

Report No. 16/2005

**Arbeitsgemeinschaft mit aktuellem Thema:  
Algebraic Cobordism**

Organised by  
Marc Levine (Boston)  
Fabien Morel (München)

April 4th – April 8th, 2005

ABSTRACT. The aim of this Arbeitsgemeinschaft was to present the theory of Algebraic Cobordism due to Marc Levine and Fabien Morel through the lines of their original articles:

Inspired by the work of Quillen on complex cobordism, one first introduces the notion of oriented cohomology theory on the category of smooth varieties over a field  $k$ . Grothendieck's method allows one to extend the theory of Chern classes to such theories. When  $\text{char}(k) = 0$ , one proves the existence of a universal oriented cohomology theory  $X \rightarrow \Omega^*(X)$ . Localisation and homotopy invariance are then proved for this universal theory. For any field  $k$  of characteristic 0 one can prove for algebraic cobordism the analogue of a theorem of Quillen on complex cobordism: the cobordism ring of the ground field is the Lazard ring  $\mathbb{L}$  and for any smooth  $k$ -variety  $X$ , the algebraic cobordism ring  $\Omega^*(X)$  is generated, as an  $\mathbb{L}$ -module, by elements of non negative degree. This implies Rost's conjectured degree formula. One also gives a relation between the Chow ring, the  $K_0$  of a smooth  $k$ -variety  $X$  and  $\Omega^*(X)$ . The technical construction of pullbacks is the subject of two talks. At the end one presents the state of advances on the conjectural isomorphism between Levine-Morel construction of algebraic cobordism and the "homotopical algebraic cobordism", the cohomology theory represented by motivic Thom spectrum in the Morel-Voevodsky  $\mathbb{A}^1$ -stable homotopy category. The *Arbeitsgemeinschaft* was organised by Marc Levine (Boston) and Fabien Morel (München). It was well attended with over 40 participants.

## Introduction by the Organisers

Over the years, many different types and flavors of cohomology theories for algebraic varieties have been constructed. Theories like étale cohomology or de Rham cohomology provide algebraic versions of the topological theory of singular cohomology. The Chow ring and algebraic  $K_0$  are other (partial) examples, more directly tied to algebraic geometry.

The partial theory  $K_0^{alg}$  was extended to a full theory with the advent of Quillen's higher algebraic  $K$ -theory. It took considerably longer for the Chow ring to be extended to motivic cohomology. In the process of doing so, Voevodsky developed his category of motives, and this construction was put in a more general setting with the development by Morel-Voevodsky of  $\mathbb{A}^1$  homotopy theory. This enabled a systematic construction of cohomology theories on algebraic varieties, with algebraic  $K$ -theory and motivic cohomology being only two fundamental examples. These two cohomology theories have in common the existence of a good theory of push-forward maps for projective morphisms. Not all cohomology theories have this structure, those that do are called *oriented*. In the Morel-Voevodsky stable homotopy category, the universal oriented theory is represented by the  $\mathbb{P}^1$ -spectrum  $M\mathbb{G}\ell$ , an algebraic version of the classical Thom spectrum  $MU$ . The corresponding cohomology theory  $M\mathbb{G}\ell^{*,*}$  is called *higher algebraic cobordism*.

In an attempt to better understand the theory  $M\mathbb{G}\ell^{*,*}$ , Levine and Morel constructed a theory of *algebraic cobordism*  $\Omega^*$ . This is (conjecturally) related to  $M\mathbb{G}\ell^{*,*}$  as the classical Chow ring  $CH^*$  is to motivic cohomology and like  $CH^*$ ,  $\Omega^*$  has a purely algebro-geometric description. In addition to giving some insight into  $M\mathbb{G}\ell^{*,*}$ ,  $\Omega^*$  gives a simultaneous presentation of both  $CH^*$  and  $K_0$ , exhibiting  $K_0$  as a deformation of  $CH^*$ .  $\Omega^*$  has also been used to give conceptually simple proofs of various "degree formulas" first formulated by Rost. These degree formulas have been used in the study of Pfister quadrics and norm varieties, properties of which are used in the proofs of the Milnor conjecture and the Bloch-Kato conjecture.

In this workshop, we describe aspects of the topological theory of complex cobordism which are important for algebraic cobordism (Lectures 1-3) and give the construction of  $\Omega^*$  and proofs of its fundamental properties (Lectures 4-7). In lectures 8-11, we show how  $K_0$  and  $CH^*$  are described by  $\Omega^*$ , how  $\Omega^*$  recovers the universal formal group law, give the proof the generalized degree formula for  $\Omega^*$  and use this to prove the degree formula for the Segre class. Additional applications to Steenrod operations, further degree formulas and the use of these in the study of quadrics and other varieties is given in lectures 12 and 13. Lectures 14 and 15 concern the construction of functorial pull-backs in algebraic cobordism. The two concluding lectures (16 and 17) give a quick sketch of the Morel-Voevodsky  $\mathbb{A}^1$  stable homotopy category and describe what we know about  $M\mathbb{G}\ell$  and its relation to motivic cohomology and  $\Omega^*$ .

The workshop *Algebraic Cobordism*, organised by Marc Levin (Boston) and Fabien Morel (München) was held April 4th–April 8th, 2005. This meeting was well attended with 55 participants.

**Arbeitsgemeinschaft mit aktuellem Thema: Algebraic Cobordism****Table of Contents**

Alexander Nenashev	
1. <i>Introduction to Classical Cobordism</i> . . . . .	885
Bernhard Hanke	
2. <i>Quillen's work on MU</i> . . . . .	888
Serge Yagunov	
3. <i>Oriented cohomology theories over a field.</i> . . . .	889
Jörg Schürmann	
4. <i>Survey of basic properties of algebraic cobordism</i> . . . . .	892
Franziska Heinloth	
5. <i>The construction of algebraic cobordism</i> . . . . .	895
Uwe Jannsen	
6. <i>Localization for algebraic cobordism</i> . . . . .	896
Ivan Panin	
7. <i>Homotopy invariance property and projective bundle theorem</i> . . . . .	899
Joël Riou	
8. <i>Universal property of K-theory</i> . . . . .	900
Annette Huber-Klawitter	
9. <i><math>\Omega^*(k)</math> and the Lazard ring</i> . . . . .	902
Alexander Schmidt	
10. <i>Degree formulas</i> . . . . .	903
Christian Serpé	
11. <i>Cobordism and Chow groups</i> . . . . .	905
Jens Hornbostel	
12. <i>Steenrod operations and other degree formulas</i> . . . . .	906
Stefan Schröer	
13. <i>Some applications</i> . . . . .	908
Ania Otwinowska	
14. <i>Construction of pull-backs in algebraic cobordism, part1</i> . . . . .	910
Jörg Wildeshaus	
15. <i>Construction of pull-backs in algebraic cobordism, part 2</i> . . . . .	911
Oliver Röndigs	
16. <i>The <math>\mathbb{A}^1</math>-homotopy approach to algebraic cobordism: Part One</i> . . . . .	912

Joseph Ayoub

17. *The motivic Thom spectrum  $M\mathbb{G}\ell$  and the algebraic cobordism  $\Omega^*(-)$*  916

## Abstracts

### 1. Introduction to Classical Cobordism

ALEXANDER NENASHEV

The objective of this lecture is to remind the audience of the main definitions, tools, methods, and results of the cobordism theory of topological manifolds. We consider in parallel three cases: smooth ( $C^\infty$ ) manifolds with no extra structure (the non-oriented case), with orientation, and with quasi-complex structure (see [3, Ch.12] for comments). These will be denoted by (i), (ii), (iii) or by using the respective group notation:  $O, SO, U$ ; the same applies to non-oriented, oriented, and complex vector bundles. The main reference is [1].

**1. Definition of the cobordism ring. Elementary computations.** In each of (i-iii), consider the set of diffeomorphism classes of closed manifolds of dimension  $n$  with the structure in question. Call two such manifolds  $M$  and  $M'$  *cobordant* if there exists an  $(n+1)$ -dimensional compact  $N$ , endowed with a structure of the same type, such that  $dN \cong M' \amalg \bar{M}$ , where the bar refers to the reverse structure. The set of cobordism classes,  $\Omega_n^O / \Omega_n^{SO} / \Omega_n^U$  respectively, becomes an abelian group with the addition  $[M] + [M'] = [M \amalg M']$  and  $-[M] = [\bar{M}]$ . Direct product of closed manifolds makes  $\Omega_*^{xxx} = \bigoplus_{n \geq 0} \Omega_n^{xxx}$  a commutative ring with  $1 = [pt]$ . We obviously have:

(i)  $\Omega_0^O = \mathbb{Z}/2$  (every element of  $\Omega_n^O$  is of order two since  $M = \bar{M}$  in this case, whence  $2[M] = [M \amalg \bar{M}]$ , which bounds the cylinder  $M \times I$ ),  $\Omega_1^O = 0$  since the circle bounds,  $\Omega_2^O = \mathbb{Z}/2$  generated by the Klein surface which does not bound anything three-dimensional.

(ii)  $\Omega_0^{SO} = \mathbb{Z}$  (an oriented point is a point endowed with a plus or minus),  $\Omega_1^{SO} = 0$  since the circle bounds an oriented two-dimensional manifold, e.g. a disk (observe that saying that  $S^1$  bounds a Moebius band proves  $\Omega_1^O = 0$  but does not work here),  $\Omega_2^{SO} = 0$  since every closed oriented surface (a sphere with handles) bounds an oriented 3-dimensional manifold; it is known but not elementary that  $\Omega_3^{SO} = 0$ .

(iii) Here we only mention the obvious fact that  $\Omega_0^U = \mathbb{Z}$ .

**2. Classifying spaces / Grassmannians.** Let  $G_n(\mathbb{R}^{n+k})$  (resp.  $\tilde{G}_n(\mathbb{R}^{n+k})$ ,  $G_n(\mathbb{C}^{n+k})$ ) denote the Grassmannian whose points are identified with the  $n$ -planes in  $\mathbb{R}^{n+k}$  (resp. oriented  $n$ -planes in  $\mathbb{R}^{n+k}$ , complex  $n$ -planes in  $\mathbb{C}^{n+k}$ ), and let  $G_n = \text{colim } G_n(\mathbb{R}^{n+k})$  (resp.  $\tilde{G}_n, G_n^{\mathbb{C}}$ ) denote the (resp. oriented, complex) infinite Grassmannian of  $n$ -planes. These are also denoted  $BO(n), BSO(n), BU(n)$ , respectively. Denote  $\gamma^n, \tilde{\gamma}^n, \gamma_{\mathbb{C}}^n$  the canonical vector  $n$ -bundle on the corresponding infinite Grassmannian, where  $n$  refers to the complex dimension in the latter case.

**Main Fact.** For any paracompact space  $B$ , there is a one-to-one correspondence

$$\left( \begin{array}{l} \text{(homotopy classes)} \\ \text{of maps } B \rightarrow G_n \end{array} \right) \leftrightarrow \left( \begin{array}{l} \text{(isomorphism classes of vector} \\ \text{bundles of rank } n \text{ on } B \end{array} \right) [f : B \rightarrow G_n] \mapsto [f^* \gamma^n],$$

and the same for oriented (resp. complex) vector bundles of rank  $n$  and maps to  $BSO(n)$  (resp.  $BU(n)$ ). For this reason  $BO(n)$  (resp.  $BSO(n), BU(n)$ ) is referred to as the *classifying space* for vector bundles (resp. oriented, complex) of rank  $n$ .

**3. Characteristic classes.** (i) To a non-oriented vector bundle  $\xi$  of rank  $n$  over a paracompact base  $B$  one assigns its Stiefel-Whitney classes  $w_i(\xi) \in H^i(B; \mathbb{Z}/2)$ ,  $0 \leq i \leq n$ .

(ii) To a complex vector bundle  $\omega/B$  of (complex) rank  $n$  one assigns its Chern classes  $c_i(\omega) \in H^{2i}(B; \mathbb{Z})$ ,  $0 \leq i \leq n$ ;  $c_n(\omega) = e(\omega_{\mathbb{R}})$ , the Euler class of  $\omega$  considered as a real oriented bundle.

(iii) To a real  $\xi/B$  of rank  $n$  one also assigns its Pontrjagin classes defined as  $p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B; \mathbb{Z})$ ,  $0 \leq i \leq [n/2]$ . Though these are defined for any real  $\xi$ , they are actually used for oriented bundles/cobordism.

The characteristic classes of each of the three types satisfy standard properties including the product formula, e.g.  $c(\omega \oplus \omega') = c(\omega)c(\omega')$ , where  $c$  refers to the total Chern class. (For  $p_i(\xi)$  the formula is only true modulo 2-torsion). See [1, Chs.4,14,15] for details. Characteristic classes are used to define invariants of cobordism classes of manifolds known as

**4. Characteristic numbers.** Let  $I$  denote a partition of  $n$ , i.e., an unordered representation  $n = i_1 + \dots + i_k$ , where all  $i_s \geq 1$  are integers.

(i) For a closed non-oriented manifold  $M$  of dimension  $n$ , let  $\tau_M$  denote its tangent bundle and  $w_I(\tau_M) = w_{i_1}(\tau_M) \cdots w_{i_k}(\tau_M) \in H^n(M; \mathbb{Z}/2)$ . The *I-Stiefel-Whitney number* of  $M$  is  $w_I(M) = \langle w_I(\tau_M), [M] \rangle \in \mathbb{Z}/2$ , where  $[M] \in H_n(M; \mathbb{Z}/2)$  is the fundamental class of  $M$  and  $\langle \cdot, \cdot \rangle$  is the Kronecker index (cap product).

(ii) For an oriented  $M$  of dimension  $4n$ , let  $p_I(M) = \langle p_I(\tau_M), [M] \rangle \in \mathbb{Z}$ , the *I-Pontrjagin number* of  $M$ . Here  $p_I(\tau_M) \in H^{4n}(M; \mathbb{Z})$ ,  $[M] \in H_{4n}(M; \mathbb{Z})$ .

(iii) For a (true) complex manifold  $K$  of complex dimension  $n$ , let  $c_I(K) = \langle c_I(\tau_K), [K] \rangle \in \mathbb{Z}$ . One has  $c_n(K) = e(K)$ , the Euler characteristics of  $K$ .

The following fact, due to Pontrjagin, admits a simple proof: if  $M$  and  $M'$  are cobordant closed manifolds of dimension  $n$ , then  $w_I(M) = w_I(M')$  for all partitions  $I$  of  $n$ ; see [1, Thm.4.9]. The same is true in (ii-iii) with appropriate changes.

**5. Thom spaces/spectra.** ([1, Ch.18]) For a vector bundle  $\xi$  with a Euclidean metric, define the *Thom space*  $Th(\xi) = E(\xi)/A$ , where  $E(\xi)$  is the total space of  $\xi$  and  $A = \{v \in E \mid |v| \geq 1\}$ ;  $t_0 = A/A$  denotes the infinite point of  $Th(\xi)$ . There is an explicit construction that assigns to every oriented vector bundle  $\xi$  of rank  $k$  over a smooth oriented manifold  $B$  and every continuous  $f : S^{n+k} \rightarrow Th(\xi)$ , a cobordism class in  $\Omega_n^{SO}$  (move  $f$  to make it transverse to the zero section  $B$  and then take the intersection  $f(S^{n+k}) \cap B$ ). This yields a homomorphism  $\pi_{n+k}(Th(\xi), t_0) \rightarrow \Omega_n^{SO}$ . Considering it for the canonical oriented bundle  $\tilde{\gamma}^k$  over  $\tilde{G}_k$ , we state the following

**Fundamental Theorem (Thom).** For  $k > n + 1$ ,  $\pi_{n+k}(Th(\tilde{\gamma}^k), t_0) \rightarrow \Omega_n^{SO}$  is an isomorphism.

One can define a spectrum  $MSO$  out of the spaces  $MSO(k) = Th(\tilde{\gamma}^k)$ ; in the non-oriented and complex cases we get the spectra  $MO$  and  $MU$  out of  $Th(\gamma^k)$  and  $Th(\gamma_{\mathbb{C}}^k)$ . The above theorem holds in all the three cases and can be stated shortly as  $\Omega_*^O \cong \pi_*(MO)$ ,  $\Omega_*^{SO} \cong \pi_*(MSO)$ ,  $\Omega_*^U \cong \pi_*(MU)$ .

**6. Computation of cobordism rings.** All the three cobordism rings in question are isomorphic to polynomial rings as follows:

(i)  $\Omega_*^O \cong \mathbb{Z}/2[x_2, x_4, x_5, \dots]$ , where we have one variable  $x_n$  in each degree  $n \neq 2^r - 1$ . It is known that we can take  $x_{2k} = [\mathbb{P}_{\mathbb{R}}^{2k}]$ , while the odd degree generators are given by more sophisticated varieties. In elementary terms:  $[M] = [M']$  in  $\Omega_n^O$  if and only if  $w_I(M) = w_I(M')$  for all partitions  $I$  of  $n$  (the ‘if’ part is hard and due to Thom).

(ii)  $\Omega_*^{SO}/(2 - \text{torsion}) \cong \mathbb{Z}[x_4, x_8, x_{12}, \dots]$ ;  $[M] = [M']$  in  $\Omega_n^{SO}$  if and only if  $w_I(M) = w_I(M')$  and  $p_I(M) = p_I(M')$  for all partitions  $I$  of  $n$  ([5]).

(iii)  $\Omega_*^U \cong \mathbb{Z}[x_2, x_4, x_6, \dots]$ , where one can put  $x_{2k} = [\mathbb{P}_{\mathbb{C}}^k]$  if  $p = k + 1$  is prime. Two complex manifolds are cobordant if and only if their Chern numbers coincide.

**7. Divisibility properties of characteristic numbers.** (i) For a smooth complex algebraic curve  $K$  of genus  $g$  we have  $c_1(K) = e(K) = 2 - 2g$ , hence  $c_1(K)$  is divisible by 2 for any such  $K$ .

If  $K$  is a non-singular algebraic hypersurface in  $\mathbb{P}_{\mathbb{C}}^{n+1}$  of degree  $d$ , then for its Segre number  $s_n$  we have  $s_n(K) = d(n + 2 - d^n)$ . (Recall that  $s_n$  is in the same relation to  $c_1, \dots, c_n$  as the polynomial  $t_1^n + \dots + t_r^n$  to the symmetric functions  $\sigma_1, \dots, \sigma_n$  in the variables  $t_i$ .) In particular, if  $Q$  is a smooth  $n$ -dimensional quadric, then  $s_n(Q) = 2(n + 2 - 2^n)$ . If in addition  $n = 2^r - 1$ , then  $s_n(Q) = 2(2^r + 1 - 2^n)$ ; the fact that this number is divisible by two but not by four was important in Voevodsky’s proof of Milnor conjecture.

(ii) Arithmetical properties of Pontrjagin numbers can be deduced from Hirzebruch’s signature formula. If  $M$  is a closed oriented manifold of dimension 4, then its signature (the signature of the intersection index form on the middle homology) is equal to  $p_1(M)/3$ , which is an integer. We conclude that  $p_1(M)$  is divisible by 3 for any 4-dimensional  $M$ .

For  $M$  of dimension 8 the signature is given by  $(7p_2(M) - p_1^2(M))/45$ , from which we deduce that  $7p_2(M) - p_1^2(M)$  is divisible by 45 for any 8-dimensional  $M$ . See [1, Ch. 19].

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## 2. Quillen's work on $MU$

BERNHARD HANKE

The geometrically constructed unitary bordism functor defines a (generalized) homology theory on the category of CW-complexes. By way of the Pontrjagin-Thom construction, this homology theory is represented by the unitary bordism spectrum  $MU$ . The associated cohomology theory, if evaluated on smooth manifolds, also has a geometric description, due to Quillen. This is based on the existence of push forward maps  $MU^*(X) \rightarrow MU^{*-d}(Y)$  for complex oriented proper maps  $X \rightarrow Y$  of relative dimension  $d$  between smooth manifolds. The cohomology theory  $MU^*(-)$  is *complex oriented* (in the sense of Adams): There is a distinguished class in  $\widetilde{MU}^2(\mathbb{C}P^\infty)$  pulling back to the standard generator of  $\widetilde{MU}^2(\mathbb{C}P^1)$ . This class is considered as a universal Euler class (alias first Chern class) for line bundles. Using it, a theory of higher Chern classes can be developed along the usual lines. Other examples of complex oriented theories are ordinary cohomology and complex  $K$ -theory. However,  $MU^*(-)$  turns out to be the universal such theory:

**Theorem 1.** *Let  $h^*$  be an oriented commutative multiplicative cohomology theory. Then there is a unique natural transformation*

$$MU^*(-) \rightarrow h^*(-)$$

*preserving Chern classes.*

In contrast to the familiar formula

$$c(E \oplus F) = c(E) \cup c(F)$$

calculating the total Chern class (in some complex oriented theory) of a Whitney sum of two vector bundles, there is no such simple relation for the tensor product of two bundles. However, based on the computation of  $MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  as a power series ring over  $MU^*$  in two indeterminates, Quillen observed that the expression for the  $MU^*$ -theoretic first Chern class  $c_1(L_1 \otimes L_2)$  (where  $L_i$  is the tautological bundle on  $\mathbb{C}P^\infty$  pulled back via either of the two projections) leads to a formal group law over  $MU^*$ . There is a universal graded formal group law, supported by the Lazard ring  $\mathbb{L}^*$ , and hence, we obtain an induced ring map

$$\delta : \mathbb{L}^* \rightarrow MU^*$$

carrying the universal formal group law to the one over  $MU^*$  constructed before. The following is one of the main results in Quillen's paper.

**Theorem 2.** *The map  $\delta$  is a ring isomorphism.*

Surjectivity of this map is proven by a clever use of  $MU^*$ -Steenrod operations and leads to the conclusion that  $MU^*$  is generated as a ring by the coefficients occurring in the formal group law over  $MU^*$ . For showing injectivity, Quillen considers the composition of  $\delta$  with the Boardman (=Hurewicz) map

$$\beta : MU^* \rightarrow \mathbb{Z}[t_1, t_2, \dots]$$



( $\deg t_i = -2i$ ) and shows that the inverse of the map defined by the power series  $\sum_{i=0}^{\infty} t_i T^{i+1}$  is a logarithm of the image of the universal group law under  $\beta \circ \delta$ . Theorem 2 and the calculation of  $\mathbb{L}^*$  (due to Lazard) imply that there is a ring isomorphism

$$MU^* \cong \mathbb{Z}[x_1, x_2, x_3, \dots]$$

where  $\deg x_i = -2i$ .

Analogous results exist for unoriented bordism  $MO^*$ . For instance, we have

**Theorem 3.**  *$MO^*$  is a polynomial ring over  $\mathbb{Z}/2$  in indeterminates  $x_i$ , where  $i$  runs through all positive integers not of the form  $2^k - 1$  and the degree of  $x_i$  is equal to  $-i$ .*

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## 3. Oriented cohomology theories over a field.

SERGE YAGUNOV

The purpose of this talk was to “transplant” topological definitions to the context of algebraic geometry and give several motivating examples playing important roles in the sequel.

### 1. DEFINITION

Fix a base-field  $k$  and denote by  $Sch/k$  the category of separated schemes of finite type over  $k$  and by  $Sm/k$  the full subcategory of smooth quasi-projective  $k$ -schemes. Let  $\mathcal{R}^*$  be a category of commutative graded rings with unit. (We do not assume the rings to be necessary graded commutative.)

A functor  $A: (Sm/k)^{op} \rightarrow \mathcal{R}^*$  is called additive if  $A(\emptyset) = 0$  and  $A(X \amalg Y) = A(X) \times A(Y)$ . We define oriented cohomology theories following Quillen’s paper [6] ( see also Levine–Morel [3]).

**Definition 1.** *An oriented cohomology theory on  $Sm/k$  is given by:*

- (1) *An additive functor  $A^*: (Sm/k)^{op} \rightarrow \mathcal{R}^*$ ;*
- (2) *For each **projective** morphism  $f: Y \rightarrow X$  in  $Sm/k$  of relative dimension  $d$ , a homomorphism of graded  $A^*(X)$ -modules  $f_*: A^*(Y) \rightarrow A^{*+d}(X)$  called push-forward homomorphism or transfer.*

*A morphism of oriented cohomology theories is a natural transformation of functors  $(Sm/k)^{op} \rightarrow \mathcal{R}^*$  which commutes with transfer maps.*

*These data should satisfy the axioms below.*

**Axiom 1** (Functoriality).  $(\text{id}_X)_* = \text{id}_{A^*(X)}$ ; for projective morphisms  $Z \xrightarrow{g} Y \xrightarrow{f} X$  in  $\text{Sm}/k$ , one has:  $(f \circ g)_* = f_* \circ g_*: A^*(Z) \rightarrow A^{*+d+e}(X)$ , where  $d$  and  $e$  are relative dimensions of the morphisms  $f$  and  $g$ , respectively.

**Axiom 2** (Transversal base-change). Let  $f$  and  $g$  be transverse morphisms and the square

$$\begin{array}{ccc} W & \xrightarrow{g'} & Y \\ f' \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Z \end{array}$$

is Cartesian in the category  $\text{Sm}/k$ .

Then  $g^*f_* = f'_*(g')^*$ , provided that  $f$  (and therefore  $f'$ ) is projective.

**Axiom 3** (Extended homotopy property). Let  $E \rightarrow X$  be a vector bundle over  $X \in \text{Sm}/k$  and let  $p: V \rightarrow X$  be an  $E$ -torsor. Then  $p^*: A^*(X) \rightarrow A^*(V)$  is an isomorphism.

**Axiom 4** (Projective bundle formula). Let  $E \rightarrow X$  be a rank  $n$  vector bundle over  $X \in \text{Sm}/k$  and  $\mathbb{P}(E) = \text{Proj}(\text{Sym}^*(E))$  be its projectivization. Let also  $s$  denote the zero-section of the canonical line bundle over  $\mathbb{P}(E)$  and  $1 \in A^0(\mathbb{P}(E))$  be the multiplicative unit element. Set  $\xi = s^*s_*(1) \in A^1(\mathbb{P}(E))$ . Then  $A^*(\mathbb{P}(E))$  is a free  $A^*(X)$ -module with basis  $(1, \xi, \xi^2, \dots, \xi^{n-1})$ .

**Corollary 1.** If  $E$  is a trivial vector bundle then  $\xi^n = 0$ .

The projective bundle formula enables us to define, following Grothendieck [2], Chern classes as the coefficients of the equation:

$$\xi^n - c_1(E)\xi^{n-1} + \dots + (-1)^n c_n(E) = 0.$$

It is often useful to consider the Chern polynomial:

$$c_t(E) = t^n - c_1(E)t^{n-1} + \dots + (-1)^n c_n(E).$$

The Chern classes satisfy the following natural properties:

1. Let  $\mathcal{L} \rightarrow X$  be a line bundle. Then  $c_1(\mathcal{L}) = s^*s_*(1) \in A^1(X)$ , where  $s: X \rightarrow \mathcal{L}$  is the zero-section.
2. Functoriality. If  $E \cong E'$  are isomorphic vector bundles over  $X$  then  $c_t(E) = c_t(E')$ ; for any morphism  $f: Y \rightarrow X$  and any vector bundle  $E$  over  $X$ , one has:  $f^*c_t(E) = c_t(f^*(E))$ .
3. Additivity formula. For an exact sequence  $0 \rightarrow E'' \rightarrow E \rightarrow E' \rightarrow 0$  of vector bundles over  $X$ , one has:  $c_t(E) = c_t(E'')c_t(E')$ .
4. Vanishing property.  $c_m(E) = 0$  for  $m > \text{rk}(E)$ .

These four properties uniquely define Chern classes.

In the opposite way (see [4]) starting from a theory supplied with an appropriate “first Chern class structure”, it is possible to (uniquely) recover transfer structure and check all the axioms above. Therefore, in the examples below we just sketch the constructions of the first Chern classes and leave checking all the axioms as

an exercise for the reader.

Following Quillen's approach [6] (and unlike the axiomatic of Grothendieck) we do not assume that  $c_1(\mathcal{L} \otimes \mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M})$ . Similarly to the topological case but with more technicalities [4], one can show that  $c_1(\mathcal{L} \otimes \mathcal{M}) = F_\omega(c_1(\mathcal{L}), c_1(\mathcal{M}))$  where  $F_\omega(U, V)$  is a formal group law corresponding to the choice of an orientation for the theory  $A$ . (Different orientations yield to automorphisms of the corresponding formal group.)

All appearing formal power series actually become polynomials due to the following fact:

**Theorem 1.** *Chern classes are nilpotent.*

## 2. EXAMPLES

**1. Chow groups.** We set  $A^p(X) = CH^p(X)$ . This is an orientable cohomology theory with additive formal group  $F_\omega(U, V) = U + V$  (such theories called *ordinary*). The canonical choice of an orientation is given by the relation  $c_1(L(D)) = [D] \in CH^1(X)$ , where  $L(D)$  denotes the line bundle corresponding to the divisor  $D$ . This theory is universal additive (at least in characteristic 0) in the sense that for any theory  $A^*$  with additive formal group law there exists a unique morphism of oriented theories  $\Theta_{CH}^A: CH^* \rightarrow A^*$ .

**2.** Another example of an ordinary cohomology theory is the even part of **étale cohomology**:  $A^p(X) = H_{\text{ét}}^{2p}(X, \mu_n^{\otimes p})$ , where  $(n, \text{Char } k) = 1$ . For a line-bundle  $\mathcal{L} \in \text{Pic}(X) \cong H_{\text{ét}}^1(X, \mathbb{G}_m)$  one sets:  $c_1(\mathcal{L}) = \partial(\mathcal{L}) \in H_{\text{ét}}^2(X, \mu_n)$ , where  $\partial$  is the differential corresponding to the short exact sequence of sheaves:

$$0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \rightarrow 0.$$

**3.** For every smooth variety  $X \in Sm/k$  consider the group  $K_0(X)$ . Formally adding the invertible Bott element  $\beta$  of degree -1 one gets the orientable cohomology theory with multiplicative group law  $F_\omega(U, V) = U + V - \beta UV$ . To obtain this formal group law one chooses the first Chern class for a line-bundle  $\mathcal{L}$  as  $c_1(\mathcal{L}) = (1 - [\mathcal{L}^\vee])\beta^{-1}$ . We call a theory with the multiplicative group law  $U + V - bUV$  periodic if  $b$  is a unit in  $A^*(k)$ . The theory  $K_0[\beta, \beta^{-1}]$  is the universal multiplicative periodic theory.

**4. Algebraic Cobordism.** For this example we shall assume that our functor  $A^*$  is defined on the much wider category of  $T$ -spectra. Set  $A^p(X) = M\mathbb{G}l^{2p,p}(X)$ , where the algebraic cobordism theory  $M\mathbb{G}l$  is represented by the  $T$ -spectrum  $M\mathbb{G}l$  (see [7]). The identical morphism of the spectrum  $M\mathbb{G}l$  being restricted to its term  $M\mathbb{G}l(1)$  determines the morphism  $\Sigma_T^\infty M\mathbb{G}l(1) \rightarrow T \wedge M\mathbb{G}l$  of the  $T$ -suspension spectrum of  $M\mathbb{G}l(1)$  to the shifted spectrum of algebraic cobordism. This gives us the canonical orienting element  $\tilde{e} \in M\mathbb{G}l^{2,1}(M\mathbb{G}l(1))$ . Since the space  $M\mathbb{G}l(1)$  is the Thom space of the canonical line bundle  $L_\infty$  over the infinite projective space  $\mathbb{P}^\infty$ , one has the morphisms:

$$M\mathbb{G}l(1) \cong Th(L_\infty) \xleftarrow{\pi_\infty} L_\infty \xleftarrow{s} \mathbb{P}^\infty,$$

where  $s$  is the zero-section. We set  $e = s^* \pi_\infty^* (\tilde{e}) \in M\mathbb{G}\ell^{2,1}(\mathbb{P}^\infty)$ . In order to check that the orienting element  $e$  induces first Chern classes, it is sufficient to verify that  $e$  restricted to  $\mathbb{P}^1$  coincides (up to the sign) to  $\sigma(1)$ , i.e. the suspension of 1 in the group  $M\mathbb{G}\ell^{2,1}(\mathbb{P}^1)$ . The computation in [5] shows that  $e = -\sigma(1)$ . This construction of the Chern classes is in parallel to the topological one, given by Conner and Floyd [1].

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### 4. Survey of basic properties of algebraic cobordism

JÖRG SCHÜRMAN

Let  $Sch_k$  be the category of separated schemes of finite type over the base field  $k$ , with  $Sm_k$  the full subcategory of smooth schemes. Smooth morphism are by definition quasi-projective. The following results are due to Levine and Morel [2, 3].

An *oriented Borel-Moore (weak) Homology* theory  $A_*$  on  $Sch_k$  or  $Sm_k$  associates to  $X$  a graded group  $A_*(X)$ , together with:

- (1) A functorial *pushdown*  $f_* : A_*(\cdot) \rightarrow A_*(\cdot)$  for a projective morphism  $f$ , which is additive.
- (2) A functorial *pullback*  $g^* : A_*(\cdot) \rightarrow A_{*+d}(\cdot)$  for a lci.- (or a smooth) morphism  $g$  of relative dimension  $d$ .
- (3) An *exterior product*  $\times : A_i(\cdot) \times A_j(\cdot) \rightarrow A_{i+j}(\cdot \times \cdot)$ , which is commutative, associative and with unit  $1_k \in A_0(k)$ .
- (4) A base change property  $g^* f_* = f'_* g'^*$  for transversal (i.e. tor-independent) cartesian squares.
- (5)  $f_*$  and  $g^*$  commute with exterior products.

**Example 1.** *An universal example is given by the group  $M_*^+(X)$  associated to the semigroup of isomorphism classes of projective morphism  $f : Y \rightarrow X$  with  $Y$  smooth. Here addition is given by the disjoint union, with the grading induced by the dimension of  $Y$ , and  $1_k = [id_k]$ . These groups have an obvious projective pushdown, smooth pullback and exterior product.*

Let us continue with the defining properties of an  $OBM(W)H$ :

- (6) Extended homotopy property.
- (7) Projective bundle formula. Here the *first Chern class operator*  $\tilde{c}_1(L)$  of a line bundle  $L \rightarrow X$  is defined in the case

*OBMH*: by  $\tilde{c}_1(L) = i^*i_*$ , with  $i$  the zero section. For an *OBMH* on  $Sch_k$  one assumes also a technical axiom *CD*.

*OBMWH*: by an additional datum, with  $\tilde{c}_1(L) : A_*(X) \rightarrow A_{*-1}(X)$ .

It should only depend on the isomorphism class of  $L$ , with  $\tilde{c}_1(\cdot)$  commuting with each other and with pushdown, pullback and exterior products. Finally  $\tilde{c}_1(\cdot)$  has to satisfy the axioms (*Sect*) and (*FGL*).

As the notion suggests,  $OBMH \Rightarrow OBMWH$ , and an  $OBM(W)H$  on  $Sch_k$  induces one on  $Sm_k$  by restriction. Finally an *OBMH* on  $Sm_k$  is the same as an *oriented cohomology theory* in the sense of the talk before. Here one uses  $A_*(X) = A^{d-*}(X)$ ,  $\tilde{c}_1(L) = c_1(L) \cdot$  and  $c_1(L) = \tilde{c}_1(L)(1_X)$  for  $X$  of pure dimension  $d$  (together with additivity and  $1_X := const^*1_k$ ).

**Example 2.**  $G_0(\cdot)[\beta, \beta^{-1}]$ , with  $\deg(\beta) = 1$  and  $G_0(\cdot)$  the Grothendieck group of coherent sheaves, and Chow groups  $CH_*(\cdot)$  are *OBMH* on  $Sch_k$ . A complex oriented (co)homology theory in topology induces an *OBMWH* on  $Sch_{\mathbb{C}}$ , where one uses the corresponding Borel-Moore homology in even degrees.

$\jmath$ From now on we assume  $\text{char}(k) = 0$ , since resolution of singularities is used.

**Theorem 1.** *There exists an universal OBMH on  $Sch_k$ , called algebraic cobordism  $\Omega_*(\cdot)$  such that  $\Omega_*(\cdot)$  is also an universal OBM(W)H on  $Sch_k$  and on  $Sm_k$ .*

Moreover, its construction implies the

- Theorem 2.**
- (1)  $M_*^+(X) \rightarrow \Omega_*(X)$  is surjective for all schemes  $X$ .
  - (2) For  $i : Z \hookrightarrow X$  a closed embedding with open complement  $j : U = X \setminus Z \rightarrow X$  one has an exact localization sequence

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \longrightarrow 0 .$$

- (3) The ring homomorphism from the Lazard ring  $\phi_{\Omega} : \mathbb{L}_* = \mathbb{L}^{-*} \rightarrow \Omega^{-*}(k) = \Omega_*(k)$  classifying the formal group law  $F_{\Omega}$  of  $\Omega_*(\cdot)$  is an isomorphism.

The first two properties reflect the algebraic nature of  $\Omega_*(\cdot)$  (they also hold for  $G_0(\cdot)[\beta, \beta^{-1}]$  and  $CH_*(\cdot)$ ). The last property depends on the *weak factorization theorem*. Its topological counterpart for complex cobordism  $\Omega_{2*}^U(pt)$  is due to Quillen [4]. It implies the

- Corollary 2.**
- (1) Let  $k \subset k'$ . Then  $\text{can} : \Omega_*(k) \rightarrow \Omega_*(k')$  is an isomorphism, with  $\text{can}$  induced from the *OBM(W)H*:  $X \mapsto \Omega_*(X \times_k k')$ .
  - (2) Let  $k = \mathbb{C}$ . Then  $\text{can} : \Omega_*(\mathbb{C}) \rightarrow \Omega_{2*}^U(pt)$  is an isomorphism, with  $\text{can}$  induced from the *OBM(W)H*:  $X \mapsto \Omega_{2*}^U(X(\mathbb{C}))$ .

Consider in the following cartesian diagram a projective morphism  $f$  on the smooth scheme  $Y$ , which is transversal to the closed inclusion  $i$ :

$$\begin{array}{ccc} Y_0 \uplus Y_1 & \longrightarrow & X \times \{0, 1\} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \times \mathbb{A}^1 \end{array}$$

Then  $f_0 = f : Y_0 \rightarrow X$  and  $f_1 = f : Y_1 \rightarrow X$  are called *elementary cobordant*, and one gets an induced epimorphism  $M_*^+(X)/\{\text{elem.cob.}\} \rightarrow \Omega_*(X)$ . But this is in general *not* injective, e.g. for  $X = k$  and  $* = 1$ . If two smooth irreducible projective curves are elementary cobordant, then they have the same *arithmetic genus*.

Let  $A_*(\cdot)$  be an  $OBM(W)H$  and  $\phi_A : \Omega_*(\cdot) \rightarrow A_*(\cdot)$  the classifying transformation coming from the universal property. Let  $\mathbb{L}_* \rightarrow A_*(k)$  classify the formal group law of  $A_*(\cdot)$ . Since  $\phi_A$  commutes with first Chern class operators and exterior products, one gets an induced transformation of  $OBM(W)H$ :

$$\phi_A : \Omega_*(\cdot) \otimes_{\mathbb{L}_*} A_*(k) \rightarrow A_*(\cdot).$$

**Theorem 3.** (1)  $\phi_K : \Omega^*(\cdot) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \rightarrow K^0(\cdot)[\beta, \beta^{-1}]$  is an isomorphism of  $OBMH$  on  $Sm_k$ . Here  $K^0(\cdot) \simeq G_0(\cdot)$  is the Grothendieck group of coherent locally free sheaves.

(2)  $\phi_{CH} : \Omega_*(\cdot) \otimes_{\mathbb{L}_*} \mathbb{Z} \rightarrow CH_*(\cdot)$  is an isomorphism of  $OBMH$  on  $Sch_k$ .

The first isomorphism is up to now not known for  $Sch_k$ , and its topological counterpart is due to Conner-Floyd [1]. The second isomorphism depends once more on the *weak factorization theorem*, and its topological counterpart is only true for rational coefficients. The induced transformation

$$CH_*(\cdot) \simeq \Omega_*(\cdot) \otimes_{\mathbb{L}_*} \mathbb{Z} \rightarrow \Omega_{2*}^U(\cdot(\mathbb{C})) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

for  $k = \mathbb{C}$  is due to Totaro [5].

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## 5. The construction of algebraic cobordism

FRANZISKA HEINLOTH

Summary: On the category of separated schemes of finite type over a base field  $k$  (more generally, on each admissible subcategory) there is a universal oriented Borel–Moore  $\mathbb{L}_*$ –functor  $\Omega_*$  of geometric type, called *algebraic cobordism*. An *oriented Borel–Moore functor with products*  $A_*$  assigns to each separated scheme of finite type over  $k$  (or each object of an admissible subcategory) a graded group (in an additive way with respect to disjoint union). It has push–forwards for proper morphisms, pull–backs for smooth equidimensional morphisms (increasing degrees by the relative dimension) and first Chern class operators  $\tilde{c}_1(L)$  (decreasing degrees by one) for each line bundle  $L$ , together with an commutative associative external product and a unit element  $1$  in the coefficients  $A_*(k)$  (which hence form a commutative ring with unit), such that all these data are compatible.

An oriented *Borel–Moore  $\mathbb{L}$ –functor* is an oriented Borel–Moore functor with products and a homomorphism of graded rings from  $\mathbb{L}_*$  to the coefficients, where  $\mathbb{L}_*$  denotes the Lazard ring with the homological grading.

For smooth irreducible  $Y$  we denote by  $1_Y$  the pull–back of  $1$  along the structure morphism  $Y \rightarrow \text{Spec}(k)$ .

An oriented Borel–Moore  $\mathbb{L}$ –functor  $A_*$  is said to be *of geometric type*, if it satisfies the following three axioms:

- (Dim) If  $Y$  is smooth and irreducible,  $L_1, \dots, L_r$  are line bundles on  $Y$  with  $r > \dim Y$ , then  $\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)(1_Y) = 0$ .
- (Sect) If  $Y$  is smooth and irreducible and  $i : Z \hookrightarrow Y$  is a smooth divisor, then  $\tilde{c}_1(\mathcal{O}_Y(Z))(1_Y) = i_*(1_Z)$ .
- (FGL) If  $Y$  is smooth and irreducible and  $L, M$  are line bundles on  $Y$ , then  $\tilde{c}_1(L \otimes M)(1_Y) = F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y)$ , where  $F_A$  is the formal group law induced by  $\mathbb{L}_* \rightarrow A_*(k)$ .

There is a universal oriented Borel–Moore functor with products  $Z_*$ :

$Z_d(X)$  is the free abelian group on isomorphism classes of *cobordism cycles over  $X$*  ( $f : Y \rightarrow X, L_1, \dots, L_r$ ), where  $f$  is projective,  $Y$  smooth and irreducible,  $L_1, \dots, L_r$  are line bundles on  $Y$  and  $\dim Y - r = d$  ( $r \geq 0$ ). Isomorphisms of cobordism cycles are allowed to permute the line bundles. Push–forward is given by composition, pull–back by fiber product, and for a line bundle  $L$  on  $X$  the first Chern class  $\tilde{c}_1(L)$  maps the class  $[f : Y \rightarrow X, L_1, \dots, L_r]$  to  $[f : Y \rightarrow X, L_1, \dots, L_r, f^*L]$ . The product structure is given by the product over  $k$ .

$\Omega_*$  is then constructed from  $Z_*$  by imposing the relations (Dim) and (Sect), tensoring with  $\mathbb{L}_*$  and finally imposing the relations (FGL).

A calculation using (FGL), (Sect) and (Dim) for  $\mathcal{O}(1, 1)$  on  $\mathbb{P}^n \times \mathbb{P}^m$  shows that the images of the coefficients of the universal formal group law under  $\mathbb{L}_* \rightarrow \Omega_*(k)$  lie in the subring generated by the classes of projective spaces and Milnor hypersurfaces.

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## 6. Localization for algebraic cobordism

UWE JANNSEN

Given the definition and formalism of algebraic cobordism as introduced in the previous lecture, the aim of this talk was to explain the following result contained in section 6 of [1]. Let  $k$  be a field.

**Theorem 1.** ([1] Theorem 6.7) *Assume that  $k$  has characteristic 0 (or that resolution of singularities holds over  $k$ ). Let  $X$  be a separated scheme of finite type over  $k$ , let  $i : Z \hookrightarrow X$  be a closed subscheme, and let  $j : U \hookrightarrow X$  be the open complement. Then one has an exact sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \longrightarrow 0 .$$

The proof uses the cycles class of a divisor with normal crossings. Let  $W$  be a smooth quasiprojective variety over  $k$ , and let  $E = \sum_{i=1}^m E_i$  be a divisor with normal crossings on  $W$ . This means that for any  $I \subset \{1, \dots, m\}$ ,

$$E_I := \bigcap_{i \in I} E_i$$

is a smooth variety of pure codimension  $|I|$ .

**Lemma 1.** (see [1] Lemma 5.4) *For any ring  $R$  and any power series  $F \in R[[u_1, \dots, u_m]]$ , there is a unique decomposition*

$$F(u_1, \dots, u_m) = \sum_I u^I F_I(u_1, \dots, u_m)$$

where the sum is over all subsets  $I \subseteq \{1, \dots, m\}$ ,  $u^I = \prod_{i \in I} u_i$ , and where the power series  $F_I$  is such that only the  $u_i$  with  $i \in I$  occur.

**Example 1.** *For the universal formal group law in  $\mathbb{L}[[u, v]]$  ( $\mathbb{L}$  = Lazard ring) we have*

$$F(u, v) = u + v + \sum_{i \geq 1, j \geq 1} a_{ij} u^i v^j ,$$

so that  $F_{\{1\}} = 1 = F_{\{2\}}$  and  $F_{\{1,2\}} = \sum_{i \geq 1, j \geq 1} a_{ij} u^{i-1} v^{j-1}$ .

Writing  $u +_F v = F(u, v)$ ,  $[n] \cdot_F u = u +_F u +_F \dots +_F u$  ( $n$  times, for  $n \geq 1$ ), define the power series in  $\mathbb{L}[[u_1, \dots, u_m]]$

$$G^{n_1, \dots, n_m}(u_1, \dots, u_m) := [n_1] \cdot_F u_1 +_F [n_2] \cdot_F u_2 +_F \dots +_F [n_m] \cdot_F u_m ,$$

and thus power series  $G_I^{n_1, \dots, n_m}(u_1, \dots, u_m) \in \mathbb{L}[[u_1, \dots, u_m]]$  for all  $I$ .



**Definition 1.** ([1] Definition 5.6) For the divisor with normal crossings  $E = \sum_{i=1}^m n_i E_i$  define its class in  $\Omega_*(|E|)$  (with  $|E| = \cup_{i=1}^m E_i = \text{support}(E)$ ) as

$$[E \rightarrow |E|] := \sum_{I \neq \emptyset} [E_I \hookrightarrow |E|, G_I^{n_1, \dots, n_m}(\mathcal{O}_W(E_1)|_{E_I}, \dots, \mathcal{O}_W(E_m)|_{E_I})] .$$

Here we write  $[Y \rightarrow X, L_1 L_2 \dots L_r] := [Y \rightarrow X, L_1, L_2, \dots, L_r]$  for a formal product of line bundles, and extend this to polynomials/power series by linearity. This class lifts the class of the line bundle  $\mathcal{O}_W(E)$  in  $\Omega_*(W)$ :

**Proposition 1.** ([1] Proposition 5.9) With the inclusion  $i : |E| \hookrightarrow W$  one has

$$[E \rightarrow W] := i_* [E \rightarrow |E|] = [id_W, \mathcal{O}_W(E)] .$$

In particular,  $[E \rightarrow W] = [E' \rightarrow W]$  if  $\mathcal{O}_W(E) \cong \mathcal{O}_W(E')$ .

The *proof of the localization theorem* is long and involved, using resolution of singularities in a crucial way. We give a sketch, keeping the notation of the theorem.

0) The property  $j^* i_* = 0$  is obvious, because  $Z \cap U = \emptyset$ .

1) For showing the surjectivity of  $j^*$  it is shown that already every cobordism cycle  $[f : Y \rightarrow U, L_1, \dots, L_n] \in \mathcal{Z}_*(U)$  (with  $Y$  smooth,  $f$  projective, and  $L_1, \dots, L_n$  line bundles on  $Y$ ), can be lifted to a cycle  $[\tilde{f} : \tilde{Y} \rightarrow X, \tilde{L}_1, \dots, \tilde{L}_n] \in \mathcal{Z}_*(X)$ . In fact, one can choose a factorization  $f : U \hookrightarrow \mathbb{P}_U^N$  and a resolution of singularities  $\pi : \tilde{Y} \rightarrow \bar{Y}$  of the closure  $\bar{Y}$  of  $Y$  in  $\mathbb{P}_X^N$  which is an isomorphism over  $Y$ . Then  $\tilde{f} = \pi \circ f$  lifts  $f$ , and the line bundles  $L_i$  can be extended to line bundles  $\tilde{Y}_i$  on  $\tilde{Y}$ .  
2) Next one shows that the relations for cobordism can also be lifted from  $U$  to  $X$ . For the relation (FGL), generated by differences

$$[Y \rightarrow U, L_1, \dots, L_r, L \otimes M] - [Y \rightarrow U, L_1, \dots, L_r, F(L, M)] ,$$

we only have to lift line bundles from  $Y$  to  $\tilde{Y}$ , and there is no problem.

3) For the relation (Sect) one has to show that a generating element of  $\langle Sect \rangle(U)$ , i.e., a difference

$$z_1 - z_2 = [Y \rightarrow U, L_1, \dots, L_{r-1}, \mathcal{O}_Y(T)] - [T \rightarrow X, \mu^* L_1, \dots, \mu^* L_{r-1}] ,$$

where  $\mu : T \hookrightarrow Y$  is a smooth divisor, can be lifted to a similar difference over  $X$ . Lifting  $z_1$  to a cycle  $[\tilde{Y} \rightarrow X, \tilde{L}_1, \dots, \tilde{L}_r]$  as in 1), by a further (embedded) resolution one can achieve that the closure  $\tilde{\mu} : \tilde{T} \hookrightarrow \tilde{Y}$  of  $T \hookrightarrow Y$  is again smooth. Then we may assume  $\tilde{L}_r = \mathcal{O}_{\tilde{Y}}(\tilde{T})$ , and so  $\tilde{z}_1 - \tilde{z}_2$ , with  $\tilde{z}_2 = [\tilde{T} \rightarrow X, \tilde{\mu}^* \tilde{L}_1, \dots, \tilde{\mu}^* \tilde{L}_{r-1}]$ , lifts  $z_1 - z_2$  and lies in  $\langle Sect \rangle(X)$ .

4) For the relation (Dim) one has to lift every generating element in  $\langle Dim \rangle(U)$ , i.e.,

$$z = [Y \rightarrow U, \pi^* M_1, \dots, \pi^* M_r, L_1, \dots, L_s] ,$$

where  $\pi : Y \rightarrow T$  is smooth,  $T$  is smooth and irreducible, and  $r > \dim(T)$ . Let  $\tilde{z}$  be an arbitrary lift as in 1). By steps 2) and 3) it then suffices to show

**Lemma 2.** ([1] Lemma 6.6) *The class of  $\tilde{z}$  in  $\Omega_*(X)$  lies in  $i_*\Omega_*(Z)$ .*

In fact, we then may modify  $\tilde{z}$  via (FGL) and (Sect) to get a lift in  $\langle \text{Dim} \rangle(X)$ . For the proof of this lemma we may assume, by pulling the situation back to  $\tilde{Y}$ , that  $\tilde{Y} = X$  and  $Y = U$ . In other words, we already have  $\pi : U \rightarrow T$ . By taking a smooth projective compactification of  $T$  and further applying resolution of singularities we may assume that  $\pi$  extends to a (not necessarily smooth) morphism  $\tilde{\pi} : X \rightarrow T$ , and that we have a projective birational morphism  $\mu : \tilde{X} \rightarrow X$  with  $\tilde{X}$  smooth,  $\mu$  an isomorphism over  $U$ , and  $\mu^{-1}(Z)$  a divisor with strict normal crossings on  $\tilde{X}$ . In this situation we have (cf. talk 14):

**Proposition 2.** ([1] Proposition 6.4)  $[\mu : \tilde{X} \rightarrow X] - i_*[id : X \rightarrow X] \in i_*\Omega_*(Z)$ .

By this result we may replace  $X$  by  $\tilde{X}$  and  $Z$  by  $\tilde{Z}$  (keeping  $U$ ) to assume that  $X$  is smooth and  $Z$  is a divisor with normal crossings. Let  $L'_i = \tilde{\pi}^*M_i$  for  $i = 1, \dots, r$ , so that  $j^*L'_i = j^*L_i$  for all  $i$ .

**Lemma 3.** ([1] Lemma 6.5)  $[id_X, L_1, \dots, L_r] - [id_X, L'_1, \dots, L'_r] \in i_*\Omega_*(Z)$ .

For the proof of this lemma we may assume  $r = 1$  and omit the indices. The assumption implies  $L' \cong L \otimes \mathcal{O}_X(A)$  for a divisor  $A$  supported on  $Z$ , and by writing  $A = A_1 - A_2$  with effective divisors  $A_i$  we reduce to the case that  $A$  is effective. By assumption on  $Z$  it is then a strict normal crossings divisor on  $X$ . Now, by the relation (FGL) we then have (omit  $id_X$ )

$$[L'] = [L \otimes \mathcal{O}_X(A)] = [F(L, \mathcal{O}_X(A))] = [L] + [g(L, \mathcal{O}_X(A)), \mathcal{O}_X(A)]$$

writing  $F(u, v) = u + g(u, v)v$ . So it suffices to show  $[\mathcal{O}_X(A)] \in \text{Im}(i_*)$ , but in fact, by Proposition 1 this class is  $i_*[A \rightarrow |A| \rightarrow Z]$ . By this lemma, we may assume, for the proof of Lemma 2, that we have already  $L_i = \tilde{\pi}^*M_i$  for all  $i$ . We now proceed by induction on  $\dim(T)$ . If  $\dim(T) = 0$  then we even have  $z = 0$ , because every cobordism cycle with a trivial line bundle is 0. Let  $\dim(T) > 0$ . Writing  $M_1 = M \otimes N^{-1}$  with very ample  $M, N$  and using the relation (FGL) we may assume that  $M_1$  is very ample and then, by Bertini, that  $M_1 = \mathcal{O}_T(T_1)$  for a smooth divisor  $T_1 \hookrightarrow T$ . Let  $X' = \tilde{\pi}^{-1}(T_1)$  be the preimage of  $T_1$  in  $X$ . Then  $X' = X_1 + A$  with effective divisors  $X_1$  and  $A$ , with  $\text{supp}(A)$  in  $Z$  and no component of  $X_1$  having support in  $Z$ . Since  $U_1 = U \cap X_1$  is smooth ( $\pi$  is smooth) and dense in  $X_1$ , we may, after replacing  $X$  by a resolution, assume that  $X_1$  is smooth. Writing then  $L_1 = \tilde{\pi}^{-1}(M_1) = \mathcal{O}_X(X') = \mathcal{O}_X(X_1) \otimes \mathcal{O}_X(A)$ , we see as above that it suffices to show that  $z = [id_X, \mathcal{O}_X(X_1), L_2, \dots, L_r] \in \text{Im}(i_*)$ . But, by the relation (Sect), this element is equal to  $(i_1)_*(z_1)$ , with  $z_1 = [id_{X_1}, i_1^*L_2, \dots, i_1^*L_r]$ , where  $i_1 : X_1 \hookrightarrow T$  is the immersion. By induction, since  $\dim(T_1) < \dim(T)$ , the element  $z_1$  lies in the image of  $\Omega_*(X_1 \setminus U_1)$ . By applying  $(i_1)_*$  we get  $z \in \text{Im}(i_*)$  and have proved the Lemma 2.

5) By step 4), the kernel of  $j^* : \Omega_*(X) \rightarrow \Omega_*(U)$  is generated by differences  $\alpha - \alpha' = [f : Y \rightarrow X, L_1, \dots, L_r] - [f' : Y' \rightarrow X, L'_1, \dots, L'_r]$  with  $\alpha|_U = \alpha'|_U$ .

6) We thus have to show that a difference  $\alpha - \alpha'$  as in step 5) lies in the image of  $i_*$ . By resolution of singularities, the two morphisms  $f$  and  $f'$  can be covered by morphisms  $Y \leftarrow Y'' \rightarrow Y'$  with smooth  $Y''$ , which are the identity on  $U$ . Thus we may assume there is a morphism  $\mu : Y' \rightarrow Y$  extending  $id_U$ . Then we are left with two cases:

$$1) [id_{Y'}, L'_1, \dots, L'_r] - [id_Y \mu^* L_1, \dots, \mu^* L_r], \quad 2) [Y = Y] - [Y' \rightarrow Y].$$

But case 1) follows with Lemma 3, and case 2) with Lemma 1.

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## 7. Homotopy invariance property and projective bundle theorem

IVAN PANIN

The main aim of the lecture is to sketch the proof of the homotopy invariance, of the projective bundle and of the so called extended homotopy invariance property of the algebraic cobordism functor of Levine-Morel. Let  $k$  be a field. We will assume that  $\text{char}(k) = 0$  for the lecture. The following transversality lemma will be used below. Let  $i : Z \hookrightarrow W$  be a smooth closed subvariety of a smooth variety  $W$ . Then  $\Omega_*(W)$  is generated by standard cobordism cycles of the form  $[f : Y \rightarrow W]$  with  $f$  transversal to  $i$ . With this Lemma in hand it is easy to prove the surjectivity part of the homotopy invariance, that is the smooth pull-back map  $p^* : \Omega_*(X) \rightarrow \Omega_*(X \times \mathbb{A}^1)$  is surjective. The injectivity is proved with the use of the first Chern class and the localization sequence. Thus for a finite type  $k$ -scheme  $X$  the smooth pull-back  $p^* : \Omega_*(X) \rightarrow \Omega_*(X \times \mathbb{A}^N)$  is an isomorphism for all  $N$ . This statement is a part of tools used to prove the projective bundle theorem, which we are going to state right now. For a finite type  $k$ -scheme  $X$  and a rank  $n + 1$  vector bundle and the associated projective bundle of lines  $q : \mathbb{P}(E) \rightarrow X$  in  $E$  and the tautological line bundle  $\mathcal{O}(-1)$  on  $\mathbb{P}(E)$  denote  $\mathcal{O}(1)$  the dual of  $\mathcal{O}(-1)$  and write  $\xi$  for the operator  $\tilde{c}_1(\mathcal{O}(1))$ . Set  $\phi_j = \xi^j \circ q^* : \Omega_{*-n+j}(X) \rightarrow \Omega_*(\mathbb{P}(E))$ . Then the homomorphism

$$\sum_{j=0}^n \phi_j : \bigoplus_{j=0}^n \Omega_{*-n+j}(X) \rightarrow \Omega_*(\mathbb{P}(E))$$

is an isomorphism. The extended homotopy invariance property is deduced from the projective bundle theorem.

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## 8. Universal property of $K$ -theory

JOËL RIOU

If  $A$  is a ring and  $b$  an element of  $A$ , we can define a formal power series in two variables  $F(X, Y) = X + Y - bXY$ ; one can check that this series provides a formal group law structure of dimension 1 on the ring  $A$ . If  $F$  is a formal group law, we say that it is multiplicative if it is of the previous form, and if it is, we say that it is periodic if  $b$  (which is uniquely determined) is invertible. We recall the fact that the set of morphisms from the Lazard ring  $\mathbb{L}$  to  $A$  is in bijection with formal group law structures on  $A$ . As a result, we have a map  $\mathbb{L} \rightarrow \mathbb{Z}[\beta, \beta^{-1}]$  corresponding to  $F(X, Y) = X + Y - \beta XY$ .

Let  $k$  be any field, we denote by  $Sm_k$  the category of smooth quasi-projective  $k$ -schemes.

**Definition 1.** *On  $Sm_k$ , an oriented Borel-Moore  $\mathbb{L}$ -functor with products  $A^*$  consists of a graded  $\mathbb{L}$ -module  $A^*(X)$  for any  $X \in Sm_k$  with smooth pull-backs, projective push-forwards, external products and Chern operators  $\tilde{c}_1(L)$  associated to line bundles satisfying some reasonable properties. We then say that an oriented weak cohomology theory is an oriented Borel-Moore  $\mathbb{L}$ -functor with products satisfying additional axioms: the projective bundle formula (PB), the extended homotopy axiom (H) and the following axioms (Dim), (Sect) and (FGL).*

If  $A^*$  is an oriented Borel-Moore functor, we say that it satisfies the (Dim) axiom if for any family of line bundles  $L_1, \dots, L_n$  over  $X \in Sm_k$ , we have  $\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_n)(1_X) = 0$  if  $n > \dim X$ , where  $1_X$  is the pull-back of  $1 \in A^0(k)$  by the structural morphism of  $X$ . The (Sect) axiom says that if  $D$  is a smooth divisor in  $X$  that is the zero locus of a section of a line bundle  $L$  on  $X$  then  $i_*(1_D) = \tilde{c}_1(L)(1)$  where  $i: D \rightarrow X$  is the inclusion. Provided the (Dim) axiom is true, it makes sense to require that if  $L$  and  $L'$  are line bundles on  $X \in Sm_k$ , then

$$\tilde{c}_1(L \otimes L')(1_X) = F(\tilde{c}_1(L), \tilde{c}_1(L'))(1_X)$$

where  $F \in A^*(k)[[X, Y]]$  is the formal power series associated to the formal group law on  $A^*(k)$  corresponding to the morphism  $\mathbb{L} \rightarrow A^*(k)$  which is part of the data, this is the (FGL) axiom.

For any  $X \in Sm_k$ , we consider the Laurent polynomials with coefficients in the Grothendieck  $K$ -group of algebraic vector bundles on  $X$ , we denote it by  $K(X)[\beta, \beta^{-1}]$  (we will only use  $K_0$ -groups, so we drop the index). The ring  $K(X)[\beta, \beta^{-1}]$  is graded so that the cohomological degree of  $\beta$  is  $-1$ . One can check that we get an oriented weak cohomology theory  $K(-)[\beta, \beta^{-1}]$  on  $Sm_k$ , the Chern operator  $\tilde{c}_1(L)$  is the multiplication by the element  $c_1(L) = (1 - [L^\vee])\beta^{-1}$ . The formula for the first Chern class obviously implies that the formal group law for this theory is given by the formal power series  $F(X, Y) = X + Y - \beta XY$ , so that this formal group law is multiplicative and periodic, the following theorem states that it is the universal one:

**Theorem 1.** *Let  $k$  be any field. For any oriented weak cohomology theory  $A^*$  on  $Sm_k$  with multiplicative and periodic formal group law, there exists a unique morphism of oriented weak cohomology theories:*

$$ch_A: K(-) [\beta, \beta^{-1}] \rightarrow A^*(-)$$

**Corollary 3.** *Let  $k$  be a field admitting the resolution of singularities. Then, for any  $X \in Sm_k$ , the natural morphism is an isomorphism:*

$$\Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z} [\beta, \beta^{-1}] \xrightarrow{\cong} K(X) [\beta, \beta^{-1}]$$

Under the assumption of the corollary, the universal property of algebraic cobordism, the fact that it satisfies the projective bundle formula and extended homotopy invariance and the previous theorem imply that the two theories considered here are universal, and then canonically isomorphic.

**Corollary 4** (Grothendieck’s Riemann-Roch). *Let  $f: X \rightarrow S$  be a projective morphism between smooth  $k$ -schemes. Then we have a commutative diagram:*

$$\begin{CD} K(X) @>ch>> CH^*(X)_{\mathbb{Q}} \\ @Vf_*VV @VVx \mapsto f_*(x \cdot Td(T_X)) \cdot Td(T_S)^{-1}V \\ K(S) @>ch>> CH^*(S)_{\mathbb{Q}} \end{CD}$$

where  $Td(T_Y)$  denotes the Todd class of the tangent bundle of a smooth scheme  $Y$  and  $ch$  the Chern character.

The proof of this corollary consists in defining a weak cohomology theory  $A^*$  such that  $A^*(X) = CH^*(X)_{\mathbb{Q}} [\beta, \beta^{-1}]$  by twisting the additive formal group law on Chow group to get a multiplicative one, this is achieved by keeping the same smooth pull-backs as for Chow groups, the projective push-forwards are defined using the formula given in the statement of the corollary; for this new theory, the first Chern class of a line bundle  $X$  is  $c_1^A(L) = (1 - \exp(-[L]))\beta^{-1}$  where  $[L]$  is the class of  $L$  in the Picard group of  $X$ , if we apply the theorem 1 to this theory, we will get a map  $ch_A: K(-) [\beta, \beta^{-1}] \rightarrow A^*$  compatible with push-forwards, so we only have to check that  $ch_A$  is the Chern character which is easy to do. The first

step in the proof of theorem 1 is to define a map  $ch_A: K(X) [\beta, \beta^{-1}] \rightarrow A^*(X)$  for any smooth  $k$ -scheme  $X$ . If one uses the compatibility that  $ch_A$  should have with Chern operators  $\tilde{c}_1$ , one can prove that  $ch_A$  must be unique and finds a candidate for  $ch_A$ . In the second step, one has to check that this  $ch_A$  is compatible with all the data: smooth pull-backs, Chern operators  $\tilde{c}_1$ , external products and projective push-forwards; this is easy except for projective push-forwards. In the third step, one checks the compatibility of  $ch_A$  with push-forwards, we can split this in two cases: projections from projective spaces  $\mathbb{P}_X^n \rightarrow X$  and closed immersions between smooth schemes. The case of projective spaces turns out to follow from the following proposition:

**Proposition 1.** *Let  $k$  be a field and  $A^*$  an oriented weak cohomology theory on  $Sm_k$  with multiplicative formal group law (i.e.  $F(X, Y) = X + Y - bXY$ , with  $b \in A^{-1}(k)$ ). Let  $X$  be an object of  $Sm_k$  and  $E$  be a vector bundle of rank  $n + 1$  on  $X$ . We denote by  $p: \mathbb{P}(E) \rightarrow X$  the projection from the projective bundle associated to  $E$ . Then, in  $A^{-n}(X)$ , we have:*

$$p_*(1) = b^n$$

The compatibility of  $ch_A$  with push-forwards associated to a closed immersion is checked in the same manner than in Fulton's book [1]: using the deformation to the normal cone, one can reduce oneself to checking the compatibility for closed immersions of the form  $s: X \rightarrow \mathbb{P}(E \oplus \mathcal{O}_X)$  where  $E$  is a vector bundle on  $X$  and  $s$  is given by the point of homogeneous coordinates  $[0 : 1]$ , which finishes this sketch of proof.

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### 9. $\Omega^*(k)$ and the Lazard ring

ANNETTE HUBER-KLAWITTER

The aim of the talk was to give an exposition of the proof of the following result of Levine and Morel:

**Theorem 1.** ([1, Theorem 12.6]) *Let  $k$  be a field of characteristic 0. Let  $\mathbf{L}_*$  denote the Lazard ring. Let  $\Omega_*$  be algebraic cobordism over  $k$ . Then the structural map*

$$\mathbf{L}_* \longrightarrow \Omega_*(k)$$

*is an isomorphism.*

Injectivity is shown along the lines of the topological case. A natural map  $\Omega_*(k) \rightarrow \mathbf{Z}[t_1, t_2, \dots]$  is constructed. (It is the map to a twisted version of Chow theory given by the universal property of algebraic cobordism.) Its composition with the map of the theorem agrees with the standard embedding of the Lazard ring into the polynomial ring (see [1, Lemma 12.2]).

Hence the real issue is surjectivity. The Theorem is easily seen to follow from the following two Lemmas:

**Lemma 1.** ([1, Theorem 4.16]) *Let  $k$  be a field. Then the natural map*

$$\mathbf{Z} \rightarrow \Omega_0(k)$$

*is an isomorphism.*

**Lemma 2.** *Define additive cobordism as*

$$\Omega_*^{ad} = \Omega_* \otimes_{\mathbf{L}_*} \mathbf{Z} .$$

*Let  $k$  be a field of characteristic 0. Then*

$$\Omega_{>0}^{ad}(k) = 0 .$$

The proof of Lemma 2 can be reduced to proving the vanishing of the class  $[Y]$  of a smooth projective variety of dimension  $d > 0$  in additive cobordism, i.e., to expressing  $[Y]$  in terms of the coefficients of the formal group law in  $\Omega_*(k)$ . These coefficients are known. In particular,  $[\mathbf{P}^1]$  occurs.

In the special case  $Y$  a smooth hypersurface of  $\mathbf{P}^{d+1}$  of degree  $n$  the axioms of algebraic cobordism together with the formulas for the class of a normal crossings divisor imply easily that  $[Y] = n[\mathbf{P}^1] = 0$ . The general case is carefully reduced to this special case via resolution of singularities. The key property of additive cobordism which is used in the argument is birational invariance:

**Proposition 1.** ([1, Proposition 12.5]) *Let  $\text{char}(k) = 0$ . Let  $W, W'$  be birationally isomorphic smooth projective varieties. Then  $[X] = [X']$  in  $\Omega_*^{ad}(k)$ .*

The proof of this proposition uses the deep fact ([2], [3]) that any birational map can be factored by a sequence of blow-ups and blow-downs with smooth centers. The case of such blow-ups can be computed directly.

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## 10. Degree formulas

ALEXANDER SCHMIDT

In the whole talk the letter  $k$  will denote a field of characteristic zero. The reference for all the results below is [1] where you also find a discussion for the case when  $k$  has positive characteristic. We denote by  $Sch/k$  the category of separated schemes of finite type over  $k$  and by  $Sm/k$  its full subcategory consisting of smooth quasi-projective  $k$ -schemes. Let  $A_*$  be an oriented Borel-Moore weak homology theory on  $Sch/k$ . Suppose that  $A_*$  is generically constant ([1], Definition 13.1). Then, for  $X \in Sch/k$  reduced with irreducible components  $X_1, \dots, X_r$ , we obtain ([1], Definition 13.4) degree maps

$$\text{deg}_i: A_*(X) \rightarrow A_{*-\dim(X_i)}(k), \quad i = 1, \dots, r.$$

Assume, in addition, that  $A_*$  satisfies the localization property ([1], Definition 13.5). Then we have the following theorem, called the generalized degree formula.

**Theorem 1.** *Let  $X \in \text{Sch}/k$  be reduced. Assume that, for each closed integral subscheme  $Z \subset X$ , we are given a projective birational morphism  $\tilde{Z} \rightarrow Z$  with  $\tilde{Z} \in \text{Sm}/k$ . Then the  $A_*(k)$ -module  $A_*(X)$  is generated by the classes  $[\tilde{Z} \rightarrow X]$ . More precisely, let  $X_1, \dots, X_r$  be the irreducible components of  $X$ . Let  $\alpha$  be an element in  $A_*(X)$ . Then, for each closed integral subscheme  $Z \subset X$  with  $\text{codim}_X Z > 0$  (i.e.  $Z$  does not contain a generic point of  $X$ ), there is an element  $\omega_Z \in A_{*-\dim(Z)}(k)$ , all but finitely many zero, such that*

$$\alpha - \sum_{i=1}^r \text{deg}_i(\alpha) \cdot [\tilde{X}_i \rightarrow X] = \sum_{Z, \text{codim}_X Z > 0} \omega_Z \cdot [\tilde{Z} \rightarrow X].$$

By [1], Corollary 13.3, algebraic cobordism is generically constant, and for a morphism  $f: Y \rightarrow X$  of smooth, projective irreducible varieties over  $k$ , we obtain the element  $\text{deg}(f) \in \Omega_d(k)$ ,  $d = \dim(Y) - \dim(X)$ . For  $X \in \text{Sm}/k$ , projective and irreducible, we consider the ideal

$$M(X) \subset \Omega_*(k)$$

generated by the classes of smooth, projective  $k$ -schemes  $Y$  with  $\dim(Y) < \dim(X)$ , such that there exists a  $k$ -morphism  $Y \rightarrow X$ . The following theorem follows immediately from the generalized degree formula.

**Theorem 2.** *For a morphism  $f: Y \rightarrow X$  between smooth projective irreducible  $k$ -schemes, one has*

$$[Y] - \text{deg}(f) \cdot [X] \in M(X).$$

Finally, we deduce from the generalized degree formula a degree formula due to M. Rost. For  $X$  smooth projective of dimension  $d$ , we set

$$s_d(X) := -\text{deg } N_d(c_1, \dots, c_d)(T_X) \in \mathbb{Z},$$

where  $N_d$  is the  $d$ -th Newton polynomial,  $T_X$  is the tangent bundle of  $X$  and  $\text{deg}: \text{CH}_0(X) \rightarrow \mathbb{Z}$  is the usual degree map. If  $d$  is of the form  $p^n - 1$  where  $p$  is a prime number and  $n \geq 1$ , then  $s_d(X)$  is divisible by  $p$ . We have the following

**Theorem 3.** *Let  $f: Y \rightarrow X$  be a morphism between smooth projective varieties of dimensions  $d > 0$ . Assume that  $d = p^n - 1$  where  $p$  is a prime number and  $n \geq 1$ . Then there exists a zero-cycle on  $X$  with integral coefficients whose degree is the integer*

$$\frac{s_d(Y)}{p} - \text{deg}(f) \cdot \frac{s_d(X)}{p}.$$

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## 11. Cobordism and Chow groups

CHRISTIAN SERPÉ

Evidently the theory  $\Omega_*^{ad} := \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}$  is the universal additive Borel-Moore weak homology theory on  $Sch/k$ . The main goal of this talk is to show that this theory is isomorphic to the Chow theory. In the second part a filtration on cobordism is given. We closely follow section 14 of [1]. For the hole talk we assume that  $k$  is a field of characteristic zero. By the universality of  $\Omega_*$  and the

fact that the group law of  $CH_*$  is additive we get a morphism

$$\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z} \rightarrow CH_*.$$

**Lemma 1.** *Let  $X \in Sch/k$ ,  $Y \in Sm/k$  irreducible and  $f : Y \rightarrow X$  be a projective morphism. Furthermore, let  $\widetilde{f(Y)} \rightarrow f(Y)$  be a resolution of singularities (i.e. the morphism  $\widetilde{f(Y)} \rightarrow f(Y)$  is birational, projective and  $\widetilde{f(Y)} \in Sm/k$ ). Then we have*

$$[Y \xrightarrow{f} X]_{\Omega_*^{ad}} = \begin{cases} \deg(Y/f(Y))[\widetilde{f(Y)} \rightarrow X]_{\Omega_*^{ad}} & \text{if } \dim Y = \dim f(Y) \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Apply the generalised degree formula to  $[Y \rightarrow f(Y)]$  and use the fact that  $\Omega_*^{ad} \simeq \mathbb{Z}$ .  $\square$

We denote by  $\mathbb{Z}_*(X)$  the free abelian group on the set of closed integral subschemes for a scheme  $X \in Sch/k$ , graded by dimension. We define a morphism

$$\phi : \mathbb{Z}_*(X) \rightarrow \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

by  $\phi(Z) := [\widetilde{Z} \rightarrow Z \rightarrow X]_{\Omega_*^{ad}}$ , where  $\widetilde{Z} \rightarrow Z$  is a resolution of singularities of  $Z$ . From Lemma 1 it follows that this is independent of the chosen resolution. We have the following observation for the morphism  $\phi$ :

- The composition  $\phi : \mathbb{Z}_*(X) \rightarrow \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \rightarrow CH_*(X)$  is the canonical morphism.
- $\phi$  is surjective (by Lemma 1)
- $\phi$  is compatible with projective push forward (by Lemma 1)

By using resolution of singularities and the fact that  $\Omega_*^{ad}$  has by definition the additive group law one can deduce

**Proposition 1.** *The morphism  $\phi$  factors through rational equivalence.*

So all together we have proven

**Theorem 1.** *The canonical morphism*

$$\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z} \rightarrow CH_*$$

*is an isomorphism of oriented Borel-Moore weak homology theories on  $Sch/k$  and  $\phi$  induces the inverse.*

From this one can deduce the following two corollaries.

**Corollary 5.** For  $X \in \text{Sch}/k$  the canonical morphism

$$\Omega_0(X) \rightarrow \text{CH}_0(X)$$

is an isomorphism.

**Corollary 6.** Let  $X$  be a smooth scheme over  $k$ .

- (1) If  $\dim(X) = 1$  we have  $\Omega_1(X) \simeq \widetilde{K_0(X)}$ .
- (2) If  $\dim(X) = 2$  we have  $\Omega_1(X) \simeq K_0(X) := \ker(K_0(X) \xrightarrow{rk} H_{Zar}^0(X, \mathbb{Z}))$ .

To get a filtration on  $\Omega_*(X)$  for a  $X \in \text{Sch}/k$  we define  $F^{(n)}\Omega_*(X)$  as the subgroup of  $\Omega_*(X)$  which is generated by classes  $[Y \xrightarrow{f} X]$  with  $Y$  smooth, irreducible, and  $\dim(Y) - \dim(f(Y)) \geq n$ . We have the following observations:

- $F^{(n)}\Omega_*(X)$  is a  $\Omega^*(k)$  submodule of  $\Omega_*(X)$
- $F^{(n)}\Omega_*(k) \simeq \Omega_{*\geq n}(k)$
- $F^{(1)}\Omega_*(X) \simeq \ker(\Omega_*(X) \rightarrow \text{CH}_*(X))$  (by theorem 1)

From the generalised degree formula one can deduce

**Theorem 2.** Let  $X \in \text{Sch}/k$  and  $n \geq 0$ . Then we have

$$F^{(n)}\Omega_*(X) \simeq \mathbb{L}_{*\geq n}\Omega_*(X).$$

Now for the associated graded it follows

**Corollary 7.** For  $X \in \text{Sch}/k$  there is a surjection

$$\mathbb{L}_* \otimes_{\mathbb{Z}} \text{CH}_* \rightarrow \text{Gr}^*\Omega_*(X)$$

of bigraded abelian groups.

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## 12. Steenrod operations and other degree formulas

JENS HORNBOSTEL

Consider an Eilenberg-Steenrod cohomology theory  $E^*(\ )$  defined on topological spaces such as singular cohomology  $H^*(\ , \mathbb{Z})$ , topological  $K$ -theory  $K^*(\ )$  or complex cobordism  $MU^*(\ )$ . A *stable cohomology operation* of degree  $d$  is a natural transformation of functors  $E^* \rightarrow E^{*+d}$  compatible with the suspension isomorphism. By the Brown representability theorem, the cohomology theory  $E^*$  is representable in the stable homotopy category  $\mathbf{SH}$  by a spectrum  $\mathbf{E}$ . It follows that the ring of all stable  $E$ -cohomology operations is isomorphic to  $[\mathbf{E}, S^* \wedge \mathbf{E}]_{\mathbf{SH}}$ . When  $E^* = H^*(\ , \mathbb{Z}/p)$  is singular cohomology with *mod*  $p$ -coefficients, then the spectrum is called the *mod*  $p$ -Eilenberg-Mac Lane spectrum and the ring of cohomology operations is called the *mod*  $p$ -Steenrod algebra and denoted by  $\mathcal{A}_p^*$ .

**Theorem 1.** *If  $p$  is a prime number, then  $\mathcal{A}_p^*$  is isomorphic to the  $\mathbb{Z}/p$ -algebra generated by elements  $P^i$  for all  $i \geq 1$  of degree  $2i(p-1)$  and  $\beta$  of degree 1 modulo the Adem relations.*

The element  $\beta$  is called the *Bockstein*. If  $p = 2$ , then the generators  $P^i$  are traditionally denoted by  $Sq^{2^i}$ .

The action of stable cohomology operations on cohomology groups provides a refinement of cohomological invariants. Stable cohomology operations appear in computations of stable homotopy groups of spheres via the Adams-Novikov spectral sequence and in the construction of other orientable cohomology theories from  $MU^*$  via the Landweber exact functor theorem. See standard textbooks on homotopy theory such as [6] for more details. Morel and Voevodsky ([5],

[7]) have constructed the stable  $\mathbb{A}^1$ -homotopy category  $\mathbf{SH}(k)$  which is the algebraic analogue of  $SH$  for algebraic varieties over a base field  $k$ . Moreover they construct a motivic Eilenberg-Mac Lane spectrum that represents bigraded motivic cohomology  $H_{mot}^{*,*}(\_, \mathbb{Z})$  and similar for finite coefficients (see [5],[7] and lectures 16,17). Voevodsky [8] constructs stable bigraded cohomology operations  $P^i : H_{mot}^{*,*}(\_, \mathbb{Z}/p) \rightarrow H_{mot}^{*+2i(p-1), *+i(p-1)}(\_, \mathbb{Z}/p)$  which conjecturally generate the bigraded endomorphism algebra over  $H_{mot}^{*,*}(k, \mathbb{Z}/p)$  of the *mod p*-motivic Eilenberg-Mac Lane spectrum in  $\mathbf{SH}(k)$  if  $p$  does not divide  $\text{char}(k)$ . He also establishes an isomorphism  $CH^n(X) \cong H_{mot}^{2n,n}(X, \mathbb{Z})$ . The induced action of the  $P^i$  on  $\bar{C}H_* := CH_* \otimes \mathbb{Z}/p$  conjecturally coincides with the Steenrod operations which Brosnan [1] constructed using equivariant Chow groups. They have been used by Merkurjev to prove a certain degree formula [4, Theorem 6.4] and reprove some results on quadrics (see lecture 13). In this lecture, we sketch another definition due

to Levine [2] of the action of Steenrod operations on  $\bar{C}H^*$  in  $\text{char}(k) = 0$  based on algebraic cobordism (which coincides with Brosnan's in this case). *From now on, we fix once and for all a prime  $p$ .* Consider the functor  $CH_*[b] := CH_*[b_1, b_2, \dots]$

where  $b_i$  is a variable of degree  $p^i - 1$ . Then there is a procedure to twist this theory, that is its first Chern class and its pull-backs as in [2, section 4]. The formal group law of this twisted theory  $CH_*[b]^{(b)}$  is no longer additive, but it is additive on the *mod p*-theory  $\bar{C}H_*[b]^{(b)}$  by [2, Proposition 9]. Consequently, the transformation  $S : \Omega_* \rightarrow CH_*[b]^{(b)}$  factors as  $\bar{S} : CH_* \rightarrow \bar{C}H_*[b]^{(b)}$ . Considering series of non-negative integers  $R = (r_1, r_2, \dots)$ , we may write  $S = \sum_R S_R b^R$  and similar *mod p*. (Observe that  $S_{(i,0,0,\dots)}$  corresponds to  $P^i$ .) The factorization of  $S$  *mod p* implies [2, Lemma 11]:

**Lemma 1.** *For all  $R \neq (0, 0, \dots)$ ,  $S_R(\Omega_{>0}(k))$  is contained in  $p \cdot CH_*(k) (\cong p \cdot \mathbb{Z})$ .*

Hence for any smooth projective variety  $X$  of dimension  $|R| > 0$  (where  $|R| := r_1(p-1) + r_2(p^2-1) + \dots$ ), we see that  $S_R(X)$  is divisible by  $p$ . This applies in particular to  $S_{(0,\dots,0,1,0,\dots)}$  with 1 in degree  $i$  which can be shown to coincide with the  $s_{p^i-1}$  of [3, section 13] and lecture 10. We set  $s_R := \frac{S_R}{p}$  and show that  $s_R(X) = \frac{1}{p} \text{deg}(c_R(-T_X)) \pmod{p}$  where  $c_R$  is the twist of the first chern

class and  $T_X$  is the tangent bundle of  $X$ . (This formula shows that Levine's  $s_R$  equals Merkurjev's  $R^p$ .) As  $S(X \times Y) = S(X)S(Y)$ , it follows [2, Lemma 12] that  $s_R(X \times Y) = 0$  if  $X$  and  $Y$  are smooth projective varieties of dimension  $\geq 1$ . Combining all this with the generalized degree formula [3, Theorem 13.6] (see lecture 10), Levine [2, Theorem 13] can give a new proof of the degree formula of Merkurjev mentioned above:

**Theorem 2.** *Let  $f : Y \rightarrow X$  be a morphism of smooth projective varieties over  $k$  of dimension  $|R|$  for some  $R = (r_1, r_2, \dots)$  as above. Then*

$$s_R(Y) \equiv \deg(f) \cdot s_R(X) \pmod{(p, I(X))}$$

where  $I(X) \subset \mathbb{Z}$  is the ideal generated by the degrees of all field extensions of all closed points of  $X$ .

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## 13. Some applications

STEFAN SCHRÖER

Let  $k$  be a ground field,  $X$  a smooth proper scheme,  $d = \dim(X)$ . Let  $n_X$  be the greatest common divisor for the numbers  $\deg(x)$ , where  $x \in X$  ranges over the closed points. Merkurjev considers numbers  $R^p(X) \in \mathbb{Z}/n_X\mathbb{Z}$  defined as follows [5]. Let  $p$  be a prime different from the characteristic of the ground field  $k$ , and  $R = (r_1, r_2, \dots, r_n)$  be a finite sequence of integers  $r_i > 0$ . Let  $c_R$  be the symmetrization of the polynomial

$$(X_1 \dots X_{r_1})^{p-1} (X_{r_1+1} \dots X_{r_1+r_2})^{p^2-1} \dots$$

which has degree  $\deg(R) = \sum r_i(p^i - 1)$ . We may view the variables  $X_i$  as Chern roots and obtain for  $\dim(X) = \deg(R)$  a zero cycle  $c_R(-T_X) \in \text{CH}_0(X)$ . The degree of  $c_R(-T_X)$  is always divisible by  $p$ . Merkurjev defines

$$R^p(X) \equiv \frac{\deg(c_R(-T_X))}{p} \text{ modulo } n_X$$

and calls the scheme  $X$   $R^p$ -rigid if  $R^p(X) \not\equiv 0$ . One should think of  $c_R(-T_X)$  as an action of an element in the Milnor basis of the Steenrod algebra. Using Brosnan's definition of Steenrod operations on Chow groups [1], the definition extends to proper schemes that are not necessarily smooth. Using the degree formula  $R^p(X) = \deg(f)R^p(Y)$  modulo  $n_Y$  for any  $f : X \rightarrow Y$ , Merkurjev proves the following:

**Theorem 1.** *Let  $\alpha$  be a correspondence from  $X$  to  $Y$ , for example the graph of a rational map. Suppose that  $X$  is  $R^p$ -rigid, that  $v_p(n_X) \leq v_p(n_Y)$  holds for the  $p$ -adic valuations, and that  $p$  does not divide the degree of  $\alpha$  over  $X$ . Then  $\dim(X) \leq \dim(Y)$ .*

Using this result, Merkurjev gives short elegant new proofs for three results on quadratic forms. The first result is due to Hoffman [2]:

**Proposition 1.** *Let  $X$  and  $Y$  be two smooth anisotropic quadrics. Suppose  $\dim(X) \geq 2^n - 1$ , and that  $Y$  becomes isotropic over the function field  $k(X)$ . Then  $\dim(Y) \geq 2^n - 1$ .*

The second result is due to Izhboldin [3]:

**Proposition 2.** *Suppose in addition that  $\dim(Y) = 2^n - 1$ . Then  $X$  becomes isotropic over the function field  $k(Y)$ .*

The last application due to Karpenko [4]. Suppose  $X$  is a Brauer-Severi variety admitting an  $\mathcal{O}_X(2)$ , and let  $Y \subset X$  be a smooth divisor with ideal isomorphic to  $\mathcal{O}_X(-2)$ . Such  $Y$  is a twisted form of a smooth quadric, which are also called involution varieties.

**Proposition 3.** *Suppose the Brauer-Severi variety  $X$  contains no nontrivial linear subvarieties. Then the twisted quadric  $Y$  remains anisotropic over the function field  $k(X)$ .*

Here a subscheme  $L \subset X$  is called linear if  $L \otimes \bar{k} \subset X \otimes \bar{k}$  is a linear subscheme in the classical sense.

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### 14. Construction of pull-backs in algebraic cobordism, part1

ANIA OTWINOWSKA

The motivation for this talk (and the next one) was to prove that  $\Omega^\star$  is the universal oriented cohomology theory on the category  $Sm/k$  of smooth schemes over  $k$ , or equivalently that  $\Omega_\star$  is the universal oriented Borel-Moore homology theory. This reduces to proving the existence of functorial pull-back maps  $f^\star : \Omega_\star(X) \rightarrow \Omega_{\star+\dim f}(Y)$  for *non-necessarily smooth* maps  $f : Y \rightarrow X$  between smooth schemes over  $k$ . As any such  $f$  is the composition of its graph  $\Gamma_f : Y \rightarrow X \times Y$  followed by the smooth projection  $pr_1 : X \times Y \rightarrow X$ , one reduces to the case where  $f$  is a closed embedding between smooth schemes, or more generally a locally complete intersection map. Mimicking Fulton’s deformation to the normal cone, one proves (see next talk) that it is enough to deal with the case where  $f : D \rightarrow X$  is a strict normal crossing divisor (SNC) : the irreducible components of  $D$  are smooth, with smooth intersections of expected dimensions. Given  $D = (L, s)$  a strict normal crossing divisor on  $X$  with support  $|D|$  (where  $L$  denotes a line bundle on  $X$  and  $s : X \rightarrow L$  a section) one would like to define a pull-back

$$\begin{aligned} Z_\star(X) & \xrightarrow{D(\cdot)} \Omega_{\star-1}(|D|) \\ (Y \xrightarrow{\phi} X, L_1, \dots, L_r) & \mapsto (\phi^\star D \rightarrow |D|, L_1, \dots, L_r) \end{aligned}$$

The following two problems can occur :

- 1) if  $\phi(Y) \subset |D|$  then the divisor  $\phi^\star D$  of  $|D|$  is not defined.
- 2) even if  $\phi(Y) \not\subset |D|$ , the divisor  $\phi^\star D$  may not be smooth, or even SNC. To solve these problems one first defines a *modified cycle group* :

**Definition 1.**

$$Z_\star(X)_D = \left\langle \{ (Y \xrightarrow{\phi} X, L_1, \dots, L_r) \in Z_\star(X) \mid \text{either } \phi(Y) \subset |D| \text{ or } \phi^\star D \text{ is SNC} \} \right\rangle$$

This is the subgroup of  $Z_\star(X)$  spanned by cycles in good position w.r.t.  $D$ . Hence one can define the pull-back :

**Definition 2.**

$$\begin{aligned} Z_\star(X)_D & \xrightarrow{D(\cdot)} \Omega_{\star-1}(|D|) \\ (Y \xrightarrow{\phi} X, L_1, \dots, L_r) & \mapsto \begin{cases} (Y \xrightarrow{\phi} D, L_1, \dots, L_r) & \text{if } \phi(Y) \subset D \\ f_\star^D(c_1(L_1^D), \dots, c_1(L_r^D))([f^\star D \rightarrow |f^\star D|]) & \text{if } \phi(Y) \not\subset D \end{cases} \end{aligned}$$

In the first part of the talk, one defines a *modified cobordism group*  $\Omega_*(X)_D$  with a natural map  $\Omega_*(X)_D \xrightarrow{\lambda} \Omega_*(X)$ . The quotient group  $\Omega_*(X)_D$  is obtained from  $Z_*(X)_D$  by mimicking the construction of  $\Omega_*(X)$  from  $Z_*(X)$ . In the second part we consider the diagram

$$\begin{array}{ccc}
 & & \Omega_*(X) \\
 & \nearrow \lambda & \vdots \\
 Z_*(X)_D & \twoheadrightarrow \Omega_*(X)_D & \Omega_{*-1}(|D|) \\
 & \searrow D(\cdot) & \downarrow \\
 & & \Omega_{*-1}(|D|)
 \end{array}$$

and prove the

**Theorem 1.** *The triangle*

$$\begin{array}{ccc}
 Z_*(X)_D & \twoheadrightarrow \Omega_*(X)_D & \\
 & \searrow D(\cdot) & \downarrow \\
 & & \Omega_{*-1}(|D|)
 \end{array}$$

*is well-defined and commutative.*

In part 2 the map  $\lambda$  is shown to be an isomorphism, completing the proof of existence of functorial pull-backs.

The key point is deformation to the normal cone.

## 15. Construction of pull-backs in algebraic cobordism, part 2

JÖRG WILDESHAUS

The purpose of this talk was to construct general pull-backs for  $\Omega_*$ , using the construction of the previous talk. It was first observed that this can be done in various settings, of which the two most important ones are the following: (1) arbitrary pull-backs for  $\Omega_*$  on the category  $Sm_k$ , (2) *l.c.i.*-pull-backs for  $\Omega_*$  on the category  $Sch_k$ . For (1), we just need the definition and basic properties of  $\Omega_*(X)_D$  as developed in talk no. 14. For (2), we need to accept that a similar definition, with analogous properties is still possible when  $X$  is only in  $Sch_k$ , and the effective divisor  $D$  is not necessarily an *NC*-divisor.

We first stated the Moving Lemma:

**Theorem 1.** *The canonical map*

$$\vartheta_X : \Omega_*(X)_D \longrightarrow \Omega_*(X)$$

*is an isomorphism.*

Using this, we then proceeded to construct *l.c.i.*-pull-backs. Given the definition of *l.c.i.*, and the presence of smooth pull-backs, what remained to be done was to construct pull-backs for regular closed immersions. The obvious solution for a codimension one regular immersion  $i_D : D \hookrightarrow X$  is to define the pull-back  $i_D^*$  as

$$i_D^* : \Omega_*(X) \xrightarrow{\vartheta_X^{-1}} \Omega_*(X)_D \xrightarrow{D(\cdot)} \Omega_{*-1}(D) ,$$

where  $D(\cdot) : \Omega_*(X)_D \rightarrow \Omega_{*-1}(D)$  is the “intersection with the divisor  $D$ ” defined in talk no. 14.

For a regular closed immersion  $i : Z \hookrightarrow Y$ , one follows Fulton’s method of deformation to the normal cone. Consider the blow-up  $W$  of  $Y \times \mathbb{P}^1$  along  $Z \times 0$ , and remove the proper transform of  $Y \times 0$ , to get an open subset  $U$  of  $W$ .  $W$  projects to  $Y$ , and it was explained how to produce elements in  $\Omega_{*+1}(U)$  from an element  $\eta \in \Omega_*(Y)$ . This involves the surjectivity of the projective push-forward

$$\Omega_{*+1}(W) \rightarrow \Omega_{*+1}(Y \times \mathbb{P}^1) .$$

In fact, using localization, one shows that the push-forward is surjective for morphisms associated to a blow-up along a regular immersion. Furthermore, one can control the kernel. This shows that the elements in  $\Omega_{*+1}(U)$  produced from  $\eta$  all have the same pull-back under  $i_V$ , where  $V$  is the intersection of  $U$  and the exceptional divisor in  $W$ . ( $i_V^*$  is defined since  $V$  is a divisor in  $U$ .) One observes that  $V$  is the normal bundle of the immersion  $i$ . This bundle is of rank  $d$ , if  $d$  denotes the codimension of  $i$ . The homotopy property then tells us that the pull-back induces an isomorphism between  $\Omega_{*-d}(Z)$  and  $\Omega_*(V)$ . Combining everything, one gets

$$i^* : \Omega_*(Y) \longrightarrow \Omega_{*-d}(Z) .$$

Nothing was said about the properties needed to check that this construction indeed gives rise to a well-defined pull-back for *l.c.i.*-morphisms. In the last part of the talk, we sketched the proof of the Moving Lemma.

## 16. The $\mathbb{A}^1$ -homotopy approach to algebraic cobordism: Part One

OLIVER RÖNDIGS

Let  $k$  be a field. The goal, motivated by algebraic topology, is to represent cohomology theories on  $\mathbf{Sm}/k$  by objects in a triangulated category, the stable motivic homotopy category. It is obtained by formally inverting certain maps of stable versions of *spaces* over  $k$ . Smooth schemes over  $k$  are not flexible enough to be used as spaces over  $k$ , since quotient space constructions do not make sense.

**Definition 1.** *Let  $\mathbf{sSet}$  be the category of simplicial sets (set-valued presheaves on the category of finite non-empty ordinals and monotone maps). A space over  $k$  is a functor  $A : (\mathbf{Sm}/k)^{\text{op}} \rightarrow \mathbf{sSet}$  which is a sheaf for the Nisnevich topology on  $\mathbf{Sm}/k$ . Let  $\mathbf{Spc}(k)$  be the category of spaces over  $k$ .*



The choice of the Nisnevich topology will be justified below. Any smooth scheme  $X$  over  $k$  defines a representable space over  $k$ , denoted by  $X: (\mathbf{Sm}/k)^{\text{op}} \rightarrow \mathbf{sSet}$ . The value of  $X$  at  $Y \in \mathbf{Sm}/k$  is the discrete simplicial set  $\text{Hom}_{\mathbf{Sm}/k}(Y, X)$ . This gives a full embedding  $\mathbf{Sm}/k \hookrightarrow \mathbf{Spc}(k)$ . Any simplicial set  $K$  defines a constant space over  $k$ , the Nisnevich sheafification of the functor  $Y \rightarrow K$ . There is a notion of *pointed* space over  $k$ , where a basepoint in  $A$  is a map  $\text{Spec}(k) \rightarrow A$ . Adding a disjoint basepoint to a space  $A$  over  $k$  produces  $A_+ = A \amalg \text{Spec}(k)$ . Let  $\mathbf{Spc}_\bullet(k)$  be the category of pointed spaces over  $k$ . It is closed symmetric monoidal under the smash product  $A \wedge B = A \times B / A \vee B$ , with unit  $T^0 = \text{Spec}(k)_+$ .

**Example 1.** Let  $p: X \rightarrow Y$  be a vector bundle in  $\mathbf{Sm}/k$ , with zero section  $z: Y \hookrightarrow X$ . The Thom space  $\text{Th}(p)$  of  $p$  is the pointed space over  $k$  obtained as the quotient of the inclusion  $X - z(Y) \hookrightarrow X$  of spaces over  $k$ . Let  $T := \text{Th}(\mathbb{A}^1 \rightarrow \text{Spec}(k))$ . Note that  $\text{Th}(p \times q) = \text{Th}(p) \wedge \text{Th}(q)$ . In particular,  $T^{\wedge n}$  is the Thom space of the trivial bundle  $\mathbb{A}^n \rightarrow \text{Spec}(k)$ .

Once the correct class of *weak equivalences* in  $\mathbf{Spc}(k)$  is set up, Thom spaces behave as in topology – see the analog of the tubular neighborhood theorem 1.

**Definition 2.** A map  $f: A \rightarrow B$  of spaces over  $k$  is a *stalkwise equivalence* if the maps  $f_x: A_x \rightarrow B_x$  induced on stalks are weak equivalences of simplicial sets. Recall that any simplicial set  $K$  has a nice topological space  $|K|$  as a geometric realization, and a map  $g$  of simplicial sets is a weak equivalence if the induced map  $|g|$  is a homotopy equivalence. Recall also that the stalks in the Nisnevich topology are the henselizations of the local rings.

The *stalkwise* homotopy category  $\mathbf{H}_s(k)$  of spaces over  $k$  is obtained by formally inverting the stalkwise equivalences. It is not good enough for our goal, since  $\mathbf{Sm}/k$  still embeds fully in the stalkwise homotopy category. To see that  $\mathbf{H}_s(k)$  is indeed a category, one shows that stalkwise equivalences are part of a model structure on spaces over  $k$  [2]. The internal hom on  $\mathbf{Spc}(k)$  is compatible with the stalkwise equivalences and thus induces an internal hom  $\underline{\text{Hom}}_{\mathbf{H}_s(k)}(-, -)$  on  $\mathbf{H}_s(k)$  [1].

**Definition 3.** A space  $C$  over  $k$  is  $\mathbb{A}^1$ -local if the projection  $\mathbb{A}_X^1 \rightarrow X$  induces an isomorphism

$$\underline{\text{Hom}}_{\mathbf{H}_s(k)}(X, C) \rightarrow \underline{\text{Hom}}_{\mathbf{H}_s(k)}(\mathbb{A}_X^1, C).$$

for every  $X \in \mathbf{Sm}/k$ . A map  $f: A \rightarrow B$  of spaces over  $k$  is a *weak equivalence* if, for every  $\mathbb{A}^1$ -local space  $C$  over  $k$ , the induced map

$$\underline{\text{Hom}}_{\mathbf{H}_s(k)}(B, C) \rightarrow \underline{\text{Hom}}_{\mathbf{H}_s(k)}(A, C)$$

is an isomorphism.

The *motivic unstable homotopy category*  $\mathbf{H}(k)$  of  $k$  is obtained by formally inverting the weak equivalences in  $\mathbf{Spc}(k)$ . The pointed version is denoted  $\mathbf{H}_\bullet(k)$ . As above,  $\mathbf{H}(k)$  is a category, because weak equivalences belong to a model structure on  $\mathbf{Spc}(k)$  [4]. Any stalkwise equivalence is a weak equivalence. The map  $\mathbb{A}^1 \rightarrow \text{Spec}(k)$  is not a stalkwise equivalence, but a weak equivalence.

**Example 2.** In  $\mathbf{H}(k)$ ,  $\mathbb{P}^1$  is isomorphic to a suspension. The diagram

$$\begin{array}{ccc} \mathbb{A}^1 - \{0\} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

is a homotopy pushout square in which  $\mathbb{A}^1$  is contractible. Hence  $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ , where  $S^1$  is the simplicial circle and  $\mathbb{G}_m$  is the space over  $k$  represented by  $\mathbb{A}^1 - \{0\}$  and pointed by 1. By collapsing  $\mathbb{A}^1$  to the point  $\mathrm{Spec}(k)$ , one sees that  $\mathbb{P}^1 \simeq T$ .

There are at least two good reasons for choosing the Nisnevich topology. One is that any smooth connected scheme of dimension  $d$  has vanishing Nisnevich cohomology in degrees  $> d$ . The other is the Homotopy Purity Theorem, which is essentially due to the fact that any smooth pair  $Z \hookrightarrow X$  Nisnevich-locally looks like  $\mathbb{A}^d \hookrightarrow \mathbb{A}^{d+c}$ . It fails in the Zariski topology.

**Theorem 1** (Morel-Voevodsky). *Let  $i: Z \hookrightarrow X$  be a closed embedding in  $\mathrm{Sm}/k$ , and let  $q$  denote the normal bundle of  $i$ . In  $\mathbf{H}_\bullet(k)$  there is an isomorphism*

$$X/X - i(Z) \cong \mathrm{Th}(q).$$

One consequence of theorem 1 is the Gysin long exact sequence for any cohomology theory represented by a spectrum over  $k$  – see corollary 8. Another consequence is a motivic Pontrjagin-Thom construction: Any projective connected  $X \in \mathrm{Sm}/k$  of dimension  $d$  admits a vector bundle  $p$  of rank  $n$  over  $X$  such that

- $\mathrm{Tan}_X + p = \mathcal{O}^{n+d}$  in  $K_0(X)$ , and
- there is a non-trivial map  $T^{n+d} \rightarrow \mathrm{Th}(p)$  in  $\mathbf{H}_\bullet(k)$ .

Details may be found in [7, Section 2]. The latter indicates that a situation in which smashing with  $T$  is invertible is desirable.

**Definition 4.** A spectrum  $E$  over  $k$  consists of a sequence  $(E_0, E_1, \dots)$  of pointed spaces over  $k$ , together with structure maps  $\sigma_n: T \wedge E_n \rightarrow E_{n+1}$  for every  $n \geq 0$ . A map of spectra over  $k$  is the obvious thing.

**Example 3.**

- The sphere spectrum over  $k$  is given by the sequence  $\mathbb{I} = (T^0, T, T^{\wedge 2}, \dots)$ . Every structure map is the identity. Similarly, any smooth scheme  $X$  over  $k$  has a suspension spectrum  $\Sigma^\infty X = (X_+, T \wedge X_+, T^{\wedge 2} \wedge X_+, \dots)$ .
- Let  $\mathrm{Spc}(k)^{\mathrm{tr}}$  be the category of simplicial Nisnevich sheaves with transfers (additive contravariant functors from the category of smooth finite correspondences over  $k$  to the category of simplicial abelian groups [5]). Forgetting the addition induces a functor  $u: \mathrm{Spc}^{\mathrm{tr}}(k) \rightarrow \mathrm{Spc}(k)$  which has a left adjoint  $\mathbb{Z}^{\mathrm{tr}}: \mathrm{Spc}(k) \rightarrow \mathrm{Spc}^{\mathrm{tr}}(k)$ . Set  $\mathrm{HZ}_n := u\mathbb{Z}^{\mathrm{tr}}(T^{\wedge n})$ . Structure maps of  $\mathrm{HZ}$  are induced by the fact that  $u$  and  $\mathbb{Z}^{\mathrm{tr}}$  are monoidal:

$$T \wedge \mathrm{HZ}_n \longrightarrow u\mathbb{Z}^{\mathrm{tr}}(T) \wedge u\mathbb{Z}^{\mathrm{tr}}(T^{\wedge n}) \longrightarrow u\mathbb{Z}^{\mathrm{tr}}(T \wedge T^{\wedge n}) = \mathrm{HZ}_{n+1}.$$

where the first map is  $\mathrm{unity} \wedge \mathrm{id}$

One may also deduce that  $\mathrm{HZ}$  is a commutative ring spectrum over  $k$ .

- As in topology, there is a universal bundle  $\gamma_n$  of rank  $n$  obtained as the colimit over  $m$  of the tautological bundles  $\gamma_n^m \rightarrow \mathrm{Gr}_m(\mathbb{A}^{m+n})$ . Set  $M\mathrm{Gl}_n := \mathrm{colim}_m \mathrm{Th}(\gamma_n^m)$ . The bundle pulls back to the direct sum  $\gamma_n^m \oplus \mathcal{O}$ . By example 1, this induces a structure map  $T \wedge M\mathrm{Gl}_n \rightarrow M\mathrm{Gl}_{n+1}$ . Again  $M\mathrm{Gl}$  is a ring spectrum over  $k$ .

There is an obvious notion of *levelwise equivalence* of spectra over  $k$ , which produces a levelwise homotopy category  $\mathbf{H}_{\mathrm{level}}(k)$ . It is not stable, because the suspension functor  $S^1 \wedge -$  does not induce an equivalence on  $\mathbf{H}_{\mathrm{level}}(k)$ . To remedy this, a coarser notion of equivalence of spectra over  $k$  has to be introduced.

**Definition 5.** Abbreviate  $\underline{\mathrm{Hom}}_{\mathbf{H}_\bullet(k)}(T, -)$  by  $\Omega_T$ . A spectrum  $G$  over  $k$  is an  $\Omega_T$ -spectrum over  $k$  if for any  $n \geq 0$ , the morphism  $G_n \rightarrow \Omega_T G_{n+1}$  induced by the structure map is an isomorphism in  $\mathbf{H}_\bullet(k)$ . A map  $f: E \rightarrow F$  of spectra over  $k$  is a stable equivalence if, for any  $\Omega_T$ -spectrum  $G$  over  $k$ , the induced map

$$\mathrm{Hom}_{\mathbf{H}_{\mathrm{level}}(k)}(F, G) \rightarrow \mathrm{Hom}_{\mathbf{H}_{\mathrm{level}}(k)}(E, G)$$

is an isomorphism.

Formally inverting the stable equivalences in the category of spectra over  $k$  produces the *stable motivic homotopy category*  $\mathbf{SH}(k)$  of  $k$ . Again stable equivalences are part of a model structure, so  $\mathbf{SH}(k)$  is a decent category. Almost by construction  $T \wedge -: \mathbf{SH}(k) \rightarrow \mathbf{SH}(k)$  is an equivalence. Since  $T$  is itself a suspension by example 2,  $\mathbf{SH}(k)$  is then a triangulated category. The suspension  $S^1 \wedge -$  induces the shift functor, and the triangles are induced by cofiber sequences  $E \hookrightarrow F \rightarrow F/E$  of spectra over  $k$ . The cyclic permutation on  $T^{\wedge 3}$  is homotopic to the identity, thus  $\mathbf{SH}(k)$  has a closed symmetric monoidal product which is compatible with the triangulated structure [3]. The unit is the sphere spectrum. Due to its excellent properties,  $\mathbf{SH}(k)$  is the perfect place to shop around for cohomology theories on  $\mathrm{Sm}/k$ .

**Corollary 8.** Let  $E$  be a spectrum over  $k$  and  $X \in \mathrm{Sm}/k$ . Then

$$E^{p,q}(X) := \mathrm{Hom}_{\mathbf{SH}(k)}(\Sigma^\infty X, S^{p-2q} \wedge T^{\wedge q} \wedge E)$$

defines a cohomology theory on  $\mathrm{Sm}/k$  having the extended homotopy property and long exact sequences of Gysin, Mayer-Vietoris and blow-up type.

Here is one example of a representable cohomology theory. See [6] for a proof.

**Theorem 2** (Voevodsky). The cohomology theory represented by  $\mathrm{HZ}$  coincides with Bloch's higher Chow groups.

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## 17. The motivic Thom spectrum $M\mathbb{G}\ell$ and the algebraic cobordism

$$\Omega^*(-)$$

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This is a report on a "work in progress" of F. Morel and M. J. Hopkins. Their work is a step toward the identification of the motivically defined theory  $M\mathbb{G}\ell^{2*,*}(-)$  with the geometrically defined one  $\Omega^*(-)$ . Namely, they prove:

**Theorem 1.** *For any smooth  $k$ -variety  $X$  the natural graded homomorphism  $M\mathbb{G}\ell^{2*,*}(X) \rightarrow \Omega^*(X)$  is surjective.*

The plan of the lecture was:

- (1) Some basic properties of  $M\mathbb{G}\ell$ .
- (2) The computation of  $M\mathbb{G}\ell^{2*,*}(k)$ .
- (3) Proof of the main theorem.

From now on, the base field  $k$  is fixed and all our varieties will be  $k$ -varieties. For simplicity we shall assume  $k$  to be of characteristic zero.

### 1. SOME BASIC PROPERTIES OF $M\mathbb{G}\ell$

In this first part, we transpose from the topological to the motivic context some classical properties of the Thom spectrum. We denote by  $T = \mathbb{A}^1/\mathbb{G}_m$  one of the motivic spheres. When speaking about spectra, we shall always mean  $T$ -spectra. The  $\mathbb{A}^1$ -homotopy category of spectra is a triangulated category denoted by  $\mathbf{SH}(k)$  (cf. Morel [1]).

Let us recall that as in algebraic topology, the motivic Thom spectrum is defined by the collection:  $(\mathbb{S}^0, Th(\gamma_1), \dots, Th(\gamma_n), \dots)$  together with the usual assembly maps. Here  $\gamma_n$  is the tautological vector bundle on the infinite Grassmanian of  $n$ -planes. For a smooth variety  $X$ , we put  $M\mathbb{G}\ell^{p,q}(X) = [X_+, M\mathbb{G}\ell \wedge T^q[p - 2q]]$ .

**Lemma 1.**  *$M\mathbb{G}\ell$  is an oriented ring spectrum.*

The proof is exactly the same as the classical one. It is based on the identification of  $Th(\gamma_1)$  with the pointed space  $(\mathbb{P}^\infty, *)$ .

As a consequence, we can define for a line bundle  $\mathcal{L}$  on  $X$  a first Chern class  $c_1(\mathcal{L}) \in M\mathbb{G}\ell^{2,1}(X)$  by the composition:  $X \xrightarrow{[\mathcal{L}]} \mathbb{P}^\infty \rightarrow M\mathbb{G}\ell \wedge T$ . Using this, one obtains a projective bundle formula by the usual method, and then the other Chern classes for vector bundles. This can be used to define the Thom classes:

**Definition-Construction 1.** Let  $\mathcal{V}/X$  be a vector bundle of rank  $r$ . The Thom class  $t(\mathcal{V})$  of  $\mathcal{V}$  lives in  $M\mathbb{G}l^{2r,r}(Th(\mathcal{V}))$ . It is defined in the following manner: Recall that one model of  $Th(\mathcal{V})$  is  $\mathbb{P}(\mathcal{V} + 1)/\mathbb{P}(\mathcal{V})$ . Thus one have a long exact sequence (which breaks into short ones):

$$\begin{array}{ccccc}
 M\mathbb{G}l^{*,*}(Th(\mathcal{V})) & \longrightarrow & M\mathbb{G}l^{*,*}(\mathbb{P}(\mathcal{V} + 1)) & \longrightarrow & M\mathbb{G}l^{*,*}(\mathbb{P}(\mathcal{V})) \\
 & & \parallel & & \parallel \\
 & & M\mathbb{G}l^{*,*}(k)[1, u, \dots, u^r] & & M\mathbb{G}l^{*,*}(k)[1, u, \dots, u^{r-1}]
 \end{array}$$

We then define  $t(\mathcal{V})$  to be the element of the middle group equal to  $u^r - c_1.u^{r-1} + \dots + (-1)^r.c_r$  where  $c_i$  are such that the image of  $t(\mathcal{V})$  became zero in the last group. The exactness of the sequence give us a unique anteced ent of  $t(\mathcal{V})$  in the first group. This is the Thom class.

A consequence of this construction is:

**Lemma 2.**  $M\mathbb{G}l$  is the universal oriented ring spectrum.

Indeed let  $E$  be such a spectrum. The construction above still make sens for  $E$ . In particular if we take the Thom classes of  $\gamma_n$  we get maps:  $Th(\gamma_n) \mapsto E \wedge T^n$  yielding the unique map of spectra  $M\mathbb{G}l \mapsto E$ . Later on, we shall apply this to  $E = H\mathbb{Z}$ , the motivic cohomology spectrum, to get the morphism:  $M\mathbb{G}l \mapsto H\mathbb{Z}$ . The next step of our study is the Thom isomorphism. Let  $\mathcal{V}/X$  be a vector bundle of rank  $r$ . Define (as in topology) the reduced diagonal:  $Th(\mathcal{V}) \mapsto Th(\mathcal{V}) \wedge X_+$  using the pull-back square:

$$\begin{array}{ccc}
 \mathcal{V} & \longrightarrow & \mathcal{V} \times 0 \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\Delta} & X \times X
 \end{array}$$

**Theorem-Definition 1.** For any oriented ring spectrum  $E$ , the following composition:

$$E \wedge Th(\mathcal{V}) \longrightarrow E \wedge Th(\mathcal{V}) \wedge X_+ \longrightarrow E \wedge E \wedge T^r \wedge X_+ \longrightarrow E \wedge T^r \wedge X_+$$

is an isomorphism. It is called the Thom isomorphism.

Roughly speaking, the above result says that an oriented ring spectrum does not make the difference between the Thom space of a non trivial vector bundle and the Thom space of a trivial one with the same rank. A consequence of that is a natural isomorphism:  $E^{*,*}(Th(\mathcal{V})) = E^{*-2r,*-r}(X)$ .

We end this section by constructing transfers map for  $M\mathbb{G}l^{2*,*}(-)$ . It is sufficient to consider the case of a closed immersion and the projection of a projective space over  $X$ . The second case follow easily from the projective bundle formula. For a closed immersion we need to use the Thom isomorphism. Indeed, let  $i : Y \subset X$  be a closed immersion. We denote by  $\nu_i, \nu_X$  and  $\nu_Y$  the normal bundles of  $i, X$  and  $Y$ . Note that  $\nu_X$  and  $\nu_Y$  are not vector bundles in the usual sens but only virtual

one (that is of negative rank). As in topology, we can form the composition in  $\mathbf{SH}(k)$ :  $Th(\nu_X) \longrightarrow Th(i^*\nu_X \oplus \nu_i) \xlongequal{\quad} Th(\nu_Y)$  When applying  $E^{*,*}$  we get a map in the opposite direction:  $E^{*,*}(Th(\nu_Y)) \longleftarrow E^{*,*}(Th(\nu_X))$ . Now using the Thom isomorphism, we have the identifications

$$E^{*,*}Th(\nu_Y) \simeq E^{*+2d_Y, **+d_Y}(Y) \quad \text{and} \quad E^{*,*}Th(\nu_X) \simeq E^{*+2d_X, **+d_X}(X)$$

Where  $d_X$  and  $d_Y$  are the dimension of  $X$  and  $Y$ . Then denoting  $c = d_X - d_Y$  the codimension of  $Y$  in  $X$ , we obtain the wanted transfer map:  $E^{*,*}(Y) \longleftarrow E^{*+2c, **+c}(X)$ . As a consequence,  $E^{2*,*}(-)$  is an oriented Borel-Moore cohomology theory. In particular using the universality of  $\Omega^*(-)$  we get the natural homomorphism in Theorem 1.

## 2. THE COMPUTATION OF $M\mathbb{G}\ell^{2*,*}(k)$

The main step of the proof of Theorem 1 is the following proposition:

**Proposition 1.** *The canonical homomorphism given by the formal group law:  $\mathbb{L}_* \longleftarrow M\mathbb{G}\ell^{-2*, -*}(k)$  is an isomorphism.*

The injectivity of the above homomorphism is easy: one use for example a complex realization. There is also a purely algebraic proof based on a Quillen trick... The main difficulty is to show the surjectivity. For this one need a difficult lemma:

**Lemma 3.** *The canonical morphism of spectra:  $M\mathbb{G}\ell \longleftarrow H\mathbb{Z}$  induce an isomorphism<sup>1</sup>:*

$$M\mathbb{G}\ell/(x_1, \dots, x_n, \dots) \xrightarrow{\sim} H\mathbb{Z}$$

Where  $x_i$  are generator of the Lazard ring.

Assuming Lemma 3, the proof of Proposition 1 goes by induction on  $*$ . The point is that for  $N > 0$ , one have  $[T^N, H\mathbb{Z}] = 0$  by Voevodsky cancellation theorem. Using a stability argument, the Lemma 7 implies that

$$[T^N, M\mathbb{G}\ell/(x_1, \dots, x_N)] = 0.$$

Then if we apply  $[T^N, -]$  to the distinguished triangle:

$$M\mathbb{G}\ell/(x_1, \dots, x_{N-1}) \wedge T^N \xrightarrow{x_N} M\mathbb{G}\ell/(x_1, \dots, x_{N-1}) \rightarrow M\mathbb{G}\ell/(x_1, \dots, x_N) \rightarrow$$

we get a surjection:  $x_N : \mathbb{Z} \longleftarrow M\mathbb{G}\ell^{-2N, -N}/(x_1, \dots, x_{N-1})(k)$ . Using the induction hypothesis, one deduce that:  $\mathbb{L}_N \longleftarrow M\mathbb{G}\ell^{-2N, -N}(k)$  is indeed a surjection.

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<sup>1</sup>The quotient ring spectrum  $M\mathbb{G}\ell/(x_1, \dots, x_n, \dots)$  is not so easy to construct. Some serious technical difficulties arise if one try to do this naively. One way to overcome these difficulties is to work in a category of  $M\mathbb{G}\ell$ -modules.

## 3. PROOF OF THE MAIN THEOREM

A consequence of Proposition 1 is that  $M\mathbb{G}\ell^{2*,*}(-)$  is generically constant. Moreover, we have a weak form of the localization property, namely: *given a smooth pair  $(Y \subset X)$  with  $Y$  of codimension  $c$ , one has an exact sequence:*

$$M\mathbb{G}\ell^{2*-2c,*-c}(Y) \longrightarrow M\mathbb{G}\ell^{2*,*}(X) \longrightarrow M\mathbb{G}\ell^{2*,*}(X - Y)$$

These properties suffice to derive a generalized degree formula (see [2], [3]) for  $M\mathbb{G}\ell^{2*,*}(-)$ . In particular, this implies that  $M\mathbb{G}\ell^{2*,*}(X)$  is generated as a  $M\mathbb{G}\ell^{-2*,-*}(k) = \mathbb{L}_*$ -module by cobordism cycles:  $[Z \hookrightarrow X]$  with  $Z$  a desingularization of a closed subset of  $X$ . This clearly implies Theorem 1.

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