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Graph Theory

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Abstract. This is the report on an Oberwolfach conference on graph theory, held 16-22 January 2005. There were three main components to the event: 5-minute presentations, lectures, and workshops. All participants were asked to give a 5-minute presentation of their interests on the first day, and subsequent days were divided into lectures and workshops. The latter ranged over many different topics, but the main three topics were: infinite graphs, topological methods and their use to prove theorems in graph theory, and Rota's conjecture for matroids.

Mathematics Subject Classification (2000): 05Cxx.

Introduction by the Organisers

This conference was one of a series of Oberwolfach conferences, held every two years or so, with focus on graph structure, decomposition, and representation. There were 49 participants, including over a dozen graduate students and postdocs.

At the request of the Oberwolfach Director, the conference schedule was designed to promote informal collaboration. In particular, there were fewer formal talks than usual, and instead there were a number of discussion groups or "workshops". Also, the first day (except for one plenary talk) was devoted to having the participants introduce themselves – we asked all participants to give a five-minute presentation of their current interests.

We were fortunate in that several of the plenary talks described major new results. For instance, Ron Aharoni and Eli Berger have just solved the Erdős-Menger conjecture; Bertrand Guenin has proved a major extension of the four-colour theorem; and Stephan Brandt and Stéphan Thomassé have settled a long-standing question about the chromatic number of dense graphs.

But probably the most distinctive feature of the meeting were the workshops. Some of these were planned before the conference, and others were held spontaneously. They were each on a topic with a chairman, but made as informal as possible. Some were more or less a sequence of talks on the topic, some were monologues, and some were genuine discussions. There were several different topics: infinite graphs and Ramsey theory, matroid theory, connectivity, graph minors and width, and topological methods. Three topics in particular gave rise to particularly active and long-running workshops: the proof of the Erdős-Menger conjecture, the prospects of extending the graph minors project to matroids, and the use of topological methods for combinatorial problems.

Our thanks to the organizers of the workshops for making them run successfully, to the Director for encouraging us to try out new ways of informal collaboration, and to all the participants for making this a highly stimulating meeting.

Workshop: Graph Theory

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Abstracts

Dense triangle-free graphs are four colorable.

Stéphan Thomassé
(joint work with Stephan Brandt)

For every $\varepsilon > 0$, Hajnal provided examples of triangle-free graphs with arbitrarily large chromatic number and minimum degree greater than $(1/3 - \varepsilon)n$, where n is the number of vertices. The construction involves a large bipartite graph (to increase the minimum degree) to which is attached a small Kneser graph (to increase the chromatic number). Any further attempt to find triangle-free graphs with unbounded chromatic number and minimum degree greater than $n/3$ failed.

It was known that triangle-free graphs with minimum degree greater than $2n/5$ are bipartite, following a result of Andrásfai, Erdős and Sós [1].

This motivated P. Erdős and M. Simonovits [4] to ask whether a triangle-free graph with minimum degree greater than $n/3$ is always three colorable.

Using a suitable weight function on the set of vertices of the Grőtzsch graph- the smallest triangle-free four-chromatic graph - R. Häggkvist gave a counterexample to this question. The minimum degree of this counterexample being $10n/29$. Later on, G.P. Jin [5] proved that every triangle-free graph with minimum degree greater than $10n/29$ is indeed 3-colorable.

The only gap to fill in was then to describe what could happen between minimum degree $n/3$ and $10n/29$.

The finiteness of the bound came from a result of C. Thomassen [6] who proved that for every $\varepsilon > 0$, if the minimum degree is at least $(1/3 + \varepsilon)n$, then the chromatic number is bounded by some constant (depending on ε).

By the same time, S. Brandt proved that every triangle-free graph which is regular of degree $> n/3$ has chromatic number at most four.

Our main result with S. Brandt is that the regularity hypothesis can be dropped in the previous statement. Our proof is in three steps.

The first one consists of a characterization of the regular triangle-free graphs of degree $> n/3$ with chromatic number four. An automatic search performed by Brandt and Pisanski gave rise to an infinite family of such graphs, called *Vega graphs*, named after the computer program used to generate the first examples. In fact, we proved that the regular triangle-free graphs are exactly the Vega graphs. The methods we use in this part of the proof are basically these of [2].

The second step of the proof is a direct application of the complementary slackness lemma of linear programming. Basically, if a graph is endowed with a weight function w which insures minimum degree, and no vertex is weighted zero, then the dual weight function which insure maximum degree must be regular. Since the regular case was settled in step one, we can assume that there exists a vertex x of the graph with $w(x) = 0$. In other words, removing the vertex x leaves a graph which still has minimum degree greater than $n/3$. So one can apply induction.

The last step, very technical, shows that adding a vertex to a Vega graph gives a Vega graph.

These three steps together give the result. And just to sum-up, and highlight the threshold let us observe that:

A triangle-free graph with min degree $0.3333333n$ can have arbitrarily large χ.
A triangle-free graph with min degree $0.3333334n$ has $\chi \leq 4$.

We do not know what happens when the minimum degree is exactly $n/3$.

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Proof of the Erdős-Menger conjecture

Eli Berger
(joint work with Ron Aharoni)

We prove that Menger's theorem is valid for infinite graphs, in the following strong form: given two sets of vertices, A and B, in a possibly infinite digraph, there exist a set $\mathcal{P}$ of disjoint A-B paths, and a set S of vertices separating A from B, such that S consists of a choice of precisely one vertex from each path in $\mathcal{P}$. This settles an old conjecture of Erdős.

History of the problem. In 1927 Karl Menger proved the following:

Theorem 1. *For any two sets A and B in a finite digraph, the minimal size of an A-B-separating set is equal to the maximal size of a family of vertex-disjoint paths from A to B.*

Soon thereafter, Erdős proved that, with the very same formulation, the theorem is also valid for infinite graphs. The idea of the proof is this: take a maximal family $\mathcal{P}$ of A-B-disjoint paths. The set $S = \bigcup\{V(P) : \; P \in \mathcal{P}\}$ is then A-B-separating, since an A-B-path avoiding it could be added to $\mathcal{P}$, contradicting the maximality of $\mathcal{P}$. Since every path in $\mathcal{P}$ is finite, if $\mathcal{P}$ is infinite then $|\mathcal{P}| = |S|$, proving the theorem. If $\mathcal{P}$ is finite then one of the proofs known for the finite case of the theorem applies.

Of course, there is some "cheating" here. Equality of cardinalities does not provide much information in the infinite case, and the separating set produced in the case that $\mathcal{P}$ is infinite is obviously too "large". Erdős, who realized this, proposed the following conjecture, better grasping the spirit of the finite theorem, known as the Erdős-Menger Conjecture. Since it is now proved, we state it as a theorem:

Theorem 2. *Given two sets of vertices, A and B, in a (possibly infinite) digraph, there exists a family $\mathcal{P}$ of disjoint A-B-paths , and a separating set consisting of the choice of precisely one vertex from each path in $\mathcal{P}$.*

The earliest reference we have for this conjecture is from 1964 (Problem 8, p. 159 in [10]. See also [7]).

The first to be tackled was of course the bipartite case, and the first breakthrough was made by Podewski and Steffens [8], who proved the countable bipartite case of the conjecture.

Podewski and Steffens [9] made yet another important progress: they proved the conjecture for countable digraphs containing no infinite paths. Later, in [1], it was realized that this case can be easily reduced to the bipartite case, by the familiar device of doubling vertices in the digraph and turning it into a bipartite graph. The bipartite reduction is very useful, and some of the insights leading to the solution of the conjecture are derived from it.

At that point in time there were two obstacles on the way to the proof of the conjecture - uncountability and the existence of infinite paths. The first of the two to be overcome was that of uncountability. In 1982 the marriage problem was solved for general cardinalities, in [6]. Soon thereafter, this was used to prove the bipartite case of the Erdős-Menger Conjecture. Let us state it explicitly:

Theorem 3. *In any bipartite graph there exists a matching F and a cover C, such that C consists of the choice of precisely one vertex from each edge in F.*

By the result of [1], we can deduce Theorem 2 from Theorem 3 for all graphs containing no infinite (unending or non-starting) paths. Thus there remained the problem of infinite paths. The difficulty they pose is that when one tries to "grow" the disjoint paths desired in the conjecture, they may end up being infinite, instead of being A-B-paths . In fact, in [1] it is proved that the Eredős-Menger Conjecture is true, if one allows in $\mathcal{P}$ not only A-B-paths, but any paths that if they start at all, they do so at A, and if they end they do so at B.

The first breakthrough in the fight against infinite paths was made in [2], where the countable case of the conjecture was proved.

The main tools in [2] are hindrances and waves. A *wave* is a set of disjoint paths starting at A whose set of ending vertices is A-B-separating. A wave is called a *hindrance* if it avoids some vertex from A. The following conjecture, which is equivalent to the Erdős-Menger Conjecture, was formulated and proved for countable graphs.

Conjecture 4. *If there is no hindrance then there is a set of disjoint paths linking* all *of A to B.*

The tool used in [2] to overcome infinite paths was the following lemma, which was proved there only for countable graphs, but is stated here generally, since it is now known for all graphs:

Lemma 5. *If there is no hindrance, then any point in A can be linked to B by a path, whose removal does not yield a hindrance.*

This lemma is quite easy to prove in the bipartite case and also in graphs containing no unending paths, but in the countable case it requires new tools and methods. Later, Conjecture 4 was proved for graphs in which all but countably many points of A are linked to B [3], and the Erdős-Menger Conjecture was proved for such graphs in [5].

In [4] there was given a reduction of the $\aleph_1$ case of the Erdős-Menger Conjecture to Lemma 5.

Main ideas of the proof. The breakthrough leading to the solution of the general case was indeed the proof of Lemma 5 for general graphs. As claimed in [4], the way from the lemma to the general proof indeed follows the same outline as in the $\aleph_1$ case. But the general case demands quite a bit more effort.

The notion lying at the core of the proof of the uncountable case is that of *κ-hindrances*, for regular uncountable cardinals κ. A *κ-hindrance* is a ladder-like structure having κ rungs, each "carving off" another part of the graph, and in which "many" rungs are of the form of a hindrance in the graph remaining at the present stage. The notion of "many" is captured by a well known set-theoretical notion - that of a "stationary set" of ordinals.

The proof of the theorem is divided into two stages.

(1) Showing that if there is no hindrance and no κ-hindrance for any uncountable regular cardinality κ, then there is a set of disjoint paths linking all of A to B.
(2) Proving that the existence of a κ-hindrance implies the existence of a hindrance.

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The topological cycle space of a locally finite graph

Henning Bruhn

(joint work with Reinhard Diestel, Maya Stein)

Almost all of the more advanced theorems about the cycle space, such as Tutte's generating theorem and MacLane's planarity criterion, are considerably weakened or fail completely in locally finite graphs. The reason for this, Diestel and Kühn [5, 6] realised, is that the traditional cycle space has too "few" cycles in an infinite graph, and in particular that what is missing are suitable infinite cycles. To remedy this, they defined cycles in a topological way, namely as the homeomorphic images of the unit circle in the Freudenthal compactification of the graph by its ends. This definition allows not only the usual finite cycles but also infinite cycles. The resulting cycle space, the topological cycle space $\mathcal{C}(G)$, has been almost surprisingly successful: suddenly all of the finite theorems carry over to locally finite graphs. See Diestel [3, 4] for a survey.

So far there was neither a satisfactory concept of duality nor a characterisation of the elements of $\mathcal{C}(G)$ in terms of degrees. We present results in this direction.

Using only finite cycles, Thomassen [7, 8] defined dual graphs for infinite graphs and found that a necessary condition for a graph to have a dual is that

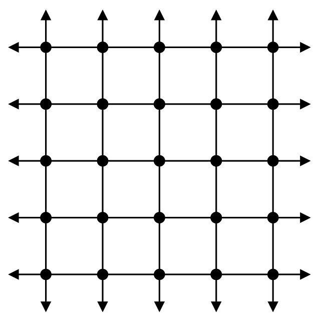
($*$) *no two vertices are joined by infinitely many edge-disjoint paths.*

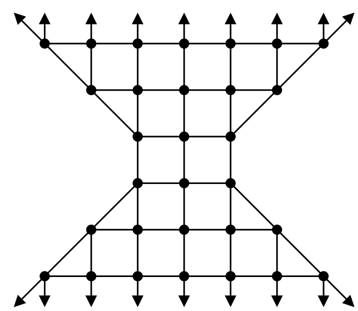
While, within the class of graphs satisfying ($*$), Thomassen proved an infinite version of Whitney's planarity criterion, that a graph is planar if and only if it has a dual, his duals differ in several points from their finite counterparts. In a finite graph, going to the dual is a symmetric operation, i.e. if G^* is the dual of G then G is a dual of G^*. Also, a finite 3-connected planar graph has a unique dual. Both of these properties, symmetry and uniqueness, are lost for Thomassen's duals. This turns out to be because the duals are defined with regard to only finite cycles and cuts. If, in contrast, we also consider infinite cycles and cuts we regain symmetry and uniqueness, while Whitney's planarity criterion remains true, although we use a more restrictive defintion of duals.

Duality can also be expressed in terms of trees. Let G and G^* be finite graphs with a bijection * of their edge sets. Then G and G^* are duals if and only if for every spanning tree T of G, $(V^*, E^* \setminus E(T)^*)$ is a spanning tree of G^*. This, too, fails for locally finite graphs if only finite cycles are considered but becomes true once we work within the topological cycle space.

In a finite graph, the elements of the cycle space are precisely those edge sets that induce a eulerian subgraph. In an infinite graph, however, looking at the vertex degrees is not enough: Indeed, although each vertex of a double ray R has even degree, its topological cycle space $\mathcal{C}(R)$ consists of only the empty set. Therefore, we need to impose a degree condition on the ends as well.

For an end ω of a locally finite graph, let $\deg(\omega) \in \mathbb{N} \cup \{\infty\}$ denote the maximal number of edge-disjoint rays in ω. If $\deg(\omega) < \infty$ we say that ω is even if $\deg(\omega)$ is even, and if $\deg(\omega)$ is odd, we call ω odd. This leaves the case when $\deg(\omega) = \infty$. The figure shows that in that instance we need a slightly more sophisticated parity concept.





Each end of the two graphs contains infinitely many edge-disjoint rays. However, for the left graph it clearly is that $E(G_l) \in \mathcal{C}(G_l)$, so its single end should better be even, while for the right graph it holds that $E(G_r) \notin \mathcal{C}(G_r)$, which means that both of its ends need to be odd. We offer a definition of parity for an end, which yields the desired answers in these two examples and which coincides with the parity of $\deg(\omega)$ if $\deg(\omega) < \infty$. Then the following statements are equivalent:

(1) $E(G) \in \mathcal{C}(G)$;
(2) all vertices and all ends of G are even; and
(3) G admits a topological Euler tour,

where a topological Euler tour is a continuous mapping of the unit circle in the Freudenthal compactification of G such that each edge is traversed exactly once.

We have only indicated necessary and sufficient conditions for the whole edge set of a graph to be in its topological cycle space. Characterising arbitrary elements of $\mathcal{C}(G)$ in terms of vertex and end degrees seems to be difficult and remains so far unsolved. With a notion of the end degree that is adapted to subgraphs we can, however, recognise cycles.

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An algorithm for source location in directed graphs

András Frank
(joint work with Mihály Bárász, Johanna Becker)

Source location is a new type of location problems where the flow-amount or edge-connectivity rather than the distance between facilities and customers is taken into consideration. In their paper Hiro Ito, Kazuhisa Makino, Kouji Arata, Shoji Honami, Yuichiro Itatsu, and Satoru Fujishige, (Source location problem with flow requirements in directed networks, Optimization Methods and Software, Vol. 18, No. 4, August 2003, pp. 427-435) considered and analysed the Flow-constrained Directed Source Location (FDSL) problem. They proved a min-max theorem for the minimum cardinality of a subset R of nodes of an edge-capacitated digraph $D = (V, A)$ so that, for every node $v \in V - R$, the maximum flow-amount from R to v is at least k and from v to R is at least l. Based on this, they described an algorithm for computing such a minimum set R whose running time depends polynomially on the size of D but exponentially on k and l.

In the present work, we describe a strongly polynomial algorithm for solving the FDSL problem. A crucial idea is the introduction of the new concept of solid sets. Given a digraph $D = (V, A)$, we call a nonempty subset Z of V **in-solid** (respectively, **out-solid**) if $\varrho(X) > \varrho(Z)$ (respectively, $\delta(X) > \delta(Z)$) for every nonempty proper subset X of Z. An in- or out-solid set is called **solid**. Singletons are always in- and out-solid, and a minimal k-in-deficient set is in-solid (where k-in-deficient means that the indegree is smaller than k). Let $H_D = (V, \mathcal{E}_D)$ denote the hypergraph of all solid sets. The set of in-solid sets is exactly the union of all k-in-deficient sets ($k = 1, 2, \ldots$). An analogous statement holds for out-solid and solid sets. The algorithm is based on the following results.

Theorem 1. *For every directed graph $D = (V, A)$, there is a spanning tree on the groundset V such that each solid set of D induces a subtree, that is, $H_D = (V, \mathcal{E}_D)$ is a subtree hypergraph.*

Theorem 2. *For every node sV, the family of maximal s-avoiding solid sets is a partition of $V - s$.*

We call this partition the **solid partition** of $V - s$. Let H'_D denote hypergraph of subsets appearing in a solid partition for some v.

Theorem 3. *If T is a basic tree for H'_D, then T is basic for the hypergraph H_D of all solid sets (and, in particular, for its subhypergraph H_{kl} of deficient sets).*

For the full paper, see Operations Research Letters, 33 (2005) 221-230.

Graphs that have rank 2 matrices

Hein van der Holst
(joint work with Wayne Barrett, Raphael Loewy)

Let F be a field. For any graph $G = (V, E)$ on n vertices (all graphs are undirected and simple), let $S(F, G)$ be the set of all symmetric $n \times n$ matrices over F whose off-diagonal entries occur in exactly the positions corresponding to the edges of G. On the diagonal entries there is no restriction. Let $\mathrm{mr}(F, G) = \min\{\mathrm{rank} A \mid A \in S(F, G)\}$. Fix a nonnegative integer k. The problem is to identify, for any field F, those graphs G such that $\mathrm{mr}(F, G) \leq k$. For $k = 1$ this is easy: a graph G has $\mathrm{mr}(F, G) \leq 1$ if and only if G is the union of a complete graph K_r and an independent set of vertices. We have given a characterization of those graphs G with $\mathrm{mr}(F, G) \leq 2$ for infinite fields in [1] and for finite fields in [2]. The characterization depends on the field. The results for infinite fields are stated below.

The complement of a graph G is denoted by G^c. If G_1 and G_2 are graphs, we denote the disjoint union of G_1 and G_2 by $G_1 \cup G_2$. The join of G_1 and G_2 is the graph $(G_1^c \cup G_2^c)^c$. We denote the join of G_1 and G_2 by $G_1 \vee G_2$.

Theorem 1. *Let F be an infinite field with $\mathrm{char} F \neq 2$ and let G be a graph. Then $\mathrm{mr}(F, G) \leq 2$ if and only if G^c is of the form $(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for nonnegative integers $s_1, s_2, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.*

Theorem 2. *Let F be an infinite field with $\mathrm{char} F = 2$ and let G be a graph. Then $\mathrm{mr}(F, G) \leq 2$ if and only if G^c is either of the form $(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r)$ or of the form $(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for nonnegative integers $k, s_1, s_2, \ldots, s_k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.*

If F is a finite field, there are similar results. The proofs of these two theorems show how to construct a matrix $A \in S(F, G)$ with $\mathrm{rank} A \leq 2$.

If A is a principal submatrix of B, then the rank of A is at most the rank of B. Hence, if H is a induced subgraph of G, then $\mathrm{mr}(F, H) \leq \mathrm{mr}(F, G)$. We can therefore characterize the class of graphs G with $\mathrm{mr}(F, G) \leq 2$ in terms of forbidden induced subgraphs. We say that a graph G is H-free if G does not contain H as an induced subgraph. If $\mathcal{F}$ is a set of graphs, we say that G is $\mathcal{F}$-free if G is H-free for each $H \in \mathcal{F}$. One of the forbidden induced subgraphs of the class of graphs G with $\mathrm{mr}(F, G) \leq 2$ is P_4, the path with four vertices. Graphs which do not contain P_4 as an induced subgraph can be constructed by the following rules. The graph with one vertex and no edges is a P_4-free graph. If G_1 and G_2 are P_4-free graphs, then G_1^c and $G_1 \cup G_2$ are P_4-free graphs. Hence each of the other forbidden induced subgraphs for the class of graphs G with $\mathrm{mr}(F, G) \leq 2$ can recursively be constructed by these rules. Some of these other forbidden induced subgraphs are $P_3 \cup K_2$, $3K_2$, $\ltimes := (\text{diamond} \cup K_1)^c$, and dart $:= (K_1 \cup \text{paw})^c$, where paw $:= (K_1 \cup P_3)^c$ and diamond $:= (2K_1 \cup K_2)^c$. There are also forbidden induced subgraphs depending on the field F.

Theorem 3. *Let F be an infinite field with* $\mathrm{char}F \neq 2$ *and let G be a graph. Then* $\mathrm{mr}(F,G) \leq 2$ *if and only if G is* $(P_4, \ltimes, \mathrm{dart}, P_3 \cup K_2, 3K_2, K_{3,3,3})$*-free.*

Theorem 4. *Let F be an infinite field with* $\mathrm{char}F = 2$ *and let G be a graph. Then* $\mathrm{mr}(F,G) \leq 2$ *if and only if G is* $(P_4, \ltimes, \mathrm{dart}, P_3 \cup K_2, 3K_2, (P_3 \cup 2K_3)^c)$*-free.*

For finite fields we need some additional forbidden induced subgraphs. Each of them depends on the number of elements in the field.

Theorem 5. *Let F be a finite field with p^t elements, p prime and $p \neq 2$, and let G be a graph. Then* $\mathrm{mr}(F,G) \leq 2$ *if and only if G is* $(P_4, \mathrm{dart}, \ltimes, P_3 \cup K_2, 3K_2, K_{3,3,3}, ((m+2)K_2 \cup K_1)^c, (K_2 \cup 2K_1 \cup mP_3)^c, (K_1 \cup (m+1)P_3)^c)$*-free, where* $m = (p^t - 1)/2$.

Theorem 6. *Let F be a finite field with 2^t elements and let G be a graph. Then* $\mathrm{mr}(F,G) \leq 2$ *if and only if G is* $(P_4, \mathrm{dart}, \ltimes, P_3 \cup K_2, 3K_2, (P_3 \cup 2K_3)^c, ((2^{t-1}+1)K_2 \cup (2^{t-1}+1)K_1)^c, (P_3 \cup 2^{t-1}K_2 \cup K_1)^c, (2K_3 \cup 2^t K_1)^c, (2^{t-1}P_3 \cup 2K_1)^c)$*-free.*

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Even Pairs in Perfect Graphs

Frédéric Maffray
(joint work with Nicolas Trotignon)

An *even pair* in a graph is a pair of vertices such that every chordless path between them has even length. A graph G is *perfect* if, for every induced subgraph H of G, the chromatic number $\chi(H)$ of H is equal to its maximum clique size $\omega(H)$. Fonlupt and Uhry [5] proved that if $\{x, y\}$ is an even pair in any graph G, then the graph G/xy obtained by contracting x, y into one vertex satisfies $\chi(G/xy) = \chi(G)$ and $\omega(G/xy) = \omega(G)$. In particular it is possible to obtain an optimal coloring of G from any optimal coloring of G/xy by assigning to x and y the color of the contracted vertex and maintaining the color of the other vertices. This suggests a conceptually simple algorithm for coloring optimally the vertices of (some) perfect graphs, and indeed variants of such an algorithm have worked for a number of classical families of perfect graphs, see [4]. Unfortunately, not all perfect graphs have even pairs. This has led to several questions, based on the following definitions.

Say that a graph G is a *quasi-parity graph* [8] if, for every induced subgraph H of G, either H or $\overline{H}$ has an even pair or $|V(H)| = 1$. Say that a graph G is *even contractile* [1] if it can be turned into a clique by a sequence of contractions of even pairs, and *perfectly contractile* [1] if every induced subgraph of G is even-contractile. A *prism* is a graph that consists in two vertex-disjoint triangles and three vertex-disjoint chordless paths between the triangles, with no other edge.

A prism is odd (even) if these three paths are odd (resp. even), and *long* if one of the three paths has length at least 2. A graph G is *bipartisan* [3] if G and $\overline{G}$ contain no odd hole, no long prism, no line-graph of $K_{3,3} - e$ and no *double diamond* (a self complementary graph on eight vertices); bipartisan graphs form "Class F_6" in [2].

- Conjecture 1 (Everett and Reed [4, 9]): A graph that contains no odd hole, no antihole and no prism is perfectly contractile.
- Conjecture 2 (Everett and Reed [4, 9]): A graph is perfectly contractile if and only if it contains no odd hole, no antihole and no odd prism.
- Conjecture 3 (Hougardy [6]): There exists a class C of line-graphs of bipartite graphs such that every minimally even pair-free graph is either an odd hole, an antihole or a graph in C.
- Conjecture 4 (Hougardy [6]): There exists a class C' of line-graphs of bipartite graphs such that every minimally non-quasi-parity graph is either an odd hole, an odd antihole, a graph in C' or the complement of a graph in C'.
- Conjecture 5 (Thomas [10]): If G is bipartisan then either G or $\overline{G}$ has an even pair or $|V(G)| = 1$.

Conjecture 1 was proved recently by Maffray and Trotignon [7], who established that every graph G in the class A described in Conjecture 1 either is a clique or has an even pair whose contraction yields a graph in A and that such a pair can be found in polynomial time. The coloring algorithm suggested above can then be implemented to work in time $O(|V(G)|^2|E(G)|)$ for every graph G in class A [11].

The other conjectures have been proved partially only, for claw-free graphs, bull-free graphs, diamond-free graphs, planar graphs, etc — see [4] for a more detailed account of these results. A proof of Conjecture 5 would provide an alternate way for the last and arguably most complex fifty pages of the strong perfect graph theorem [2].

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The Roots of the Stable Set Polynomial of a Clawfree Graph

Maria Chudnovsky

(joint work with Paul Seymour)

A *stable set* in a graph is a set of pairwise non-adjacent vertices. The *stable set polynomial* of a graph G is the polynomial

$$S(G)(x) = \Sigma_{i\geq 0} a_i x^i$$

where a_i is the number of stable sets in G of size i.

Given a graph H, its *line graph* $L(H)$ is the graph whose vertex set is the set of edges of H, and two vertices are adjacent if they share an end in H. In [2] Heilmann and Lieb proved that if G is a line graph, then all the roots of $S(G)$ are real. This property does not hold for all graphs, since the stable set polynomial of a claw is

$$1 + 4x + 3x^2 + x^3$$

and not all its roots are real (a *claw* is the graph with vertex set $\{v_1, v_2, v_3, v_4\}$ and three edges v_1v_2, v_1v_3, v_1v_4.)

A graph G is said to be *clawfree* if no induced subgraph of it is a claw. We answer a question of Hamidoune [1] that was later posed as a conjecture by Stanley [3].

Theorem 1. *If G is clawfree then all roots of $S(G)$ are real.*

Since all line graphs are clawfree, this extends the result of [2].

The proof of 1 consists of two parts. First we prove a lemma about polynomials, that allows us to deduce that non-negative linear combinations of certain polynomials have all roots real. Then we find a recursion formula for $S(G)$, describing $S(G)$ as a non-negative linear combination of polynomials, satisfying the hypotheses of the lemma. We then combine the two parts, applying the lemma to the recursive formula for $S(G)$, to conclude that all roots of $S(G)$ are real.

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Certifying non-representability of matroids

Geoff Whittle
(joint work with Jim Geelen, Bert Gerards)

Seymour [4] showed that, for a matroid M given by a rank oracle, it requires in the worst case exponentially many calls in the size of M to prove that M is binary. It is straightforward to extend this result to all other fields. What about proving that a matroid is *not* binary? We know that $U_{2,4}$ is the only excluded minor for representability over $GF(2)$, and the existence of a $U_{2,4}$-minor can be shown in eight rank evaluations, so it can be proved in a constant number of rank evaluations that a matroid is not binary. Rota bravely conjectured that for each prime power q, the set of excluded minors for representability over $GF(q)$ is finite. This conjecture, which is arguably the most famous in matroid theory, would imply that there is a constant c_q such that there exists a proof that a matroid is not representable over $GF(q)$ that uses at most c_q calls to the rank oracle.

While Rota's Conjecture has not been verified for fields other than $GF(2)$, $GF(3)$ and $GF(4)$, it turns out that it is still possible to give a short proof that a matroid is not representable over $GF(q)$. In particular we prove

Theorem 1: For any prime power q, proving non-representability over $GF(q)$ for an n-element matroid requires only $O(n^2)$ rank values.

Note that the theorem tells us nothing about how we might find such a proof. Establishing Theorem 1 relies crucially on a study of inequivalent representations of matroids and we discuss this now. Two representations of a matroid over a field are *equivalent* if one can be obtained from the other by elementary row operations and column scaling. A major obstacle to proving Rota's Conjecture and to giving short proofs of non-representability is the existence of inequivalent representations of matroids. This problem does not arise for $GF(2)$ and $GF(3)$ and Kahn [2] showed that it arises in only a limited way for $GF(4)$. Moreover, in that paper, Kahn conjectured that for a fixed finite field, there was a bound on the number of inequivalent representations of a 3-connected matroid over a given field. While this conjecture turns out to be true for $GF(5)$, examples are given in [3] that show that Kahn's Conjecture fails for all fields with at least seven elements. Now we are challenged for a short proof of non-representability. I can convince you that a certain matrix does not represent M—that just requires one call to the oracle—but this is of no use if there is a potentially unbounded set of matrices that are plausible candidates for representations of M.

A natural way to try to get past this obstacle is to raise connectivity. Unfortunately matroid 4-connectivity is a very restrictive condition. There are satisfactory chain theorems for 3-connected matroids. Moreover, via the 2-sum decomposition, it is easily seen that one can reduce the problem of proving non-representability to 3-connected matroids. Neither of these desirable properties hold for 4-connected matroids. These problems are overcome by considering a notion of connectivity that is intermediate between 3-connectivity and 4-connectivity that we call *k-coherence*.

The idea of k-coherence is that we allow 3-separations in matroids, but we explicitly forbid the type of structures that lead to counterexamples to Kahn's Conjecture. The counterexamples to Kahn's Conjecture described in [3] belong to two very specific classes. For each $k \geq 5$, there is a member of each class with a partition into k subsets such that these subsets form a particular pattern of interlocking non-degenerate 3-separations. For one type, any union of blocks of the partition is 3-separating; for the other type, any union of blocks that respects a certain cyclic ordering is 3-separating. Loosely speaking, a matroid is k-coherent if it is 3-connected and there is no partition into k-parts of either of the above types. Unlike 4-connectivity it is possible to prove reasonable chain theorems for k-coherence. In particular it can be shown that if M is k-coherent and is not a wheel or a whirl, then there is an element e such that either $M \backslash e$ or M/e is k-coherent.

We prove that Kahn's Conjecture does hold for k-coherent matroids. Specifically we have

Theorem 2: For any prime power q and integer $k \geq 2$, there exists a constant $c_{q,k}$ such that each k-coherent matroid has at most $c_{q,k}$ inequivalent representations over $GF(q)$.

A very coarse outline of the techniques used to prove Theorem 2 follows. Two elements of a matroid M are *clones* if the function that interchanges them and is the identity on all other elements is an automorphism of M. An element x of M is *fixed* if it is not possible to extend M by an element x' to obtain a matroid M' in which x and x' are independent clones. It is the existence of elements that are not fixed (and dually, not *cofixed*) that leads to inequivalent representations. On the other hand, it is easily proved that, if x is fixed in M, then a representation of $M \backslash x$ that extends to a representation of M does so uniquely. Thus the number of inequivalent representations of M is at most the number of inequivalent representations of $M \backslash x$.

For a fixed $k \geq 5$, a matroid N is a *k-skeleton* if it is k-coherent and, for each element x of N, if x is fixed, then $N \backslash x$ is not k-coherent and if x is cofixed, then N/x is not k-coherent. We have proved that for any positive integer n, there are only a finite number of k-skeletons that do not contain the n-point line or its dual as a minor. The proof is long and technical; it uses, amongst other things, the grid theorem for $GF(q)$-representable matroids [1].

As the $(q+2)$-point line and its dual are not $GF(q)$-representable, it follows that, for any prime power q, there are a finite number of $GF(q)$-representable k-skeletons. Thus there exists a bound on the number of inequivalent $GF(q)$-representations of a k-skeleton. Moreover, for any k-coherent matroid M, there exists a k-skeleton minor N of M such that M can be built from N via a sequence of single-element extensions and coextensions with the property that the extensions are fixed and the coextensions are cofixed. This establishes Theorem 2. This theorem and the techniques used to prove it provide us with the tools needed to prove Theorem 1.

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Recognizing rank-width

Sang-il Oum

(joint work with Bruno Courcelle, Paul Seymour)

Some algorithmic problems, NP-hard on general graphs, are known to be solvable in polynomial time when the input graph admits a decomposition into trivial pieces by means of a tree-structure of cutsets of bounded order. However, it makes a difference whether the input graph is presented together with the corresponding tree-structure of cutsets or not. We have in mind two kinds of decompositions, "*tree-width*" and "*clique-width*" decompositions. There are many results known for graphs of bounded tree-width, but less progress had been made for graphs of bounded clique-width. We have a linear-time algorithm to decide whether an input graph has tree-width at most k for fixed k by Bodlaender [1], but for clique-width, the existence of polynomial-time algorithms to recognize graphs of clique-width at most k is only shown for $k \leq 3$ by Corneil, Perl, and Stewart [4] for $k = 2$ and by Corneil, Habib, Lanlignel, Reed, and Rotics [3] for $k = 3$.

Open problem For fixed $k > 3$, is there a polynomial-time algorithm that decides whether an input graph has clique-width at most k?

We define the *rank-width* [7], that is a graph parameter approximately equal to clique-width. More precisely, we have the following inequality for every graph G:

$$\text{rank-width} \leq \text{clique-width} \leq 2^{\text{rank-width}+1} - 1.$$

Moreover in polynomial time we can transform the tree-structure of rank-width to that of clique-width and vice versa.

Clique-width has nice algorithmic properties, but no good "minor" relation was known analogous to graph minors for *tree-width*. But for rank-width, we have a *vertex-minor* relation of graphs [8]. For a graph G and a vertex v of G, *local complementation* at v is an operation on G, replacing a subgraph induced on the neighbors of v by its complement graph. The graph obtained by applying local complementation at v to G is denoted by $G * v$. A graph H is a *vertex-minor* of G if H can be obtained by applying a sequence of local complementations and deletions of vertices to G. Vertex-minors were called *l-reductions* by Bouchet [2].

We show the following three theorems.

Theorem 1 (Oum and Seymour [7]**)** *For fixed k, there is an algorithm that with input an n-vertex graph G, either decides that G has rank-width at least $k+1$, or outputs a decomposition of G with rank-width at most $3k+1$. Its running time is $O(n^9 \log n)$.*

Theorem 2 ([8] **or** [9]**)** *For fixed k, there is a finite list of graphs $G_1, G_2, \ldots, G_m$ such that for every graph H, rank-width of H is at most k if and only if G_i is not isomorphic to a vertex-minor of H for all i.*

Theorem 3 (Courcelle and Oum [5]**)** *For every graph H, there is a closed modulo-2 counting monadic second-order logic formula φ_H expressing that a given graph contains a vertex-minor isomorphic to H.*

In [9], Theorem 2 was proved by showing much stronger statement on vertex-minors; we show that a set of graphs of bounded rank-width are *well-quasi-ordered* by the vertex-minor relation. In other words, for every infinite sequence of graphs $G_1, G_2, \ldots$ of bounded rank-width, there exist i and j such that $i < j$ and G_i is isomorphic to a vertex-minor of G_j. This well-quasi-ordering theorem is analogous to the well-quasi-ordering theorem [10, 6] for graphs and matroids of bounded tree-width, branch-width respectively.

It is known that given (counting) monadic second-order logic formula φ on graphs, there is a linear-time algorithm to evaluate φ for graphs of bounded clique-width if an input graph is given by the tree-structures of clique-width, called *k-expressions*. Since an algorithm in Theorem 1 can output the k-expression in $O(n^9 \log n)$ time, we may eliminate the need of an explicit input of k-expressions. Therefore, Theorem 3 implies that if an input graph has bounded rank-width, then for fixed graph H, there is a polynomial-time algorithm that answers whether the input graph contains a vertex-minor isomorphic to H. For fixed k, Theorem 2 states that only a finite number of vertex-minor testing is enough to show that a graph has rank-width at most k. By combining with Theorem 1, we obtain a $O(n^9 \log n)$-time algorithm that decides whether an input graph has rank-width at most k for fixed k.

Note. At this workshop at Oberwolfach, J. Geelen suggested an idea based on blocking sequences*, that would improve the running time of Theorem 1. Theorem 1 uses submodular function minimization algorithms as a black box. However by* *taking advantages of some properties of* cut-rank *functions of graphs, this can be done much faster and now the algorithm of Theorem 1 can run in $O(n^4)$ time.*

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The local chromatic number and topological properties of graphs

Gábor Tardos
(joint work with Gábor Simonyi)

The local chromatic number of graphs is a coloring type graph parameter that was introduced about 20 years ago by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [3]. This talk was based on the three upcoming papers [11, 12, 13] that try to better understand the properties of this graph parameter. We think that the local chromatic number deserves more attention than what it has obtained so far.

The definition of the local chromatic number is as follows.

Definition 1. *([3]) The local chromatic number $\psi(G)$ of a graph G is*

$$\psi(G) := \min_{c} \max_{v \in V(G)} |\{c(u) : u \in N(v)\}| + 1,$$

where $N(v) = \{u : uv \in E(G)\}$ and the minimum is taken over all proper colorings c of G.

In short, $\psi(G)$ is the fewest number of colors we can have in the most colorful closed neighborhood of a vertex in a proper coloring of the graph. It is obvious that the chromatic number $\chi(G)$ is an upper bound on $\psi(G)$. At first sight it is quite surprising, however, that $\psi(G) < \chi(G)$ is also possible, moreover, the gap between these two parameters can be arbitrarily large, cf. [3].

In [4] it was observed that the fractional chromatic number $\chi_f(G)$ can serve as a lower bound for $\psi(G)$. This motivates the study of the local chromatic number of graphs where the fractional and ordinary chromatic numbers are far apart. Not very many different families of such graphs are known. Here we discuss the local chromatic number of some of the standard examples for this gap. These standard examples have the other common feature that the topological technique introduced by Lovász [5] to bound the chromatic number from below is relevant for them in the sense that the bound it gives is sharp for these graphs. It turns out that the same kind of topological information that results in a lower bound for the chromatic number can also be used to bound the local chromatic number from below, and this bound is also sharp in many cases.

The main examples of graphs with a large gap between their fractional and ordinary chromatic number given in the book [9] are Kneser graphs and Mycielski graphs. More important for us are two variants of these families that clearly provide at least the same large gap between the two mentioned coloring parameters. The first of these variants is the family of Schrijver graphs $SG(n,k)$ discovered by Schrijver [10] as vertex color-critical induced subgraphs of Kneser graphs. The second is the family of so-called generalized Mycielski graphs, see their definition, e.g., in [7] or [14].

The chromatic number of $SG(n,k)$ is determined by Schrijver [10] to be $n-2k+2$ by generalizing the topological argument of Bárány [2] that provided a short proof for the earlier result of Lovász [5] determining the chromatic number of Kneser graphs.

For the local chromatic number of Schrijver graphs we have the following result.

Theorem 2. *(*[11]*)* If $t = n - 2k + 2 > 2$ is odd and $n \geq 4t^2 - 7t$ then

$$\psi(SG(n,k)) = \left\lceil \frac{t}{2} \right\rceil + 1.$$

The proof of the lower bound in this result uses topological methods. The same argument applies to all graphs that satisfy a certain topological criterion which implies that the chromatic number of the graph is at least t.

The upper bound part of Theorem 1 is given by a combinatorial construction that also can be formulated in a more general setting. As a result we can prove similar statements determining the local chromatic number of generalized Mycielski graphs and Borsuk graphs (for the definition of the latter see [6]) of certain parameters.

For 4-chromatic Schrijver graphs we have:

Theorem 3. *(*[12]*)*

$$\psi(SG(2k+2,k)) = 4.$$

This theorem is again true in a more general setting, namely, for all graphs G satisfying a topological criterion that implies $\chi(G) \geq 4$ we also have $\psi(G) \geq 4$. For this statement, however, we need a somewhat stronger topological criterion than the one used for Theorem 1. See [12] for the definition of the two criteria and for discussion on the different implications.

For even $t \geq 6$ we do not know the minimal local chromatic number of a t-chromatic Schrijver (or generalized Mycielski, etc.) graph: it is either $t/2+1$ or $t/2+2$.

It turns out that 4-chromatic Schrijver graphs are closely related to quadrangulations of the Klein bottle. The chromatic number of surface quadrangulations is a widely investigated topic, see [1, 8, 15], and the above mentioned connection suggests that analogs of Theorem 2 may be true for certain quadrangulations of non-orientable surfaces. Indeed, one can show that non-bipartite quadrangulations of the projective plane have local chromatic number 4, generalizing a celebrated result of Youngs [15] stating that such graphs are 4-chromatic. We also prove that certain quadrangulations of the Klein bottle that are shown to be 4-chromatic in [1] and [8] have local chromatic number 4. Surprisingly, however, one can construct graphs that quadrangulate other non-orientable surfaces, have chromatic number 4, and local chromatic number only 3. For further details we refer the reader to [13].

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The intersection of a matroid and a simplicial complex

Ron Aharoni
(joint work with Eli Berger)

A classical theorem of Edmonds from 1970 relates the maximal size of a set in the intersection of a pair of matroids with a "covering number" of the pair. We prove a generalization of this theorem, in which one of the matroids is replaced by a general simplicial complex (i.e., a hypergraph closed down with respect to containment), and its rank function is replaced by the topological connectivity of the complex.

As is well known, a complex can be realized geometrically, in a unique way. For example, every graph can be embedded, without generating new intersection between edges, in $\mathbb{R}^3$. A complex $\mathcal{C}$ is said to be *k-connected* if for every $i \le k$, every continuous function from the i-dimensional sphere into $\mathcal{C}$ can be extended to a continuous function from the $i+1$-dimensional ball to $\mathcal{C}$. As a matter of definition, -1-connectedness means being non-empty. A parameter $\eta(\mathcal{C})$ of the complex $\mathcal{C}$ is defined as the largest k for which $\mathcal{C}$ is k-connected, plus 2. If $\mathcal{C}$ is k-connected for every k, we write $\eta(\mathcal{C}) = \infty$.

For matroids, η is basically the rank function - the two are equal if η is finite.

Definition 1. *Let Γ be a bipartite graph with sides A and B, and let $\mathcal{C}$ be a simplicial complex on B. A $\mathcal{C}$-ISR is a function $f : A \to B$ using only edges of Γ, such that $f[A] \in \mathcal{C}$.*

The following generalization of Rado's theorem (which states the same for matroids) was proved by Aharoni and Haxell:

Theorem 2. *If $\eta(\mathcal{C} \restriction N[X]) \ge |X|$ for every $X \subseteq A$ then there exists a $\mathcal{C}$-ISR.*

(Here $N[X]$ is the set of neighbors of X.)

We use this theorem to prove the aforementioned generalization of Edmonds' theorem:

Theorem 3. *Let $\mathcal{M}, \mathcal{C}$ be a matroid and a simplicial complex, respectively, on the same ground set S. Then*

$$\max\{|\sigma| : \ \sigma \in \mathcal{M} \cap \mathcal{C}\} \ge \min\{\rho_{\mathcal{M}}(A) + \eta(\mathcal{C} \restriction (S \setminus A) : \ A \subseteq S\}$$

.

One application is an extension of Edmonds' theorem to the case of three matroids:

Theorem 4. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be matroids on the same ground set S. Then* $\max\{|\sigma| : \ \sigma \in \bigcap_{1 \le i \le 3} \mathcal{M}_i\} \ge \frac{1}{2} \min\{\sum_{1 \le i \le 3} \rho_{\mathcal{M}_i}(A_i) : \ \bigcup_{1 \le i \le 3} A_i = S\}$.

Here is one of numerous other applications:

Theorem 5. *If $\mathcal{M}, \mathcal{N}$ are two matroids on S, and S can be partitioned into k sets belonging to $\mathcal{M}$ and into k sets belonging to $\mathcal{N}$, then it can be partitioned into $2k$ sets belonging to $\mathcal{M} \cap \mathcal{N}$.*

Intrinsic metric and 2-cell embeddings of graphs

Bojan Mohar
(joint work with Matt DeVos)

By taking a collection of disjoint polygons in the Euclidean plane and identifying pairs of their sides of equal length, one obtains a *polyhedral surface* S, assuming each side of every polygon is identified with precisely one other side. The identified sides determine the edges, and the corners of polygons give rise to vertices of certain graph G. We say that G is *2-cell embedded* in S. The polygons determine the *2-cells* or *faces* of this embedding.

By viewing each 2-cell as a subset of the plane, we obtain a metric on S that is called the *polyhedral metric* of the embedding of G. The distances in this metric space correspond to lengths of shortest rectifiable curves in S connecting the corresponding points.

Let v be a vertex of a polyhedral surface S, and let $\alpha_1, \alpha_2, \ldots, \alpha_d$ (where d is the degree of v) be the incident angles in the faces containing v. Then we define the *Gaussian curvature* at v as

$$\kappa(v) = 2\pi - \sum_{i=1}^{d} \alpha_i.$$

For such notion of the discrete curvature, an analogue of the Gauss-Bonnet theorem holds, whose version restricted to convex polyhedra in the 3-space is known as the Descartes Lost Theorem, see [1]:

$$\sum_{v \in V(G)} \kappa(v) = 4\pi.$$

If all 2-cells are regular polygons with side length 1, then we have

$$\phi(v) = \frac{1}{2\pi}\kappa(v) = 1 - \frac{1}{2}\deg(v) + \sum_{v \sim f} \frac{1}{|f|},$$

where the summation runs over all faces f incident with v, and $|f|$ denotes the number of sides of f. As this notion of curvature can be defined without any reference to angles and actual polygons, it is called the *combinatorial curvature*.

In [5], Higuchi made a conjecture equivalent to the one given below concerning graphs with everywhere positive combinatorial curvature.

Conjecture 1 (Higuchi)**.** *Let G be a graph which is 2-cell embedded in a surface S so that every vertex and face has degree ≥ 3. If S is homeomorphic to a subset of the 2-sphere and the combinatorial curvature ϕ is everywhere positive, then G is finite.*

A complete solution of this conjecture follows from the following theorem whose proof can be found in [4].

Theorem 2 (DeVos and Mohar). *Let G be a graph which is 2-cell embedded in a surface S so that every vertex and face has degree ≥ 3. If ϕ is everywhere positive, then S is homeomorphic to either the 2-sphere or the projective plane and G is finite. Furthermore, if G is not a prism, antiprism, or the projective planar analogue of one of these, then $|V(G)| \leq 3444$.*

Although all of our results concern polygonal metric spaces, many of these results can be extended to more general spaces by way of approximation. As noted by Higuchi, Theorem 2 can be viewed as a discrete analogue of a result of Myers [7] who proved that every complete Riemannian manifold with Ricci curvature bounded below by a positive constant κ_0 is compact, has volume bounded in terms of κ_0 and has finite fundamental group. On the other hand, results in [4] show other possible directions for improvements of such results. One such extension may be a conjecture of Milnor [6] that every complete Riemannian manifold with non-negative Ricci curvature has finitely generated fundamental group.

If a polyhedral surface S is the 2-sphere, its graph G is 3-connected, and every vertex of G has nonnegative Gaussian curvature, then a theorem of Alexandrov [2, 3] states that the polyhedral metric of S can be obtained from some polyhedral realization in the 3-space corresponding to a cell complex that is obtained from the given embedding by diagonal flips. Besides the presentation of this beautiful and, unfortunately, less known result, several applications of the intrinsic metric were presented. In particular, an extension of Theorem 2 about positively curved spherical complexes and a structural characterization of planar triangulations with maximum degree 6 was discussed. The latter result was obtained earlier by Thurston [8] by means of other methods.

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Wide embedded graphs behave chromatically like plane or projective plane graphs

Luis A. Goddyn

An *embedded graph* $G \hookrightarrow X$ is a 2-cell embedding of a graph G on a *surface* X. (Here X is piecewise-linear and homeomorphic to either a sphere with g handles S_g, or a sphere with k cross-caps, N_k.) We often write G instead of $G \hookrightarrow X$. Since faces of G are open 2-cells, if $X \neq S_0$, then G contains graph cycles which are not contractible (homotopicly nontrivial), The *edgewidth*, $\mathrm{ew}(G)$, is the length of a shortest noncontractible cycle in G. Embedded graphs of very high edgewidth are (informally) said to be *wide*. Thomassen [6] proved that for every X there exists w such that every $G \hookrightarrow X$ with $\mathrm{ew}(G) \geq w$ satisfies $\chi(G) \leq 5$. This result may be informally stated "wide embedded graphs behave chromatically almost like plane graphs."

The point of this talk is to promote the view that this informal notion is not quite accurate. It is perhaps better to make a statement such as in the title of this talk. This becomes apparent when considering the *circular chromatic number* $\chi_c(G)$. This graph invariant is now widely studied (eg. [8]) since it is a refinement of the chromatic number $\chi(G)$. We define $\chi_c(G)$ to be the least possible value of

$$\max\left\{\frac{|C|}{|C^+|}, \frac{|C|}{|C^-|} \mid C \subseteq E(G) \text{ is a circuit in } G\right\} \tag{1}$$

among all orientations of G. Here (C^+, C^-) is the natural partition of C induced by the orientation of G. By replacing "circuit" with "cocircuit" in (1) we may define the dual invariant $\phi_c(G)$, the *circular flow number* of G. By replacing "circuit" with "contractible circuit" in (1) we may define the *local chromatic number* $\chi_{loc}(G)$ of an embedded graph G. It is known [2] that if $G \hookrightarrow X$ and X is orientable, then

$$\phi_c(G^*) = \chi_{loc}(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G). \tag{2}$$

Here G^* denotes the *surface dual* of $G \hookrightarrow X$ The inequalities (2) also hold when X is nonorientable, but we must replace $\phi_c(G^*)$ by the *circular biflow* number $\beta_c(G^*)$, which we do not define here. (Biflow numbers are signed-graph invariants as discussed in [1].) The relationship between $\phi_c(G^*)$ and $\chi_{loc}(G)$ seems to be unexplored where X is not orientable.

It is natural to define the following invariant in order to characterize the chromatic properties of wide embedded graphs. The *wide chromatic number* of a surface X is the real number

$$\chi_w(X) = \lim_{w\to\infty} \sup\{\chi_c(G) \mid G \hookrightarrow X \text{ and } \mathrm{ew}(G) \geq w\}. \tag{3}$$

By Thomassen's upper bound, every surface X satisfies

$$4 \leq \chi_w(X) \leq 5. \tag{4}$$

Replacing $\chi_c(G)$ with $\chi(G)$ in (3) would result in an uninteresting definition, since every surface different from S_0 embeds arbitrarily wide graphs with chromatic number 5. However, such graphs may have *circular* chromatic numbers very close to 4, which makes $\chi_w(X)$ somewhat more interesting, and perhaps a truer measure of "chromatic properties" of wide embedded graphs.

For every nonorientable surface $X = N_k$ we have $\chi_w(X) = 5$. This follows from the existence [2, Example 6.5] of arbitrarily wide embedded graphs $G \hookrightarrow N_k$ with $\chi_c(G) = 5$, for any $k \geq 1$.

Perhaps surprisingly, for orientable surfaces $X = S_g$, $g > 0$ we only know that $4 \leq \chi_w(S_g) \leq 5$. I now propose the following.

Conjecture 1. *For any $g \geq 0$, we have $\chi_w(S_g) = 4$.*

In [2] it is proved that for any surface X and $\epsilon > 0$, there exists $w > 0$ such that every $G \hookrightarrow X$ with $\mathrm{ew}(G) > w$ satisfies

$$\chi_c(G) \leq \chi_{loc}(G) + \epsilon. \tag{5}$$

In view of (4) this implies that we may replace "$\chi_c(G)$" with "$\chi_{loc}(G)$" in the definition (3). If, further, X is orientable, then we may replace "$\chi_c(G)$" with "$\phi_c(G^*)$" in definition (3). Thus Conjecture 1 is equivalent to the assertion that for any g and $\epsilon > 0$, wide enough embedded graphs $G \hookrightarrow S_g$ satisfy $\phi_c(G) < 4 + \epsilon$. By standard "lifting arguments" (such as in [7]) it suffices to verify this assertion for cubic graphs G.

Grünbaum [4] has proposed something much stronger than Conjecture 1.

Conjecture 2. *For any $G \hookrightarrow S_g$ such that $ew(G^*) \geq 3$ we have $\phi_c(G) = 4$.*

Equivalently, Grünbaum asserts that every cubic graph $G \hookrightarrow S_g$ with $\mathrm{fw}(G) \geq 3$ is three edge colourable. (Here, the *facewidth* $\mathrm{fw}(G)$ is the least number of points in which a noncontractible curve in S_g meets G. For cubic embedded graphs we have $\mathrm{fw}(G) = \mathrm{ew}(G^*)$.)

The following weak form of Grünbaum's conjecture would suffice to prove Conjecture 1.

Conjecture 3. *For any g there exists w such that every cubic graph $G \hookrightarrow S_g$ with $ew(G) \geq w$ is 3-edge colourable.*

Conjecture 3 may be informally stated, "snarks on S_g have bounded facewidth".

On the negative side, it seems to be difficult to find a nontrivial class of graphs for which any of these conjectures can be proven. Even worse, I have not been able to demonstrate $\chi_w(S_g) < 5$ for some $g > 0$.

On the positive side, wide embedded graphs $G \hookrightarrow S_g$ for which $\chi_c > 4$ seem to be difficult to construct. Steve Fisk [3] proposed an a class of graphs (called *Fisk graphs*) with this property. He showed that if every face of $G \hookrightarrow S_g$ is a triangle, and there are exactly two vertices of odd degree, and these two vertices are adjacent, then $\chi(G) \geq 5$. Fisk graphs of arbitrary edgewidth exist on every $S_g \neq S_0$. If Conjecture 3 holds, then by (5), $\chi_c(G)$ is only very slightly greater than 4 for any wide Fisk graph $G \hookrightarrow S_g$. To date, the best I have been able to show [unpublished] is that $\chi_{loc}(G) < 5$ for a very special subclass of Fisk triangulations on the torus.

This state of knowledge is embarrassing, compared to our supposed intuition of chromatic properties of wide embedded graphs. The fact $\chi_w(N_k) = 5$, justifies the statement that wide embedded graphs on a nonorientable surface behave at worst very much like projective plane graphs. I propose that it is of central importance to work toward proving the "obvious" oriented analogue: "wide embedded graphs on S_g are chromatically very similar to plane graphs."

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Finding large planar subgraphs

Deryk Osthus

(joint work with Daniela Kühn and Anusch Taraz)

Planar subgraphs. In [6], we studied the following extremal question: Given a function $m = m(n)$, how large does the minimum degree of a graph G of order n have to be in order to guarantee a planar subgraph with at least $m(n)$ edges?

For example, we proved the following result, which gives the threshold for a planar subgraph which is almost a spanning triangulation:

Theorem 1. *For every $\gamma > 0$ there exists $C = C(\gamma)$ such that every graph G of order n and minimum degree at least $(1/2 + \gamma)n$ contains a planar subgraph with at least $3n - C$ edges.*

This is best possible in the sense that the constant C has to depend on γ and the additional term γn in the bound on the minimum degree cannot be replaced by a sublinear one.

The following result from [4] improves an earlier one from [6], which had an additional error term in the minimum degree condition.

Theorem 2. *There exists an integer n_0 such that every graph G of order $n \geq n_0$ and minimum degree at least $2n/3$ contains a triangulation as a spanning subgraph.*

The bound on the minimum degree is best possible: for all integers n there are graphs of order n and minimum degree $\lceil 2n/3 \rceil - 1$ without a spanning triangulation. Komlós, Sárközy and Szemerédi [3] proved the related result that every graph of sufficiently large order n and minimum degree at least $2n/3$ contains the square of a Hamilton cycle.

Our work was partly motivated by the maximum planar subgraph problem: In a given graph G, it asks for a planar subgraph with the maximum number of edges. Călinescu et al. [2] showed that this problem is Max SNP-hard. On the other hand, our proof of Theorem 2 implies that the maximum planar subgraph problem can be solved in polynomial time for graphs with minimum degree at least $2/3n$. Similarly, the proofs of our results in [6] give improved approximation algorithms for graphs whose minimum degree is sufficiently large for the respective results to apply.

Simultaneous partition of graphs. Given a graph G with m edges, the Max cut problem is to determine (the size of) the maximum cut in G. For complete graphs, the largest cut has size $m/2 + o(m)$. On the other hand, it is well known that a cut of size at least $m/2$ in a graph G can be found using the natural greedy algorithm. Now consider two graphs G_1 and G_2 on the same vertex set V and suppose that G_i has m_i edges. The aim now is to find a partition of V so that this induces a large cut in both of the G_i. In [5] we proved the following result (where for a given graph G and disjoint subsets A, B of its vertex set, $e_G(A, B)$ denotes the number of edges between A and B):

Theorem 3. *Suppose that G_1 and G_2 are two graphs on the same vertex set V, where G_i has m_i edges. There is a bipartition of V into two classes A and B so that for both $i = 1, 2$ we have*

$$e_{G_i}(A, B) \geq m_i/2 - \sqrt{m_i}.$$

This is clearly best possible up to the error term $\sqrt{m_i}$ and answers a question of Bollobás and Scott [1]. We also proved analogues of this result for partitions into more than two vertex classes. D. Rautenbach drew our attention to the problem during the workshop and the probabilistic argument leading to its solution was also found (and announced) during the same workshop.

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Matchings and Hamilton cycles in uniform hypergraphs

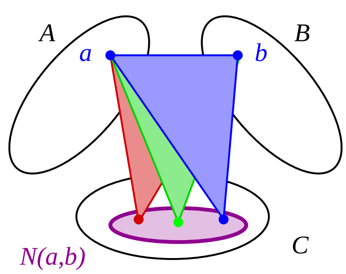
Daniela Kühn

(joint work with Deryk Osthus)

Matchings in uniform hypergraphs. The so called 'marriage theorem' of Hall provides a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. For hypergraphs there is no analogue of this result—up to now only partial results are known. For example, Conforti et al. [3] extended Hall's theorem to so-called balanced hypergraphs and Haxell [5] extended Hall's theorem to a sufficient condition for the existence of a hypergraph matching which contains a given set of vertices. Moreover, there are many results about the existence of almost perfect matchings in hypergraphs which are pseudo-random in some sense. Most of these are based on an approach due to Rödl (see e.g. [1] for an introduction to the topic or Vu [11] for more recent results). For random r-uniform hypergraphs, the threshold for a perfect matching is still not known. There are several partial results, see e.g. Kim [6].

A simple corollary of Hall's theorem for graphs states that every bipartite graph with vertex classes A and B of size n whose minimum degree is at least $n/2$ contains a perfect matching. This can also be easily proved directly by considering a matching of maximum size. In [7] we proved an analogue of this result for uniform hypergraphs. For simplicity, I will only describe the situation for 3-uniform hypergraphs here, but we have proved analogous results for r-uniform hypergraphs.

So let us consider a 3-partite 3-uniform hypergraph $\mathcal{H}$ with vertex classes A, B and C where $|A| = |B| = |C| = n$. Let E denote the set of hyperedges of $\mathcal{H}$. Thus the elements of E are triples abc with $a \in A$, $b \in B$ and $c \in C$. One way to define the minimum degree of $\mathcal{H}$ is the following. Given vertices $x, y \in A \cup B \cup C$, the *neighbourhood* $N(x, y)$ *of* x *and* y in $\mathcal{H}$ is the set of all those vertices z which form a hyperedge together with x, y, i.e. for which $xyz \in E$. The *minimum degree* $\delta_2(\mathcal{H})$ is then defined to be the minimum $|N(x, y)|$ over all pairs x, y which lie different vertex classes of $\mathcal{H}$.



Theorem 1. *Every* 3*-uniform* 3*-partite hypergraph* $\mathcal{H}$ *whose three vertex classes have size* $n \geq 1000$ *and whose minimum degree* $\delta_2(\mathcal{H})$ *is at least* $n/2 + \sqrt{2n \log n}$ *has a perfect matching.*

Theorem 1 is best possible up to the error term $\sqrt{2n \log n}$. The proof relies on a probablistic argument based on the number of perfect matchings in a bipartite graph with given degrees (the latter is given by Brégman's proof [2] of the Minc conjecture on the permanent of a 0-1 matrix).

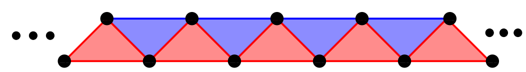
Surprisingly, a simple argument already shows that a significantly smaller minimum degree guarantees a matching which covers all but at most 3 vertices of $\mathcal{H}$:

Theorem 2. *Every* 3*-uniform* 3*-partite hypergraph* $\mathcal{H}$ *whose three vertex classes have size* n *and whose minimum degree* $\delta_2(\mathcal{H})$ *is at least* $n/3$ *has a matching which covers all but at most* 3 *vertices of* $\mathcal{H}$*.*

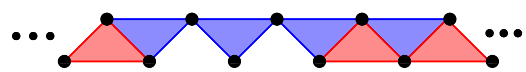
The bound on the minimum degree in Theorem 2 is again best possible. Both Theorems 1 and 2 can be used to prove analoguous results about matchings in 3-uniform hypergraphs which are not required to be 3-partite.

Hamilton cycles in 3-uniform hypergraphs. A classical theorem of Dirac states that every graph on n vertices with minimum degree at least $n/2$ contains a Hamilton cycle. If one seeks an analogue of this result for 3-uniform hypergraphs $\mathcal{H}$, then several alternatives suggest themselves. We define the *minimum degree* $\delta(\mathcal{H})$ of $\mathcal{H}$ to be the minimum $|N(x,y)|$ over all pairs of distinct vertices $x, y \in \mathcal{H}$ (where $N(x,y)$ is defined as in the previous section).

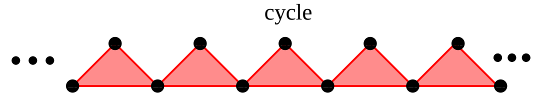
We say that a 3-uniform hypergraph $\mathcal{C}$ is a *cycle of order* n if there a exists a cyclic ordering $v_1, \dots, v_n$ of its vertices such that every consecutive pair $v_i v_{i+1}$ lies in a hyperedge of $\mathcal{C}$ and such that every hyperedge of $\mathcal{C}$ consists of 3 consecutive vertices. Thus the cyclic ordering of the vertices of $\mathcal{C}$ induces a cyclic ordering of its hyperedges. A cycle is *tight* if every three consecutive vertices form a hyperedge. A cycle of order n is *loose* if it has the minimum possible number of hyperedges among all cycles on n vertices.



tight cycle



cycle



loose cycle of even order

A *Hamilton cycle* of a 3-uniform hypergraph $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$ which is a cycle containing all its vertices. In [8] we proved the following result.

Theorem 3. *For each* $\sigma > 0$ *there is an integer* $n_0 = n_0(\sigma)$ *such that every* 3*-uniform hypergraph* $\mathcal{H}$ *with* $n \geq n_0$ *vertices and minimum degree at least* $n/4 + \sigma n$ *contains a loose Hamilton cycle.*

The bound on the minimum degree in Theorem 3 is best possible up to the error term σn. In fact, if the minimum degree is less than $\lceil n/4 \rceil$, then we cannot even guarantee *any* Hamilton cycle.

Recently, Rödl, Ruciński and Szemerédi [10] proved that if the minimum degree is at least $n/2 + \sigma n$ and n is sufficiently large, then one can even guarantee a tight Hamilton cycle. Their bound is best possible up to the error term σn.

The proofs of both our Theorem 3 and the result of Rödl, Ruciński and Szemerédi [10] rely on the Regularity Lemma for 3-uniform hypergraphs due to Frankl and Rödl [4]. However, Rödl, Ruciński and Szemerédi make extensive use of the fact that the intersection of the neighbourhoods of any two pairs of vertices is nonempty, which is far from true in our case. For this reason, our argument has a rather different structure. (In fact, if we assume that our hypergraph has minimum degree $n/2 + \sigma n$ and the number of vertices is divisible by four, then our result is much easier to prove). Instead, we prove and use a 'blow up' type result: every 'pseudo-random' hypergraph contains a loose Hamilton cycle. This in turn uses a probabilistic argument based on results about random perfect matchings in pseudo-random graphs [9].

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Lifts of graphs - The state of the art

Nathan Linial

This talk was a survey of *lifts of graphs.* The most developed part of this theory concerns *random* lifts. We also mention some extremal problems in this area, as well as a recent application to the construction of expander graphs with near-optimal spectral gap. These papers were written jointly with (in alphabetical order)
Alon Amit, Yonatan Bilu, Yotam Drier, Jirka Matousek, and Eyal Rozenman.
Covering maps are fundamental objects of study in topology. They apply to graphs as well, since graphs are one-dimensional simplicial complexes. It is well-known and easy to show that if G is a finite connected graph, then any covering map $\varphi : H \to G$ has a degree, or *fold number.* This is an integer n such that the inverse

image of every vertex $x \in V(G)$ consists of n vertices in $V(H)$, and likewise for edges $|\varphi^{-1}(e)| = n$ for every $e \in E(G)$. This allows us to define the class $L_n(G)$ of all those (labelled) graphs H that have an n-fold cover map onto G. This set has a natural structure of a probability space. Indeed, the most well developed part of the theory deals with the asymptotic almost-sure properties of graphs in $L_n(G)$. The main question is how these properties are affected by the features of the base graph G.

Here and below, G is a finite connected graph. The first question to ask is how likely is it for a graph in $L_n(G)$ to be connected. This turns out to be easy to answer

Proposition. *Let G be a finite connected graph.*

- *If G is a tree, then* none *of the graphs in $L_n(G)$ is connected for any $n \geq 2$.*
- *If G in unicyclic, then a random graph in $L_n(G)$ is connected with probability $\frac{1}{n}$.*
- *If G has more edges than vertices, then a random graph in $L_n(G)$ is connected with probability $1-o(1)$. (Henceforth we state this briefly as "almost all lifts of G are connected").*

The degree of connectivity is a more complicated matter. In [1] we first observe that if $\delta = \delta(G)$ is the smallest vertex degree in G, then no graph in $L_n(G)$ has connectivity exceeding δ. On the other hand, we prove:

Theorem. *If $\delta(G) \geq 3$, then almost every lift is δ-connected.*

We also stated the following zero-one law from [5] about the largest matching in graph lifts:

Theorem. *Every G falls into one of the following mutually exclusive four categories:*

- *Every lift of G has a perfect matching.*
- *In every lift of G, the largest matching misses at least a fraction ϵ of the vertices for some fixed $\epsilon(G) > 0$.*
- *The largest matching in almost every n-lift of G almost surely misses $\Theta(\log n)$ vertices.*
- *Some lifts of G have no perfect matching, but almost all of them do.*

Given G, we are able to efficiently tell the category to which it belongs.

We also mentioned our work [2] in which the typical chromatic number in lifts of G is analyzed. In [4] we consider typical and extremal Hadwiger numbers of lifts of graphs. In this context we ask:

Problem. *Do there exist lifts of the complete graphs K_r in which the Hadwiger number is $o(r)$?*

Finally in [3] we use random lifts to construct regular graphs with nearly maximal spectral gaps. This is a very desirable property in the theory of expander graphs. In this context we mentioned the following conjecture, the proof of which would show, among others, that d-regular *Ramanujan Graphs* exist for every integer $d \geq 3$. A *signing* of a symmetric $0,1$ matrix A is a symmetric $0,1,-1$ matrix B such that $b_{ij} = 0$ iff $a_{ij} = 0$.

Conjecture. *Let A be the adjacency matrix of a d-regular graph. Then A has a signing B with a spectral radius $\leq 2\sqrt{d-1}$.*

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Reducibility for the Four-Color Theorem

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(joint work with Serguei Norine)

A proof of the Four-Color Theorem (4CT) was given by Appel, Haken and Koch in [2] and [3], and was later reprinted in [4]. The proof is computer-assisted, but even the non-computer part is extremely complicated, and to the author's knowledge has never been independently verified. A simpler proof was obtained by Robertson, Sanders, Seymour and Thomas in [6]. While this proof *has* been independently verified, it is still computer-assisted. The purpose of the present research was to come closer to a computer-free proof.

Both known proofs of the 4CT proceed in two steps—reducibility and discharging. The proof in [6] uses computers for both steps, but the discharging argument has been completely written down (by a computer) on approximately $13,000$ lines. Each of those lines takes some thought to verify, but, in principle, they can all be checked by a human.

Thus the main bottleneck is reducibility, and so we focus exclusively on that part of the proof. A *near-triangulation* is a non-null connected planar drawing G such that every finite region is a triangle. A *configuration* K consists of a near-triangulation $G(K)$ and a map $\gamma_K \colon V(G(K)) \to \mathbb{Z}$ with the following properties:

(i) for every vertex v, $G(K)\backslash v$ has at most two components, and if there are two, then $\gamma_K(v) = d(v) + 2$,

(ii) for every vertex v, if v is not incident with the infinite region, then $\gamma_K(v) = d(v)$, and otherwise $\gamma_K(v) > d(v)$; and in either case $\gamma_K(v) \geq 5$,

(iii) K has ring-size ≥ 2, where the *ring-size* of K is defined to be $\sum_v (\gamma_K(v) - d(v) - 1)$, summed over all vertices v incident with the infinite region such that $G(K)\backslash v$ is connected.

Let T be a triangulation. A configuration K *appears* in T if $G(K)$ is an induced subgraph of T, every finite region of $G(K)$ is a region of T, and $\gamma_K(v) = d_T(v)$ for every vertex $v \in V(G(K))$. The reducibility part of the 4CT consists of showing that no member of an explicit set $\mathcal{U}$ of 633 configurations appears in a minimal counterexample to the 4CT. This is done by running the same program on each of the 633 configurations in $\mathcal{U}$ to check that each of those configurations is "reducible". We have studied the concept of "exterior" (in the sense of [1]) of certain configurations and proved various theorems. Due to space limitations we are not able to state them here.

What is needed and is sorely missing is a theory of reducibility, which should imply statements of the form "if a certain configuration K is reducible, then so is another configuration obtained from K by means of some well-defined rules". There are several conjectures along those lines, most notably [5, Vermutung 1a]. Unfortunately, that conjecture is too strong—it implies that the configuration K, where $G(K)$ is a triangle and $\gamma_K(v) = 5$ for every vertex v of $G(K)$, cannot appear in a minimal counterexample to the 4CT. That would be a fantastic result to prove, but there is currently no hope, because the existing methods are not sufficiently strong. Thus our best hope is to prove something weaker, but the right statement eludes us at the moment. We managed to prove the following.

Let $G_1, G_2, \ldots, G_k$ be disjoint graphs, each isomorphic to K_4 with one edge deleted; let $x_i y_i$ be the deleted edge of G_i. Let G be obtained from $G_1 \cup G_2 \cup \ldots \cup G_k$ by identifying x_i and y_{i+1} for all $i = 1, 2, \ldots, k-1$. Let K_k be the configuration with $G(K_k) = G$ and $\gamma_{K_k}(v) = 6$ if $v = x_i = y_{i+1}$ for some $i = 1, 2, \ldots, k-1$, and $\gamma_{K_k}(v) = 5$ otherwise. We were able to show that K_k is reducible for all $k \geq 1$. This is the first nontrivial example of an infinite sequence of reducible configurations. The proof is computer-free and not too hard. However, we are still far away from a computer-free proof of the 4CT.

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Bricks and Pfaffian orientations.

Serguei Norine
(joint work with Robin Thomas)

A *labeled graph* is a graph with vertex-set $\{1, 2, \ldots, n\}$ for some n. Let G be a directed labeled graph and let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of G. Define the *sign* of M to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges are written. We say that a labeled graph G is *Pfaffian* if there exists an orientation D of G such that the signs of all perfect matchings in D are positive, in which case we say that D is a *Pfaffian orientation* of G. An unlabeled graph G is *Pfaffian* if it is isomorphic to a labeled Pfaffian graph. It is well-known and that in that case every labeling of G is Pfaffian. Pfaffian orientations have been introduced by Kasteleyn [5, 6, 7], who demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time.

Matching decomposition procedure by Kotzig, Lovász and Plummer [9] allows us to reduce characterization of Pfaffian graphs to two special classes: braces, which are bipartite, and bricks. Pfaffian bipartite graphs were characterized in terms of forbidden subgraphs by Little [8]. A structural characterization of Pfaffian bipartite graphs was given by Robertson, Seymour and Thomas [14] and independently by McCuaig [10]. No satisfactory characterization is known for Pfaffian bricks.

We have discovered substantial obstructions to implementing both structural and forbidden minor approaches. We have found examples of Pfaffian bricks on $2n - 2$ vertices, $(n^2 + 5n - 12)/2$ edges and a complete graph on n vertices as a subgraph. This implies that most likely there is no structural characterization of Pfaffian bricks similar to the characterization of Pfaffian braces in [14], because such a characterization would imply a linear upper bound on the number of edges in Pfaffian bricks. We have also found an infinite family of bricks, which are minimally non-Pfaffian. In fact, this family contains exponentially many elements with given number of vertices. However, we believe that the graphs in this family, $K_{3,3}$, the Petersen graph and twinplex are the only minimal non-Pfaffian graphs.

In [12] I was able to obtain a characterization of Pfaffian graphs in terms of their drawing in the plane.

Theorem 1. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

There are several ways to generalize Theorem 1. In [] For a labeled graph G, an orientation D of G and a perfect matching M of G, denote the sign of M in the directed graph corresponding to D by $D(M)$. We say that a graph G is *k-Pfaffian* if there exist, a labeling of G, orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i D_i(M) = 1.$$

For surfaces of higher genus the following result was mentioned by Kasteleyn [6] and proved by Galluccio and Loebl [4] and independently by Tesler [15].

Theorem 2. *Every graph that can be embedded on a surface of genus g is 4^g-Pfaffian.*

I was able to prove the following analogue of Theorem 1 for the torus [13].

Theorem 3. *Every 3-Pfaffian graph is Pfaffian. A graph G is 4-Pfaffian if and only if there exists a drawing of G on the torus such that $cr(M)$ is even for every perfect matching M of G. Every 5-Pfaffian graph is 4-Pfaffian.*

The theorem above suggest that the following conjecture might hold.

Conjecture 4. *For a graph G and a non-negative integer g the following are equivalent*

(1) *There exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G.*
(2) *G is 4^g-Pfaffian.*
(3) *G is $(4^{g+1} - 1)$-Pfaffian.*

In [11] we generalize Pfaffian orientations to Pfaffian labelings. Let Γ be an Abelian group, denote by 1 an identity of Γ and denote by -1 some element of order two in Γ. Let G be a graph with $V(G) = \{1, 2, \ldots, 2n\}$. We say that $l : E(G) \to \Gamma$ is a *Pfaffian labeling* of G if for every perfect matching $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$, such that $u_i < v_i$ for every $1 \leq i \leq k$ we have

$$\prod_{e \in M} l(e) = sgn \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

The definition of Pfaffian labelings is motivated by the list-edge coloring conjecture. In a k-regular multigraph one can define a sign for every k-edge coloring (see [1]). A powerful algebraic technique developed by Alon and Tarsi [2] implies that if in a k-edge-colorable k-regular multigraph G all k-edge colorings have the same sign then G is k-list-edge-colorable. In [11] we prove the following theorem, which generalizes a theorem by Ellingham and Goddyn [3] and settles a conjecture by Goddyn.

Theorem 5. *A multigraph G admits a Pfaffian labeling if and only if for all k all the k-edge colorings of every k-regular multigraph G' with the same underlying simple graph as G have the same sign.*

We also give two characterizations of graphs that admit a Pfaffian labeling.

Theorem 6. *A graph admits a Pfaffian labeling if and only if every brick and every brace in its tight cut decomposition is either Pfaffian or isomorphic to the Petersen graph. If a graph admits a Pfaffian Γ-labeling for some Abelian group Γ then it admits a Pfaffian $\mathbb{Z}_4$-labeling.*

Theorem 7. *A graph admits a Pfaffian labeling if and only if there exists a drawing of it in the projective plane (possibly with crossings) and a representation of this drawing so that every perfect matching intersects itself an even number of times and goes through the crosscap an even number of times.*

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Dense minors in highly connected graphs

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(joint work with Thomas Böhme, John Maharry and Bojan Mohar)

Let a be an integer. It is proved that for any s and k, there exists a constant $N = N(s, k, a)$ such that every $16a$-connected graph with at least N vertices either contains a subdivision of $K_{a,sk}$ or a minor isomorphic to s disjoint copies of $K_{a,k}$. In fact, we prove that the connectivity $3a + 2$ and the minimum degree at least

$16a$ are enough. The condition "a subdivision of $K_{a,sk}$" is necessary since G could be a complete bipartite graph $K_{16a,m}$, where m could be arbitrarily large. The requirement on $N(s,k,a)$ vertices is necessary since there exist graphs without K_a-minor whose connectivity is $\Theta(a\sqrt{\log a})$.

When $s = 1$ and $k = a$, this implies that every $16a$-connected graph with at least $N(a)$ vertices has a K_a-minor. This is the first result where a linear lower bound on the connectivity in terms of a forces a K_a-minor. This was also conjectured in [4, 5].

Our result together with the recent result in [3] also implies that there exists an absolute constant c such that there are only finitely many ck-contraction-critical graphs without K_k as a minor. This result is related to the well-known conjecture of Hadwiger [2].

Our result was also motivated by the well-known result of Erdős and Pósa [1]. Our result may be stated as follows. Suppose that G is $16a$-connected and without a subdivision of $K_{a,t}$. Then there exists an integer $F(s,k,a,t)$ such that either there are s disjoint copies of $K_{a,k}$-minor in G, or G has a vertex set F of order at most $F(s,k,a,t)$ such that $G - F$ has no minor isomorphic to $K_{a,k}$.

Our result also implies that there exist absolute constants c_1 and c_2 with $c_1 \geq c_2$ such that there are only finitely many c_1k-connected c_2k-color-critical graphs without K_k as a minor. This fact is related to Thomassen's result [6] which says that there are only finitely many 6-color-critical graphs on a fixed surface. Notice that the set of graphs embeddable on a fixed surface is closed under taking minors. More generally, Mohar conjectured that there are only finitely many 3-connected k-color-critical graphs without K_k as a minor.

Our result implies the following, as well.

There is a constant $c > 0$ and a polynomial time algorithm for deciding either

(1) a given graph G is k-colorable, or
(2) G contains K_{ck}-minor, or
(3) there is a graph H without K_{ck}-minor and with no k-coloring.

Observe that if c would be 1, then H in (3) would be a counterexample to Hadwiger's conjecture

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Global Rigidity of Graphs

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(joint work with Tibor Jordán)

A *framework* is a pair (G,p), where $G=(V,E)$ is a graph and p is a map from V to $\mathbb{R}^2$. We consider the framework to be a straight line realisation of G in $\mathbb{R}^2$ in which the *length* of an edge $uv \in E$ is given by the Euclidean distance between the points $p(u)$ and $p(v)$.

Let (G,p) and (G,q) be frameworks. We say that

- (G,p) and (G,q) are *equivalent* if $|p(u)-p(v)| = |q(u)-q(v)|$ for all $uv \in E$.
- (G,p) and (G,q) are *congruent* if $|p(u)-p(v)| = |q(u)-q(v)|$ for all $u,v \in V$.
- (G,p) is *rigid* if there exists an $\epsilon > 0$ such that every framework (G,q) which is equivalent to (G,p) and satisfies $|p(v)-q(v)| < \epsilon$ for all $v \in V$, is congruent to (G,p).
- (G,p) is *globally rigid* if every framework (G,q) which is equivalent to (G,p), is congruent to (G,p).

It can be seen that (G,p) is rigid if and only if there is no 'continuous deformation' of (G,p) which preserves the lengths of all its edges, see [1].

Saxe [9] showed that it is NP-hard to determine if a given framework is globally rigid. General feeling is that the problem of deciding when a given framework is rigid is also NP-hard, although no proof is yet known. We obtain problems of a different complexity, however, if we prohibit algebraic dependencies between the points of the framework. We say that (G,p) is a *generic framework* if the coordinates of all the points $p(v)$, $v \in V$, are algebraically independent over $\mathbb{Q}$. Gluck [4] showed that if a particular generic framework (G,p) is rigid then all generic frameworks (G,q) are rigid. Thus the rigidity of a generic framework (G,p) depends only on the graph G and not the map p. We say that a graph G is *rigid* if (G,p) is rigid for some (or equivalently, all) generic frameworks (G,p). The graph G is *minimally rigid* if it is rigid and $G-e$ is not rigid for all $e \in E$. Minimally rigid graphs were characterised in 1970 by Laman.

Theorem 1. [7] *Let $G=(V,E)$ be a graph. Then G is minimally rigid if and only if $|E| = 2|V|-3$ and $|E(H)| \le 2|V(H)|-3$ for all subgraphs H of G with $|V(H)| \ge 2$.*

The inequality in Laman's theorem can be used to define a matroid R_G on the edge set E of a graph G: we say that a subset $F \subseteq E$ is *independent* in R_G if, for all $\emptyset \neq F' \subseteq F$, the number of vertices covered by F', $v(F')$, satisfies $|F'| \le 2v(F)-3$. It follows from Laman's theorem that a graph G is rigid if and only if R_G has rank $2|V|-3$. Lovász and Yemini determined the rank function of R_G by using the fact that the function which defines independence in R_G is intersecting submodular. In particular they obtained the following characterization of rigid graphs.

Theorem 2. [8] *Let $G = (V, E)$ be a graph. Then G is rigid if and only if for all families $\{H_1, H_2, \ldots, H_t\}$ of subgraphs of G which cover E, we have*

$$\sum_{i=1}^{t}(2|V(H_i)| - 3) \geq 2|V| - 3.$$

This theorem is used in [8] to show that every 6-connected graph is rigid.

In order to describe the characterization of globally rigid frameworks we need some further concepts. A graph $G = (V, E)$ is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. Hendrickson showed in 1992 that the redundant rigidity and 3-connectivity of the graph G are necessary conditions for the global rigidity of any generic framework (G, p).

Theorem 3. [5] *Suppose (G, p) is a generic framework. If (G, p) is globally rigid then either G is a complete graph with at most three vertices, or G is 3-connected and redundantly rigid.*

Hendrickson conjectured that these conditions are also sufficient to imply the global rigidity of a generic framework. An important step in resolving this conjecture is the following result of Connelly which was announced in the early 1990's but its proof is only currently due to appear in print. The result uses the operation of a 1-extension: given graphs G and H we say that G is a 1-*extension* of H if G can be obtained from H by deleting an edge uw and then adding a new vertex v of degree three joined to u, w and some other vertex x of H.

Theorem 4. [3] *Let (G, p) be a generic framework. If G can be obtained from K_4 by a sequence of 1-extensions and edge additions then (G, p) is globally rigid.*

The missing step in proving Hendrickson's conjecture was to show that every 3-connected redundantly rigid graph can be obtained from K_4 by a sequence of edge additions and 1-extensions. This was first verified for 3-connected redundantly rigid graphs which have the minimum number of edges, by Berg and Jordán.

Theorem 5. [2] *Let $G = (V, E)$ be a 3-connected and redundantly rigid graph with $|E| = 2|V| - 2$. Then G can be obtained from K_4 by a sequence of 1-extensions.*

Their proof was extended in [6] to all 3-connected redundantly rigid graphs.

Theorem 6. *Let $G = (V, E)$ be a 3-connected and redundantly rigid graph. Then G can be obtained from K_4 by a sequence of edge additions and 1-extensions.*

Hendrickson's conjecture now follows.

Theorem 7. *Let (G, p) be a generic framework. Then (G, p) is globally rigid if and only if either $G = K_2, K_3$, or G is 3–connected and redundantly rigid.*

We may also deduce the following extension of the above mentioned result of Lovász and Yemini.

Theorem 8. *Let (G, p) be a generic framework. If G is 6-connected then (G, p) is globally rigid.*

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Odd-K_5-free graphs are 4-colourable

Bertrand Guenin

A *colouring* of a graph G is an assignment of colours to the vertices of G. A colouring is *proper* if adjacent vertices are assigned different colours. We say that G is *k-colourable* if there exists a proper colouring of G with k colours. The Four-Colour theorem [1, 3] states that every (loopless) planar graph is 4-colourable.

A graph G contains a graph H as a minor if H can be obtained from G by deleting and contracting edges of G. A graph G is *K_5-free* if it does not contain K_5 as a minor. Wagner [4] proved that K_5-free graphs are essentially planar, i.e. they can be constructed from planar graphs and a special fixed graph by pasting on a vertex, an edge, or a triangle. It is straightforward to see that this structural result, together with the Four-Colour theorem, implies that K_5-free graphs are 4-colourable.

We say that G contains H as an *odd minor* if H can be obtained from G by first deleting edges and then contracting every edge on some cut. A graph G is *odd-K_5-free* if it does not contain K_5 as an odd minor. Clearly if a graph is K_5-free it is odd-K_5-free. However the converse is not true in general as the graph obtained from K_5 by replacing a single edge by two series edges illustrates. Bert Gerards [2] conjectured the following result which is now a theorem:

Theorem 1. *Odd-K_5-free graphs are 4-colourable.*

We say that a graph G is a *minimum counterexample* if G contradicts the theorem but no graph with fewer vertices does. We say that a vertex v of G is *saturated* if for every pair of edges f, g, where f, g are incident to v, there exists a triangle using v which avoids both f and g. Theorem 1 is a consequence of the following two propositions:

Proposition 2. *Let G be a minimum counterexample. Then*

(1) *all vertices of G are saturated,*
(2) *G is 4-connected,*
(3) *G has a K_5 minor.*

Proposition 3. *If a graph G satisfies properties (1),(2), (3) then G contains K_5 as an odd minor.*

Graphs which are K_5-free are 4-colourable thus (3) holds. The proof of (1) is short so we give the main idea next. An (odd) circuit C is said to be *spanned* by a cut $\delta(U)$ if all edges of C except one, are contained in $\delta(U)$. Then (3) is obtained by restricting the result in the next proposition to cuts of the form $\delta(v)$.

Proposition 4. *Let G be a minimum counterexample and let $\delta(U)$ be a non-empty cut. Then there is no pair of edges in $\delta(U)$ which intersect all circuits spanned by $\delta(U)$.*

Proof (sketch): Suppose, for a contradiction, there is a pair of edges f, g which intersect all the circuits spanned by $\delta(U)$. Delete f, g and contract every edge in $\delta(U)$. Observe that the resulting graph is loopless. By minimality we can 4-colour that graph. Uncontract all contracted edges and extend the colouring (if edge uv was contracted to a single vertex, then both u, v are assigned the same colour as that vertex). Now by suitably permuting the colour classes of vertices in U we obtain a proper colouring for the original graph. □

The main challenge in the paper is to show Pr. 3. To indicate the strategy we need to introduce the following definition: A *signed graph* (G, Σ) is a pair which consists of a graph G and a subset of the edges Σ called the signature. We may think of the edges of Σ as having odd length and the edges outside Σ, even length. Two signed graphs (G, Σ) and (G, Γ) are *equivalent* if $\Gamma = \Sigma \triangle \delta(U)$ for some cut $\delta(U)$. We say that the signed graph (G, Σ) contains the signed graph (H, Γ) as a *signed minor*, if we can obtain (H, Γ) from (G, Σ) by a sequence of the following operations: (i) delete an edge (and remove it from the signature), (ii) contract an edge which is not in the signature, (iii) replace the signed graph by an equivalent signed graph. It is easy to verify that: G contains K_5 as an odd minor if and only if (G, EG) contains (K_5, EK_5) as a signed minor. Hence, it suffices to show that if G satisfies properties (1),(2),(3) then (G, EG) contains (K_5, EK_5) as a signed minor. In Figure 1 we list all signed graphs (K_5, Σ) up to equivalence. Consider a minimum counterexample G. Since G contains K_5 as a minor, (G, EG) contains (K_5, Σ) as a signed minor where (K_5, Σ) is equivalent to one of the signed graphs in figure 1. Among all such signed minors (K_5, Σ) we choose one which corresponds to a signed graph which is as far down the list as possible (where the order is given from (a) to (g)). If (K_5, Σ) is equivalent to (K_5, EK_5) then we are done. Otherwise we use the fact that every vertex is saturated (1) and connectivity (2) to find another signed minor (K_5, Σ) which appears further down the list.

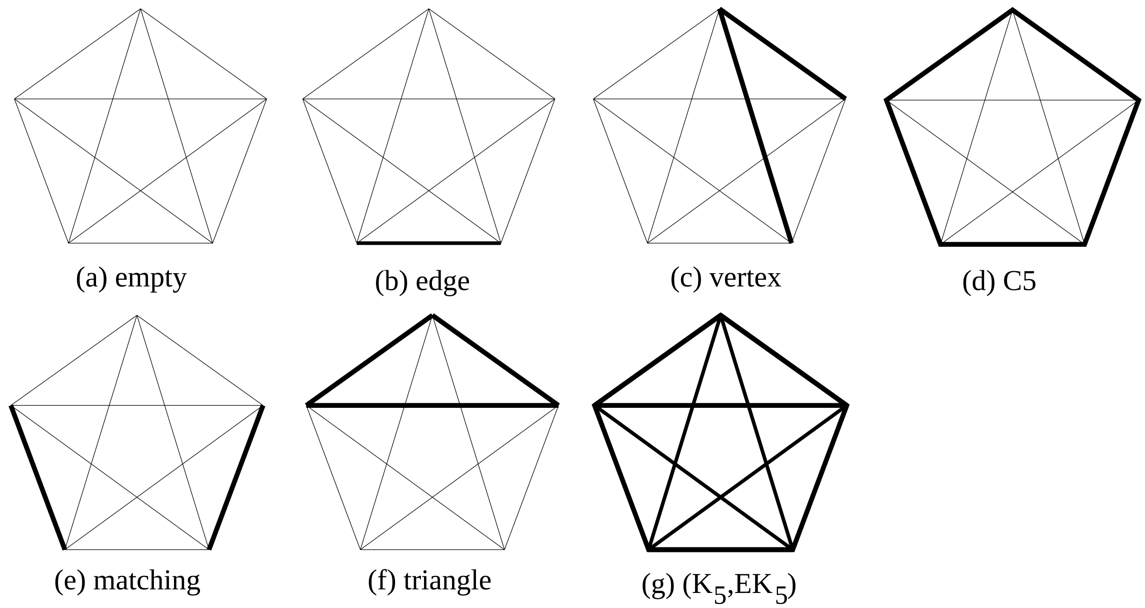

FIGURE 1. All signatures of K_5 (bold edges are in the signature).

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