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Introduction by the Organisers

This was the most recent in a long series of annual conferences in Oberwolfach covering all areas of algebraic and geometric topology, and the last such conference before going over to the new two-year cycle of meetings. According to the records kept in the library of the institute, the first topology meeting was held in 1963, and meetings have been held every year since then except for 1968. None of the participants in the first meeting is still active in research. Of the people present at this meeting, it was Rainer Vogt who has been attending this series for the longest period: since 1969.

Every year for the last twelve years, a "keynote speaker" has been chosen to give some focus to the topology meeting. Thus while we do have talks which cover all areas of algebraic and geometric topology, we try to focus on one particular area of current interest. This year, the keynote speaker was Yair Minsky, who talked about the classification of non-compact hyperbolic 3-manifolds N with finitely generated fundamental group.

The two main conjectures (now proved) in this area are Marden's Tameness Conjecture and Thurston's Ending Lamination Conjecture. The Tameness Conjecture is about the topology of N, and asserts that any end of N is *topologically tame*, i.e., is homeomorphic to $S \times \mathbb{R}$ for some closed surface S. The Ending Lamination Conjecture is about the geometry of N, and concerns the data needed to determine N up to isometry. The most interesting case is that of a geometrically infinite (and topologically tame) end ε of N. Thurston showed how to associate to such an ε a geodesic lamination λ on the surface S, the *ending lamination* of ε, and conjectured that ε is determined up to isometry by λ.

The Ending Lamination Conjecture was recently proved by Minsky, partly in joint work with Brock and Canary, and making use of the earlier result of Masur and Minsky that the curve complex of a surface is hyperbolic in the sense of Gromov. At the meeting, Minsky gave the keynote series of three lectures on the background to and proof of the Ending Lamination Conjecture.

Within the last year, the Tameness Conjecture has also been proved, by Agol (an independent proof has also been announced by Calegari and Gabai). In his talk at the meeting Agol sketched some of the ideas of the proof and outlined several applications to other problems in 3-dimensional topology.

The Tameness and Ending Lamination Conjectures together give a complete parametrization of the set $AH(M)$ of (non-compact) hyperbolic 3-manifolds homotopy equivalent to a given compact 3-manifold M with non-empty boundary. However, if $AH(M)$ is given the natural *algebraic topology*, i.e., that coming from its containment in the $\mathrm{PSL}_2(\mathbb{C})$ character variety of $\pi_1(M)$, then the classifying data is not continuous. As a consequence, the topological structure of $AH(M)$, in other words the deformation theory of these hyperbolic structures, is not completely clear. This was the topic of Canary's talk. He described how, although (in the case that ∂M is incompressible) the components of the interior of $AH(M)$ are open topological cells, their closures can intersect in unexpected and wild ways. Probably the main problem left in this whole area is to better understand the topology of this deformation space $AH(M)$. For example, what is the Hausdorff dimension of its boundary?

In contrast, if M is a closed hyperbolizable 3-manifold, then $AH(M)$ is simply a pair of points, by Mostow rigidity. However, even in this case the relation between the geometry of M (i.e., its hyperbolic metric) and its topology is not well understood. Souto's talk addressed an interesting question in this context, namely the relation between the Heegaard genus of M and the lengths of geodesics in M. Specifically, if S is a genus g Heegaard surface in M, then, although it is easy to see that there is no lower bound on the lengths of closed geodesics in M that depends only on g, Souto showed that there exists $\varepsilon_g > 0$ such that the set of primitive closed geodesics in M of length $\leq \varepsilon_g$ is unknotted in the sense that it can be isotoped to lie on parallel copies of S.

The other talks were chosen to cover as many different areas of topology as possible, and hence it is difficult to find an overall theme to describe them. On the more geometric side, Nathalie Wahl described her work on diffeomorphism groups of 3-manifolds obtained by attaching certain types of handles to S^3, and their connection to groups of self equivalences of certain graphs. Among the applications of this work are new proofs of homological stability of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ (F_n a free group), the vanishing of $H_*(\mathrm{Aut}(F_n);\mathbb{Z}^n)$ in a range, and the construction of an infinite loop map from $\mathbb{Z} \times B\Gamma^+_\infty$ (the limit of mapping class groups of surfaces) to $\mathbb{Z}\times B\mathrm{Aut}^+_\infty$ (the limit of the $\mathrm{Aut}(F_n)$). Ian Hambleton described his recent proof that for any pair of finite periodic groups G and G', the product $G \times G'$ acts freely and smoothly on $S^n \times S^n$ for some n — even in the cases (already well known) when G and G' themselves do not act freely on any spheres. Stefan Bauer described his recent work on invariants of 4-manifolds, including a refinement of the Seiberg-Witten invariants due to him and Furuta. Hyam Rubinstein talked about an interesting generalization of the class of small Seifert fibered 3-manifolds, in which the three solid tori whose union is the manifold are replaced by handlebodies of genus 2.

In a more algebraic direction, Jesper Grodal described some of the latest developments in the field of 2-compact groups — spaces which are complete at the prime 2, and whose loop space has finite mod 2 cohomology (i.e., looks like the 2-completion of a finite complex). The goal is to classify all simply connected 2-compact groups (this has already been done at odd primes), and understand how close they are to being 2-completions of classifying spaces of compact connected Lie groups. In his talk, Grodal focused on the problem of defining root systems for 2-compact groups, and some of the problems which arise at the prime 2 and did not arise for odd primes.

Among the other algebraic talks, Kathryn Hess described new algorithms for describing the Hopf algebra structure on the homology of the loop space of a space X, in terms of an appropriate model for chains on X. In the field of geometric group theory, Martin Bridson talked about subgroups of direct products of hyperbolic groups, and described his counterexample with Grunewald to a conjecture of Grothendieck, where they construct a homomorphism of finitely presented, residually finite groups which induces an equivalence of representation categories but is not an isomorphism.

Workshop: Topologie

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Abstracts

Ends and the classification of hyperbolic 3-manifolds

Yair Minsky

I gave a series of three talks aimed at discussing the classification and deformation theory of hyperbolic 3-manifolds.

The first lecture was an overview and "tour" of hyperbolic 3-manifolds. I discussed a number of examples (knot complements, Fuchsian groups, manifolds formed by amalgamations) and their basic structure: convex core, limit sets, boundary at infinity. For finite volume manifolds Mostow-Prasad rigidity [18, 19] tells us that their hyperbolic structure is unique, but in the infinite volume case there is typically a high-dimensional parameter space of deformations (Bers [1, 2]).

I also introduced cusps and Margulis tubes, which are standard tubular neighborhoods of short geodesics (see Thurston [23] or Kapovich [10] for general structure of hyperbolic 3-manifolds).

In the second talk I began with a discussion of how interesting examples are constructed by perturbing cusps. Thurston's Dehn-filling theorem (and its various generalizations by Bonahon-Otal, Comar and Bromberg [7]) allow a manifold with a cusp to be perturbed slightly in a way that turns the cusp into a Margulis tube with very short geodesic core. Constructions of Kerckhoff-Thurston [11] and Bonahon-Otal [4] used this idea to generate Kleinian surface groups with many (even infinitely many) very short geodesics. More exotic geometric limits are constructed by Brock [5] and Soma [20].

A Kleinian surface group is a discrete Kleinian group isomorphic to $\pi_1(S)$ for a closed surface S (the case of surfaces with cusps can also be treated). Classically these are constructed by Ahlfors and Bers as quasiconformal deformations of Fuchsian groups, hence called quasifuchsian groups. These are parameterized, by Bers' simultaneous uniformization theorem [1], by a product of Teichmüller spaces. In general there are also invariants called "ending laminations", introduced by Thurston (see Minsky [15, 14]), which serve to describe ends of the manifold that are not associated to Teichmüller data. I discussed the definition of ending laminations and a few more examples of this type.

In the third talk I sketched Thurston's "Intersection number lemma" which is responsible for the definition of ending laminations and, in improved form, for Bonahon's theorem [3] that surface groups are tame and hence admit ending laminations when they are not geometrically finite. I then discussed Thurston's Ending Lamination Conjecture [22] which states that these invariants suffice to determine the surface group uniquely.

I introduced the notion of pleated surfaces (see Canary-Epstein-Green [8]) and of the complex of curves (see Harvey [9]), and indicated very briefly how these tools are used in the proof of the Ending Lamination Conjecture. Fixing a hyperbolic 3-manifold N homotopy-equivalent to S (without cusps for simplicity), we can "realize" any essential simple closed curve in S totally geodesically in an intrinsically hyperbolic surface in N, and extracting from this surface its shortest simple curve, we obtain a coarse self-map of the complex of curves. This map is shown in [16] to be contracting in a strong sense, and together with hyperbolicity of the complex of curves (Masur-Minsky [12]) this is the first step in obtaining geometric control of the set of "short curves" in N based on its ending invariants. In Minsky [17] this argument is carried out to provide a "Lipschitz model" for the manifold N and its thick-thin decomposition, based on the work in Masur-Minsky [13]. In Brock-Canary-Minsky [6] this model is upgraded to a *bilipschitz* model, which together with Sullivan's rigidity theorem [21] gives a proof of the conjecture.

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Short geodesics in hyperbolic 3-manifolds are not knotted

Juan Souto

Following Otal [3] we say that a simple closed curve in a 3-manifold N is *unknotted* with respect to a closed embedded surface S if γ can be isotoped into S; equivalently, γ is contained in an embedded surface S' isotopic to S. More generally, a finite collection $\{\gamma_1, \dots, \gamma_n\}$ of disjoint simple closed curves in N is *unlinked* with respect to S if there is a collection of disjoint embedded parallel surfaces $S_1, \dots, S_n$ isotopic to S and with $\gamma_i \subset S_i$ for $i = 1, \dots, n$.

Otal [3] proved that for all g there is some positive ϵ_g such that if N is a complete hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$ then the set of all primitive geodesics in N shorter than ϵ_g is unlinked with respect to $\Sigma_g \times \{0\}$. Here Σ_g is the closed surface of genus g.

Theorem 1. *For all g there is $\epsilon_g > 0$ such that for every closed hyperbolic manifold N and for every genus g Heegaard surface $S \subset N$ the following holds: The collection of primitive simple closed geodesics in N which are shorter than ϵ_g is unlinked with respect to S.*

If one is interested only in studying closed hyperbolic manifolds with volume less than some constant C then Theorem 1 follows from a result due to Moriah-Rubinstein [2]. However, it is an important feature of Theorem 1 that the constant ϵ_g depends only on the genus of the Heegaard surface. As in [2], the proof of Theorem 1 relies on the relation between Heegaard splittings and minimal surfaces due to Pitts-Rubinstein [4].

The relation between Heegaard splittings and short geodesics established by Theorem 1 is one of the crucial arguments needed in Brock-Souto [1], where we give upper and lower bounds on the volume of a hyperbolic 3-manifold in terms of the combinatorics of one of its Heegaard splittings.

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Quasi-isometry invariance of the cohomology ring of nilpotent groups

Roman Sauer

We report on some results concerning the invariance of certain group homological invariants under quasi-isometry, in particular the quasi-isometry invariance of the cohomology algebra of nilpotent groups. The focus is on the general induction technique which allows to prove these results. The starting point of the method described below is Gromov's alternative description of groups being quasi-isometric [Gro93, $0.2.C_2'$].

Theorem (Gromov's Dynamic Criterion)**.** *Two finitely generated groups* Λ, Γ *are quasi-isometric if and only if there exists a non-empty locally compact space, called a coupling of* Λ *and* Γ*, on which both groups act continuously, proper and cocompactly in such a way that the actions commute.*

The first use of the dynamic criterion as a basis for defining an induction map between the group cohomology of quasi-isometric groups is in Shalom's fundamental paper on the geometry of amenable groups [Sha]. His definition is very explicit in terms of the standard bar resolutions of the groups. We provide a new, "homological-algebra"-flavoured description of the induction map thereby generalizing and sharpening some of Shalom's results. As already in Shalom's work, this induction exists also in the more general setting of a uniform embedding of groups, a notion encompassing subgroup inclusions and quasi-isometric embeddings. Furthermore, the existence of a uniform embedding $\Lambda \to \Gamma$ is equivalent to the existence of a coupling as in Gromov's criterion with the exception that the Λ-action on the coupling is not required to be cocompact.

The essentials of the induction map are being described now. Suppose that there exists a coupling of the groups Λ and Γ; it may come from a quasi-isometry of Λ, Γ or from a uniform embedding $\Lambda \to \Gamma$. Let R be a commutative ring. Then there is a compact topological space Y (constructed from the coupling) with a continuous Λ-action, a functor $\bar{I} : \{R\Lambda\text{-modules}\} \to \{R\Gamma\text{-modules}\}$ and a homomorphism, called the **induction**, in cohomology

$$\text{(1)} \qquad \mathrm{I}^n : \mathrm{H}^n(\Lambda, M) \longrightarrow \mathrm{H}^n(\Lambda, \mathcal{F}(Y;R) \otimes_R M) \xrightarrow{\cong} \mathrm{H}^n(\Gamma, \bar{I}(M))$$

for every $R\Lambda$-module M. Here $\mathcal{F}(Y;R)$ is the ring of functions $Y \to R$ with the property that the preimage of any $r \in R$ is open and closed; it carries a natural Λ-action. We consider $\mathcal{F}(Y;R) \otimes_R M$ as an $R\Lambda$-module by the diagonal Λ-action. The first map in (1) is induced by the inclusion $M \hookrightarrow \mathcal{F}(Y;R) \otimes_R M$, $m \mapsto \mathrm{id}_Y \otimes m$. The second map in (1) is always an isomorphism. A similar discussion applies to group homology. If we can prove that the first map and hence I^n are injective under certain assumptions, then we get the estimate $\mathrm{cd}_R(\Lambda) \leq \mathrm{cd}_R(\Gamma)$ for the cohomological dimensions over R. So we do not need to care so much about the actual definitions of $\bar{I}(M)$ and the second map in (1) in this case. The following theorem is obtained by analyzing the first map.

Theorem. *Let R be a commutative ring, and suppose Λ embeds uniformly into Γ where Λ and Γ are discrete, countable groups. Then the following two statements hold.*

(i) *If $\mathrm{cd}_R(\Lambda)$ is finite, then we have $\mathrm{cd}_R(\Lambda) \leq \mathrm{cd}_R(\Gamma)$.*
(ii) *If Λ is amenable and $\mathbb{Q} \subset R$, then we have $\mathrm{cd}_R(\Lambda) \leq \mathrm{cd}_R(\Gamma)$.*

Furthermore, $(i), (ii)$ hold true if cd_R is replaced by the homological dimension hd_R.

Here statement (ii) for the cohomological dimension is already proved in [Sha, theorem 1.5] and was conjectured for the homological dimension in [Sha, section 6.4]. An important point is that we can deal with non-amenable groups by imposing a finiteness condition. The theorem above also generalizes a result of Gersten [Ger93]. By a result of Stammbach [Sta70], the rational homological dimension of a solvable group equals its Hirsch number. Hence we obtain the following corollary which was known before only under additional finiteness conditions on the groups (see [BG96], [Sha]).

Corollary. *Let Γ be a solvable group. Let Λ be a solvable group quasi-isometric to Λ. Then the Hirsch ranks of Γ and Λ coincide.*

Finally, we mention a generalization of Shalom's result saying that the Betti numbers of quasi-isometric nilpotent groups are quasi-isometry invariants.

Theorem. *If Γ and Λ are quasi-isometric nilpotent groups, then the real cohomology rings $\mathrm{H}^*(\Gamma, \mathbb{R})$ and $\mathrm{H}^*(\Lambda, \mathbb{R})$ are isomorphic as graded rings.*

We remark that the corresponding statement for rational coefficients does *not* hold. By a theorem of Nomizu the cohomology algebra of a nilpotent group and that of its Malcev Lie algebra are isomorphic. So this theorem sheds some positive light on a long standing question in geometric group theory: Are the Malcev completions of quasi-isometric nilpotent groups isomorphic?

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On the stabilization of the mapping class group of 3-manifolds

Nathalie Wahl

Let N be a compact, connected, oriented 3-manifold (whose universal cover satisfies the Poincaré conjecture and with no summand a copy of $\mathbb{R}P^3$) and consider the manifold

$$M^s_{n,k} := N \# (\#_n\ S^1 \times S^2) \# (\#_k\ S^1 \times D^2) \# (\#_s\ D^3)$$

obtained from N by attaching n handles, removing k tori and removing s balls. Let $A^s_{n,k}$ denote the mapping class group $\pi_0\mathrm{Diff}(M^s_{n,k};\partial)$, the group of components of diffeomorphisms fixing the boundary pointwise, with twists along 2-spheres factored out. In a joint work with A. Hatcher [3], we show that the homology group $H_i(A^s_{n,k};\mathbb{Z})$ is independent of n,k and s when $n \geq 3i+3$. This is proved by studying complexes of embedded spheres and discs in the manifold $M^s_{n,k}$.

We describe several applications of this theorem.

When $N = S^3$, the groups $A^s_{n,k}$ can be interpreted as "automorphisms of free groups with boundaries", otherwise defined as the group of components of the homotopy equivalences of a certain graph $G^s_{n,k}$ fixing k boundary circles and s boundary points in the graph (see [4, 7]). When $k = 0$, we have $A^0_{n,0} \cong Out(F_n)$, $A^1_{n,0} \cong Aut(F_n)$ and $A^2_{n,0} \cong F_n \rtimes Aut(F_n)$, where F_n denotes the free groups on n generators and $Aut(F_n)$ and $Out(F_n)$ its automorphism and outer automorphism groups. So the homological stability of $Aut(F_n)$ and $Out(F_n)$ is a special case of the above stability theorem (although a better stability range is known [1, 2]). Another direct consequence of our theorem is that the twisted homology group $H_i(Aut(F_n), H_1(F_n)) = 0$ when $n \geq 3i+9$. This is obtained by considering the spectral sequence for the short exact sequence $F_n \to A^2_{n,0} \to A^1_{n,0}$.

Our main application is to the study of the map from the mapping class groups of surfaces to the automorphism groups of free groups given by the action of diffeomorphisms on the fundamental group of the surface (which is a free group if the surface has at least one boundary component). Both the stable mapping class group of surfaces Γ_∞ and the stable automorphism group of free groups Aut_∞ give rise to infinite loop spaces $B\Gamma^+_\infty$ and $BAut^+_\infty$ when taking the plus-construction of their classifying spaces. By the work of Madsen and Weiss [5], $B\Gamma^+_\infty$ is now well understood, whereas $BAut^+_\infty$ remains rather mysterious. In [7], we use the stability theorem for the groups $A^1_{n,k}$ to show that the natural map $B\Gamma^+_\infty \to BAut^+_\infty$ is a map of infinite loop spaces. The infinite loop space structure on $B\Gamma^+_\infty$, as discovered by Tillmann [6], comes from considering disjoint union on a cobordism category $\mathcal{S}$ with the property that $\Omega B\mathcal{S} \simeq \mathbb{Z} \times B\Gamma^+_\infty$. We use automorphisms of free groups with boundaries to construct a larger cobordism category $\mathcal{T}$ built with graphs as well as surfaces. The stability theorem is then the main ingredient in showing that $\Omega B\mathcal{T} \simeq \mathbb{Z} \times BAut^+_\infty$, and the infinite loop map is induced by the inclusion of categories $\mathcal{S} \to \mathcal{T}$.

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Finite domination and nonsingular closed 1-forms

Dirk Schütz

Given a closed connected and smooth manifold M, one can ask the question which elements of $H^1(M;\mathbb{R})$ can be represented by nonsingular closed 1-forms. In the case that $\dim M \geq 6$ Latour [3] gave necessary and sufficient conditions for the existence of a nonsingular closed 1-form representing ξ. These conditions are:

- M is $(\pm\xi)$-contractible;
- a K-theoretic obstruction $\tau_L(M,\xi) \in \mathrm{Wh}(\pi_1(M);\xi)$ vanishes.

Let us describe these terms. Assume we have a finite connected CW-complex X and let $\xi \in H^1(X;\mathbb{R})$ be nonzero. We can think of ξ as a homomorphism $\xi : \pi_1(X) \to \mathbb{R}$. Let $\rho : \overline{X} \to X$ be the covering space corresponding to $\ker\xi$. Then $\rho^*\xi = 0$. Notice that $\pi_1(X)$ acts on $\overline{X}$ by covering transformations and on $\mathbb{R}$ by $g \cdot x = x + \xi(g)$. We can find an equivariant map $h : \overline{X} \to \mathbb{R}$, that is, a map with $h(gx) = h(x) + \xi(g)$.

Now we say that X is ξ-*contractible* if given $\varepsilon > 0$ and an equivariant $h : \overline{X} \to \mathbb{R}$ there is an equivariant homotopy $H : \overline{X} \times [0,1] \to \overline{X}$ with $H_0(x) = x$ and $hH_1(x) - h(x) \leq -\varepsilon$ for all $x \in \overline{X}$. This definition does not depend on ε or h. Also the notion of ξ-contractibility is a homotopy invariant. Furthermore if X is ξ-contractible, the Novikov complex $C_*(X;\widehat{\mathbb{Z}\pi_1(X)}_\xi)$ is acyclic. Here $\widehat{\mathbb{Z}\pi_1(X)}_\xi$ is a completion of the group ring $\mathbb{Z}\pi_1(X)$. So if M is $(\pm\xi)$-contractible, the obstruction $\tau_L(M,\xi)$ is simply the Whitehead torsion of the Novikov complex in an appropriate quotient of $K_1(\widehat{\mathbb{Z}\pi_1(M)}_\xi)$.

Let us now discuss the special case when a positive multiple of ξ sits in $H^1(M;\mathbb{Z})$. Such ξ can be represented by a map $f : M \to S^1$ and the question whether ξ can be represented by a nonsingular closed 1-form turns into the question whether there is a smooth fibre bundle map $f : M \to S^1$ representing ξ. For $\dim M \geq 6$ necessary and sufficient conditions where given by Farrell [2]. They are:

- the covering space $\overline{M}$ corresponding to $\ker\xi$ is finitely dominated;

- a K-theoretic obstruction $\tau_F(M,\xi) \in \mathrm{Wh}(\pi_1(M))$ vanishes.

For $\xi \in H^1(M;\mathbb{Z})$ the conditions that M is $(\pm\xi)$-contractible and that $\overline{M}$ is finitely dominated are equivalent, and in that case $i_*\tau_F(M,\xi) = \tau_L(M,\xi)$. Here $i_* : \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(M);\xi)$ is induced by the inclusion of rings $\mathbb{Z}\pi_1(M) \subset \widehat{\mathbb{Z}\pi_1(M)}_\xi$. This was shown by Ranicki [4].

Easy examples show that in general M being $(\pm\xi)$-contractible is not equivalent to $\overline{M}$ being finitely dominated. Instead the relation between these notions is given by

Theorem 1. *Let X be a finite connected CW-complex and $N \leq \pi_1(X)$ a normal subgroup such that $\pi_1(X)/N \cong \mathbb{Z}^k$ for some $k \geq 1$. Denote $\rho : \overline{X} \to X$ the regular covering space corresponding to N. Then $\overline{X}$ is finitely dominated if and only if X is ξ-contractible for all nonzero $\xi \in H^1(X;\mathbb{R})$ with $\rho^*\xi = 0$.*

As mentioned above this was proven by Ranicki for $k = 1$. If X is aspherical the theorem also follows from work of Bieri and Renz [1].

An unpublished result of Farrell states that if a $\mathbb{Z}^2$-covering space $\rho : \overline{M} \to M$ is finitely dominated, then $\tau_F(M,\xi) = \tau_F(M,\xi')$ for all $\xi, \xi' \in H^1(M;\mathbb{Z})$ with $\rho^*\xi = 0 = \rho^*\xi'$. So all such ξ can be represented by fibre bundle maps or none of them. Because of Theorem 1 we expect an impact on the obstructions $\tau_L(M,\xi)$ for all $\xi \in H^1(M;\mathbb{R})$ with $\rho^*\xi = 0$ as well. Indeed it turns out that one obstruction $\tau_F(M,\xi')$ determines every such $\tau_L(M,\xi)$ via $i_* : \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(M);\xi)$. Combining this with the result of Latour we get

Theorem 2. *Let M be a closed connected smooth manifold with $\dim M \geq 6$, $N \leq \pi_1(M)$ a normal subgroup with $\pi_1(M)/N \cong \mathbb{Z}^k$ for some $k \geq 1$ and such that the covering space $\rho : \overline{M} \to M$ corresponding to N is finitely dominated. Then the following are equivalent.*

(1) There is a nonzero $\xi \in H^1(M;\mathbb{R})$ with $\rho^*\xi = 0$ which can be represented by a nonsingular closed 1-form.
(2) Every nonzero $\xi \in H^1(M;\mathbb{R})$ with $\rho^*\xi = 0$ can be represented by a nonsingular closed 1-form.

Proofs of Theorem 1 and 2 can be found in [5].

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A canonical enriched Adams-Hilton model: theory and applications

Kathryn Hess

The canonical enriched Adams-Hilton model for simplicial sets. The goal of the work described in this section, carried out in collaboration with P.-E. Parent, J. Scott and A. Tonks [HPST], was to construct a canonical chain Hopf algebra, free as an algebra, weakly equivalent to the normalized chains on the Kan loop group on a given 1-reduced simplicial set K. We were motivated to search for such a model by the need for precise input data for constructing algebraic models for other spaces built from K, such as its free loop space ([BH]) or double loop space (see next section) or of homotopy fibers ([H]).

Let C_* denote the normalized chain functor on simplicial sets, and G the Kan loop group construction on the category of reduced simplicial sets. Given a simply-connected chain coalgebra (C, ∂, Δ), let ΩC denote its cobar construction, which is a chain algebra, free on $s^{-1}C_+$, the desuspension of the positive-degree part of C. Milgram proved in [M] that there is a natural quasi-isomorphism of chain algebras $q\colon \Omega(C \otimes C') \to \Omega C \otimes \Omega C'$ for any pair of chain coalgebras C, C'. On the other hand, in [S] Szczarba defined a natural quasi-isomorphism of chain algebras

$$\theta_K \colon \Omega C_*(K) \xrightarrow{\simeq} C_*(GK)$$

for any 1-reduced simplicial set K.

Given two chain coalgebras (C, ∂, Δ) and (C', ∂', Δ'), it is extremely important for our purposes to consider chain maps $f\colon (C, \partial) \to (C', \partial')$ that may not be coalgebra maps but that nonetheless induce chain algebra maps $\widetilde{\Omega} f\colon \Omega C \to \Omega C'$ with "germ" f, i.e., $\widetilde{\Omega} f(s^{-1}c) = s^{-1}\big(f(c)\big) + \text{higher-order terms}$. Such chain maps are called *strongly homotopy coalgebra (SHC) maps* and are coalgebra maps up to an infinite family of coherent homotopies. There is an analogous definition of *strongly homotopy comodule maps*.

Let

$$C_*(K) \otimes C_*(L) \underset{f_{K,L}}{\overset{\nabla_{K,L}}{\rightleftarrows}} C_*(K \times L) \circlearrowleft \varphi_{K,L}$$

denote the usual simplicial (and natural) Eilenberg-Zilber and Alexander-Whitney equivalences, where K and L are 1-reduced simplicial sets. In [GM] Gugenheim and Munkholm proved that, since $\nabla_{K,L}$ is a coalgebra map, $f_{K,L}$ is a naturally an SHC map, i.e., it is the "germ" of a chain algebra quasi-isomorphism

$$\widetilde{\Omega} f_{K,L} \colon \Omega C_*(K \times L) \xrightarrow{\simeq} \Omega\big(C_*(K) \otimes C_*(L)\big).$$

We can now define a canonical coproduct ψ_K on $\Omega C(K)$ to be the composition

$$\Omega C_*(K) \xrightarrow{\Omega(\Delta_K)_\sharp} \Omega C_*(K \times K) \xrightarrow{\widetilde{\Omega} f_{K,K}} \Omega\big(C_*(K)^{\otimes 2}\big) \xrightarrow{q} \Omega C_*(K)^{\otimes 2}$$

where $(\Delta_K)_\sharp$ is the chain coalgebra map induced on the normalized chains by the simplicial diagonal $\Delta_K \colon K \to K \times K$.

Theorem 1.

(1) ψ_K *is homotopy cocommutative (in fact , $E(\infty)$-cocommutative).*
(2) ψ_K *is strictly coassociative.*
(3) Szczarba's equivalence θ_K is an SHC map with respect to ψ_K and the usual coproduct on $C_*(GK)$. If K is a suspension, the θ_K is a strict coalgebra map.

Part (1) of this theorem is proved by giving explicit formulas for the necessary homotopies in terms of the maps in the Eilenberg-Zilber equivalences, while part (2) depends on the naturality of all the maps involved, as well as on the coassociativity of the natural equivalence f.

An acyclic models argument proves part (3), once we have established the following technically difficult result. If C and C' are Hopf algebras such that C is free as an algebra and $f\colon C \to C'$ is a chain algebra map, then to construct $\widetilde{\Omega} f\colon \Omega C \to \Omega C'$, it suffices to define $\widetilde{\Omega} f$ on the (desuspended) algebra generators of C.

In [B] Baues defined a natural, strictly coassociative, homotopy cocommutative coproduct on $\Omega C_*(K)$, in a strictly combinatorial manner, but did not show that there was a weak equivalence linking the Hopf algebra he obtained to the chains on the loop group. We have shown that Baues's coproduct agrees with ours.

An application: modelling double loop spaces. In joint work with R. Levi, I have constructed a canonical chain algebra, the homology of which is isomorphic to that of the double loop space on a given 2-reduced simplicial set K [HL]. The construction is based on the canonical Adams-Hilton model of the previous section.

Let $C = C_*(K)$, and let ∂ and Δ denote its differential and coproduct, respectively. Let $\overline{C} = s^{-1}C$. Define a differential $\widetilde{d}$ and a coproduct $\widetilde{\Delta}$ on $C \oplus \overline{C}$ by

$$\widetilde{d}c = dc - \bar{c} \qquad \widetilde{d}\bar{c} = -\overline{dc}$$
$$\widetilde{\Delta}c = \Delta c \qquad \widetilde{\Delta}\bar{c} = \bar{c}_i \otimes c^i + (-1)^{c_i} c_i \otimes \bar{c}^i$$

where $\Delta c = c_i \otimes c^i$.

Theorem 2. *The natural projection $\pi\colon (C \oplus \overline{C}, \widetilde{d}, \widetilde{\Delta}) \to (C, d, \Delta)$ is map of chain coalgebras. Furthermore, $\widetilde{\Delta}$ is an SHC map.*

Consequently, the projection induces a chain algebra map $\Omega\pi\colon \Omega(C \oplus \overline{C}) \to \Omega C$. We can show that $\Omega\pi$ is a coalgebra map as well, with respect to the coproduct ψ_K defined in the previous section and to a coproduct $\widetilde{\psi}_K$ on $\Omega(C \oplus \overline{C})$ defined as the composition

$$\Omega(C \oplus \overline{C}) \xrightarrow{\widetilde{\Omega}\widetilde{\Delta}} \Omega\big((C \oplus \overline{C})^{\otimes 2}\big) \xrightarrow{q} \Omega(C \oplus \overline{C})^{\otimes 2}.$$

The next step in the double loop space model construction consists in lifting the Szczarba equivalence to $\Omega(C \oplus \overline{C})$. We observe first that there is a natural chain algebra and right $C_*(GK)$-comodule structure on the acyclic cobar construction $\Omega C_*(GK) \otimes_{t_\Omega} C_*(GK)$.

Theorem 3. *There is a canonical quasi-isomorphism*

$$\widetilde{\theta}_K : \Omega(C \oplus \overline{C}) \to \Omega C_*(GK) \otimes_{t_\Omega} C_*(GK)$$

of chain algebras that is also a strongly homotopy comodule map. Furthermore, the following diagram commutes exactly.

$$\begin{array}{ccc} \Omega(C \oplus \overline{C}) & \xrightarrow{\widetilde{\theta}_K} & \Omega C_*(GK) \otimes_{t_\Omega} C_*(GK) \\ \Big\downarrow{\scriptstyle \Omega\pi} & & \Big\downarrow{\scriptstyle p} \\ \Omega C & \xrightarrow{\theta_K} & C_*(GK) \end{array}$$

where p is the obvious projection.

We can now define the canonical double loop space model. Let $\square$ denote cotensor product.

Theorem 4. *Let $B = \Omega(C \oplus \overline{C})\square_{\Omega C}\mathbb{Z}$ with its natural chain algebra structure. Then $H_*(B) \cong H_*(G^2K)$ as graded algebras.*

The heart of the proof of this theorem is the following sequence of isomorphisms, in which one has to be a bit careful about the relevant multiplicative structure.

$$\mathrm{cotor}^{\Omega C}(\Omega(C \oplus \overline{C}), \mathbb{Z}) \cong \mathrm{cotor}^{C_*(GK)}(\Omega C_*(GK) \otimes_{t_\Omega} C_*(GK), \mathbb{Z}) \cong H_*(G^2K)$$

We are in the process of developing interesting applications of this model. Observe that when K is of finite type, then B is also of finite type, i.e., has a finite number of generators in each degree.

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Waldhausen's stable parametrized h-cobordism theorem

John Rognes

The theorem in the title provides the fundamental link between high-dimensional geometric topology and the algebraic K-theory of rings or topological spaces. It asserts that for a compact manifold M the stable h-cobordism space $\mathcal{H}(M)$ of M (which is defined in terms of geometric topology) is homotopy equivalent to the looped Whitehead space $\Omega Wh(M)$ of M (which is defined in terms of algebraic K-theory). In particular, for connected M of sufficiently high dimension, $\pi_0\mathcal{H}(M)$ equals the set of isomorphism classes of h-cobordisms on M, and is in bijective correspondence with $\pi_0\Omega Wh(M)$, which is the Whitehead group $Wh(\pi) = K_1(\mathbb{Z}[\pi])/(\pm\pi)$ of the fundamental group $\pi = \pi_1(M)$. This recovers S. Smale's classical h-cobordism theorem (On the structure of manifolds, Amer. J. Math. 84 (1962), 387–399), including its non-simply connected generalizations. To be precise, there is one stable h-cobordism space and one Whitehead space for the topological and piecewise linear category, and one for the differentiable category, and $\mathcal{H}(M) \simeq \Omega Wh(M)$ in either category. The parametrized theorem was first asserted by A.E. Hatcher (Higher simple homotopy theory, Ann. of Math. (2), 102 (1975), 101–137), but not given adequate proof. Waldhausen developed a new proof ca. 1982, but publication was delayed. The speaker, B. Jahren and F. Waldhausen are now jointly preparing a manuscript for publication, seeking to fill this gap in the literature.

In more detail, an h-cobordism W on M is a compact manifold with boundary $\partial W = M \coprod M'$ such that both inclusions $M \to W$ and $M' \to W$ are homotopy equivalences. The h-cobordism space $H(M)$ is defined as a simplicial set, with q-simplices the locally trivial bundles of h-cobordisms on M parametrized over the affine q-simplex $|\Delta^q|$. The stable h-cobordism space $\mathcal{H}(M)$ is the homotopy colimit of the spaces $H(M \times I^k)$ over suitable stabilization maps, which become more highly connected as $\dim(M) + k$ grows by the stability theorem of K. Igusa (The stability theorem for smooth pseudoisotopies, K-Theory 2 (1988), 1–355).

The Whitehead spaces are defined in terms of Waldhausen's algebraic K-theory of spaces functor $X \mapsto A(X)$. The latter is a homotopy functor, with an associated assembly map α that fits into the following fiber sequence:

$$\Omega Wh^{PL}(X) \to \Omega^\infty(A(*) \wedge X_+) \xrightarrow{\alpha} A(X) \to Wh^{PL}(X)\,.$$

This defines the PL (and topological) Whitehead space. There is different fiber sequence

$$\Omega Wh^{DIFF}(X) \to \Omega^\infty\Sigma^\infty(X_+) \xrightarrow{\iota} A(X) \to Wh^{DIFF}(X)$$

defining the differentiable Whitehead space. The algebraic K-theory of spaces also has an interpretation in terms of the algebraic K-theory of structured ring spectra: writing $X \simeq BG$ for a topological group G there is a group S-algebra $S[G] = \Sigma^\infty(G_+) \simeq \Sigma^\infty(\Omega X_+)$ and then $A(X) = K(S[G])$. This makes $A(X)$ and the Whitehead spaces partially accessible to computation (J. Rognes, Two-primary algebraic K-theory of pointed spaces, Topology 41 (2002), 873–926, and J. Rognes, The smooth Whitehead spectrum of a point at odd regular primes, Geom. Topol. 7 (2003), 155-184).

By triangulation theory the PL and topological stable h-cobordism spaces are homotopy equivalent, and by smoothing theory the stable parametrized h-cobordism theorem in the differentiable category is equivalent to the one in the PL category. It therefore suffices to prove the theorem in the PL case.

The proof can be divided into three parts. First, for a simplicial set X the algebraic K-theory $A(X)$ can be defined in terms of the abstract K-theory of a category with cofibrations and weak equivalences, and the assembly map α and its homotopy fiber $\Omega Wh^{PL}(X)$ can be modeled in similar terms. The latter model admits a simplification, as the nerve of the category $s\mathcal{C}^h(X)$ with objects the finite cofibrations $y\colon X \to Y$ that are weak homotopy equivalences, and morphisms the maps $f\colon Y \to Y'$ under X that are simple, i.e., whose geometric realization $|f|\colon |X| \to |Y|$ has contractible point inverses. This part of the argument has already been published by Waldhausen (Algebraic K-theory of spaces, Algebraic and geometric topology, Springer Lecture Notes in Math. 1126 (1985), 318–419).

The second, non-manifold part relates the category $s\mathcal{C}^h(X)$ defined in terms of simplicial sets to a corresponding simplicial category $s\mathcal{E}^h_\bullet(|X|)$ defined in terms of (Euclidean) polyhedra. The geometric realization of a simplicial set does not in general have a canonical polyhedral structure, but the subcategory of non-singular simplicial sets, i.e., those where each non-degenerate simplex $\bar{x}\colon \Delta^n \to X$ is embedded, does admit a polyhedral realization functor. Let $s\mathcal{D}^h(X)$ be the full subcategory of $s\mathcal{C}^h(X)$ with objects such that Y is non-singular. Proposition: For non-singular X the inclusion $s\mathcal{D}^h(X) \to s\mathcal{C}^h(X)$ is a homotopy equivalence of categories.

Let $s\mathcal{E}^h(|X|)$ be the category with objects compact PL embeddings $|X| \to P$ of polyhedra that are homotopy equivalences, and simple maps $P \to P'$ under $|X|$ as morphisms. Then it is not known whether the geometric realization functor $s\mathcal{D}^h(X) \to s\mathcal{E}^h(|X|)$ is a homotopy equivalence. Instead we need to introduce an extra simplicial direction. Let $s\mathcal{D}^h_\bullet(X)$ be the simplicial category that in degree q has objects the finite cofibrations of non-singular simplicial sets $X \times \Delta^q \to Z$ over Δ^q that are weak homotopy equivalences, such that the projection map $p\colon Z \to \Delta^q$ becomes a locally trivial PL bundle after geometric realization. The morphisms are simple maps $Z \to Z'$ under $X \times \Delta^q$ and over Δ^q. Proposition: For non-singular X the inclusion $s\mathcal{D}^h(X) \to s\mathcal{D}^h_\bullet(X)$ is a homotopy equivalence.

Let $s\mathcal{E}^h_\bullet(|X|)$ be the similarly defined simplicial category of polyhedra. Proposition: The polyhedral realization functor $s\mathcal{D}^h_\bullet(X) \to s\mathcal{E}^h_\bullet(|X|)$ is a homotopy equivalence.

Taken together, these three propositions prove the non-manifold part of the theorem: For non-singular X there is a chain of homotopy equivalences $s\mathcal{C}^h(X) \simeq s\mathcal{E}^h_\bullet(|X|)$.

The third, manifold part of the proof, asserts that for a compact combinatorial manifold X, i.e., a finite non-singular simplicial set X such that $M = |X|$ is a PL manifold, the functor $H(M) \to s\mathcal{E}^h_\bullet(M)$ that takes an h-cobordism $M \subset W$ to the underlying weak equivalence of polyhedra $M \to W$, and likewise in parametrized families over $|\Delta^q|$, stabilizes to a homotopy equivalence $\mathcal{H}(M) \to s\mathcal{E}^h_\bullet(M)$. This will then complete the proof of Waldhausen's stable parametrized h-cobordism theorem.

Free actions of finite groups on products of spheres

Ian Hambleton

The study of free finite group actions on products of spheres is a natural continuation of the spherical space form problem [7]. In this work, we show that certain products of finite groups do act freely and *smoothly* on a product of spheres, even though the individual factors can't act freely and smoothly (or even topologically) on a single sphere. This verifies a conjecture of Elliott Stein [10]. The method involves a detailed analysis of the product formulas in surgery theory, and a refinement of Dress induction for surgery obstructions. The following outline is mostly taken from the introduction to my recent preprint [6].

If a finite group G acts freely on S^n, then (i) every abelian subgroup of G is cyclic, and (ii) every element of order 2 is central. In [7], Madsen, Thomas and Wall proved that these conditions are sufficient to imply the existence of a free topological action on some sphere. Actually, these two conditions have a very different character. By the work of P. A. Smith and R. Swan [11], condition (i) is necessary and sufficient for a free simplicial action of G on a finite-dimensional *simplicial complex* which is homotopy equivalent to a sphere. The finite groups G satisfying condition (i) are exactly the groups with periodic Tate cohomology, or equivalently those for which every subgroup of order p^2, p prime, is cyclic (the p^2-conditions). On the other hand, Milnor [8] proved that condition (ii) is necessary for a free G-action by homeomorphisms on any closed, topological *manifold* which has the mod 2 homology of a sphere. The groups with periodic cohomology satisfying condition (ii) are just those which have no dihedral subgroups, or equivalently those for which every subgroup of order $2p$, p prime, is cyclic (the $2p$-conditions). Milnor's result shows for example that the periodic dihedral groups do not act topologically on S^n, although they do act simplicially on a finite complex homotopy equivalent to S^n.

For free finite group actions on a product of spheres, the analogue of condition (i) was suggested by P. Conner [5]: if G acts freely on a k-fold product of spheres $(S^n)^k := S^n \times \cdots \times S^n$, is every abelian subgroup of G generated by at most k elements? Conner proved this statement for $k = 2$, and a lot of work [9], [4], [1], [3] has been done to determine what additional conditions are necessary to produce free simplicial actions on a finite-dimensional simplicial complex homotopy equivalent to a product of spheres. The picture is now almost completely clarified, at least for elementary abelian groups and spheres of equal dimension: Adem and Browder [1] and Carlsson [4] showed that $G = (\mathbb{Z}/p)^r$ acts freely on $(S^n)^k$, for p a prime, implies $r \leq k$ provided that $n \neq 1, 3, 7$ in the case $p = 2$ (the restriction $n \neq 1$ for $p = 2$ was recently removed in [12]). The same result is conjectured to hold for finite-dimensional G-CW complexes homotopy equivalent to a product of spheres of possibly *unequal* dimensions (see [2] for some recent progress). Carlsson has also proposed an interesting generalization of the Conner conjecture:

Question (G. Carlsson). *Suppose that $G = (\mathbb{Z}/p)^k$ acts freely on a finite complex X, then is*

$$\sum_{i\geq 0} \operatorname{rank}_p H_i(X;\mathbb{Z}/p) \geq 2^k \quad ?$$

Much less seems to be known at present about the additional conditions needed to produce free actions by homeomorphisms or diffeomorphisms on the closed manifolds $(S^n)^k$ for $k > 1$. Let D_q denote the dihedral group of order $2q$, with q an odd prime. Elliott Stein [10] proved that, for every $n = 4j + 3$ and any $k \geq 2$, there exist free, orientation-preserving piece-wise linear actions of $(D_q)^k$ on $(S^n)^k$. Many of these actions are smoothable.

These examples show that a direct generalization of Milnor's condition (ii) is *not* necessary for actions on products of spheres. In [6] we verify a conjecture of Stein's:

Theorem. *If G_1 and G_2 are finite groups with periodic Tate cohomology, then $G_1 \times G_2$ acts freely and smoothly on some product $S^n \times S^n$.*

The techniques used to prove this statement also show that any product of periodic groups $G_1 \times \cdots \times G_k$, with $k > 1$, acts freely and smoothly on $(S^n)^k$ for some n. Of course there are groups G satisfying Conner's condition which are not the direct product of periodic groups, so these examples are just the simplest case. The surgery techniques need to be developed further to study more general groups. I wonder what is known about the following:

Question. *If there is a finite free G-CW complex $X \simeq S^n \times S^n$, then does G act freely and smoothly on $S^n \times S^n$?*

I would like to thank Alejandro Adem for reminding me about E. Stein's paper and this open question.

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Tameness of hyperbolic 3-manifolds

Ian Agol

A tame manifold is the interior of a compact manifold with boundary. The Marden conjecture states that every hyperbolic 3-manifold with finitely generated fundamental group is tame. We prove this conjecture. An independent proof has also been announced recently by Calegari and Gabai. The main tools we use in the proof is an orbifold version of a branched covering trick of Canary, and results on end-reductions due to Myers, as well as generalizations of known results on tameness from the hyperbolic category to the category of pinched negatively curved manifolds. In fact, we prove tameness of manifolds with a complete pinched negatively curved metric, hyperbolic cusps, and finitely generated fundamental group.

The tameness result has applications to many open questions in 3-manifold topology and the structure of Kleinian groups. The main geometric consequence is Thurston's geometric tameness conjecture, which implies that the ends of Kleinian groups are either simply degenerate or geometrically finite. As Thurston showed, this conjecture implies the Ahlfors measure conjecture. We also get a new, purely geometric proof of Ahlfors finiteness theorem. Along with the ending lamination conjecture of Thurston, which has a proof announced last year by Brock, Canary and Minsky, the Marden conjecture gives a complete classification of Kleinian groups in terms of topological type, conformal data for the geometrically finite ends, and topological data (ending laminations) for the simply degenerate ends. In effect, this is a vast generalization of Mostow rigidity for non-cocompact Kleinian groups.

The main applications to 3-manifolds are obtained via Canary and Thurston's covering theorem. This implies that every finitely generated subgroup of the fundamental group of a finite volume 3-manifold is either geometrically finite, or contains the fiber subgroup of a virtual fibration of index at most two. Consequences are Simon's conjecture for 3-manifolds satisfying the geometrization conjecture, which says that covers of compact 3-manifolds with finitely generated group are tame (or almost compact, in the case that there is boundary). Also, it follows from work of Agol-Long-Reid and Wise that the fundamental group of the figure 8 knot complement is LERF, that is, finitely generated subgroups are the intersection of finite index subgroups containing them.

Absolute torsion and multiplicativity of signature mod 4

Andrew Ranicki

In 2001 Ian Hambleton formulated the conjecture that the signatures of manifolds in a fibre bundle of orientable manifolds are multiplicative mod 4. The first attempt at a proof was based on the K_1-valued round torsion invariant and additive L-theory of Ranicki [4, 5], and the round L-theory of Hambleton, Ranicki and Taylor [2]. Unfortunately, the round torsion is not a round L-theory invariant, as claimed in these papers. This mistake has now been fixed by my student Andrew Korzeniewski [3], who defined a new torsion invariant of an n-dimensional Poincaré complex X

$$\tau^{NEW}(X) = \tau^{NEW}([X] \cap - : C(X)^{n-*} \to C(X)) \in K_1(\mathbb{Z}) = \mathbb{Z}_2 \ .$$

(In general, the new torsion is defined in $\widehat{H}^n(\mathbb{Z}_2; K_1(\mathbb{Z}[\pi_1(X)]))$, but only the simply-connected component in $\widehat{H}^n(\mathbb{Z}_2; K_1(\mathbb{Z})) = K_1(\mathbb{Z})$ is required for the signature application). Unlike Whitehead torsion, the new torsion can take nontrivial values on manifolds, e.g. $\mathbb{CP}^2 \# \mathbb{CP}^2$. For round Poincaré complexes X (i.e., those with $\chi(X) = 0$) the new torsion is a round Poincaré bordism invariant, and is nontrivial e.g. for S^1.

The mod 4 reduction of the signature $\sigma(X) \in \mathbb{Z}$ of a $4k$-dimensional Poincaré complex X is expressed in terms of the new torsion and the Euler characteristic $\chi(X) \in \mathbb{Z}$ by

$$\sigma(X) = 2\tau^{NEW}(X) + (2k+1)\chi(X) \in \mathbb{Z}_4$$

with $2 : \mathbb{Z}_2 \to \mathbb{Z}_4; 1 \mapsto 2$. This expression is a lift of

$$\sigma(X) \equiv \chi(X) \pmod 2$$

and is a generalization of the classical congruence

$$\text{signature} \equiv \text{rank} + \det - 1 \pmod 4$$

for a nonsingular symmetric form over $\mathbb{Z}$.

The new torsion invariant is used to prove the conjecture:

Theorem (Hambleton, Korzeniewski, Ranicki [1]). *The signatures of manifolds in a fibre bundle $F \to E \to B$ are multiplicative mod 4*

$$\sigma(E) = \sigma(B)\sigma(F) \in \mathbb{Z}_4$$

with $\sigma = 0$ *for* $\dim \not\equiv 0 \pmod 4$.

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Subdirect products of hyperbolic groups

Martin Bridson

One may reasonably take the view that the most basic of finitely presented groups are the finite groups and that the next class worthy of mention is formed by the virtually cyclic groups. What comes next?

One might start taking direct products and pass to consideration of virtually abelian groups, or one might allow free products and pass to consideration of virtually free groups; proceeding in the former vein, one might enlarge the class of groups considered progressively to include virtually nilpotent, solvable, then amenable groups — "the amenable side of the universe"; proceeding in the latter vein one maps out the hyperbolic/nonpositively curved side of the universe.

Each of the above-named subclasses of amenable groups is closed under the formation of subdirect products, but what are the subdirect products of hyperbolic groups? This is the basic question addressed in this lecture. The first point that I want to make is that for the most basic hyperbolic groups — virtually free groups — the classification of subdirect products is non-trivial but remarkably restricted in the presence of suitable finiteness hypotheses. Next I shall explain that this restricted behaviour extends to subdirect products in the class of surface groups and the groups with the same elementary theory as the free groups [13].

In contrast to this controlled behaviour, there is a great diversity of behaviour among the finitely presented subgroups of subdirect products of more general hyperbolic groups, even 2-dimensional hyperbolic groups and the fundamental groups of closed hyperbolic n-manifolds with $n \geq 3$.

The *tame part* of this discussion finds applications in the study of Kähler manifolds thanks to beautiful recent work of Delzant and Gromov [8]. As an application of the *wild part* of the discussion I shall describe my work with Grunewald [6] in which we provide counterexamples to a question of Grothendieck concerning representations of groups.

If F is a non-abelian free group, then the collection of finitely generated subgroups of $F \times F$ is rather wild: such subgroups need not be finitely presentable, one can construct examples that have an unsolvable conjugacy problem, and the isomorphism problem is unsolvable in the class of all finitely generated subgroups of $F \times F$. Each of these statements can be proved by encoding facts about arbitrary finitely-presented groups into facts about the subgroup structure of $F \times F$ via a fibre product construction [11].

Example 1. Let $p : F_2 \to Q$ be the map from the free group on the generators to $Q = \langle a, b \mid R \rangle$. Let $P = \{(x, y) \mid p(x) = p(y)\} \subset F \times F$.

P is finitely generated, but if Q is infinite then P is not finitely presented. If Q has unsolvable word problem, then P has unsolvable conjugacy problem.

There are also non-obvious *finitely presented* subdirect products of free groups:

Example 2. Let F be a finitely generated free group, let D_n be the direct product of n copies of F and let $\pi_n : D_n \to \mathbb{Z}$ be a homomorphism whose restriction to each direct factor is surjective. Stallings [14] shows that $\mathrm{SB}_3 = \ker \pi_3$ is finitely presented but $H_3(\mathrm{SB}_3, \mathbb{Q})$ is infinite dimensional.

Bieri [3] later proved SB_3 is of type $\mathcal{F}_{n-1}$ but dim $H_n(\mathrm{SB}_n, \mathbb{Q}) = \infty$.

A discrete group Γ is said to be of type $\mathcal{F}_n$ if there is a classifying space $K(\Gamma, 1)$ with a compact n-skeleton (e.g., type $\mathcal{F}_2$ means finitely presentable).

In contrast to the above examples, Baumslag and Roseblade [2] proved that the only finitely presented subgroups of $F \times F$ are "the obvious ones". In [4] Howie, Miller, Short and I proved that the Baumslag-Roseblade result does extend to general direct products if one adjusts the finiteness assumption appropriately. We were also able to extend the result to direct products of surface groups. Let us take here "surface group" to mean the fundamental group of a compact surface (with boundary allowed, to include free groups).

Theorem 1. *Let D be the direct product of n surface groups. If $S \subseteq D$ is a subgroup with $H_i(S, \mathbb{Z})$ finitely generated for $i \leq n$, then S has a subgroup of finite index that is a direct product of at most n surface groups.*

On-going work of myself and C.F. Miller III adds further to the impression that the finitely presented subdirect products of free groups are tamer than previously thought. For example, with three factors, Stallings-type examples are the only non-obvious finitely presented subgroups:

Theorem 2. *If D is a product of three free groups and $S \subset D$ is a finitely presented subdirect product*[1] *then either $H_3(S, \mathbb{Z})$ is finitely generated, or else has S a subgroup of finite index that is normal in D with abelian quotient.*

Theorem 1 does not extend to the class of hyperbolic small cancellation groups (cf. Theorem 5), nor to cocompact lattices in $\mathrm{SO}(n, 1)$ (see [5]):

Theorem 3. *If Γ is the fundamental group of a closed hyperbolic 3-manifold that fibres over the circle, and D_n is the direct product of $n \geq 2$ copies of Γ, then there is a subgroup $S \subset D_n$ that is π_1 of a closed aspherical manifold N of dimension $3n - 1$, but neither S nor any of its subgroups of finite index can be expressed as a direct product of hyperbolic groups.*

There is an interesting class of groups to which Theorem 1 *does* extend, as Jim Howie and I proved recently [7]. We call a finitely generated group Γ is a *Sela Group* if it is a subgroup of a group that has the same first order logic as the free group (see [13]); this includes surface groups.

[1] a subgroup whose projection to each direct factor is surjective.

Theorem 4. *Let D be the direct product of n Sela groups. If $S \subseteq D$ is a subgroup with $H_i(S, \mathbb{Z})$ finitely generated for $i \leq n$, then S has a subgroup of finite index that is a direct product of at most n Sela groups.*

Turning back to the idea that subdirect products of more general hyperbolic groups are much more diverse than those described above, I close this lecture with a brief outline of my construction with Fritz Grunewald of counterexamples to the following question of Alexander Grothendieck [10]: *Let $Rep_A(\Gamma)$ denote the category of finitely generated A-modules with a Γ-action. Suppose that $u : \Gamma_1 \to \Gamma_2$ is a homomorphism of finitely presented, residually-finite groups such that restriction of scalars functor $u_A^* : Rep_A(\Gamma_2) \to Rep_A(\Gamma_1)$ is an equivalence of categories for every commutative ring A. Does it follow that u is an isomorphism?*

Our negative solution to this problem belongs in this lecture because a key ingredient in the proof is the fact that one can combine the 1-2-3 Theorem of [1] and Wise's version [15] of the Rips construction [12] to construct somewhat pathological finitely presented subgroups in the direct product of two small-cancellation groups (in contrast to Theorem 1).

Theorem 5. *There exist 2-dimensional, residually-finite hyperbolic groups H with finitely presented subgroups $u : P \hookrightarrow H \times H$, such that P is not isomorphic to $H \times H$, but u_A^* is an equivalence of categories for every commutative ring A.*

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Deformation theory of hyperbolic 3-manifolds

Richard Canary

We will discuss recent results on the topology of the deformation space $AH(M)$ of (marked) hyperbolic 3-manifolds homotopy equivalent to a fixed compact hyperbolizable 3-manifold M with non-empty boundary. The recent resolutions of Thurston's Ending Lamination Conjecture (by Brock, Canary and Minsky [23, 8, 9]) for topologically tame hyperbolic 3-manifolds and Marden's Tameness Conjecture (by Agol [1] and Calegari-Gabai [13]) give a complete classification of the hyperbolic 3-manifolds in $AH(M)$ in terms of their (marked) topological type and their end invariants (which capture the asymptotic geometry of their ends.) However, the topological types is not locally constant on $AH(M)$ (see [4]) and the end invariants vary discontinuously on $AH(M)$ (see [6]), so one does not obtain a parameterization of $AH(M)$ and recent work indicates that the global topology of $AH(M)$ can be quite complicated.

If $N = \mathbf{H}^3/\Gamma$ is a (orientable) hyperbolic 3-manifold, then N is the quotient of $\mathbf{H}^3$ by a discrete subgroup Γ of $\mathrm{Isom}_+(\mathbf{H}^3)$ (which may be identified with $\mathbf{PSL}_2(\mathbf{C})$.) The "marking" of a hyperbolic 3-manifold is a homotopy equivalence from M to N (well-defined only up to homotopy), so gives rise to an identification of $\pi_1(M)$ with Γ (well-defined up to conjugation.) So, one obtains a discrete, faithful representation of $\pi_1(M)$ into $\mathbf{PSL}_2(\mathbf{C})$ (well-defined up to conjugation). So, we may identify $AH(M)$ with a subset of the character variety $X(M)$ which is the Mumford quotient of $\mathrm{Hom}(\pi_1(M), \mathbf{PSL}_2(\mathbf{C}))$ by $\mathbf{PSL}_2(\mathbf{C})$ acting by conjugation. $AH(M)$ inherits the resulting quotient topology.

If $MP(M)$ is the interior of $AH(M)$, then Marden [20] and Sullivan [26] showed that $MP(M)$ consists entirely of geometrically finite hyperbolic 3-manifolds. (Geometrically finite hyperbolic 3-manifolds are the most easily understood hyperbolic 3-manifolds.) Combining work of Ahlfors [2], Bers [3], Kra [18], Marden [20], Maskit [21], and Thurston [24] one sees that the components of $MP(M)$ are in one-to-one correspondence with the set $\mathcal{A}(M)$ of marked homeomorphism types of compact hyperbolic 3-manifolds homotopy equivalent to M and that each component is parameterized by analytical data. (See [14] for a discussion of this parameterization in topological language.) Canary and McCullough [14] completely characterize the situations where $\mathcal{A}(M)$ is infinite.

Combining the proofs of Thurston's Ending Lamination Conjecture and Marden's Tameness Conjecture with convergence results of Thurston [27, 28], Ohshika [25], Kleineidam-Souto [17], and Lecuire [19], one obtains a proof of the Bers-Sullivan-Thurston Density conjecture which asserts that $AH(M)$ is the closure of its interior $MP(M)$, i.e., that every hyperbolic 3-manifold with finitely generated fundamental group is an algebraic limit of geometrically finite hyperbolic 3-manifolds. (Brock and Bromberg [10, 7] previously proved important special cases of this density result using cone-manifold techniques.)

Anderson and Canary [4] showed that the homeomorphism type of the limit of a sequence of hyperbolic 3-manifolds in $AH(M)$ can differ from that of the approximates. Their construction showed that two components of $MP(M)$ can "bump," i.e., have intersecting closures. Anderson, Canary and McCullough [5] completely characterized when two components of $MP(M)$ can bump if M has incompressible boundary. Roughly, two components bump if and only if the associated homeomorphism types differ exactly by removing solid torus components of the characteristic submanifold and regluing them after shuffling the order of the attaching annuli. Combining the results of [5] with the Bers-Sullivan-Thurston Density conjecture one gets an enumeration of the components of $AH(M)$ in purely topological terms. In particular, applying work of Canary-McCullough [14], one sees that if M has incompressible boundary, then $AH(M)$ has infinitely many components if and only if there is a thickened torus component V of the characteristic submanifold of M which intersects the boundary ∂M in at least two annuli. Holt [15, 16] further showed that arbitrarily many components of $MP(M)$ can bump at a single point. Moreover, he shows that if any collection of components has connected closure, then there is a common point where all the components bump.

McMullen [22] used the construction from [4] to show that quasifuchsian space $QF(S) = MP(S \times I)$ self bumps, i.e., there is a point x in the boundary of $QF(S)$ such that the intersection of any sufficiently small neighborhood of x with $QF(S)$ is disconnected. Bromberg and Holt [12] showed that if M contains an essential annulus (whose core curve is primitive in $\pi_1(M)$) then every component of $MP(M)$ self-bumps. Notice that this implies, in particular, that $AH(M)$ is not a manifold in any of these cases.

Bromberg [11] has shown that the space of punctured torus groups is not locally connected. The space of punctured torus groups is the relative deformation space $AH(S \times I, \partial S \times I)$ (where S is the punctured torus) consisting of hyperbolic 3-manifolds which are homotopy equivalent to $S \times I$ and have a cusp in the homotopy class of ∂S. His proof suggests that it is quite often the case that $AH(M)$ is not locally connected.

Despite all the pathological behavior described in this brief note, we hope that the techniques developed in the proof of the Ending Lamination Conjecture will allow us to develop a much better insight into the topology of $AH(M)$.

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Root systems for p-compact groups

Jesper Grodal

A p-compact group is a homotopy version of a compact Lie group, but with all the structure concentrated at a single prime p. More precisely, a p-compact group is a triple (X, BX, e) where X is a space such that $H^*(X; \mathbf{F}_p)$ is finite, BX is a pointed connected p-complete space, and $e : X \xrightarrow{\simeq} \Omega BX$ is a homotopy equivalence.

The classification of p-compact groups for p odd by Andersen-Grodal-Møller-Viruel [2] states that there is a 1-1-correspondence between connected p-compact groups and finite reflection groups over the p-adic integers $\mathbf{Z}_p$. This statement does not carry over verbatim to $p = 2$ even conjecturally, and it appears that $\mathbf{Z}_p$-reflection groups have to be replaced with certain "$\mathbf{Z}_p$-root data".

The goal of my talk, which was a report on the paper [1] with K. Andersen, was to introduce these $\mathbf{Z}_p$-root data and explain their relation to the maximal torus normalizer. I put particular emphasis on explaining the relationship between automorphisms of the root data and automorphisms of the maximal torus normalizer. I also explained the relationship to earlier work by Tits [4] and Dwyer-Wilkerson [3].

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Monopoles and merges in dimension four

Stefan Bauer

A restricted version of a topological quantum field theory is constructed for K-oriented rational homology 3-spheres. It takes values in the homotopy category of equivariant spectra in the sense of algebraic topology. Restriction to closed 4-manifolds, considered as cobordisms between empty 3-manifolds, recovers the refined Seiberg-Witten invariants. The object $N(Y)$ associated to a 3-manifold Y in some sense can be viewed as a topological spectrum which encodes the analytical flow of the Dirac operator over arbitrary families. Conjecturally, equivariant Borel-homology of these spectra recovers the Floer-homologies of Ozsvath-Szabo and the Seiberg-Witten-Floer-homology.

Triple handlebody decompositions of 3-manifolds

Hyam Rubinstein

This is a report on the PhD thesis of my student James Coffey, who is currently being completing at the University of Melbourne. James has been studying the class of 3-manifolds which can be built by gluing three handlebodies together by homeomorphisms between regions on their boundaries. The idea is to generalise the class of Seifert fibred spaces with infinite fundamental group, orbit surface S^2 and three exceptional fibres of multiplicity (p, q, r). Such Seifert fibred spaces provide interesting examples of non-Haken 3-manifolds, i.e., 3-manifolds which are irreducible (every embedded 2-sphere bounds a 3-ball) but do not have embedded incompressible surfaces (i.e., closed orientable surfaces different from the 2-sphere, whose fundamental group injects into the fundamental group of the 3-manifold). Waldhausen observed that these Seifert spaces are non Haken if and only if they have finite first homology. As is well known, the Seifert fibred spaces have infinite fundamental group exactly when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$.

Definitions. We say that a handlebody H with a collection of disjoint essential simple closed curves Γ in ∂H has a boundary pattern Γ. The boundary pattern satisfies the n-disk condition, if every meridian disk for H meets Γ at least n times. A 3-manifold M has a triple handlebody decomposition if there are handlebodies H_i, $1 \leq i \leq 3$, with boundary patterns Γ_i so that each Γ_i separates ∂H_i into two regions U_i, V_i (not necessarily connected) and after a homeomorphism gluing U_1 to U_2, V_1 to U_3 and V_2 to V_3 we get M.

Now to summarise the main results of the thesis.

Theorem 1. *If the boundary patterns Γ_i satisfy the n_i disk conditions, where $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \geq \frac{1}{2}$, then $\pi_1(H_i)$ injects into $\pi_1(M)$, for $1 \leq i \leq 3$.*

Theorem 2. *For a 3-manifold M with triple handlebody decomposition satisfying the same conditions as in Theorem 1, the universal covering is homeomorphic to $\mathbf{R}^3$.*

Theorem 3. *For a 3-manifold M with triple handlebody decomposition satisfying the same conditions as in Theorem 1, the characteristic variety or JSJ decomposition can be explicitly constructed.*

Theorem 4. *For a 3-manifold M with triple handlebody decomposition satisfying the same conditions as in Theorem 1, the word problem in the fundamental group of M is solvable.*

Theorem 5. *Explicit examples of 3-manifolds with triple handlebody decompositions satisfying the conditions in Theorem 1 can be constructed by Dehn surgery on knots with free spanning surfaces with no quadrilateral meridian disks in the complementary handlebody to the spanning surface, so long as the surgery is of the form $\frac{p}{q}$ where $p \geq 3$. Other examples can be found by 2-fold branched coverings of certain families of knots and links in the 3-sphere. Finally a collection of pretzel* *knots are described for which Dehn surgery produces infinitely many non Haken 3-manifolds with triple handlebody decompositions as in Theorem 1.*

Nielsen coincidence theory in arbitrary dimensions

Ulrich Koschorke

Let $f_1, f_2 : M \longrightarrow N$ be two (continuous) maps between smooth connected manifolds M and N without boundary, of strictly positive dimensions m and n, resp., M being compact. We are interested in making the coincidence locus

$$C(f_1, f_2) := \{x \in M \mid f_1(x) = f_2(x)\}$$

as small (or simple in some sense) as possible after possibly deforming f_1 and f_2 by a homotopy.

Question. *How large is the* **m***inimum number of* **c***oincidence* **c***omponents*

$$MCC(f_1, f_2) := \min\{\#\pi_0(C(f_1', f_2')) \mid f_1' \sim f_1, f_2' \sim f_2\} \ ?$$

In particular, when does this number vanish, i.e. when can f_1 and f_2 be deformed away from one another?

This is a very natural generalization of one of the central problems of classical fixed point theory (where $M = N$ and $f_2 =$ identity map): determine the minimum number of fixed points among all maps in a given homotopy class (see [Br] and [BGZ, proposition 1.5]). Note, however, that in higher codimensions $m - n > 0$ the coincidence locus is generically a closed $(m-n)$-manifold so that it makes more sense to count *pathcomponents* rather than points. Also the methods of (first order, singular) (co)homology will no longer be strong enough to capture the subtle geometry of coincidence manifolds.

In this lecture I used the language of normal bordism theory (and a nonstabilized version thereof) to define and study lower bounds $N(f_1, f_2)$ (and $N^{\#}(f_1, f_2)$) for $MCC(f_1, f_2)$.

After performing an approximation we may assume that the map $(f_1, f_2) : M \to N \times N$ is smooth and transverse to the diagonal $\Delta = \{(y, y) \in N \times N \mid y \in N\}$. Then the coincidence locus

$$C \ = \ C(f_1, f_2) \ = \ (f_1, f_2)^{-1}(\Delta)$$

is a closed smooth $(m-n)$-dimensional manifold, equipped with

i) maps

$$\begin{array}{ccc} & & E(f_1, f_2) := \left\{ (x, \theta) \in M \times N^I \;\middle|\; \begin{array}{c} \theta(0) = f_1(x); \\ \theta(1) = f_2(x) \end{array} \right\} \\ & \overset{\tilde g}{\nearrow} & \Big\downarrow \mathrm{pr} \\ C & \xrightarrow[g = \mathrm{incl}]{} & M \end{array}$$

where $\widetilde{g}$ is the natural lifting which adds the constant path at $f_1(x) = f_2(x)$ to $g(x) = x \in C$; and

ii) a stable vector bundle isomorphism

$$\overline{g} \; : \; TC \oplus g^*(f_1^*(TN)) \; \cong \; g^*(TM)$$

deduced from the isomorphism

$$\overline{g}^{\#} \; : \; \nu(C, M) \; \cong \; (f_1, f_2)^*(\nu(\Delta, N \times N)) \; \cong \; f_1^*(TN)\,|C$$

of (nonstable) normal bundles.

The triple $(C, \widetilde{g}, \overline{g})$ gives rise to a well-defined bordism class

$$\widetilde{\omega}(f_1, f_2) \; := \; [C, \widetilde{g}, \overline{g}] \; \in \; \Omega_{m-n}(E(f_1, f_2); \; \widetilde{\varphi})$$

in the normal bordism group of such triples (here the virtual coefficient bundle is defined by

$$\widetilde{\varphi} \; := \; pr^*(f_1^*(TN) - TM) \; ;$$

e.g., if M and N are stably parallelized, then $\widetilde{\varphi}$ is trivial and we are dealing with (stably) framed bordism).

Keeping track also of the fact that C is a smooth submanifold of M with (nonstabilized) normal bundle described by $\overline{g}^{\#}$, we obtain a sharper invariant

$$\omega^{\#}(f_1, f_2) \; \in \; \Omega^{\#}(f_1, f_2)$$

which, however, lies in general only in a suitable bordism *set* (not group).

A crucial ingredient of both the $\widetilde{\omega}$- and the $\omega^{\#}$-invariant is the map $\widetilde{g}$. Indeed, the path space $E(f_1, f_2)$ has a very rich topology. Already its set $\pi_0(E(f_1, f_2))$ of pathcomponents can be huge — it corresponds bijectively to the so called Reidemeister set $R(f_1, f_2)$, a well-studied set-theoretic quotient of the fundamental group $\pi_1(N)$. This leads to a natural decomposition

$$C(f_1, f_2) \; = \coprod_{A \in \pi_0(E(f_1, f_2))} \widetilde{g}^{-1}(A) \quad .$$

Let $N(f_1, f_2)$, and $N^{\#}(f_1, f_2)$, resp., denote the corresponding number of nontrivial contributions by the various pathcomponents A of $E(f_1, f_2)$ to $\widetilde{\omega}(f_1, f_2)$ and $\omega^{\#}(f_1, f_2)$, resp.

Theorem 1.

(i) The integers $N(f_1, f_2)$ and $N^{\#}(f_1, f_2)$ depend only on the homotopy classes of f_1 and f_2;

(ii) $N(f_1, f_2) = N(f_2, f_1)$ and $N^{\#}(f_1, f_2) = N^{\#}(f_2, f_1)$;

(iii) $0 \leq N(f_1, f_2) \leq N^{\#}(f_1, f_2) \leq MCC(f_1, f_2) < \infty$;

(iv) if $m = n$ then $N(f_1, f_2) = N^{\#}(f_1, f_2)$ coincides with the classical Nielsen number (which has a standard definition at least if both M and N are orientable or if f_2 is the identity map).

Recall the decisive progress made by J. Nielsen on the classical minimizing problem when he decomposed fixed point sets into equivalence classes. In our interpretation this is just the decomposition of a 0-dimensional bordism class according to the pathcomponents of its target space. In higher (co)dimensions $m-n$ the map $\widetilde{g}$ into $E(f_1, f_2)$ and the twisted framing $\overline{g}^{\#}$ contain much more information. E.g., if $M = S^m$ and $n \geq 2$, then $\Omega^{\#}(f_1, f_2)$ can be identified with the homotopy group $\pi_m(S^n \wedge \Omega(N)^+)$, and $\omega^{\#}(f_1, f_2)$ is closely related to a Hopf-Ganea invariant. This allows us to reduce many aspects of our problem to questions in standard homotopy theory.

Details of definitions, proofs, and applications will be given elsewhere (compare e.g. [K 3] and [K 2]). Here we present just one sample result.

Theorem 2. *Let N be an odd-dimensional spherical space form (i.e., the quotient of S^n by a free action of a finite group). Then we have for all $f_1, f_2 : S^m \to N$:*

$$MCC(f_1, f_2) = N^{\#}(f_1, f_2) = \begin{cases} 0 & \text{if } f_1 \sim f_2 \text{ or } m < n; \\ \#\pi_1(N) & \text{if } f_1 \not\sim f_2 \text{ and } m > 1; \\ |d^0(f_1) - d^0(f_2)| & \text{if } m = n = 1. \end{cases}$$

(Here $d^0(f_i) \in \mathbb{Z}$ denotes the usual degree).

Finally note that our approach applies also to over- and undercrossings of link maps into a manifold of the form $N \times \mathbb{R}$. This yields unlinking obstructions which often settle unlinking questions and which, in addition, turn out to distinguish a great number of different link homotopy classes. In certain cases they even allow a complete link homotopy classification. Moreover, our approach leads also to the notion of Nielsen numbers of link maps (cf. [K 4]).

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