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Nichtkommutative Geometrie

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Abstract. These notes contain the extended abstracts of talks given at the meeting on 'noncommutative geometry' in September 2004. The range of topics includes index theory, algebraic and topological K-theory, cyclic homology, quantum groups, spectral triples, the Baum-Connes conjecture as well as number theory, dynamical systems and mathematical physics.

Mathematics Subject Classification (2000): 11F, 19D, 19K, 22D, 46L, 59B2X, 81T.

Introduction by the Organisers

Noncommutative geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. Within mathematics, it is a highly interdisciplinary subject drawing ideas and methods from many areas of mathematics and physics. Natural questions involving noncommuting variables arise in abundance in many parts of mathematics and quantum mathematical physics.

On the basis of ideas and methods from algebraic topology and Riemannian geometry, as well as from the theory of operator algebras and from homological algebra, an extensive machinery has been developed which permits the formulation and investigation of the geometric properties of noncommutative structures. Areas of intense research in recent years are related to topics such as index theory, K-theory, cyclic homology, quantum groups and Hopf algebras, the Novikov- and Baum-Connes conjectures as well as to the study of specific questions in other fields such as number theory, modular forms, topological dynamical systems, renormalization theory, theoretical high-energy physics and string theory.

The meeting was attended by 45 participants, including a fair number of young mathematicians but also many of the leading experts in the field. The exchange of ideas was very lively and quite a few significant new results were presented in the talks, which covered many of the aspects of noncommutatve geometry mentioned above.

Workshop: Nichtkommutative Geometrie

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Abstracts

Spectral Flow and resolvent cocycles

Alan L. Carey
(joint work with John Phillips, Adam Rennie, Fyodor A. Sukochev)

The odd local index theorem of Connes and Moscovici [CoM] may be thought of as a far reaching generalisation of the classical index theorem for Toeplitz operators. It is thus a natural question to ask whether the index theorem of Coburn, Douglas, Schaeffer and Singer [CDSS, CMX] proved in the setting of semifinite von Neumann algebras and giving a topological formula for the Breuer-Fredholm index of Weiner-Hopf operators with almost periodic symbol is the prototype for a von Neumann algebra version of the local index theorem. This question was answered in the affirmative by our noncommutative geometry calculation of the index of Toeplitz operators with noncommutative symbol [CPS2, L]. In both cases there is a clear interpretation of the index as computing spectral flow along a certain path of unbounded self-adjoint Breuer-Fredholm operators. (This follows from recent work of some of us in [CP1, CP2] interpreting the Breuer-Fredholm index of the [CDSS] Weiner-Hopf operators as 'type II' spectral flow.) This motivated the present general study of the local index formula of Connes and Moscovici in the setting of semifinite von Neumann algebras via a computation of spectral flow along a path of self-adjoint unbounded Breuer-Fredholm operators.

This line of reasoning touches on a more general program outlined by [BeF] and applied there to foliations. Other examples include differential operators with almost periodic symbol [Sh], the L^2-index theorem (see [M] and references therein).

The starting point is a Hilbert space $\mathcal{H}$ on which there is an unbounded densely defined self adjoint Breuer-Fredholm operator $\mathcal{D}$ affiliated to a semifinite von Neumann algebra $\mathcal{N}$. In this setting Carey-Phillips introduced an integral formula for the spectral flow along the linear path joining $\mathcal{D}$ to a unitarily equivalent operator $u\mathcal{D}u^*$, [CP1, CP2], where $u \in \mathcal{N}$ is such that $[\mathcal{D}, u]$ extends to a bounded operator on H. The natural framework for this formula is that of odd spectral triples (generalised to the von Neumann setting as in [CP1, BeF, CPS1, CPS2]).

We exploit the fact the spectral flow formula is an integral of an exact one-form (this idea was inspired by [G, Si]) hence the path of integration may be changed to obtain a new formula which is amenable to perturbation theory methods. We employ the perturbation technique of the resolvent expansion to write spectral flow in terms of a 'function-valued cochain' in the (b, B) bicomplex of cyclic cohomology. Our function-valued cochain is reminiscent of, though distinct from, Higson's 'improper cocycle' [H]. Our cochain is a finite (b, B) cocycle modulo functions holomorphic in a half-plane. We refer to this cochain as the resolvent cocycle and it should be thought of as a substitute for the JLO cocycle (which is the starting point for the argument of Connes-Moscovici).

The resolvent cocycle can be further expanded employing the quantised pseudo-differential calculus of Connes-Moscovici, [CoM]. The end result is an expression for the spectral flow in terms of a sum of generalised zeta functions of the form

$$\zeta_b(z) = \tau(b(1+\mathcal{D}^2)^{-z}), \quad b \in \mathcal{N}.$$

This sum of zeta functions is meromorphic in a half-plane, with (at worst) only a single simple pole at the 'critical point'in this half-plane. The residue at this pole is precisely the spectral flow.

Under the assumption that the individual zeta functions in this sum analytically continue to a deleted neighbourhood of the critical point, we may take residues of the individual terms at the critical point. The resulting formula, when $\mathcal{N} = \mathcal{B}(H)$, is essentially that which is obtained by pairing the (odd, renormalised) resolvent cyclic cocycle obtained by Connes and Moscovici, [CoM], with the Chern character of the unitary u^*. The only difference between the two formulae is that we cannot assume $\mathcal{D}$ is invertible (it may have zero in the continuous spectrum) and hence use inverse powers of $(1+\mathcal{D}^2)^{1/2}$, whereas Connes-Moscovici assume that $\mathcal{D}$ is invertible and use inverse powers of $|\mathcal{D}|$.

The novel aspects of our approach are:

• Our result calculates spectral flow in semifinite spectral triples generalising part of the type I theory of [CoM]. Specifically, our formula for spectral flow is given in terms of a cyclic cocycle, which is the generalisation to semi-finite von Neumann algebras of the residue cocycle of [CoM].

• Only the final step of our proof requires the analytic continuation property of the generalised zeta functions. Indeed, we express spectral flow as the *residue* of a sum of zeta functions without invoking *any* analytic continuation hypothesis.

• Assuming the individual zeta functions in the above sum have analytic continuations to a deleted neighbourhood of the critical point allows us to write spectral flow as a sum of residues of zeta functions. This gives a si,ple proof that the residues of these zeta functions then assemble to form a finite (b, B) cocycle for the algebra $\mathcal{A}$ of the spectral triple.

• We make no assumptions on the decay of our zeta functions along vertical lines in the complex plane thus reducing the side conditions that need to be checked when applying the local index formula of [CoM].

• Our proof that the residue cocycle [CoM] is indeed a (b, B)-cocycle is quite simple even in the general semifinite case by virtue of using our resolvent cocycle.

• Except for the need to verify a number of estimates, the strategy of our proof is straightforward, and is identical in both the type I and type II cases

• We remark that there is an unrenormalised version of the residue cocycle in [CoM] containing an infinite number of terms in the case that one of the terms in the expansion has an essential singularity, whereas their renormalised version always has a bounded number of terms. The unrenormalised version presents an issue of convergence which is difficult to address. Since we do not pass through an intermediate step where the cocycle contains a potentially infinite number of terms, we are free to allow essential singularities from the outset.

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Equivariant Spectral Triples for $SU_q(n), n > 2$: nonexistence results

Partha Sarathi Chakraborty

A natural question in the context of Noncommutative Geometry is what should be candidates of Lie groups in the noncommutative framework. It was believed that the C* algabraic theory of quantum groups do not go hand in hand with the theory laid up by Alain Connes. Earlier we proved that this believe is ill founded. A proper formulation of left invariant geometry was given and it was shown that the prime example of quantum SU(2) fits well in the framework of NCG ([1]).

Examples of Noncommutative spaces were created and was further analyzed by Alain Connes ([2]). In the present work we describe a general technique to study Dirac operators on noncommutative spaces under the additional assumptions of equivariance governed by a quantum group. The Dirac operator D comes with two restrictions on it, namely, it has to have compact resolvent, and must have bounded commutators with algebra elements. Various analytic consequences of the compact resolvent condition (growth properties of the commutators of the algebra elements with the sign of D) have been used in the past by various authors. Here we will take a new approach that will help us exploit it from a combinatorial point of view. The idea is very simple. Given a selfadjoint operator with compact resolvent, one can associate with it a certain graph in a natural way. This makes it possible to do a detailed combinatorial analysis of the growth restrictions (on the eigenvalues of D) that come from the boundedness of the commutators, and to characterize the sign of the operator D completely. Using this, we then prove that for $SU_q(n), n > 2$ the L_2-space does not admit any equivariant Dirac operator with nontrivial sign acting on it. This is joint work with **Arupkumar Pal** of Indian Statistical Institute, Delhi.

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Renormalization and motivic Galois theory

Alain Connes

(joint work with Matilde Marcolli)

We describe our joint work [7] on the interpretation of renormalization as a Riemann-Hilbert correspondence thus clarifying the role of the Birkhoff decomposition appearing in the joint work with Kreimer [5], [6] which provided a conceptual understanding of perturbative renormalization in terms of the Birkhoff decomposition of loops in a pro-unipotent Lie group G determined by the physical theory, through the Hopf algebra of Feynman graphs [14], [5]. It however left open the interpretation of renormalization in the context of the Riemann–Hilbert correspondence, a broad term encompassing, in various forms and levels of generalization, equivalences between geometric problems associated to differential systems with singularities and representation theoretic data associated to the monodromy. It will be used here in a local irregular singular context, similar to [16].

In our joint work [7] we find the missing Riemann–Hilbert correspondence and describe the corresponding geometric differential systems and “monodromy group". We construct the Riemann–Hilbert correspondence associated to perturbative renormalization, in the form of a classification of flat equisingular bundles in terms of representations of a *" motivic Galois group"* U^*.

The geometric problem is the classification of equisingular flat connections on a two dimensional complex space B^* which is the total space of a $\mathbb{G}_m$-principal bundle over an infinitesimal punctured disk Δ^*. This classification problem stems directly from the divergences of the physical theory at the dimension D where one would like to do physics. The base Δ^* is the space of complexified dimensions around D. The fibers of the principal $\mathbb{G}_m$-bundle B describe the arbitrariness in the normalization of integration in complexified dimension $z \in \Delta^*$, in the commonly used regularization procedure known as Dim-Reg (dimensional regularization). The $\mathbb{G}_m$-action corresponds to the rescaling $\hbar\,\partial/\partial\hbar$. The group G is the pro-unipotent Lie group whose Hopf algebra is the Hopf algebra of Feynman graphs of [14], [5].

The equisingularity condition is a geometric formulation of the fact that in quantum field theory, in the minimal substraction scheme, the counterterms are independent of the choice of a unit of mass. An equisingular connection is a $\mathbb{G}_m$-invariant G-valued connection, singular on the fiber over the origin of Δ, and satisfying the following property: the equivalence class of the singularity of the pullback of the connection by a section of the principal $\mathbb{G}_m$-bundle only depends on the value of the section at the origin.

The group playing the role of the monodromy group is in fact a " motivic Galois group" U^* which is uniquely defined through the Riemann–Hilbert correspondence. Its representations classify equisingular flat vector bundles. As an algebraic group scheme, U^* is a semi-direct product by the multiplicative group $\mathbb{G}_m$ of a pro-unipotent group scheme U whose Lie algebra is the free graded Lie algebra

$$\mathcal{F}(1,2,3,\cdots)_{\bullet}$$

generated by elements e_{-n} of degree n, $n > 0$.

Thus, there are three different levels at which Hopf algebra structures enter the theory of perturbative renormalization. First, there is Kreimer's Hopf algebra of rooted trees [14], which is adapted to the specific physical theory by decorations of the rooted trees. There is then the Connes–Kreimer Hopf algebra of Feynman graphs, which is dependent on the physical theory by construction, but which does not require decorations. There is then the algebra associated to the group U^*, which is universal with respect to the set of physical theories.

We show that the divergences of quantum field theory provide the data allowing one to define an action of the group U on the set of dimensionless coupling constants of physical theories, through the map of the corresponding group G to formal diffeomorphisms constructed in [6]. In particular, this exhibits the renormalization group as the action of a one parameter subgroup $\mathbb{G}_a \subset U$ of the above Galois group.

We then construct a specific universal singular frame on principal U-bundles over B. When using in this frame the dimensional regularization technique of QFT, all divergences disappear and one obtains a finite theory, which only depends upon the choice of a local trivialization for the principal $\mathbb{G}_m$-bundle B. When computed as iterated integrals, its coefficients are certain rational numbers that appear in the local index formula of [9].

This means that we can view equisingular flat connections on finite dimensional vector bundles as endowed with arithmetic structure. We show that they can be organized into a Tannakian category with a natural fiber functor to the category of vector spaces, over any field of characteristic zero. The Tannakian category obtained this way is equivalent to the category of finite dimensional representations of the affine group scheme U^*, which is uniquely determined by this property.

Closely related group schemes appear in motivic Galois theory and U^* is for instance abstractly (but non-canonically) isomorphic to the motivic Galois group $G_{\mathcal{M}_T}(\mathcal{O})$ ([10], [11]) of the scheme $S_4 = \mathrm{Spec}(\mathcal{O})$ of 4-cyclotomic integers, $\mathcal{O} = \mathbb{Z}[i][\frac{1}{2}]$.

The natural appearance of the "motivic Galois group" U^* in the context of renormalization confirms a suggestion made by Cartier in [1], that in the Connes–Kreimer theory of perturbative renormalization one should find a hidden "cosmic Galois group" closely related in structure to the Grothendieck–Teichmüller group. The question of relations between my work with Kreimer, motivic Galois theory, and deformation quantization was further emphasized by Kontsevich in [13]. At the level of the Hopf algebra of rooted trees, relations between renormalization and motivic Galois theory were also investigated by Goncharov in [12].

This also realizes the hope formulated in [3] of relating concretely the renormalization group to a Galois group. Here we are dealing with the Galois group dictated by renormalization and the renormalization group appears as a canonical one parameter subgroup $\mathbb{G}_a \subset U$.

The appearance of multiple polylogarithms in the coefficients of divergences in QFT, discovered by Broadhurst and Kreimer, as well as recent considerations of Kreimer on analogies between residues of quantum fields and variations of mixed Hodge–Tate structures associated to polylogarithms (*cf.* [15]), suggest the existence for the above category of equisingular flat bundles of suitable Hodge-Tate realizations given by a specific choice of Quantum Field Theory.

These facts altogether indicate that the divergences of Quantum Field Theory, far from just being an unwanted nuisance, are a clear sign of the presence of totally unexpected symmetries of geometric origin. This shows, in particular, that one should understand how the universal singular frame "renormalizes" the geometry of space-time using the Dim-Reg minimal substraction scheme and the universal counterterms.

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Algebraic and topological K-theory of locally convex algebras

Guillermo Cortiñas

We consider complete locally convex algebras over the field $\mathbb{C}$ of complex numbers. We wish to compare various topological and algebraic K-theories of such algebras. If A and B are two such algebras and $p \in [1,\infty]$, there is defined a Bott-periodic bivariant K-theory $kk_*^{(p)}(A,B)$ [2]; we write

$$K_*^{top,p}(A) := kk_*^p(\mathbb{C},A) \qquad (* \in \mathbb{Z}).$$

It is shown in [2] that

$$kk^{(p)}(A,B) = kk^{(1)}(A,B) \qquad (p \in [1,\infty))$$

Thus there are essentially two distinct $kk^{(p)}$-theories, $kk^{(1)}$ and $kk^{(\infty)}$. It is further shown in [2] that

$$K_0^{top,p}(A) = K_0(A \otimes_\pi \mathcal{L}^p) \qquad (p \in (1,\infty]) \tag{1.1}$$

Here K_0 is the algebraic K-group, $\otimes_\pi$ is the projective tensor product, and $\mathcal{L}^p$ is the p-Schatten ideal if $p < \infty$, and the ideal of all compact operators if $p = \infty$. On the algebraic side we have the (Quillen-Gersten-Karoubi-Wagoner) K-theory groups

$$K_n(A) = \pi_n \mathbb{K}(A) \qquad (n \in \mathbb{Z}). \tag{1.2}$$

Here $\mathbb{K}(A)$ is the nonconnective K-theory spectrum of [3], [5] and [7]. We shall also consider Weibel's algebraic homotopy K-groups ([8])

$$KH_*(A) := \pi_n(\operatorname*{hocolim}_{\Delta^{op}}([n] \mapsto \mathbb{K}(A \otimes \mathbb{C}[t_0, \dots, t_n]/ < 1 - (t_0 + \cdots + t_n) >)) \tag{1.3}$$

It is shown in [8] that KH is invariant under polynomial homotopy and excisive. One can also consider a version of this which is invariant under C^∞-homotopy, namely

$$K^\Delta(A) := \pi_n(\operatorname*{hocolim}_{\Delta^{op}}([n] \mapsto \mathbb{K}(A \otimes_\pi C^\infty(\Delta^n))) \tag{1.4}$$

Our main result compares, for $p \in (1,\infty]$, the groups $K_*^{top,p}(A)$ with $KH_*(A \otimes_\pi \mathcal{L}^p)$, $K_*^\Delta(A \otimes_\pi \mathcal{L}^p)$ and $K_*(A \otimes_\pi \mathcal{L}^p)$. To state the theorem we need one more piece of notation. We write

$$HC_*(A) := HC_*^{alg}(A/\mathbb{Q}).$$

for the algebraic cyclic homology groups of A as a $\mathbb{Q}$-algebra.

Theorem 1.1. *Let A be a locally convex algebra, and $p \in (1,\infty]$. Then*

i) There are isomorphisms

$$KH_*(A \otimes_\pi \mathcal{L}^p) \cong K_*^{top,p}(A) \cong K_*^\Delta(A \otimes_\pi \mathcal{L}^p).$$

ii) There is a natural long exact sequence ($n \in \mathbb{Z}$)

$$K_{n+1}^{top,p}(A) \to HC_{n-1}(A \otimes_\pi \mathcal{L}^p) \to$$
$$K_n(A \otimes_\pi \mathcal{L}^p) \to K_n^{top,p}(A) \to HC_{n-2}(A \otimes_\pi \mathcal{L}^p).$$

The two main ingredients of the proof are the excision theorem for infinitesimal K-theory ([1]), the identity (1.1), and the variant of Kasparov's homotopy invariance theorem ([4]) for locally convex algebras proved in [2]. Apart from this, the method of proof follows some standard arguments such as those discussed in the survey paper [6].

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Algebraic K-theory and the coefficient ring in bivariant K-theory for locally convex algebras

Joachim Cuntz

(joint work with Andreas Thom)

In this note we report on joint work with A.Thom. In [2] and [3] the author had developed a bivariant K-theory for different categories of locally convex algebras. An important problem that remained open for the variant $kk^{\rm alg}$ of the theory, that applies to arbitrary locally convex algebras, was the determination of the coefficient ring $R = kk^{\rm alg}(\mathbb{C}, \mathbb{C})$. It was shown in [3] that the unital commutative ring R admits a unital homomorphism into $\mathbb{C}$ and is therefore in particular nontrivial. Also, in [3] a modified theory was discussed for which the coefficient ring reduces to $\mathbb{Z}$. However, this theory seemed hard to manage in general.
We can now prove that the coefficient ring can be reduced to $\mathbb{Z}$ simply by stabilizing the algebra in the second variable by a Schatten ideal $\mathcal{C}_p$. This stabilization gives a very natural theory $kk^{(p)}$ that still admits a bivariant Chern character into periodic cyclic theory. The present note contains an outline of the construction and of the ideas involved in the computation of the coefficient ring for the stabilized theory. More generally, for any locally convex algebra A, $kk^{(p)}(\mathbb{C}, A)$ can be determined in terms of algebraic K-theory.
A key ingredient in our proof is an old result by Higson (based on previous work by Kasparov and Cuntz) showing that every stable split exact functor on the category of C^*-algebras is homotopy invariant, [4]. The proof of this result can be adapted to a setting with differentiable homotopies ("diffotopies") and to locally convex algebras.

1. The stabilized bivariant theory

In [3], a bivariant K-theory $kk^{\rm alg}$ had been defined on the category of locally convex algebras using noncommutative stable homotopy in a way analogous to [2].

The theory has the usual properties of a bivariant topological K-theory (in particular it defines an additive category where the objects are locally convex algebras, the morphism set for two objects A and B is the abelian group $kk^{\mathrm{alg}}(A,B)$ and this category is triangulated).
Given $p \in [1,\infty]$, let $\mathcal{C}_p$ denote the Schatten ideal of compact operators with p-summable singular values in a separable Hilbert space. We include the case $p = \infty$, where by definition $\mathcal{C}_\infty$ is the ideal of all compact operators. Given a locally convex algebra A, we denote by $\mathcal{C}_p \hat{\otimes} A$ the completed projective tensor product.

Definition 1.1. *Let A and B be locally convex algebras, $p \in [1,\infty]$ and $n \in \mathbb{Z}$. We define*

$$kk_n^{(p)}(A,\, B\,) = kk_n^{\mathrm{alg}}(A,\, \mathcal{C}_p \hat{\otimes} B\,)$$

where kk_n^{alg} is defined as in [3].

Proposition 1.2. *For $1 \leq p < \infty$, the natural map $kk_n^{(1)}(A,\, B\,) \to kk_n^{(p)}(A,\, B\,)$ defines an isomorphism for all A, B and n.*

Proof. By an argument from [2], the Schatten ideal $\mathcal{C}_p$ is isomorphic to $\mathcal{C}_1$ for all $1 \leq p < \infty$ in the category kk^{alg}. □

As a consequence we see that, in the definition of $kk_n^{(p)}$, we can restrict to the cases $p = 1$ and $p = \infty$.

2. Diffotopy invariance of split-exact functors on the category of locally convex algebras

Definition 2.1. *Let E be a (covariant) functor from the category of locally convex algebras to the category of abelian groups. We say that*

- *E is diffotopy invariant, if the maps $E(\mathcal{C}^\infty([0,1],A)) \to E(A)$ induced by the different evaluation maps for $t \in [0,1]$ are all the same (it is easy to see that this is the case if and only if the map induced by evaluation at $t = 0$ is an isomorphism).*
- *E is split-exact, if, for every extension $0 \to I \to A \to B \to 0$ of locally convex algebras with a homomorphism splitting $B \to A$, the induced sequence $0 \to E(I) \to E(A) \to E(B) \to 0$ is exact (and then automatically also split).*
- *E is M_2-stable if the map $E(A) \to E(M_2(A))$ induced by the natural inclusion $j : A \to M_2(A)$ into 2×2-matrices is an isomorphism for each locally convex algebra A.*
- *E is weakly $\mathcal{C}_p$-stable, if there is a map $E(\mathcal{C}_p \hat{\otimes} A) \to E(A)$, such that the composition $E(A) \to E(\mathcal{C}_p \hat{\otimes} A) \to E(A)$ with the map induced by the natural inclusion $A \to \mathcal{C}_p \hat{\otimes} A$ is the identity for each locally convex algebra A.*

Definition 2.2. *Let A and I be locally convex algebras. An abstract Kasparov (A,I)-module is a triple (φ, U, P) where*

- *φ is a continuous homomorphism from A into a unital locally convex algebra D containing I as an ideal.*
- *U is an invertible element and P is an idempotent element in D such that the following commutators are in I for all $x \in A$:*

$$[U, \varphi(x)],\ [P, \varphi(x)],\ [U, P]$$

Let E be a functor on the category of locally convex algebras that is split exact and M_2-stable. Then every abstract Kasparov module (φ, U, P) induces a quasihomomorphism, in the sense of [1], from A to $M_2(I)$ and therefore a map $E(\varphi, U, P) : E(A) \to E(I))$.

Theorem 2.3. *Every functor from the category of locally convex algebras to the category of abelian groups which is split exact and weakly $\mathcal{C}_p$-stable for some $p > 1$, is diffotopy invariant.*

Sketch of proof. Let $\mathrm{ev}_0, \mathrm{ev}_1$ be the two evaluation maps $\mathcal{C}^\infty([0,1], A) \to A$. As in the argument by Kasparov-Higson one constructs two abstract $(\mathcal{C}^\infty([0,1], A), A)$-Kasparov modules (φ, U_0, P) and (φ, U_1, P) such that $E(\mathrm{ev}_t) = E(\varphi, U_t, P)$, $t = 0, 1$ and such that $U_1 = e^{ih}U_0$ for a selfadjoint element h in a suitable smooth subalgebra of a C^*-algebra. The assumptions on E then imply (using a technical lemma from [5]) that $E(\varphi, U_0, P) = E(\varphi, U_1, P)$ as in [4]. □

3. K_0 of a stable algebra and $kk^{(p)}(\mathbb{C}, A)$.

In this section we consider the functor E, defined on the category of locally convex algebras by

$$E(A) = K_0(\mathcal{C}_p \hat{\otimes} A)$$

where $_0$ is the usual algebraic K_0-group. E is M_2-stable in the sense of 2.1 and thus also invariant under inner automorphisms. E is also split exact and weakly $\mathcal{C}_p$-stable. Thus by 2.3, E is diffotopy invariant. This is the key to proving the following

Theorem 3.1. *For every locally convex algebra A and for $1 < p \leq \infty$ one has $kk_0^{(p)}(\mathbb{C}, A) = K_0(\mathcal{C}_p \hat{\otimes} A)$. In particular, $kk_0^{(p)}(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$.*

Sketch of proof. It follows from the definition of kk^{alg} in that it is the universal diffotopy functor into an additive category which is stable under tensoring with the algebra $\mathcal{K}$ of smooth compact operators in both variables and has long exact sequences in both variables, see [3]. The functor E satisfies these properties and by what has been said above it also is diffotopy invariant. Therefore there exist maps

$$kk^{\mathrm{alg}}(A, B) \to \mathrm{Hom}\,(E(\mathbb{C}), E(\mathcal{C}_p \hat{\otimes} A)) \to E(A)$$

The fact that the composition of these maps is an isomorphism can be proved following the lines of the corresponding argument in [2], 7.2 and 7.4. □

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Spectral Triples on Quantum Spheres

Ludwik Dabrowski

Spectral triple $(A, \mathcal{H}, D)$ is the key notion of the most recent 'layer' of noncommutative differential geometry a la Connes, apt to encode the concept of a noncommutative Riemannian spin$_c$ manifold. It consists of a (unital) $*$-algebra A of bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D = D^\dagger$ on $\mathcal{H}$ with
• compact resolvent $(D - \lambda)^{-1}$, $\forall\lambda \in \mathbb{C} \setminus \mathrm{spec}D$
• bounded commutators $[D, \alpha]$, $\forall\alpha \in A$.
In the classical (commutative) situation the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ canonically associated with a Riemannian spin$_c$ manifold consists of the algebra $\mathcal{A}$ of smooth functions on M represented (by pointwise multiplication) on the Hilbert $\mathbb{H}$ space of ($\mathcal{L}^2$) Dirac spinors, and of the Dirac operator D, constructed from the Levi-Civita' connection (metric preserving and torsion-free) plus a $U(1)$-connection.
These data are of great importance both in Mathematics and Physics. Together with the real structure and gradation operators J and γ (known also as charge conjugation and parity) they satisfy certain further seven properties which guarantee that the underlying differential, metric and spin structure can be reconstructed back from them.
A (noncommutative) spectral geometry (a la Connes), a generalization of these concepts to noncommutative algebras A, has already found a plentiful of applications. But the whole *zoo* of q-deformed spaces coming from the quantum group theory, was commonly believed not to match well the Connes' approach. This was supported by some apparent "no-go" hints such as that exponentially growing spectrum of the quantum Casimir operator would prevent bounded commutators with the algebra, some known differential calculi seemed not to come as bouned commutators with any D, an early classification of equivariant representations of A missed the spinorial ones and also on some deformation theory grounds.

However, the intense recent activity indicated a possibility to reconcile these two lines of mathematical research. In the talk I review some of the simplest studied examples of quantum spheres of lowest dimension (2 and 3). More precisely, I shall

be concerned mainly with (the algebra of) the underlying space of the quantum group $SU_q(2)$ and its two homogeneous spaces known as the standard and the equatorial Podleś sphere.
Essential requirement will be the *equivariance* with respect to some Hopf $*$-algebra U acting on A and the first order condition.

Existence of γ-elements

Heath Emerson
(joint work with Ralf Meyer)

Let G be a discrete group with a G-finite model for its classifying space $\mathcal{E}G$ for proper actions of G. The Dirac dual Dirac method, invented by Mischenko and Kasparov, and utilized since in numerous instances (*e.g.* [6], [7], [8]), is a powerful means of proving, in specific cases, the Strong Novikov Conjecture (SNC), *i.e.* injectivity of the analytic assembly map

$$\mathrm{K}_*^{\mathrm{top}}(G) = \mathrm{KK}_*^G(C_0(\mathcal{E}G), \mathbb{C}) \to \mathrm{K}_*\big(C_r^*(G)\big).$$

It presents as sufficient conditions for SNC the existence of a proper G-C^*-algebra P, and elements $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ called the *Dirac class* and $\eta \in \mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ called the *dual Dirac class*, such that $\mathsf{D} \otimes_{\mathbb{C}} \eta = 1_{\mathsf{P}}$. If the Dirac dual Dirac method applies to G, *i.e.* if P, D and η as above exist, then we say that G has a γ-element (the latter referring to the class $\eta \otimes_{\mathsf{P}} \mathsf{D} \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$.) As SNC is not known in general, it is not known whether every group G has a γ-element.

Another method of attaching SNC which has had some considerable success, proceeds via coarse geometry. In this approach, one utilizes the *Descent Principle*, (see *e.g.* [5]) which asserts that, at least if G is torsion-free, then isomorphism of the coarse Baum-Connes assembly map

$$\mathrm{KX}_*(|G|) \to \mathrm{K}_*\big(C^*(|G|)\big)$$

implies injectity of the analytic assembly map, and thus SNC. The coarse Baum-Connes assembly map for $|G|$ (by the symbol $|G|$ we mean the metric, or *coarse* space, underlying G) only depends on G up to coarse equivalence of metric spaces.

We show that, actually, the Dirac-dual-Dirac method is related to the coarse geometric method, and in fact, that if G is torsion-free, then the existence or nonexistence of a γ-element for a group G is geometric, *i.e.* depends only on the large-scale geometry of G.

To do this, we utilize a variant on a construction of Higson and Roe (see [10].) For every metric space X we construct a C^*-algebra $\mathfrak{c}(X)$ which we term the *stable Higson corona of* X. The latter is by definition the quotient of the C^*-algebra of continuous, bounded, vanishing variation functions on X with values in the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space, by the ideal of those such functions vanishing at infinity. An obvious extension gives a map

$$\mu_X^* : \tilde{\mathrm{K}}_{*+1}\big(\mathfrak{c}(X)\big) \to \mathrm{KX}^*(X), \tag{1.1}$$

where $\mathrm{KX}^*(X)$ is a K-theoretic analog of the coarse K-homology $\mathrm{KX}_*(X)$ of X. We call this map the 'coarse co-assembly map for X.' Its domain, co-domain, and the map itself, all depend only on the large-scale geometry of X. See [4] for details.

Our first theorem is then:

Theorem 1.1. *Let G be a discrete, torsion-free group with a G-finite model for $\mathcal{E}G$ (equivalently, G has a finite model for BG.) Then the Dirac dual Dirac method applies to G if and only if the coarse co-assembly map*

$$\mu^*_{|G|} : \tilde{\mathrm{K}}_*\big(\mathfrak{c}(|G|)\big) \to \mathrm{KX}^*(|G|) \tag{1.2}$$

for $|G|$ is an isomorphism. In particular, the existence or non-existence of a γ-element for such a group G depends only on the large-scale geometry of G.

Similar results are available for groups with torsion, and for groups without G-finite models for $\mathcal{E}G$, but they are more complicated to state. In the case where G has torsion, the finite subgroups of G must be taken into account in (1.2); if no G-finite model for $\mathcal{E}G$ is available, we work with the system of G-compact subsets of $\mathcal{E}G$ and take inverse limits. See [2] for the statements and proofs.

To extend the analysis further, we require a slightly modified version of the stable Higson corona. If X is a metric (or more generally, a coarse) space, let $\mathfrak{c}^{\mathfrak{red}}(X)$ be the quotient of the C^*-algebra of bounded continuous functions of vanishing variation from X to the C^*-algebra of bounded operators on a Hilbert space which are constant modulo compacts, by the ideal of those such functions which are zero modulo the compacts, and which vanish at infinity. It is easy to check that $\tilde{K}_*\big(\mathfrak{c}(X)\big) \cong \tilde{\mathrm{K}}_*\big(\mathfrak{c}^{\mathfrak{red}}(X)\big)$. Again $\mathfrak{c}^{\mathfrak{red}}(X)$ depends only on the large-scale geometry of X.

Any group which acts isometrically on X also acts by automorphisms of $\mathfrak{c}(X)$ and of $\mathfrak{c}^{\mathfrak{red}}(X)$. For example, if we set $X = |G|$, with G a group satisfying the hypotheses stated at the beginning, then, since G acts isometrically on $|G|$ (in the coarse sense), $\mathfrak{c}^{\mathfrak{red}}(|G|)$ becomes in a natural way a G-C^*-algebra. There is an evident G-equivariant extension yielding a canonical map

$$\mu^*_{|G|,G} : \mathrm{K}^{\mathrm{top}}_{*-1}\big(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\big) \to \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C}). \tag{1.3}$$

The codomain of the map (1.3) is directly related to the classical Novikov conjecture; elements of it correspond bijectively (rationally) to *higher signatures for* G. The classical Novikov conjecture asserts that all such higher signatures are homotopy invariant, in a slightly technical sense which we do not explain here.

Our results above can be refined by the following theorem.

Theorem 1.2. *Let G be a discrete group with G-finite $\mathcal{E}G$.*

(1) *Any element of* $\mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ *in the range of the map* (1.3) *corresponds to a homotopy invariant higher signature.*
(2) *The unit class* $1_{\mathcal{E}G} \in \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ *lies in the range of map* (1.3) *if and only if G has a γ-element.*

Next, using a theorem of [9] we can analyze in geometric terms the *dual Dirac class itself*, and not merely the question of its existence. According to *loc.cit.*, that part of the Dirac dual Dirac method (see first paragraph) which involves the G-C^*-algebra P and the Dirac class $\mathsf{D} \in \mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ *always applies*; that is, for every group G there is a P and a D such that the Dirac dual Dirac method applies to G if and only if there exists $\eta \in \mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ such that $\mathsf{D} \otimes_{\mathbb{C}} \eta = 1_{\mathsf{P}}$. Moreover, D and P are unique up to KK^G-equivalence, and for every G-C^*-algebra A, the left hand side of the Baum-Connes conjecture with coefficients in A is isomorphic to $\mathrm{K}_*(\mathsf{P} \otimes A \rtimes G)$, while the Baum-Connes map itself can be reformulated as the map $\mathrm{K}_*(\mathsf{P} \otimes A \rtimes G) \to \mathrm{K}_*(A \rtimes_r G)$ induced by Kasparov product with D.

Attempts to write down sufficient conditions for a an individual class to lie in the range of the map (1.3) led us to the rather surprising conclusion that the group $\mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ discussed in the previous paragraph can actually be *calculated* in terms of the K-theory of the stable Higson corona (by means of spectral sequences computing topological K-theory groups):

Theorem 1.3. *Let G be a discrete group with G-finite model for $\mathcal{E}G$. Then there is a natural isomorphism*

$$\mathrm{K}_*^{\mathrm{top}}\big(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\big) \cong \mathrm{KK}_*^G(\mathbb{C}, \mathsf{P}). \tag{1.4}$$

Under this isomorphism, a dual Dirac class corresponds to any element of the domain of (1.4) *mapping to $1_{\mathcal{E}G}$ under the map* (1.3).

Theorem 1.3, from which all previously stated theorems can be rapidly deduced, has various other applications. For instance it can be used to show that proper Lipschitz classes in the sense of [1] lie in the range of the map (1.3), and that a dual Dirac class for a hyperbolic group must come (in a precise sense) from its Gromov boundary.

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Duality and Index Theory

Jerome Kaminker

In commutative topology there are two standard types of duality—Poincaré duality and Spanier-Whitehead duality. If M is a closed Spin^c manifold then there are K-theoretic versions which are related to each other and to the Thom isomorphism by the following diagram.

$$\begin{array}{ccc} K^{N+i-n}(\nu M) & \xrightarrow{Spanier-Whitehead} & K_{n-i}(M) \\ \Phi\Big\uparrow & & \Big\downarrow \cap[M]_K \\ K^i(M) & = & K^i(M) \end{array}$$

Here, νM is the normal bundle of an embedding of M in a a high dimensional sphere and it is known that this can be taken as a Spanier-Whitehead dual of M.

There are natural extensions of this to the noncommutative setting which have been considered by several people, [2, 6, 5, 4]. Given a C^*-algebra, A, a Spanier-Whitehead dual is an algebra $\mathcal{D}A$ such that there are classes $\Delta \in K^i(A \otimes \mathcal{D}A)$ and $\delta \in K_i(A \otimes \mathcal{D}A)$ satisfying that the respective Kasparov products induce inverse isomorphisms

$$\Delta : K_j(A) \rightleftarrows K^{j+i}(\mathcal{D}A) : \delta. \tag{1.1}$$

If one can take $\mathcal{D}A$ to be A itself, then this is the definition of Poincaré duality which Connes has introduced in his definition of a noncommutative manifold. The notion of noncommutative Spanier-Whitehead duality arises in several settings.

1) If Γ is a cocompact lattice in a semi-simple Lie group G, then the Baum-Connes conjecture for Γ is equivalent to $C(B\Gamma)$ being a Spanier-Whitehead dual for $C_r^*(\Gamma)$. The classes Δ and δ are given by the Mishchenko line bundle and the dual Dirac class respectively.

2) The Cuntz-Krieger algebras O_A and O_{A^t} are Spanier-Whitehead dual, [4].

3) If Γ is a torsion free hyperbolic group, then $C(\partial\Gamma) \rtimes \Gamma$ is its own Spanier-Whitehead dual. Thus, it satisfies Poincaré duality, [3]. This duality is closely related to the Baum-Connes isomorphism for hyperbolic groups. Indeed, there is a commutative square relating them.

4) The Baum-Connes map for the group $\Gamma = \mathbb{Z}^n$ is related to the Fourier-Mukai transform by the following diagram.

$$\begin{array}{ccc} K_*(B\mathbb{Z}^n) & \xrightarrow{\text{Baum-Connes}} & K_*(C_r^*(\Gamma)) \\ \text{Poincaré duality}\Big\downarrow & & \Big\downarrow\text{Fourier transform} \\ K^*(B\mathbb{Z}^n) & \xrightarrow{\text{Mukai transform}} & K^*(B\mathbb{Z}^n). \end{array} \tag{1.2}$$

Note that $B\mathbb{Z}^n = T^n$ and we are taking n even. The genuine Fourier-Mukai transform is an isomorphism between the derived category of coherent sheaves on an abelian variety with that of its dual variety. However, the explicit definition of the transform can be used to induce a map on K-theory and that is what is intended in the diagram. One may observe that the proof that the actual Mukai transform is an isomorphism can be based on application of the Stone-von Neumann theorem, [7], and with some care, that result can be used to deduce that the Baum-Connes map is an isomorphism for $\mathbb{Z}^n$. It would be interesting to obtain a direct proof of Baum-Connes for $\mathbb{Z}^n$ or for nilpotent groups based on Stone-von Neumann.

These examples suggest a relation between noncommutative Spanier-Whitehead duality and noncommutative transversality. By the latter we mean that two algebras, represented on the same Hilbert space, are transverse if products of elements from the two algebras are compact. This is more apparent in the next example.

5) Let $A \in SL(2n, \mathbb{Z})$ have no eigenvalues of absolute value 1. The matrix A induces an expansive automorphism of the torus, T^{2n}, and its stable and unstable foliations are transverse. Assume that they are actually orthogonal in an appropriate sense. Then one can show that the C^*-algebras of the foliations are transverse in the above sense and, moreover, are Morita equivalent, hence isomorphic since they are stable. It would be interesting to know when two noncommutative torii which are Morita equivalent, have this property because they can be obtained as the stable and unstable algebras for a toral automorphism for which they are orthogonal.

These algebras themselves are not the ones exhibiting Spanier-Whitehead duality, but the matrix A induces automorphism of the two algebras and the resulting crossed-products do satisfy Spanier-Whitehead duality.

6) There are several cases of T-duality in string theory, as studied by V. Mathai and his collaborators, which are instances of fiberwise Spanier-Whitehead duality, e.g. [1].

While the notion of noncommutative Spanier-Whitehead duality is a natural and simple one, and much more common than Poincaré duality, it arises in some complicated and interesting settings and is related to generalized transforms.

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Cup products in Hopf-Cyclic Cohomology and Connes-Moscovici Characteristic Map

Masoud Khalkhali

1. Hopf-cyclic cohomology with coefficients

One of the most interesting developments in cyclic cohomology theory in the last few years was the introduction of a cyclic cohomology theory for Hopf algebras by Connes and Moscovici [6, 7, 8]. This theory reduces to Lie algebra homology and group homology for Hopf algebras associated to Lie algebras and groups, respectively and together with Connes-Moscovici's Hopf algebra $\mathcal{H}_1$, enormously simplifies index theory computations in the context of transverse index theory for foliations. An interesting feature here was the introduction of the so called *modular pairs in involution* (δ, σ), consisting of a character δ and a grouplike element σ on a given Hopf algebra that satisfy certain compatibility conditions. For commutative or cocommutative Hopf algebras one can choose $\delta = \varepsilon$ and $\sigma = 1$. Soon after a dual theory was proposed by Khalkhali and Rangipour in [11] with many computations.

To understand the algebraic underpinnings of these cohomology theories, we found it convenient to go beyond Hopf algebras and develop theories for an algebra or coalgebra endowed with action or coaction of a Hopf algebra, and also to allow general coefficients in the theory [12]. The two theories alluded above, where found to be special cases where one considers a Hopf algebras acting or coacting on itself via multiplication or comultiplication, respectively. It was also understood that a guiding principle here is that Hopf-cyclic cohomology theories can be understood as generalizations of equivariant de Rham cohomology theory. In [12] modular pairs in involution were shown to be one dimensional coefficients that one can introduce in the theory. The question of identifying the most general coefficient systems was left open and was later completely solved in [9, 10]. In my talk in Oberwolfach I sketched a proof of existence of cup products, in fact of two rather different types, for Hopf-cyclic cohomology [13]

An *stable anti Yetter-Drinfeld* (SAYD) module over a Hopf algebra H, introduced by Hajac-Khalkhali-Rangipour-Sommerhaeuser [9] is a left H-module and a left H-comodule M where the module and comodule actions satisfy certain compatibility conditions. Given an H-module coalgebra C and an SAYD H-module M, the *Hopf-cyclic cohomology* of C with coefficients in M defined in [10] is denoted by $HC^p_H(C, M)$. For $C = H$ and $M = k$ a one dimensional SAYD module we recover the Connes-Moscovici Hopf-cyclic cohomology $HC^p_{(\delta,\sigma)}(H)$ [6, 7, 8]. Given an H-comodule algebra B and an SAYD module H we denote the Hopf-cyclic cohomology of B with coefficients in M by $HC^{p,H}(B, M)$. It is shown in [10] that for $B = H$ and $M = k$ a one-dimensional SAYD H-module we have a natural isomorphism with the dual theory of Khalkhali-Rangipour [11] for Hopf algebras: $HC^{p,H}(B, M) = \widetilde{HC}^p_{(\delta,\sigma)}(H)$. Finally if A is an H-module algebra and M is an SAYD H-module we denote the Hopf-cyclic cohomology of A with coefficients in

M by $HC^p_H(A,M)$. For $M = H$ with conjugation action one obtains the Hopf algebra equivariant cyclic cohomology of Akbarpour-Khalkhali [1, 2], while for $H =$ the Hopf algebra of Laurent polynomials and M a one dimensional module we obtain the *twisted cyclic cohomology*.

2. Cup products of the first kind

Let B be an H-comodule algebra, M be an $SAYD$ H-module, and A an H-module algebra. In [13] we construct a pairing

$$HC^{p,H}(B,M) \otimes HC^q_H(A,M) \longrightarrow HC^{p+q}(B \rtimes_H A),$$

where on the right hand side we have the ordinary cyclic cohomology of the H-*twisted tensor product* algebra $B \rtimes_H A$. For $H = k$ the ground field we obtain Connes' cup product [4]. For $H = B = kG$ the group algebra of a discrete group, and $M = k$ it is shown in [11] that the cohomology groups $HC^{p,H}(B,M)$ are isomorphic to the direct sum of group cohomology of G with trivial coefficients and we recover the map

$$H^p(G) \otimes HC^q_G(A) \longrightarrow HC^{p+q}(A)$$

first defined by Connes in [5]. Here $HC^q_G(A)$ denotes the G-invariant part of the cyclic cohomology of A.

Our method is an extension of the method used by Connes in [4] in the *untwisted* case. First we realize all cyclic cocycles as (M,H)-*twisted closed graded traces* on differential graded (DG) H-algebras. We derive a necessary and sufficient condition for these coycles to be trivial in the cyclic complex. The required pairing is induced from naturally defined traces over the twisted tensor product of two DG H-algebras.

3. Cup products of the second kind

Let C be an H-module coalgebra, A an H-module algebra, and M an $SAYD$ H-module. We say that C acts on A if there is a linear map $C \otimes A \to A$ such that for all $c \in C, a, b \in A$ and $h \in H$ we have $c(ab) = c^{(1)}(a)c^{(2)}(b)$, $c(1) = \varepsilon(c)1$, and $h(ca) = (hc)a$. In [13] we construct a pairing

$$HC^p_H(C,M) \otimes HC^q_H(A,M) \longrightarrow HC^{p+q}(A),$$

for all p and q. For $p = 0$ or $q = 0$ this pairing was already defined in [10] and shown to coincide with the Connes-Moscovici characteristic map when $C = H$.

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Principal fibrations from θ-deformations

Giovanni Landi
(joint work with Walter van Suijlekom)

The ADHM construction [1, 2] of instantons in Yang-Mills theory has at its heart the theory of connections on principal and associated bundles. A central example is the basic $SU(2)$-instanton on S^4 which is described by the well-known Hopf $SU(2)$-principal bundle $S^7 \to S^4$ and connections thereon.

We consider a noncommutative version of this Hopf fibration, in the framework of the isospectral deformations introduced in [5], while trying to understand the structure behind the noncommutative instanton bundle found there.

We first review the construction of θ-deformed spheres where θ is an anti-symmetric real-valued matrix. Apart from the noncommutative spheres S^m_θ, we also introduce differential calculi $\Omega(S^m_\theta)$ as quotients of the universal differential calculi. On the sphere S^m_θ one constructs a noncommutative Riemannian spin geometry $(C^\infty(S^m_\theta), D, \mathcal{H})$ in which the Dirac operator D is the classical one and $\mathcal{H} = L^2(S^m, \mathcal{S})$ is the usual Hilbert space of spinors. Then the deformations are isospectral, as mentioned. Furthermore, one also constructs a Hodge star operator $*_\theta$ acting on the differential calculus $\Omega(S^m_\theta)$ and which is most easily defined using the so-called splitting homomorphism [4].

Then we focus on two noncommutative spheres S^4_θ and $S^7_{\theta'}$ starting from the algebras $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ of polynomial functions on them. The latter algebra carries an action of the (classical) group $SU(2)$ by automorphisms in such a way that its invariant elements are exactly the polynomials on S^4_θ. The anti-symmetric 2×2 matrix θ is given by a single real number also denoted by θ. On the other hand, the requirements that $SU(2)$ act by automorphisms and that S^4_θ makes the algebra of invariant functions, give the matrix θ' in terms of θ. This yields a one-parameter family of noncommutative Hopf fibrations.

For each irreducible representation $V^{(n)} := \mathrm{Sym}^n(\mathbb{C}^2)$ of $SU(2)$ we construct the noncommutative vector bundles $E^{(n)}$ associated to the fibration $S^7_{\theta'} \to S^4_\theta$. By dualizing the classical construction, these bundles are described by the module of coequivariant maps from $\mathbb{C}^2$ to $\mathcal{A}(S^7_{\theta'})$. As expected, these modules are finitely generated projective and we construct explicitly the projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ such that these modules are isomorphic to the image of $p_{(n)}$ in $\mathcal{A}(S^4_\theta)^{4^n}$. Then, one defines connections $\nabla_{(n)} = p_{(n)}\mathrm{d}$ as maps from $\Gamma(S^4_\theta, E^{(n)})$ to $\Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$, where $\Omega^*(S^4_\theta)$ is the quotient of the universal differential calculus mentioned above. The corresponding connection one-form A turns out to be valued in a representation of the Lie algebra $su(2)$.

By using the projection $p_{(n)}$, the Dirac operator with coefficients in the noncommutative vector bundles $E^{(n)}$ is given by $D_{p_{(n)}} := p_{(n)} D p_{(n)}$. In order to compute its index, we first show that the local index theorem of Connes and Moscovici [6] takes a very simple form in the case of isospectral deformations. When applied to the projections $p_{(n)}$ on S^4_θ, we obtain exactly as in the classical case,

$$\mathrm{Ind}\, D_{p_{(n)}} = \frac{1}{6} n(n+1)(n+2). \tag{1.1}$$

Finally, we show that the fibration $S^7_{\theta'} \to S^4_\theta$ is a 'not-trivial principal bundle with structure group $SU(2)$'. This means that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft Hopf-Galois extension; in fact, it is a principal extension [3]. On this extension, we find an explicit form of the (strong) connection which induces connections on the associated bundles $E^{(n)}$ as maps from $\Gamma(S^4_\theta, E^{(n)})$ to $\Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(\mathcal{A}(S^4_\theta))$, where $\Omega^*(\mathcal{A}(S^4_\theta))$ is the universal differential calculus on $\mathcal{A}(S^4_\theta)$. We show that these connections coincide with the Grassmannian connections $\nabla = p_{(n)}\mathrm{d}$ on the quotient $\Omega(S^4_\theta)$ of the universal differential calculus alluded to before.

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Compact quantum metric spaces from ergodic actions of compact quantum groups

Hanfeng Li

We introduce a notion of length function for compact quantum groups. Every separable compact quantum group admits a length function. Given a length function on a co-amenable compact quantum group G, we show how to induce a quantum metric on any ergodic action of G. This generalizes Rieffel's work for compact group case.

Equivariant Chern characters in K and L-theory

Wolfgang Lück

We construct for G-equivariant K-homology an equivariant Chern character as follows. Let C be a finite cyclic group. The *Artin defect* is the cokernel of the map

$$\bigoplus_{D\subset C, D\neq C} \operatorname{ind}_D^C\colon \bigoplus_{D\subset C, D\neq C} R_{\mathbb{C}}(D) \to R_{\mathbb{C}}(C).$$

For an appropriate idempotent

$$\theta_C \in R_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$$

the Artin defect becomes after inverting the order of $|C|$ canonically isomorphic to

$$\theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

Theorem: *Let X be a proper G-CW-complex. For a finite cyclic subgroup $C \subset G$ let (C) be its conjugacy class, $N_G C$ its normalizer, $C_G C$ its centralizer and $W_G C = N_G C / C_G C$. Then there is a natural isomorphism called* equivariant Chern character

$$\operatorname{ch}^G\colon \bigoplus_{(C)} K_p(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_p^G(X) \otimes_{\mathbb{Z}} \Lambda^G.$$

We show that the Baum-Connes Conjecture implies a modification of the Trace Conjecture due to Baum and Connes, which says that the image of the standard trace $K_0(C_r^*(G)) \to \mathbb{R}$ takes values in Λ^G.

The Chern character mentioned above is a special case of an equivariant Chern character which can be applied to any equivariant (co-)homology theory, for instance also to those appearing as sources of the assembly maps in the Farrell-Jones Isomorphism Conjecture whose target are K-and L-groups of group rings. They also enter in the extension of the proof of the K-theoretic Novikov Conjecture due to Bökstedt-Hsiang-Madsen from the family of the trivial subgroup to the family of finite subgroups thus detecting a much larger portion in the algebraic K-theory of an integral group ring. It yields also a good understanding of the passage from the algebraic K-theory of the complex group ring to the topological K-theory of the reduced C^*-algebra of a group.

Theorem: *Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G) \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n^{\mathrm{top}}(C_r^*(G)) \end{array}$$

where $C_G\langle g\rangle$ is the centralizer of the cyclic group generated by g in G and the vertical arrows come from the obvious change of ring and of K-theory maps The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and in the Baum-Connes Conjecture. If these conjectures are true for G, then the horizontal arrows are isomorphisms.

There are also cohomological versions which will appear in the proof of the following result about the topological K-theory of the classifying space BG of a discrete group G.

Theorem: *Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions. Then there is a $\mathbb{Q}$-isomorphism*

$$K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \left(\prod_{i\in\mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \ prime} \ \prod_{(g)\in \mathrm{con}_p(G)} \left(\prod_{i\in\mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\widehat{\mathbb{Q}_p})\right),$$

where $\mathrm{con}_p(G)$ is the set of conjugacy classes (g) of elements $g \in G$ of order p^d for some integer $d \geq 1$ and $C_G\langle g\rangle$ is the centralizer of the cyclic subgroup $\langle g\rangle$ generated by g.

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Quantum Statistical Mechanics of Q-lattices

Matilde Marcolli
(joint work with Alain Connes)

In [4], [5], we constructed noncommutative spaces with a rich arithmetic structure, associated to commensurability classes of $\mathbb{Q}$-lattices. We studied extensively the quantum statistical mechanical system associated to 2-dimensional $\mathbb{Q}$-lattices and its relation to the Galois theory of the field of modular functions.

A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ consists of a pair (Λ, ϕ) of a lattice $\Lambda \subset \mathbb{R}^n$ (a cocompact free abelian subgroup of $\mathbb{R}^n$ of rank n) together with a system of labels of its torsion points given by a homomorphism of abelian groups $\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$.

Two $\mathbb{Q}$-lattices are commensurable, $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$, iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$.

A $\mathbb{Q}$-lattice is *invertible* iff ϕ is an isomorphism. Notice that most $\mathbb{Q}$-lattices are not commensurable to an invertible one. The space resulting from the quotient of the space of $\mathbb{Q}$-lattices by the equivalence relation of commensurability has the typical properties that make it best described through the language of noncommutative geometry.

Any 2-dimensional $\mathbb{Q}$-lattice can be written in the form $(\Lambda, \phi) = (\lambda(\mathbb{Z}+\mathbb{Z}\tau), \lambda\rho)$, for some $\lambda \in \mathbb{C}^*$, some $\tau \in \mathbb{H}$, and some $\rho \in M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$. Thus, the space of 2-dimensional $\mathbb{Q}$-lattices up to the scale factor $\lambda \in \mathbb{C}^*$ and up to isomorphisms, is given by $M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \mod \Gamma = \mathrm{SL}_2(\mathbb{Z})$. The commensurability relation is implemented by the partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$. The corresponding noncommutative algebra of coordinates is given by the convolution algebra of continuous compactly supported functions on the quotient of the space $\mathcal{U} := \{(g, \rho, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} | g\rho \in M_2(\hat{\mathbb{Z}})\}$ by the action of $\Gamma \times \Gamma$.

This has a natural time evolution $\sigma_t(f)(g, \rho, z) = \det(g)^{it} f(g, \rho, z)$ and representations π_L, for $L = (\Lambda, \phi) = (\rho, z)$ a 2-dimensional $\mathbb{Q}$-lattice, on the Hilbert space $\ell^2(\Gamma\backslash G_\rho)$ for $G_\rho := \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}})\}$. The convolution algebra has a C^*-algebra completion $\mathcal{A}_2$, where the norm is the sup over all representations π_L.

Invertible $\mathbb{Q}$-lattices correspond to finite energy representations, with Hamiltonian $H\,\epsilon_m = \log\det(m)\,\epsilon_m$. The Hilbert space is, in this case, identified with $\ell^2(\Gamma\backslash M_2^+(\mathbb{Z}))$. The partition function is of the form $Z(\beta) = \zeta(\beta)\zeta(\beta-1)$, where ζ is the Riemann zeta function. This suggests that the resulting quantum statistical mechanical system will have phase transitions at $\beta = 1$ and $\beta = 2$.

The equilibrium states of a quantum statistical mechanical system at inverse temperature β are described by the KMS condition [2]. At zero temperature we assume as notion of KMS_∞ the weak limit of KMS_β states as $\beta \to \infty$. Symmetries of the system act on extremal KMS states. We need to consider both symmetries given by automorphisms and by endomorphisms, whenever the latter act on KMS states. In particular the action on KMS_∞ states is defined as a limit – via warming up and cooling down of the system.

Our main result on the structure of KMS states is the following.

- In the range $\beta \leq 1$ there are no KMS states.
- In the range $\beta > 2$ the set of extremal KMS states is given by the classical Shimura variety $\mathcal{E}_\beta \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$.
- At zero temperature, a state $\varphi_{\infty,L} \in \mathcal{E}_\infty$ with $L = (\rho, \tau)$ generic, when evaluated on an arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$, takes values in an embedding F_τ of the modular field in $\mathbb{C}$. There is an isomorphism $\theta_\varphi : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_f)$ that intertwines the Galois action on the values of the state with the action of symmetries,

$$\gamma^{-1}\,\varphi(f) = \varphi(\theta_\varphi(\gamma)f), \quad \forall f \in \mathcal{A}_{2,\mathbb{Q}}, \;\; \forall \gamma \in \mathrm{Gal}(F_\tau/\mathbb{Q}).$$

In fact, here the symmetry group of the system is the quotient $\mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_f)$, where in $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}})$, the subgroup $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts by automorphisms while $\mathrm{GL}_2^+(\mathbb{Q})$ acts by endomorphisms.

The arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ is an algebra of unbounded multipliers of $\mathcal{A}_2$, which naturally contains the modular functions. It is acted upon by the symmetries of the system since the quotient $\mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_f)$ is isomorphic to the automorphism group of the modular field $\mathrm{Aut}(F)$, by a result of Shimura [10].

The noncommutative geometry of the space of $\mathbb{Q}$-lattices modulo the equivalence relation of commensurability provides a setting that unifies several phenomena involving the interaction of noncommutative geometry and number theory. These include, in the 1-dimensional case, the Bost–Connes (BC) system [1] with arithmetic spontaneous symmetry breaking and its dual space under the duality given by taking the crossed product with the time evolution. The latter is the noncommutative space underlying the construction of the spectral realization of the zeros of the Riemann zeta function in [3]. The corresponding space in the 2-dimensional case contains in its algebra of coordinates the modular Hecke algebras of [7]. The noncommutative compactifications of modular curves of [9] also appear here as a stratum in the compactification of the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices.

Moreover, an interesting and difficult problem is the generalization of the results of [1] to other number fields. The space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices provides a new approach to the problem, for the case of imaginary quadratic fields, since it is closely related to the Galois theory of the modular field. A quantum statistical mechanical system that recovers the explicit class field theory construction for an imaginary quadratic field $\mathbb{K}$, starting from the noncommutative algebra of coordinates of the space of commensurability classes of 1-dimensional $\mathbb{K}$-lattices, is being investigated in our joint work with N. Ramachandran [6]. The fact that the noncommutative modular curves of [9] appear in the compactification suggests the possible existence of a path towards the case of real quadratic fields, along the lines of Manin's real multiplication program [8].

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Polynomial growth cohomology for combable groups

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Group cohomology of polynomial growth is defined for any finitely generated, discrete group, using cochains that have polynomial growth with respect to the word length function. We give a geometric condition that guarantees that the group cohomology of polynomial growth agrees with the usual group cohomology and verify it for a class of combable groups. Our sufficient condition involves a chain complex that is closely related to exotic cohomology theories studied by Allcock and Gersten and by Mineyev. It can be formulated for arbitrary discrete metric spaces and is a quasi-isometry invariant.

Let G be a finitely generated, discrete group and let ℓ be a word length function on G. We may use the length function to define variants of the usual group cohomology $H^n(G)$ with complex coefficients. Roughly speaking, we take one of the standard chain complexes that compute group cohomology and restrict attention to cochains that have, say, polynomial growth. This yields the *group* *cohomology of polynomial growth* of G, which we denote by $H^n_{\mathrm{pol}}(G)$. Similarly, we can study group cohomology of subexponential or exponential growth. In this introduction, we only consider the case of polynomial growth. The question we study is whether the canonical maps $H^n_{\mathrm{pol}}(G) \to H^n(G)$ are isomorphisms for all $n \in \mathbb{N}$ for a given group G. This question came up in the work of Alain Connes and Henri Moscovici on the Novikov conjecture for hyperbolic groups in [2] and has also been studied by Ronghui Ji in [6].

Our main contribution to this problem is that we introduce a certain chain complex $\mathcal{S}^1\tilde{C}_\bullet(G)$ of bornological vector spaces relevant to it. We prove two main theorems. First, this chain complex has a bounded contracting homotopy if G is a combable group in the notation of [4] or, possibly more generally, if G has a synchronous combing of polynomial growth in the notation of [5]. Secondly, if the chain complex $\mathcal{S}^1\tilde{C}_\bullet(G)$ has a bounded contracting homotopy, then $H^n_{\mathrm{pol}}(G) \cong H^n(G)$. An important feature of our construction is that the homotopy type of the chain complex $\mathcal{S}^1\tilde{C}_\bullet(G)$ is a quasi-isometry invariant of G. In contrast, the original problem of whether $H^n_{\mathrm{pol}}(G) \cong H^n(G)$ does not appear to be invariant under quasi-isometry.

The chain complex $\mathcal{S}^1\tilde{C}_\bullet(G)$ is related to the convolution algebra

$$\mathcal{S}^1(G) := \Big\{ f\colon G \to \mathbb{R} \;\Big|\; \sum_{g\in G} |f(g)|(\ell(g)+1)^k < \infty \quad \forall k \in \mathbb{N} \Big\}.$$

If we use the Banach algebra $\ell^1(G)$ instead, we obtain a complex that is homotopy equivalent to one constructed by Daniel J. Allcock and Stephen M. Gersten in [1] (for groups for which the classifying space BG has finite type). However, already the chain complex $\ell^1\tilde{C}_\bullet(\mathbb{Z})$ has non-trivial homology and hence cannot be contractible. The best one can say is that the range of the differential in this complex is dense in the kernel, that is, the "reduced homology" vanishes. Igor Mineyev shows in [8] that this happens for groups with a sufficiently nice combing. However, this seems too weak to compare the bounded cohomology and the usual group cohomology.

The complex $\mathcal{S}^1\tilde{C}_\bullet(G)$ only depends on the large scale geometry of G. More precisely, we construct a complex $\mathcal{S}^1\tilde{C}_\bullet(X)$ for any metric space X, which is functorial for quasi-Lipschitz maps and has the property that the chain maps induced by close quasi-Lipschitz maps are chain homotopic. Thus $\mathcal{S}^1\tilde{C}_\bullet(X)$ is a quasi-isometry invariant of X up to chain homtopy equivalence. We now explain the construction of $\mathcal{S}^1\tilde{C}_\bullet(X)$.

To any set X, we associate a simplicial set $S(X)$ whose n-simplices are all $n+1$-tuples $(x_0,\dots,x_n) \in X^{n+1}$. The jth face and degeneracy maps leave out or double x_j, respectively. We let $C_\bullet(X)$ be the reduced simplicial chain complex associated to $S(X)$. Thus $C_n(X)$ can be identified with the space of compactly supported functions $X^{n+1} \to \mathbb{R}$ that vanish on $(x_0,\dots,x_n)$ if $x_j = x_{j+1}$ for some $j \in \{0,\dots,n-1\}$. We define the reduced subcomplex $\tilde{C}_\bullet(X)$ by $\tilde{C}_n(X) = C_n(X)$ for $n \geq 1$ and $\tilde{C}_0(X) = \ker(\alpha\colon C_0(X) \to \mathbb{R})$, where α is the augmentation map defined on basis vectors by $x \mapsto 1$. Any map $f\colon X \to Y$ induces a chain map

$\tilde{C}_\bullet(X) \to \tilde{C}_\bullet(Y)$. There is also an evident formula for a chain homotopy between the maps $\tilde{C}_\bullet(X) \to \tilde{C}_\bullet(Y)$ induced by two maps $X \to Y$. Hence $\tilde{C}_\bullet(X)$ is always contractible because it is homotopy equivalent to $\tilde{C}_\bullet(\star) = 0$.

Now we let X be a discrete, proper metric space. Let $\mathcal{S}^1C_n(X)$ be the space of functions $f\colon X^{n+1} \to \mathbb{R}$ with the following properties: there is $R > 0$ such that $f(x_0, \ldots, x_n) = 0$ if $d(x_i, x_j) \geq R$ for some $i, j \in \{0, \ldots, n\}$, or if $x_j = x_{j+1}$ for some $j \in \{0, \ldots, n-1\}$;

$$\sum_{x_0,\ldots,x_n \in X} |f(x_0, \ldots, x_n)| \cdot (d(x_0, \star) + \cdots + d(x_n, \star) + 1)^k < \infty$$

for all $k \in \mathbb{N}$, for some fixed point $\star \in X$. A subset $S \subseteq \mathcal{S}^1C_n(X)$ is considered bounded if we can choose the parameter R above uniformly for all $f \in S$ and if the sums above are uniformly bounded for $f \in S$. Thus $\mathcal{S}^1C_n(X)$ becomes a bornological vector space that contains $C_n(X)$ as a dense subspace. The differential of $C_\bullet(X)$ extends and turns $\mathcal{S}^1C_\bullet(X)$ into a chain complex of bornological vector spaces. Let $\mathcal{S}^1\tilde{C}_\bullet(X)$ be the kernel of the augmentation map $\mathcal{S}^1C_\bullet(X) \to \mathbb{R}$.

A combing on a metric space (X, d) with a chosen base point $\star$ is a sequence of maps $f_n\colon X \to X$ with the following properties; $f_0(x) = \star$ for all $x \in X$, and for any $x \in X$ there exists $n \in \mathbb{N}$ such that $f_N(x) = x$ for all $N \geq n$; the maps f_n are uniformly quasi-Lipschitz; and the pairs of maps (f_n, f_{n+1}) are uniformly close in the sense that the set of $d(f_n(x), f_{n+1}(x))$ for $x \in X$, $n \in \mathbb{N}$ is bounded. We say that the combing has polynomial growth if the number of $n \in \mathbb{N}$ with $f_n(x) \neq f_{n+1}(x)$ is controlled by a polynomial in $d(x, \star)$. Such combings exist for all hyperbolic groups, for automatic groups, and for groups that are combable in the sense of [4]. However, it seems that such combings do not exist for non-Abelian nilpotent groups.

Our first main result is that $\mathcal{S}^1\tilde{C}_\bullet(X)$ has a bounded contracting homotopy if X has a combing of polynomial growth. The proof is actually quite simple. Since the maps f_n, f_{n+1} are close for each $n \in \mathbb{N}$, there is an explicit chain homotopy $H(f_n, f_{n+1})\colon \mathcal{S}^1\tilde{C}_\bullet(X) \to \mathcal{S}^1\tilde{C}_\bullet(X)$ between the maps induced by f_n and f_{n+1}. The sum $H = \sum_{n\in\mathbb{N}} H(f_n, f_{n+1})$ is the desired contracting homotopy. The hypotheses on (f_n) guarantee that H is a bounded linear operator on $\mathcal{S}^1\tilde{C}_\bullet(X)$.

Our second main result is that $H^n_{\mathrm{pol}}(G) \cong H^n(G)$ if $\mathcal{S}^1\tilde{C}_\bullet(G)$ is contractible. This requires some homological algebra with bornological modules over the convolution algebras $\mathbb{R}[G]$ and $\mathcal{S}^1(G)$. These two are bornological algebras, and the embedding $i\colon \mathbb{R}[G] \to \mathcal{S}^1(G)$ is a bounded algebra homomorphism. We work with bornologies instead of topologies because this gives better results for spaces that are built out of $\mathbb{R}[G]$ and $\mathcal{S}^1(G)$. The right category of modules over a bornological algebra A (like $\mathbb{R}[G]$ or $\mathcal{S}^1(G)$) is the category $\mathsf{Mod}(A)$ of bornological left A-modules. Such a module is defined by a bounded homomorphism $A \to \mathrm{End}(M)$ or, equivalently, a bounded bilinear map $A \times M \to M$ satisfying the usual properties. The morphisms are the bounded A-linear maps.

Homological algebra in $\mathsf{Mod}(A)$ works as usual if we only admit extensions with a bounded linear section. That is, we only allow resolutions of a module that have a bounded linear contracting homotopy. Let $\hat{\otimes}$ be the completed bornological tensor product. A bornological left A-module of the form $A \mathbin{\hat{\otimes}} X$ with the obvious module structure is called *free*. One checks easily that free modules are projective with respect to extensions with a bounded linear section. Therefore, the usual argument in homological algebra that shows that two free resolutions of the same module are homotopy equivalent still works in our setting.

The group cohomology of G can be defined as $H^n(G) \cong \operatorname{Ext}^n_{\mathbb{R}[G]}(\mathbb{R},\mathbb{R})$, where $\mathbb{R}$ is equipped with the trivial representation of G and the resulting module structure over $\mathbb{R}[G]$. That is, it is the cohomology of the chain complex $\operatorname{Hom}_{\mathbb{R}[G]}(P_\bullet,\mathbb{R})$, where $P_\bullet \to \mathbb{R}$ is some free $\mathbb{R}[G]$-module resolution of $\mathbb{R}$. If we let G act diagonally on $C_n(G)$, then the chain complex $C_\bullet(G)$ constructed above becomes such a free $\mathbb{R}[G]$-module resolution. Similarly, the group cohomology of polynomial growth is isomorphic to the cohomology $\operatorname{Ext}^n_{\mathcal{S}^1(G)}(\mathbb{R},\mathbb{R})$ of the chain complex $\operatorname{Hom}_{\mathcal{S}^1(G)}(P'_\bullet,\mathbb{R})$, where $P'_\bullet \to \mathbb{R}$ is some free $\mathcal{S}^1(G)$-module resolution of $\mathbb{R}$.

Let $i^*\colon \mathsf{Mod}(\mathcal{S}^1(G)) \to \mathsf{Mod}(\mathbb{R}[G])$ be the functor induced by the embedding $i\colon \mathbb{R}[G] \to \mathcal{S}^1(G)$. There exists a functor $i_!\colon \mathsf{Mod}(\mathcal{S}^1(G)) \to \mathsf{Mod}(\mathbb{R}[G])$ that is left adjoint to i^*, that is,

$$\operatorname{Hom}_{\mathcal{S}^1(G)}(i_!(M),N) \cong \operatorname{Hom}_{\mathbb{R}[G]}(M,i^*(N)) \tag{0.1}$$

for all $M \in \operatorname{Ob}\mathsf{Mod}(\mathbb{R}[G])$, $N \in \operatorname{Ob}\mathsf{Mod}(\mathcal{S}^1(G))$. More explicitly, we have $i_!(M) \cong \mathcal{S}^1(G) \mathbin{\hat{\otimes}}_{\mathbb{R}[G]} M$. It is easy to see that $i_!(C_\bullet(G)) \cong \mathcal{S}^1 C_\bullet(G)$. This is always a chain complex of free $\mathcal{S}^1(G)$-modules. If $\mathcal{S}^1\tilde{C}_\bullet(G)$ has a bounded contracting homotopy, then $i_!(C_\bullet(G))$ is a free $\mathcal{S}^1(G)$-module resolution of $\mathbb{R}$. Hence we may use it to compute $\operatorname{Ext}^n_{\mathcal{S}^1(G)}(\mathbb{R},\mathbb{R})$. The adjointness relation (0.1) yields

$$\begin{aligned}\operatorname{Ext}^n_{\mathcal{S}^1(G)}(\mathbb{R},\mathbb{R}) &\cong H^n(\operatorname{Hom}_{\mathcal{S}^1(G)}(i_!(C_\bullet(G)),\mathbb{R}))\\ &\cong H^n(\operatorname{Hom}_{\mathbb{R}[G]}(C_\bullet(G),\mathbb{R})) \cong \operatorname{Ext}^n_{\mathbb{R}[G]}(\mathbb{R},\mathbb{R})\end{aligned}$$

because $i^*(\mathbb{R}) = \mathbb{R}$. Thus $H^n_{\mathrm{pol}}(G) \cong H^n(G)$ if $\mathcal{S}^1\tilde{C}_\bullet(G)$ is contractible.

Actually, contractibility of $\mathcal{S}^1\tilde{C}_\bullet(G)$ has much stronger consequences: it implies that $\operatorname{Ext}^n_{\mathcal{S}^1(G)}(M,N) \cong \operatorname{Ext}^n_{\mathbb{R}[G]}(M,N)$ for all $\mathcal{S}^1(G)$-modules M, N, and all $n \in \mathbb{N}$. Even more, the functor on the derived categories $i^*\colon \mathsf{Der}(\mathcal{S}^1(G)) \to \mathsf{Der}(\mathbb{R}[G])$ induced by the embedding $i\colon \mathbb{R}[G] \to \mathcal{S}^1(G)$ is fully faithful. Following the notation of [7], this means that $\mathcal{S}^1(G)$ is an isocohomological convolution algebra on G. I introduced this concept in [7] to facilitate the computation of a rather subtle coinvariant space.

The concept of an isocohomological embedding seems to apply frequently in the following situation. Let A be a C^*-algebra defined by generators and relations, or just a Banach algebra like $\ell^1(G)$. These generators and relations also define a polynomial subalgebra $\mathcal{P}(A) \subseteq A$. In nice cases, we get a smooth subalgebra $\mathcal{S}(A) \subseteq A$ by taking polynomials in the generators with sufficiently rapidly decreasing entries. Here smooth means, say, that $\mathcal{S}(A)$ is closed under holomorphic functional calculus in A, so that it has the same K-theory as A. In such situations, one may hope that the embedding $\mathcal{S}(A) \to \mathcal{P}(A)$ induces a fully faithful functor between the derived categories of $\mathcal{P}(A)$ and $\mathcal{S}(A)$. There probably are no general theorems that guarantee this, one has to verify the condition on a case by case basis.

For instance, for non-commutative tori this is equivalent to a known assertion: it simply means that we can compute derived functors for modules over $\mathcal{P}(A)$ and $\mathcal{S}(A)$ by *the same* Koszul type complex. This is the crucial step in the computation of the cyclic homology groups for $\mathcal{S}(A)$ by Alain Connes. However, this example already shows that we do not get particularly strong ties between the Hochschild and cyclic homology of $\mathcal{P}(A)$ and $\mathcal{S}(A)$: whereas the result for $\mathcal{P}(A)$ is always simple, the result for $\mathcal{S}(A)$ depends in a subtle way on diophantine approximation properties of the parameter θ. In the case of groups, knowing that $\mathbb{R}[G] \to \mathcal{S}^1(G)$ is isocohomological implies that the homogeneous parts of the Hochschild homology of $\mathcal{S}^1(G)$ and $\mathbb{R}[G]$ agree. However, it is not clear in general what happens at conjugacy classes with infinitely many elements because there the passage from $\mathcal{S}^1(G)$ to $\mathbb{R}[G]$ changes also the module, not just the algebra that acts.

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On the assembly map via localization of categories

Ryszard Nest

(joint work with Ralf Meyer)

Let G be a second countable locally compact group. Let us introduce some notation. The *G-equivariant Kasparov category* is the additive category whose objects are the G-C^*-algebras and whose group of morphisms $A \to B$ is $\mathrm{KK}_0^G(A, B)$. The composition in KK^G is the Kasparov product.
In order to apply the standard methods of homological algebra we need some extra structure. For our purposes, the following is sufficient.

Let $A \to A[1]$ denote the desuspension functor, i.e. $A[-1] = C_0(\mathbb{R}) \otimes A$ and, given an element $\phi \in \mathrm{KK}^G(A, B)$, let

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \nwarrow & & \swarrow [1] \\ & C_\phi & \end{array}$$

be the exact triangle associated to the cone of ϕ - here we use the fact that every element of KK^G can be represented by an equivariant *-homomorphism of appropriate C*-algebras KK^G-equivalent to A (resp. B).

Theorem 1.1. *The above structure gives* KK^G *a structure of triangulated category.*

Let A be a G-C^*-algebra. We call A *weakly contractible* if A is KK^H-equivalent to 0 for all compact subgroups $H \subseteq G$. We let $\mathcal{CC} \subseteq \mathrm{KK}^G$ be the full subcategory of weakly contractible objects. This is a localizing subcategory of KK^G, that is, it is triangulated and closed under direct sums. We call $f \in \mathrm{KK}^G(A, B)$ a *weak equivalence* if it is invertible in $\mathrm{KK}^H(A, B)$ for all compact subgroups $H \subseteq G$. The weakly contractible objects and the weak equivalences determine each other: a morphism is a weak equivalence if and only if its "mapping cone" is weakly contractible and an object is weakly contractible if and only if the zero map $0 \to A$ is a weak equivalence.

A good model for the "quotient" $\mathrm{KK}^G/\mathcal{CC}$ is given as follows. We call $A \in \mathrm{KK}^G$ *compactly induced* if it is KK^G-equivalent to $\mathrm{Ind}_H^G A'$ for some compact subgroup $H \subseteq G$ and some H-C^*-algebra A'. We let $\mathcal{CI} \subseteq \mathrm{KK}^G$ be the full subcategory of compactly induced objects and $\langle \mathcal{CI} \rangle$ the localizing subcategory generated by it. We show that $B \in \mathcal{CC}$ if and only if $\mathrm{KK}^G(A, B) \cong 0$ for all $A \in \langle \mathcal{CI} \rangle$. Hence $f \in \mathrm{KK}^G(B, B')$ is a weak equivalence if and only if the induced map $\mathrm{KK}^G(A, B) \to \mathrm{KK}^G(A, B')$ is an isomorphism for all $A \in \langle \mathcal{CI} \rangle$.

In particular, we will call a $\mathcal{CI}$-simplicial approximation for $A \in \mathrm{KK}^G$ a weak equivalence $f \in \mathrm{KK}^G(\tilde{A}, A)$ with $\tilde{A} \in \langle \mathcal{CI} \rangle$.

Theorem 1.2. *For every* $A \in \mathrm{KK}^G$ *there exists a weak equivalence* $\tilde{A} \to A$ *with* $\tilde{A} \in \langle \mathcal{CI} \rangle$.

The construction of $\mathcal{CI}$-simplicial approximations is functorial and also has good exactness properties. Therefore, if $F\colon \mathrm{KK}^G \to \mathfrak{C}$ is any homological functor into an Abelian category, then its *localization* $\mathbb{L}F(A) := F(\tilde{A})$ is again a homological functor $\mathrm{KK}^G/\mathcal{CC} \to \mathfrak{C}$. It comes with a natural transformation $\mathbb{L}F(A) \to F(A)$. This map for the functor $F(A) := \mathrm{K}(G \ltimes_{\mathrm{r}} A)$ is naturally isomorphic to the Baum-Connes assembly map. In particular,

Proposition 1.3. $\mathrm{K}^{\mathrm{top}}(G, A) \cong \mathbb{L}F(A)$.

Let $\star \in \mathrm{KK}^G$ be the trivial G-module given by complex numbers. The tensor product operation in KK^G is compatible with the subcategories $\mathcal{CC}$ and $\mathcal{CI}$.

Therefore, if $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ is a $\mathcal{CI}$-simplicial approximation for $\star$, then $\mathsf{D} \otimes \mathrm{id}_A \in \mathrm{KK}^G(\mathsf{P} \otimes A, A)$ is a $\mathcal{CI}$-simplicial approximation for $A \in \mathrm{KK}^G$. Thus we can describe the localization of a functor more explicitly as $\mathbb{L}F(A) := F(\mathsf{P} \otimes A)$.

We call D a *Dirac morphism* for G. The existence of the Dirac morphism is equivalent to the representability of a certain functor, proved by applying a suitable version of Brown's Representability Theorem.

A *dual Dirac morphism* is an element $\eta \in \mathrm{KK}^G(\star, \mathsf{P})$ that is a one-sided inverse to the Dirac morphism. Suppose that it exists. Then $\gamma = \mathsf{D}\eta$ is an idempotent in $\mathrm{KK}^G(\star, \star)$. By exterior product, it acts on any $A \in \mathrm{KK}^G$. We have $A \in \mathcal{CC}$ if and only if $\gamma_A = 0$ and $A \in \langle \mathcal{CI} \rangle$ if and only if $\gamma_A = 1$. The category KK^G splits as a direct product

$$\mathrm{KK}^G \cong \mathcal{CC} \times \langle \mathcal{CI} \rangle.$$

Therefore, the assembly map is split injective for any covariant functor. If also $\gamma = 1$, then $\mathcal{CC} \cong 0$, that is, any weakly contractible object is already isomorphic to 0. Hence $\mathbb{L}F = F$ for any functor F. The latter actually happens for groups with the Haagerup property and, in particular, for amenable groups.

Another useful idea of the approach above allows the following. Instead of deriving the functor $A \mapsto \mathrm{K}(G \ltimes_{\mathrm{r}} A)$, one can derive the crossed product functor $A \mapsto G \ltimes_{\mathrm{r}} A$ itself. Its localization $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A$ is a triangulated functor from KK^G to KK. It can be described explicitly as $G \ltimes_{\mathrm{r}}^{\mathbb{L}} A = G \ltimes_{\mathrm{r}} (A \otimes \mathsf{P})$ if $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \star)$ is a Dirac morphism. The usual Baum-Connes conjecture asks $\mathsf{D}_* \in \mathrm{KK}(G \ltimes_{\mathrm{r}}^{\mathbb{L}} A, G \ltimes_{\mathrm{r}} A)$ to be an isomorphism on K-theory. Instead, we can ask it to be a KK-equivalence. The latter condition insures that the Baum-Connes conjecture holds for $F(G \ltimes_{\mathrm{r}} A)$ for any split exact, stable homotopy functor F on C^*-algebras because such functors descend to the category KK. Examples are given by local cyclic (co)homology and K-homology.

Suppose that G is a locally compact quantum group. Under certain natural conditions KK^G as a triangulated category and both induction and restriction functors make perfect sense. As the result, one can ask for the analogue of localisation in an appropriate triangulated subcategory. The easiest example is the case of coactions of a compact connected group G. This turns out to be fairly tractable. In fact, the obvious "analogue" of $\mathcal{CI}$ is the triangulated subcategory generated by a single element $C^*(G)$ with its canonical coaction of G. The appropriate notion of proper coaction is given by

$$\{\text{A is a proper G-coalgebra }\} \Leftrightarrow \left\{ \begin{array}{c} KK^{\hat{G}}(A, B) = 0 \\ \text{for any KK-contractible G-comodule B} \end{array} \right\}.$$

Similarly to the group case there exists a best "simplicial approximation" to any G-coation, the assembly map makes perfect sense and, in fact, its being an isomorphism is equivalent to the implication

$$\forall_{G-algebra A}\ K_*(A \ltimes_{\mathrm{r}} G) = 0 \Rightarrow K_*(A) = 0$$

As it turns out, the $KK^{\hat{G}}$-category of coactions of a sufficiently regular locally compact quantum group G admits an analogue of tensor product. In fact, by using $KK^{\hat{G}}$-equivalence of A with $A \ltimes_{\mathrm{r}} \hat{G} \ltimes_{\mathrm{r}} G$ one can assume that all objects admit an inner action of $\hat{G}$ and this allowes one to define a twisted tensor product of two G-comodules which is again a G-comodule. As a direct consequence the existence and universal properties of the "Dirac" element in the general theory go through.

Bordism, rho-invariants and the Baum-Connes conjecture

Paolo Piazza
(joint work with Thomas Schick)

Statement of results. Let Γ be a finitely generated discrete group. In this talk I have explained how to prove vanishing results for rho-invariants associated to

(i) the spin-Dirac operator of a spin manifold with positive scalar curvature and fundamental group Γ

(ii) the signature operator of the disjoint union of a pair of homotopy equivalent oriented manifolds with fundamental group Γ.

The invariants we consider are more precisely

- the Atiyah-Patodi-Singer ($\equiv$ APS) rho-invariant associated to a pair of finite dimensional unitary representations $\lambda_1, \lambda_2 : \Gamma \to U(d)$ [1]
- the L^2-rho invariant of Cheeger-Gromov [3]
- the delocalized eta invariant of Lott for a non-trivial conjugacy class of Γ which is finite [10].

We prove that all these rho-invariants vanish if the group Γ is *torsion-free* and the Baum-Connes map for the maximal group C^*-algebra is bijective [4]. This condition is satisfied, for example, by torsion-free amenable groups or by torsion-free discrete subgroups of $SO(n,1)$ and $SU(n,1)$. For the delocalized invariant we only assume the validity of the Baum-Connes conjecture for the *reduced* C^*-algebra. In addition to the examples above, this condition is satisfied e.g. by Gromov hyperbolic groups or by cocompact discrete subgroups of $SL(3,\mathbb{C})$.

Let us now consider the signature operator; then we observe that the *vanishing* of these rho invariants for the disjoint union $X \amalg -X'$, with X and X' homotopy equivalent, is equivalent to their *homotopy invariance* on the single manifolds, since each of these rho invariants is certainly additive under disjoint union. Thus what we have proved is that the three rho-invariants associated to the signature operator are, for such groups, *homotopy invariant.* For the APS and the Cheeger-Gromov rho-invariants the latter result had been established by Navin Keswani in [5] [6]. Our proof re-establishes this result and also extends it to the delocalized eta-invariant of Lott. The proof exploits in a fundamental way results from bordism theory [2] [8] as well as various generalizations of the APS-index theorem [9]; it also embeds these results in general vanishing phenomena for degree zero higher rho invariants (taking values in $A/\overline{[A,A]}$ for suitable C^*-algebras A).

We shall now give more details about the proofs.

General principle of the proofs. In order to simplify the exposition we shall concentrate on the case where Γ is the fundamental group of our manifold and the Γ-covering is the universal covering. We shall be only concerned with vanishing results. To establish these results , we apply the following *general principle*. To avoid undo repetitions, let ρ stand for any of the ρ-invariants we want to investigate.

(1) We first define a *stable* variant ρ^s of ρ. This will be defined as the invariant of a perturbation of our generalized Dirac operator. Such perturbations do not always exist, we need the vanishing of the index class of the generalized Dirac operator. This very strong assumption is satisfied for geometric reasons if one looks at the Dirac operator of a spin manifold with positive scalar curvature, as well as for the signature operator on the disjoint union $X \amalg -X'$ of two homotopy equivalent manifolds.

 We study the main properties of ρ^s. Most important is that it appears as the correction term in an index theorem for manifolds with boundary for suitably perturbed Dirac operators. We use this fact and the assumed *surjectivity* of the Baum-Connes map in order to show that the stable rho-invariant is *well defined*, independent of the chosen perturbation (we are always under the assumption that the index class of our operator is zero).

(2) Then we use our injectivity assumption on the Baum-Connes map and fundamental results in bordism theory in order to show that the stable invariant ρ^s, whenever it is defined, is equal to the stable invariant of a particularly nice manifold. Once again, suitable index theorems on manifolds with boundary play a crucial role for this step. For this nice manifold we compute the stable invariant and show that *it vanishes*.

 To this point, we have therefore shown that in certain special situations one can define an invariant ρ^s which turns out to be zero.

(3) As a last step we show that in the two geometric situations we are studying, the stable invariant ρ^s coincides with the unstable invariant ρ. This will be done by constructing very special perturbations (used in the definition of the stable invariant) which make the direct comparison of the stable and unstable invariant possible. For the signature operator on $X \amalg -X'$ we use perturbations that are inspired by the work of Hilsum-Skandalis [7].

 In fact, in the case of a spin manifold with positive scalar curvature, we don't have to perturb at all, so the last step is trivial.

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Holomorphic bundles on noncommutative toric orbifolds

Alexander Polishchuk

Let A_θ be the algebra of smooth functions on the noncommutative 2-torus T_θ associated with an irrational number θ. Recall that its elements are expressions of the form $\sum_{m,n} a_{m,n}U_1^mU_2^n$, where the coefficients $(a_{m,n})_{(m,n)\in\mathbb{Z}^2}$ rapidly decrease at infinity, and the generators satisfy the commutation relation

$$U_1U_2 = \exp(2\pi i\theta)U_2U_1$$

. Consider the crossed product algebra $B_\theta = A_\theta * \mathbb{Z}/2\mathbb{Z}$, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on A_θ by the *flip automorphism* :

$$\phi : \sum_{m,n} a_{m,n}U_1^mU_2^n \mapsto a_{m,n}U_1^{-m}U_2^{-n}.$$

Abusing the notation we denote the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ (sitting inside B_θ) also by ϕ. The algebra B_θ was studied in the papers [1], [2], [3] and [6]. In particular, it is known that it is simple, has a unique tracial state, and is an AF-algebra. Also its K-theory has been computed: one has $K_0(B_\theta) = \mathbb{Z}^6$ and $K_1(B_\theta) = 0$. Furthermore, in [6] it is shown that the positive cone in $K_0(B_\theta)$ coincides with the preimage of the positive cone in $K_0(A_\theta)$ under the natural homomorphism $K_0(B_\theta) \to K_0(A_\theta)$ (in other words, it consists of all elements $x \in K_0(B_\theta)$ such that $\mathrm{tr}_*(x) > 0$, where $\mathrm{tr}_* : K_0(B_\theta) \to \mathbb{R}$ is the homomorphism induced by the trace).

By a *vector bundle* on the noncommutative toric orbifold $T_\theta/(\mathbb{Z}/2\mathbb{Z})$ we mean a finitely generated projective right B_θ-module. We want to define what is a holomorphic structure on such a vector bundle. As in [5], [4], let us consider a complex structure on T_θ associated with a complex number $\tau \in \mathbb{C} \setminus \mathbb{R}$. It is given by a derivation

$$\delta : A_\theta \to A_\theta : \sum_{m,n} a_{m,n} U_1^m U_2^n \mapsto 2\pi i \sum_{m,n} (m\tau + n) U_1^m U_2^n$$

of A_θ that we view as an analogue of the $\overline{\partial}$-operator. Recall that in [5], [4] we studied the category $\mathrm{Hol}(T_{\theta,\tau})$ of holomorphic bundles on T_θ. By definition, these are pairs $(P, \overline{\nabla})$ consisting of a finitely generated projective right A_θ-module P and an operator $\overline{\nabla} : P \to P$ satisfying the Leibnitz identity

$$\overline{\nabla}(f \cdot a) = f \cdot \delta(a) + \overline{\nabla}(f) \cdot a,$$

where $f \in P$, $a \in A_\theta$. Now we extend δ to a *twisted derivation* $\widetilde{\delta}$ of B_θ by setting

$$\widetilde{\delta}(a_0 + a_1\phi) = \delta(a_0) - \delta(a_1)\phi,$$

where $a_0, a_1 \in A_\theta$. This extended map satisfies the twisted Leibnitz identity

$$\widetilde{\delta}(b_1 b_2) = b_1 \widetilde{\delta}(b_2) + \widetilde{\delta}(b_1)\kappa(b_2),$$

where κ is the automorphism of B_θ given by $\kappa(a_0 + a_1\phi) = a_0 - a_1\phi$, where $a_0, a_1 \in A_\theta$. We define a *holomorphic structure* on a vector bundle P on $T_\theta/(\mathbb{Z}/2\mathbb{Z})$ as an operator $\overline{\nabla} : P \to P$ satisfying the similar twisted Leibnitz identity

$$\overline{\nabla}(f \cdot b) = f \cdot \widetilde{\delta}(b) + \overline{\nabla}(f) \cdot \kappa(b),$$

where $f \in P$, $b \in B_\theta$. By definition, a *holomorphic bundle* is a pair $(P, \overline{\nabla})$ consisting of a vector bundle P equipped with a holomorphic structure $\overline{\nabla}$. One can define morphisms between holomorphic bundles in a natural way, so we obtain the category $\mathrm{Hol}(T_{\theta,\tau}/(\mathbb{Z}/2\mathbb{Z}))$ of holomorphic bundles.

Recall that the combined results of [5] and [4] give the following theorem.

Theorem. *The category* $\mathrm{Hol}(T_{\theta,\tau})$ *is abelian and one has an equivalence of bounded derived categories*

$$D^b(\mathrm{Hol}(T_{\theta,\tau})) \simeq D^b(\mathrm{Coh}(E)),$$

where $\mathrm{Coh}(E)$ *is the category of coherent sheaves on the elliptic curve* $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

Furthermore, the image of the abelian category $\mathrm{Hol}(T_{\theta,\tau})$ in the derived category $D^b(\mathrm{Coh}(E))$ can be described as the heart of the tilted t-structure associated with a certain torsion pair in $\mathrm{Coh}(E)$ (depending on θ).

Our main result is a similar explicit description of the category of holomorphic bundles on $T_\theta/(\mathbb{Z}/2\mathbb{Z})$.

Theorem. *The category* $\mathrm{Hol}(T_{\theta,\tau}/(\mathbb{Z}/2\mathbb{Z}))$ *is abelian and one has an equivalence of bounded derived categories*

$$D^b(\mathrm{Hol}(T_{\theta,\tau}/(\mathbb{Z}/2\mathbb{Z}))) \simeq D^b(\mathrm{Rep}-Q),$$

where $\mathrm{Rep}-Q$ *is the category of finite-dimensional representations of the quiver with relations* Q *defined as follows:*

(i) Q *has* 6 *vertices named* u_1, u_2, u_3, u_4, v *and* w*;*

(ii) Q has 6 arrows: 2 arrows $v \overset{x_0,x_1}{\to} w$ *and 4 arrows* $u_i \overset{e_i}{\to} v$, $i = 1, \ldots, 4$;
(iii) Q has 4 quadratic relations: $x_0 e_0 = 0$ *and* $(x_1 - \lambda_i x_0)e_i = 0$, $i = 1, 2, 3$ *with*

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{\wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{1+\tau}{2})\},$$

where $\wp$ *is the Weierstrass* $\wp$*-function associated with the lattice* $\mathbb{Z} + \mathbb{Z}\theta \subset \mathbb{C}$.

Furthermore, the image of $\mathrm{Hol}(T_{\theta,\tau}/(\mathbb{Z}/2\mathbb{Z}))$ in $D^b(\mathrm{Rep} - Q)$ again corresponds to the tilted t-structure for a certain explicit torsion pair in $\mathrm{Rep} - Q$ depending on θ. An additional feature of the above equivalence is that the natural forgetful map

$$K_0(\mathrm{Hol}(T_{\theta,\tau}/(\mathbb{Z}/2\mathbb{Z}))) \to K_0(B_\theta)$$

is, in fact, an isomorphism and the positive cones are the same.

These results can also be generalized to crossed products of A_θ with finite cyclic subgroups in $\mathrm{SL}_2(\mathbb{Z})$. In particular, we can calculate $K_0(A_\theta * \mathbb{Z}/4)$ for all irrational θ, where the action of $\mathbb{Z}/4$ is induced by the Fourier automorphism of A_θ. This solves the problem raised in [7]

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Characters of K-cycles and a problem of Connes

Michael Puschnigg

In a well known series of papers A. Connes introduced various explicit character formulas for (sufficiently regular) K-cycles. These characters take values in suitable cyclic cohomology theories. The most general of them is the character of Θ-summable (infinite dimensional) K-cycles with values in entire cyclic cohomology. It provides explicit index formulas for index problems in K-homology.

More recently an abstract bivariant Chern-Connes character on Kasparov's bivariant K-theory with values in bivariant local cyclic cohomology has been constructed (P.). In particular it yields an abstract Chern character on K-homology. This character cannot be given by explicit formulas but has pleasant naturality properties.

Theorem 1.1. *The local cyclic cohomology class of Connes' character of a Θ-summable K-cycle coincides with the abstract character of the corresponding K-homology class.*

An important example of a Θ-summable K-cycle is discussed in Connes' book "Géométrie Noncommutative" (pp 81-83). For a lattice Γ in a semisimple Lie group G, this K-cycle represents Kasparov's canonical element $\gamma \in KK^{\Gamma}(\mathbb{C},\mathbb{C})$. The K-cycle is given by an elliptic differential operator on the symmetric space associated to G. In his book A. Connes poses the problem of showing that the character of this K-cycle over $C_r^*(\Gamma)$ is cohomologous to the canonical trace on $C_r^*(\Gamma)$. As a consequence of Theorem 0.1 and some known calculations of the abstract character one obtains

Theorem 1.2. *Connes' problem has an affirmative answer for cocompact lattices in simple Lie groups of real rank one.*

This result yields another proof of the Kadison-Kaplansky conjecture for such groups.

K-theory for boundary actions on euclidean buildings

Guyan Robertson

Let k be a non-archimedean local field with finite residue field $\overline{k}$ of order q. Let G be the group of k-rational points of an absolutely almost simple, simply connected linear algebraic k-group. Then G acts on its Bruhat-Tits building Δ, and on its Furstenberg boundary Ω. Here $\Omega = G/B$, where B is a Borel subgroup.

Theorem 1. $\mathcal{A}(\Gamma)$ depends only on the group Γ and is a pisun C^*-algebra. In fact $\mathcal{A}(\Gamma)$ is a higher rank Cuntz-Krieger algebra in the sense of Robertson-Steger and the class $[\mathbf{1}]$ of the identity in $K_0(\mathcal{A}(\Gamma))$ has finite order, with explicit bounds on the order.

If G is not one of the exceptional types $\widetilde{E}_8, \widetilde{F}_4$ or $\widetilde{G}_2$, then the order of $[\mathbf{1}]$ is less than covol(Γ), where the Haar measure μ on G is normalized so that an Iwahori subgroup of G has measure 1.

A rank 1 example : $G = \mathrm{SL}_2(k)$**.** $\mathcal{A}(\Gamma)$ is a Cuntz-Krieger algebra and

$$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^{n+1} \oplus \mathbb{Z}/n\mathbb{Z}$$

where $n = -\chi(\Gamma) = \frac{(q-1)}{\gcd(2,q-1)} \cdot \#\{\text{vertices of } \Gamma\backslash\Delta\}$ is the order of $[\mathbf{1}]$.

The rank 2 cases. Here the order of $[\mathbf{1}]$ is strictly less than $\chi(\Gamma)$.

Example. $G = \mathrm{SL}_3(k)$: $\mathcal{A}(\Gamma)$ is a higher rank Cuntz-Krieger algebra and the order of $[\mathbf{1}]$ is bounded by

$$\frac{1}{3}\gcd(3, q-1) \cdot (q^2-1) \cdot \#\{\text{vertices of } \Gamma\backslash\Delta\}.$$

Strong numerical evidence suggests that the order of $[\mathbf{1}]$ is actually :

$$\frac{(q-1)}{\gcd(3, q-1)} \cdot \#\{\text{vertices of } \Gamma\backslash\Delta\}.$$

This number is a lower bound for the order of $[\mathbf{1}]$.

Since $C_r^*(\Gamma)$ embeds in $\mathcal{A}_\Gamma$, there is a natural homomorphism

$$K_*(C_r^*(\Gamma)) \to K_*(\mathcal{A}_\Gamma).$$

This homomorphism is not injective, since $[\mathbf{1}]$ does have finite order in $K_0(\mathcal{A}_\Gamma)$. It is therefore worth comparing the K-theories of these two algebras. If the building is type $\widetilde{A}_2$, everything can be calculated explicitly.

Theorem 2. Let Γ be a torsion free cocompact lattice in $G = \mathrm{SL}_3(k)$. Then

$$K_0(C_r^*(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)} \quad \text{and} \quad K_1(C_r^*(\Gamma)) = \Gamma_{ab}. \tag{0.1}$$

This result is an immediate consequence of the Baum-Connes Theorem of V. Lafforgue.

In [CMSZ] a detailed study was undertaken of groups of type rotating automorphisms of $\widetilde{A}_2$ buildings, subject to the condition that the group action is free and transitive on the vertex set of the building. For $\widetilde{A}_2$ buildings of orders $q = 2, 3$, the authors of that article give a complete enumeration of the possible groups with this property. These groups are called $\widetilde{A}_2$ groups. Some, but not all, of the $\widetilde{A}_2$ groups are cocompact lattices in $\mathrm{PGL}_3(k)$ for some local field k with residue field of order q. For those that are, it is an empirical fact that either $k = \mathbb{Q}_p$ or $k = \mathbb{F}_q((X))$ in all the examples constructed so far.

For each $\widetilde{A}_2$ group $\widetilde{\Gamma} < \mathrm{PGL}_3(k)$, consider the unique type preserving subgroup $\Gamma < \widetilde{\Gamma}$ of index 3. Then

$$\chi(\Gamma) = (q-1)(q^2-1)$$

Remark There are eight such groups Γ if $q = 2$, and twenty-four if $q = 3$.

One checks that in all these examples,

$$\operatorname{rank} K_0(\mathcal{A}_\Gamma) = 2 \cdot \operatorname{rank} H_2(\Gamma, \mathbb{Z}) = \begin{cases} 4 & \text{if } q = 2, \\ 30 & \text{if } q = 3. \end{cases}$$

Furthermore, the class of $[\mathbf{1}]$ in the $K_0(\mathcal{A}_\Gamma)$ has order $q-1$. Note that for $q = 2$ this means that $[\mathbf{1}] = 0$.

These values also appear to be true for higher values of q. In particular, they have been verified for a number of groups with $q = 4, 5, 7$. Here is an example with $q = 4$.

Example Consider the Regular $\widetilde{A}_2$ group Γ_r, with $q = 4$. This is a torsion free cocompact subgroup of $\mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is the Laurent series field $F_4((X))$ with coefficients in the field F_4 with four elements. Its embedding in $\mathrm{PGL}_3(F_4((X)))$ is essentially unique, by the Strong Rigidity Theorem of Margulis. The group Γ_r

is torsion free and has 21 generators $x_i, 0 \le i \le 20$, and relations (written modulo 21):

$$\begin{cases} x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & 0 \le j \le 6, \\ x_j x_{j+3} x_{j-6} = 1 & 0 \le j \le 20. \end{cases}$$

Let $\Gamma < \mathrm{PSL}_3(\mathbb{K})$ be the type preserving index three subgroup of Γ_r. The group Γ has generators $x_j x_0^{-1}$, $1 \le j \le 20$. Using the results of [RS] one obtains

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{88} \oplus (\mathbb{Z}/2\mathbb{Z})^{12} \oplus (\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/7\mathbb{Z})^4 \oplus (\mathbb{Z}/9\mathbb{Z}),$$

and the class of $[\mathbf{1}]$ in $K_0(\mathcal{A}_\Gamma)$ is $3 + \mathbb{Z}/9\mathbb{Z}$, which has order $q - 1 = 3$. It also follows from [RS] that $K_0(\mathcal{A}_\Gamma) = K_1(\mathcal{A}_\Gamma)$. Now

$$K_0(C_r^*(\Gamma)) = \mathbb{Z}^{45} = \mathbb{Z}^{44} \oplus \langle [\mathbf{1}] \rangle \quad \text{and} \quad K_1(C_r^*(\Gamma)) = (\mathbb{Z}/2\mathbb{Z})^6 \oplus (\mathbb{Z}/3\mathbb{Z}).$$

This, and similar, examples suggest that the *only* reason for failure of injectivity of the natural homomorphism

$$K_0(C_r^*(\Gamma)) \to K_0(\mathcal{A}_\Gamma)$$

is the fact that $[\mathbf{1}]$ has finite order in $K_0(\mathcal{A}_\Gamma)$.

Example For completeness, here are the results of the computations for one of the groups with $q = 3$. The Regular group 1.1 of [CMSZ], with $q = 3$, has 13 generators x_i, $0 \le i \le 12$, and relations (written modulo 13):

$$\begin{cases} x_j^3 = 1 & 0 \le j \le 13, \\ x_j x_{j+8} x_{j+6} = 1 & 0 \le j \le 13. \end{cases}$$

Let Γ be the type preserving index three subgroup. The group Γ has generators $x_j x_0^{-1}$, $1 \le j \le 12$. Note that the group 1.1 has torsion, but its type preserving subgroup Γ is torsion free. One obtains

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{30} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^6 \oplus (\mathbb{Z}/13\mathbb{Z})^4,$$

and the class of $[\mathbf{1}]$ in $K_0(\mathcal{A}_\Gamma)$ is $1 + \mathbb{Z}/2\mathbb{Z}$, which has order $q - 1 = 2$. It also follows that

$$K_0(C_r^*(\Gamma)) = \mathbb{Z}^{16} \quad \text{and} \quad K_1(C_r^*(\Gamma)) = (\mathbb{Z}/3\mathbb{Z})^3 \oplus (\mathbb{Z}/13\mathbb{Z}).$$

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From surgery to analysis

John Roe
(joint work with Nigel Higson)

This talk describes joint work with Nigel Higson, also of Penn State. The papers "Mapping surgery to analysis I–III" are available on my web site at

`http://www.math.psu.edu/roe/writings/roepapers.html`

1. The analytic signature

Let M be a compact (smooth) oriented manifold. Recall that there is defined an *(analytic) higher signature*

$$\mathrm{Sign}(M) \in K_n(C_r^*(\Gamma)) \tag{1.1}$$

where $\Gamma = \pi_1 M$ and $n = \dim M$.

One can define it to be the 'higher index' of the signature operator on M, i.e. the equivariant index of the signature operator on the universal cover.

Theorem 1.2. *Let $f\colon M' \to M$ be an orientation-preserving homotopy equivalence. Then $\mathrm{Sign}(M') = \mathrm{Sign}(M)$ as elements of $K_n(C_r^*(\Gamma))$, i.e., the analytic higher signature is homotopy invariant.*

Objective: gain the 'best possible' understanding of this result.

2. The assembly map

For a compact M as above there is an *analytic assembly map*

$$K_i(M) \to K_i(C_r^*(\Gamma)) \tag{2.1}$$

which sends the homology class of an elliptic operator to its higher index. Thus we can reformulate Theorem 1.2 as follows: given $f\colon M' \to M$ an orientation-preserving homotopy equivalence, the homology class

$$f_*[D_{M'}] - [D_M] \tag{2.2}$$

lies in the kernel of the assembly map. (D_M is the signature operator on M.)

Need an explicit understanding of the kernel (and cokernel) of assembly.

3. Surgery theory

Definition 3.1. *The* manifold structure set $\mathcal{S}(X)$ *is the collection of eqiuvalence classes (modulo diffeomorphism) of homotopy equivalences* $M \to X$*, where* M *is a smooth manifold.*

For instance, $\mathcal{S}(S^n)$ is the Milnor-Kervaire group Θ^n of exotic spheres. *Surgery theory* computes the structure set by embedding it in a long exact sequence

$$\ldots \to L_{n+1}(\mathbf{Z}\Gamma) \to \mathcal{S}(X) \to [X, G/O] \to L_n(\mathbf{Z}\Gamma) \tag{3.2}$$

The rightmost map here is the *assembly map* of surgery theory. It takes an element of $[X, G/O]$, represented by a (degree one normal) map $M \to X$, into the difference of the L-theory higher signatures of M and X.

Note that the homotopy invariance of these higher signatures is now a formal consequence of exactness.

4. The analytic surgery sequence

Project now is: fit the analytic assembly map into a long exact sequence analogous to the surgery exact sequence, and then produce a natural transformation from one to the other.

It will be the K-theory long exact sequence associated to a certain C^*-algebra extension.

Need to consider spaces which are *non-compact* in various ways:

- noncompact (proper) metric spaces, e.g. complete manifolds;
- Γ-*presented spaces*, i.e. a cover of X is given and we work equivariantly on the cover.

Choose a Hilbert space H which is an (ample) X-module, i.e. a module over $C_0(X)$ (or the cover in the Γ-presented case).

One can use C^**-categories* to handle matters more functorially here.

Define $C^*(X)$ to be the C^*-algebra generated by the locally compact, finite propagation, (equivariant) operators on H.

Define $D^*(X)$ to be the C^*-algebra generated by the pseudolocal, finite propagation, (equivariant) operators on H.

Then there is an exact sequence

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0 \tag{4.1}$$

Definition 4.2. *The* analytic surgery sequence *is the long exact sequence in K-theory associated to this short exact sequence of* C^**-algebras.*

Lemma 4.1. *The K-theory group* $K_{n+1}(D^*(X)/C^*(X)$ *is isomorphic to the K-homology group* $K_n(X)$*.* (Paschke duality)

Lemma 4.2. *If* X *is compact (and* Γ*-presented) then* $C^*(X)$ *is Morita equivalent to* $C^*_r(\Gamma)$*, and in particular they have the same K-theory.*

(Consider the example of a one-point space.)

Thus the boundary map in the long exact sequence becomes a homomorphism $K_n(X) \to K_n(C^*_r(\Gamma))$, and this is one definition of the assembly map. $K_{n+1}(D^*(X))$ is therefore an analytic counterpart to the structure set.

Example: when Γ is trivial, this is *reduced* K-homology.

5. Rigidity and $D^*(X)$

By 'rigidity' we mean an invariance or vanishing theorem for an index (e.g. homotopy invariance for the signature, positive scalar curvature vanishing for Dirac). A 'stable reason' for rigidity should correspond to an element of $K_{n+1}(D^*(X))$.

Example 5.1. *If M is a compact spin manifold of positive scalar curvature, then there is defined a class in $K_{n+1}(D^*(M))$ which maps to the homology class of the Dirac operator (and thus gives a 'reason' for the vanishing of the index). Proof — the Dirac operator is invertible!*

In fact the construction gives a map from the space of concordance classes of positive scalar curvature metrics (Hajduk, Stolz) to $K_{n+1}(D^*(X))$.

Warning: there are also 'unstable reasons' for rigidity. For instance, it is a theorem that the (ordinary) index of the Dirac operator on a spin manifold of nonnegative sectional curvature is zero. (Exercise!) This *does not* translate to our context, as one can see e.g. by noting that the the *higher* index of Dirac on a torus is not zero.

6. Detecting manifold structures

How can we detect the structure set analytically — i.e. detect that a homotopy equivalence is not homotopic to a diffeomorphism?

Use the difference between *signature* and *signature operator*. One can define a *signature* (global notion) for any (suitable) space with Poincaré duality — it does not have to be a manifold. (Such nonmanifold Poincaré spaces arise by patching manifolds together by homotopy equivalences.) However the signature *operator* (local notion) exists only for manifolds.

Local	Global
Diffeomorphism	Homotopy equivalence
Manifold	Poincaré space
Signature operator	Signature

7. Defining analytic signatures

Papers I and II set up technology to associate *signatures* in $K_*(C^*(X))$ to various kinds of analytic and geometric Poincaré complexes controlled over X.

No algebraic surgery is involved.

These signatures are invariant under suitable notions of homotopy and bordism, and in the case of a complete manifold the signature constructed by this process agrees with the higher index of the signature operator.

Paper III uses these constructions to map the surgery exact sequence to the analytic surgery sequence.

8. The structure invariant

Let $f\colon M' \to M$ be a homotopy equivalence. We want to measure 'how far' f is from a diffeomorphism.

Build a non-compact Poincaré space W by joining together the open metric cones on M and M' using f. Take the signature $\mathrm{Sign}(W) \in K_{n+1}(C^*(W))$.

Its image in $K_{n+1}(D^*(W))$ is an analytic obstruction to f being a diffeomorphism.

Problem: We want an invariant in $K_{n+1}(D^*(M))$.

9. Cones and boundaries

Let X be a space with a conelike end based on Y. Let X_Y be obtained by compactifying the end (gluing on a copy of Y at infinity.) Every X-module is canonically also a X_Y-module.

Proposition 9.1. *If we regard H as a X_Y-module in this way, then every operator in $D^*(X)$ is also in $D^*(X_Y)$. Moreover, this 'compression' process induces a natural transformation from the analytic surgery sequence of X to the long exact K-homology sequence of the pair (X_Y, Y).*

(The point is that X_Y-pseudolocality is *a priori* a stronger condition than X-pseudolocality. The proof uses Kasparov's Lemma.)

Apply this as follows: the Poincaré space W of the previous slide comes with a natural map to the open double cone bM on M. The compactification of both ends of bM gives $M \times [0,1]$, which projects to M.

Apply the following series of maps to $\mathrm{Sign}(W)$ (W was formed from the homotopy equivalence f) to get an invariant $\sigma(f)$ in $K_{n+1}(D^*(M))$:

$$\begin{array}{ccc}
K(C^*(W)) & & \\
\downarrow & & \\
K(C^*(bM)) & \longrightarrow & K(D^*(bM)) \\
 & & \downarrow \\
 & & K(D^*(M \times [0,1])) \\
 & & \downarrow \\
 & & K(D^*(M))
\end{array}$$

This defines a map $\sigma\colon \mathcal{S}(M) \to K_{n+1}(D^*(M))$.

10. The main theorem

Theorem 10.1. *The map σ defined above fits into a diagram relating the classical surgery exact sequence to the analytic surgery exact sequence. This diagram commutes, except for a multiplicative factor of 2 in every sixth square.*

The factors of 2 comes from the fact that the *boundary* of the K-homology class of the signature operator on a manifold M is the K-homology class of the signature operator on the boundary ∂M times a constant which is either 1 or 2 depending on the parity of the dimension.

11. An example: multisignatures

See Wall, *Surgery on compact manifolds*, Chapter 13B

Finite-dimensional representations of Γ give homomorphisms $C_r^*(\Gamma) \to M_m(\mathbf{C})$ (at least if Γ is amenable — actually, consider *finite* groups). Let $\{\rho_1, \ldots, \rho_k\}$ be a finite set of such representations, of dimensions $m_1, \ldots, m_k$. Each induces a K-theory homomorphism

$$\tau_j\colon K_0(C_r^*(\Gamma)) \to K_0(M_{m_j}(\mathbf{C})) = \mathbf{Z} \tag{11.1}$$

Define $\tau\colon K_0(C_r^*(\Gamma)) \to \mathbf{Q}^k$ by

$$a \mapsto \left(\tfrac{1}{m_1}\tau_1(a), \ldots, \tfrac{1}{m_k}\tau_k(a)\right). \tag{11.2}$$

Proposition 11.3. *If a belongs to the image of the assembly map, then $\tau(a)$ is a multiple of $(1, \ldots, 1)$.*

Proof — see Atiyah's L^2 index theorem.

Thus the multisignature τ gives a well defined map from the cokernel of assembly (in $K_0(D^*(X))$ to $\mathbf{Q}^k/\mathbf{Q}$. This then gives a map on part of the structure set of (odd-dimensional) manifolds, corresponding to Wall's multisignature.

Pseudo-multiplicative unitaries on C^*-modules associated to locally compact groupoids

Thomas Timmermann

In this talk, we prove a duality theorem for actions of groupoids on C^*-algebras. As a tool, we develop techniques for pseudo-multiplicative unitaries on C^*-modules, borrowing on the theory of (pseudo-)multiplicative unitaries on Hilbert spaces developed by S. Baaj, G. Skandalis, M. Enock and J.-M. Vallin [1, 10]. These results form part of my on-going PhD work.

Let G be a locally compact groupoid with a left Haar system λ. Denote by G^0 its space of units. Our starting point is the C^*-module $L^2(G, \lambda)$ over $C_0(G^0)$, obtained by completing $C_c(G)$ with respect to the inner product $\langle f|g\rangle := \int \overline{f} g \, d\lambda$. The range and source map of G induce representations π_r and π_s of $C_0(G^0)$ on $L^2(G, \lambda)$, respectively. Denote by $L^2(G, \lambda)_{\pi_s} \otimes L^2(G, \lambda)$ and $L^2(G, \lambda) \otimes_{\pi_r} L^2(G, \lambda)$ the internal tensor products taken with respect to the action on the first and second factor, respectively. Our main actor is the pseudo-multiplicative unitary $W\colon L^2(G,\lambda)_{\pi_s} \otimes L^2(G,\lambda) \to L^2(G,\lambda) \otimes_{\pi_r} L^2(G,\lambda)$ defined by the formula $(Wf)(x,y) := f(x, x^{-1}y)$.

The definition of the left leg $\hat{S} \subset L_{C_0(G^0)}(L^2(G,\lambda))$ of W, of its coproduct and of its coactions carries over from [1] without problems. One recovers the C^*-algebra $C_0(G)$ with the coproduct transpose to the multiplication map of G and the notion of actions of G on $C_0(G^0)$-algebras.

In contrast, generally, the right leg of W can not be defined as a C^*-subalgebra of $L_{C_0(G^0)}(L^2(G,\lambda))$ because it corresponds to the left regular representation which does not even commute with the module structure on $L^2(G,\lambda)$. However, if G is r-*discrete*, it is spanned by α-*twisting operators* – maps T on $L^2(G,\lambda)$ possessing an α-adjoint T^* satisfying $\langle \eta | T\xi\rangle = \alpha(\langle T^*\eta|\xi\rangle)$ for all $\eta, \xi \in L^2(G,\lambda)$, where α is a partial automorphism of $C_0(G^0)$. The right leg S of W can then be defined as a C^*-*family* $(S^\alpha)_\alpha$ of twisting operators on $L^2(G,\lambda)$. Introducing the notion of a twisted tensor product of such C^*-families, we can follow the development in [1] and define a coproduct on S as well as the notion of coactions of S on C^*-algebras.

Theorem 1.1. *There exists a bijection between injective coactions of S and Fell bundles on G.*

Once the nature of the right leg S has been clarified, we can define reduced crossed products and carry over the Baaj-Skandalis duality theorem.

Theorem 1.2. *Let C be a $C_0(G^0)$-algebra with a coaction of $\hat{S}$. Then the reduced crossed product $C \rtimes_r S$ carries a coaction of S, and the iterated crossed product $C \rtimes_r S \rtimes_r \hat{S}$ is $\hat{S}$-equivariantly isomorphic to $C \otimes_B K_{C_0(G^0)}(L^2(G,\lambda))$. A similar result holds for coactions of S.*

There are interesting examples of r-discrete groupoids which are not globally but only locally Hausdorff, see [3]. For such groupoids, M. Khoshkam and G. Skandalis constructed the left regular representation on a canonical Hilbert module $L^2(G,\lambda)_{KS}$, see [4]. Following their approach, we define a related Hilbert module $L^2(G,\lambda)$ and exhibit a pseudo-multiplicative unitary $W\colon L^2(G,\lambda)_{\pi_s} \otimes L^2(G,\lambda) \to L^2(G,\lambda) \otimes_{\pi_r} L^2(G,\lambda)$, using the same formula $(Wf)(x,y) := f(x, x^{-1}y)$ as before.

Theorem 1.3. *Let G be a locally Hausdorff r-discrete groupoid. Then the set $\mathfrak{H}G$ of limit sets of primitive nets in G carries a natural structure of a Hausdorff r-discrete groupoid. The Hilbert module $L^2(G,\lambda)$ is isomorphic to the Hilbert module associated to $\mathfrak{H}G$. Under this isomorphism, W corresponds to the pseudo-multiplicative unitary associated to $\mathfrak{H}G$.*

Let us end with comments on on-going work and open questions. A generalisation of the theory of multiplicative unitaries to pseudo-multiplicative unitaries on C^*-modules and a proof of the duality theorem for a C^*-module analog of weak Kac systems is going to appear in my PhD thesis. This generalisation entails a duality theorem for coactions of general locally compact Hausdorff groupoids which are not necessarily r-discrete. Second, in the von Neumann-algebraic setting, pseudo-multiplicative unitaries have received much interest in connection with inclusions of factors. It would be interesting to test the techniques developed so far in the C^*-algebraic setting on the unitaries associated by O'uchi Moto to certain inclusions of C^*-algebras.

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On C^*-Algebras and K-theory for Fredholm Manifolds

Jody Trout

(joint work with Dorin Dumitrasçu)

This project [3] is concerned with constructing C^*-algebras and computing the K-theory for a particular class of infinite-dimensional Hilbert manifolds, namely *Fredholm manifolds* [4, 5]. This is part of a research program to introduce concepts and techniques from Alain Connes' noncommutative geometry into the study of infinite-dimensional Fredholm manifolds.

Recall that if M is a smooth *finite-dimensional* Riemannian manifold, then there are two C^*-algebras naturally associated to M. One is the commutative C^*-algebra $C_0(M)$ of all continuous complex-valued functions which vanish at infinity on M. By the Serre-Swan theorem, we have the well-known fact that $K^j(M) \cong K_j(C_0(M))$, where $K^j(M)$ is the (reduced) topological K-theory of M. Furthermore, if M has a spin (or spin^c) structure, there is a Poincaré duality isomorphism $K^{n-j}(M) \cong K_j^c(M)$, where $K_j^c(M)$ denotes the dual (compactly supported) K-homology of M and n is the dimension of M. The other C^*-algebra is $\mathcal{C}(M) = C_0(M, \mathrm{Cliff}(TM))$ of continuous sections of the Clifford algebra bundle $\mathrm{Cliff}(TM) \to M$ vanishing at infinity. This C^*-algebra was used by Kasparov in studying the Novikov Conjecture [8] . If M is even-dimensional and has a spin structure then this C^*-algebra is Morita equivalent to $C_0(M)$. (In general, $\mathcal{C}(M)$ is Morita equivalent to $C_0(TM)$.)

If M is an *infinite-dimensional* Hilbert manifold, modeled on a separable infinite-dimensional Euclidean (i.e., real Hilbert) space $\mathcal{E}$, then these two constructions do not work. For example, $C_0(\mathcal{E}) = \{0\}$ since there are no compactly supported continuous functions on $\mathcal{E}$ which are non-zero. However, the Clifford C^*-algebra has been generalized by Higson-Kasparov-Trout [7] to the case $M = \mathcal{E}$, by a direct limit construction. The component C^*-algebras in the direct limit are given by $\mathcal{A}(E^a) = C_0(\mathbb{R})\hat{\otimes}\mathcal{C}(E^a) \cong C_0(\mathbb{R})\hat{\otimes}C_0(E^a, \mathrm{Cliff}(E^a))$ where $\hat{\otimes}$ denote the $\mathbb{Z}_2$-graded tensor product and $C_0(\mathbb{R})$ is graded by even and odd functions. Since the map $E^a \mapsto \mathcal{A}(E^a)$ is functorial with respect to inclusions of finite-dimensional subspaces, they can construct a non-commutative direct limit C^*-algebra (in the better notation of [6]):

$$\mathcal{A}(\mathcal{E}) = \varinjlim \mathcal{A}(E^a)$$

where the direct limit is taken over *all finite-dimensional subspaces* $E^a \subset \mathcal{E}$. This C^*-algebra was used to prove an equivariant Bott periodicity theorem for infinite-dimensional Euclidean spaces [7] and has had applications to proving cases of the Novikov Conjecture and, more generally, the Baum-Connes Conjecture [6, 11].

Now, suppose the Hilbert manifold M is fibered as the total space of a smooth infinite rank Euclidean vector bundle $p : F \to X$, with fiber $\mathcal{E}$ and compatible affine connection ∇, over a finite-dimensional Riemannian manifold X. Let $p_a : F^a \to X$ be a *finite rank* subbundle of F. Using the connection ∇ and the metrics on F and X, one can give the total space F_a a canonical structure of a Riemannian manifold and define the component C^*-algebra

$$\mathcal{A}(F^a) = C_0(\mathbb{R})\hat{\otimes}\mathcal{C}(F^a) \cong C_0(\mathbb{R})\hat{\otimes}C_0(F^a, \mathrm{Cliff}(TF^a)).$$

Since the map $F^a \mapsto \mathcal{A}(F^a)$ is functorial with respect to inclusions of finite-dimensional subbundles [10], one can then construct a direct limit C^*-algebra:

$$\mathcal{A}(F, \nabla) = \varinjlim \mathcal{A}(F^a)$$

where the direct limit is taken over *all finite rank subbundles* $p_a : F^a \to X$ of F. Trout [10] used this C^*-algebra to prove an equivariant Thom isomorphism theorem for infinite rank Euclidean bundles, which reduces to the Higson-Kasparov-Trout Bott periodicity theorem when the base manifold X is a point.

For a more general *curved* Hilbert manifold M, with Riemannian metric g, there does not seem to be a natural generalization of the previous constructions. Based on the above, one would be tempted to construct a direct limit C^*-algebra

$$\text{“}\mathcal{A}(M) = \varinjlim_{M_a \subset M} \mathcal{A}(M_a)\text{”}$$

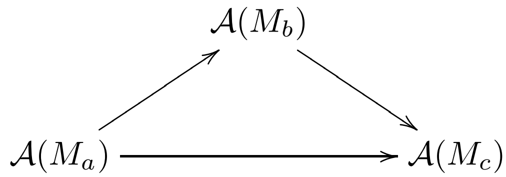
where the component C^*-algebras should be given by

$$\mathcal{A}(M_a) = C_0(\mathbb{R})\hat{\otimes}\mathcal{C}(M_a) = C_0(\mathbb{R})\hat{\otimes}C_0(M_a, \mathrm{Cliff}(TM_a))$$

and the direct limit is taken over *all finite-dimensional submanifolds* $M_a \subset M$. The problem is that, even though the component C^*-algebras have many functoriality properties, if we have smooth (isometric) inclusions

$$M_a \subset M_b \subset M_c$$

of finite-dimensional submanifolds of M, there is no obvious way to define a commuting diagram needed to construct the corresponding direct limit:



However, if the Hilbert manifold M has a *Fredholm structure*, then we can construct a direct limit C^*-algebra by choosing an appropriate *countable sequence* $\{M_n\}_{n=k}^{\infty}$ of expanding, topologically closed, finite-dimensional submanifolds of $\dim(M_n) = n$. The sequence $\{M_n\}_{n=k}^{\infty}$ is called a *Fredholm filtration* of M. The countability of this sequence of submanifolds clearly simplifies the direct limit construction since we only needs to define each "Gysin" map $\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$, which require connections and normal bundles to construct.

Equip the Riemannian Fredholm manifold (M, g) with an *augmented* Fredholm filtration $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$ where U_n is a total open tubular neighborhood of $M_n \hookrightarrow M_{n+1}$. We define the C^*-algebra for the triple $(M, g, \mathcal{F})$ as the direct limit:

$$\mathcal{A}(M, g, \mathcal{F}) = \varinjlim \mathcal{A}(M_n)$$

which is a separable, $\mathbb{Z}_2$-graded, nuclear C^*-algebra.

Using Mukherjea [9], we define the *topological K-theory groups* of $(M, \mathcal{F})$ as:

$$K^{\infty - j}(M, \mathcal{F}) = \varinjlim K^{n-j}(M_n), \quad j = 0, 1,$$

where the connecting map $K^{n-j}(M_n) \to K^{(n+1)-j}(M_{n+1})$ is the Gysin (or shriek) map of the embedding $M_n \hookrightarrow M_{n+1}$, and the inspiration for our connecting map $\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$. Note that this definition does, in general, depend on the choice of Fredholm filtration.

However, using appropriate notions of Spin_q-structures for Riemannian Fredholm manifolds, originally investigated by Anastasiei [1] and de la Harpe [2], we obtain the following Serre-Swan and Poincaré duality isomorphism theorem:

Theorem 0.1. *Let (M, g) be a smooth Fredholm manifold with oriented Riemannian q-structure $(1 \leq q \leq \infty)$. If M has a* Spin_q*-structure then there are isomorphisms*

$$K^{\infty - j}(M, \mathcal{F}) \cong K_{j+1}(\mathcal{A}(M, g, \mathcal{F})) \cong K_j^c(M), \quad j = 0, 1,$$

where $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$ is any augmented Fredholm filtration of M.

Thus, these K-theory groups do not depend on the choice of the Riemannian metric g or the (augmented) Fredholm filtration $\mathcal{F}$. The dimension shift and the relation with Poincaré duality for finite-dimensional spin manifolds then justifies our interpretation of $\mathcal{A}(M, g, \mathcal{F})$ as an appropriate non-commutative (suspension of the) "algebra of functions vanishing at infinity" on M. As an example, we show that certain based loop groups ΩG, where G is a compact Lie group, have Fredholm Spin_q-structures.

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Induction of C^*-algebra coactions

Stefaan Vaes

We provide an overview of induction and imprimitivity results in the setting of locally compact quantum groups. In particular, we discuss induction of coactions on C*-algebras. To obtain these C*-algebraic results, we essentially use von Neumann algebraic techniques. Our results unify and extend the existing results for actions and coactions of locally compact groups [4, 5, 7, 8, 14, 15, 17]. Details on this subject will appear in [18].

1. Preliminaries and notations

Based on several theories (see e.g. [1, 6]), the theory of locally compact (l.c.) quantum groups was developed in [11, 12].

Definition 1.1. *A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group when*

- *M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation : $(\Delta\otimes\iota)\Delta = (\iota\otimes\Delta)\Delta$;*

- *there exist normal semi-finite faithful weights φ and ψ on M such that*
  - *φ is left invariant: $\varphi\big((\omega \otimes \iota)\Delta(x)\big) = \varphi(x)\omega(1)$ when $\varphi(x) < +\infty$,*
  - *ψ is right invariant: $\psi\big((\iota \otimes \omega)\Delta(x)\big) = \psi(x)\omega(1)$ when $\psi(x) < +\infty$.*

Ordinary l.c. groups appear in this theory as $M = L^\infty(G)$ with $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ defined by $\Delta(F)(p,q) = F(pq)$.

Fix a l.c. quantum group in the sense of definition 1.1. Using the Haar weight, one can construct the left regular representation. It is a multiplicative unitary in the sense of [1]. This allows to define a C*-subalgebra $A \subset M$ of 'continuous functions vanishing at infinity'. It also allows to define a dual object, on three different levels: a von Neumann algebraic dual $\hat{M}$, a reduced C*-algebraic dual $\hat{A}$ and a full C*-algebraic dual $\hat{A}^{\mathrm{u}}$ (see the table below). Finally, it is possible to define a co-inverse and co-unit and to prove the uniqueness of the Haar weights. Summarizing, we get the following operator algebras associated with a l.c. quantum group. For the reader's convenience, we also indicated there classical counterparts.

Quantum	Classical	Quantum	Classical
M	$L^\infty(G)$	$\hat{M}$	$\mathcal{L}(G)$
A	$C_0(G)$	$\hat{A}$	$C^*_r(G)$
A^{u}	$C_0(G)$	$\hat{A}^{\mathrm{u}}$	$C^*_f(G)$

Here we used the following notations: $\mathcal{L}(G)$ denotes the group von Neumann algebra of G generated by the left regular representation, $C^*_r(G)$ is the reduced group C*-algebra, while $C^*_f(G)$ is the full group C*-algebra. We finally remark that, in the classical picture, there is no difference between A and A^{u}. This is due to the commutativity of the algebra A (which implies an amenability property).

2. Closed quantum subgroups, coactions, crossed products

Before discussing induction and imprimitivity, we introduce the following basic ingredients: closed quantum subgroups, unitary corepresentations, coactions and crossed products. A *morphism* between the l.c. quantum groups (M, Δ) and (M_1, Δ_1) is a non-degenerate *-homomorphism $\pi : A^{\mathrm{u}} \to \mathcal{M}(A_1^{\mathrm{u}})$ satisfying $(\pi \otimes \pi)\Delta = \Delta_1\pi$. We say that (M_1, Δ_1) is a *closed quantum subgroup* of (M, Δ) if there is given such a morphism π such that there exists a *faithful, normal* *-homomorphism $\hat{M}_1 \to \hat{M}$ making the following diagram commutative. Here $\hat{\pi}$ denotes the dual morphism from $(\hat{M}_1, \hat{\Delta}_1)$ to $(\hat{M}, \hat{\Delta})$.

$$\begin{array}{ccc} \hat{A}_1^{\mathrm{u}} & \xrightarrow{\hat{\pi}} & \mathcal{M}(\hat{A}^{\mathrm{u}}) \\ \big\downarrow & & \big\downarrow \\ \hat{M}_1 & \longrightarrow & \hat{M} \end{array}$$

This definition comes down to an operator algebraic characterization of closed subgroups. Indeed, given a continuous morphism $\pi : G_1 \to G$, we know that G_1 is a closed subgroup of G if and only if there exists a faithful, normal *-homomorphism $\mathcal{L}(G_1) \to \mathcal{L}(G)$ mapping λ_p to $\lambda_{\pi(p)}$.

A *unitary corepresentation* of a l.c. quantum group (M, Δ) on a Hilbert space K is a unitary operator $U \in M \otimes \mathrm{B}(K)$ satisfying $(\Delta \otimes \iota)(U) = U_{13}U_{23}$. Observe that, automatically, $U \in \mathcal{M}(A \otimes \mathcal{K}(K))$. A *continuous coaction* of (M, Δ) on a C*-algebra B is a non-degenerate *-homomorphism $\alpha : B \to \mathcal{M}(A \otimes B)$ such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$ and such that the closed linear span of $\alpha(B)(A \otimes 1)$ is $A \otimes B$. It is possible to define analogously coactions on von Neumann algebras. Given a continuous coaction, we can define the full and reduced crossed products denoted by $\hat{A} \;{}_r\!\ltimes B$ and $\hat{A}^{\mathrm{u}} \;{}_f\!\ltimes B$ respectively.

In particular, one considers the coaction of (A, Δ) on itself by left translation. If $\hat{A} \;{}_r\!\ltimes A \cong \mathcal{K}(H)$, the l.c. quantum group is said to be *regular*. If $\hat{A}^{\mathrm{u}} \;{}_f\!\ltimes A \cong \mathcal{K}(H)$, the l.c. quantum group is said to be *strongly regular*. Examples of strongly regular l.c. quantum groups include all l.c. groups and their duals, Kac algebras, compact and discrete quantum groups, certain bicrossed products as well as algebraic quantum groups in the sense of Van Daele. It is not known whether there exist regular but non strongly regular l.c. quantum groups. On the other hand, there are non-regular quantum groups: for instance the quantum $E(2)$ group, the quantum $ax+b$ group or certain bicrossed products. The latter can even be non-semi-regular, see [2].

3. Quantum homogeneous spaces and Mackey imprimitivity

We fix a l.c. quantum group (M, Δ) with closed quantum subgroup (M_1, Δ_1). Associated with this data, we get a right coaction of (M_1, Δ_1) on M by right translation denoted by $\alpha : M \to M \otimes M_1$. It is then obvious to define the quantum counterpart of $L^\infty(G/G_1)$. We write

$$Q := \{x \in M \mid \alpha(x) = x \otimes 1\}$$

and we consider Q as the *measured homogeneous space*. We then want to find inside Q the l.c. homogeneous space as a dense C*-subalgebra. Observe that $\Delta : Q \to M \otimes Q$ defines a coaction of (M, Δ) on Q by left translation.

Theorem 3.1. *Suppose that (A, Δ) is a regular l.c. quantum group. There exists a unique C*-subalgebra $D \subset Q$ satisfying*

- *D is strongly dense in Q,*
- *$\Delta : D \to \mathcal{M}(A \otimes D)$ is a continuous coaction of (A, Δ) on D,*
- *$\Delta(Q) \subset \mathcal{M}(\mathcal{K}(H) \otimes D)$ and the map $\Delta : Q \to \mathcal{M}(\mathcal{K}(H) \otimes D)$ is strict.*

We call D the quantum homogeneous space*.*

We remark that the *strictness* of $\Delta : Q \to \mathcal{M}(\mathcal{K}(H) \otimes D)$ means that Δ is continuous on bounded subsets of Q when we equip Q with the strong* topology and $\mathcal{M}(\mathcal{K}(H) \otimes D)$ with the strict topology.

The relevance of the quantum homogeneous space comes from the following Mackey type imprimitivity theorem.

Theorem 3.2. *Suppose that (A, Δ) is a strongly regular l.c. quantum group. There exist canonical covariant Morita equivalences*

$$\hat{A} \;{}_r\!\ltimes D \underset{\text{Morita}}{\sim} \hat{A}_1 \quad \textit{and} \quad \hat{A}^{\mathrm{u}} \;{}_f\!\ltimes D \underset{\text{Morita}}{\sim} \hat{A}^{\mathrm{u}}_1 .$$

The covariance of these Morita equivalences is with respect to the dual coactions on the crossed products. Of course, the Morita equivalence $\hat{A}^{\mathrm{u}} \;{}_f\!\ltimes D \sim \hat{A}^{\mathrm{u}}_1$ implements the *induction of corepresentations* from (M_1, Δ_1) to (M, Δ). Observe that the strong regularity assumption is necessary: if (M_1, Δ_1) is the one-point subgroup, a covariant Morita equivalence $\hat{A}^{\mathrm{u}} \;{}_f\!\ltimes A \sim \mathbb{C}$ comes down to strong regularity.

Idea of the proof. Observe that a 'Morita equivalence' of von Neumann algebras is given by an *imprimitivity correspondence*, i.e. a bimodule on which both algebras are each other's commutant. It is an easy exercise to check that the von Neumann algebra $\hat{M} \ltimes Q$ is the commutant of $\hat{M}_1$, when both are represented in a natural way on the L^2-space of (M, Δ). The key point in the proof of theorems 3.1 and 3.2 is to use this easy observation. In order to do so, we have to treat unitary corepresentation theory in a von Neumann algebraic language. This is done through the use of correspondences. We just illustrate this approach for l.c. groups, due to Connes [3]. Indeed, a unitary representation $u : G \to \mathrm{B}(H)$ gives rise to an $\mathcal{L}(G)$-$\mathcal{L}(G)$ correspondence on the Hilbert space $L^2(G) \otimes H$, where g acts on the left by $\lambda_g \otimes u_g$ and on the right by $\rho_g \otimes 1$. One can characterize these $\mathcal{L}(G)$-$\mathcal{L}(G)$ correspondences as those for which there exists a *bicovariant* representation of $L^\infty(G)$, i.e. a representation which is covariant with respect to both the left and the right action of G.

In this setting, induction of unitary corepresentations comes down to taking the *internal tensor product* with the imprimitivity correspondence between $\hat{M}_1$ and $\hat{M} \ltimes Q$. Finally, in order to go up to the stated C*-algebraic results, one has to replace Hilbert spaces by C*-modules and treat correspondences on C*-modules. One performs as such an induction procedure for unitary corepresentations on Hilbert C*-modules. In particular, one can induce the regular corepresentation on the Hilbert $\hat{A}_1$-module $\hat{A}_1$. This yields an induced Hilbert $\hat{A}_1$-module $\mathcal{E}$ together with a unitary corepresentation of (M, Δ). Using a quantum version of Landstad's theorem [13], we deduce that $\mathcal{K}(\mathcal{E}) \cong \hat{A} \;{}_r\!\ltimes D$ to prove the existence of D.

4. Induction of C*-algebra coactions

Suppose that (M, Δ) is a strongly regular l.c. quantum group with closed quantum subgroup (M_1, Δ_1) given by the morphism π. Suppose that $\eta : C \to \mathcal{M}(A_1 \otimes C)$ is a continuous coaction of (A_1, Δ_1) on the C*-algebra C. We define an *induced C*-algebra* $\operatorname{Ind} C$ together with an *induced coaction* $\operatorname{Ind} \eta$ of (A, Δ).

Exactly as it was easy to define the measured homogeneous space as a fixed point algebra, it is easy to define an algebra containing $\operatorname{Ind} C$, but which is too big. Recall that we denoted by α the right coaction of (M_1, Δ_1) on M by restricting the comultiplication.

$$\widetilde{C} = \{X \in \mathcal{M}(\mathcal{K}(H) \otimes C) \mid X \in (M' \otimes 1)' \quad \text{and} \quad (\alpha \otimes \iota)(X) = (\iota \otimes \eta)(X)\} \, .$$

Observe that in the case where $C = \mathbb{C}$, we get $\widetilde{C} = Q$.

Theorem 4.1. *There exists a unique* C^**-subalgebra* $\operatorname{Ind} C$ *of* $\widetilde{C}$ *satisfying*

- $\Delta \otimes \iota : \operatorname{Ind} C \to \mathcal{M}(A \otimes \operatorname{Ind} C)$ *is a continuous coaction of* (A, Δ),
- $\Delta \otimes \iota : \widetilde{C} \to \mathcal{M}(\mathcal{K}(H) \otimes \operatorname{Ind} C)$ *is strictly continuous on unit ball of* $\widetilde{C}$,
- $\operatorname{Ind} C \subset \widetilde{C}$ *is non-degenerate:* $\operatorname{span}(\operatorname{Ind} C)(H \otimes C)$ *is dense in* $H \otimes C$.

There exist canonical covariant Morita equivalences

$$\hat{A}^{\mathrm{u}} \;{}_f\!\ltimes \operatorname{Ind} C \underset{\text{Morita}}{\sim} \hat{A}_1^{\mathrm{u}} \;{}_f\!\ltimes C \quad \textit{and} \quad \hat{A} \;{}_r\!\ltimes \operatorname{Ind} C \underset{\text{Morita}}{\sim} \hat{A}_1 \;{}_r\!\ltimes C \, .$$

5. Final remarks

Induction of unitary corepresentations was first considered by Kustermans in [9], but his concrete approach – defining the underlying Hilbert space for the induced corepresentation in the spirit of Mackey – does not allow to prove imprimitivity theorems. We want to remark that, of course, our induction is unitarily equivalent to his. A proof of this fact uses unavoidably modular theory, since this theory is an essential part of Kustermans' approach.

If G_1 is a closed subgroup of a l.c. group G, one can characterize the G-C*-algebras that are induced from a G_1-action as those that admit a G-equivariant inclusion of $C_0(G/G_1)$ into the center. Of course, we cannot hope for the same result in the quantum setting, since it requires the commutativity of the quantum homogeneous space D. It is nevertheless possible to characterize induced C*-algebras in a similar way, see [18] for a precise formulation.

It is possible as well to describe what happens if we first restrict a coaction to a closed quantum subgroup and then induce it up again. In the classical case, this comes down to tensoring with $C_0(G/G_1)$ and taking the diagonal action. Again we cannot hope for the same result, since the notion of a diagonal coaction does not make sense in the non-commutative world. After restriction and induction, we find instead of a tensor product a kind of *twisted product* of the quantum homogeneous space and the original C*-algebra, together with a 'diagonal' coaction. These twisted products resemble the purely algebraic Yetter-Drinfeld modules. Again, see [18] for details.

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T-duality in string theory via noncommutative geometry

Mathai Varghese

(joint work with Jonathan Rosenberg)

The T-dual of a type II string theory compactified on a circle, in the presence of a topologically nontrivial NS 3-form H-flux, was analyzed in special cases in the literature, where it was observed that T-duality changes not only the H-flux, but also the spacetime topology. A general formalism for dealing with T-duality for compactifications arising from a free circle action was developed in [BEM]. This formalism was shown to be compatible with two physical constraints: (1) it respects the local Buscher rules, and (2) it yields an isomorphism on twisted K-theory, in which the Ramond-Ramond charges and fields take their values. It was shown in [BEM] that T-duality *exchanges the first Chern class with the fiberwise integral of the H-flux*, thus giving a formula for the T-dual spacetime topology.

In this talk, we will present an account of the results in [MR], consisting of a formula for the T-dual of a toroidal compactification, that is a theory compactified via a free torus action, with H-flux. One striking new feature that occurs in the higher rank case is that not every toroidal compactification with H-flux has a T-dual; moreover, even if it has a T-dual, then the T-dual need not be another toroidal compactification with H-flux. A big puzzle has been to explain these mysterious "missing T-duals", and our work presents a solution to this problem using noncommutative geometry. We also show that the generalized T-duality group $GO(n, n; \mathbb{Z})$, n being the rank of the torus, acts to generate the complete list of T-dual pairs related to a given toroidal compactification with H-flux. We will explain these results by providing examples and applications.

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Chern character for totally disconnected groups

Christian Voigt

Let G be a (second countable, locally compact) totally disconnected group. Using equivariant sheaf theory, P. Baum and P. Schneider introduced a bivariant equivariant cohomology theory $H^*_G(X, Y)$ for pairs of locally compact G-spaces X and Y [2]. One virtue of this bivariant theory is that it generalizes and unifies several constructions which appeared earlier in the literature. In particular, it contains as a special case the cosheaf homology groups considered by P. Baum, A. Connes and N. Higson in connection with the Baum-Connes conjecture for p-adic groups [1]. Alternatively, for proper actions on simplicial complexes, one can view cosheaf homology as equivariant Bredon homology [3], [4] for a certain coefficient system.
In their paper Baum and Schneider conjectured that there exists a Chern character for equivariant KK-theory with values in bivariant equivariant cohomology which becomes an isomorphism after tensoring the equivariant KK-groups with $\mathbb{C}$. However, they could prove this conjecture only in the case of profinite groups. In our talk we outline the proof of the following theorem which in some sense completes the work of Baum and Schneider.

Theorem 1.1. *Let G be a totally disconnected group and let X and Y be finite dimensional locally finite G-simplicial complexes. If the action of G on X is proper and X is G-finite there exists an equivariant Chern character*

$$ch^G_* : KK^G_*(C_0(X), C_0(Y)) \to \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_G(X, Y)$$

which becomes an isomorphism after tensoring the left hand side with $\mathbb{C}$.

To clarify our terminology, we point out that a G-simplicial complex is a simplicial complex with a smooth and type-preserving simplicial action of the group

G. This is a natural simplical analogue of the notion of a G-CW-complex.
We remark that W. Lück has obtained the corresponding result, actually in a much more general setting, in the case of discrete groups [4]. However, the method of proof in [4] is completely different from ours. Moreover, it seems to be unclear if the approach of Lück can be extended to arbitrary totally disconnected groups.
Our proof of theorem 1.1 is based on equivariant cyclic homology. Equivariant cyclic homology is a noncommutative generalization of equivariant de Rham cohomology [10]. A crucial feature is that the basic ingredient in the theory is not a complex in the usual sense of homological algebra. For the proof of theorem 1.1 we use equivariant periodic cyclic theory $HP^G_*(A,B)$ as well as equivariant local cyclic homology $HL^G_*(A,B)$. The definition of the latter theory is due to M. Puschnigg [7] in the non-equivariant case.
The Chern character is obtained by a sequence of maps and isomorphisms

$$KK^G_*(C_0(X), C_0(Y) \to HL^G_*(\mathfrak{Smooth}(C_0(X)), \mathfrak{Smooth}(C_0(Y))) \\ \cong HL^G_*(C^\infty_c(X), C^\infty_c(Y)) \cong HP^G_*(C^\infty_c(X), C^\infty_c(Y)) \cong \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_G(X,Y).$$

Here $\mathfrak{Smooth}$ denotes the smoothing functor for representations of the totally disconnected group G [5]. In particular, this functor does not show up if G is discrete.
The first arrow is a consequence of the universal property of equivariant KK-theory [9]. The functor KK^G is the universal functor on the category of separable G-C^*-algebras which is homotopy invariant, stable and split exact. Equivariant local cyclic homology shares these properties. The first isomorphism is due to the fact that the canonical homomorphism $C^\infty_c(X) \to \mathfrak{Smooth}(C_0(X))$ is an isoradial subalgebra [6]. Such homomorphisms induce invertible elements in the local theory. To construct the second isomorphism we need the assumption on X being proper, in fact there is no bivariant transformation between equivariant local and periodic cyclic homology in general. The proof of the last isomorphism is based on an equivariant version of the classical Hochschild-Kostant-Rosenberg theorem.

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Automorphic representations of real and p-adic Lie groups

Antony Wassermann

If M is a factor, usually the hyperfinite type II_1 factor, and G is a polish group, an automorphic representation is a homomorphism $G \to \mathrm{Out}(M) = \mathrm{Aut}(M)/\,\mathrm{Inn}(M)$ which lifts to a Borel map $g \mapsto \alpha_g$ into $\mathrm{Aut}(M)$. Thus $\alpha_g \alpha_h = \mathrm{Ad}\, u(g,h)\alpha_{gh}$ for a Borel map $u : G \times G \to U(M)$. So $\alpha_x u(y,z)u(x,yz) = w(x,y,z)u(x,y)u(xy,z)$ for $w : G \times G \times G \to \mathbb{T}$ Borel. The class $[w] \in H^3_{Borel}(G, \mathbb{T})$ is called the Connes-Jones invariant.
We conjectured that is G is locally compact then any 3-cocycle can be realized in the hyperfinite type II_1 factor. We described few constructions of automorphic representations. The first for discrete groups, due to Connes and Jones. The second using loop groups for compact Lie groups. The third, implicit in Connes' work on periodic automorphisms, giving a canonical construction for compact metric groups leaving a Cartan subalgebra invariant. The last extended this method to real and p-adic Lie groups with arithmetic discrete subgroups of finite covolume.
Finally we discussed the computation of $H^3_{Borel}(G, \mathbb{T})$ for $G = SL_n(k)$ where $\mathbb{Q}_p \subset k$ is a finite extension. Generalising Moores' computation of 2-cohomology, we explained how the spectral sequence of Prasad and Raganuthan could be used to reduce the computation to that of compact subgroups, in particular the unipotent radical U of the Iwahori subgroup. For $n \geq 6$, work of Suslin, Markujew, Levine and Quillen shows that $H^3(SL_n(k), \mathbb{T}) = K_3(k)_{torsion}$ = roots of unity in K^{cycl} fixed by all σ^2 for $\sigma \in Gal(k^{cycl}/k)$. When $k = \mathbb{Q}_p$, $H^3(U, \mathbb{Z}/p)$ can also be computed as the Lie algebra cohomology of a graded finite-dimensional nilpotent Lie algebra over $\mathbb{F}_p$.

Quantum $ax + b$ group

Stanislaw L. Woronowicz

Let a, b be selfadjoint operators, $a > 0$ and $\hbar \in]0, \frac{\pi}{2}[$. We say that $a \overset{\hbar}{\longrightarrow\!\!\circ}\, b$ if $a^{it} b a^{-it} = e^{\hbar t} b$ for all $t \in \mathbb{R}$. $\overset{\hbar}{\longrightarrow\!\!\circ}$ is called Zakrzewski relation, It is an analytical counterpart of the algebraic relation $ab = q^2 ba$, where $q^2 = e^{-\hbar}$.

Definition: A read number $\hbar$ is called admissible if there exists a locally compact quantum group (A, Δ) (in the sense of Kustermans and Vaes) with selfadjoint elements a, b affiliated with A such that

1. $a > 0, a \overset{\hbar}{\longrightarrow\!\!\circ}\, b$
2. $\Delta(b) \supset a \otimes b + b \otimes I$

3. $\Delta(b)$ strongly commutes with $I \otimes |b|$
4. A is of minimal size.

Remark:

1. For classical $ax + b$ group we have the equality $\Delta(b) = a \otimes b + b \otimes I$ and consequently $\Delta(b)$ commutes with $I \otimes b$. However, in the quantum case $a \otimes b + b \otimes I$ is symmetric, but not selfadjoint and the most one can have is the inclusion $\Delta(b) \supset a \otimes b + b \otimes I$. For the same reason $\Delta(b)$ no longer commutes with $I \otimes b$. It does not commute with $I \otimes \operatorname{sign} b$.
2. If π is a representation of A such that $\ker \pi(b) = \{0\}$ then the commutant of $(\pi(a), \pi(b))$ is a factor of type $I_N (N = 2, 3, 4, \ldots, \infty)$. N is called the multiplicity of π. We say that A is of minimal size if there exists a faithful representation of A of multiplicity 2.

Theorem: Let $\hbar \in]0, \frac{\pi}{2}]$. Then

$$\Big(\hbar \text{ is admissible }\Big) \Longleftrightarrow \Big(\hbar = \frac{\pi}{2k+3},\ k = 0, 1, 2, \ldots\Big)$$

Renormalisation of noncommutative ϕ^4-theory to all orders

Raimar Wulkenhaar

(joint work with Harald Grosse)

In recent years there has been considerable interest in quantum field theories on the Moyal plane characterised by the $\star$-product (in D dimensions)

$$(1.1) \quad (a \star b)(x) := \int d^D y \frac{d^D k}{(2\pi)^D} a(x + \tfrac{1}{2}\theta \cdot k) b(x+y)\, \mathrm{e}^{\mathrm{i}ky}\,, \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}\,.$$

The interest was to a large extent motivated by the observation that this kind of field theories arise in the zero-slope limit of open string theory in presence of a magnetic background field [1]. A few months later it was discovered [2] (first for scalar models) that these noncommutative field theories are not renormalisable beyond a certain loop order. The argument is that non-planar graphs are finite but their amplitude grows beyond any bound when the external momenta become exceptional. When inserted as subgraphs into bigger graphs, these exceptional momenta are attained in the loop integration and result in divergences for any number of external legs. This problem is called UV/IR-mixing.

The UV/IR-mixing contains a clear message: If we make the world noncommutative at very short distances, we must at the same time modify the physics at large distances. The required modification is, to the best of our knowledge, unique: It is given by an harmonic oscillator potential for the free field action. In fact, we can prove the following

Theorem 1.1. *The quantum field theory associated with the action*

(1.2)
$$S=\int d^4x\Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi+\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)+\frac{\mu_0^2}{2}\phi\star\phi+\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\;,$$

for $\tilde{x}_\mu:=2(\theta^{-1})_{\mu\nu}\,x^\nu$, ϕ-real, Euclidean metric, is perturbatively renormalisable to all orders in λ.

The proof is given in [3] and [4]. A summary of the main ideas and techniques can be found in [5].

Compared with the commutative ϕ^4-model, the bare action of relevant and marginal couplings contains necessarily an additional term: an harmonic oscillator potential for the free scalar field action. This is a result of the renormalisation proof. It entails a discrete spectrum of the corresponding differential operator: Renormalisation induces a compactification of the underlying noncommutative geometry.

Our proof rests on two concepts:

(1) *The representation of the ϕ^4-action in the harmonic oscillator base of the Moyal plane.* Then, the action describes a matrix model the kinetic term of which is neither constant nor diagonal. We have derived a closed formula for the resulting propagator, using Meixner polynomials in an essential way.
(2) *The renormalisation group approach for dynamical matrix models, the core of which is a flow equation for the effective action.* The renormalisation proof is now reduced to the verification that the flow equation—a non-linear first-order differential equation—admits a regular solution which depends on finitely many initial data. In the perturbative regime, the flow equation is solved by ribbon graphs drawn on Riemann surfaces.

We have proven a power-counting theorem which relates the power-counting behaviour of ribbon graphs to their topology and to the asymptotic scaling dimensions of the cut-off propagator. As a result, only planar graphs with two or four external legs can be relevant or marginal. These graphs are labelled by an infinite number of matrix indices. There exists a discrete Taylor expansion which decomposes the (infinite number of) planar two- and four-leg graphs into a linear combination of four relevant or marginal base functions and an irrelevant remainder. These four universal base functions have the same index dependence as the original action in matrix formulation, which implies the renormalisability of the model.

We have also computed in [6] the one-loop β-functions of the model which describe the dependence of the bare coupling constant and the bare oscillator frequency on the cut-off matrix size. It turned out that $\frac{\lambda}{\Omega^2}$ remains constant under the renormalisation flow. Starting from given small values for Ω_R,λ_R at an initial matrix size $\mathcal{N}_R$, the frequency Ω grows in a small region around $\ln\frac{\mathcal{N}}{\mathcal{N}_R}=\frac{48\pi^2}{\lambda_R}$ to

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Group actions on Banach spaces and K-theory of operator algebras associated to groups

Guoliang Yu
(joint work with Gennadi Kasparov)

In this first part of this talk, I discussed when a group admits a proper affine isometric action on a uniformly convex Banach space. In particular, I showed that every hyperbolic group admits a proper affine isometric action on l^p-space for some $p \geq 2$. It remains an open question whether $SL(n, Z)$ admits a proper affine isometric action on some uniformly convex Banach space if $n \geq 3$.

In the second part of my talk, I explained how proper affine isometric actions on uniformly convex Banach spaces can be used to compute K-theory of operator algebras associated to groups. In particular, I explained the construction of an infinite dimesional Bott element associated to a uniformly convex Banach space. This is joint work with Gennadi Kasparov.

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