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Komplexe Analysis

Organised by
Jean-Pierre Demailly (Grenoble)
Klaus Hulek (Hannover)
Thomas Peternell (Bayreuth)

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Introduction by the Organisers

The Komplexe Analysis workshop, organised by J.P. Demailly (Grenoble), K. Hulek (Hannover) and Th. Peternell (Bayreuth), was held August 22-27. The meeting was attended by over 40 participants from many European countries, USA, Japan and Korea, among them quite a number of young mathematicians. The spectrum of the meeting was very broad, ranging from analytic question, e.g. Levi flat hypersurfaces and non-Kaehler geometry to classification theory, classical algebraic geometry and numerical aspects of algebraic geometry.

Workshop: Komplexe Analysis

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Abstracts

Application of complex analysis to oscillating integrals

Daniel Barlet

Let $(X_{\mathbb{R}}, 0)$ be a germ of real analytic subset in $(\mathbb{R}^N, 0)$ of pure dimension $n+1$ with an isolated singularity at 0. Let

$$(f_{\mathbb{R}}, 0) : (X_{\mathbb{R}}, 0) \longrightarrow (\mathbb{R}, 0)$$

a real analytic germ with an isolated singularity at 0, such that its complexification $f_{\mathbb{C}}$ vanishes on the singular set S of $X_{\mathbb{C}}$. We also assume that $X_{\mathbb{R}} - \{0\}$ is orientable.

To each $A \in H^0(X_{\mathbb{R}} - f^{-1}(0), \mathbb{C})$ we associate a $n-$cycle $\Gamma(A)$ ("explicitly" described) in the complex Milnor fiber of $f_{\mathbb{C}}$ at 0 such that the non trivial terms in the asymptotic expansions of the oscillating integrals $\int_A e^{i\tau f(x)}\varphi(x)$ when $\tau \to \pm\infty$ can be read from the spectral decomposition of $\Gamma(A)$ relative to the monodromy of $f_{\mathbb{C}}$ at 0.

The use of the Gauss-Manin connection associated to $f_{\mathbb{C}}$ in this classical problem already appears in [M.74]. Special cases of the results decribed here were obtained in [B.M. 02] (with also a "complex version") and [B.03].

We consider also the case where $\partial A = \{0\}$ (corresponding to an A coming from $H^0(X_{\mathbb{R}} - \{0\}, \mathbb{C})$). The terms in the asymptotic expansion of $\int_A e^{i\tau f(x)}\varphi(x)$ which are negative powers of τ are also describe in this case (they are not interesting without the hypothesis $\partial A = \{0\}$) by the mean of a "natural" lift of the (closed) cycle $\Gamma(A)$ to a compact cycle in the Milnor'fiber of $f_{\mathbb{C}}$ at 0. This last result generalizes [B.99].

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Beauville surfaces without real structures and group theory

Ingrid C. Bauer

(joint work with F. Catanese and F. Grunewald)

In [Bea] (see p. 159) A. Beauville constructed a new surface of general type with $K^2 = 8,\ p_g = 0$ as a quotient of the product of two Fermat curves of degree 5 by the action of the group $(\mathbb{Z}/5\mathbb{Z})^2$. Inspired by this construction, in the article [Cat00], dedicated to the geometrical properties of varieties which admit an unramified covering biholomorphic to a product of curves, the following definition was given:

Definition. *A* Beauville surface *is a compact complex surface S which*
1) is rigid*, i.e., it has no nontrivial deformation,*
2) is isogenous to a higher product*, i.e., it is a quotient $S = (C_1 \times C_2)/G$ of a product of curves of resp. genera ≥ 2 by the free action of a finite group G.*

Moreover, the following was proven:

Theorem. *(Catanese) Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S. The corresponding moduli space $M_S^{top} = M_S^{diff}$ is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

If S is a Beauville surface this implies: $X \cong S$ or $X \cong \bar{S}$.
We adress the following problems:

Question 1. *Existence and classification of Beauville surfaces, i.e., a) which finite groups G can occur?*
b) classify all possible Beauville surfaces for a given finite group G.

Question 2. *Is the Beauville surface S biholomorphic to its complex conjugate surface $\bar{S}$?*
Is S real (i.e., does there exist a biholomorphic map $\sigma : S \to \bar{S}$ with $\sigma^2 = id$)?

Remark. *Beauville surfaces which are not biholomorphic to their complex conjugate surfaces are counterexamples to the* Friedman-Morgan speculation *(1987) that two algebraic surfaces are diffeomorphic if and only if they are in the same connected component of the moduli space.*
For different counterexamples to the above compare [Cat03]*,* [C-W04]*,* [K-K02]*,* [Man01].

In order to reduce the description of Beauville surfaces to some group theoretic statement, we need to recall that surfaces isogenous to a higher product belong to two types:
1) S is of **unmixed type** if the action of G does not mix the two factors, i.e., it is the product action of respective actions of G on C_1, resp. C_2.
2) S is of **mixed type**, i.e., C_1 is isomorphic to C_2, and the subgroup G^0 of transformations in G which do not mix the factors has index precisely 2 in G.
The datum of a Beauville surface can be described group theoretically.

Definition. *Let G be a finite group.*
1) A quadruple $v = (a_1, c_1; a_2, c_2)$ of elements of G is an unmixed Beauville structure for G *if and only if*
(i) the pairs a_1, c_1, and a_2, c_2 both generate G,

(ii) $\Sigma(a_1, c_1) \cap \Sigma(a_2, c_2) = \{1_G\}$, *where*

$$\Sigma(a, c) := \bigcup_{g \in G} \bigcup_{i=0}^{\infty} \{ga^i g^{-1}, gc^i g^{-1}, g(ac)^i g^{-1}\}.$$

We write $\mathbb{U}(G)$ *for the set of unmixed Beauville structures on* G.

2) A mixed Beauville quadruple for G *is a quadruple* $M = (G^0; a, c; g)$ *consisting of a subgroup* G^0 *of index* 2 *in* G, *of elements* $a, c \in G^0$ *and of an element* $g \in G$ *such that*

i) G^0 *is generated by* a, c,

ii) $g \notin G^0$,

iii) for every $\gamma \in G^0$ *we have* $g\gamma g\gamma \notin \Sigma(a, c)$.

iv) $\Sigma(a, c) \cap \Sigma(gag^{-1}, gcg^{-1}) = \{1_G\}$.

We call $\mathbb{M}(G)$ *the set of mixed Beauville quadruples on the group* G.

Remark. *1) Every Beauville structure on a finite group* G *gives rise to a Beauville surface. From (1) i) we obtain two Galois coverings* $\lambda_i : C(a_i, c_i) \to \mathbb{P}^1$ *(Riemann's existence theorem) and Condition (1), ii) assures that the action of* G *on* $C(a_1, c_1) \times C(a_2, c_2)$ *is free.*

2) Let be $\iota(a_1, c_1; a_2, c_2) = (a_1^{-1}, c_1^{-1}; a_2^{-1}, c_2^{-1})$. *Then* $S(\iota(v)) = \overline{S(v)}$. *(Note that* $\bar{\alpha} = \alpha^{-1}$, $\bar{\gamma} = \gamma^{-1}$.*)*

Proposition. *Let* G *be a finite group and*

$v = (a_1, c_1; a_2, c_2) \in \mathbb{U}(G)$.

Assume that $\{\mathrm{ord}(a_1), \mathrm{ord}(c_1), \mathrm{ord}(a_1c_1)\} \neq \{\mathrm{ord}(a_2), \mathrm{ord}(c_2), \mathrm{ord}(a_2c_2)\}$ *and that* $\mathrm{ord}(a_i) < \mathrm{ord}(a_ic_i) < \mathrm{ord}(c_i)$. *Then* $S(v) \cong \overline{S(v)}$ *if and only if there are inner automorphisms* ϕ_1, ϕ_2 *of* G *and an automorphism* $\psi \in Aut(G)$ *such that, setting* $\psi_j := \psi \circ \phi_j$, *we have* $\psi_1(a_1) = a_1^{-1}$, $\psi_1(c_1) = c_1^{-1}$, *and* $\psi_2(a_2) = {a_2}^{-1}$, $\psi_2(c_2) = {c_2}^{-1}$.

In particular $S(v)$ *is isomorphic to* $\overline{S(v)}$ *if and only if* $S(v)$ *has a real structure.*

Remark. *Dropping the assumption on the orders of* a_i, c_i, *we can define a finite permutation group* $\mathrm{A}_{\mathbb{U}}(G)$*such that for* $v, v' \in \mathbb{U}(G)$ *we have :* $S(v) \cong S(v')$ *if and only if* v *is in the* $\mathrm{A}_{\mathbb{U}}(G)$*-orbit of* v'.

Theorem. *The following groups admit unmixed Beauville structures* v *such that* $S(v)$ *is not biholomorphic to* $\overline{S(v)}$*:*

1. the symmetric group $\mathfrak{S}_n$ *for* $n \geq 8$ *and* $n \equiv 2$ *mod* 3,

2. the alternating group $\mathfrak{A}_n$ *for* $n \geq 16$ *and* $n \equiv 0$ *mod* 4, $n \equiv 1$ *mod* 3, $n \not\equiv 3, 4$ *mod* 7.

The following result shows that there are Beauville surfaces which are biholomorphic to their complex conjugate surface, but do not have a real structure.

Theorem. *Let* $p > 5$ *be a prime with* $p \equiv 1$ *mod* 4, $p \not\equiv 2, 4$ *mod 5,* $p \not\equiv 5$ *mod* 13 *and* $p \not\equiv 4$ *mod* 11. *Set* $n := 3p + 1$. *Then there is an unmixed Beauville surface* S *with group* $\mathfrak{A}_n$ *which is biholomorphic to the complex conjugate surface* $\bar{S}$, *but is not real.*

For **mixed** Beauville surfaces the situation is more complicated, but we succeed to give a general construction for finite groups admitting a mixed Beauville structure.

Let H be non-trivial group. Let $\Theta : H \times H \to H \times H$ be the automorphism defined by $\Theta(g,h) := (h,g)$ $(g,h \in H)$. We consider the semidirect product $H_{[4]} := (H \times H) \rtimes \mathbb{Z}/4\mathbb{Z}$ where the generator 1 of $\mathbb{Z}/4\mathbb{Z}$ acts through Θ on $H \times H$. Since Θ^2 is the identity we find $H_{[2]} := H \times H \times 2\mathbb{Z}/4\mathbb{Z} \cong H \times H \times \mathbb{Z}/2\mathbb{Z}$ as a subgroup of index 2 in $H_{[4]}$.

We have now

Lemma. *Let H be a non-trivial group and let $a_1, c_1,\ a_2, c_2$ be elements of H. Assume that*

1. the orders of a_1, c_1 are even,
2. a_1^2, a_1c_1, c_1^2 generate H,
3. a_2, c_2 also generate H,
4.$(\mathrm{ord}(a_1) \cdot \mathrm{ord}(c_1) \cdot \mathrm{ord}(a_1c_1), \mathrm{ord}(a_2) \cdot \mathrm{ord}(c_2) \cdot \mathrm{ord}(a_2c_2)) = 1$.

Set $G := H_{[4]}$, $G^0 := H_{[2]}$ as above and $a := (a_1, a_2, 2)$, $c := (c_1, c_2, 2)$. Then $(G^0; a, c)$ is a mixed Beauville structure on G.

As an application we find the following examples

Theorem. *Let p be a prime with $p \equiv 3$ mod 4 and $p \equiv 1$ mod 5 and consider the group $H := \mathbf{SL}(2, \mathbb{F}_p)$. Then $H_{[4]}$ admits a mixed Beauville structure u such that $S(u)$ is not biholomorphic to $\overline{S(u)}$.*

Remark. *Note that the smallest prime satifying the above congruences is $p = 11$ and we get that G has order equal to* 6969600.

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Extension of equivariant vector bundles

Michel Brion

Consider a homogeneous space $X_0 = G/K$, where G is a complex linear algebraic group and K is a complex algebraic subgroup. Let $\mathcal{V}_0$ be a G-equivariant complex vector bundle on X_0 and let X be a G-equivariant compactification of X_0 (i.e., X is a compact complex algebraic variety where G acts with an open orbit isomorphic to X_0). Does $\mathcal{V}_0$ extend to an equivariant vector bundle on X ?

This question arises naturally when studying equivariant vector bundles on quasi-homogeneous varieties. It was raised by Kostant in the setting where G is an adjoint semisimple group, K is the fixed point subgroup of an involutive automorphism θ of G, and X is the wonderful compactification of the symmetric space G/K (as introduced in [3]). In fact, Kostant asked for a canonical extension, in view of applications to representation theory of real reductive groups. I refer to Syu Kato's paper [5] for a precise formulation of his question, and a positive answer in the case of an adjoint semisimple group K regarded as the symmetric space $K \times K/diag(K)$.

Returning to the general setting, it is easy to see that $\mathcal{V}_0$ extends to an equivariant vector bundle on some equivariant compactification $X(\mathcal{V}_0)$ (depending on $\mathcal{V}_0$); then $\mathcal{V}_0$ extends to those compactifications which have an equivariant map onto $X(\mathcal{V}_0)$. As a consequence, $\mathcal{V}_0$ extends to an arbitrary compactification, but only as a coherent G-linearized sheaf. However, $\mathcal{V}_0$ may admit no extension as a vector bundle on certain "small" compactifications.

For example, consider the homogeneous space $X_0 := \mathbb{C}^n \setminus \{0\}$ under $G := GL_n$, and its equivariant compactification $X := \mathbb{P}^n$. Let $\mathcal{V}_0$ be the pull-back of the tangent bundle of $\mathbb{P}^{n-1}$ under the projection $\mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$. Then $\mathcal{V}_0$ is a homogeneous vector bundle on X_0. Further, if $n \geq 3$ then $\mathcal{V}_0$ does not extend to a vector bundle on the open subset $X_0 \cup \{0\} = \mathbb{C}^n$ of X, as follows from a result of Horrocks [4].

In my talk, I presented an affirmative answer to Kostant's question for certain complex adjoint symmetric spaces. Any such space G/K satisfies the inequality

$$rk(G/K) \geq rk(G) - rk(K),$$

where $rk(G)$ (resp. $rk(K)$) denotes the rank of G (resp. K), i.e., the dimension of a maximal subtorus, and likewise, $rk(G/K)$ denotes the rank of the symmetric space G/K, i.e., the dimension of a maximal subtorus S of G such that $\theta(x) = x^{-1}$ for any $x \in S$. Let us say that G/K is of minimal rank if this inequality is an equality. With this definition, the main result of the talk is

Theorem 1. *Let X be the wonderful compactification of a complex adjoint symmetric space G/K of minimal rank. Then any equivariant vector bundle on G/K extends to an equivariant vector bundle on X, generated by its global sections and having trivial higher cohomology groups.*

Here is an outline of the proof; details will be given in [2]. Recall that the equivariant vector bundles on any homogeneous space G/K are in bijection with the K-modules M, via $M \mapsto \mathcal{L}_{G/K}(M)$. For a symmetric space G/K, the group K is reductive. Thus, all the K-modules are semisimple, and it suffices to extend $\mathcal{L}_{G/K}(M)$, where M is simple. By the Borel-Weil theorem, M is the space of global sections of some K-equivariant line bundle $\mathcal{L}_{K/B_K}(\mu)$, where B_K is a Borel subgroup of K, and μ is a character of B_K identified with the corresponding one-dimensional B_K-module. In other words, $\mathcal{V}_0$ is the direct image of the equivariant line bundle $\mathcal{L}_{G/B_K}(\mu)$ under the equivariant fibration

$$\pi_0 : G/B_K \to G/K$$

with fiber being the flag variety K/B_K. To obtain the desired extension, I will construct an equivariant compactification of π_0 over X, where $\mathcal{L}_{G/B_K}(\mu)$ extends to an equivariant line bundle.

There exists a Borel subgroup B of G such that $B \cap K = B_K$. Then the orbit $Y_0 := B/B_K$ is closed in G/K, since K/B_K is closed in G/B. Let Y be the closure of Y_0 in X, this is a B-stable subvariety. Consider the "induced" G-variety $G \times^B Y$, an equivariant compactification of $G \times^B B/B_K \cong G/B_K$. This variety is provided with G-equivariant morphisms

$$\pi : G \times^B Y \to X$$

(a compactification of π_0) and

$$f : G \times^B Y \to G/B$$

(a compactification of the natural map $f_0 : G/B_K \to G/B$). Further, for any character λ of B, there is an equivariant line bundle $f^*\mathcal{L}_{G/B}(\lambda)$ on $G \times^B Y$; it extends $\mathcal{L}_{G/B_K}(\mu)$ if and only if λ extends μ. Under this assumption,

$$\mathcal{V} := \pi_*(f^*\mathcal{L}_{G/B}(\lambda))$$

is a coherent G-linearized sheaf on X which restricts to $\mathcal{V}_0$ on X_0.

To show that $\mathcal{V}$ is indeed locally free and satisfies the assertions of Theorem 1, one applies the theorem on cohomology and base change to the morphism π. Indeed, under the assumptions of Theorem 1, π turns out to be flat with reduced fibers. In fact, it follows from [1] that the fibers of π realise a flat degeneration of the flag variety K/B_K to a union of Schubert varieties in the larger flag variety G/B. Further, again under the assumptions of Theorem 1, the dominant weight μ of K turns out to extend to a dominant weight λ of G. So the proof is completed by using known homological properties of the line bundle $\mathcal{L}_{G/B}(\lambda)$ on unions of Schubert varieties, together with the semicontinuity theorem for π.

Among the adjoint symmetric spaces, those of minimal rank form a rather restricted class: they consist of the products of homogeneous spaces $K \times K/diag(K)$ (where the group K is simple), PSL_{2n}/PSp_{2n}, PSO_{2n}/PSO_{2n-1}, and E_6/F_4. The argument of Theorem 1 extends to further examples of symmetric spaces, e.g., to $G_{(m+n}/GL_m \times GL_n$ and SO_{2n}/GL_n; indeed, these spaces G/K contain several closed B-orbits, but all of them are multiplicity-free, and any dominant weight of K extends to a dominant weight of G. This argument also extends to the spherical homogeneous spaces of minimal rank, as introduced and classified by Ressayre in [6].

It would be interesting to describe the category of equivariant vector bundles on wonderful compactifications of symmetric spaces of minimal rank, generalizing Syu Kato's description [5] in the case of group compactifications, and to obtain an intrinsic characterization of those bundles constructed above. Also, it is easily shown that the rational cohomology ring of the wonderful compactification of any adjoint symmetric space is generated by Chern classes of equivariant vector bundles, but finding explicit generators of this ring is an open problem. The above construction is one step towards its solution for spaces of minimal rank.

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Polygon spaces, tangents to quadrics and special Lagrangians

Ciprian S. Borcea

Special Lagrangians have a recognized importance in Mirror Symmetry, most emphatically in relation to the Strominger-Yau-Zaslow conjecture [SYZ]. Yet, there's only a sparse collection of explicit examples with known topology in the compact (projective) case [Br2]. We propose two series of examples, based, respectively, on polygon spaces and spaces of common tangent k-planes to quadrics. The first series is explored here in more detail due to wider connections with toric geometry, root systems, and Batyrev-Borisov duality.

Polygon spaces: We consider planar polygons with prescribed length for all edges, up to equivalence under Euclidean motions. Self-intersection or degeneration to a line is permitted. With edge-length-vector $q = (q_1, ..., q_n)$ normalized by $\sum_{i=1}^n q_i = 2$, the corresponding *configuration space* V_q^{n-3} has the expected real dimension $(n-3)$ for any q in the interior of the second hypersimplex:

$$\Delta(2, n) = \{q \in R^n \mid 0 \le q_i \le 1, \ \sum_{i=1}^n q_i = 2 \ \}$$

Singularities occur only when all vertices are collinear. In the parameter space this corresponds to 'walls': $\sum_{i=1}^n \epsilon_i q_i = 0, \quad \epsilon_i = \pm 1$, which divide $\Delta(2, n)$ into 'chambers'. Topology may vary only upon crossing some wall, and chambers equivalent under permutations $\sigma \in \mathcal{S}_n$ give smooth diffeomorphic *polygon spaces* $V_q^{n-3} \approx V_{\sigma(q)}^{n-3}$. (Double) tori can be obtained by starting with a triangle and then proceeding with small enough truncations at the vertices.

Rational complexifications (mod reflection): When dividing further by reflection ι in a line, V_q^{n-3}/ι has a natural complexification via the corresponding space of *spatial polygons*. In particular, for rational q, the complexification can be interpreted as the Geometric Invariant Theory quotient $(P_1)^n//_q PSL(2)$, which is a *rational complex projective variety*. This direction is well illustrated in the literature: e.g. [Kly] [HK]. Here, we pursue:

Calabi-Yau complexifications: With $z_i \in C, z_i\bar{z}_i = 1$, unit vectors along the edges, V_q^{n-3} is described (mod S^1) by conjugate equations: $\sum_{i=1}^n q_i z_i = 0$ and $\sum_{i=1}^n q_i \bar{z}_i = 0$, which propose the complexification:

$$\sum_{i=1}^n q_i z_i = 0, \quad \sum_{i=1}^n q_i \frac{1}{z_i} = 0, \quad z \in P_{n-1}(C)$$

The resulting *complex projective variety* DV_q^{n-3} will be called a *Darboux variety*, since the case $n = 4$ of articulated quadrilaterals is treated in this manner in [D]. However, all cases $n > 4$ require desingularization.

Theorem 1. *The standard Cremona blow-up $\tilde{P}_{n-1}^C \to P_{n-1}$ which resolves the indeterminacies of $(z_i) \mapsto (1/z_i)$ induces a resolution $\tilde{DV}_q^{n-3}$ of the Darboux variety DV_q^{n-3}, for generic $q \in P_{n-1}$. $\tilde{DV}_q^{n-3}$ is a Calabi-Yau manifold (i.e. has trivial canonical bundle) and is given as a codimension two complete intersection in the toric variety $P_{\Pi_{n-1}} = \tilde{P}_{n-1}^C$.*

Here Π_{n-1} stands for a *permutohedron* of dimension $(n-1)$, obtained as a 'well truncated simplex'. $P_{\Pi_{n-1}}$ denotes the toric variety defined by the normal fan of Π_{n-1}. This is related to the root system A_{n-1} since $P_{\Pi_{n-1}}$ gives a resolution of the toric variety defined by the normal fan of the polytope of roots. Using Batyrev-Borisov duality [BB], this leads to:

Theorem 2. *The mirror family for the family of resolved Darboux varieties $\tilde{DV}_q^{n-3}$ is given by complete intersections of type $(1,...,1)$ $(1,...,1)$ in $(P_1)^{n-1}$.*

Considering that $P_{\Pi_{n-1}}$ contracts to $(P_1)^{n-1}$, and real points of Calabi-Yau manifolds defined over R correspond to special Lagrangians [Br1], we find (planar) polygon spaces as special Lagrangians for adequate parameters in both families.

The polygon space interpretation explains a wealth of birational properties of Darboux varieties: birational automorphisms corresponding to reflections in diagonals, fiber product presentations for any diagonal etc. There's also:

Proposition 3. *DV_q^{n-3} is the Hessian of the 'diagonal' cubic:*

$$\{\sum_{i=1}^n \frac{1}{q_i^2} w_i^3 = 0\} \subset P_{n-2} = \{w \mid \sum_{i=1}^n w_i = 0\} \subset P_{n-1}$$

The toric considerations can be extended to other centrally symmetric polytopes; in particular, there's a similar scenario for the 'well truncated cube', i.e. the BC type of root systems.

Tangents to quadrics: Let $\mathcal{Q} = \{Q_1, ..., Q_{d+1}\}$ be $d+1$ smooth quadrics in $P_{2d+1}(C)$, in general position. Let $T^{(k)}(\mathcal{Q}) \subset G(k+1, 2d+2)$ denote the variety of projective k-subspaces tangent to all quadrics Q_i, for $0 \leq k \leq 2d$.

Theorem 4. *$T^{(k)}(\mathcal{Q})$ are of Calabi-Yau type i.e. can be resolved to smooth projective Calabi-Yau manifolds $\tilde{T}^{(k)}(\mathcal{Q})$ of dimension $d + k(2d-k)$.*

For $\mathcal{Q}^* = \{Q_1^*, ..., Q_{d+1}^*\}$ the dual quadrics, we have $T^{(k)}(\mathcal{Q}) = T^{(2d-k)}(\mathcal{Q}^*)$, and adequate birational models $T^{[k]}(\mathcal{Q})$ allow inclusions:

$$T^{[0]}(\mathcal{Q}) \subset T^{[1]}(\mathcal{Q}) \subset ... \subset T^{[d]}(\mathcal{Q})$$

For $k = 0$, some possibilities for the real points can be found in [LdM].

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Immersed Levi-flat hypersurfaces into non negatively curved complex surfaces

Bertrand Deroin

We prove that there is no Levi-flat immersion of a Riemann surface foliation of class C^1 of a 3-dimensional compact manifold into the complex projective space, if the foliation carries a harmonic current absolutely continuous with respect to Lebesgue measure, with a density bounded from above and below. We give also rigidity results for Levi-flat immersion of class C^1 of such Riemann surface foliation into complex surfaces of non negative curvature.

The qualitative properties of singular holomorphic foliations by curves of the complex projective plane are, as far as I know, not already well understood. However, in affine coordinates x and y there are given by the very simple differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y),$$

where P and Q are complex polynomials in x and y.

In the real case, the classical Poincaré-Bendixson theorem asserts that every solution of such an equation accumulates in the real projective plane to a singularity or to a cycle. The analog of this theorem in the complex domain in not known and is called the *exceptionnal minimal set* conjecture: it states that every leaf of a holomorphic foliation of the complex projective plane accumulates on a singularity. If such a leaf were not accumulating on a singularity, then its closure would be a *compact lamination by holomorphic curves* of the complex projective plane. Conjecturally, the only one are the algebraic curves.

Several authors have made contributions to this problem when the total space of the lamination is a compact 3-dimensional manifold. A real hypersurface of a complex surface is said to be *Levi-flat* if it is foliated by holomorphic curves. In chronological order, Siu [Si], Iordan [Io] and Cao/Shaw/Wang [C-S-W] have proved that there is no compact Levi-flat hypersurface in $\mathbf{C}P^2$ of class C^8, C^4 and C^2. For that purpose, they study the inverse of the operator $\overline{\partial}$ on the exterior of the Levi-flat, and the regularity of the solution near the boundary.

In this talk I would like to explain a dynamical proof that there is no compact Levi-flat hypersurface in the complex projective plane. The regularity that I need is very strong. However under this regularity I show that there is not even an *immersed* compact Levi-flat in $\mathbf{C}P^2$, and I prove some rigidity properties of immersed compact Levi-flat hypersurfaces in complex surfaces of non negative curvature.

A *harmonic current* is a linear operator on the space of smooth 2-forms on the leaves of $\mathcal{F}$,

positive and $\partial\overline{\partial}$-closed. Garnett [Ga] proved the existence of a harmonic current, showing that it appears naturally as an invariant measure by the diffusion along the leaves with respect to a conformal metric. Our regularity assumption is the existence of a harmonic current which is absolutely continuous with respect to the Lebesgue measure, with a density bounded from above and below. Such a harmonic current is said to be AC. Our main result is the following.

Theorem. *Let S be a complex surface carrying a metric of non negative Ricci curvature Ω, and $\mathcal{F}$ be a Riemann surfaces foliation of class C^1 of a 3-dimensional closed manifold M. Then if $\mathcal{F}$ has an AC harmonic current C then every Levi-flat immersion* [1] *$\pi : M \to S$ of class C^1 satisfies $\pi^*\Omega = 0$ or $\mathcal{F}$ is a quotient of the horizontal foliation of $\mathbf{C}P^1 \times \mathbf{S}^1$.*

[1] A *Levi-flat immersion* is an immersion of class C^1 which is holomorphic along the leaves.

Because on the complex projective plane the Fubini-Study metric has strictly positive curvature, we obtain the following corollary.

Corollary. *Let $\mathcal{F}$ be a Riemann surface foliation of class C^1 of a 3-dimensional closed manifold M. Then if $\mathcal{F}$ has an AC harmonic current there is no Levi-flat immersion $\pi : M \to \mathbf{C}P^2$ of class C^1.*

There are many other examples of surfaces of non negative curvature. First, we have the Del Pezzo surfaces, for which there exists a metric of positive curvature. These surfaces are completely understood: there is only $\mathbf{C}P^1 \times \mathbf{C}P^1$, and the blow-up of $\mathbf{C}P^2$ at $d = 0, \dots, 8$ points. Our theorem states that if a Riemann surface foliation of class C^1 immerses holomorphically along the leaves into a Del Pezzo surface and possesses a AC harmonic current, then it is the quotient of the horizontal foliation of $\mathbf{C}P^1 \times \mathbb{S}^1$. When the curvature can vanish, the complete classification seems to be unknown. For instance, among the blow-up of the complex projective plane in 9 points, the author does not know the surfaces carrying a hermitian metric of non negative curvature, unless it is an elliptic fibration. The description of the directions where the curvature vanishes seems to be very difficult too.

Idea of the proof. The general idea is to think of an immersed Levi-flat hypersurface as a 1-parameter family of holomorphic curves, and to extend the classical notions coming from algebraic geometry that are available on compact curves. For instance, in [H-M] they study the self-intersection of a foliation cycle and they prove that it vanishes for a Levi-flat hypersurface.

Against a harmonic current, numerical invariants of this kind has been introduced by Candel (see [Gh]); namely the *Chern-Candel class* of a holomorphic line bundle over a Riemann surface foliation. If $E \to \mathcal{F}$ is a holomorphic line bundle over a Riemann surface foliation, then the Chern-Candel class of E against a harmonic current C is the real number $c_1(E, C) := \frac{1}{2\pi} C(\Omega)$, where Ω is the Chern curvature along the leaves of a hermitian metric $|.|$ on E.

One invariant of special interest for us is the Chern-Candel class of the normal bundle of the leaves of a Levi-flat hypersurface, and is called the *normal class.* Our main contribution is to prove that when the harmonic current is AC, the normal class is bounded by the opposite of the Euler characteristic [2], or the foliation is a quotient of the horizontal foliation of $\mathbf{C}P^1 \times \mathbf{S}^1$. We obtain this result by identifying the normal class to the *action*, which has been introduced by Frankel to measure how far is the harmonic current from a closed $(1,1)$-current. We then generalize Frankel's inequality proved in [Fr]. The theorem follows from an application of adjunction formula.

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[2] The Euler characteristic is the Chern-Candel class of the tangent bundle to the leaves.

lecturing. I thank Étienne Ghys, who suggested to me this problem. This work has partially been done while the author was a host of the UNAM in Cuernavaca.

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Indices of 1-forms on singular varieties

Wolfgang Ebeling
(joint work with Sabir M. Gusein-Zade and José Seade)

Let $(X,0)$ be the germ of a complex analytic variety of pure dimension n and ω be a (complex and, generally speaking, continuous) 1-form on $(X,0)$ with an isolated singularity at the origin. Three different notions of an index of the isolated singular point 0 of ω are discussed:

- the *radial index* $\mathrm{ind}_{\mathrm{rad}}(\omega; X, 0)$ [1, 2],
- the *homological index* $\mathrm{ind}_{\mathrm{hom}}(\omega; X, 0)$ of a holomorphic 1-form on the germ of a complex analytic variety with an isolated singularity (inspired by X. Gómez-Mont and G.-M. Greuel) [2],
- and the *index* $\mathrm{ind}(\omega; X, 0)$ (analogue of the GSV-index for vector fields, defined by the speaker and S. M. Gusein-Zade) for a holomorphic 1-form on an isolated complete intersection singularity.

Let ω be holomorphic. If $(X,0)$ is smooth, then all three indices coincide. If $(X,0)$ is an isolated complete intersection singularity , then it follows from Greuel's work that

$$\begin{aligned} \mathrm{ind}_{\mathrm{hom}}(\omega; X, 0) &= \mathrm{ind}(\omega; X, 0) \\ \mathrm{ind}(\omega; X, 0) - \mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) &= \mu, \end{aligned}$$

where μ is the Milnor number of $(X, 0)$.

Let $(X, 0)$ be a germ of a complex analytic space of pure dimension n with an isolated singular point at the origin. Then we can show that the difference

$$\mathrm{ind}_{\mathrm{hom}}(\omega; X, 0) - \mathrm{ind}_{\mathrm{rad}}(\omega; X, 0)$$

between the homological and the radial indices does not depend on the 1-form ω. Therefore we can consider the difference

$$\nu(X, 0) = \mathrm{ind}_{\mathrm{hom}}(\omega; X, 0) - \mathrm{ind}_{\mathrm{rad}}(\omega; X, 0)$$

as a generalized Milnor number of the singularity $(X, 0)$.

Consider the absolute De Rham complex of $(X, 0)$

$$0 \longrightarrow \mathcal{O}_{X,0} \xrightarrow{d} \Omega^1_{X,0} \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_{X,0} \longrightarrow 0$$

and let

$$\bar{\chi}(X, 0) := \sum_{i=0}^{n} (-1)^{n-i} h_i(\Omega^\bullet_{X,0}, d) - 1$$

be the reduced Euler characteristic of this complex.

Theorem 2. [2] *One has*

$$\nu(X, 0) = \bar{\chi}(X, 0)$$

if

(i) $(X, 0)$ *is a curve singularity,*
(ii) $(X, 0) \subset (\mathbb{C}^{d+1}, 0)$ *is the cone over the rational normal curve in* $\mathbb{CP}^d$.

Statement (i) implies that the invariant $\nu(X, 0)$ is different from the Milnor number introduced by R.-O. Buchweitz and G.-M. Greuel for such singularities. Statement (ii) was obtained with the help of H.-Ch. von Bothmer and R.-O. Buchweitz. For $d = 4$ this is Pinkham's example of a singularity which has smoothings with different Euler characteristics.

Question Is $\nu(C, 0) = \bar{\chi}(X, 0)$ always true?

J.-P. Brasselet, D. Massey, A. J. Parameswaran, and J. Seade introduced the notion of the local Euler obstruction $\mathrm{Eu}_{f,X}(0)$ of a holomorphic function $f : (X, 0) \to (\mathbb{C}, 0)$ with an isolated critical point on the germ of a complex analytic variety $(X, 0)$. We adapt the definition to the case of a 1-form and define the *local Euler obstruction* $\mathrm{Eu}(\omega; X, 0)$ of ω [1].

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an arbitrary germ of an analytic variety with a Whitney stratification $X = \bigcup_{i=0}^q V_i$, $V_0 = \{0\}$. For a stratum V_i, $i = 0, \ldots, q$, let N_i be the normal slice in the variety X to the stratum V_i ($\dim N_i = \dim X - \dim V_i$) at a point of the stratum V_i, let $\ell : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function, and let n_i be the radial index of the 1-form $d\ell$ on N_i.

Theorem 3. [1] *One has*

$$\mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) = \sum_{i=0}^{q} n_i \cdot \mathrm{Eu}(\omega; \overline{V_i}, 0)\,.$$

We also have an "inverse" of this formula. For the differential of a function the radial index is related to the Euler characteristic of the Milnor fibre of the function. This gives an expression for the local Euler obstruction of the differential of a function in terms of Euler characteristics of some Milnor fibres.

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Anti-self-dual hermitian metrics on Inoue surfaces

Akira Fujiki

(joint work with Massimiliano Pontecorvo)

Let S be a compact smooth complex surface. If S admits an anti-self-dual (*abbr.* asd) hermitian metric which is not conformal to a Kähler metric, then it is known that S must be a surface of class VII, i.e., the fist betti number $b_1(S) = 1$. So far very little is known as to which surfaces of class VII may admit asd hermitian metrics and what their moduli should look like.

The only exception is the case $b_2(S) = 0$, where the complete classification is known by Pontecorvo [6]: S is necessarily a Hopf surface and all the surfaces and metrics are explicitly describable. This classification also shows that every Hopf surface does not admit an asd hermitian metric; indeed, the existence property of such metrics is not stable under small deformation of complex structures.

When $b_2(S) > 0$, if we assume further that S is minimal, only examples so far known are those constructed explicitly by LeBrun [5], where S is a parabolic Inoue surface for which the unique elliptic curve on it has a "real" period and the metrics are invariant by the unique holomorphic circle action on the surface.

The purpose of our study now is to produce further asd hermitian metrics on various surfaces of class VII such as Inoue and Enoki surfaces, by a variant of the method of Donaldson-Freedman [1] and Kim-Pontecorvo [4]. Namely we first fix an asd metric (which is never hermitian) on the connected sum $m\boldsymbol{P}^2$ of m copies of complex projective plane $\boldsymbol{P}^2$ (with reversed oritentation) for some $m > 0$. We then pass to the associated twistor space and use the complex geometric method to produce asd hermitian metrics on surfaces of class VII with global spherical shell. As the initial metrics we take the asd metrics constructed by Joyce [3] on $m\boldsymbol{P}^2$. Indeed, the structure of the associated twistor space Z has been studied in detail in [2]. The most typical applicatin of our method is the following

Theorem. On any hyperbolic or half Inoue surface S there exists a real m-dimensional family of asd hermitian metrics on S, where $m = b_2(S)$.

A similar, but somewhat weaker result also holds true for parabolic Inoue surfaces and Enoki surfaces. One difference is that these surfaces depend on continuous parameters, while the surfaces in Theorem depend only on discrete parameters.

For instance recall that among minimal surfaces of class VII a hyperbolic Inoue surface is characterized by the existence of two cycles of rational curves on it. Moreover, each hyperbolic Inoue surface is completely determined by the cyclic sequences of self-intersection numbers of the irreducible components of these two cycles, up to "transpositions". On the other hand, a parabolic Inoue surface contains a unique elliptic curve, whose period gives its continuous moduli. Here the asd hermitian metric is supposed to exist only when this period is "real". In any case it is interesting to compare our asd metrics with those constructed by LeBrun [5].

The proof of Theorem roughly goes as follows. Let Z be one of the Joyce twistor spaces as above. Take a pair of pair of "elementary" surfaces $(S_i^+, S_i^-), i = 1, 2$, on Z. $S_i^\pm$ are smooth toric surfaces and $L_i := S_i^+ \cap S_i^-$ are twistor lines in Z. Take the blowing-up $\mu : \tilde{Z} \to Z$ with center the disjoint union of L_i. The exceptional divisors $Q_i := \mu^{-1}(L_i)$ are isomorphic to each other and by identifying Q_i by a suitable isomorphism $\varphi : Q_1 \to Q_2$ inside $\tilde{Z}$ we obtain a singular complex 3-space $\hat{Z} := \tilde{Z}/\varphi$ together with a pair of disjoint rational surfaces $(\hat{S}_1^+, \hat{S}_1^-)$ with ordinary double curves on $\hat{Z}$ as the proper transform of (S_1^+, S_1^-). We then construct a smoothing (by deformations) of the pair $(\hat{Z}; (\hat{S}_1^+, \hat{S}_1^-)) \to (Z_t; (S_{1t}^+, S_{1t}^-))$ with some complex parameter t, where Z_t and $S_{1t}^\pm$ are smooth for general values of t.

The real structure $\sigma : Z \to Z$ lifts or extends to all the above objects (under certain conditions), and for some 'real' value of the parameter t, Z_t turns out to be a twistor space of an asd hermitian metric on the complex surface $S_{1t}^\pm$, which turns out to be a hyperbolic Inoue surface if we choose φ and t suitably. Moreover, we can show that in this way all the hyperbolic Inoue surfaces are obtained by a suitable choice of the initial Joyce metric.

Note finally that Joyce metrics depend on $m - 1$ real parameters, which contribute as part of the parameters of asd hermitian metrics on hyperbolic Inoue surfaces mentioned in Theorem. It is interesting to try to understand the nature of these parameters and the whole moduli space of such metrics on a fixed hyperbolic Inoue surface.

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Addition formulas for theta functions, and linear systems on abelian varieties

Samuel Grushevsky

In this talk we prove a conjecture of Buchstaber and Krichever that a certain addition formula for theta functions characterizes Jacobians among all abelian varieties, obtain cubic equations for the hyperelliptic locus, and explain how this is related to trisecants and translates of the 2Θ linear system. This report is based on the results that appeared as [Gr].

The unique holomorphic solution of

$$\{f(x+y)=f(x)f(y)\mid \forall x,y\in\mathbb{C}^n\}$$

is $f(x)=\exp(a\cdot x)$ for some $a\in\mathbb{C}^n$. If we try to generalize this functional equation and consider $\{1=\phi(x+y)f(x)f(y)\}$, where f and ϕ are different functions, the exponent remains the only solution. However, it was observed by Buchstaber and Krichever in [BK] that if one considers a further generalization

$$1=\sum_{i=0}^{n}\phi_i(x+y)f_i(x)f_i(y),$$

then the properly adjusted theta functions of Jacobian varieties of genus n provide solutions. More precisely, Buchstaber and Krichever prove the following statement (here we have rewritten their formula in terms of theta functions rather than Baker-Akhiezer functions and homogenized the equation):

Addition formula ([BK]). Let C be a curve of genus g and let $A_0,\dots,A_{g+1}\in C\subset J(C)$ be arbitrary $g+2$ points on C, considered embedded into its Jacobian. Then for appropriately chosen functions ϕ_i (which are computed explicitly in [BK]) the following holds $\forall x,y\in\mathbb{C}^g$:

$$0=\sum_{i=0}^{g+1}\phi_i(x+y)\theta(A_i+x)\theta(A_i+y), \qquad (*)$$

where θ denotes the usual Riemann's theta function.

Buchstaber and Krichever conjectured that this addition formula characterizes Jacobians among all principally polarized abelian varieties. We prove their conjecture under a (necessary) general position assumption.

Theorem ([Gr]). If for some $g+2$ points $A_0,\dots,A_{g+1}\in X$ on a g-dimensional abelian variety X, which are in general position in a certain sense[1], there exist functions ϕ_i such that the addition formula $(*)$ is satisfied $\forall x,y\in\mathbb{C}^g$, then X

[1] We thank G. Pareschi and M. Popa for pointing out that a general position assumption is necessary and providing a counterexample to the statement without the general position assumption.

is the Jacobian variety of some curve, and the points $\{A_i\}$ lie on an Abel-Jacobi image of this curve inside X.

Proof (Idea). We use Riemann's bilinear addition theorem $\theta(2x)\theta(2y) = K(x+y)\cdot K(x-y)$, where K denotes the Kummer theta map $K : X \to \mathbb{P}^{2^g-1}$ sending a point $x \in X$ to the set of the values of theta functions of the second order $\{\Theta[\varepsilon](x)\}$ at this point. Upon rescaling x and y by 2, the formula $(*)$ thus becomes

$$0 = \sum_{i=0}^{g+1} \phi_i(x+y)K(A_i+x+y)\cdot K(x-y).$$

We recall that the theta functions of the second order form a basis for sections of the bundle 2Θ over X. Note now that as a function of $x-y$ the above is a linear combination $0 = \vec{v}\cdot K(x-y)$, where $\vec{v}$ does not depend on $(x-y)$. Since this is a linear combination of the basis, it vanishes identically iff $\vec{v}=0$, so we obtain a vector identity. $0 = \sum_{i=0}^{g+1}\phi_i(x+y)K(A_i+x+y)$. This is simply saying that for all $z \in \mathbb{C}^g$ the $g+2$ points $K(A_i+z)$ are collinear.

Now this is exactly the Gunning's multisecant formula from [Gu]. Moreover, if some $g-1$ of $\phi_i(z)$ are zero at some point z, then the remaining three Kummer images are collinear, and thus we get a trisecant of the Kummer image. Generically (and this is where we need the general position assumption), the locus of such z should be of dimension $g-(g-1)=1$, and thus we can apply the trisecant criterion of Fay-Gunning-Welters stating that if there is a 1-dimensional family of trisecants, the abelian variety is a Jacobian. □

The multisecant condition or the addition formula $(*)$ can also be interpreted as follows. Consider any section of the linear system $2\Theta_a$ — as a function of $z \in X$, it is a linear combination $\vec{v}\cdot K(z+a)$ for some vector $v \in \mathbb{C}^{2^g}$. Then the number of conditions imposed by a set of points $\{A_i\}$ on $2\Theta_a$ is equal exactly to the dimension of the linear span of $\{K(A_i+a)\}$. In this reformulation a result similar to the theorem above, but with a different general position assumption, was recently independently obtained by Pareschi and Popa in [PP] as the first step in trying to develop a theory for abelian varieties paralleling the Castelnuovo theory for the projective space.

Writing down the coefficients ϕ_i in $(*)$ explicitly, following [BK], allows one to obtain explicit equations for theta functions characterizing Jacobians. However, it is only possible to get equations involving only theta constants, and not arbitrary shifts, for the case of hyperelliptic curves.

Theorem. An abelian variety X is the Jacobian of a hyperelliptic curve with some special choice of the basis of cycles if and only if for some fixed azygetic set of characteristics $[a_i, b_i]$ the following cubic equations

$$\begin{aligned}
&\sum_{\varepsilon}\Theta[\varepsilon](z)\Theta[\varepsilon](z)\Theta[\sigma](z)\\
=\;&\sum_{\varepsilon}\sum_{k=0}^{g}(-1)^{(\varepsilon+\sigma,a_{k+1})}\Theta[\varepsilon](z)\Theta[\varepsilon+b_k]\Theta[\sigma+b_k](z),
\end{aligned}$$

are satisfied for all z and all characteristics σ, and moreover a certain general position assumption is satisfied.

Remark: It would be interesting to understand the locus of abelian varieties with $g+2$ chosen points satisfying the addition formula $(*)$, but *not* the general position assumption. If this locus were small or tractable, then one would see that the Jacobian or hyperelliptic locus is defined by relatively few equations, which would lead to interesting results on the cohomological dimension of the moduli space of abelian varieties.

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Cartan decomposition of the moment map

Peter Heinzner
(joint work with Gerald Schwarz)

Let Z be a complex space with a holomorphic action of the complex reductive group $U^{\mathbb{C}}$, where $U^{\mathbb{C}}$ is the complexification of the compact Lie group U. We assume that Z admits a smooth U-invariant Kähler structure and a U-equivariant moment mapping $\mu\colon Z \to \mathfrak{u}^*$, where $\mathfrak{u}$ is the Lie algebra of U and $\mathfrak{u}^*$ its dual. We assume that $G \subset U^{\mathbb{C}}$ is a (for simplicity closed) Lie subgroup such that the Cartan decomposition $U^{\mathbb{C}} = U\exp(i\mathfrak{u}) \simeq U \times i\mathfrak{u}$ induces a Cartan decomposition $G = K\exp(\mathfrak{p}) \simeq K \times \mathfrak{p}$ where $K = U \cap G$ and $\mathfrak{p} \subset i\mathfrak{u}$ is an $(\mathrm{Ad}K)$-stable linear subspace. We have the subspace $i\mathfrak{p} \subset \mathfrak{u}$ and by restriction an induced "moment" mapping $\mu_{i\mathfrak{p}}\colon Z \to (i\mathfrak{p})^*$. We define $\mathcal{M}_{i\mathfrak{p}}$ to be the zeroes of $\mu_{i\mathfrak{p}}$, and we define $\mathcal{M}$ to be the zeroes of μ. For a given μ we have the set $\mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M}) := \{z \in Z;\ \overline{U^{\mathbb{C}} \cdot z} \cap \mathcal{M} \neq \emptyset\}$ of semistable points with respect to μ and the $U^{\mathbb{C}}$-action on Z. We call $\mathcal{S}_G(\mathcal{M}_{i\mathfrak{p}}) := \{z \in Z;\ \overline{G \cdot z} \cap \mathcal{M}_{i\mathfrak{p}} \neq \emptyset\}$ the set of semistable points of Z with respect to $\mu_{i\mathfrak{p}}$ and the G-action on Z. Most of our results have the hypothesis that $Z = \mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$. In general, $\mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$ is a proper open subset of Z, but we can force equality by replacing Z by $\mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$. If Z is a Stein space then it admits a smooth strictly plurisubharmonic U-invariant exhaustion function ρ. Associated with ρ is a U-invariant Kähler structure and a moment mapping μ. Moreover, for any such μ, the equality $Z = \mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$ holds automatically. Another interesting example of equality is the case where Z is the set of semistable points (in the sense of geometric invariant theory) given by a $U^{\mathbb{C}}$-linearized ample line bundle. Also in this case there exists a U-invariant Kähler structure on Z and a μ such that $Z = \mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$.

We show that $\mathcal{M}_{i\mathfrak{p}}$ is the correct analogue of the usual Kempf-Ness set $\mathcal{M}$ when considering the action of G. Specifically, we have the following.

Theorem 1. *Let Z, G, $\mathcal{M}_{i\mathfrak{p}}$ and $\mathcal{M}$ be as above and assume that $Z = \mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$. Then $Z = \mathcal{S}_G(\mathcal{M}_{i\mathfrak{p}})$ and the G-action has the following properties.*

1. *Let $z \in \mathcal{M}_{i\mathfrak{p}}$. Then $G \cdot z \cap \mathcal{M}_{i\mathfrak{p}} = K \cdot z$ and $G_z = K_z \cdot \exp(\mathfrak{p}_z)$ where $\mathfrak{p}_z$ denotes the elements of $\mathfrak{p}$ such that the corresponding vector field on Z vanishes at z.*
2. *An orbit $G \cdot z$ is closed if and only if $G \cdot z \cap \mathcal{M}_{i\mathfrak{p}} \neq \emptyset$.*
3. *There is a quotient space $Z/\!\!/G$ which parameterizes the closed G-orbits in Z. The inclusion $\mathcal{M}_{i\mathfrak{p}} \to Z$ induces a homeomorphism $\mathcal{M}_{i\mathfrak{p}}/K \simeq Z/\!\!/G$.*
4. *Let $z \in Z$ such that $G \cdot z$ is closed. Then there is a locally closed real analytic G_z-stable subset S of Z, $z \in S$, such that the natural map $G \times^{G_z} S \to Z$ is a real analytic G-isomorphism onto the open set $G \cdot S$. Moreover, S can be chosen such that $G \cdot S$ is saturated with respect to the quotient map $Z \to Z/\!\!/G$.*
5. *Let $z \in Z$ and suppose that $Y \subset \overline{G \cdot z}$ is closed and G-stable. Then there is a Lie group homomorphism $\lambda\colon \mathbb{R} \to G$ such that $\lim_{t\to+\infty} \lambda(t) \cdot z$ exists and is a point in Y.*

The space $Z/\!\!/G$ is rather nice. For example, it follows from (3) that it is Hausdorff and from (4) one can deduce that it is locally homeomorphic to real semi-analytic sets.

Of course, there is much earlier work on quotients and slice theorems for actions of complex reductive groups, and there is also earlier work for actions of real groups. In particular, in the latter case, there are the papers of Richardson-Slodowy [RiSl90] and Luna [Lu75]. Here one has a complex representation space V of $U^{\mathbb{C}}$ and real forms $V_{\mathbb{R}}$ of V and G of $U^{\mathbb{C}}$. One considers the action of G on $V_{\mathbb{R}}$. Our results are more general in that our actions are not necessarily algebraic, the group G is not necessarily a real form of $U^{\mathbb{C}}$ and we consider the quotient of Z, not just of a real form of Z.

Perhaps the most interesting new observation is that the existence of $Z/\!\!/G$ is closely related to $\mu_{i\mathfrak{p}}$ and is given by $\mathcal{M}_{i\mathfrak{p}}/K$. In the case where G is a real form there are also results about the structure of the G-action on Lagrangian submanifolds X of Z using moment map techniques (see, e.g., O'Shea and Sjamaar [O'SSj00] and references therein). This case is also rather special. The $\mu_{i\mathfrak{p}}$-component of μ on X is completely determined by μ. One establishes results concerning $\mathcal{M}$ and the $U^{\mathbb{C}}$-action on Z and then restricts to X. This works because $X \cap \mathcal{M} = X \cap \mathcal{M}_{i\mathfrak{p}}$ and because the map $\mu_{\mathfrak{k}}\colon Z \to \mathfrak{k}^*$ obtained by restricting μ to $\mathfrak{k}$ is constant on X.

Besides the results mentioned above, we also consider several topics pertaining to proper actions and compact isotropy groups. In particular we show the following.

Theorem 2. *Assume that $Z = \mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$. Let X be a G-stable closed subset of Z such that the G-action on X is proper. Then the natural map $G \times^K (\mathcal{M}_{i\mathfrak{p}} \cap X) \to X$ is a homeomorphism and a real analytic isomorphism if X and $\mu_{i\mathfrak{p}}$ are real analytic.*

We have a similar decomposition for the subset $\mathrm{Comp}_{i\mathfrak{p}}(Z)$ of points $z \in Z$ such that $G \cdot z$ is closed and G_z is compact. The results on proper actions are applied to obtain decompositions, due to Mostow, for groups and homogeneous spaces. The application relies on properties of a very special strictly plurisubharmonic exhaustion of $U^{\mathbb{C}}$ related to the Cartan decomposition.

Most of our results rely upon the notion of $\mu_{i\mathfrak{p}}$-adapted sets. A $\mu_{i\mathfrak{p}}$-adapted subset of Z is a K-invariant subset A of Z such that for all $z \in Z$ and $\xi \in i\mathfrak{p}$, the curve $(\exp it\xi) \cdot z$ lies in A for a connected set J of $t \in \mathbb{R}$. Moreover, we require that if $t_+ := \sup J < \infty$, then $\mu_{i\mathfrak{p}}(\exp(it_+\xi) \cdot z)(i\xi) > 0$ and a similar negativity condition if $t_- := \inf J > -\infty$. The main technical point is to show that every K-orbit in $\mathcal{M}_{i\mathfrak{p}}$ has a neighborhood basis of open $\mu_{i\mathfrak{p}}$-adapted sets. The $\mu_{i\mathfrak{p}}$-adapted sets have very nice properties. For example, if A_1 and A_2 are $\mu_{i\mathfrak{p}}$-adapted, then $G \cdot A_1 \cap G \cdot A_2 = G \cdot (A_1 \cap A_2)$.

If we drop our usual assumption that $Z = \mathcal{S}_{U^{\mathbb{C}}}(\mathcal{M})$ but retain that every K-orbit in $\mathcal{M}_{i\mathfrak{p}}$ has a basis of open $\mu_{i\mathfrak{p}}$-adapted neighborhoods, then we can show that $\mathcal{S}_G(\mathcal{M}_{i\mathfrak{p}})$ is open in Z and that Theorem 1 holds for Z replaced by $\mathcal{S}_G(\mathcal{M}_{i\mathfrak{p}})$. If U (hence G) is commutative, then the condition on $\mu_{i\mathfrak{p}}$-adapted neighborhoods is automatic (and $\mathcal{S}_G(\mathcal{M}_{i\mathfrak{p}})$ is open). It would be interesting to know if $\mathcal{S}_G(\mathcal{M}_{i\mathfrak{p}})$ is always open in Z!

This results should be considered as a first part of a project where actions of real semisimple Lie groups G on complex manifolds will be investigated from the Hamiltonian point of view. Semisimplicity ensures that $G^{\mathbb{C}} = U^{\mathbb{C}}$. For a general complex manifold Z, where G acts on Z by holomorphic transformations, there is no hope that $G^{\mathbb{C}}$ acts holomorphically, e.g., if Z is a bounded domain in some $\mathbb{C}^n$. But $G^{\mathbb{C}}$ always acts locally on Z and several of the observations in this paper are valid also in this case. Moreover, if G is a compact Lie group and Z is a Stein space, then one has the powerful concept of the existence of a universal complexification of the G action on Z (see [He91] and [HeIa97]). We plan to consider this kind of question later. But even in the case where $G^{\mathbb{C}} = U^{\mathbb{C}}$ acts on Z holomorphically a lot of questions remain open. Examples indicate that it should be extremely interesting to clarify the interplay of the various geometric objects associated with μ, $\mu_{i\mathfrak{p}}$ and $\mu_{\mathfrak{k}}$.

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Bound on the number of curves of a given degree through a general point of a projective variety

Jun-Muk Hwang

This work was motivated by the following result of J.M. Landsberg's.

Theorem 1 ([L2, Theorem 1]) *Let X be an irreducible projective variety of dimension n in a projective space and let $x \in X$ be a general point. Then the number of lines lying on X and passing through x is either infinite or bounded by $n!$.*

It is remarkable that the bound $n!$ is optimal: it is achieved when X is a smooth hypersurface of degree n in $\mathbf{P}_{n+1}$. However, even if we disregard the optimality of the bound, the uniformity of the bound is already quite remarkable. Namely, the fact that the bound depends only on the dimension n of X is worth noticing. When interpreted as such a uniform boundedness result, Theorem 1 naturally leads to the following questions.

Question 1 What about curves of higher degree? Is the number of curves of degree $d > 0$ lying on X and passing through a general point $x \in X$ either infinite or bounded by a number depending only on d and n?

Question 2 What about the case when there are infinitely many lines through a general point x? Is the number of components of the space of lines lying on X and passing through a general point $x \in X$ bounded by a number depending only on n?

Question 3 What about non-general points? Is the number of lines lying on X through any given point of X either infinite or bounded by a number depending only on n?

In Landsberg's proof, the uniformity comes from his earlier result [L1] that a line osculating to order $n+1$ at a general point of X must be contained in X. The differential-geometric argument of [L1] using the moving frame method seems difficult to be generalized to handle above questions.

We introduce an approach to these questions, using tools from the study of uniform lower bounds for the Seshadri numbers of an ample line bundle at general points of a variety ([EKL],[HK]). Our result on Question 1 and Question 2 can be stated as follows. Let us denote by $\mathrm{Curves}_d(X, x)$ the space of curves of degree d lying on a projective variety X and passing through a point $x \in X$.

Theorem 2 *Let n and d be two positive integers. Then there exists a positive real number $\mu_{n,d}$ determined by n, d with the following property. For any irreducible projective variety of dimension n in a projective space and any general point $x \in X$, the number of components of* $\mathrm{Curves}_d(X, x)$ *is bounded by $\mu_{n,d}$.*

Regarding Question 3, there is a counter-example. Let k be an odd integer and consider the Fermat surface $X_0^k + X_1^k + X_2^k + X_3^k = 0$ in $\mathbf{P}_3$. Then through the point $(1, -1, 0, 0)$ there are at least k distinct lines defined by $X_0 + X_1 = 0$ and $X_2 + e^{\frac{2\pi j \sqrt{-1}}{k}} X_3 = 0, 1 \leq j \leq k$. Given any dimension $n \geq 2$ and an integer $M > 0$, by taking the Segre product of the Fermat surface with a smooth variety of dimension $n - 2$ containing no lines, we get an example of a smooth variety of dimension n where the number of lines through any point in a codimension 2 subset is finite, but larger than M. This example suggests that the following result of ours gives a more or less optimal answer to Question 3.

Theorem 3 *Let n and d be two positive integers. Then there exists a positive real number $\nu_{n,d}$ determined by n, d with the following property. Let X ba an irreducible projective variety of dimension n in a projective space. Then there exists a subvariety R of codimension ≥ 2 in the smooth locus of X such that for* *any smooth point of X off R, the number of the components of* $\mathrm{Curves}_d(X, x)$ *is bounded by* $\nu_{n,d}$.

The rough idea of the proofs of Theorem 2 and Theorem 3 is the following. First it is easy to get a bound depending on the degree of X. This is possible by the effective bound on the number of components of Chow varieties obtained in recent works on effective bounds on the number of maps dominating varieties of general type, e.g. [Gu] and [Ts]. Now to prove Theorem 2 and Theorem 3, the strategy is to construct a foliation on X 'generated by curves of degree d'. This foliation has the property that its general leaf contains all curves of degree d lying on X passing through a general point of the leaf, and it is the foliation of minimal rank with this property. This construction is motivated by the construction of rationally connected fibration in the study of uniruled varieties ([Ko]). To prove Theorem 2, we may replace X by a leaf of the foliation. The heart of the proof of Theorem 2 is to show that the degree of the leaf can be bounded in terms of n and d. This is achieved by using an argument from Ein-Kčhle-Lazarsfeld's work on Seshadri numbers ([EKL]). The proof of Theorem 3 is by an induction argument using Theorem 2 and by a study of the foliation in codimension 1.

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Threefolds with big and nef anticanonical bundles

Priska Jahnke

(joint work with Thomas Peternell and Ivo Radloff)

We start the classification of almost Fano threefolds, i.e., of smooth threefolds X where the anticanonical divisor $-K_X$ satisfies the two conditions

$$-K_X.C \geq 0 \text{ for all irreducible curves } C \subset X, \text{ and } (-K_X)^3 > 0.$$

We will always assume that there is equality $-K_X.C = 0$ for at least one curve, i.e., X not Fano.

In the surface case, the second Hirzebruch surface Σ_2 is an easy example. The anticanonical system $|-K_{\Sigma_2}|$ is spanned, but not ample, mapping Σ_2 to the quadric cone $xy = z^2$ in $\mathbb{P}_3$. We may think of Σ_2 either as a deformation of $\mathbb{P}_1 \times \mathbb{P}_1$, or as a desingularisation of the quadric cone, obtained by deforming the smooth quadric $Q_2 \simeq \mathbb{P}_1 \times \mathbb{P}_1$.

The picture is in fact always quite similar. By the Base Point Free Theorem, on any almost Fano threefold, $|-mK_X|$ is spanned for $m \gg 0$. The corresponding map (with connected fibers)

$$\psi : X \longrightarrow X'$$

contracts all anticanonically trivial curves. By assumption, ψ is not an isomorphism. The resulting X' is a Gorenstein Fano threefold with canonical singularities. We call X' an *anticanonical model* of X. Note that the map is crepant, i.e.,

$$K_X = \psi^*(K_{X'}).$$

In this sense, our X plays the role of a terminal modification of X' in the sense of [R83]. One of the questions we are interested in is whether any X is a deformation of a smooth Fano threefold as in our example above.

The classification of smooth Fano threefolds is due to Iskovskikh and Mori and Mukai. The singular case, allowing canonical or terminal singularities, is a still ongoing project which is far from being complete. The first steps are the following:

(1) Boundedness: $(-K_X)^3 \leq 72$ ([P04]), implying finiteness of the classification problem ([Ma70], [B01]);
(2) Generatedness: complete list of all canonical Gorenstein Fano threefolds X', where the anticanonical system $|-K_{X'}|$ itself is not base point free ([JR04]);
(3) Anticanonical embedding: complete list of all canonical Gorenstein Fano threefolds X', where the anticanonical system $|-K_{X'}|$ is base point free, but not very ample, called *hyperelliptic* ([CSP04]).

The three steps allow us to essentially focus on those X where $|-K_X|$ is spanned and where the map defined by this system has indeed connected fibers, i.e., already conicides with ψ.

In the first step of our classification, we moreover restrict to the case where the Picard group is as simple as possible. In the Fano case this means Pic $\simeq \mathbb{Z}$, whereas in our situation

$$\mathrm{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Then ψ either contracts a finite number of curves, or a single irreducible divisor D to a point or a curve. In the first case it is known that X' (now having terminal singularities) admits a smoothing, i.e., is a deformation of a smooth Fano threefold ([N97]). In the divisorial case this question is essentially open.

Our classification uses Mori theory (see [Mo82]). Since K_X is not nef, there exists an extremal contraction

$$\phi : X \longrightarrow Y$$

with $-K_X$ is ϕ–ample. The target Y in our case is one of the following:

- $Y \simeq \mathbb{P}_1$ and ϕ is a Del Pezzo fibration,

- $Y \simeq \mathbb{P}_2$ and ϕ is a conic bundle,
- Y is a (terminal) Fano threefold and either
  - $X = Bl_C(Y)$ with a smooth curve C and Y is smooth, or
  - $X = Bl_p(Y)$ and $Y_{sing} \subset \{p\}$.

For our classification we compare ψ and ϕ. For example in the case ψ contracts a divisor D to a point or a curve, we obtain 15 families of del Pezzo fibrations, 14 families of conic bundles and 41 families of blow-up's at the moment (see [JPR04]). It is already clear by now that some of them cannot be deformations of smooth Fanos. Finding the true reason for this is what we are trying at the moment.

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A complex ball uniformization for the moduli spaces of del Pezzo surfaces via periods of K3 surfaces

Shigeyuki Kondo

We give a complex ball uniformization of the moduli spaces of del Pezzo surfaces by using the theory of periods of $K3$ surfaces. First of all we recall the work of Allcock, Carlson, Toledo [ACT] in which they gave a complex ball uniformization of the moduli space of smooth cubic surfaces by using the periods of abelian varieties. Let S be a smooth cubic surface given by $f_3(x_0, x_1, x_2, x_3) = 0$ in $\mathbf{P}^3$ where f_3 is a homogeneous polynomial of degree 3. Note that S is a del Pezzo surface of degree 3. Then consider the cubic threefold V given by $t^3 = f_3$ on which a projective transformation g of order 3 naturally acts. The intermediate Jacobian $J(V)$ of V is a 5-dimensional abelian variety with an automorphism ι of order 3 induced by g whose period is contained in the Siegel upper half plane. The fact that the period of $J(V)$ is an eigen-vector of ι implies that the period domain of $J(V)$ is a 4-dimensional complex ball.

Now let S be a del Pezzo surface of degree d. By definition, S is a smooth quadric or the blowing ups of $\mathbf{P}^2$ at $9 - d$ points in general position. We consider only the cases $d = 1, 2, 3, 4$ because in other cases del Pezzo surface is rigid. Instead of abelian varieties, we consider a $K3$ surface X with an automorphism σ of finite order associated to a smooth del Pezzo surface S. For example, if S is of degree 2, its ani-canonical map is a double covering of $\mathbf{P}^2$ branched along a smooth quartic curve C (We remark that this correspondence gives us an isomorphism between the moduli of smooth del Pezzo surfaces of degree 2 and that of smooth non-hyperelliptic curves of genus 3). Then by taking a 4-cyclic cover of $\mathbf{P}^2$ branched along C, we have a $K3$ surface with an automorphism σ of order 4 (if C is given by $f_4(x_0, x_1, x_2) = 0$, then X is a quartic surface given by $t^4 = f_4$). The period domain of the above $K3$ surfaces is a bounded symmetric domain of type IV and of dimension 12. The fact that the period of X is an eigen-vector of σ implies that the period domain of the pairs (X, σ) is a 6-dimensional complex ball B. The image of the period map is the complement of the union H of hypersurfaces in B. We call H the *discriminant locus*. By taking the quotient by an arithmetic subgroup Γ, we have a coarse moduli space $(B \setminus H)/\Gamma$ of smooth del Pezzo surfaces of degree 2 (Here we use the Torelli type theorem for $K3$ surfaces). We remark that H/Γ consists of two components. A generic member of them corresponds to a plane quartic curve with a node or a hyperelliptic curve of genus 3. For more details we referee the reader to [K1].

In cases of $d = 1, 4$, see Remarks 4–6 in [K2]. In case of $d = 3$, the result is a joint work with Igor Dolgachev and Bert van Geemen [DGK]. In cases of $d = 1, 3$ and 4, these complex ball uniformizations are closely related to the work of Deligne and Mostow [DM]. By the same way, we can give a complex ball uniformization of the moduli of curves of genus 4 ([K2]) and a uniformization of the moduli of curves of genus 6 by a symmetric domain of type IV ([K2], Remark 3).

It is an interesting problem to construct an automorphic form on B vanishing exactly on the discriminant locus H, if it exists. In fact, Allcock and Freitag [AF] found it in case of cubic surfaces ($d = 3$) by using Borcherds theory [B] on automorphic forms on a bounded symmetric domain on type IV.

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Braid monodromy of hypersurface singularities

Michael Lönne

Reporting on main results of [7] we introduce braid monodromy invariants associated to any right-equivalence class of hypersurface singularities. These invariants are investigated and determined in case of hypersurface singularities of Brieskorn Pham type.

A holomorphic function, more precisely a holomorphic function germ is usually studied by means of versal unfoldings, e.g. given by a function

$$F(x,z,u) = f(x) - z + \sum b_i u_i.$$

In case of a semi universal unfolding the unfolding dimension is given by the Milnor number $\mu = \mu(f)$ and we get a diagram

$$\begin{array}{ccccccc} z, u_1, ..., u_{\mu-1} & \mathbb{C}^\mu & \supset & \mathcal{D} & = & \{(z,u)|F(0,z,u) = 0 = \nabla F(0,z,u)\} \\ \downarrow & \downarrow & & & & \\ u_1, ..., u_{\mu-1} & \mathbb{C}^{\mu-1} & \supset & \mathcal{B} & = & \{u|F(\_,0,u) \text{ is not Morse}\} \end{array}$$

The restriction $p|_\mathcal{D}$ of the projection to the discriminant $\mathcal{D}$ is a finite map, such that the branch set coincides with the bifurcation set $\mathcal{B}$.

The key observation for the present work is, that a suitable restriction of p to a subset of $p^{-1}(\mathbb{C}^{\mu-1} \setminus \mathcal{B}) \setminus \mathcal{D}$ is a fibre bundle in a natural way. Its fibres are diffeomorphic to the μ-punctured disc and its isomorphism type depends only on the right equivalence class of f.

Thanks to Moishezon the study of complements of plane curves by the methods of Zariski and van Kampen has been revived [9], and has found a lot of applications, e.g [10, 11]. Conceptionally recast as braid monodromy theory it has been successfully generalized to the complements of hyperplane arrangements and it has found an interesting new interpretation in the theory polynomial coverings by Hansen, [2, 4].

Based on this interpretation the fibre bundle obtained from $p|_\mathcal{D}$ naturally gives rise to a braid monodromy homomorphism, which is in fact given by the Lyashko Looijenga map, [8], up to an inner automorphism of Br_μ and is an invariant of the unfolded function.

As in the case of plane curves the method of van Kampen, [6], leads to an explicit presentation of the fundamental group of the discriminant complement $\mathbb{C}^\mu \setminus \mathcal{D}$ in terms of generators and relations.

We address the problem to find the invariants and the group presentations for $\pi_1(\mathbb{C}^\mu \setminus \mathcal{D})$ in case of polynomial functions of a special kind:

Definition: A polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ is called *Brieskorn Pham polynomial*, if with $l_i \in \mathbf{Z}^{>0}$

$$f(x_1, ..., x_n) = x_1^{l_1+1} + \cdots + x_n^{l_n+1}.$$

Our main results are most naturally stated referring to the geometrically distinguished Dynkin diagram associated to f by Pham, Gabrielov and Hefez & Lazzeri, [12, 3, 5].

Definition: Let $I_n := \{i_1...i_n \,|\, 1 \leq i_\nu \leq l_\nu,\ 1 \leq \nu \leq n\}$ be a set of multiindices ordered lexicographically,

(1) Multiindices $i, j \in I_n$ are called *correlated*, if $i < j$ and $j_\nu \in \{i_\nu, i_\nu + 1\}$,
(2) Multiindices $i, j, k \in I_n$ are called *correlated*, if $i < j$, $i < k$, $j < k$ are.

Then up to sign the intersection graph is given by the set I_n of vertices and the set $\{(i, j) | \, i < j \text{ corr.}\}$ of edges.

Theorem. The braid monodromy group of a Brieskorn-Pham polynomial $x_1^{l_1+1} + \cdots + x_n^{l_n+1}$ is generated by the following twist powers:

$$\begin{array}{ccll} \sigma_{i,j}^3 & : & i < j & \text{correlated} \\ \sigma_{i,j}^2 & : & i < j & \text{not correlated} \\ \sigma_{j,k}\sigma_{i,j}^2\sigma_{j,k}^{-1} & : & i < j < k & \text{correlated} \end{array}$$

The most important corollary drawn from this theorem is a presentation of the fundamental group of the discriminant complement which can be computed by the method of Zariski and van Kampen and provides a partial answer to a problem of the list [1] of Brieskorn.

Theorem. The fundamental group $\pi_1(\mathbb{C}^k - \mathcal{D})$ for a Brieskorn Pham polynomial $x_1^{l_1+1} + \cdots x_n^{l_n+1}$ has a finite presentation

$$\begin{array}{llll} \langle t_i,\, i \in I & | & t_i t_j t_i = t_j t_i t_j, & i < j \text{ corr.}, \\ & & t_i t_j = t_j t_i, & i < j \text{ not corr.}, \\ & & t_i t_j t_k t_i = t_j t_k t_i t_j, & i < j < k \text{ corr.} \rangle \end{array}$$

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A foliation of S^5 by complex surfaces and its moduli space

Laurent Meersseman
(joint work with Alberto Verjovsky)

Let X be a smooth (that is C^∞) manifold of dimension $2n+1$ equipped with a smooth foliation $\mathcal{F}$. The foliation $\mathcal{F}$ is called a *foliation by complex manifolds* if it has a foliated atlas (U_α, ϕ_α) modelled on $\mathbb{C}^n \times \mathbb{R}$ such that the changes of charts

$$(z,t) \in \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \times \mathbb{R} \stackrel{\phi_\beta \circ \phi_\alpha^{-1}}{\longrightarrow} (\xi_{\alpha\beta}(z,t), \zeta_{\alpha\beta}(t)) \in \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \times \mathbb{R}$$

are holomorphic along the leaves, that is $\xi_{\alpha\beta}$ is a biholomorphism for fixed t.

In other words, the leaves of $\mathcal{F}$ are complex manifolds and $T_{\mathcal{F}}$, the tangent bundle to the foliation, is endowed with a smooth and integrable almost complex operator J. In fact, $(T_{\mathcal{F}}, J)$ is a smooth integrable and Levi-flat CR-structure.

As an example, take $X = \mathbb{S}^3$ endowed with the Reeb foliation. It has a compact leaf diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. The other leaves are diffeomorphic to $\mathbb{R}^2$. Fix an orientation and a smooth riemannian metric on X. By restriction, it defines an orientation and a metric on each 2-dimensional leaf of $\mathcal{F}$. Then define J as the rotation of angle $+\pi/2$. This defines a smooth almost complex operator on $T_{\mathcal{F}}$. Since the leaves have (real) dimension 2, it is automatically integrable. We thus obtain a smooth foliation of $\mathbb{S}^3$ by complex surfaces. The compact leaf is an elliptic curve and the other leaves are biholomorphic to $\mathbb{C}$.

This leads to the following question.

Question 3. *For which values of n does there exist a foliation by complex manifolds on $\mathbb{S}^{2n-1}$?*

In this abstract, I will describe such a foliation on $\mathbb{S}^5$ and give some of its properties. In particular, I will define the notion of a coarse moduli space of a foliation by complex manifolds and compute it in this case.

Theorem 2. *There exists a foliation by complex surfaces on $\mathbb{S}^5$.*

Let $\mathcal{F}_0$ be the foliation of [2]. As a smooth foliation, it is a variation of the Lawson foliation of $\mathbb{S}^5$ [1]; however it is topologically different. There are two compact leaves which are primary Kodaira surfaces. The non-compact leaves are $\mathbb{C}^*$-bundles over an elliptic curve, or line bundles over an elliptic curve or biholomorphic to the affine surface $\{z_1^3 + z_2^3 + z_3^3 = 1\}$ of $\mathbb{C}^3$.

Let me give some properties of $\mathcal{F}_0$. I first need some general definitions. Let X be a smooth manifold of dimension $2n+1$ equipped with a smooth foliation $\mathcal{F}^{diff}$. Define the set $\mathcal{C}(X, \mathcal{F}_{diff})$ as the set of foliations by complex manifolds on X which are diffeomorphic to $\mathcal{F}_{diff}$ (modulo CR-isomorphisms). We call this set *the set of complex structures on the foliation* $\mathcal{F}_{diff}$. It can be empty, and if not, can be finite or infinite dimensional (see [3]).

Assume now that $\mathcal{C}(X, \mathcal{F}_{diff})$ has a structure of a complex manifold M. Let $\pi : (W, J) \to B$ be a *deformation family of* $\mathcal{F}_{diff}$. This means that B is a complex manifold, that W is a smooth manifold equipped with a CR structure J and finally that π is a CR submersion such that

(i) the level sets W_t of π are diffeomorphic to X.
(ii) the CR structure J defines a foliation by complex manifolds $\mathcal{F}_t$ on W_t which is diffeomorphic to $\mathcal{F}_{diff}$.

Then there exists a natural map from B to M: it maps t to the point of M corresponding to $\mathcal{F}_t$ in $\mathcal{C}(X, \mathcal{F}_{diff})$.

In this context, we may adapt the notion of coarse moduli space. We say that M is *a coarse moduli space* for $\mathcal{F}_{diff}$ if, given any deformation family $\pi : (W, J) \to B$ of $\mathcal{F}_{diff}$, the natural map $B \to M$ is holomorphic (more a technical condition about stability by pull-backs).

We have

Theorem 3. *The set* $\mathcal{C}(\mathbb{S}^5, (\mathcal{F}_0)_{diff})$ *can be identified with* $\mathbb{C}^3$. *Moreover* $\mathbb{C}^3$ *is a coarse moduli space for* $(\mathcal{F}_0)_{diff}$.

This result has to be understood has a rigidity result. The key property is that, if $\mathcal{F}$ and $\mathcal{F}'$ are diffeomorphic to $(\mathcal{F}_0)_{diff}$ and have biholomorphic compact leaves, then they are CR isomorphic. In other words, a complex structure on $(\mathcal{F}_0)_{diff}$ is completely determined by the data of the complex structures on the two compact leaves.

Moreover, the two compact leaves are always primary Kodaira surfaces, that is elliptic bundles over an elliptic curve. Indeed one shows that both are quotient of a fixed $\mathbb{C}^*$-bundle over the same elliptic curve $\mathbb{E}_\alpha$ by a complex homothety (which may be different for each compact leaf). Therefore, the complex structure on the two compact leaves is entirely described by a triple $(\alpha, \beta, \beta') \in \mathbb{C}^3$ representing the modulus of the common base and the moduli of the fibers. This explains the $\mathbb{C}^3$ appearing in the statement of the Theorem.

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Automorphisms of hyperkäher manifolds

Keiji Oguiso

In [Mc], McMullen has found a very interesting K3 automorphism, namely, an automorphism having a Siegel disk. One of remarkable properties of such a K3 automorphism is that it is of positive entropy but has no dense orbit in the Euclidean topology. My surprise is that the target K3 surface is necessarily of algebraic dimension 0 (though it admits an automorphism of infinite order) and, contrary to the projective case, the character of some automorphism on the space of the two forms is not a root of unity. One can also make a simply-connected 4-dimensional counterexample of Kodaira's problem about algebraic approximation of compact Kähler manifolds from his K3 surface [Og1], as a supplement of a work of Voisin [Vo].

The rest is a review of my work about automorphisms of a hyperkähler manifold [Og1, 2], which is inspired by McMullen's K3 automorphism.

Let M be a compact Kähler manifold. We denote the biholomorphic automorphism group of M by $\mathrm{Aut}\,(M)$. Due to the works of Yomdin, Gromov and Friedland ([Yo], [Gr], [Fr]), the *topological entropy* $e(g)$ of an automorphism $g \in \mathrm{Aut}\,(M)$ can be defined by $e(g) := \log \delta(g)$. Here $\delta(g)$ is the spectral radius, i.e. the maximum of the absolute values of eigenvalues, of $g^*|H^*(M)$. One has $e(g) \geq 0$, and $e(g) = 0$ iff the eigenvalues of g^* are on the unit circle S^1. A subgroup G of $\mathrm{Aut}\,(M)$ is said to be of *null-entropy* (resp. of *positive-entropy*) if $e(g) = 0$ for $\forall g \in G$ (resp. $e(g) > 0$ for $\exists g \in G$).

A hyperkähler manifold (*a HK mfd*, for short) is a compact complex simply-connected Kähler manifold M admitting an everywhere non-degenerate global holomorphic 2-form ω_M such that $H^0(M, \Omega_M^2) = \mathbf{C}\omega_M$. Recall that $H^2(M, \mathbf{Z})$ admits a natural $\mathbf{Z}$-valued symmetric bilinear form called *BF-form* or *Bogomolov-Beauville-Fujiki's form* ([Be], see also an excellent survey [Hu, Section 1]). The existence of BF-form sometimes allows one to study HK mfds as if they were K3 surfaces (= 2-dimensional HK mfds).

The next two theorems are proved in [Og1, 2]:

Theorem 4. *Let M be a HK mfd. Let $\rho(M)$ be the Picard number of M. Then:*

(1) If M is not projective, then $\mathrm{Aut}\,(M)$ *is almost abelian of finite rank. More precisely, if the Néron-Severi group $NS(M)$ is of negative definite w.r.t. BF-form (resp. if otherwise, i.e. if BF-form on $NS(M)$ is degenerate), then* $\mathrm{Aut}\,(M)$ *is almost abelian of rank at most one (resp. at most $\rho(M) - 1$). Moreover, in the first case, it is of rank one iff* $\mathrm{Aut}\,(M)$ *has an element of positive entropy. In the second case,* $\mathrm{Aut}\,(M)$ *is always of null-entropy.*

(2) Let $G <$ $\mathrm{Aut}\,(M)$*. Assume that M is projective and G is of null-entropy. Then G is almost abelian of rank at most $\rho(M) - 2$.*

Here, a group G is called *almost abelian of finite rank r* if there are a normal subgroup $G^{(0)}$ of G of finite index, a finite group K and a non-negative integer r

which fit in the exact sequence $1 \longrightarrow K \longrightarrow G^{(0)} \longrightarrow \mathbf{Z}^r \longrightarrow 0$. The rank r is well-defined.

Theorem 5. *Let X be a K3 surface, $G < \mathrm{Aut}\,(X)$, and $g \in \mathrm{Aut}\,(X)$. Then:*

(1) G is of null-entropy iff either G is finite or G makes an elliptic fibration on X, say $\varphi : X \longrightarrow \mathbf{P}^1$, stable. Moreover, the estimate in the first theorem (2) is optimal for K3 surfaces.

(2) g is of positive entropy iff g has a Zariski dense orbit.

The second theorem gives an algebro-geometric characterization of (the positivity of) the topological entropy of a K3 automorphism. This result is also inspired by the following question of McMullen [Mc]:

Question 4. *Does a K3 automorphism g have a dense orbit (in the Euclidean topology) when a K3 surface is projective and g is of positive entropy?*

Note that McMullen's automorphism has a Zariski dense orbit but no dense orbit in the Euclidean topology.

Let us return back to the first theorem. In the statement (1), the first (resp. second) case exactly corresponds to the case $a(M) = 0$, $a(M) = 1 (= \dim M/2)$ when M is a K3 surface. Here $a(*)$ is the algebraic dimension of $*$. From this and a work of Matsushita [Ma] about fiber space structures on a projective HK mfd (which says that the dimension of the base space is either 0, $\dim M/2$ or $\dim M$), I cannot help posing the following question here:

Question 5. *Is the algebraic dimension $a(M) \in \{0, \dim M/2, \dim M\}$ for a HK mfd M?*

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The non-Petri locus for pencils

Edoardo Sernesi
(joint work with A. Bruno)

On a smooth projective curve Γ of genus γ we consider the following conditions on a pair $(L,x) \in \mathrm{Pic}^{d-1}(\Gamma) \times \Gamma$ for any d such that $\frac{\gamma+1}{2} + 1 \leq d \leq \gamma$:

(*) $h^0(L(-x)) \neq 0, \quad h^0(L(x)) \geq 2, \quad h^1(L^2) \neq 0$

We prove the following:

Theorem 1. *If Γ is a general curve then for all d as above the set of (L,x)'s satisfying conditions (*) is finite and non empty.*

Consider the coarse moduli space M_g of nonsingular curves of genus $g \geq 3$ and for each pair of positive integers d,r let $P^r_{g,d} \subset M_g$ be the locus of curves C carrying a g^r_d $\mathcal{L}$ for which the Petri map

$$H^0(\mathcal{L}) \otimes H^0(\omega\mathcal{L}^{-1}) \to H^0(\omega)$$

is not injective. It is a proper closed subset which we call the *non-Petri locus*. We apply the previous theorem to prove the following:

Theorem 2. *For each g,d such that $\frac{g}{2} + 1 \leq d \leq g-1$ the locus $P^1_{g,d}$ has a divisorial component.*

This theorem has been recently proved by G. Farkas using different methods. Our proof consists in associating to a pair (L,x) on Γ the cuspidal curve $C = \Gamma/2x$ endowed with the invertible sheaf $\mathcal{L}$ of degree d which pulls back to $L(x)$ on Γ. If (L,x) satisfies conditions (*) then $\mathcal{L}$ has non-injective Petri map. From this fact and from theorem 1 one derives the proof of theorem 2.

Numerically decomposing the intersection of algebraic varieties

Andrew J. Sommese
(joint work with Jan Verschelde and Charles W. Wampler)

In [6] an approach was formulated to numerically compute solution sets of systems of complex polynomials. The approach was by representing an irreducible i-dimensional component X of a polynomial system

$$f(x) := \begin{bmatrix} f_1(x_1,\ldots,x_N) \\ \vdots \\ f_n(x_1,\ldots,x_N) \end{bmatrix} \tag{1}$$

by "generic" points, where these are modeled by the intersection of X with a "random" $(N-i)$-dimensional linear subspace of $\mathbb{C}^N$. If such a set of points could be found, homotopy continuation would generate as many "widely-spaced" points as desired on X.

In a series of articles, [1, 2, 3, 4], an efficient algorithm was developed and implemented in software that, when given as input a system f as in Eq.1, outputs the dimensions and degrees of the irreducible components of $f^{-1}(0)$, and gives generic points of the form described above for each component.

A new algorithm from the upcoming article [5] was presented. This algorithm has for input two sets of "generic points" of two irreducible components (A a component of $f(x) = 0$ and B a component of a possibly identical system $g(x) = 0$) and as output, it has the dimensions, degrees, and sets of "generic points" of the components of $A \cap B$.

As one application of this method, a new "equation-by-equation" method of finding isolated solutions of polynomial systems was presented. This algorithm has the theoretical potential, which has been supported by some preliminary runs, to solve polynomial systems that are orders of magnitude beyond what can currently be solved.

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Complex geometric applications of Gauge Theory

Andrei Teleman
(joint work with Matei Toma)

A classical problem in Complex Geometry asks:

Question: *Let X be a complex surface. Which differentiable vector bundles E on X do allow holomorphic structures?*

Using the classification of differentiable vector bundles on 4-manifolds, the problem can be reformulated as follows:

For which triples $(r, c_1, c_2) \in \mathbb{N} \times H^2(M, \mathbb{Z}) \times \mathbb{Z}$ exists there a holomorphic rank r vector bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = c_1$, $c_2(\mathcal{E}) = c_2$?

A result of Schwarzenberger solves the problem in the algebraic case:

Answer: *A vector bundle E on an algebraic surface X admits holomorphic structures if and only if $c_1(E) \in \mathrm{NS}(X)$.*

In the non-algebraic case the problem is not completely solved. We refer to [ABT], [To1], [To2] for results in particular cases.

Let E be a differentiable rank r bundle on X and $c \in \mathrm{NS}(X)$. Following the conventions in [BLP], we put:

$$\Delta(E) := 2rc_2(E) - (r-1)c_1(E)^2 \ , \ m(r,c) := r \inf \left\{ -\sum_{i=1}^{r} \left(\frac{c}{r} - \mu_i\right)^2 \mid \mu_i \in \mathrm{NS}(X) \right\} .$$

Note that $m(r,c) \geq 0$ if X is non-algebraic, because in this case the intersection form is negative on the Neron-Severy group. A holomorphic rank r bundle $\mathcal{E}$ is called *filtrable* if it admits a filtration

$$0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}$$

by subsheaves of ranks $\mathrm{rk}(\mathcal{E}_i) = i$. The main result in the non-algebraic case is the following theorem of Bănică&Le Potier.

Theorem: [BLP] *Let X be a non-algebraic surface and a differentiable rank r bundle on X.*

1. *If E admits holomorphic structures, then $\Delta(E) \geq 0$.*
2. *E admits filtrable holomorphic structures if and only if $c_1(E) \in NS(X)$ and $\Delta(E) \geq m(r, c_1(E))$, excepting the case when X is a K3 surface with $a(X) = 0$, $\Delta(E) = 2r$ and $c_1(E) \in rNS(X)$.*

In the excepted case, E admits no holomorphic structure.

Therefore, it remains to decide whether bundles E with $0 \leq \Delta(E) < m(r, c_1(E))$ admit holomorphic structures or not. Such holomorphic structures cannot be filtrable, so it is very difficult to detect them. To every class $c \in \mathrm{NS}(X)$ we associate the set

$$\mathrm{Hol}(r,c) := \{\Delta(\mathcal{E}) | \ \mathcal{E} \text{ is a holomorphic } r\text{-bundle on } X \text{ with } c_1(\mathcal{E}) = c\} ,$$

which is a subset of $[-(r-1)c^2 + 2r\mathbb{Z}] \cap [0, \infty)$. Our problem reduces to the computation of the sets $\mathrm{Hol}(r,c)$. The smallest element in $\mathrm{Hol}(r,c)$ will be denoted by $\Delta_{\min}(r,c)$.

In [TT1] we solved completely the existence problem for holomorphic structures on rank 2 vector bundles over non-algebraic K3 surfaces:

Theorem: [TT1] *Let X be a non-algebraic K3 surface. For every $c \in \mathrm{NS}(X)$ one has*

$$\mathrm{Hol}(2,c) = [-c^2 + 4\mathbb{Z}] \cap [\min(6, m(2,c)), \infty) ,$$

excepting the case when $a(X) = 0$ and $c \in 2\mathrm{NS}(X)$. In this case

$$\mathrm{Hol}(2,c) = \mathrm{Hol}(2,0) = \{0\} \cup (4\mathbb{Z}_{>0} + 4) .$$

The proof uses the Kobayashi-Hitchin correspondence in the non-algebraic case ([DK], [LT]) and the explicit form of the Donaldson polynomial invariants of K3 surfaces, as given in [KM].

These methods apply to any non-algebraic Kählerian surface. For a closed, connected, oriented 4-manifold M, we denote by $w_2(\mathrm{Hom}(\pi_1(M), PU(2)))$ the set of Stiefel-Whitney classes of flat $PU(2)$ bundles over M. If P is a $PU(2)$-bundle over M with $w_2(P) \notin w_2(\mathrm{Hom}(\pi_1(M), PU(2)))$, then the Uhlenbeck compactification $\overline{\mathcal{M}}_g^{\mathrm{ASD}}(P)$ of the moduli space of g-ASD connection does not contain flat instantons, and all strata of this compactification are regular, for a generic Riemannian metric g on M. Therefore, the Donaldson polynomial invariant

$$q_P : H^2(M, \mathbb{Z})^{d_P} \longrightarrow \mathbb{Z}$$

associated with P is well defined. Here we assume of course that the *complex expected dimension*

$$d_P := -p_1(P) + \frac{3}{2}(b_1(M) - b_+(M) - 1)$$

is an integer.

Our result for a general Kählerian surface X is:

Theorem: [TT2] *Let X be a Kählerian surface, $c \in \mathrm{NS}(X)$ and $\bar{c} \in H^2(M, \mathbb{Z}_2)$ its reduction modulo 2. Then one of the following holds:*

(1) $\bar{c} \in w_2(\mathrm{Hom}(\pi_1(M), PU(2)))$. *In this case* $\Delta_{\min}(2,c) = 0$.
(2) $\bar{c} \notin w_2(\mathrm{Hom}(\pi_1(M), PU(2)))$ *and* $\Delta_{\min}(2,c)$ *is realised by a filtrable holomorphic 2-bundle.*
(3) $\bar{c} \notin w_2(\mathrm{Hom}(\pi_1(M), PU(2)))$ *and* $\Delta_{\min}(2,c)$ *is realised by a non-filtrable holomorphic 2-bundle* $\mathcal{E}$ *with* $H^2(\mathcal{E}nd_0(\mathcal{E})) \neq 0$.
(4) $\Delta_{\min}(2,c) = \min\{-p_1(P) |\ w_2(P) = \bar{c},\ q_P \neq 0\}$ *hence, in this case,* $\Delta_{\min}(2,c)$ *coincides with a differential topological invariant of* X.

The proof is based on a version of Donaldson's non-vanishing theorem [DK].

Note that, for a holomorphic 2-bundle $\mathcal{E}$ with $H^2(\mathcal{E}nd_0(\mathcal{E})) \neq 0$, there exists a dominant morphism $p : Y \to X$ defined on a Kählerian surface Y, such that $p^*(\mathcal{E})$ is filtrable. In most cases Y is a 2-sheeted cover of X branched along a bicanonical divisor of X. Therefore, such bundles, as the filtrable ones, can be constructed "by classical methods". Our result shows that:

When $\Delta_{\min}(2, c)$ cannot be obtained using holomorphic bundles given by classical methods, then it coincides with a differential topological invariant.

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Reporter: Ivo Radloff

Participants

Prof. Dr. Daniel Barlet

Departement de Mathematiques
Universite de Nancy I
Boite Postale 239
F-54506 Vandoeuvre les Nancy Cedex

Dr. Ingrid Bauer

Mathematisches Institut
Universität Bayreuth
95440 Bayreuth

Dr. Thomas Rudolf Bauer

Mathematisches Institut
Universität Bayreuth
95440 Bayreuth

Dr. Laurent Bonavero

Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Ciprian Borcea

Department of Mathematics
Rider University
2083 Lawrenceville Road
Lawrenceville, NJ 08648-3099
USA

Prof. Dr. Hans-Christian von Bothmer

Institut für Mathematik
Universität Hannover
Postfach 6009
30060 Hannover

Dr. Sebastien Boucksom

Institut de Mathematiques
Equipe Topologie, Geometrie et Dyn.
Universite de Paris VII; Case 7012
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Michel Brion

Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Frederic Campana

Dept. de Mathematiques
Universite de Nancy I
B.P. 239
F-54506 Vandoeuvre-les-Nancy Cedex

Prof. Dr. Fabrizio Catanese

Lehrstuhl für Mathematik VIII
Universität Bayreuth
NW - II
95440 Bayreuth

Dr. Bertrand Deroin

Mathematiques
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Prof. Dr. Wolfgang Ebeling

Institut für Mathematik
Universität Hannover
Postfach 6009
30060 Hannover

Dipl.-Math. Cord Erdenberger

Institut für Mathematik (C)
Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Akira Fujiki

Department of Mathematics
Graduate School of Science
Osaka University
1-1 Machikaneyama-cho,Toyonaka
Osaka 560-0043
JAPAN

Dr. Andreas Gathmann

Fachbereich Mathematik
Universität Kaiserslautern
67653 Kaiserslautern

Dr. Samuel Grushevsky

Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
USA

Dr. Gordon Heier

NA 4/ 72
Fakultät für Mathematik
Ruhr-Uni Bochum
Universitätsstraße 150
44780 Bochum

Prof. Dr. Peter Heinzner

Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

Prof. Dr.Dr.h.c. Alan T. Huckleberry

Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Klaus Hulek

Institut für Mathematik (C)
Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Jun-Muk Hwang

School of Mathematics
Korea Inst. for Advanced Study
207-43 Cheongnyangni-dong
Dongdaemun-gu
Seoul 130-722
Korea

Dr. Priska Jahnke

Universität Bayreuth
Mathematisches Institut
Universitätsstr. 30
95440 Bayreuth

Prof. Dr. Yujiro Kawamata

Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Stefan Kebekus

Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln

Prof. Dr. Shigeyuki Kondo

Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
Japan

Prof. Dr. Manfred Lehn

Fachbereich Mathematik und
Informatik
Universität Mainz
55099 Mainz

Dr. Michael Lönne

Institut für Mathematik (C)
Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Eduard J.N. Looijenga

Faculteit Wiskunde
Universiteit Utrecht
Postbus 80.010
NL-3508 TA Utrecht

Prof. Dr. Jon Ingolfur Magnusson

Dept. of Mathematics
Science Institute
University of Iceland
Dunhaga 3
IS-107 Reykjavik

Dr. Laurent Meersseman

U. F. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Jan Nagel

UFR Math. Pures Appl.
Universite de Lille
F-59655 Lille

Prof. Dr. Karl Oeljeklaus

LATP, UMR-CNRS 6632
Centre de Mathematiques et Inform.
Universite de Provence, Aix-Mars.I
39, rue Joliot Curie
F-13453 Marseille Cedex 13

Prof. Dr. Keiji Oguiso

Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Thomas Peternell

Fakultät für Mathematik und Physik
Universität Bayreuth
95440 Bayreuth

Dr. Ivo Radloff

Universität Bayreuth
Mathematisches Institut
Universitätsstr. 30
95440 Bayreuth

Prof. Dr. Frank-Olaf Schreyer

FB Mathematik und Informatik
Geb. 27
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken

Prof. Dr. Edoardo Sernesi

Dipartimento di Matematica
Universita degli Studi "Roma Tre"
Largo S. Leonardo Murialdo 1
I-00146 Roma

Prof. Dr. Andrew J. Sommese

Department of Mathematics
University of Notre Dame
Notre Dame IN 46556-4618
USA

Prof. Dr. Mina Teicher

Dept. of Mathematics
Bar-Ilan University
52 900 Ramat-Gan
ISRAEL

Prof. Dr. Andrei Teleman

Centre de Mathematiques et
d'Informatique
Universite de Provence
39, Rue Joliot-Curie
F-13453 Marseille Cedex 13

Prof. Dr. Misha Verbitsky

Department of Mathematics
University of Glasgow
University Gardens
GB-Glasgow, G12 8QW

Prof. Dr. Jörg Winkelmann

Institut Elie Cartan
-Mathematiques-
Universite Henri Poincare, Nancy I
Boite Postale 239
F-54506 Vandoeuvre les Nancy Cedex

Prof. Dr. Mikhail Zaidenberg

Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Kang Zuo

Fachbereich 17
AG Algebraische Geometrie
Johannes-Gutenberg-Universität
55099 Mainz