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Mini-Workshop: Analytical and Numerical Methods in Image and Surface Processing

Organised by
Gerhard Dziuk (Universität Freiburg)
Ulrich Reif (TU Darmstadt)
Martin Rumpf (Universität Bonn)
Peter Schröder (Caltech)

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Abstract. The workshop successfully brought together researchers from mathematical analysis, numerical mathematics, computer graphics and image processing. The focus was on variational methods in image and surface processing such as active contour models, Mumford-Shah type functionals, image and surface denoising based on geometric evolution problems in image and surface fairing, physical modeling of surfaces, the restoration of images and surfaces using higher order variational formulations.

Mathematics Subject Classification (2000): 11 (Partial Differential Equations), 16 (Numerical Analysis and Scientific Computing).

Introduction by the Organisers

In different areas of mathematics and computer science significant progress has been achieved with respect to geometric modeling, the processing of detailed surface models and geometric methods in morphological image processing. In contrast to classical tools in approximation theory such as spline curves and surfaces these new approaches rely on methods from the calculus of variations and geometric evolution problems. As examples we would like to mention here:

- Variational methods in image and surface processing such as active contour models, Mumford-Shah type functionals,
- image and surface denoising based on geometric evolution problems,
- physical modeling of surfaces based on shell models,
- the restoration of images and surfaces using higher order variational formulations with curvature dependent energy integrants.

Recently it has become clear that even though the applications areas differ significantly the methodological overlap is enormous. E. g. currently Willmore flow is one of the hot topics in geometric analysis and at the same time it turned out to be an adequate approach for the restoration of destroyed surfaces. Simultaneously the Willmore energy is a fundamental building block in variational methods of morphological image processing. Indeed as for surfaces it can be used for restoration and the disocclusion of images. Furthermore, it enters important fairness criteria in geometry processing and the generation of C^1 smooth surface models.

The progress in this new, active and already very successful research field has its roots in originally completely different disciplines of mathematics and computer science. Hence, an exchange of different methodologies and a joining of activities is very promising. This was the intention of the mini workshop. To achieve this goal researchers from completely different areas of mathematics and mathematically oriented computer science were invited to share their point of view and their recent achievements with others. The intention of this mini-workshop was to initiate stronger exchange, to encourage the exploration of synergies and the building of bridges between the different disciplines and the different types of approaches for closely related research fields. In particular we intended to bring together people from

- geometric analysis related to second and fourth order problems,
- numerical geometric analysis for geometric variational problems and geometric gradient flows,
- geometric approximation theory in particular subdivision and spline surfaces,
- engineering applications of geometric variational problems,
- variational problems in computer graphics,
- image processing and in particular morphological methods in this field.

Let us unroll one of these synergy fields:

- Tools in surface modeling are at the same time tools in morphological image processing. Indeed image morphology can be regarded as the set of all level sets and thus morphological processing of images coincides with the geometric processing of the level sets. Vice versa, image processing methods are of increasing importance and success in surface processing applications. Image edges and image textures find their counterparts in edge structures on surfaces and the color and structure textures on complicated geometries.
- Neither explicit nor implicit methods have been able to establish as the approved standard in geometric processing. On the one hand explicit methods directly operate on triangular grids as a wide spread standard for the representation of discrete surfaces and enable extremely detailed models, not beatable by implicit - even sparse - surface models. On the other hand implicit surface representations by level sets are more flexible. They allow weaker notions of solutions which in particular enable topological chances.

- Practical algorithms can be derived via a discretization of variational formulations of continuous models making use of the general finite element procedure. Or one directly asks for discrete counterparts of geometric quantities. The latter approach is usually based on invariance principles and qualitative properties to be reproduced from the continuous setting. The notion of discrete curvatures is a good example for this. These different discretization approaches were compared and we were aiming to stimulate progress making use of synergy.
- The numerical analysis of the geometric evolution problems discusses adequate discretizations and concerns about convergence and stability issues. Recent results in this direction were presented and underline the practical usability of these methods. Here close relations between models for parametric surfaces, graph surfaces and implicit surface representation have been exploited.
- Various approaches to treat free discontinuity problems in image and surface processing have been proposed. Level set methods and phase field models were discussed and compared during the workshop.
- The construction of higher order methods for the approximation of surfaces and for the solution of partial differential equations on surfaces have been proposed. In particular subdivision techniques and so called WEB splines were presented. Subdivision surfaces proved in particular very useful in computer graphics and turned out to be applicable for the discretization of fourth order shell models. On the other hand tensor product B-splines - which are one of the established standards in industrial surface modeling - can be modified using appropriate weighting functions to be applicable on general domains and surfaces with complicated boundaries. Again the usability of such approximation schemes for different applications such as shell models for surfaces and the discretization of deformations in image matching were discussed.

The organizers contributed introductory talks into the different fields to stimulate discussion and interaction.

In parallel to this mini workshop another mini workshop on fluid interfaces took place. Together with the organizers of the latter workshop we arranged for several joined talks on the common interest field of geometric evolution problems by participants of both workshops. This interplay turned out to be very fruitful and inspiring.

One Thursday evening we had a longer evening talk on a new approach for the discretization of fluid motion given by Peter Schröder.

Mini-Workshop: Analytical and Numerical Methods in Image and Surface Processing

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Abstracts

Shell Computations

Peter Schröder
(joint work with Fehmi Cirak, Mathieu Desbrun, Eitan Grinspun, Anil Hirani, Michael Ortiz)

The physical behavior of thin flexible structures such as car hoods, leaves, or felt hats can be modeled with the so-called thin-shell equations. Assuming that the thickness is far smaller than the local curvature radius the Kirchhoff-Love theory gives a stored energy density function for the deformation from the rest to the deformed configuration which depends only on the surface and its first and second fundamental form (see for example [1]). Consequently a fourth order non-linear PDE must be solved to evolve the surface shape under some set of given loads. In a finite element treatment this requires shape functions which have square integrable curvatures. More commonly this is referred to as the need to have C^1 shape functions. In a classical finite element framework this could be resolved by using hermite bases over each triangle. In this case the corners of the triangle carry not only position constraints but also tangent and curvature information to ensure that the individual elements meet with C^1 continuity. Such elements are possible, but they are of high polynomial order which tends to lead to oscillations in practice. They are also exceedingly cumbersome to implement and lead to a formulation with degrees of freedom beyond the usual displacements. These additional degrees of freedom (capturing the tangent and curvature constraints at vertices) do not possess a natural physical interpretation leading to difficulties in setting them correctly to achieve a well defined solution.

All these difficulties can be overcome if the surface is modeled with subdivision basis functions [10, 8, 9]. Subdivision generalizes knot insertion ideas from classical splines to the arbitrary topology surface setting. The degrees of freedom are the control points of a subdivision surface (equivalently the coefficients of the underlying basis functions). In the case of splines it is well known that a given spline can be written as a linear combination of translates and dilates of itself. This refinement relation is the key to subdivision. A given surface can be written as a linear combination of basis functions each weighted by an appropriate coefficient. Applying the refinement relation to each basis function results in a new set of coefficients with respect to basis functions which are dilated (but of the same type). Considering only the coefficients and the resulting operation on these one sees that the refinement relation is realized through a topological refinement operation of the control mesh with the new control points being given as finite linear combinations (according to the refinement relation) of old control point. In the limit of this refinement process the basis functions themselves are recovered. In the case of thin-shell simulation one may express all the necessary stiffness integrals in terms of evaluations of the subdivision surface (as well as its partial derivatives up to order two) at quadrature points. The smoothness of subdivision surfaces is sufficient to justify their use in a finite element treatment of the thin-shell equations [7].

Because subdivision surfaces are very attractive for geometric modeling using the underlying basis functions in a thin-shell treatment leads to an integrated modeling and simulation environment without the need to mesh a given geometry only to make it accessible for finite element analysis [2].

When using the subdivision element method a new issue arises, however. Traditional approaches towards adaptive computations, such as element refinement are not applicable anymore. This is so because the basis functions whose support overlaps a given domain triangle are non-local: they include the three corners as well as all vertices reachable by traversing one mesh edge (the so-called 1-ring of a triangle). While this results in no conceptual difficulties (local stiffness matrices are not three by three anymore but somewhat larger), it makes it so that a given triangle cannot be split in isolation. Instead one must take a different approach to adaptive computations: instead of splitting elements one must refine basis functions [5, 6]. A function which is refineable by definition allows for this. The underling mesh plays only a supporting role in the implementation of such a method and serves only to resolve the necessary quadratures. Procedures to manage the basis functions are simple expression which in essence manage the associated index sets. Such operations are efficiently supported by modern data structures. Additionally, the necessary functions for (un-)refinement as well as managing the tesselations of the overlaps of the corresponding supports are quite short and easily verifiable, leading to a straightforward implementation.

In summary the subdivision element method allows for a unified treatment of geometric modeling and thin-shell (or other) physical simulation all the while supporting adaptive refinement in a natural and efficient way.

These ideas were first explored in 2000 and have since been applied to many different settings with great success (see the talk of Fehmi Cirak in these same proceedings). Since then we have pursued an entirely different approach to the simulation of thin-shells by reconsidering the very foundations of the formulation of stored energy functions for surfaces in an entirely discrete setting. This approach is based on discrete differential geometry [3].

Here one asks what measures can be evaluated on a simplicial mesh. An old theorem of Hadwiger's asserts that the space of rigid motion invariant additive measures of convex bodies which are continuous (in the Haussdorff sense as a sequence of convex sets converges to a limit set) is spanned by the Minkowski functionals (or equivalently Cauchy Quermaß integrals). For a triangle mesh this implies that a stored energy function can only depend on length, area, and dihedral angles (between neighboring triangles). An approach to the simulation of discrete shells based on these observations was proposed by Grinspun et al. [4]. The advantage of these methods is their simplicity since they do not rely on smooth basis functions anymore but instead use a given triangulation of a surface directly. While this approach works very well in practice as yet little is known about questions such as accuracy and convergence. Analysing these questions is subject of ongoing research and first results indicate that the method is consistent and converges in the limit of finer and finer triangulations (subject to standard aspect ratio requirements) to the desired continuous quantities (integrals of first and second fundamental forms).

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Surface Representations of Higher Order – A Survey

Ulrich Reif

Surfaces can be represented either in parametric or in implicit form. While both approaches have their pros and cons, parametrized models are the method of choice in many applications, like FE analysis of shells or industrial modeling of surfaces. Here, we focus on parametrized models of higher order, meaning that we exclude piecewise linear geometry, which is widespread in Computer Graphics and standard FE methods.

Within the variety of smooth parametrizations of surfaces, a basic distinction is between *adapted* and *trimmed* representations. Adapted representations use a tessellation of the given domain which typically makes the boundary of the surface at least locally a parameter line. Among these methods, we find the following:

- **Macro elements.** Here, an appropriately refined standard triangulation of the domain is used to define a space of piecewise polynomial splines. In 2d, certain C^k-constructions are available [1], while in 3d, no general principles beyond C^1 have been found yet. Macro elements are used in

FE-simulations, where higher regularity or superior rates of convergence are requested.

- **Singular parametrization.** Here, the partitioning of the domain is almost regular in the sense that there are only few extraordinary vertices. At these vertices, where parametric smoothness conditions seem to be too restrictive, singularities in the parametrization are employed to circumvent the notorious compatibility problems. Additional non-linear constraints have to be imposed to guarantee the geometrical smoothness of the surfaces [2]. So far, the concept of singular parametrization has not found universal acceptance in applications.
- **Geometric continuity.** Starting from a similar topology as in the preceding case, smooth surfaces are constructed by relaxing the parametric smoothness conditions. Geometric smoothness conditions refer to the idea that at common edges of neighboring patches not transversal derivatives, but geometric quantities (normal vector, principal curvatures and directions, etc.) are required to coincide [3]. In applications, also the concept of geometric continuity plays a minor role.
- **Subdivision.** Subdivision surfaces solve the problems at extraordinary vertices by iterative refinement. The algorithms are simple, and the generated surfaces are visually smooth, though not C^2 everywhere [4]. While subdivision surfaces have become a standard in Computer Graphics, they are rarely used in industrial applications, so far. A possible explanation for that situation are small-scale shape artifacts, which can be observed for all subdivision algorithms currently in use.

Trimmed representations use a periodic grid to define the surface. Boundaries are incorporated by simply restricting the surface to the desired domain. Trimmed tensor product NURBS have become a standard in the industrial modeling of surfaces. The main problems related with this approach concern fitting of boundary data, and stability of spline bases. Recently, the concept of web-splines was introduced to account for these problems [5]. By using a weight function, homogeneous boundary conditions can be incorporated, while an extension procedure removes instability. web-splines are geared to FE-applications, but can also be used for geometric modeling.

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Image Processing $\stackrel{?}{=}$ Surface Processing

Martin Rumpf
(joint work with U. Diewald, G. Dziuk, M. Droske, N. Litke, O. Nemitz, P. Schröder)

In surface and in image processing methods based on partial differential equations and variational approaches are widespread. The talk underlines the huge overlap of these two disciplines. It is shown that morphological methods in image processing naturally involve concepts from differential geometry and can be understood as a processing of the entity of level sets of the image intensity map. On the other hand surfaces are often considered in an implicit description to allow for a flexible manipulation. Furthermore, it is demonstrated that image processing methodology can directly be applied to the processing of surfaces via their parameter maps and surface describing functions on these parameterizations. In particular we derive variational problems and geometric evolutions problems for image and surface fairing, restoration and matching.

In addition, we present a general approach for the formulation of geometric evolution problems in level set representation. This allows the easy translation of many surface processing methods into the image processing environment.

1. Surface and image smoothing

Frequently due to the data acquisition images or surfaces are characterized by significant noise. We aim for a smoothing with a simultaneous enhancement of important features such as edges or corners. Based on the definition of an anisotropic surface energy we consider the corresponding anisotropic mean curvature motion as a feature sensitive smoothing process. A scale of successively smoothed representations is generated, where time is considered as the scale parameter.

2. Surface and image restoration

In surface restoration usually a damaged region of the surface has to be replaced by a surface patch which restores the region in a suitable way. In particular one aims for C^1 continuity at the patch boundary. The Willmore energy is considered to measure fairness and appropriate boundary conditions enable to ensure continuity of the normal field. The corresponding L^2 gradient flow as the actual restoration process leads to a fourthed order partial differential equations.

3. Morphological image matching

A variational method to non rigid registration of multi-modal image data is presented. A suitable deformation will be determined via the minimization of a morphological, i.e., contrast invariant, matching functional along with an appropriate regularization energy. As already discussed above the morphology of images can be described by the entity of level sets of the image and hence by its Gauss map. A class of morphological matching functionals is presented which measures the defect of the template Gauss map in the deformed state with respect to the deformed Gauss map of the reference image. The problem is regularized by considering a nonlinear elastic regularization energy.

4. Matching surfaces via image matching on parameter domains

We present a new variational method for matching surfaces. Instead of matching two surfaces via a non-rigid deformation directly in $I\!R^3$, we apply well established matching methods from *image processing* in the parameter domains of the surfaces. A matching energy is introduced which may depend on curvature, feature demarcations or surface textures, and a regularization energy controls length and area changes in the induced deformation between the two surfaces. The metric on both surfaces is properly incorporated into the formulation of the energy. This approach reduces all computations to the 2D setting while accounting for the original geometries.

5. Level set formulation of geometric evolution problems

A general approach for the integration of geometric gradient flows over level sets ensembles is presented. It enables to derive a variational formulation for the level set formulation of various second and fourth order evolution problems. Starting from single embedded surfaces and the corresponding gradient flow, energy and metric are generalized to sets of level set surfaces using the co-area formula and the identification of normal velocities and variations of the level set function in time via the level set equation.

Suppose a general energy $e[\mathcal{M}] := \int_{\mathcal{M}} f \,\mathrm{d}a$ on surfaces $\mathcal{M}$ and a metric $g_{\mathcal{M}}(\cdot,\cdot)$ on normal variations of the surfaces are given. Now, we consider the gradient flow for a surface $\mathcal{M}$ with respect to the energy $e[\mathcal{M}]$:

$$\partial_t x = -\mathrm{grad}_{g_{\mathcal{M}}} e[\mathcal{M}]$$

That is, we assume the speed of propagation of the surface $\mathcal{M}$ in normal direction to be the representation of normal variations of the energy $e[\cdot]$. Let us assume that we simultaneously want to evolve all level sets $\mathcal{M}_c$ of a given level set function ϕ. We denote by $l[\phi] := \{\mathcal{M}_c[\phi] \,|\, c \in I\!R\}$ the ensemble of all level sets of a function ϕ. At first, we take into account the co-area formula and define a global energy

$$E[\phi] := \int_{I\!R} e[\mathcal{M}_c] \mathrm{d}c = \int_{\Omega} \|\nabla\phi\| f \,\mathrm{d}x\,,$$

where, we set $e[\mathcal{M}_c] = 0$ if $\mathcal{M}_c = \emptyset$. Now, we identify a function ϕ with the corresponding level set ensemble $l[\phi]$ and regard it as an element of the manifold $\mathcal{L} := \{l[\phi] \,|\, \phi : \Omega \to I\!R\}$ of all level set ensembles on a domain Ω. This manifold carries a trivial linear structure, because we so far do not impose any constraints. A variation of the level set function ϕ induces a variation of the level sets $\mathcal{M}_c$ in the level set ensemble $l[\phi]$. Thus, a tangent vector $s := \partial_t \phi$ on $\mathcal{L}$ can be identified with a motion velocity v of the corresponding level sets $\mathcal{M}_c$. Actually, it is the classical level set equation, which quantifies this identification:

$$s + \|\nabla\phi\|\, v = 0\,.$$

Now, we are able to define the metric $g_{\mathcal{L}}$ on $\mathcal{L}$ via integration of the metric $g_{\mathcal{M}_c}$ defined on normal variations of a single level set surface over all level sets. We suppose a metric

$$g_{\mathcal{M}}(v_1, v_2) := \int_{\mathcal{M}} \mu_{\mathcal{M}}(v_1, v_2)\, \mathrm{d}a$$

to be given, where $\mu_{\mathcal{M}}(\cdot,\cdot)$ is the density of this metric. By definition the metric $g_{\mathcal{L}}$ operates on tangent vectors s_1, s_2, which are variations of a level set function ϕ. Based on our above observation they correspond to normal velocities of the level sets. Thus, we obtain

$$g_{\mathcal{L}}(s_1, s_2) := \int_{I\!R} g_{\mathcal{M}_c}(v_1, v_2)\, \mathrm{d}c = \int_{\Omega} \mu_{\mathcal{M}_\phi}\left(\frac{s_1}{\|\nabla\phi\|}, \frac{s_2}{\|\nabla\phi\|}\right) \|\nabla\phi\|\, \mathrm{d}x$$

for two tangent vectors s_1, s_2 on $\mathcal{L}$ with corresponding normal velocities $v_1 = -\frac{s_1}{\|\nabla\phi\|}$, $v_2 = -\frac{s_2}{\|\nabla\phi\|}$. Here, for the actual integration over all level sets $\mathcal{M}_c$ we have again applied the co-area formula. Now, we are able to rewrite the simultaneous gradient flow of all level sets in terms of the level set function: $\partial_t\phi = -\mathrm{grad}_{g_{\mathcal{L}}} E[\phi]$. Finally, we obtain as a level formulation of our geometric gradient flow

$$\int_{\Omega} \mu_{\mathcal{M}_\phi}\left(\frac{\partial_t\phi}{\|\nabla\phi\|}, \frac{\vartheta}{\|\nabla\phi\|}\right) \|\nabla\phi\|\, \mathrm{d}x = -\left.\frac{d}{d\delta} E[\phi + \delta\vartheta]\right|_{\delta=0}$$

for all functions $\vartheta \in C_0^\infty(\Omega)$.

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Discrete geometric evolution problems

Gerhard Dziuk

Geometric Flows. Transient geometric partial differential equations are used in a wide range of applications. They appear in discrete surface modeling problems such as smoothing of noisy surfaces, surface restauration and image segmentation [1, 2]. Geometric evolution equations appear in physical models such as phase transitions and motion of grain boundaries. Last but not least the motion of surfaces under geometric flows is interesting by itself.

A general form of the geometric evolution of a twodimensional surface Γ is given by the law

$$V = f(\cdot, \nu, H, K, \Delta_\Gamma H),$$

where V is the normal velocity of Γ with normal ν, mean curvature $H = \kappa_1 + \kappa_2$ and Gauß curvature $K = \kappa_1 \kappa_2$. Δ_Γ denotes the Laplace-Beltrami operator. The main geometric flow problems are Mean Curvature Flow, Willmore Flow and Surface Diffusion. The methods of analysis, discretization and numerical analysis depend on the mathematical model for the surface: parametric model, implicit model or graph. Analysis and numerical analysis of level set methods are closely related to those for graphs.

The second order model problem for a geometric flow problem is *Mean Curvature Flow*. It is an $L^2(\Gamma)$-gradient flow for isotropic surface area: $V = -H$. It leads to the intrinsic heat equation

$$x_t - \Delta_\Gamma x = 0$$

for a parametrisation $x = x(p,t)$ with $p \in M$ where M is a suitable parameter surface. $H^{-1}(\Gamma)$-gradient flow for area leads to isotropic *Surface Diffusion* $V = \Delta_\Gamma H$. *Willmore Flow* is defined as the $L^2(\Gamma)$-gradient flow for the classical bending energy, the Willmore functional $W = \frac{1}{2}\int_\Gamma H^2$. Here, $V = \Delta_\Gamma H - 2HK + \frac{1}{2}H^3$. For parametrized surfaces the intrinsic parabolic plate equation is

$$x_t + \Delta_\Gamma^2 x + (2|\nabla_\Gamma \nu|^2 - \frac{1}{2}|\Delta_\Gamma x|^2)\Delta_\Gamma x = 0,$$

where for twodimensional surfaces $|\nabla_\Gamma \nu|^2 = H^2 - 2K$. A typical sixth order parabolic geometric PDE is obtained as the $H^{-1}(\Gamma)$-gradient flow of the energy $E = \int_\Gamma \gamma(\nu) + \frac{\delta}{2}\int_\Gamma H^2$ with a small parameter $\delta > 0$ and with an anisotropy function γ. For applications in image processing as well as in physical models anisotropic energies play an important role.

Discretization of Geometric Flows. For a survey we refer to [4] and [6]. All geometric flows lead to highly nonlinear degenerate parabolic PDEs for which discretization is a quite subtle task. The discretization heavily depends on the mathematical model for the surface/interface which is chosen. For mean curvature flow there is a satisfactory analysis and numerical analysis for several surface representations. For the fourth order flows the numerical analysis just has begun, [5].

The discretization of *parametric models* is based on a discretization of the Laplace-Beltrami operator Δ_Γ introduced in [7, 8]. This together with the differential geometric fact that $H\nu = -\Delta_\Gamma id$ on the surface Γ leads to a weak formulation of the mean curvature vector:

$$\langle H\nu, \varphi \rangle = \int_\Gamma \nabla_\Gamma id \cdot \nabla_\Gamma \varphi$$

for test functions φ. Also higher order geometric flows can be treated with this ansatz - see [9] for parametric Willmore Flow. The main feature of this method is that it is not necessary to have pointwise mean curvature vectors but that it is sufficient to have a discrete formulation of curvature as a functional on $H^1(\Gamma)$. Time discretization is done semi-implicitly. These schemes lead to linear systems in each time step which can be solved by a conjugate gradient method efficiently.

For *implicit surface models (level set models)* finite element methods again use a weak form of curvature and a semi-implicit time discretization. Convergence results for graphs are contained in [3]. For a complete survey over the field we refer to [4, 6].

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Numerical analysis for the Willmore flow of graphs

Klaus Deckelnick
(joint work with Gerhard Dziuk)

The Willmore flow problem consists in finding a family of smooth, oriented surfaces $\Gamma(t) \subset \mathbb{R}^3$ $0 \le t < T$, which satisfy

$$V = \Delta_\Gamma H + \frac{1}{2}H^3 - 2HK \qquad \text{on } \Gamma(t). \tag{1}$$

Here V is the normal velocity and $H = \kappa_1 + \kappa_2$, $K = \kappa_1\kappa_2$ denote mean and Gauss curvature respectively. The law (1) can be interpreted as the L^2–gradient flow of the Willmore functional $W(\Gamma) = \frac{1}{2}\int_\Gamma H^2 dA$, which, apart from being of geometric interest, occurs in models for the bending energy of thin elastic plates or membranes. In the case of closed compact surfaces, results on the existence and uniqueness of solutions to (1) have been obtained in in [11], [6], [7], while [8], [10] present numerical approaches. Surfaces satisfying suitable boundary conditions occur e.g. in problems from surface restoration, [1].

Let us assume that the surfaces $\Gamma(t)$ are graphs over some base domain $\Omega \subset \mathbb{R}^2$, i.e. $\Gamma(t) = \{(x, u(x,t)) \,|\, x \in \Omega\}$. Abbreviating $Q = \sqrt{1 + |\nabla u|^2}$, we can translate the evolution law (1) into a PDE for the height function u, namely

$$u_t = -Q\,\nabla \cdot (E(\nabla u)\nabla(QH)) + \frac{1}{2}Q\,\nabla \cdot \left(\frac{H^2}{Q}\nabla u\right) \qquad \text{in } \Omega \times (0,T), \tag{2}$$

where

$$E(p)_{ij} := \frac{1}{\sqrt{1+|p|^2}}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right), \qquad i,j = 1,2,\ p \in \mathbb{R}^2.$$

We prescribe the boundary and initial conditions

$$u = g, \quad H = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{4}$$

Eqn. (2) is a highly nonlinear parabolic equation of fourth order for u. However, it has (after division by Q) a nice divergence structure in which the Gauss curvature K no longer appears. Introducing $w = -QH$ (cf. [5]) as a new variable one derives the following variational formulation of (2):

$$\begin{aligned} \int_\Omega \frac{u_t\varphi}{Q} + \int_\Omega E(\nabla u)\nabla w \cdot \nabla\varphi + \frac{1}{2}\int_\Omega \frac{w^2}{Q^3}\nabla u \cdot \nabla\varphi &= 0 \quad \forall \varphi \in H^1_0(\Omega) \\ \int_\Omega \frac{w\zeta}{Q} - \int_\Omega \frac{\nabla u \cdot \nabla\zeta}{Q} &= 0 \quad \forall \zeta \in H^1_0(\Omega), \end{aligned}$$

where the second relation follows from the fact that $w = -QH = -Q\nabla\cdot\left(\frac{\nabla u}{Q}\right)$. This variational formulation can now easily be used in order to discretize the problem in space. Suppose that $\mathcal{T}_h$ is a quasiuniform family of triangulations (allowing curved elements on the boundary) of Ω, X_h the space of linear finite elements and $X_{h0} := X_h \cap H^1_0(\Omega)$. Then the semi–discrete problem consists in finding a pair

$(u_h(t), w_h(t)), 0 \leq t \leq T$, such that $u_h(t) - I_h g \in X_{h0}$, $w_h(t) \in X_{h0}$, $u_h(0) = u_h^0$ and

$$\begin{aligned} \int_\Omega \frac{u_{ht}\varphi_h}{Q_h} + \int_\Omega E(\nabla u_h)\nabla w_h \cdot \nabla\varphi_h + \frac{1}{2}\int_\Omega \frac{w_h^2}{Q_h^3}\nabla u_h \cdot \nabla\varphi_h &= 0 \quad \forall \varphi_h \in X_{h0} \\ \int_\Omega \frac{w_h \zeta_h}{Q_h} - \int_\Omega \frac{\nabla u_h \cdot \nabla \zeta_h}{Q_h} &= 0 \quad \forall \zeta_h \in X_{h0}. \end{aligned}$$

Here I_h is the usual Lagrange interpolation operator and $u_h^0 \in X_h$ a suitable approximation of u_0. Our main result are the following quasioptimal error estimates:

Theorem: Suppose that (2)–(4) has a smooth solution on the interval $[0, T]$. Then

$$\begin{aligned} \sup_{0\leq t\leq T} \|(u - u_h, w - w_h)(t)\| + \Big(\int_0^T \|u_t - u_{ht}\|^2 dt\Big)^{\frac{1}{2}} &\leq ch^2|\log h|^2 \\ \sup_{0\leq t\leq T} \|\nabla(u - u_h)(t)\| + \Big(\int_0^T \|\nabla(w - w_h)\|^2 dt\Big)^{\frac{1}{2}} &\leq ch. \end{aligned}$$

The *proof* is presented in [4] and is built on the basic energy estimate for Willmore flow. Further ingredients are suitably chosen interpolation operators and a careful use of geometric quantities (see [2], [3] for corresponding techniques and results in the case of mean curvature flow).

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A Sharp Diffuse Interface Tracking Method for Approximating Evolving Interfaces

Charlie Elliott

This survey talk concerned the use of phase field models in approximating interface evolution involving curvature. The classical model is the Allen-Cahn equation,[1],

$$\epsilon\phi_t - \epsilon\Delta\phi + \frac{1}{\epsilon}W'(\phi) = C_W g$$

where ϕ is a phase field function whose zero level set approximates a surface which evolves according to forced motion by mean curvature, $V = -H + g$ where V is the normal velocity, H is the mean curvature and g is a prescribed forcing. The homogeneous free energy $W(\cdot)$ is symmetric with two equal minima at ± 1. The canonical smooth double well energy is $W(r) := (r^2 - 1)^2/4$. The Allen-Cahn equation is gradient flow for the Cahn-Hilliard functional

$$\int_\Omega (\epsilon|\nabla\phi|^2/2 + W'(\phi)/\epsilon - C_W g\phi)$$

The phase field function has the profile $\phi(x,t) = \psi(d(x,t)/\epsilon)$, up to an error of $O(\epsilon)$, across the interface where $\psi(r) = tanh(r)$. Using the double obstacle potential, [3, 4, 6, 21],

$$W(\phi) = \begin{cases} \frac{1}{2}(1-\phi^2) & \text{for } |\phi| \leq 1 \\ \infty & \text{for } |\phi| > 1 \end{cases}$$

instead leads to the profile $\psi(r) = sin(r)$ for $|r| \leq \pi/2$ and $\psi(r) = sign(r)$ for $|r| \geq \pi/2$. The double obstacle potential has the advantage of requiring the calculation of the phase field function only in the sharply defined diffuse interface. When combined with the adaptive refinement and coarsening of a finite element mesh yielding a fine mesh in the narrow $O(\epsilon)$ sharply defined diffuse interface this forms the basis of the *sharp diffuse interface tracking method*, [10, 11, 19, 14, 2]. The numerical analysis of the Allen-Cahn equation and the computation of curvature interface motion in general is considered in [7, 20, 9, 8].

The approach can be generalized to kinetic and gradient anisotropy together with advection of the interface. For example the anisotropic Allen-Cahn equation

$$\epsilon\beta(\nabla\phi)\phi_t - \epsilon\nabla\cdot DA(\nabla\phi) - \epsilon q.\nabla\phi + W'(\phi)/\epsilon = C_W g$$

with

$$A(p) = \frac{1}{2}(\gamma(p))^2, \quad p \in \mathbb{R}^{n+1}$$

approximates

$$\frac{\beta(\nu)}{\gamma(\nu)}V = -H_\gamma + q.\nu + g$$

where ν is the outward pointing unit normal to the surface, H_γ is the anisotropic curvature associated with the energy density $\gamma(\nu)$, $\beta(\nu)$ is an anisotropic kinetic mobility and q is an advection velocity, [12, 13, 17].

The fourth order surface diffusion flow $V = \Delta_\Gamma H_\gamma$ (with Δ_Γ denoting the Laplace-Beltrami operator for the surface Γ) may be approximated by the fourth order Cahn-Hilliard equation with degenerate mobility, [3, 5, 2]. The phase field approach may also be used for Willmore flow, [18].

Coupling the Navier-Stokes and Cahn-Hilliard equations yields a phase field model for free surface flow for viscous fluids incorporating surface tension at the interface between the fluids, [16]

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B-Splines as Finite Elements

Bernhard Mössner
(joint work with Ulrich Reif)

B-splines have optimal approximation order with a minimal number of degrees of freedom. Therefore using B-splines seems a natural choice in Finite Element applications. On the other hand, there are two problems: Dirichlet boundary conditions and stability.

Weight extended B-splines (web-splines), developed by Höllig, Reif and Wipper [2] solve these problems. To satisfy homogeneous boundary conditions, all basis functions are multiplied with a so called weight function w. This idea was already used in [4] to use polynomials as basis functions on an arbitrary bounded domain.

To obtain a stable basis, is to identify those B-splines, which are responsible for the instability. A B-spline b_i is called an inner B-spline, if there exists a cell $Q \subseteq \operatorname{supp} b_i$, which lies completely inside the domain $\Omega \subseteq \mathbb{R}^d$. All other B-splines, which intersect Ω, are called outer B-splines. The outer B-splines can cause instability. The idea is to link these outer B-splines to inner B-splines:

$$B_i := b_i + \sum_{j \in I_i} e_{ij} b_j$$

This new basis is stable. It remains to determine the coefficients e_{ij}, such that this new basis has the same approximation order as splines. Therefore the basis functions must have the following properties:

- local support
- representation of polynomials

Both properties can by easily achieved. Let n be the order of the B-splines. To get local support, for an outer B-spline b_j a nearest inner cell Q is determined. On Q there are n^d inner B-splines b_i, $i = i_1, \ldots, i_{n^d}$, which are not zero. The outer B-spline b_j is linked to all these inner B-splines, i.e. $e_{ij} = 0$ for all $i \neq i_1, \ldots, i_{n^d}$. The inner B-splines b_i restricted to Q form a basis of the space of polynomials on Q. Consider a fixed inner B-Splines b_i. The polynom $p^{(i)}$ defined by b_i on Q can be extended to Ω. This polynomial can be represented by the B-splines, i.e. $p^{(i)} = \sum_k p_k^{(i)} b_k$. Now e_{ij} is exactly the coefficient of b_j in this representation, i.e. $e_{ij} = p_j^{(i)}$.

The potential of the method is illustrated by solving the Helmholtz eigenvalue problem. This is joint work with A. Richter (TU Darmstadt) [5]. The eigenvalues are to be computed for a statistical analysis of a quantum mechanical experiment.

Besides many favorable properties, web-spline spaces have the drawback that, because of the extension, they are not nested. Surprisingly there is an easier way to get a stable basis: The B-splines are normalized. In the second part of the talk, this new idea, which substantially extends standard results on the stability of B-splines, is investigated. By the normalization nearly all B-splines from a stable basis. The exceptions are B-splines, which intersect the boundary only a small part, but no boundary knot of the support of this B-spline lies inside Ω. To deal with these exceptional B-splines, three possible solutions are presented and briefly discussed.

First numerical results on the Stokes equation conclude the talk.

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A Level Set Based Adaptive Finite Element Algorithm for Image Segmentation

Michael Fried

We present an adaptive finite element algorithm for segmentation with denoising of multichannel images. It is based on a level set formulation of the Mumford–Shah approach proposed by Chan and Vese in [1], [2], [3]. The aim is to find homogeneous regions Ω^i and their boundaries Γ inside a given, possibly noisy image $g : \Omega \to [0,1]^{N_c}$ and a piecewise smooth approximation u to g such, that u is smooth inside the segments Ω^i but may (and usually will) jump across the boundary Γ. The Mumford–Shah approach to the segmentation problem is to minimize the following functional (compare [5])

$$F_{MS}(u,\Gamma) = \int_{\Omega} \frac{1}{N_c} \sum_{k=1}^{N_c} (g_k - u_k)^2 \; + \int_{\Omega\setminus\Gamma} \frac{1}{N_c} \sum_{k=1}^{N_c} \lambda_k |\nabla u_k|^2 + \; \mu|\Gamma|.$$

Here, the first term assures that the channels u_k of u approximate the corresponding channels g_k of the given image g, the second term, the smoothness term, revers to the denoising properties of the approximation u, while the last term $\mu|\Gamma|$

requires Γ to be as short as possible. In what follows, we restrict the maximal number of different segments Ω^i to be $N = 2^M$, which means that there may be still some edges of g inside the segments. Hence, due to the heat equation like diffusion of the classical Mumford–Shah method, this 'interior' edges will be blurred and eventually lost. To avoid this, we slightly change the functional F_{MS} to

$$F_{TV}(u,\Gamma) = \int\limits_{\Omega} \frac{1}{N_c}\sum_{k=1}^{N_c}(g_k - u_k)^2 \;+\; \int\limits_{\Omega\setminus\Gamma} \frac{1}{N_c}\sum_{k=1}^{N_c}\lambda_k|\nabla u_k| + \;\mu|\Gamma|,$$

which leads to a TV like denoising inside the segments, hence is able to keep more details and edges while it is still denoising. The presented algorithm is able to switch between both methods such allowing a direct comparison of them and an easy adaption of the denoising properties to the actual given situation.

Following the approach of Chan and Vese, we represent the segments Ω^i by M level set functions $\Phi = (\phi_{M-1}, \ldots, \phi_0)$:

$$\Omega^i := \left\{x \in \Omega \mid \Pi^i(\Phi(x)) = 1\right\},$$

using the indicator functions

$$\Pi^i(\Phi(x)) = \prod_{j\in I(i)} H(\phi_j(x)) \prod_{j\in \overline{I}(i)} (1 - H(\phi_j(x))),$$

where the products are taken over the index set $I(i)$ consisting of all the indices $j \in \{0, \ldots, M-1\}$ where the jth digit of i in binary representations equals 1, respectively over its complement $\overline{I}(i)$ and $H(z)$ denotes the Heaviside function. Replacing the length term $|\Gamma|$ by $|\Gamma| = \sum_{j=0}^{M-1} \int\limits_{\Omega} |\nabla H(\phi_j)|$, the functionals we are dealing with are now

$$F(\Phi) = \sum_{i=0}^{N-1} \int\limits_{\Omega} \frac{1}{N_c}\sum_{k=1}^{N_c}\left[(g_k - u_k^i)^2 + \lambda_k|\nabla u_k^i|^p\right]\Pi^i(\Phi) + \mu\sum_{j=0}^{M-1}\int\limits_{\Omega}|\nabla H(\phi_j)|.$$

We assume here for fixed Φ (i. e. for fixed segments Ω^i) u to be given as the solution of the Poisson equation on the segments Ω^i (in case of $p = 2$), respectively the corresponding pde for $p = 1$. In order to proceed, we regularize the functional, replacing $H(z)$ by $H_\rho(z) = \frac{1}{2} + \frac{1}{\pi}\arctan(\frac{z}{\rho})$ for $\rho > 0$, wherever it appears in $F(\Phi)$. The steepest descent for the regularized functional $F_\rho(\Phi)$ leads to a system of coupled evolution equations for the level set functions Φ_l, $\quad l = 0, \ldots, M-1$:

$$
\begin{aligned}
\frac{\partial_t \phi_l}{\delta_\rho(\phi_l)} - \mu \nabla \cdot \frac{\nabla \phi_l}{|\nabla \phi_l|} &= \sum_{i=0}^{N-1} f_l(u^i, \nabla u^i) \Pi^i_{l,\rho}(\Phi) \quad && \text{in } \Omega \times (0,T], \\
\frac{\delta_\rho(\phi_l)}{|\nabla \phi_l|} \frac{\partial \phi_l}{\partial \nu} &= 0 && \text{on } \partial\Omega \times (0,T], \\
\phi_l(\cdot, 0) &= \phi_{l\,0}(\cdot) && \text{in } \overline{\Omega}
\end{aligned}
$$

where $f_l(u^i, \nabla u^i) = \frac{(-1)^{(1-a_l(i))}}{N_c} \sum_{k=1}^{N_c} \left[(g_k - u^i_k)^2 + \lambda_k |\nabla u^i_k|^p \right]$ with sign, depending on l and i, and $\Pi^i_{l,\rho}(\Phi) = \prod_{j \in I(i) \setminus \{l\}} H_\rho(\phi_j) \prod_{j \in \overline{I}(i) \setminus \{l\}} (1 - H_\rho(\phi_j))$.

For the special situation of the Minimal Partition Problem of gray-scale images , which is included in the above formulation, and under some additional assumptions, a solution Φ of the evolution problem is known. For $t \to \infty$, it behaves like $c\,t^{\frac{1}{3}}$ with sign depending on the initially given level set function Φ as well as on the given image g and immediately develops discontinuities wherever g jumps. Numerical experiments show this behavior also in the general case.

To develop an algorithm for the evolution problem, we regularize once more, this time replacing $\|\nabla \phi_l\|$ by $Q_\varepsilon(\nabla \phi_l) := \sqrt{\varepsilon^2 + |\nabla \phi_l|^2}$, $\varepsilon \in (0,1)$. Discretization is done by piecewise linear finite elements on a conforming simplicial triangulation in space and a semi– implicit scheme in time: $\forall \varphi_h \in X_h$

$$
\frac{1}{\tau} \int_\Omega \frac{\phi^m_{h,l} - \phi^{m-1}_{h,l}}{\delta_\rho(\phi^{m-1}_{h,l})} \varphi_h + \mu \int_\Omega \frac{\nabla \phi^m_{h,l}}{Q_\varepsilon(\nabla \phi^{m-1}_{h,l})} \cdot \nabla \varphi_h
= \int_\Omega \sum_{i=0}^{N-1} f_l(u^i, \nabla u^i) \Pi^i_{l,\rho}(\Phi^{m-1}_h) \varphi_h
$$

Given the level set functions Φ and thus the segments Ω^i, we compute u^i also using a finite element discretization for Poisson's equation on Ω^i (respectively the corresponding pde in case of $p = 1$) and extending u^i to $\Omega \setminus \Omega^i$ via solving an additional Laplace problem. This extension is needed, as the product $\Pi^i_{l,\rho}(\Phi^{m-1}_h)$ does not vanish on all of Ω.

Several numerical results including a test for convergence to the mentioned known solution, minimal partition of 3D images and a comparison of the denoising qualities of both approaches ($p = 1,\ p = 2$) were presented. Figure 1 shows the segmentation and the resulting approximation u of a RGB image g obtained for $p = 2$ using the Mumford–Shah like method.

Figure 1. segmentation of a RGB image, original with boundaries (left), piecewise smooth approximation (right)

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The interdependency of segmentation and image matching: a coupled free discontinuity approach

Marc Droske

(joint work with Martin Rumpf, Wolfgang Ring)

It is evident that the fundamental tasks of *segmentation*, *image matching (registration)* and *image restoration* depend on each other. The richness of available information about the local image structure influences the quality and robustness of feature extraction. Many simple feature extraction techniques are however an unstable and ambiguous processes. In the case of multichannel data, this ambiguity is somewhat reduced. Let us suppose, that we have an exact registration of an MR and a CT image of the same patient available. In that case, ambiguous edges in the MR image, e. g., along the boundary of a bone structure might be clearly detectable in the other image. Hence, in the same fashion as feature detection in color images, the detection of features in the MR-CT pair is much more robust.

Another way of feature detection is *segmentation*. It is most often also based on some kind of feature detector, which allows to devise *external forces* which attract the contour towards dominant edges. In contrast to pure feature extraction,additional forces depending on the curve itself (*internal forces*) play an important rôle, for instance controlling the length or the curvature of the contour.

Segmentation can be performed by minimizing the Mumford-Shah functional

$$E_{\mathrm{MS}}(u,\Gamma) = \int_\Omega (u-u_0)^2 dx + \frac{\mu}{2}\int_{\Omega\setminus\Gamma} \|\nabla u\|^2 \,\mathrm{d}\mu + \nu \mathscr{H}^{d-1}(\Gamma).$$

by an optimization over a space of contour sets and an appropriate set of images.

Conversely, imagine two images that are not yet registered, but that a precise segmentation of a particular object is available. The alignment of such segments is not hard to achieve. What remains is to perform a registration which already receives valuable hints about the position of certain features.

In order to rule out the interdependency we consider a combined approach, which exploits complementary feature information and is hence significantly more robust than solving the feature detection and the alignment of feature sets separately. We aim at the minimization of

$$\begin{aligned}\tilde{E}_{\mathrm{MS}}(\Gamma,\phi,u_R,u_T) = {} & \frac{1}{2}\int_\Omega (u_R-u_{R,0})^2 \,\mathrm{d}\mu + \frac{\mu}{2}\int_{\Omega\setminus\Gamma} \|\nabla u_R\|^2 \,\mathrm{d}\mu + \frac{\nu}{2}\mathscr{H}^{d-1}(\Gamma) \\ & + \frac{1}{2}\int_\Omega (u_T-u_{T,0})^2 \,\mathrm{d}\mu + \frac{\mu}{2}\int_{\Omega\setminus\Gamma^\phi} \|\nabla u_T\|^2 \,\mathrm{d}\mu + \frac{\nu}{2}\mathscr{H}^{d-1}(\Gamma^\phi).\end{aligned}$$

We have derived a level set algorithm for the solution of this *coupled free discontinuity problem* and first restrict ourselves to edge sets which are the union of finitely many Jordan-curves. In this case, the feature set can be viewed as the boundary of detected segments, which are mapped to similar segment boundaries in the second image. For a large class of images, this is a very suitable and convenient approach, since images can often be decomposed into a finite set of independent objects.

In a shape optimization framework, we would start with an initial feature set and evolve it according to a regularized energy minimization method. The curve may be elegantly described and propagated by the level set approach of Osher and Sethian. Hintermüller & Ring have derived a level set based Newton-type regularized optimization algorithm for minimizing the Mumford-Shah [4] functional. That work is the algorithmical basis for our joint free discontinuity problem for registration. The details of the combined matching algorithm can be found in [3].

As an alternative to the level set algorithm we introduce a corresponding phase-field approach. Let us consider the approximation, that has been proposed by Ambrosio&Tortorelli in [1]. They have shown the Γ-convergence of an elliptic approximation E_ϵ to the Mumford-Shah functional.

We suggest an coupled phase-field formulation by again introducing an auxiliary variable v, describing the singularity set S_T of the image u_T, but at the same time

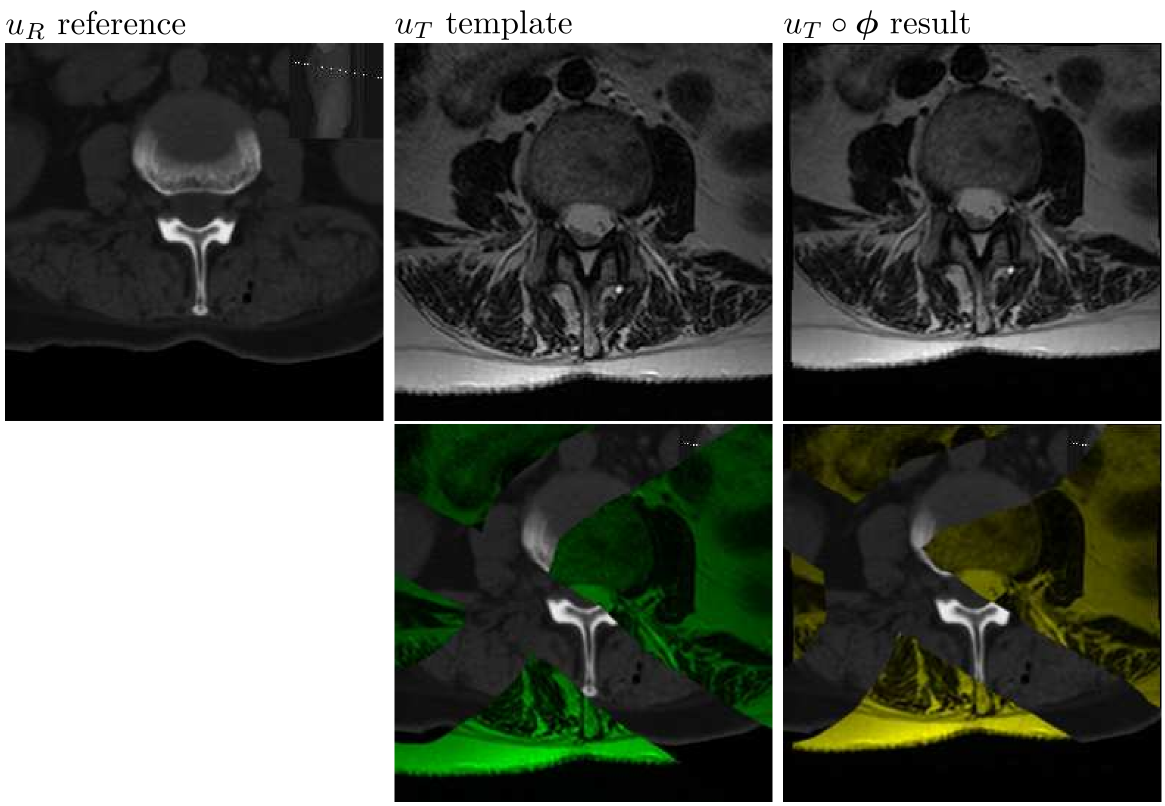

FIGURE 1. *Image registration by the phase field Mumford-Shah approach of CT and MR slices of a human vertebra.* BOTTOM ROW: *The left image shows the initial misfit by overlaying $u_{R,0}$ and $u_{T,0}$ (green), while the image in the right shows the alignment of $u_{T,0} \circ \boldsymbol{\phi}$ (yellow) in comparison with $u_{R,0}$.*

$v \circ \boldsymbol{\phi}$ should energetically describe the edge set S_R in the image u_R. A corresponding energy formulation is then given by the minimization of (see Figure 1)

$$E_{\mathrm{ATreg},\epsilon}[u_R, u_T, v, \boldsymbol{\phi}] := \frac{1}{2}\int_\Omega \left\{ (u_R - u_{R,0})^2 + (u_T - u_{T,0})^2 \right\} \mathrm{d}\mu + \frac{\mu}{2}\int_\Omega \left\{ (v^2 \circ \boldsymbol{\phi} + k_\epsilon)\|\nabla u_R\|^2 + (v^2 + k_\epsilon)\|\nabla u_T\|^2 \right\} \mathrm{d}\mu + \frac{\nu}{2}\int_\Omega \left\{ \epsilon\|\nabla v\|^2 + \frac{1}{4\epsilon}(v-1)^2 \right\} \mathrm{d}\mu.$$

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Interactive High-Quality Shape Modeling

Leif Kobbelt
(joint work with Mario Botsch)

In this talk, I'm presenting recent results on how interactive shape modeling functionality can be achieved for complex geometric objects represented by high-resolution polygonal (triangle) meshes. We target at application scenarios where a detailed 3D model is given by some per-process (e.g. 3D scanning or CAD system export) and some user- controlled global freeform deformations should be applied. This scenario is quite natural in numerical simulation and conceptual design applications where variants of a base-geometry have to be generated in an efficient manner without building physical prototypes or using complicated CAD systems.

In principle there are two major approaches to shape modeling: surface-based and volumetric. In surface-based approaches the deformation is computed with respect to a parametrization over a 2-dimensional domain while in volumetric approaches the surrounding 3-dimensional space is deformed.

We present a surface-based freeform modeling framework for unstructured triangle meshes which is based on constraint shape optimization. The goal is to simplify the user interaction even for quite complex freeform or multiresolution modifications. The user first sets various boundary constraints to define a custom tailored (abstract) basis function which is adjusted to a given design task. The actual modification is then controlled by moving one single 9-dof manipulator object. The technique can handle arbitrary support regions and piecewise boundary conditions with smoothness ranging continuously from C^0 to C^2. To more naturally adapt the modification to the shape of the support region, the deformed surface can be tuned to bend with anisotropic stiffness. We are able to achieve real-time response in an interactive design session even for complex meshes by precomputing a set of scalar-valued basis functions that correspond to the degrees of freedom of the manipulator by which the user controls the modification.

These surface-based methods for interactive freeform editing of high resolution 3D models are very powerful, but at the same time require a certain minimum tessellation or sampling quality in order to guarantee sufficient robustness. In contrast to this, space deformation techniques do not depend on the underlying surface representation and hence are affected neither by its complexity nor by its quality aspects. However, while analogously to surface-based methods high quality deformations can be derived from variational optimization, the major drawback lies in the computation and evaluation, which is considerably more expensive for volumetric space deformations.

To compensate this drawback, we present a new technique which allows us to use triharmonic radial basis functions for real-time freeform shape editing. An incremental least-squares method enables us to approximately solve the involved linear systems in a robust and efficient manner and by precomputing a special set of deformation basis functions we are able to significantly reduce the per-frame costs. Moreover, evaluating the resulting linear basis functions on the GPU finally allows us to deform highly complex polygon meshes or point-based models at a rate of 25M vertices or 13M splats per second, respectively.

I will show examples where these modeling techniques were applied in the context of an industrial evaluation. This will demonstrate the flexibility of the underlying modeling metaphor according to which the user intuitively controls a localized global deformation of the polygon mesh by prescribing boundary constraints to a shape optimization problem.

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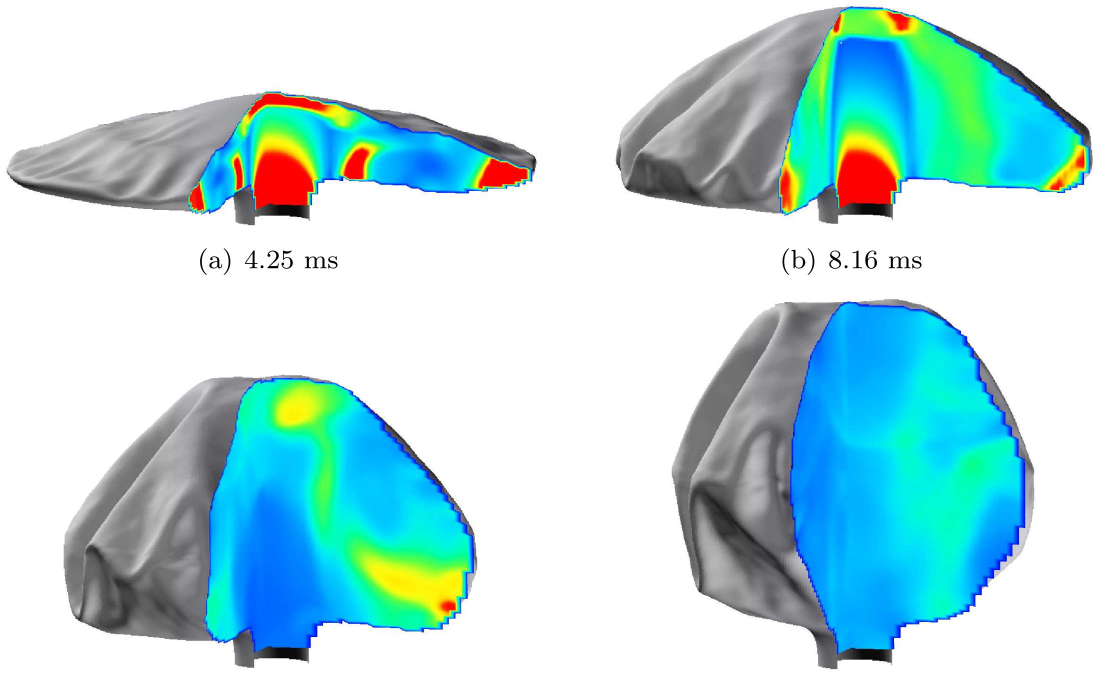
The Subdivision Element Framework for Thin Shell Mechanics

Fehmi Cirak

We introduce a new paradigm for thin-shell finite-element analysis based on the use of subdivision surfaces for: i) describing the geometry of the shell in its undeformed configuration, and ii) generating smooth interpolated displacement fields possessing bounded energy within the framework of the Kirchhoff-Love theory of thin shells. The particular subdivision strategy adopted here is Loop's scheme, with extensions such as required to account for creases and displacement boundary conditions. The displacement fields obtained by subdivision are in H^2 and, consequently, have a finite Kirchhoff-Love energy. The displacement field of the shell is interpolated from nodal displacements only. In particular, no nodal rotations are used in the interpolation. The interpolation scheme induced by subdivision is nonlocal, i. e., the displacement field over one element depend on the nodal displacements of the element nodes and all nodes of immediately neighboring elements. However, the use of subdivision surfaces ensures that all the local displacement fields combine conformingly to define one single limit surface.

One sample application of the subdivision element framework is showcased in Figure 1. The simulation corresponds to the deployment of an initially-flat airbag made of an elastic fabric with a compressed gas inflator. Important features of the airbag deployment process can be observed in these snapshots, including the high-frequency wrinkling modes of the airbag fabric and the shock reflections of the gas on the deforming airbag walls. The ability of the developed framework to



(a) 4.25 ms (b) 8.16 ms

(c) 12.13 ms (d) 18.02 ms

FIGURE 1. Coupled thin shell-fluid simulation of airbag deployment. The airbag fabric is modeled using subdivision thin-shells and the inflowing gas is modeled using a gas dynamics solver. The coupling is performed using the level sets and the "ghost-fluid" technique (see [5] for details). Snapshots show the deformed membrane configurations and density iso-contours of the enclosed fluid.

capture these complex features of the coupled interaction between the flow and the highly flexible airbag fabric is noteworthy.

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Capillarity and calibrability of sets in crystalline mean curvature flow

Maurizio Paolini

In this talk we would like to point out an interesting connection between the capillarity problem in a cylindrical domain in presence of microgravity [4] and the bending phenomenon for crystalline mean curvature flow [1].

1. Capillarity problem

Let $\Omega = F \times (-M, M)$ with a bounded cross section $F \subseteq \mathbf{R}^2$, we seek the equilibrium configuration of a contained fluid with volume constraint and tangential contact with the boundary, and such that the occupied volume A is the subgraph of some function $u : F \to \mathbf{R}$, i.e. $A = \{(x, y) \in \Omega : y < u(x)\}$. By applying standard variational techniques the solution, if it exists, will have constant mean curvature, say λ, for the graph of u.

Setting $\xi(x) = \nu_x$ (the x component of the upward unit normal to the graph of u at $(x, u(x))$), obviously $|\xi| \leq 1$ in F and it coincides with the exterior normal to ∂F at the boundary of F. Moreover div ξ is the mean curvature of the graph of u and hence is constant. By integrating over F we then also readily get $\lambda = |\partial F|/|F|$.

Existence of a vector field ξ with such properties is precisely the definition of calibrability for F.

2. Crystalline anisotropy

Anisotropy is introduced by means of a function $\varphi : \mathbf{R}^3 \to \mathbf{R}^+$ that satisfies the usual properties of a norm: positivity, positive homogeneity and convexity. Associated to φ we define the dual norm $\varphi^o(\xi^\star) = \max_{\xi \in W_\varphi} \xi \cdot \xi^\star$ and introduce the unit balls $W_\varphi = \{\varphi(\xi) \leq 1\}$ (Wulff shape) and $F_\varphi = \{\xi : \varphi^o(\xi) \leq 1\}$ (Frank diagram).

We say that φ is regular when W_φ is smooth and strictly convex; φ is crystalline is W_φ is a polyhedron. Other choices are possible, and of particular interest is the case when W_φ is a cylinder circumscribed to the unit sphere.

For a regular anisotropy we introduce the nonlinear duality mapping $T^o : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T^o(\xi) = \varphi^o(\xi)\nabla_\xi \varphi^o(\xi)$, which provides also a one to one mapping of the Frank diagram onto the Wulff shape; it is strictly monotone and positively homogeneous of degree one. For a crystalline anisotropy T^o can still be defined in a natural way, but becomes a multivalued maximal monotone graph: for example it will typically map a vertex of the Frank diagram onto a whole face of the Wulff shape.

Given a surface $\Sigma = \partial A$ we now introduce the Cahn-Hoffmann vector field n_φ, which will play the role of the euclidean normal ν in this anisotropic setting, for any $x \in \Sigma$ we let $n_\varphi = T^o(\nu_\varphi)$ where $\nu_\varphi = \frac{\nu}{\varphi^o(\nu)}$ is a rescaled version of the eucidean normal.

We can finally define the anisotropic mean curvature as $\kappa_\varphi = \mathrm{div}\ n_\varphi$, and we say that a time dependent surface $\Sigma(t) = \partial A(t)$ flows by anisotropic mean curvature if the (vector) velocity is given by $\mathbf{V} = -\kappa_\varphi n_\varphi$ or equivalently if the normal velocity is $V_\nu = -\varphi^o(\nu)\kappa_\varphi$.

In the crystalline case however the definition of n_φ is not unique since we now only have an inclusion $n_\varphi \in T^o(\nu_\varphi)$ so that n_φ itself must be treated as an unknown.

An *admissible* surface is a faceted surface such that all faces (say F) are parallel to some facet of the Wulff shape (say f). Quite often an admissible surface will just evolve staying admissible, with all faces translating parallel to themselves with velocity given by a simple low that depends unambiguously upon F and f [9], but unfortunately this is not always the case. In Figure 1 you can see an example of *bending* phenomenon: the anisotropy is described by an hexagonal prism as W_φ and the initial (admissible) surface is defined by taking the Wulff shape and just lifting the top face of a sufficiently large amount. During evolution the front and back faces will develop a bended region near the top, as described in [1], [2], [3], [10].

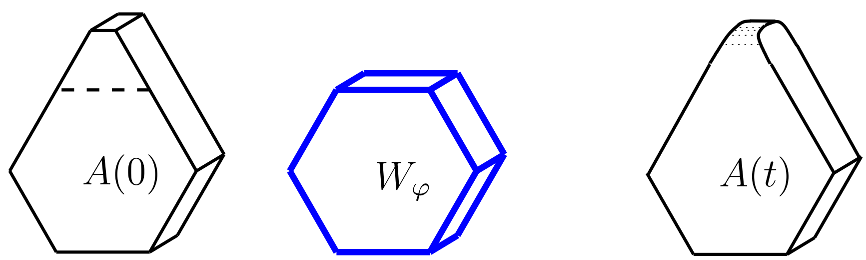

FIGURE 1. Bending example: initial admissible surface (left), Wulff shape and expected evolution with bending (right).

It turns out that *bending* of faces is controlled by an anisotropic version of the *calibrability* conditions known for the capillarity problem [2], [3], [5].

A numerical approximation of crystalline mean curvature flow can be obtained as described in [7] and [8] by first approximating the *sharp* interface using an anisotropic version of the Allen-Cahn equation and then enforcing the *dynamic mesh method* of [6]. It turns out that bending of facets is automatically reproduced by this discretization method without the need of special treatment.

A particularly interesting choice for the anisotropy consists in taking the Wulff shape W_φ as the cylinder circumscribed to the unit sphere, this is neither regular nor crystalline. A typical evolution will develop horizontal plateaus (say F) that move vertically with velocity $|\partial F|/|F|$ as long as they are calibrable in the isotropic sense that we stated at the beginning. On the contrary, if F is not calibrable it will bend in regions where its boundary has large curvature and the velocity vector field is exactly the same as under the so-called *total variation flow* in which, formally, the vertical velocity of the surface is given by the local curvature of the horizontal level line. The two evolution laws however differ on smooth parts of Σ and on vertical walls.

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On Geometric Variational Models for Inpainting Surface Holes

Gloria Haro

(joint work with Vicent Caselles, Guillermo Sapiro and Joan Verdera)

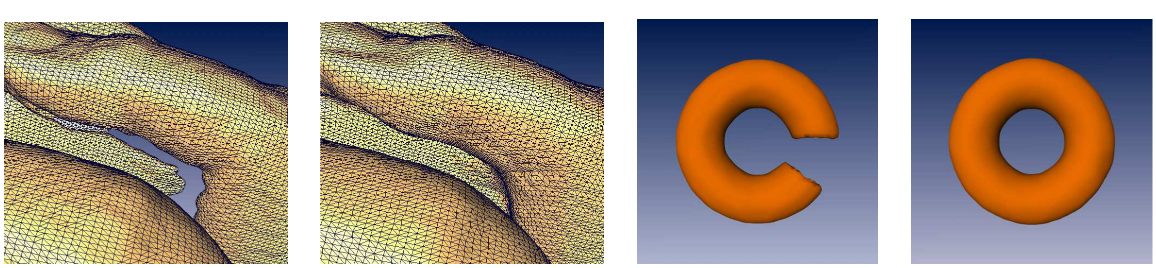
Inpainting is a term used in art to denote the modification of images (painting, photographs, etc) in a form that can not be detected by an ordinary observer. It normally refers to the filling-in of regions of missing information or the replacement of regions by a different kind of information. This is a very important topic in image processing, with applications including image coding and wireless image transmission (e.g., recovering lost blocks), special effects (e.g., removal of objects), and image restoration (e.g., scratch removal). The basic idea behind the computer algorithms that have been proposed in the literature is to fill-in these regions with available information from their surroundings. Several names have been used for this filling-in operation, including *disocclusion* in [2, 7], or *inpainting* in [3].

It turns out that images are not the only kind of data where there is a need for digital inpainting. Surfaces obtained from range scanners often have holes, regions where the 3D model is incomplete. The main cause of holes are occlusions, but these can also be due to low reflectance, constraints in the scanner placement, or simply lack of sufficient coverage of the object by the scanner. The reader is directed to the pioneering work in surface inpainting of [6] for an excellent and detailed description of the nature of holes in scanning statues and for a literature review in the subject.

Geometric approaches for filling-in surface holes are introduced and studied in this talk. The basic idea is to represent the surface of interest in implicit form, and fill-in the holes with a scalar, or systems of, geometric partial differential equations, often derived from optimization principles.

The first algorithm here proposed is an adaptation of the variational formulation for image inpainting presented in [1, 2] to the problem of surface hole filling. As in [6], the use of volumetric data (that is, the surface is represented as the zero level-set of a function) brings us topological freedom. In contrast with [6], we use a system of coupled anisotropic (geometric) partial differential equations designed to smoothly continue the isophotes of the embedding function, and therefore the surface of interest (as the zero level isophote). These equations are based on the geometric characteristics of the known surface (e.g., the curvatures), and as [6], are applied only at the holes and a neighborhood of them (being these equation anisotropic and geometry based, they lead to a slightly slower algorithm than the one reported in [6], as expected with geometric flows). A preliminary version of this (first) model was presented in [8]. We formalize this and improve it here with an automatic initialization method. This initialization is based on the computation of a conical neighborhood $\mathcal{F}$ of the known part of the surface, call it $\mathcal{S}$, where the distance function is uniquely attained. Thereby we can define the signed distance function d_s and then ∇d_s is the extension of the unit normal to $\mathcal{S}$ to a neighborhood of it. This construction also helps us to label both parts of the surface as interior and exterior, and this is useful in this first method.

We also develop additional curvature based hole surface inpainting methods. The first of them is based on a variational model which integrates the Laplacian of a distance function (i.e., a function which satisfies $|\nabla D| = 1$, and $D = d_s$ in the conical neighborhood $\mathcal{F}$), in a open set containing the hole. Recall that the Laplacian of the distance function gives the mean curvature of its level sets. We also use a similar functional but with the square of the Laplacian instead of its absolute value, this is related to the work in [5] where the authors use the Willmore flow. The second method is more heuristic and is based on the diffusion of a function ω which represents mean curvature of level sets of an underlying implicit function. Then, the function u with the prescribed curvature ω is computed solving the PDE: $u_t = |\nabla u| \left(\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \omega\right)$.



Finally, we also present simpler methods based on the Laplace equation and the so-called AMLE model [4], which permit to reconstruct a function which is distance-like near the known part of the surface and whose zero level set can be interpreted as the reconstructed surface. If our interest is just to find a smooth reconstruction, this approach may be sufficient. If one wants a reconstruction which is based on minimizing mean curvature, it can serve as an initialization.

The theoretical and computational framework, as well as examples with synthetic and real data, are presented in this talk.

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Reporter: Bernhard Mößner

Participants

Prof. Dr. Alexander I. Bobenko

Institut für Mathematik
Fakultät II - Sekr. MA 8 - 3
Technische Universität Berlin
Straße des 17.Juni 136
10623 Berlin

Prof. Dr. Fehmi Cirak

California Institue of Technology
Mail Code 158-79
Pasadena, CA 91125
USA

Klaus Deckelnick

Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg

Marc Droske

Institut für Mathematik
Gerhard-Mercator-Universität
Duisburg
Lotharstr. 65
47048 Duisburg

Prof. Dr. Gerhard Dziuk

Abteilung f. Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Str. 10
79104 Freiburg

Prof. Dr. Charles M. Elliott

Department of Mathematics
University of Sussex
Falmer
GB-Brighton BN1 9RF

Dr. Michael Fried

Abteilung f. Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Str. 10
79104 Freiburg

Gloria Haro

Departamento de Tecnologia
Universidad Pompeu Fabra
Passeig Circumvallacio, 8
E-08003 Barcelona

Prof. Dr. Leif Kobbelt

Lehrstuhl für Informatik VIII
RWTH Aachen
Ahornstraße 55
52074 Aachen

Bernhard Mößner

Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. Maurizio Paolini

Dipartimento di Matematica
Univ. Cattolica del Sacro Cuore di
Brescia
Via Musei 41
I-25121 Brescia

Prof. Dr. Ulrich Reif

Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. Martin Rumpf

Institut für Numerische Simulation
Universität Bonn
Nußallee 15
53115 Bonn

Prof. Dr. Peter Schröder

Department of Computer Science
California Institute of Technology
1200 East California Boulevard
Pasadena CA 91125
USA