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Komplexe Algebraische Geometrie

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Introduction by the Organisers

The Workshop "Komplexe algebraische Geometrie" , organized by Fabrizio Catanese (Bayreuth), Yujiro Kawamata (Tokyo), Gang Tian (Princeton), and Eckart Viehweg (Essen), drew together 49 participants.In an age range between the early 20's and the late 50's, there were young PhD students, together with established leaders of the fields which are related to the not too specialized thematic title of the workshop. There were 23 talks, each one lasting 50 minutes but followed by a lively 10-15 minutes discussion, which as usual continued outside the lecture room and throughout the day and the night. The Conference fully fulfilled its purported aim, of setting in contact mathematicians with different specializations and non uniform background, of presenting new fashionable topics alongside with new insights on long standing classical open problems, and also cross-fertilizations with other research topics as arithmetic and physics (for the last, cf. the survey of Jun Li on Gromow Witten invariants). The more detailed contents of the several expositions can be suitably grouped under the following general headings :

(1) Moduli spaces and invariant theory (Brion, Mukai, Verra, Paul, Shepherd-Barron)
(2) Hodge Theory and variations of Hodge structures (Voisin, Ekedahl, Esnault, Möller, Zuo)
(3) Surfaces (Schröer, Schreyer, Pignatelli)

(4) Classification and geometry of higher dimensional varieties (Pirola, Kovács, Reid, Kebekus, Corti)
(5) Complex analytic methods (Voisin, Jun Li, Paul)
(6) Arithmetic questions and motivic integration (Esnault, Ein, Yasuda, Hulek)
(7) Algebraic curves (Verra, Aprodu)

The style of the talks varied considerably, ranging from general overviews like the one by Corti, to "experimental" conjectures, like the ones given by Schreyer, but the variety of striking results and very interesting and challenging proposals is so great that certainly each organizer and each participant had its favourite ones. We believe however that the quality of the expositions in the abstracts collected here is such that we do not deem it necessary to further procrastinate the anticipated pleasure of the reader.

Workshop: Komplexe Algebraische Geometrie

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Abstracts

Moduli of projective spherical varieties

Michel Brion
(joint work with Valery Alexeev)

A spherical variety is a normal complex algebraic variety X equipped with an action of a connected reductive algebraic group G, and containing a dense orbit of a Borel subgroup B of G. In particular, X contains a dense G-orbit; one can show that X contains only finitely many B-orbits, and hence only finitely many G-orbits.

Spherical varieties form a restricted class of varieties with group action, characterized by several finiteness properties. For example, a normal projective G-variety X is spherical if and only if the G-modules $H^0(X, L^n)$ are multiplicity-free for any $n \gg 0$, where L is an ample G-linearized line bundle on X. Examples of spherical varieties include the rational homogeneous manifolds, the toric varieties, and the equivariant compactifications of complex symmetric spaces.

The classification of spherical varieties is a long-standing open question, where many partial results are known. Luna and Vust classified the equivariant embeddings of a fixed spherical homogeneous space, that is, the spherical varieties having a prescribed G-orbit, in terms of "colored fans" that generalize the fans classifying toric varieties (see [7]). Then the spherical homogeneous spaces under groups G of type A were classified by Luna in terms of combinatorial objects called "spherical systems" (see [6]). Recently, this was extended by Bravi and Pezzini to groups of type D, see [5].

All these combinatorial results concern "individual" spherical varieties, and do not address the question of classifying families. In work in progress with Valery Alexeev, we follow an alternative, geometric approach to the classification problem, which is valid for all groups and yields information on families.

Specifically, we introduce the notion of stable spherical varieties, analogous to stable curves and also to Alexeev's stable toric varieties (see [1]). We show the existence of two coarse moduli spaces for projective stable spherical varieties (PSSV's for brevity). The first moduli space parametrizes the PSSV's X equipped with a finite G-equivariant map $f : X \to \mathbb{P}(V)$, where V is a fixed representation of G. The second one parametrizes the pairs (X, D), where X is a PSSV, and D is an ample effective Cartier divisor on X that contains no G-orbit.

In fact, the second moduli space is a special case of the first one, M_V. Moreover, M_V is a projective scheme where the G-equivariant automorphism group of $\mathbb{P}(V)$ acts with finitely many orbits. This generalizes our earlier work concerning reductive varieties, a subclass of spherical varieties that contains all equivariant group embeddings (see [2, 3]). The main tool is a finiteness result from [4], asserting that there are only finitely many spherical subvarieties of $\mathbb{P}(V)$ up to the action of the equivariant automorphism group.

A detailed exposition is available: V. Alexeev, M. Brion: Stable spherical varieties and their moduli, arXiv: math.AG/0505673

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On the homotopy types of compact Kähler manifolds and the Kodaira problem

Claire Voisin

The celebrated Kodaira theorem [1] says that a compact complex manifold is projective if and only if it admits a Kähler form whose cohomology class is integral. This suggests that Kähler geometry is an extension of projective geometry, obtained by relaxing the integrality condition on a Kähler class. This point of view, together with the many restrictive conditions on the topology of Kähler manifolds provided by Hodge theory, would indicate that compact Kähler manifolds and complex projective ones cannot be distinguished by topological invariants. This is supported by the results known for Kähler surfaces, for which a much stronger statement was proved by Kodaira :

Theorem. [2] *A compact Kähler surface is deformation equivalent to a complex projective surface.*

Since Kodaira's result, the situation for higher dimensional manifolds remained unknown. A classical problem, sometimes called the Kodaira problem, asks whether the theorem above still holds in higher dimensions. In [3], we show however the following result, which in particular provides a negative answer to Kodaira's problem:

Theorem. *In any dimension ≥ 4, there exist compact Kähler manifolds which do not have the homotopy type of a complex projective manifold.*

Let us give a simple idea of the starting point of the construction of [5], leading quickly to the construction of a rigid Kähler compact manifold which is not projective, and thus is a counterexample to the Kodaira problem. We consider a complex torus T which admits an endomorphism ϕ_T such that :

1) The pair (T, ϕ_T) is rigid.

2) T is not projective (in fact, the presence of ϕ_T will prevent T being projective).

In order to construct from this a rigid Kähler compact manifold which is not projective, we first consider the product $T \times T$. Inside it, we blow-up the $T \times 0$ and $0 \times T$ which guarantee that we have a product, the diagonal, which guarantees that the two factors in the product are isomorphic, and the graph of ϕ_T which guarantees the presence of the endomorphism ϕ_T, once one knows that our torus is of the form $T \times T$. The resulting Kähler compact manifold X is then rigid, as its only deformations must come from a deformation of $T \times T$ preserving all the structures above, that is to deformations of T preserving ϕ_T.

It turns out that this Kaehler manifold also satisfies the conclusion of our theorem.

The construction above may seem rather artificial, as it uses blowing-ups to make rigid a Kähler manifold which was not rigid at all, and indeed admitted arbitrarily small projective deformations. The following question, which was asked to me by N. Buchdahl, F. Campana, S.-T. Yau, and can be considered as a birational version of the Kodaira problem, is thus quite natural:

Question. *Let X be a compact Kähler manifold. Does there exist a bimeromorphic model X' of X which deforms to a projective complex manifold?*

In the paper [4], we not only show that the answer to the birational Kodaira problem is again no, but also prove the following much stronger topological result:

Theorem. *In any even dimension ≥ 10, there exist compact Kähler manifolds X, such that no compact bimeromorphic model X' of X has the homotopy type of a projective complex manifold.*

The main ingredient of the proof that our manifolds are not homotopy equivalent to projective ones is the observation that, while Hodge theory provides on any compact Kaehler manifolds Hodge structures on cohomology groups which are compatible with the cup-product, it also provides for projective manifolds the supplementary constraint that these Hodge structures should admit (integral) polarizations. If the cohomology ring is complicated enough, it turns out that compatibility with the ring structure prevents the existence of polarizations.

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Maximally degenerate mixed Hodge modules

Torsten Ekedahl

Inspired by the calculations of the degeneration of a variation of Hodge structures occurring in mirror symmetry we have made a systematic study of degenerations of variations of polarised Hodge structures and in particular the action of monodromy on the rational part of the variation.

Recall that if N is a nilpotent endomorphism of a vector space V we can define a unique increasing filtration, the *weight filtration*, $W^N_\bullet$ with the property that $NW^N_i \subseteq W^N_{i-2}$ and the induced map $N^j\colon W^N_j/W^N_{j-1} \to W^N_{-j}/W^N_{-j-1}$ is an isomorphism for all j. Given a finite dimensional local commutative $\mathbf{Q}$-algebra A with maximal ideal $\mathfrak{m}$ and an A-module M we copy the construction of the weight filtration with respect to N to define an increasing filtration, the *weight filtration*, $W_\bullet M$ as follows: We put $W_kM = M$ and $W_{-k-1}M = 0$ if $\mathfrak{m}^{k+1}M = 0$ and let $W_{k-1}M$ be the annihilator of $\mathfrak{m}^k$ and $W_{-k}M = \mathfrak{m}^kM$ if k is the largest integer such that $\mathfrak{m}^kM \neq 0$. We then extend this filtration by the condition that $W_\ell M/W_{-k}M = W_\ell(W_{k-1}M/W_{-k}M)$ for $-k < \ell < k$.

Proposition 1. *The following are equivalent:*

i) There are $N_1, \dots, N_m \in \mathfrak{m}$ forming a basis for $\mathfrak{m}/\mathfrak{m}^2$ such that for each $N = \sum_i a_iN_i$, $0 < a_i \in \mathbf{Q}$, the weight filtration of N is independent of N.

ii) There is an $N \in \mathfrak{m}$ such that $N^j\colon W_jM/W_{j-1}M \to W_{-j}M/W_{-j-1}M$ is an isomorphism for all j.

Under these conditions the weight filtration of an N such as in i) is equal to the weight filtration of M.

We next recall the setup that is obtained (cf., [CK82]) by a degeneration to the origin of a polarised variation of (rational) Hodge structures over $(\Delta^*)^n$, $\Delta^* = \{z \in \mathbf{C} \mid |z| < 1\}$, with unipotent monodromy (where we have made a normalisation of weights to make them centered around weight 0):

- A $\mathbf{Q}$-vector space M with a unipotent action of the group $\mathbf{Z}^n$.
- A weight filtration $W_\bullet$ on M which is the weight filtration associated to any $\sum_i a_iN_i$, $0 > a_i \in \mathbf{Q}$ and $N_i := \log\gamma_i$, γ_i being the i'th generator of $\mathbf{Z}^n$.
- A (decreasing) Hodge filtration $F^\bullet$ of $F_{\mathbf{C}}$ making $(F, W_\bullet, F^\bullet)$ a mixed Hodge structure.

- An alternating or symmetric $\mathbf{Z}^n$-invariant form $\langle -,- \rangle$ on M inducing (with the aid of any $\sum_i a_i N_i$) a polarisation on the primitive part of any Gr_i^W.

We let A be the image of the group ring $\mathbf{Q}[\mathbf{Z}^n]$ in $\mathrm{End}_{\mathbf{Q}}(M)$. By the assumption of unipotence this is a local $\mathbf{Q}$-algebra with the N_i being contained in its maximal ideal.

Proposition 2. *There is a (unique) grading of A for which the N_i have degree* 2.

From the proposition it follows that the $\mathbf{Z}^n$-invariance of $\langle -,- \rangle$ can be reformulated as $\langle au, v \rangle = (-1)^{|a|}\langle u, av \rangle$, where $|a|$ is the degree of $a \in A$.

This leads to the notion of *mixed Hodge module*; a module M over a finite dimensional graded $\mathbf{Q}$-algebra A generated in degree 2 provided with a (symmetric or alternating) form $\langle -,- \rangle$ fulfilling the conditions of Proposition 1 (and $\langle au, v \rangle = (-1)^{|a|}\langle u, av \rangle$ for $a \in A$) and a filtration $F^\bullet$ of $M_{\mathbf{C}}$ for which $A^i F^j \subseteq F^{j-i}$ making $(M, W_\bullet, F^\bullet, \langle -,- \rangle)$ a polarised mixed Hodge structure (with respect to any $N \in A$ fulfilling the conditions of Proposition 1). We get as a consequence that multiplication by $a \in A^i$ gives an endomorphism of M of type $(-i,-i)$.

Remark 3. *In particular we get that there are $N \in \mathfrak{m}_A$ which induce isomorphisms $N^j \colon \mathrm{Gr}_j^W M \to \mathrm{Gr}_{-j}^W M$. In the case when M is a free A-module this means that A is a Gorenstein algebra fulfilling the* strong Lefschetz condition *(see for instance* [HMNW03]*).*

It is useful to consider two other filtrations on M; the *socle filtration*, $\mathrm{soc}_\bullet M$, where $\mathrm{soc}_i M$ is the annihilator of $\mathfrak{m}^i$, and the *cosocle filtration*, $\mathrm{cosoc}^\bullet M$ given by $\mathrm{cosoc}^i M = \mathfrak{m}^i M$. The common length of the socle and cosocle filtrations will be called the *degree of nilpotence of M*. It follows directly from the fact that each A^i acts by endomorphisms of M of type $(-i,-i)$ that the components of the socle and cosocle filtrations are sub-mixed Hodge structures. Note also that they are each other's duals under the pairing $\langle -,- \rangle$. By construction we get injective resp. surjective morphisms of mixed Hodge structures

$$\mathrm{soc}^{i+1} M/\mathrm{soc}^i \hookrightarrow (A^i)^* \bigotimes \mathrm{soc}^1 M, \quad A^i \bigotimes \mathrm{cosoc}^1 M \twoheadrightarrow \mathrm{cosoc}^{i+1} M/\mathrm{cosoc}^i$$

where A^i has been given the mixed Hodge structure purely of weight $(-i,-i)$.

The *weight amplitude* of a mixed Hodge module M is defined to be the interval $[i,j]$, where i is maximal for the property that $W_i M = 0$ and j is minimal for the property that $W_j M = M$. We define the *Hodge amplitude* similarly considering instead the Hodge filtration and define the *weight length* resp. *Hodge length* to be the length of the weight resp. Hodge amplitude intervals. It is a general fact that the degree of nilpotence of M is always $\leq$ the Hodge length. We declare M to be *maximally degenerate* if we have equality and the socle, $\mathrm{soc}\, M := \mathrm{soc}^1 M$, is equal to the least non-zero component of the weight filtration (that component is always contained in the socle).

Theorem 4. *Let M be a maximally degenerate mixed Hodge module over the algebra A.*

i) $\mathrm{Gr}_n^W M = 0$ if n is odd and Gr_{2n}^W is purely of type (n,n).

ii) $M_{\mathbf{C}}$ is generated as $A_{\mathbf{C}}$ module by F^i, where $i-1$ is the left end of the Hodge amplitude and $F^j = A_{\mathbf{C}}^{\leq i-j} F^i$, where $A^{\leq k} := \bigoplus_{i\leq k} A^i$.

iii) The socle, cosocle, and weight filtrations of M coincide.

The situation that seems to be the one appearing in mirror symmetry is the following.

Proposition 5. *Let M be a mixed Hodge module over the algebra A and assume that $\operatorname{soc} M$ is of dimension 1. Then M is maximally degenerate and free of rank 1 as an A-module. In particular A is a Gorenstein ring.*

Remark 6. *There are maximally degenerate Hodge modules over non-Gorenstein algebras.*

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Hodge type over the complex numbers and congruences for the number of rational points over finite fields

Hélène Esnault

We discuss Grothendieck-Deligne's philosophy establishing the correspondence between the two concepts of the title on three examples.

1. Fanos

If X is a complex Fano variety defined over $\mathbb{C}$, then Kodaira vanishing implies $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$. Thus one expects

Theorem 1 (Lang-Manin conjecture in " Varieties over a finite field with trivial Chow group of 0-cycles have a rational point, Invent. math. **151** (2003), 187-191."). *Let X be a Fano variety defined over $\mathbb{F}_q$. Then $|X(\mathbb{F}_q)| \equiv 1$ mod q.*

This is a consequence of

Theorem 2 (loc.cit.). *Let X be an absolutely irreducible, projective variety defined over $\mathbb{F}_q$ with $CH_0(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q(X)}) = \mathbb{Q}$. Then $|X(\mathbb{F}_q)| \equiv 1$ mod q.*

The link between $H^i(X, \mathcal{O}_X) = 0$ over $\mathbb{C}$ and Theorem 1 is motivic cohomology.

2. Unequal characteristic

Theorem 3 (arXiv 2004). *Let V be projective smooth over a local field K with finite rsidue field $\mathbb{F}_q$. Then if the coniveau filtration fufills $N^1H^i(X) = H^i(X)$ for all $i \geq 1$ (here H^i is ℓ-adic cohomology or if K has characteristic 0, it could be de Rham cohomology), the reduction Y mod p of a regular model fulfills $Y(\mathbb{F}_q) \equiv 1$ mod q. In particular if K has char. 0 and V is a surface, the $N^1H^i(X) = H^i(X)$ condition is equivalent to $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, thus this Hodge condition implies $Y(\mathbb{F}_q) \equiv 1$ mod q for Y as above.*

The main technical point generalizes as follows and establishes the link between char. 0 and char. $p > 0$

Theorem 4 (joint with P. Deligne, arXiv 2004). *If X is a scheme of finite type over a local field K, then a lifting $\in \mathrm{Gal}(\bar{K}/K)$ of the geometric Frobenius $\in \mathrm{Gal}(\bar{\mathbb{F}_q}/\mathbb{F}_q)$ acting on $H^i(X), H^i_c(X)$ have eigenvalues in $\bar{\mathbb{Z}}$.*

3. Slopes and theta divisors

One can compute, using "Hodge type of subvarieties of $\mathbb{P}^n$ of small degrees. Math. Ann. **288** (1990), 549 - 551", that if D, D' are two theta divisors on an abelian variety defined ove $\mathbb{C}$, then $gr_0^F H^i(D) = gr_0^F H^i(D')$.

Theorem 5 (joint with P. Berthelot and S. Bloch, Serre conjecture). *Let D, D' be two theta divisors of an abelian variety defined over $\mathbb{F}_q$. Then $D(\mathbb{F}_q) \equiv D'(\mathbb{F}_q)$ mod q.*

Here the link is provided by

Theorem 6 (joint with P. Berthelot and S. Bloch). *If X is a projective variety defined over a finite field, then the slope < 1 part of rigid cohomology $H^i(X/K)$ is computed by $H^i(X, W\mathcal{O}_X)_K$, with $K = \mathrm{Frac}(W(\mathbb{F}_q))$.*

Gromor-Witten Invariants of Toric CY threefolds

Jun Li

In [1], M. Aganagic, A. Klemm, M. Mariño and C. Vafa proposed an algorithm to compute Gromov-Witten invariants in all genera of all toric Calabi-Yau threefolds. One usually computes the Gromov-Witten invariants of toric varieties via virtual localization that reduce these invariants to Hodge integrals of which the later can be determined recursively. However, the algorithm proposed in [1] does not involve Hodge integrals and is significantly more effective. It can be summarized as follows.

1. *There exist certain open Gromov-Witten invariants counting holomorphic maps from bordered Riemann surfaces to $\mathbb{C}^3$ with boundary mapped to three particular Lagrangian submanifolds. The* topological vertex $C_{\vec{\mu}}(\lambda; \mathbf{n})$ *is a generating*

function of such invariants indexed by a triple of partitions $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$ *and a triple of integers* $\mathbf{n} = (n_1, n_2, n_3)$.

2. The Gromov-Witten invariants of any toric Calabi-Yau threefold can be expressed in terms of $C_{\vec{\mu}}(\lambda; \mathbf{n})$ *by explicit gluing algorithms.*

3. By a duality between Chern-Simons theory and Gromov-Witten theory understood via string theory, the topological vertex $C_{\vec{\mu}}(\lambda; \mathbf{n})$ *can be effectively evaluated using Chern-Simons link invariants of three manifolds.*

In the joint paper with C-C. Liu, K. Liu and J. Zhou, we developed a mathematical theory of the topological vertex based on relative Gromov-Witten theory. Our results can be summarized as follows.

A. We introduce the notation of formal toric Calabi-Yau (FTCY) graphs, which is a refinement and generalization of the graph associated to a toric Calabi-Yau threefold. An FTCY graph Γ *determines a relative FTCY threefold* $Y_\Gamma^{\mathrm{rel}} = (\hat{Y}, \hat{D})$, *where* $\hat{Y}$ *is a formal scheme with at most normal crossing singularities,* $\hat{D}$ *is a possibly disconnected smooth divisor in* $\hat{Y}$, *and*

$$\det\left(\Omega_{\hat{Y}}(\log \hat{D})\right) \cong \mathcal{O}_{\hat{Y}}.$$

B. We define formal relative Gromov-Witten invariants *for smooth relative FTCY threefolds. These invariants include the Gromov-Witten invariants of (smooth) toric Calabi-Yau threefolds as special cases.*

C. We show that the formal relative Gromov-Witten invariants so defined satisfy the degeneration formula of relative Gromov-Witten invariants of smooth projective varieties.

D. Any smooth relative FTCY threefold can be degenerated to indecomposable ones whose isomorphism classes are determined by a triple of integers $\mathbf{n} = (n_1, n_2, n_3)$. *By degeneration formula, the formal relative Gromov-Witten invariants in (B) can be expressed in terms of the generating function* $\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n})$ *of formal relative Gromov-Witten invariants of an indecomposable FTCY threefold. This degeneration formula agrees with the gluing algorithms described in (2).*

We expect that the invariants $\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n})$ defined using relative Gromov-Witten invariants coincide with that of the topological vertex of AKMV. This was proved in case when one of the partitions is empty and in all the low degree cases that have been checked.

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Geometric proof of finite generation of certain rings of invariants

Shigeru Mukai

Let $\rho : \mathbf{C}^n \downarrow S = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the standard unipotent linear action of the n-dimensional additive group $\mathbf{C}^n$ on the polynomial ring S of $2n$ variables, that is, $(t_1, \dots, t_n) \in \mathbf{C}^n$ acts by $\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}$ for $1 \le i \le n$. In 1958, Nagata[5] proved that the ring of invariants S^G with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension 3 was not finitely generated for $n = 16$. We studied this example more systematically and obtained the following:

Theorem *The ring of invariants S^G of ρ with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension r is finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} > 1.$$

This inequality is equivalent to the finiteness of the Weyl group of the Dynkin diagram $T_{2,r,n-r}$ with three legs of length 2, r and $n-r$. The 'only if' part of the theorem follows from this observation and the following geometric interpretation of S^G. (See [2] for the details.)

Proposition *Let $\mathbf{P}^{r-1}$ be the projective space $\mathbf{P}_*(\mathbf{C}^n/G)$ and $\{p_1, \dots, p_n\} \subset \mathbf{P}^{r-1}$ be the image of the standard basis of $\mathbf{C}^n$. Then the ring of invariants S^G is isomorphic to the total coordinate ring, or the Cox ring, $TC(X)$ of the blow-up $X = X_G$ of $\mathbf{P}^{r-1}$ at the n points $p_1, \dots, p_n$.*

In this talk, as a continuation of [4], I explained the proof of 'if' part in the case $\dim G = 2$. The variety X in the proposition is the blow-up of $\mathbf{P}^{n-3}$ at n points in general position. The key of our proof is that X is the moduli space of parabolic 2-bundles over an n-pointed projective line $(\mathbf{P}^1 : p_1, \dots, p_n)$ for a certain weight. This fact enables us to determine the effective cone of X, the movable cone $\operatorname{Mov} X$ (see [2]) and its chamber structure. For example, taking

$$\begin{cases} -K_X = (n-2)h - (n-4)\sum_1^i e_i \\ f_1 = h + e_1 - e_2 - \cdots - e_n \\ f_2 = h - e_1 + e_2 - \cdots - e_n \\ \vdots \\ f_n = h - e_1 - e_2 - \cdots + e_n \end{cases}$$

as a basis of $\operatorname{Pic} X \otimes \mathbf{Q}$, a divisor $D \sim -aK_X + \sum_1^n b_i f_i$ is movable if and only if

$$(n-4)a - \sum_{i \in I} b_i + \sum_{j \notin I} b_j \ge 0$$

holds for every $I \subset \{1\dots,n\}$ with $|I|$ even and $|b_i| \leq a$ holds for evey $1 \leq i \leq n$, where h is the pull-back of a hyperplane and $e_1,\dots,e_n$ are the exceptional divisors.

The cone $\mathrm{Mov}\,X$ is divided into finitely many chambers, which are rational polyhedral cones, by the flopping walls. For every movable divisor D on X, the graded ring $\bigoplus_{n\geq 0} H^0(X,nD)$ is finitely generated by the GIT-construction of the moduli spaces. Hence the $(\mathrm{Mov}\,X)$-part of $TC(X)$ is finitely generated. The total coordinate ring $TC(X)$ is finitely generated since it is generated by the equations of 2^{n-1} exceptional divisors, whose linear equivalence classes are

$$e_I = \frac{1}{4}(-K_X - \sum_{i\in I} f_i + \sum_{j\notin I} f_j)$$

with I odd, over the $(\mathrm{Mov}\,X)$-part. In the case of $\dim G = 3$ (and $n \leq 8$), the finite generation is similarly proved replacing the parabolic bundles over $\mathbf{P}^1$ by vector bundles over a del Pezzo surface.

In the case $r = 3$, the minimal (finite) set of generators of $TC(X)$, X itself being a del Pezzo surface, is determined in [1]. Other cases of 'if' part is easy.

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Kummer surfaces attached to the selfproduct of the cuspidal rational curve

Stefan Schröer

Let A be an abelian surface over the complex numbers, and $\iota : A \to A$ the sign involution. The quotient surface $Z = A/\iota$ is a normal surface with 16 rational double points of type A_1, whose minimal resolution S is a K3 surface. One also says that S is a Kummer K3 surface.

It is easy to see that the Kummer construction works in positive characteristics $p \neq 2$ as well. In contrast, Shioda [2] and Katsura [1] observed that the Kummer construction breaks down in characteristic $p = 2$ for supersingular abelian surfaces A. In this case, they showed that singularities on the quotient surface Z are elliptic, and that the minimal resolution X is a rational surface.

The goal of this paper is to give a new type of Kummer construction for the supersingular situation at $p = 2$. To explain this construction, let me discuss the situation where A is *superspecial*, that is, isomorphic to the selfproduct $E \times E$ of supersingular elliptic curves. My idea is to replace the supersingular elliptic curve E by a cuspidal rational curve C, and the group action of $\mathbb{Z}/2\mathbb{Z}$ by a suitable group scheme action of the infinitesimal group scheme α_2. In particular, we start with the nonnormal surface $Y = C \times C$. To be explicit, write the two factors of Y as

$$C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}] \quad \text{and} \quad C = \operatorname{Spec} k[v^2, v^3] \cup \operatorname{Spec} k[v^{-1}].$$

The α_2-action corresponds to the vector field

$$\delta = (u^{-2} + r)D_u + (v^{-2} + s)D_v,$$

which satisfies $\delta^2 = 0$, and depends on two parameters $r, s \in k$. It turns out that the quotient $Z = Y/\alpha_2$ is a normal surface, whose singularities are rational double points of type D_4, B_3, or D_8, provides r, s do not both vanish.

The minimal resolution of singularities $X \to Z$ is a K3-surfaces with Picard number $\rho = 22$. Such K3 surfaces are also called supersingular. The resulting family of normal surfaces $Y = Y_{r,s}$ admits a simultaneous resolution of singularities after a purely inseparable base change. I construct this simultaneous resolution by successively blowing up half fibers.

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On the rational connectivity of the moduli space of curves of low genus

Alessandro Verra

The general question afforded in the talk is the following:

When, for a smooth, irreducible complex curve D with general moduli, there exists an embedding $D \subset S$, where S is a smooth surface and D moves in a non isotrivial linear system $|D|$ of dimension ≥ 1?

As is well known the existence of such an embedding is a very special property for a curve D with general moduli. Indeed let g be the genus of D then such a property just means that the moduli space $\mathcal{M}_g$ is uniruled, which is excluded for $g \geq 23$ by the Theorem of Harris-Mumford-Eisenbud. For lower genus the uniruledness of $\mathcal{M}_g$ is somehow expected. The main reasons are two conjectures: Mumford's conjecture that to have negative Kodaira dimension is equivalent to be uniruled and the slope conjecture of Harris and Morrison, which would imply that $\mathcal{M}_g$ has negative Kodaira dimension for $g \leq 23$.

According to the type of the surface S there are some classical and some more recent answers to the question. If S is rational it is a result of Beniamino Segre

that the answer is no as soon as $g \geq 11$. The only other well known case is when S is a K3 surface: the answer is yes for $g \leq 11$, $g \neq 10$ and no for $g \geq 12$.
For various reasons the next interesting case seems to be the case where S is a regular canonical surface. Already the very few cases where S is a canonical complete intersection can be very well exploited: as described in the talk the following theorem holds.

Theorem 1. *A general curve D of genus $g \leq 15$ admits an embedding in a canonical complete intersection S. Moreover D moves on S in a non isotrivial linear system $\mid D \mid$.*

Using more geometry, it turns out that the above embeddings often put in evidence the linkage of D to another curve C in S, having lower genus and moving in a unirational Hilbert scheme. This remark is the initial step to prove the two new theorems presented in the talk:

Theorem 2. *$\mathcal{M}_g$ is unirational for $g \leq 14$.*

Theorem 3. *$\mathcal{M}_{15}$ is rationally connected.*

Theorem 2 is due to the speaker, theorem 3 to Andrea Bruno and the speaker. Previous results on the unirationality of $\mathcal{M}_g$ were known for $g \leq 13$, (Severi $g \leq 10$, Ran-Chang $g = 11, 13$, Sernesi $g = 12$). The Kodaira dimension was known to be negative also for $g = 15, 16$, (Ran-Chang).
It would be interesting to have some indications on the following question:
for which values of $g \leq 23$ is $\mathcal{M}_g$ rationally connected?

Contact loci in arc spaces

Lawrence Ein
(joint work with Robert Lazarsfeld, Mircea Mustaţă)

Let X be a smooth complex variety. Given $m \geq 0$, we denote by

$$X_m \;=\; \mathrm{Hom}\big(\,\mathrm{Spec}\ \mathbb{C}[t]/(t^{m+1})\,,\,X\,\big)$$

the space of m^{th} order arcs on X. Similarly we define the space of formal arcs on X as

$$X_\infty = \mathrm{Hom}\big(\,\mathrm{Spec}\ \mathbb{C}[[t]]\,,X\,\big)$$

Consider now a non-zero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ defining a subscheme $Y \subseteq X$. Given a finite or infinite arc γ on X, the order of vanishing of $\mathfrak{a}$ — or the order of contact of the corresponding scheme Y — along γ is defined in the natural way. [1] For a fixed integer $p \geq 0$, we define the *contact loci*

$$\mathrm{Cont}^p(Y) \;=\; \mathrm{Cont}^p(\mathfrak{a}) \;=\; \left\{ \gamma \in X_\infty \,|\, \mathrm{ord}_\gamma(\mathfrak{a}) = p \right\}.$$

These are locally closed cylinders, i.e. they arise as the common pull-back of the locally closed sets

$$\text{(1)} \qquad \mathrm{Cont}^p(Y)_m \;=\; \mathrm{Cont}^p(\mathfrak{a})_m \;=_{\mathrm{def}}\; \left\{ \gamma \in X_m \,|\; \mathrm{ord}_\gamma(\mathfrak{a}) = p \right\}$$

defined for any $m \geq p$. Let W be the closure of an irreducible component of $\mathrm{Cont}^p(\mathfrak{a})$. We can naturally associate a valuation of the function field of X to W in the following manner. Let f be a nonzero rational function of X. We define

$$\mathrm{val}_W(f) = \mathrm{ord}_\gamma(f) \;\text{ for a general } \gamma \in W.$$

Such a valuation is called a contact valuation. Suppose $\mu : X' \longrightarrow X$ be a proper birational morphism. Assume that E is an irreducible divisor in X'. We can define the valuation associated to E by $\mathrm{val}_E(f) =$ the vanishing order of f along E. A valuation on the function field of X is called a divisoriral valuation if it is of the form $m \cdot \mathrm{val}_E$ for some positive integer m. A basic invariant of val_E from higher dimensional birational geometry is the discrepancy along E which is defined as

$$k_E = \mathrm{val}_E(det(J(\mu))) \;\text{ where } J(\mu) \text{ is the Jacobian matrix of } \mu.$$

k_E is just the coefficient of the relative canonical divisor $K_{X'/X}$ along E.

Theorem 1. *Every contact valuation is a divisorial valuation. Conversely, every divisorial valuation can be realized uniquely as a contact valuation.*

In the above correspondence, suppose that a contact valuation val_W is equal to a divisorial valuation $m \cdot Val_E$. The following theorem relates the geometry between the two valuations.

Theorem 2. $\mathrm{codim}(W, X_\infty) = m \cdot (k_E + 1)$.

The above two theorems also hold for singular varieties after some minor modifications using Nash's blow-up and Mather's canonical class.

These results can be used to study singularities of pairs. Let X be a normal $\mathbb{Q}$-Gorenstein complex variety and Y be closed subscheme of X. We also fix a positive number λ. One can study the singularities of the pair $(X, \lambda \cdot Y)$ using log-resolution. Consider a divisorial valuation of the form val_E with center $c_X(E)$ in X. We consider the log discrepancy of $(X, \lambda \cdot Y)$ along E,

$$a(E, X, \lambda \cdot Y) = k_E + 1 - \lambda \cdot \mathrm{val}_E(I_Y),$$

where I_Y is the ideal of Y in X. Suppose that B is a closed subset of X. We can measure the singularities of the pair $(X, \lambda \cdot Y)$ along B using the invariant minimal log-discrepancy.

[1] Specifically, pulling $\mathfrak{a}$ back via γ yields an ideal (t^e) in $\mathbb{C}[t]/(t^{m+1})$ or $\mathbb{C}[[t]]$, and one sets

$$\mathrm{ord}_\gamma(\mathfrak{a}) \;=\; \mathrm{ord}_\gamma(Y) \;=\; e.$$

(Take $\mathrm{ord}_\gamma(\mathfrak{a}) = m + 1$ when $\mathfrak{a}$ pulls back to the zero ideal in $\mathbb{C}[t]/(t^{m+1})$ and $\mathrm{ord}_\gamma(\mathfrak{a}) = \infty$ rwhen it pulls back to the zero ideal in $\mathbb{C}[[t]]$.)

Definition 3. *Let $B \subseteq X$ be a nonempty closed subset. The minimal log discrepancy of $(X, \lambda \cdot Y)$ on W is defined by*

$$mld(B; X, \lambda \cdot Y) := \inf_{c_X(E) \subseteq W} \{a(E; X, \lambda \cdot Y)\}. \tag{2}$$

Suppose D is a normal effective Cartier divisor in X and B be a closed subset of D. The following theorem is a joint result with Mustaţă. It allows us to compute the minimal log-discrepancy $mld(B; X, D + \lambda \cdot Y)$ using $mld(B; D, \lambda \cdot Y|_D)$.

Theorem 4. *Let X be a normal, local complete intersection variety, and Y be a proper closed subscheme of X. Let λ is a positive number. Assume $D \subset X$ is a normal effective Cartier divisor such that $D \not\subseteq Y$, then for every proper closed subset $B \subset D$, we have*

$$mld(B; X, D + \lambda \cdot Y) = mld(B; D, \lambda \cdot Y|_D).$$

The theorem is first proved in the case that X is smooth in a joint paper of Ein, Mustaţă and Yasuda. In general, Kollár, and Shokurov have conjectured that the result is true when X is just normal and $\mathbb{Q}$-Gorenstein (Inversion of Adjunction). See Kollár's article for a discussion of this conjecture and related topics. The following are some geometric applications of the Theorem.

Theorem 5. *If X is a normal, local complete intersection variety, and Y is a closed subscheme. Suppose that λ is a positive number then the function $x \longrightarrow mld(x; X, \lambda \cdot Y)$, $x \in X$, is lower semicontinuous.*

Theorem 6. *Let X be a normal, local complete intersection variety. X has log canonical (canonical, terminal) singularities if and only if X_m is equidimensional (respectively irreducible, normal) for every m.*

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Comparing Shimura- and Teichmüller curves

Martin Möller

We give a Hodge-theoretic characterization of Teichmüller curves that exhibits similarities and differences to Shimura curves.

1. Teichmüller curves

The bundle ΩT_g of holomorphic 1-forms over the Teichmüller space has a natural $\mathrm{SL}_2(\mathbb{R})$-action given by postcomposing the charts of the 1-form with the linear map and providing the Riemann surface with the unique complex structure that makes the new charts holomorphic. The orbit of $(X^0, \omega^0) \in \Omega T_g$ projected to Teichmüller space T_g is a holomorphic embedding

$$\tilde{\jmath} : \mathbb{H} \to T_g$$

that is geodesic for the Teichmüller metric. Only rarely these geodesics project to algebraic curves $C = \mathbb{H}/\Gamma$ in the moduli space of curves M_g. These curves are called *Teichmüller curves* and (X^0, ω^0) is called a *Veech surface*.

Examples of Teichmüller curves are Hurwitz spaces for covers of an elliptic curve unramified outside one point. For these examples the trace field $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \Gamma)$ equals $\mathbb{Q}$. A remarkable series of Teichmüller curves in all genera g and with $r := [K : \mathbb{Q}] = g$ was discovered by Veech ([Ve89]). McMullen noticed in [McM03] that the eigenforms for real multiplication over a Hilbert modular surface intersected with M_2 are invariant under the $\mathrm{SL}_2(\mathbb{R})$-action. From that he deduced the existence of infinitely many Teichmüller curves with $r = 2$ in genus two. Although the $\mathrm{SL}_2(\mathbb{R})$-invariance of the locus of eigenforms does not hold in higher genera we will see below that Teichmüller curves always map to some locus of real multiplication in A_g.

2. Shimura curves

One can think of a *Shimura variety* as the moduli space of abelian varieties plus an additional endomorphism structure. Instead of repeating the precise definition (using the notion of Mumford-Tate groups) we propose to compare the variation of Hodge structures (VHS) of Teichmüller curves $C \to M_g \to A_g$ with the following characterization of Shimura curves:

Theorem 1. *(Viehweg, Zuo,* [ViZu04]*) The universal family of abelian varieties over a Shimura curve $g : A \to C$ of Hodge type is characterized, replacing C if necessary by an étale cover, by a decomposition of the polarized VHS as follows:*

$$\mathbb{V}_{\overline{\mathbb{Q}}} = (\mathbb{L} \otimes \mathbb{T}) \;\oplus\; \mathbb{U}$$

Here $\mathbb{T}$ and $\mathbb{U}$ are unitary local systems and $\mathbb{L}$ is defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$ and maximal Higgs.

In the theorem a rank two real VHS $\mathbb{L}$ is said to be *maximal Higgs*, if the Kodaira-Spencer mapping

$$\mathcal{E}^{1,0} \to \mathcal{E}^{0,1} \otimes \Omega^1_{\overline{C}}(\log(\overline{C} \setminus C))$$

on the graded pieces of the Hodge filtration of $\mathbb{L}$ is an isomorphism. This is equivalent to the condition that the image of the representation

$$\pi_1(C) \to \mathrm{SL}_2(\mathbb{R})$$

associated with $\mathbb{L}$ is a lattice.

The local system $\mathbb{U}$ accounts for a fixed abelian subvariety contained in all fibres of the family.

3. VHS of Teichmüller curves

In the same language we can characterize Teichmüller curves ([Mö04a]):

Theorem 2. *Let $f : X \to C$ be the universal family of a finite étale cover of a Teichmüller curve. The polarized weight one VHS on the local system $\mathbb{V} = R^1 f_* \mathbb{Z}$ splits over some number field $K \subset L \subset \mathbb{R}$ as*

$$\mathbb{V}_L = \mathbb{W}_L \oplus \mathbb{M}_L, \quad \text{where} \quad \mathbb{W}_L = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_r,$$

such that each $\mathbb{L}_i$ has rank two, the $\mathbb{L}_i$ are Galois conjugate and such that precisely $\mathbb{L}_1$ is maximal Higgs. Moreover the splitting $\mathbb{V}_L = \mathbb{W}_L \oplus \mathbb{M}_L$ is defined over $\mathbb{Q}$. Conversely if $f : X \to C$ is a family of curves such that the VHS of $\mathbb{V}_{\mathbb{R}} = R^1 f_* \mathbb{R}$ has a rank two subsystem that is maximal Higgs, then $C \to M_g$ is a Teichmüller curve (maybe precomposed by an étale cover).

We note some consequences of this description:

Corollary 3. *The family of Jacobians over a Teichmüller curve has, up to isogeny, a factor of dimension r with real multiplication by K.*

Corollary 4. *The map $C \to M_g$ is rigid, hence a Teichmüller curve is defined over a number field. Moreover the absolute Galois group acts on the set of Teichmüller curves.*

Furthermore we obtain parallel to the case of Shimura curves [Mö04b]:

Theorem 5. *For each finite étale cover of a Teichmüller curve the associated family of Jacobians has a finite Mordell-Weil group.*

This yields a characterization of so-called *periodic points* on Veech surfaces, which is used in [McM04] to obtain a complete classification of Teichmüller curves in genus $g = 2$.

For the VHS-description one deduces that a curve $C \to M_g \to A_g$ is both Shimura and Teichmüller (i.e. is geodesic for both the Teichmüller metric and the Hodge metric on A_g) only if $r = 1$ and $\mathbb{M}$ is unitary. In fact in [Mö05] we show:

Theorem 6. *The only families of curves that are both Shimura and Teichmüller are given, up to étale cover, by*

$$y^N = x(x-1)(x-t)$$

for $N = 2$ (i.e. the moduli space of elliptic curves) and $N = 4$ in genus three.

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On Green and Green-Lazarsfeld conjectures for generic d-gonal curves

Marian Aprodu

Our main object of study is Koszul cohomology of curves. The final goal of the theory is to have a cohomological tool for controlling the geometry of curves in the sense of Brill-Noether theory. We report on some refinements of previous results by Voisin, Teixidor, and Schreyer on the Green conjecture, and of previous joint results with Voisin on the Green-Lazarsfeld conjecture, all regarding generic d-gonal curves.

1. Koszul cohomology

Let X be a complex projective variety. For two integers p and q, and a line bundle L on X, the Koszul cohomology group $K_{p,q}(X, L)$ was defined in [5] as the cohomology of the complex:

$$\wedge^{p+1} H^0(L) \otimes H^0(L^{q-1}) \to \wedge^p H^0(L) \otimes H^0(L^q) \to \wedge^{p-1} H^0(L) \otimes H^0(L^{q+1}).$$

This definition, using multilinear algebra, involves contraction maps. A more geometric description, using spaces of sections in line bundles over Hilbert schemes of points, was given by Voisin in [10]. It is the breaking point which finally lead to a solution to the generic Green conjecture, [10], [11].

In the Appendix to [5], Green and Lazarsfeld proved that if X is smooth, and L, L_1, L_2 are line bundles on X such that $L = L_1 \otimes L_2$, and $r_1 := h^0(L_1) - 1 \geq 1$, $r_2 := h^0(L_2) - 1 \geq 1$, then $K_{r_1+r_2-1,1}(X, L) \neq 0$. In other words, the existence of special linear series on X reflects into the non-vanishing of certain Koszul cohomology groups. The aim of the theory is the investigate cases when this fact is revertible.

2. Green and Green-Lazarsfeld conjectures

The Green-Lazarsfeld result applied for the canonical bundle of a smooth curve X yields to $K_{g-c-2,1}(X,\omega_X) \neq 0$, where g denotes the genus, and c denotes the Clifford index of X. It was conjectured by Green that this is optimal, i.e. $K_{g-c-1,1}(X,\omega_X) = 0$. If true, Green's conjecture would provide an effective way to compute this mysterious invariant which is Clifford index.

One can also apply the Green-Lazarsfeld non-vanishing result to vector bundles of high degree on the curves X, obtaining consequently $K_{h^(L)-d-1,1}(X,L) \neq 0$, for any line bundle L of degree at least $2g+d$, where d denotes the gonality of the curve X. Green and Lazarsfeld conjectured that this is the best one can do, i.e. $K_{h^0(L)-d,1}(X,L) = 0$, after possible augmentation of the degree. At a first sight, it might appear that this conjecture be harder to establish, due to the fact that the line bundle L varies. This is actually a fake problem, as one can reduce oneself to verifying the predicted vanishing for a given line bundle:

Theorem 1 (*see* [1])**.** *If L is a nonspecial line bundle on a smooth curve X, which satisfies $K_{n,1}(X,L) = 0$, for an integer $n \geq 1$, then, for any effective divisor E of degree $e \geq 1$, one has $K_{n+e,1}(X,L+E) = 0$.*

Both Green, and Green-Lazarsfeld conjectures are known to hold for generic curves (for Green's conjecture *see* [10] and [11], and for the Green-Lazarsfeld conjecture we refer to [4] and [2]), with the notable difference that in the odd-genus case, the Green conjecture is known to hold for any curve of maximal gonality, as shown by Hirschowitz, Ramanan and Voisin, *cf.* [7] and [11].

Theorem 2 (Hirschowitz-Ramanan-Voisin)**.** *Any smooth curve X of genus $2k+1$ with $K_{k,1}(X,\omega_X) \neq 0$ carries a pencil of degree $k+1$.*

The Green conjecture is valid also for curves of non-maximal gonality which are generic in their gonality strata, and this happens for all possible gonalities *cf.* [11] and [9], *see* also [8]. It is well-known that by fixing the gonality d we obtain a stratification of the moduli space of curves with irreducible gonality strata, and thus it makes perfect sense to speak about generic d-gonal curves. Using the ideas of [10], [11], we solved jointly with Voisin the Green-Lazarsfeld conjecture for generic d-gonal curves with $d \geq g/3$, *see* [4].

3. Some recent results

The breakthrough realized by Voisin in [10], [11] opened the door for further progress in the theory. We note first that the result of Hirschowitz-Ramanan-Voisin extends to stable curves. More precisely, if Y is a singular stable curve of genus $g = 2k+1$ with very ample canonical bundle, and $K_{k,1}(Y,\omega_Y) \neq 0$ then $[Y]$ is a limit of $(k+1)$-gonal smooth curves, *cf.* [3]. In particular, using the theory of compactified relative Jacobians developed by Caporaso and Pandharipande, it follows that over such a curve Y there exists a torsion-free, ω_Y-semistable sheaf F of rank one with $\chi(F) = 1-k$ and $h^0(F) \geq 2$. This fact is used to show the following.

Theorem 3 (*see* [3]). *Let $d \geq 3$ be an integer, and X be a smooth d-gonal curve with $d < [g_X/2] + 2$, and such that* $\dim(W^1_{d+n}(X)) \leq n$ *for all* $0 \leq n \leq g_X - 2d + 2$. *Then* $\mathrm{Cliff}(X) = d - 2$, *and X verifies both Green, and Green-Lazarsfeld conjectures.*

Theorem 4 (*see* [3]). *The Green-Lazarsfeld conjecture is valid for any smooth curve X of genus $g_X = 2k - 1$ and gonality $k + 1$, with $k \geq 2$.*

The latter theorem yields to an effective version of the Hirschowitz-Ramanan-Voisin Theorem for syzygies of pluricanonical curves.

Corollary 5 (*see* [3]). *If X is a smooth curve of genus $2k - 1$, where $k \geq 2$, with*

$$K_{2(2p-1)(k-1)-(k+1),1}(X, \omega_X^{\otimes p}) \neq 0$$

for some $p \geq 2$, then X carries a g^1_k.

The proof strategy of the two theorems above is the same, namely to construct, starting from X, some suitable singular stable curve Y of genus $g = 2k + 1$ and analyze its Koszul cohomology. For the first theorem, the singular curve which we construct is irreducible, as in [10] and [4], whereas for the second theorem, it has one rational component, as in [2]. If this new singular curve Y has $K_{k,1}(Y, \omega_Y) \neq 0$, then, by what we have said above, we get a particular rank-one torsion-free sheaf F on Y, which will trace furthemore some pencil on the original curve X. An important role in this argument is played by the following Lemma of Voisin's, which can also be seen as the unifying ingredient of the two conjectures for generic d-gonal curves.

Lemma 6 (Voisin, [10], [4]). *Let Y be an irreducible stable curve, and X be its normalization. Then, for any integer $n \geq 1$, and for any two points x and y of X lying over the same node of Y, there are natural inclusions*

$$K_{n,1}(X, \omega_X) \subset K_{n,1}(X, \omega_X + x + y) \subset K_{n,1}(Y, \omega_Y).$$

Finally, we mention that, for large genera, the known cases for which the two conjectures are verified, with the exception of plane curves, have Cliff = gon − 2. As pointed out by Voisin, it would be interesting to have an example of a non-planar curve of large genus with Cliff = gon − 3 for which the Green conjecture be verified.

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Arakelov inequalities over higher dimensional base

Kang Zuo
(joint work with Eckart Viehweg)

Let Y be a complex $n-$dimensional projective manifolds, $S \subset Y$ a reduced normal crossing divisor, $U = Y \setminus S$, and let $f : V \to U$ be a family of abelian varieties. Asssum that the local system $R^1 f_* \mathbb{C}_V =: \mathbb{W}$ has unipotent monodromy along the components of S. The Deligne extension of $(R^1 f_* \mathbb{C}_V) \otimes \mathcal{O}_U$ carries a Hodge filtration. Taking the graded sheaf one obtains the logarithmic Higgs bundle

$$(F^{1,0} \oplus F^{0,1}, \tau).$$

Choose a "good" compactification $\bar{\mathcal{A}}_g^{(N)}$ of the fine moduli scheme $\mathcal{A}_g^{(N)}$ of polarized abelian varities with level N structure such that $\Omega^1_{\bar{\mathcal{A}}_g^{(N)}}(\log S)$ is big and nef. Assum that the induced morphism $\varphi : U \to \mathcal{A}_g^{(N)}$ is finite and can be extended to a finite morphism $\varphi : Y \to \bar{\mathcal{A}}_g^{(N)}$ (which implies that $\omega_Y(S)$ is nef and ample with respect to U). One defines the $\omega_Y(S)-$slope for a torsion free sheaf F on Y

$$\mu(F) = \frac{c_1(F) \cdot c_1(\omega_Y(S))^{n-1}}{\text{rank}(F)}$$

and the $\omega_Y(S)-$discriminant

$$\delta(F) = (2\text{rank}(F) \cdot c_2(F) - (\text{rank}(F) - 1) \cdot c_1(F)^2) \cdot c_1(\omega_Y(S)^{n-2}.$$

Theorem 1 *For $\mathbb{V}$ a $\mathbb{C}$-sub-variation of Hodge structures of $\mathbb{W}$ with Higgs bundle*

$$(E^{1,0} \oplus E^{0,1}, \theta),$$

and without a unitary direct factor one has

$$\mu(E^{1,0}) - (E^{0,1}) \leq \mu(\Omega^1_Y(\log S)).$$

The equality

$$\mu(E^{1,0}) - \mu(E^{0,1}) = \mu(\Omega^1_Y(\log S))$$

implies that $E^{1,0}$ and $E^{0,1}$ are both semi-stable with respect to $\omega_Y(S)$.

$\omega_Y(S)$ is nef and ample with respect to U allows to apply Yau's uniformization theorem based his solution of Kähler Einstein metric on canonically polarized manifolds and his joint work with G. Tian on log-canonically polarized manifolds. It implies that the sheaf $\Omega^1_Y(\log S)$ is poly-stable with respect to $\omega_Y(S)$. Hence one has a direct sum decomposition

$$\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_{s''} \oplus \cdots \oplus \Omega_{s'} \oplus \cdots \oplus \Omega_s$$

in stable sheaves. We choose the indices such that for $i = 1, \ldots, s''$ the sheaf Ω_i is invertible, for $i = s'' + 1, \ldots, s'$ it is of rank > 1 and for all $m > 0$ the sheaves $S^m(\Omega_i)$ remain stable, whereas for $i = s' + 1, \ldots, s$ and for some $m_i > 1$ the sheaves $S^{m_i}(\Omega_i)$ are non-stable.

Theorem 2. *Assume that $s = s'$, i.e. that for each $\mu^{(0)}$ stable direct factor Ω_i of $\Omega^1_Y(\log S)$ the sheaves $S^m(\Omega_i)$ remain μ stable, for all $m > 0$. Moreover assume that for all irreducible $\mathbb{C}$-sub-variations of Hodge structures $\mathbb{V}$ of $\mathbb{W}$ with logarithmic Higgs bundle $(E^{1,0} \oplus E^{0,1}, \theta)$*

$$\mu(E^{1,0}) - \mu(E^{0,1}) = \mu(\Omega^1_Y(\log S))$$

If one of the following conditions hold true

i. $\mathrm{Min}\{\delta(E^{1,0}), \delta(E^{0,1})\} = 0$,
ii. *$s'' = s$, hence Ω_i is invertible for $i = 1, \ldots, s$,*

U is a rigid Shimura subvariety of the moduli stack $\mathcal{A}_g^{(N)}$ and the universal covering $\tilde{U}$ is the product of s complex balls of dimensions $n_j = \mathrm{rk}(\Omega_j)$.

Effective bound of rational maps between varieties of general type

Gian Pietro Pirola
(joint work with Juan Carlos Naranjo)

Let X and Y be two smooth complex projective varieties of general type of dimension n. Let

$$M(X,Y) = \{f : X - - \to Y \mid f \text{ rational dominant map } \}.$$

Let $M_d(X,Y) = \{f \in M(X,Y),\ d = \deg f\}$. Let

$$m(X,Y) = \# M(X,Y)$$

and $m_d(X,Y) = \# M_d(X,Y)$ be their cardinality. Set

$$M(X) = \{f : X - - \to ?\};$$

where ? is any variety of general type (up to birational equivalence) and $m(X) = \# M(X)$.

We have the well known result:

Theorem 1. n=1 de Franchis (1913) [2]**, any** n **Kobayashi Ochiai (1975)** [6]. *The set $M(X,Y)$ is finite, that is $m(X,Y) < \infty$.*

Of the many generalization of the Franchis theorem we recall the classical:

Theorem 2. Severi, de Franchis. *If $n = 1$ then $m(X) < \infty$.*

It is still unknown if this holds for $n > 1$. Partial important results on this conjecture were obtained in [3] and [7]. We have (see [7]):

Conjecture 1. Iitaka, Severi. *For any n, $m(X) < \infty$.*

We are interested in effective bound of $m(X,Y)$ in terms of numerical birational invariant of X and Y. When $n = 1$ effective bounds were obtained by Martens, Horward - Sommese and Kani (see [8], [4] and [5]). In all these works one associates to $f \in M(X,Y)$ the map between the Jacobian varieties, $f^* : J(Y) \to J(X)$ or equivalently the underline morphism of Hodge structures. This gives the injection

$$M(X,Y) \hookrightarrow Hom(H^1(Y,\mathbb{Z}), H^1(X,\mathbb{Z}))$$

defined by $f \to f^*$. Actually in [5] (and [4]) effective bounds for $m(X)$ are given in terms of the genus $g(X)$ of X. The method of Kani gives, roughly, an estimate of the type:

$$m(X,Y) \le m(X) \le c^{4g(X)^2}.$$

Moreover Kani in [5] provided examples showing that $m(X)$ cannot be polynomially bounded. The following conjecture-problem is, in our opinion, the most interesting open problem in the topic.

Conjecture 2. Heier [7] (de Franchis problem). *For $n = 1$ $m(X,Y)$ is polynomially bounded in terms of the genus of X.*

The best bound in curve case was provided recently by Tanabe in [10]; he roughly (for the precise estimate see the theorem below) proved:

$$m(X,Y) \leq c^{2g(X)}.$$

The idea was to associate to $f \in M(X,Y)$ a single element in the lattice $H^1(X,\mathbb{Z})$ and then be able to bound the set of map associated to the same element. Since our work is the higher dimension version of [10], we first describe Tanabe's work in detail. We have two steps.

Step one: Geometric Lemma. Let $\omega \in H^0(\Omega^1_Y)$ $\omega \neq 0$ and $\alpha \in H^0(\Omega^1_X)$. Set

$$L = \{f \in M(X,Y) : f^*(\omega) = \alpha\}.$$

One has

Lemma 3.

$$\#L \leq 4(g(X) - 1).$$

Proof. Sketch: Denote by $Z(\omega)$ and $Z(\alpha)$ the zero loci of ω and respectively of α. For $p \in Z(\omega)$ and $q \in Z(\alpha)$ set $L(p,q) = \{f \in L : f(q) = p\}$, an easy local computation shows that

$$\#L(p,q) \leq n + 1$$

where n is the multiplicity of p as zero of ω. Since $L = \bigcup_{q \in Z(\alpha)} L(p,q)$ the result follows. □

Step two: Minimal period. Let $\gamma \neq 0$ be a minimal period of Y. That is $\gamma \in H^1(Y,\mathbb{Z})$ such that

$$|\gamma| = \min_{\alpha \in H^1(Y,\mathbb{Z}) \setminus \{0\}} |\alpha|$$

where

$$|\alpha|^2 = |\int_Y \alpha^{1,0} \wedge \alpha^{0,1}|.$$

Then take $\omega = \gamma^{1.0}$ and say that two functions f and g of $M(X,Y)$ are equivalent if $f^*(\gamma) = g^*(\gamma)$ in $H^1(X,\mathbb{Z})$, (i.e. $f^*(\omega) = g^*(\omega)$). The geometric lemma gives a linear bound for the cardinality of the equivalences classes. Then

$$f \to f^*(\gamma)$$

defines an injection:

$$M(X,Y)/\sim \hookrightarrow H^1(X,\mathbb{Z}).$$

The use of the minimal period now helps to bound the number of elements of the quotient $M_d(X,Y)/\sim$. Observe that the image belongs to the sphere of radius $\sqrt{d} \cdot |\gamma|$ centered at the origin of the real vector space $H^1(X,\mathbb{R})$ of dimension $2g(X)$. Let $f, g : X \to Y$ be two maps of degree d such that $f^*(\gamma) \neq g^*(\gamma)$. Then one obtains:

$$|f^*(\gamma) - g^*(\gamma)| \geq \frac{1}{\sqrt{d}} |\gamma|.$$

This is deduced since either $\beta_1 = f_*(f^*(\gamma)-g^*(\gamma)) \neq 0$ or $\beta_2 = g_*(f^*(\gamma)-g^*(\gamma)) \neq 0$, f_* and g_* being the Gysin maps. Then one obtains that either $|\beta_1| \geq |\gamma|$ or $|\beta_2| \geq |\gamma|$, and the norm estimate follows easily.

The bound is then obtained either by using Kani packing lemma [5] on the lattice $H^1(X,\mathbb{Z})$, or by the interesting reduction mod. r , $r > 2d$, of [10].

To give the statements of our theorem (see [9]), let us introduce the following function

$$P(a,e) = (a+1)^e - (a-1)^e, \qquad a \in \mathbb{R},\ e \in \mathbb{N}.$$

This is a polynomial on a. Its leading term is $2ea^{e-1}$. Also we denote

$$\rho = \rho(X,Y) = \frac{K_X^n}{K_Y^n}$$

where X, Y are n-dimensional varieties (if $n = 1$, then $\rho = \frac{g(X)-1}{g(Z)-1}$). Let $b_i(X)$ be the Betti number $\dim H^i(X,\mathbb{C})$.

Theorem 4. *Under the previous notations we have*

$n = 1$ **Tanabe method+Kani packing:** $m(X,Y) \leq 4(g(X)-1)\rho P(2\rho, 2g(X))$

$n = 2$ *Assume* $p_g(Y) \geq 2$, X, Y *minimal. Then*

$$m(X,Y) \leq 4(K_X^2)^2\, P(4\sqrt{2}\,\rho, 2b_2(X)-2).$$

$n = 3$ *Assume* K_X, K_Y *nef,* $p_g(Y) \geq 2$ *and the image of* Y *by the bicanonical map has dimension at least* 2*. Then*

$$m(X,Y) \leq 4\cdot 9^2\, h^2\, K_X^3\, P(36\sqrt{2}\,h, 20h+6)\cdot P(4\sqrt{2}\rho, 2b_3(X)),$$

here $h = h^0(X,\mathcal{O}(2K_X)) + h^0(X,\Omega_X^2) - p_g(X) + 1$.

$n \geq 4$ K_X, K_Y *are nef and the image of the canonical map has dimension* $\geq n-1$. *Then:* $m(X,Y) \leq 2n(K_X^n)^2(2\rho+1)^{b_n(Y)\cdot b_n(X)}$.

We remark that we do not need the restrictive hypothesis which guarantees the injectivity of the representation of the elements of $M(X,Y)$ as maps of suitable Hodge structures. This allows to find good bounds under mild hypotheses. We remark that effective bounds obtained by Chow variety method can be found in [1] and [7]. Now we explain the ideas of the proof in surface case. That is X and Y will be minimal surfaces of general type. First we generalize the geometric part of Tanabe to surfaces with p_g at least 2 by using appropriate pencil V of 2-forms on Y. To perform this we use the following elementary but very useful fact: there exists an open set where all the maps are different in all the points. Then, by using the fact that the curves are moving in a pencil, and hence they cut this open set, one can reduce to the one-dimensional case. To explain this in more detail, let us define the following equivalence relation, denoted by $\sim$, on $M(X,Y)$:

$$f \sim g \iff f^*(\omega) = g^*(\omega)\ \forall \omega \in V.$$

Lemma 5. *Let L be an equivalence class of $\sim$; then*

$$\#L \leq 4K_X^2 \cdot K_Y^2.$$

Proof. Sketch: Consider the pencil $|V|$ on Y (and respectively $f^*(V)$ on X), $f \in L$. We take the moving part M and the fixed part F (resp. G, N), and write:

$$|V| = F + |M| \qquad (resp.\ |f^*V| = G + |N|)$$

Note that G contains the ramification divisor of any $f \in L$. Fix a general element in $\omega \in V$, and set $\alpha = f^*(\omega)$. Decompose the zero divisor of ω as $F + C$ and of α as $G + D$. By construction we have

$$f(D) = C$$

for all $f \in L$. Moreover since C is general we have an injection (with an abuse of language)

$$L \hookrightarrow M(D, C),$$

and therefore, by taking the normalization, an injection $L \hookrightarrow M(\tilde{D}, \tilde{C})$. Note that the normalization $\tilde{D}$ and $\tilde{C}$ of D and C are union of smooth disconnected curves of genus greater than one. Next we take $\beta \in V$ independent of ω and construct, by using residues, the holomorphic form δ

$$\delta = Res_{\tilde{C}} \frac{\beta^2}{\omega}$$

on $\tilde{C}$. By construction one has that $f^*(\delta)$ is fixed for all $f \in L$. The result follows from Lemma 3 and a suitable bound on the genus of $\tilde{D}$. □

Finally we represent the map using couples of elements in the transcendental lattice of the source variety. Roughly speaking, the transcendental lattice is the complementary of the Neron-Severi group in the second cohomology group of the surface. The geometric part allows us to estimate the number of maps which are represented by the same couple of elements of the lattice.

Observe that, since we are not assuming that X, Y are canonical, the representation of the maps in $M(X, Y)$ as maps of transcendental Hodge structures is not injective in general.

Apparently the "inductive procedure" does not extend to higher dimensions due to the method and to the lack of a smooth minimal model in higher dimension. We need then some additional hypotheses which allow us to give the bound in higher dimensional case following a similar argument.

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An experimental approach to numerical Godeaux surfaces

Frank-Olaf Schreyer

A (numerical) Godeaux surface is a minimal surface X of general type with $K^2 = 1$ and $p_g = 0$, hence also $q = 0$ and $H_1(X, \mathbb{Q}) = 0$. So in some sense these are the surfaces of general type with smallest possible invariants. Godeaux constructed a family of such surfaces as quotients of a quintic hypersurface by fix point free action of $\mathbb{Z}_5$. Despite this fact a complete classification of these surface is open. The following is known by the work of Miyaoka

(1) The torsion group $T = H^1(X, \mathbb{Z})$ is a cyclic group of order at most 5.

(2) The pencil $|2K|$ has no fixed components on the canonical model

$$X' = Proj R_X \text{ with } R_X = \sum_{n \geq 0} H^0(X, \mathcal{O}(nK)).$$

(3) The 3-canonical system $|3K|$ has presisely b fix points, where

$$b = |\{t \in T \mid t \neq -t\}|/2.$$

(4) If $T = \mathbb{Z}_5$ then X is the quotient of a quintic hypersurface by T. In particular surface with $T = \mathbb{Z}_5$ form an irreducible 8-dimensional unirational family.

Reid classified surface with $T = \mathbb{Z}_4$ or $\mathbb{Z}_3$. Again in both cases there is one 8-dimensional irreducible family.
For $T = \mathbb{Z}_2$ or $T = 0$ much less is known. Existence of such surface was proved by Rebecca Barlow, by a complicated quotient construction.

Traditionally there are two approaches to construct numerical Godeaux surfaces: Either by a Godeaux approach as quotient of a simpler surface by a possibly non free group action, or by a Campedelli approach as a double plane branched along a curve with a specific configuration of singularities.

In this talk I present a third approach based on homological algebra following a suggestion of Miles Reid.

The basic idea is to study X via its image defined by the system $|2K, 3K|$ in the weighted projective space $P = \mathbb{P}(2, 2, 3, 3, 3, 3)$. Let $S = k[x_0, x_1, y_0, \ldots, y_3]$ be the homogeneous coordinate ring of P. We study R_X as an S-module. A Hilbert function computation shows that R_X has expected syzygies of type

$$\begin{array}{ccccccccccc} & & & S & & S(-6)^6 & & S(-9)^8 & & S(-12)^3 & \\ & & & \oplus & & \oplus & & \oplus & & \oplus & \\ 0 & \leftarrow R_X & \leftarrow & S(-4)^4 & \leftarrow & S(-7)^{12} & \leftarrow & S(-10)^{12} & \leftarrow & S(-13)^4 & \leftarrow 0 \\ & & & \oplus & & \oplus & & \oplus & & \oplus & \\ & & & S(-5)^3 & & S(-8)^8 & & S(-11)^6 & & S(-17) & \end{array}$$

Modulo x_0, x_1 this complex decomposes into the sum of the resolutions of

$$\overline{R}_i = \sum_{n \equiv i \, mod \, 3} (R_X/(x_0, x_1))_n$$

for $i = 0, 1, 2$. $\overline{R}_0$ is the resolution of a finite scheme of length 4, which we can choose the coordinate points in $\mathbb{P}^3$, if it consist of 4 distinct points. In any case there are finitely many choices up to projectivities for these complexes. The original complex can be obtained from this one by a deformation argument, that is choosing appropriate entries depending on x_0 and x_1. The main result so far is the following.

Consider the intersection $Q \subset \mathbb{P}^{11}$ of the quadrics defined by the pfaffians of the following matrices

$$\begin{pmatrix} 0 & p_3 & p_9 & p_6 \\ -p_3 & 0 & p_8 & p_{11} \\ -p_9 & -p_8 & 0 & p_5 \\ -p_6 & -p_{11} & -p_5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p_1 & p_4 & p_7 \\ -p_1 & 0 & p_8 & p_5 \\ -p_4 & -p_8 & 0 & p_2 \\ -p_7 & -p_5 & -p_2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & p_0 & p_6 & p_9 \\ -p_0 & 0 & p_{10} & p_7 \\ -p_6 & -p_{10} & 0 & p_2 \\ -p_9 & -p_7 & -p_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p_3 & p_0 & p_{10} \\ -p_3 & 0 & p_{11} & p_1 \\ -p_0 & -p_{11} & 0 & p_4 \\ -p_{10} & -p_1 & -p_4 & 0 \end{pmatrix}.$$

There is an open subvariety V of the Fano variety $grassQ$ of lines in Q and a $\mathbb{P}^3$-bundle $E \to V$ such that an open subvariety $U \subset E$ dominates an 8-dimensional component of the moduli space of numerical Godeaux surfaces with $T = 0$.

I do not know whether this component is unirational, however it is close to unirationality in the following sense:

Let $\mathbb{F}_q$ be a finite field with q elements. There exists a Las Vegas algorithm which produces random numerical Godeaux surface in our component defined over $\mathbb{F}_q$ with $O(q^2)$ field operations. (An algorithm with $O(1)$ field operations gives a proof of unirationality).

We plan to investigate this family and families lying over the complement of V or families with other base locus for $|2K|$ with the help of Computer algebra and probabilistic algorithm.

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Holomorphic one-forms on varieties of general type

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(joint work with Christopher D. Hacon)

The impact of zeros of vector fields on the geometry of the underlying variety has been studied extensively. For instance, it is known that the existence of a nowhere zero vector field on a compact complex manifold implies that all of its characteristic numbers vanish [CL77].

Carrell asked whether something similar is implied by the existence of a nowhere vanishing holomorphic one form. He proved that this is the case for surfaces, namely if S is a compact complex surface admitting a nowhere vanishing holomorphic one form, then $c_1(S)^2$ and $c_2(S)$ are zero [Car74]. On the other hand, he also gave an example of a threefold X, a $\mathbb{P}^1$-bundle over an abelian surface, for which $c_1(X)^3 \neq 0$. This suggests that one needs to treat varieties with negative Kodaira dimension differently. Therefore we restrict to the case of varieties of non-negative Kodaira dimension.

Having to pay attention to the Kodaira dimension of the variety makes it natural to approach the problem from the point of view of classification theory and first restrict to the case of *minimal* varieties. For a minimal variety X, K_X is nef, therefore $c_1(X)^{\dim X} \neq 0$ is equivalent to $K_X^{\dim X} > 0$ which is equivalent to X being of general type.

These considerations naturally lead to the following conjecture.

Conjecture 1. *Let X be a smooth projective variety of general type. Then X does not admit a nowhere vanishing holomorphic one form.*

Once we focus on varieties of general type, restricting to the case of minimal varieties is the right thing to do according to a conjecture of Carrell:

Conjecture 2 (Carrell)**.** *Let X be a smooth projective variety of general type. If X admits a nowhere vanishing holomorphic one form, then X is minimal.*

Remark 1. *This is known for surfaces and using the classification of extremal contractions one can easily see that it also holds for threefolds. This was explicitly checked in* [LZ03, Lemma 2.1]*.*

Conjecture 1 has been confirmed for canonically polarized varieties (i.e., whose canonical divisor is ample) in [Zha97] and for threefolds in [LZ03]. An immediate consequence of this conjecture is that a variety of general type does not admit any smooth morphisms onto an abelian variety.

We prove Conjecture 1 for smooth minimal varieties and for varieties whose Albanese variety is simple. We use completely different methods to deal with the different cases.

Theorem 2. *Let X be a smooth projective variety of general type. Assume that either*

(2.1) *X is minimal, or*
(2.2) *the Albanese variety of X is simple.*

Then X does not admit a nowhere vanishing holomorphic one form.

This completely confirms Conjecture 1 assuming Conjecture 2. Using [LZ03, Lemma 2.1] this also gives a new proof of the threefold case [LZ03, Theorem 1].

Next we review the idea of the proof of in the case when X is minimal (2.1).

Let us first consider the case when ω_X is ample. We want to prove that X does not admit a nowhere vanishing global holomorphic one-form. We prove this via a vanishing theorem. That is, we prove that the existence of a nowhere vanishing global holomorphic one-form on X implies that the top cohomology group of any ample line bundle on X vanishes. This easily implies that ω_X cannot be ample.

In the general case we need to prove that the existence of a nowhere vanishing global holomorphic one-form on X implies that that ω_X cannot be nef and big. However, instead of extending our vanishing theorem to cover that case, we extend it to the singular case. Namely we prove the following:

Theorem 3. *Let Y be a projective variety with only rational singularities of dimension n, and let $\phi : X \to Y$ be a resolution of singularities of Y. Let $\phi^{\#}\Omega_Y = \mathrm{Im}[\phi^*\Omega_Y \to \Omega_X]$. Assume that there exists a $\vartheta \in H^0(X, \phi^{\#}\Omega)$ such that the zero locus of ϑ is empty. Then for any ample line bundle $\mathcal{L}$ on Y, $H^n(Y, \mathcal{L}) = 0$.*

In order to prove this theorem we use the technical machinery developed in [DB81], [GNPP88] and [Kov04]. To apply the result we set Y to be the canonical model of X, prove that in that case $\phi^{\#}\Omega_Y = \Omega_X$, and argue as above.

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Motivic integration over Deligne-Mumford stacks

Takehiko Yasuda

The motivic integration, after Kontsevich [4], Denef-Loeser [2] etc, is useful to compare various invariants of birational varieties. This is basically due to the change of variables formula: Let $f : Y \to X$ be a proper birational morphism of varieties with Y smooth, $f_\infty : J_\infty Y \to J_\infty X$ the natural morphism of arc spaces, $\mathcal{I}_f \subset \mathcal{O}_Y$ the Jacobian ideal sheaf of f and F a "function" (that is, a map with values in some ring) on $J_\infty X$ with some condition. Then

$$\int F \, d\mu_X = \int (F \circ f_\infty) \mathbf{L}^{-\operatorname{ord} \mathcal{I}_f} d\mu_Y.$$

To generalize this to Deligne-Mumford (DM for short) stacks, the notion of *twisted arc* was introduced in [6]. Let $\mathcal{X}$ be a DM stack over $\mathbf{C}$ (supposed to be separated and of finite type). Then a twisted arc on $\mathcal{X}$ is a representable morphism

$$[\operatorname{Spec} \mathbf{C}[[t]]/\mu_l] \to \mathcal{X}$$

for some positive integer l. Here $\mu_l \subset \mathbf{C}$ is the cyclic group of l-th roots of unity, which naturally acts on $\operatorname{Spec} \mathbf{C}[[t]]$ and $[\operatorname{Spec} \mathbf{C}[[t]]/\mu_l]$ is the quotient stack associated to this action. There is a DM stack $\mathcal{J}_\infty \mathcal{X}$ (not of finite type) whose $\mathbf{C}$-point (that is, a morphism $\operatorname{Spec} \mathbf{C} \to \mathcal{J}_\infty \mathcal{X}$) corresponds to a twisted arc on $\mathcal{X}$. We define $|\mathcal{J}_\infty \mathcal{X}| := \mathcal{J}_\infty \mathcal{X}/isom$, which is the set of the isomorphism classes of twisted arcs. Note that stacks constitute a 2-category and that there are morphisms between morphisms of stacks. When X is a variety, then $|\mathcal{J}_\infty X| = J_\infty X(\mathbf{C})$. We can define the motivic measure $\mu_{\mathcal{X}}$ on $|\mathcal{J}_\infty X|$ as well as on the arc space of a variety, and integrals of functions with respect to the measure.

A morphism $f : \mathcal{Y} \to \mathcal{X}$ of DM stacks is said to be *birational* if there are open dense substacks $\mathcal{Y}_0 \subset \mathcal{Y}$ and $\mathcal{X}_0 \subset \mathcal{X}$ which are isomorphic by f. For example, if a finite group G effectively acts on a variety M, then the natural morphism from the quotient stack $[M/G]$ to the quotient variety M/G is birational. If $f : \mathcal{Y} \to \mathcal{X}$ is a proper birational morphism of DM stacks, then the natural map $f_\infty : |\mathcal{J}_\infty \mathcal{Y}| \to |\mathcal{J}_\infty \mathcal{X}|$ is almost bijective. (Since f is not necessarily representable, f_∞ is not generally the composition of morphisms. See [7].) We can now generalize the change of variables formula as follows.

Theorem 1 ([7]). *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper birational morphism of DM stacks and $\mathcal{I}_f \subset \mathcal{O}_{\mathcal{Y}}$ the Jacobian ideal sheaf of f. Suppose that $\mathcal{Y}$ is smooth and that $\mathcal{X}$ is either a smooth stack or a singular variety. Let F be a function on $|\mathcal{J}_\infty \mathcal{X}|$ with some condition. Then*

$$\int F\mathbf{L}^{s_{\mathcal{X}}}\, d\mu_{\mathcal{X}} = \int (F \circ f_\infty)\mathbf{L}^{-\mathrm{ord}\, \mathcal{I}_f + s_{\mathcal{Y}}}\, d\mu_{\mathcal{X}}.$$

Here when $\mathcal{X}$ is a smooth stack, for a twisted arc γ on $\mathcal{X}$, $s_{\mathcal{X}}(\gamma)$ is a rational number determined by the associated μ_l-action on the tangent space of $\mathcal{X}$ at the image of the closed point by γ. When X is a variety, then $s_X \equiv 0$. Note that $\mathrm{ord}\, \mathcal{I}_f$ *is now generally* $\mathbf{Q}$*-valued unlike in the variety case.*

Batyrev's previous results in [1] correspond to the local and much less general case of this theorem. His method was rather computational. Denef and Loeser [3] gave an alternative approach with full use of the motivic integration. The theorem above is a reformulation of their results in a full generality in terms of the birational geometry of DM stacks.

An application of this is on the discrepancy of quotient singularities. Let $G \subset GL_d(\mathbf{C})$ be a finite subgroup without reflection. For $g \in G$ of order l, choosing suitable basis, write $g = \mathrm{diag}(\zeta_l^{a_1}, \dots, \zeta_l^{a_d})$, $0 \le a_i \le l-1$, where $\zeta_l := \exp(2\pi\sqrt{-1}/l)$. Then the *age* of g, denoted $\mathrm{age}(g)$, is defined to be $\sum_{i=1}^d a_i/l \in \mathbf{Q}$.

Theorem 2. *The discrepancy (also called minimal discrepancy) of $\mathbf{C}^d/G$ is equal to $\min\{\mathrm{age}(g) | 1 \ne g \in G\} - 1$.*

This is a refinement of the Reid–Shepherd-Barron-Tai criterion for terminal or canonical quotient singularities (for example, see [5].) Maybe this is already known, but implicit in the literature as far as I know.

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Diptych varieties and $\mathbf{C}^*$ covers of Mori flips

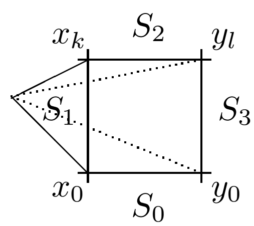
Miles Reid
(joint work with Gavin Brown)

"Diptych" is a technical term in art, meaning a picture made up of two panels joined along a hinge ("triptych" is more common). The diptych varieties we construct are affine Gorenstein 6-folds V_{ABLM}, that serve as key varieties for the $\mathbf{C}^*$ covers of Mori flips of type A. Each V_{ABLM} comes with a $(\mathbf{C}^*)^4$-action, and a regular sequence A, B, L, M of $(\mathbf{C}^*)^4$-eigenfunctions so that the two panels

$$V_{AB} : (L = M = 0) \quad \text{and} \quad V_{LM} : (A = B = 0)$$

are toric varieties. The V_{ABLM} are schemes over $\mathbf{Z}$, not themselves toric, but entirely governed by toric data. They form an infinite discrete series with 3 natural number indexes.

Each 6-fold V_{ABLM} is obtained by studying $(\mathbf{C}^*)^4$-equivariant deformations of a *closed tent*, a reducible surface $T = S_0 \cup S_1 \cup S_2 \cup S_3$ consisting of 4 toric surfaces glued together as a cycle along their coordinate axes as in the Figure.



with $S_0 \cong \mathbf{C}^2 \cong S_2$, and
$S_1 = \frac{1}{r}(1, a)$, $S_3 = \frac{1}{s}(1, b)$.

In more detail, S_1 is the cyclic quotient singularity $\frac{1}{r}(1, a)$; thus $S_1 \subset \mathbf{C}^{k+1}$ with coordinates $x_0, \ldots, x_k$ given by the continued fraction expansion $[a_1, \ldots, a_{k-1}] = \frac{r}{r-a}$, and $S_3 \subset \mathbf{C}^{l+1}$ with coordinates $y_0, \ldots, y_l$ given by $[b_1, \ldots, b_{l-1}] = \frac{s}{s-b}$. Then $T \subset \mathbf{C}^{k+l+2}$ consists of S_1, S_3 together with $S_0 = \mathbf{C}^2(x_0, y_0)$ and $S_2 = \mathbf{C}^2(x_k, y_l)$.

Our Theorem 1 determines the $(\mathbf{C}^*)^4$-equivariant deformations of T to a toric 4-fold V_{AB} that smooth the top two axes (the x_k- and y_l-axes in the figure), in terms of matrixes in $\mathrm{SL}(2, \mathbf{Z})$, and (equivalently) in terms of continued fraction expansions of 0. This is pure toric geometry, and basically elementary.

Our Theorem 2 is more entertaining: if a tent T admits two $(\mathbf{C}^*)^4$-equivariant deformations $T \subset V_{AB}$ and $T \subset V_{LM}$, one smoothing the top two axes (the x_k- and y_l-axes), and the other smoothing the bottom two axes (the x_0 and y_0-axis), then, without any further assumptions, these fit together into a 4-parameter deformation

$$\begin{array}{ccc} T & \subset & V_{AB} \\ \cap & & \cap \\ V_{LM} & \subset & V_{ABLM}. \end{array}$$

The V_{ABLM} is our new construction.

Our Theorem 3 classifies the data of Theorem 2 in terms of combinatorics; the clearest detailed treatment is in terms of 4 families, one of which depends on three natural number indexes. The other families depend on only two indexes, or they can be viewed as specialisations of the big family with indexes allowed to take small values.

The proof of Theorem 2 is based on the combinatorics of continued fraction expansions of 0 and the algebra of serial unprojection. These give rise to a beautiful calculus of 5×5 Pfaffians coming from a series of pentagrams.

Background. The current construction of the varieties V_{ABLM} makes sense without any reference to Mori flips or to higher-dimensional geometry. However, our main motivation in developing them was the application to Mori flips. This application also explains the level of generality in which we work.

A Mori flip is a diagram $X^- \to X \leftarrow X^+$ where $X^\pm$ are 3-folds with the singularities of the Mori category, and projective over X. We are only concerned with the germ of X^- around the exceptional curve C^-, so we can assume that X^- is a tubular neighbourhood of C^- and X is a Stein or formal neighbourhood of $P = f(C^-) \in X$. We assume that C^- is a single irreducible curve isomorphic to $\mathbf{P}^1$. The flip is of type A if the general elephant $S \in |-K_X|$ is a Du Val singularity of type A. This is a strong assumption; it follows at once (by an argument of Mori) that also the general hyperplane section $P \in H \subset X$ is a cyclic quotient singularity.

We believe that the $\mathbf{C}^*$ cover of every Mori flip of type A is obtained as a regular pullback of one of our diptych varieties. Mori flips of type A provide the following motivation for our construction: the local 3-fold X contains two surfaces, the elephant $S \in |-K_X|$ and the hyperplane section $H \in |m_{X,P}|$, and both are cyclic quotient singularities. On forming the $\mathbf{C}^*$ cover of X, that is,

$$V = \operatorname{Spec} \bigoplus_{i \in \mathbf{Z}} \mathcal{O}_X(iK_X),$$

we obtain an affine Gorenstein 4-fold, and the $\mathbf{C}^*$ covers of S and H are now both hyperplane sections of V, and both are toric varieties. In addition, it is elementary that deformations of the germ X^- are a priori unobstructed (because X^- has hyperquotient singularities and the fibre C^- is one dimensional, there is no room for any T^2). Thus it is reasonable to look for a key variety containing V as a variety having two toric sections.

Algebraic and Analytic K-Stability of Polarized Manifolds

Sean Timothy Paul
(joint work with Gang Tian)

Let (M^n, ω) be a Kähler Manifold. This talk revolved around the following basic question:

Under what circumstances does ω admit a metric of constant scalar curvature?

To be more specific, define $P(M, \omega) := \{\varphi \in C^\infty(M) : \omega + \partial\overline{\partial}\varphi > 0\}$. This is the usual description of all Kähler metrics in the same class as ω. Then the problem is:

$$\textit{Find } \varphi \in P(M,\omega) \textit{ such that } \operatorname{Scal}(\omega + \partial\overline{\partial}\varphi) \equiv \mu \ (*)$$

Here μ is a constant, the average of the scalar curvature, it depends only on $C_1(M)$ and $[\omega]$. $(*)$ is actually a *Variational* problem: there is a natural energy ν_ω on the space $P(M, \omega)$ whose critical points are those φ such that $\omega + \partial\overline{\partial}\varphi$ has constant scalar curvature (csc). This energy was introduced by T. Mabuchi ([6]) in the 1980's. It is called the ***K-Energy Map*** (denoted by ν_ω) and is given by the following formula

$$\nu_\omega(\varphi) := -\frac{1}{V}\int_0^1 \int_X \dot{\varphi}_t(\operatorname{Scal}(\varphi_t) - \mu)\omega_t^n dt.$$

Above, φ_t is a smooth path in $P(M, \omega)$ joining 0 with φ. The K-Energy does not depend on the path chosen. Of course, $(*)$ is an *infinite dimensional* variational problem. In fact it is a *minimization* problem. A fundamental idea is that if the Kähler class is *integral* then we may approximate this problem by an infinite sequence of *finite dimensional* problems. Integrality of ω means that there is a line bundle L over M such that $\omega = C_1(L)$. So L is a polarization on M, therefore M is an algebraic manifold. We can now reformulate our question

When does the polarized manifold (M, L) admit a csc metric in the class $C_1(L)$?

When $L = \lambda K_M$ (in particular $\lambda = -1$ so M is Fano) a csc metric is the same as a Kähler Einstien Metric. The following conjecture was inspired by Yau.

Conjecture (Tian, Donaldson [12],[5]) Assume that $Aut(M, L)$ is discrete. Then *a csc metric exists in the class $C_1(L)$ if and only if (M, L) is asymptotically K-Stable.*

The definition of (asymptotic) K-Stability appears below. To state the next result, let us first define the notion of properness. This is a slight modification of the concept introduced by Tian in [12].

Definition
*ν_ω is **proper** if there exists a strictly increasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (i.e. $\lim_{T\longrightarrow\infty} f(T) = \infty$) such that $\nu_\omega(\varphi) \geq f(||\varphi||_{C^0})$ for all $\varphi \in P(M, \omega)$.* $||\varphi||_{C^0}$ denotes the sup norm of φ over M.

Properness of the K-Energy map should be thought of as an infinite dimensional (analytic) kind of stability. This fits perfectly into the standard formal picture

relating moment maps to Geometric Invariant theory.
Theorem 1 (Paul, Tian [9])

*If the K-energy is **proper** then (M, L) is asymptotically K-Stable. Asymptotic K-Semistability follows if the K-energy is bounded from below.*

Theorem 2 (Tian [12])

Let $L = -K_M$. Assume that M has no holomorphic vector fields. Then M admits a Kähler Einstein metric if and only if the K-Energy is proper.

Theorem 3 (Chen, Tian)

Let ω be an arbitrary (not necessarily integral) Kähler metric. If there is a csc metric in $[\omega]$ then the K-Energy is bounded from below.

We should point out that in the 80's Bando and Mabuchi gave a beautiful proof of Theorem 3 when the class $[\omega] = C_1(M)$ see [1].

K-stability is related to the finite dimensional reduction alluded to in the previous paragraph. Let us now explain this reduction. Since L is positive we may embed M into a large projective space by a basis $\{S_0, S_1, \ldots, S_{N_k}\}$ of $H^0(M, L^k)$ $(k >> 0)$. This embedding induces a map

$$i_{\{S_j\}} : SL(N_k + 1, \mathbf{C}) \longrightarrow P(M, \omega)$$

defined by pulling back ω_{FS} by an automorphism σ of $\mathbf{C}P^{N_k}$. If $\sigma^*(\omega_{FS}) = \omega_{FS} + \partial\overline{\partial}\varphi_\sigma$ then we define $i_{\{S_j\}}(\sigma) := \frac{1}{k}\varphi_\sigma$. These special metrics are called the *Bergman metrics at level k. The finite dimensional reduction takes place when we restrict ν_ω to $SL(N_k + 1, \mathbf{C})$ via the map $i_{\{S_j\}}$* . This restiction is motivated by a host of *density* results concerning these metrics (see [13], [11],[14],[2]). One of the main technical issues is to let $\sigma = \lambda_t$, an algebraic one parameter subgroup, and analyze the small t asymptotics $\lim_{t\longrightarrow 0} \nu_\omega(\lambda_t)$. It is when we restrict the energies to the subspaces defined by the Bergman metrics that we make contact with Geometric Invariant Theory. The first steps in this direction were taken in [3].

Let $Hilb_m(M)$ denote the mth Hilbert point of X in $\mathbf{C}P^{N_k}$ see [7]. Let λ be an algebraic one parameter subgroup of $SL(N_k + 1, \mathbf{C})$. Then the weight $w_\lambda(\mathrm{Hilb}_m(X))$ of the action of λ on this point is given by a numerical polynomial of degree at most $n + 1$ (see [Mum])

$$w_\lambda(\mathrm{Hilb}_m(X)) = a_{n+1}(\lambda)m^{n+1} + a_n(\lambda)m^n + O(m^{n-1}).$$

Let $P(m) = h^0(M, \mathcal{O}_{\mathbb{P}^{N_k}}(m))$ be the Hilbert polynomial. Following Donaldson, define $F_1(\lambda)$ to be the coefficient of $\frac{1}{m}$ in the expansion

$$\frac{w_\lambda(Hilb_m(X))}{mP(m)} = F_0(\lambda) + F_1(\lambda)\frac{1}{m} + O(\frac{1}{m^2})$$

Now we can make the

Definition (Tian, Donaldson [12], [5])
*Let V be projective algebraic scheme, embedded in $\mathbb{P}^N$. Then V is **K-Stable** if $F_1(\lambda) < 0$ for every one parameter subgroup λ of $SL(N+1, \mathbb{C})$. Semistability allows $\leq$ in the previous inequality . A polarized scheme (V, L) is **Asymptotically K-Stable** if $i_{L^k}(V) \hookrightarrow \mathbf{C}P^{N_k}$ is K-Stable for all sufficiently large k, where i_{L^k} denotes a Kodaira embedding.*

What Tian and I actually prove is the following
Let $X \hookrightarrow \mathbb{P}^N$. There is a function $\Psi_X : SL(N+1, \mathbb{C}) \to \mathbb{R}$ depending on the embedding of X where $-\infty \leq \Psi_X \leq C$ such that

$$d\nu_\omega(\varphi_{\lambda(t)}) - \Psi_X(\lambda(t)) = 4dF_1(\lambda)\log(t) + O(1) \quad (*)$$

The main problem now is to deduce the properness of the K-Energy from the assumption of asymptotic K-Stability. This problem looks very difficult. For partial results in the case of Toric surfaces please consult [5].

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Modularity of non-rigid Calabi-Yau varieties

Klaus Hulek
(joint work with Helena Verrill)

The proof by Andrew Wiles of Fermat's Last Theorem was achieved by proving the Taniyama-Shimura-Weil conjecture which says that every elliptic curve defined over $\mathbb{Q}$ is modular. It is natural to consider also the modularity of varieties in higher dimension. For this let $X \subset \mathbb{P}^n_{\mathbb{Z}}$ be a smooth variety defined over the integers and let

$$\bar{X}_p = X(\mathbb{F}_p) \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p.$$

Assume that p is a prime of good reduction. The (geometric) Frobenius defines a morphism

$$\mathrm{Fr}_p : \bar{X}_p \to \bar{X}_p$$

which in turn induces automorphisms of the étale cohomology groups

$$\mathrm{Fr}^*_{i,p} : H^i_{\text{ét}}(\bar{X}_p, \mathbb{Q}_l) \to H^i_{\text{ét}}(\bar{X}_p, \mathbb{Q}_l)$$

where l is a prime different from p. Let

$$P_{i,p}(t) = \det(1 - t\mathrm{Fr}^*_{i,p}).$$

By Deligne's proof of the Riemann hypothesis

$$P_{i,p}(t) = \prod_j (1 - \alpha_{ij} t) \in \mathbb{Z}[t]; \quad |\alpha_{ij}| = p^{i/2}.$$

If n is the (complex) dimension of X then we shall consider the middle cohomology of X and define its L-series by

$$L(H^n_{\text{ét}}(X), s) \overset{\circ}{=} \prod_p \frac{1}{P_{n,p}(p^{-s})}$$

where $\overset{\circ}{=}$ means equality up to the finitely many factors for the primes of bad reduction.

Question: When is X modular, i.e. when is the above L-function the L-function of an automorphic form?

By the Langlands programme one expects the answer to be positive in general.

A natural class of varieties for which one can study this problem are Calabi-Yau varieties which can be seen as higher dimensional analogues of elliptic curves. The case of dimension 2, i.e. $K3$ surfaces, was considered by Shioda-Inose [SI] and Livné [L] who proved modularity results for $K3$ surfaces with maximal Picard number 20. In some sense the situation is easier for 3-dimensional Calabi-Yau varieties the most basic case being that of rigid Calabi-Yau varieties. For X to be *rigid* means that $h^{12}(X) = h^{21}(X) = 0$, i.e. that X has no complex deformations. In this case the middle cohomology is particularly simple since

$$H^3(X) = H^{30} \oplus H^{03} \cong \mathbb{C}^2.$$

The expectation is that the L-series of a rigid Calabi-Yau variety equals that of a weight 4 elliptic modular form (Fontaine-Mazur conjecture, see also Yui [Y]).

Indeed this conjecture has been proven by Dieulefait and Manoharmayum under relatively mild assumptions on the primes of bad reduction [DM].

Various authors have also found non-rigid modular Calabi-Yau threefolds [LY], [HV1], [Sch]. These examples are of two types: the L-function is of the form

$$L(X,s) \overset{\circ}{=} L(g_3 \otimes g_2, s) \prod_j L(g_2^j, s-1) \text{ or } L(X,s) \overset{\circ}{=} L(f_4, s) \prod_j L(g_2^j, s-1)$$

where g_2, g_2^j, g_3 and f_4 are modular forms of weight $2, 3$ and 4 respectively.

In this talk I discussed both types of examples. The second type can be treated in a systematic way. For this we introduce the following concept.

Definition Let X be smooth projective and define $b := h^{12}(X)$. Let $Y_j, j = 1, \dots, b$ be birational elliptic ruled surfaces contained in X. We say that these surfaces *span* $H^{21}(X) \oplus H^{12}(X)$ if the natural map

$$H^3(X, \mathbb{C}) \to \oplus_{j=1}^b H^3(Y_j, \mathbb{C})$$

is surjective.

One can then prove

Proposition Let X be a smooth projective threefold defined over $\mathbb{Q}$ with $h^{30}(X) = h^{03}(X) = 1$ and assume that X contains birational ruled surfaces $Y_j, j = 1, \dots, b$ which are defined over $\mathbb{Q}$ and which span $H^{21}(X) \oplus H^{12}(X)$. Let ρ be the 2-dimensional Galois representation given by the kernel U from the exact sequence

$$0 \to U \to H^3_{\text{ét}}(X, \mathbb{Q}_l) \to \oplus_j H^3_{\text{ét}}(Y_j, \mathbb{Q}_l) \to 0.$$

Assume that one of the following conditions holds:

(i) X has good reduction at 3 and 7 or
(ii) X has good reduction at 5 and some prime $p \equiv \pm 2 \bmod 5$ and the trace of Fr_p on U is not divisible by 5 or
(iii) X has good reduction at 3 and the trace of Fr_3 on U is not divisible by 3.

Then X is modular. More precisely

$$L(X,s)) \overset{\circ}{=} L(f_4, s) \prod_{j=1}^{b} L(g_2^j, s-1) \tag{1}$$

where f_4 is a weight 4 form and the g_2^j are the weight 2 forms associated to the base curves E_j of the birational ruled surfaces Y_j.

The idea of the proof is quite simple: one checks that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$- representation U shares the same good properties as that of a rigig Calabi-Yau threefold and thus reduces the result to [DM], resp. [D]. For details see [HV2].

One can use this approach to construct large lists of fibre products of rational elliptic fibrations for which one can thus prove modularity.

The other type of examples occcured in the work of Livné and Yui [LY]. They studied semi-stable $K3$-fibrations for which the iterated Kodaira-Spencer map is 0. By the Arakelov inequality as proved by Sun, Tan and Zuo [STZ] such a fibration must have at least 6 singular fibres. If equality holds then the base curve is modular in the sense that the regular part $\mathbb{P}^1 \setminus S$, where S is the critical locus of the fibration, is of the form $\mathbb{H}/\Gamma$ with Γ some arithmetic group acting on the upper half plane $\mathbb{H}$.

The examples of Livné and Yui are of the form $X = (S(\Gamma) \times E)/\iota$ where E is an elliptic curve and $\Gamma = \Gamma(4), \Gamma_0(3) \cap \Gamma(2), \Gamma_0(8) \cap \Gamma(2)$ or $\Gamma_1(8, 4, 12)$. The universal elliptic curve $S(\Gamma)$ is in these cases an extremal $K3$ surface and $\iota(x, y) = (-x, -y)$.

In our paper [HV3] we showed that one can prove modularity also for a number of other examples of a similar type. (The associated $K3$-fibrations, however, are no longer semi-stable.) The case we treated in detail was the variety $X = (S(\Gamma_1(7)) \times E)/\iota$. The main difference to the Livné-Yui examples lies in the fact that the curve of non-zero 2-torsion points no longer decomposes into three sections but is an irreducible elliptic curve B. This elliptic curve also appears in the motive of the Kummer varitey X making the situation slightly more complex than in the Livné-Yui examples. For details see [HV3].

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Fibrations of low genus and surfaces with $q = p_g = 1$

Roberto Pignatelli
(joint work with Fabrizio Catanese)

Let X be a projective surface, and let $f : X \to B$ be a *fibration*, *i.e.* a surjective morphism with connected fibres (of genus g) onto a smooth curve B of genus b. We may assume without loss of generality the fibration to be relatively minimal, contracting every rational curve with self-intersection (-1) contained in a fibre.

In [6] we announced classifications theorems for *low genus fibrations*, *i.e.* for $g = 2$ and 3; more precisely in the case $g = 3$ we consider only the case when the general fibre is not hyperelliptic and we moreover need to assume that every fibre is $2-$connected (excluding, *e.g.*, fibrations with double fibres).

1. Genus 2 fibrations

The canonical map of a genus 2 curve is a double cover of $\mathbb{P}^1$. Given a genus 2 fibration f one can *glue* the canonical map of the fibres to a rational map from X to a $\mathbb{P}^1$-bundle over B, more precisely $\mathbb{P}(V_1)$ where $V_1 = f_*\omega_{X|B}$ with $\omega_{X|B} := \omega_X \otimes f^*\omega_B^{-1}$.

This map allows to construct X as double cover of a ruled surface (as done by many authors, see, *e.g.* [7], [8]). One first constructs the ruled surface $\mathbb{P}(V_1)$, and then has to find a divisor on it, the branch curve of the double cover. This double cover is called *relative canonical map*. The difficulty of this construction is that one has to find a curve in a suitable linear system *with prescribed singularities*.

The problem comes from the *special* fibres, fibres that can be decomposed as $E_1 + E_2$ with $E_1E_2 = 1$. Let us consider by sake of simplicity only the easier example: two elliptic curves intersecting transversally in a point. The relative canonical map of this reducible curve blows up the intersection point (sending the exceptional divisor isomorphically to the corresponding fibre of the ruled surface) and contracts the two elliptic curves to two points.

The branch curve on $\mathbb{P}(V_1)$ is a relative sextic, *i.e.* a curve intersecting a general fibre in six points. On the special fibres corresponding to the above example, three of these points converge to the image of each elliptic curve. Therefore the branch curve contains the fibre and has two quadruple points on it.

Our approach is slightly different: we use the bicanonical map of the fibres. It induces a morphism of degree 2 from X onto a relative conic (a singular birational model of $\mathbb{P}(V_1)$) contained in the $\mathbb{P}^2$-bundle $\mathbb{P}(V_2)$ over B, where $V_2 = f_*\omega_{X|B}^{\otimes 2}$. The advantage is that the branch curve has no essential singularities.

Definition 1. *We define the* **associated 5-tuple** (B, V_1, τ, ξ, w) *of a genus* 2 *fibration* f *where:*

- B *is a curve;*
- V_1 *is a rank* 2 *vector bundle;*
- τ *is an effective divisor on* B*;*
- $\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$*;*

- *$w \in \mathbb{P}(H^0(B,\mathcal{F}))$, for a suitable vector bundle $\mathcal{F}$ on B (depending on ξ).*

We refer for a precise definition of this 5−tuple to [6]. We just recall that B is the base curve, $V_1 := f_*\omega_{X|B}$, τ is the set of the points corresponding to the *special* fibres, ξ is the extension corresponding to the exact sequence

$$0 \to S^2(V_1) \overset{\sigma_2}{\to} V_2 \to \mathcal{O}_\tau \to 0,$$

induced by the natural multiplication map σ_2: from this 4 data we can reconstruct the conic bundle. The last datum, w, depending from a vector bundle $\mathcal{F}$ obtained by ξ with a procedure we do not repeat here (see [6]), gives the branch curve.

Definition 2. *We will say that a a 5−tuple (B, V_1, τ, ξ, w) is* **admissible** *if*

- *B is a smooth curve;*
- *V_1 is a vector bundle on B of rank 2;*
- *$\tau \in Div^+(B)$;*
- *$\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ yields a vector bundle V_2;*
- *$w \in \mathbb{P}(H^0(B,\mathcal{F}))$, where $\mathcal{F}$ is obtained by ξ following the above mentioned procedure;*

and if moreover they satisfy some open conditions[1] ensuring that the associated double cover has only Rational Double Points as singularities.

Theorem 3. *The associated 5−tuple of a genus 2 fibration is admissible. Viceversa, every admissible 5−tuple is associated to a genus 2 fibration $f : X \to B$, with invariants $\chi(\mathcal{O}_X) = \deg(V_1) + (b-1), K^2 = 2\deg V_1 + \deg\tau + 8(b-1)$. Two genus 2 fibration having the same associated 5−tuple are isomorphic.*

2. Surfaces with $q = p_g = 1$

Let S be a minimal surface of general type with $q = p_g = 1$; in this case $2 \leq K_S^2 \leq 9$, and the Albanese map is a morphism $f : S \to B$ where B is a smooth elliptic curve. In fact, for $K_S^2 = 2$ it was proved in [2] that the Albanese is a genus 2 fibration, S is a double cover of $B^{(2)}$, the second symmetric power of B, and the moduli space is generically smooth, unirational of dimension 7.

Definition 4. *We will denote by $\mathcal{M}$ the family, in the moduli space of the minimal surfaces of general type, corresponding to the surfaces S with $p_g(S) = q(S) = 1$, $K_S^2 = 3$.*

$\mathcal{M}$ is studied in [3], [4]. In [3] it is proved that for this class of surfaces the Albanese is a genus g fibration with $g = 2$ or 3. The second case is completely classified in [4], where it is shown that it gives a generically smooth, unirational connected component of $\mathcal{M}$ of dimension 5.

We are left with the case $g = 2$, where we can use theorem 3. In [3] was shown the existence of this case, and conjectured that this family of surfaces should form an unirational component of the moduli space (so $\mathcal{M}$ would have two unirational connected components). By use of theorem 3 we have disproved this conjecture giving a complete description of $\mathcal{M}$ as follows:

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[1] We do not specify here the open conditions by lack of space.

Theorem 5. *$\mathcal{M}$ has 4 connected components, all unirational of dimension 5.*

We use the classification of vector bundles over elliptic curves given in [1]. In [3] is proved that the vector bundle V_1 is undecomposable, and then one can assume (up to translations) without loss of generality $V_1 = E_{[0]}(2,1)$ (in Atiyah's notation). By a result of Clemens ([5]) every irreducible component of this moduli space has dimension at least 5. We have one parameter for B, one for τ ($\deg \tau = 1$ so τ is a point of B) and one can easily compute that ξ varies in a 2-parameters space. For general choice of the above data, the resulting vector bundle $\mathcal{F}$ has $h^0(\mathcal{F}) = 2$, and therefore its projective space gives one further parameter for w: this gives the *main stream* component, unirational of dimension 5.

To understand if there are other components, one need a case-by-case analysis, since for special choice of (τ, ξ), $h^0(\mathcal{F})$ grows. Most cases give *strata* of dimension smaller than 5. There is one exception, if ξ is such that V_2 decomposes as sum of three line bundles (finitely many choices for ξ).

In this case, for general choice of τ, $h^0(\mathcal{F}) = 4$ (so 2 parameters for the pair (B, τ) and 3 for w) but the resulting linear system of branch curves has a fixed part not reduced: this case does not fulfil our open conditions. If we assume τ special ($\tau = [0]$ or τ 2$-$torsion), $h^0(\mathcal{F}) = 5$. This very special case gives two more unirational families with 5 parameters (1 for B and 4 for w).

We can show that all the three families exist and do not intersect pairwise.

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Geometry of chains of minimal rational curves

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(joint work with Jun-Muk Hwang)

Minimal degree rational curves play an essential role in the study of the geometry of Fano manifolds, see e.g. [2], [4] and [5]. It is, however, frequently important to consider not just single curves, but connected chains of them. Among others, these were used in the proof of the boundedness of the degrees of Fano manifolds of Picard number 1 in [6]. At present, not much is known about the geometry of chains of curves. A basic problem in this direction is the computation of the dimension d_k of the locus of the family of length-k chains of minimal rational curves passing through a fixed, general point of the Fano manifold. This information can then for instance be used to bound the multiplicities of divisors at a general point of the variety. As an example, we give a bound on the multiplicities of divisors on the moduli space of stable rank-2 bundles on a curve.

It turns out, however, that it is not easy to compute the dimensions d_k by a direct method even for concrete examples, such as rational homogeneous manifolds. The goal of this note is therefore to develop an *infinitesimal* approach to this problem, as introduced in [3]. Our main results, give a lower bound on the d_k, which often becomes effective. We illustrate this by computing d_k for several examples.

In addition to the intrinsic interest of computing these dimensions, which are natural geometric invariants of Fano manifolds, our argument shows a connection of our problem with the classical problem on the dimensions of higher secant varieties of a projective variety, cf. [1], [7]. This connection is provided by the theory of varieties of minimal rational tangents which is briefly recalled.

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Flips after Shokurov: an introduction

Alessio Corti

(joint work with Florin Ambro, Osamu Fujino, James M^c^Kernan, H. Takagi)

This is an introduction to the recent work of Shokurov [2] on flips, following the forthcoming book [1]. The main purpuse is to state Shokurov's finite generation conjecture and briefly indicate how the 2-dimensional case of the conjecture implies existence of 3-fold flips. Shokurov's construction of 4-fold flips is also an elaborate reduction to the 2-dimensional finite generation conjecture but I do not discuss this.

A *pl flipping contraction* is a flipping contraction $f\colon X \to Z$ for the divisor $K+S+B$, such that S is f-negative. The flip of a pl flipping contraction is called a *pl flip*.

It is known that if pl flips exist and terminate (this is ok in dimension 3 and 4), then all klt flips exist. I focus on pl flips.

A *model* of a variety X is a proper birational morphism $f\colon Y \to X$ from a (normal) variety Y. A *b-divisor* on X is an element:

$$\mathbf{D} \in \mathbf{Div} X = \lim_{Y \to X} \operatorname{Div} Y$$

where the (projective) limit is taken over all models $f\colon Y \to X$ under the push forward homomorphism $f_*\colon \operatorname{Div} Y \to \operatorname{Div} X$. A b-divisor $\mathbf{D}$ on X has a *trace* $\mathbf{D}_Y \in \operatorname{Div} Y$ on every model $Y \to X$.

A b-divisor on X gives rise to a sheaf $\mathcal{O}_X(\mathbf{D})$ of $\mathcal{O}_X$-modules in a familiar way; if $U \subset X$ is a Zariski open subset, then

$$\mathcal{O}_X(\mathbf{D})(U) = \{\varphi \in k(X) \mid \mathbf{D}_{|U} + \mathbf{div}_U \, \varphi \geq 0\}$$

The *Cartier closure* of a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor D on X is the b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^*(D)$$

on models $f\colon Y \to X$.

If $f\colon Y \to X$ is a model and D is a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor on Y, we abuse notation slightly and think of $\overline{D}$ as a b-divisor on X. Indeed f_* identifies b-divisors on Y with b-divisors on X.

Let $\mathbf{D}$ be a b-$\mathbb{Q}$-Cartier b-divisor on X and $S \subset X$ an irreducible normal subvariety of codimension 1 not contained in the support of $\mathbf{D}_X$. I define the *restriction* $\mathbf{D}^0 = \operatorname{res}_S \mathbf{D}$ of $\mathbf{D}$ to S as follows. Pick a model $f\colon Y \to X$ such that $\mathbf{D} = \overline{\mathbf{D}_Y}$; let $S' \subset Y$ be the proper transform. I define

$$\operatorname{res}_S \mathbf{D} = \overline{\mathbf{D}_{Y|S'}}.$$

In other words, $\operatorname{res}_S \mathbf{D}$ is the Cartier closure of the (ordinary) restriction $\mathbf{D}_{Y|S'}$. (Strictly speaking, $\overline{\mathbf{D}_{Y|S'}}$ is a b-divisor on S'; as already noted, b-divisors on S' are canonically identified with b-divisors on S via push forward.)

An integral b-divisor $\mathbf{M}$ is *mobile* if there is a model $f\colon Y \to X$, such that

(1) the linear system (of ordinary divisors) $|\mathbf{M}_Y|$ is free on Y, and

(2) $\mathbf{M} = \overline{\mathbf{M}_Y}$ is the Cartier closure of $\mathbf{M}_Y$.

A sequence $\mathbf{M}_\bullet$ of mobile b-divisors on X is *positive sub-additive* if $\mathbf{M}_1 > 0$ and

$$\mathbf{M}_{i+j} \geq \mathbf{M}_i + \mathbf{M}_j.$$

The associated *characteristic sequence* is the sequence $\mathbf{D}_i = (1/i)\mathbf{M}_i$ of b-$\mathbb{Q}$-Cartier b-divisors. We say that the characteristic sequence is *bounded* if there is a (ordinary) $\mathbb{Q}$-Cartier divisor D on X such that all $\mathbf{D}_i \leq \overline{D}$.

A *pbd-algebra* is the graded algebra $R = R(X, \mathbf{D}_\bullet) = \oplus H^0(X, i\mathbf{D}_i)$ where $\mathbf{D}_i = (1/i)\mathbf{M}_i$ is a bounded characteristic sequence of a positive sub-additive sequence $\mathbf{M}_\bullet$ of b-divisors.

Consider a pl flipping contraction $f\colon X \to Z$ for the divisor $K + S + B$. Let $r > 0$ be a positive integer and $D \sim r(K + S + B)$ a Cartier divisor on X; it is well known that the flip of f exists if and only if the algebra

$$R = R(X, D) = \oplus_{i \geq 0} H^0(X, iD)$$

is finitely generated (in fact in that case the flip is the Proj of this algebra). We show that R is a pbd-algebra.

Let D be a Cartier divisor on X. The *mobile b-part of* D is the divisor $\mathbf{Mob}\, D$ with trace

$$(\mathbf{Mob}\, D)_Y = \operatorname{Mob} f^* D$$

on models $f\colon Y \to X$, where $\operatorname{Mob} f^*D$ is the mobile part of the divisor f^*D (the part of D which moves in the linear system $|f^*D|$).

Choose as above a Cartier divisor $D \sim r(K+S+B)$. Denote by $\mathbf{M}_i = \mathbf{Mob}\, iD$ the mobile part and $\mathbf{D}_i = (1/i)\mathbf{M}_i$; then tautologically

$$R = R(X, D) = R(X, \mathbf{D}_\bullet)$$

is a pbd-algebra. Now I come to the punchline. As I explained above, provided that S is not contained in the support of D (which is easily arranged) it makes sense to form the restriction $\mathbf{D}_i^0 = \operatorname{res}_S \mathbf{D}_i$; we consider the associated pbd-algebra on S:

$$R(S, \mathbf{D}_\bullet^0).$$

It is easy to see, though not trivial, that $R(X, \mathbf{D}_\bullet)$ is finitely generated if and only if $R(S, \mathbf{D}_\bullet^0)$ is finitely generated.

The key issue is to state a condition on the system $\mathbf{D}_\bullet^0$ that, under suitable conditions, ensures that the pbd-algebra $R(S, \mathbf{D}_\bullet^0)$ is finitely generated.

If (X, B) is a pair of a variety X and a divisor $B \subset X$, we denote by $\mathbf{A} = \mathbf{A}(X, B)$ the b-divisor whose trace on a model $f\colon Y \to X$ is the discrepancy

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y.$$

(this is defined provided that $K_X + B$ is $\mathbb{Q}$-Cartier). Note that the pair (X, B) has klt singularities if and only if the round up $\lceil \mathbf{A}(X, B) \rceil$ is an effective b-divisor exceptional over X. The key notion is the following:

Let (X, B) be a pair of a variety X and divisor $B \subset X$. A system $\mathbf{D}_\bullet$ of b-divisors on X is *asymptotically canonically saturated* if for all i, j, there is a model $Y(i,j) \to X$ such that

$$\mathrm{Mob}\lceil (j\mathbf{D}_i + \mathbf{A})_Y \rceil \leq j\mathbf{D}_{j\,Y}$$

on all models $Y \to Y(i,j)$.

Shokurov's finite generation Conjecture: Let (X, B) be a klt pair, $f\colon X \to Z$ a birational contraction to an affine variety Z. Assume that $K + S + B$ is anti-ample on the fibres of f. If $\mathbf{D}_\bullet$ is canonically asymptotically saturated positive convex bounded characteristic system of b-divisors on X, then the pbd-algebra $R(X, \mathbf{D}_\bullet)$ is finitely generated.

It is not difficult to show that the finite generation conjecture holds in dimension 2. This implies existence of 3-fold klt flips.

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Another canonical compactification of A_g

N.I. Shepherd-Barron

The moduli space $A_{g,\mathbb{Z}}$ of principally polarized abelian g-folds has various well known compactifications. For example, there is the Satake (or minimal) compactification A_g^S, on which the bundle M of weight 1 modular forms is ample, while among the toroidal ones there is the (second) Voronoi compactification A_g^{Vor}, which, according to work of Alexeev and Nakamura, has an interpretation in terms of moduli. This talk concerned the first Voronoi compactification A_g^F, also toroidal. Details and references are on the arXiv at math.AG/0502362.

Theorem 1. *Suppose that the base is a field* k.
(1) The boundary $A_g^F - A_g$ *is an absolutely irreducible divisor* D.
(2)A_g^F *has* $\mathbb{Q}$*-factorial singularities.*
(3) $Pic(A_g^F) \otimes \mathbb{Q}$ *is generated by* M *and* D. *They are linearly independent if* $g \geq 2$.
(4) $12M - D$ *is nef, and* $aM - D$ *is ample if and only if* $a > 12$.

Assume now that $g \geq 2$.

(5) The cone of curves $\overline{NE}(A_g^F)$ *is generated by rays* $\mathbb{R}_+[C_1]$ *and* $\mathbb{R}_+[C_2]$. *Here,* C_2 *is any curve in* D *contracted by* $A_g^F \to A_g^S$ *and* $C_1 = \mathbb{P}^1 \times \{B\}$, *where* $\mathbb{P}^1$ *is the* j*-line and* B *is a fixed abelian* $(g-1)$*-fold.*

Now suppose that k *has characteristic zero.*

(6) A_g^F *has canonical (or terminal) singularities if* $g \geq 5$ *(or* $g \geq 6$*).*

(7) If $g \geq 11$, then A_g has a canonical model, in the sense of Mori and Reid; it is A_g^F if $g \geq 12$ and arises as the contraction of the ray $\mathbb{R}_+[C_1]$ if $g = 11$. A_{11}^F is a minimal model.

(1)-(3) are easy consequences of well known results. (4) and (5) are proved together. As for (6), the result for the interior A_g is due to Tai but since then has not been stated explicitly for A_g^F. (7) follows at once.

In characteristic zero, the successive efforts of Tai, Freitag and Mumford have shown that A_g is of general type for $g \geq 7$, while Grushevsky and Lehavi have recently announced that A_6 is too. Clemens and Donagi proved A_4 and A_5 to be unirational, Katsylo has shown that A_3 is rational and the rationality of A_2 is due to Igusa.

By the main theorem on toroidal compactifications A_g^F is determined by decomposing, in an admissible way, the cone C_g of semi-positive real symmetric bilinear forms, with rational radical, in g variables. To get A_g^F take the convex hull of $C_g \cap (L - \{0\})$, where L is the lattice of integral symmetric bilinear forms in g variables; then the cones over the faces of this hull give the decomposition.

According to Barnes and Cohn this hull is the same as the convex hull of the primitive semi-positive rank 1 forms in L; then, by definition, the maximal faces are defined by the *perfect* quadratic forms. Their classification does not really enter, since they are so many (more than $10,000$ when $g = 8$ and billions when $g = 9$). However, their elementary properties are used several times during the proof of the result above.

On the whole, it seems harder to come by such results for other locally symmetric varieties. For example, even for Hilbert modular surfaces, or moduli spaces of non-principally polarized abelian surfaces, where minimal and canonical models are certainly known to exist, exhibiting them has not always been done.

Much of the talk was devoted to describing problems that arise.

(1) Once ample line bundles have been identified on a projective scheme over $\mathbb{Z}$, it's automatic to ask for metrics on them, maybe with logarithmic singularities in codimension 2. For example, given that U_g, the universal abelian g-fold, has a compactification $\overline{U}_g$ that is the boundary divisor D in A_{g+1}^F, the known ample bundles on A_{g+1}^F give explicit ample divisors on $\overline{U}_g$. Moreover, the restriction L of D cuts out the 2Θ bundle on the generic fibre ϕ of $U_g \to A_g$ and is trivial along the zero section (but not its closure). Is there a metric on L whose restriction to ϕ has translation-invariant curvature form?

(2) Is the total co-ordinate ring $TC(A_g^F)$, or its subring $\bigoplus_{a,b\geq 0} H^0(A_g^F, \mathcal{O}(aM - bD))$, finitely generated? If so, how much of it can be computed? (Compare the analogous problem concerning the ring $\bigoplus_{a\geq 0} H^0(A_g^F, \mathcal{O}(aM))$ of Siegel modular forms.) Perhaps it is better to ask whether, given that there are inclusions $A_g^F \hookrightarrow A_{g+1}^F$, for example as $\{j_0\} \times A_g^F$ for a fixed value j_0 of the j-invariant, the bigraded ring $\bigoplus_{a,b\geq 0} H^0(A_g^F, \mathcal{O}(aM - bD))$ stabilizes to something accessible. (Cf. Freitag's work on stable Siegel modular forms.)

(3) (char. zero) Compute the plurigenera $P_n(A_g^F)$ and the top self-intersection number $(K_{A_g^F})^N$, where $N = g(g+1)/2$. More generally, find the intersection numbers $M^a.D^b$ for $a+b=N$. (Erdenberger, Grushevsky and Hulek have already made progress towards this.)

(4) (char. zero) For $6 \leq g \leq 10$, the ray $\mathbb{R}_+[C_1]$ defines a flipping contraction. General conjectures imply that there is a flip and, more generally, that the minimal model program can be run on A_g^F. Can this be done explicitly?

(5) Is $12M - D$ semi-ample?

(6) (Ekedahl) For which ample bundles L on A_g^F is $E \otimes L$ ample? Here E is the Hodge bundle, which extends to all toroidal compactifications. This is a question about the cone $\overline{NE}$ on a variety $\mathbb{P}(E)$ whose Picard number ρ is 3; the fact that $\rho = 2$ on A_g^F makes characterizing its ample classes much easier.

(7) Freitag and Weissauer have shown that all subvarieties in $A_{g,\mathbb{C}}$ of sufficiently small codimension (in particular, all divisors when $g \geq 9$) are of general type. Codimension r subvarieties on a smooth complex projective n-fold X are all of general type if Ω_X^{n-r} is ample; is it possible to make sense of, and prove, a statement like this for A_g^F?

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