces . Here the solution is non-trivial and depends on data which are not visible in the statement of the problem. They are motivated by Homological Mirror Symmetry and consist, roughly speaking, of an infinite collection of trees embedded into $S^2 \setminus \{x_1, ..., x_{24}\}$ with the tail vertices belonging to the set $\{x_1, ..., x_{24}\}$. The sheaf $\mathcal{O}_{S^2 \setminus \{x_1,...,x_{24}\}}^{can}$ has to be modified by means of automorphisms assigned to every edge of a tree and then glued together with certain model sheaf near each singular point x_i.

3. The relationship between K3 surfaces and singular affine structures on S^2 is of very general origin. Starting with a projective analytic Calabi-Yau manifold X over a complete non-archimedean local field K one can canonically construct a PL manifold $Sk(X)$ called the skeleton of X. If X is a generic K3 surface then $Sk(X)$ is S^2. The group of birational automorphisms of X acts on $Sk(X)$ by integral PL transformations. For $X = K3$ we obtain an action of an arithmetic subgroup of $SO(1,18)$ on S^2. Further examples should come from Calabi-Yau manifolds with large groups of birational automorphisms.

Higher genus Topological String Amplitudes on CY-3 folds

Albrecht Klemm

The supersymmetric σ-model on a Calabi-Yau threefold M admits two topological theories in which the world sheet spin operator is twisted either by the vector $U(1)_V$ or the axial $U(1)_A$ of the $N = (2,2)$ super conformal world sheet algebra, so that either $Q_A = G_+ + \bar{G}_+$ or $Q_B = G_+ + \bar{G}_-$ becomes a scalar BRST operator, which is well defined on all world sheet genera. While $U(1)_V$ exists on any sympletic manifold M the $U(1)_A$ becomes anomalous in the quantized supersymmetric σ-model unless M is Calabi-Yau manifold. One calls the topological theories in which the observables are defined by the cohomology of $Q_{A/B}$ the A- and the B-model respectively [1].

For the A-model the observables can be identifies with $H^*(M)$. The functional integral over the σ-model maps localizes on the holomorphic embeddings of

$$\Phi : \Sigma_g \to M$$

and the path integral becomes merely a integral over the moduli space of those maps. The virtual dimension of the integral is $vdim\overline{\mathcal{M}}_g(\beta) = \int_{[\mathcal{C}]} c_1(TM) + (1-g)(3 - \dim_{\mathbb{C}}(M)$ is zero for all maps in the special case of CY 3folds. That is the partition function $Z = \int \mathcal{D}\Phi \exp(-S(\Phi, G, B))$ gets, superficially discrete, contributions from all genus g curves, which makes the CY 3fold case especially interesting. The integrals

$$r^g_\beta = \int_{\overline{\mathcal{M}}_g(\beta)} c^{vir}(g,\beta)$$

over the stable compactification of maps from a genus g Riemann surface Σ_g into a curve $\mathcal{C} \in M$ with $[\mathcal{C}] = \beta \in H^2(M,\mathbb{Z})$ are called Gromow-Witten invariants. The free energy $F = \log(Z)$ of the topological A-model has a natural expansion

$$F(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \sum_\beta r^g_\beta q^\beta$$

in the string coupling λ and $q^\beta = \exp[2\pi i \int_{[\mathcal{C}]} B + iJ] = \exp[2\pi i \sum_i t_i d_i]$. Here J is the Kählerform on M and B is the two-form of the Neveu-Schwarz antisymmetric background field, t_i are complexified Kählerparameters and $d_i \in \mathbb{Z}$,

$i = 1, \ldots, h_{11}(M)$ are the degrees specifying $\beta \in H^2(M, \mathbb{Z})$. $F(\lambda, t)$ is not not completely independent of the metric G (and the background field B) on M, but is rather a family index on a complexified symplectic family of M. It is however independent of the complex structure of M. Mirror symmetry maps it to a family index on the mirror manifold W, which does not depend on the sympletic structure but only on the complex structure of W and is calculable in the topological B model, see [2] for advanced methods.

The $r^g_\beta \in \mathbb{Q}$ are sympletic invariants. Recently there have been conjectures [4] that the following sympletic invariants carry the same information: The *Gromow-Witten invariants* above, the *Gopakumar-Vafa invariants*, which are alternating sums in the dimensions of the cohomology groups of the moduli space of $D2$-$D0$ branes [3], and the *Donaldson-Thomas* invariants which are virtual integrals over the moduli space of torsions free sheaves on M [5]. The precise nature of the conjecture is as follows. Gopakumar and Vafa define their invariants n^g_β as

$$F(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{g=0}^{\infty} \sum_{\substack{\beta \\ m=1}} n^g_\beta (-1)^{g-1} \frac{[m]^{(2g-2)}}{m} q^{\beta m} ,$$

where $[x] := q_\lambda^{\frac{x}{2}} - q_\lambda^{-\frac{x}{2}}$ and $q_\lambda = e^{i\lambda}$. The observation that $n^g_\beta \in \mathbb{Z}$ in many examples fits well with the interpretation of n^g_β as counting BPS states. In simple cases the direct calculation of the n^g_β from the moduli space of $D2$-$D0$ branes have been performed [6]. The partition function $Z = \exp(F)$ has the following product form [1]

$$Z_{GW} = \prod_{\beta}^{\infty} \left[\left(\prod_{r=1}^{\infty} (1 - q_\lambda^r q^Q)^{r n^{(0)}_\beta} \right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1 - q_\lambda^{g-1-\frac{l}{2}} q^Q)^{(-1)^g \binom{2g-2}{l} n^{(g)}_\beta} \right] .$$

This product form resembles the Hilbert scheme of symmetric products written in terms of partition sums over free fermionic and bosonic fields as well as the closely related product form for the elliptic genus of symmetric products. The constant map piece can be written as $M(q_\lambda)^{-\frac{\chi(M)}{2}}$ where $M(q) = \prod_{m>0}(1 - q^m)^{-m}$ is the McMahon function. The relation with the Donaldson-Thomas invariants $\tilde{n}^m_\beta \in \mathbb{Z}$ with the expansion

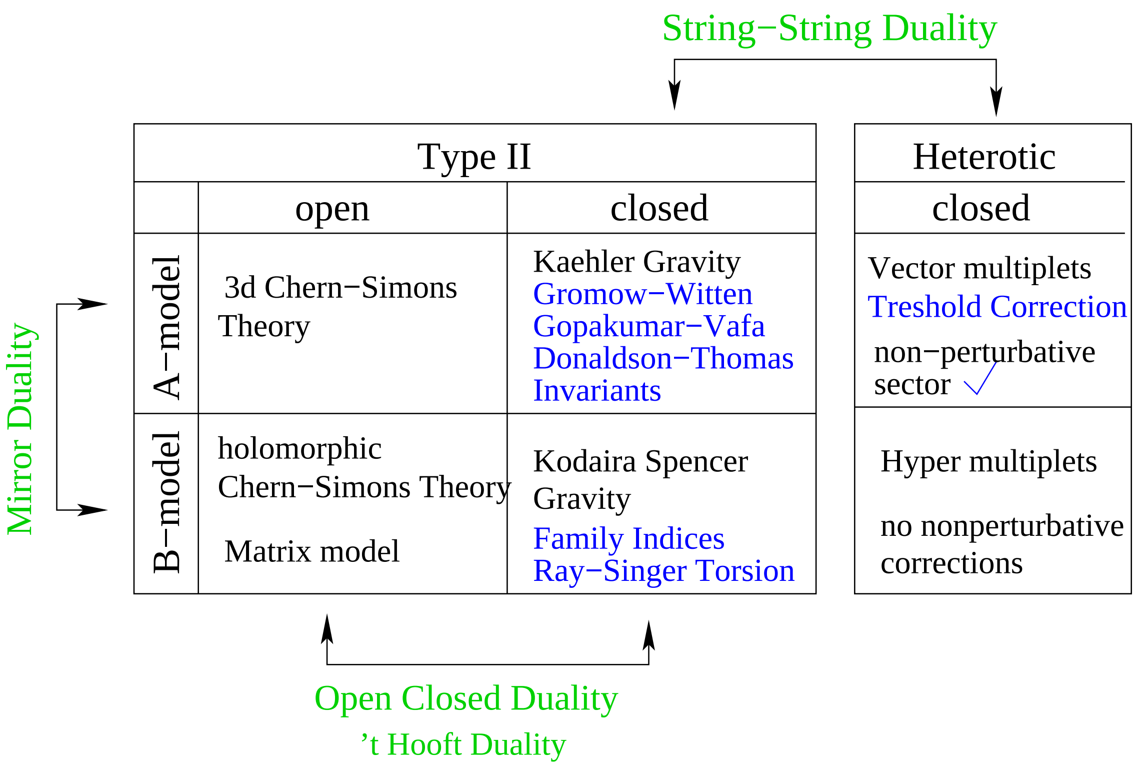
$$Z_{DT}(M, q_\lambda, q) = \sum_{\beta, m \in \mathbb{Z}} \tilde{n}^m_\beta q^m_\lambda q_\beta$$

is simply

$$Z_{GW} = (M, q_\lambda, q) M(q_\lambda)^{\frac{\chi(M)}{2}} = Z_{DT}(M, -q_\lambda, q) .$$

We next present an overview over techniques to perform the calculation of higher genus amplitudes and determine these symplectic invariants explicetly.

[1] Here we ignored an $\exp(\frac{c(t)}{\lambda^2} + l(t))$ factor.



Beside the techniques of direct evaluating of the integral over the moduli of maps by equivariant localization w.r.t. to an induced torus action in M pioneered by Kontsevich, there are methods using the dualities indicated in the above scheme. We will focus on the open closed dualities or large N transitions methods.

On non-compact toric Calabi-Yau like the total space of a line bundle $M = \mathcal{O}(K_B) \to B$ over a toric base B the problem has recently be completly solved using the topological Vertex [7]. These non-compact toric Calabi-Yau M can be patched by $\mathbb{C}^3$ patches in a way, which is compatible with a gobal T^2 fibration. The T^2 fiber and the a and b cycles in $H^1(T^2)$ are generated by the torus action in every patch and the global data are encoded by the gluing of 1-cycles between a patch and its three adjacent patches. With the vertex one can solve for the all genus partition function on an arbitrary open toric CY variety in two steps

(1) The vertex calculates the most general open string amplitude in each patch with arbitrary genus and arbitrary boundary conditions on three stacks of Harvey-Lawson Special Lagrangian 3-cycles. The answer in a specific framing can be written in the form

$$Z_{R_1,R_2,R_3}(q_\lambda) = q_\lambda^{\kappa(R_1)+\kappa(R_3)} S_{R_3^t}(q_\lambda^\rho) \sum_Q S_{R_1^t/Q}(q_\lambda^{R_3+\rho}) S_{R_2/Q}(q_\lambda^{R_3^t+\rho}) \, .$$

Here the boundary conditions of the open string are labeled by three representations R_i of $U(N_i)$, where N_i is number of branes in the it'h stack. $S_R(\underline{x})$ and $S_{R/Q}(\underline{x})$ are Schur functions and relative Schur functions respectively. Their arguments are the sets $q_\lambda^{R+\rho} = \{q^{l_i(R)_\lambda - i + \frac{1}{2}}\}$ and $q_\lambda^\rho =$

$\{q_\lambda^{\frac{1}{2}-i}\}$, where $l_i(R)$ is the length of the i'th row of the Young-Tableaux of the representation R and $\kappa(R) = |R| + \sum_i l_i(R)(l_i(R) - 2j)$ with $|R|$ the total number of boxes in the Young-Tableaux. The all genus dependence can be made explicit by expanding $Z_{R_1,R_2,R_3}(q_\lambda) = \sum_g \lambda^{2g-2+h} Z^{(g)}_{R_1,R_2,R_3}$ with h the number of holes. This open string amplitude is related by a large N transition to Chern-Simons link invariants. The framing dependence of the link invariants is reflected by an ambiguity in the definition of the compactification of the open string moduli space.

(2) The vertex amplitudes have a natural gluing rule, when two patches are joint [7]. More generally from two partition functions $Z_{\Gamma_R}(q_\lambda)_Q$ and $Z_{\Gamma_L}(q_\lambda)_Q$ for toric varieties with graphs Γ_R and Γ_L with arbitrary boundary conditions Q one can obtain the partition function of the total graph as

$$Z_{\Gamma_R \cup \Gamma_L}(q_\lambda) = \sum_Q Z_{\Gamma_L}(q_\lambda)_Q e^{-|Q|t} (-1)^{(n_Q+1)|R|} q^{n_Q \frac{\kappa(Q)}{2}} Z_{\Gamma_R}(q_\lambda)_{Q^t} \ ,$$

where t is the complexified Kähler parameter, which corresponds to the class of the edge joining the graphs. Q^t is the representation corresponding to transposed Young-Tableaux of Q and $n_Q \in \mathbb{Z}$ is the framing choice. In the gluing procedure these framings are related to the gluing of the 1-cycles of the T^2 fiber. The expression between $Z_{\Gamma_L}(q_\lambda)_Q$ and $Z_{\Gamma_R}(q_\lambda)^t_Q$ can be viewed as a "propagator" for Feynman rules for gluing the amplitudes. Summing over all open string boundaries gives a closed string amplitude.

We further discussed the the identification of the vertex with the partition function of a melting crystal [8], where the string λ gets identified with the Boltzmann weight and its relation to the 3KP hierarchy [7].

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