Mathematisches Forschungsinstitut Oberwolfach

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Discrete Geometry

Organised by
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Abstract. The workshop on Discrete Geometry was attended by 53 participants, many of them young researchers. In 13 survey talks an overview of recent developments in Discrete Geometry was given. These talks were supplemented by 16 shorter talks in the afternoon, an open problem session and two special sessions.

Mathematics Subject Classification (2000): 52Cxx.

Introduction by the Organisers

The *Discrete Geometry* workshop was attended by 53 participants from a wide range of geographic regions, many of them young researchers (some supported by a grant from the European Union). The morning sessions consisted of survey talks providing an overview of recent developments in Discrete Geometry:

- Extremal problems concerning convex lattice polygons. (Imre Bárány)
- Universally optimal configurations of points on spheres. (Henry Cohn)
- Polytopes, Lie algebras, computing. (Jesús A. De Loera)
- On incidences in Euclidean spaces. (György Elekes)
- Few-distance sets in d-dimensional normed spaces. (Zoltán Füredi)
- On norm maximization in geometric clustering. (Peter Gritzmann)
- Abstract regular polytopes: recent developments. (Peter McMullen)
- Counting crossing-free configurations in the plane. (Micha Sharir)
- Geometry in additive combinatorics. (József Solymosi)
- Rigid components: geometric problems, combinatorial solutions. (Ileana Streinu)
- Forbidden patterns. (János Pach)

- Projected polytopes, Gale diagrams, and polyhedral surfaces. (Günter M. Ziegler)
- What is known about unit cubes? (Chuanming Zong)

There were 16 shorter talks in the afternoon, an open problem session chaired by Jesús De Loera, and two special sessions: on geometric transversal theory (organized by Eli Goodman) and on a new release of the geometric software Cinderella (Jürgen Richter-Gebert). On the one hand, the contributions witnessed the progress the field provided in recent years, on the other hand, they also showed how many basic (and seemingly simple) questions are still far from being resolved. The program left enough time to use the stimulating atmosphere of the Oberwolfach facilities for fruitful interaction between the participants.

Workshop: Discrete Geometry

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Abstracts

Siegel's Lemma w. r. t. Maximum Norm and Sum–Distinct Sets

Iskander Aliev

Let $\|\cdot\|$ denote the maximum norm. We show that for any non–zero vector $\mathbf{a} \in \mathbb{Z}^n$, $n \geq 5$, there exist linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} \in \mathbb{Z}^n$ such that $\mathbf{x}_i \mathbf{a} = 0$, $i = 1, \ldots, n-1$ and

$$0 < \|\mathbf{x}_1\| \cdots \|\mathbf{x}_{n-1}\| < \frac{\|\mathbf{a}\|}{\sigma_n}, \quad \sigma_n = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^n dt\,.$$

This result implies a new lower bound on the greatest element of a sum–distinct set of positive integers (Erdös–Moser problem). The main tool is the Busemann theorem from convex geometry.

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Minimum Spanning Trees in the Unit Disk

Christoph Ambühl

The aim of this talk is to sketch a proof of the following theorem.

Theorem 1. *Let S be a set of points from the unit disk around the origin, with the additional property that the origin is in S. Let $e_1, e_2, \ldots, e_{|S|-1}$ be the edges of the Euclidean minimum spanning tree of S. Then*

$$\mu(S) := \sum_{i=1}^{|S|-1} |e_i|^2 \leq 6.$$

There is a long history of upper bounds on $\mu(S)$. Already in 1968, Gilbert and Pollack [5] gave a upper bound of $8\pi/\sqrt{3}$. In 1989, Steele gave a bound of 16 based on space filling curves [8]. The problem recently became very popular in the context of wireless networks. It is used to give an upper bound on the approximation ratio of an algorithm to compute energy efficient broadcast trees in wireless networks. In [9], Wan, Călinescu, Li, and Frieder claimed that $\mu(S) \leq 12$. Unfortunately, there is a small error in their paper. The correct analysis only yields $\mu(S) \leq 12.15$, as stated by Klasing, Navarra, Papadopoulos, and Perennes in [6]. Independently, Clementi, Crescenzi, Penna, Rossi, and Vocca showed $\mu(S) \leq 20$ [3]. Recently, Flammini, Klasing, Navarra, and Perennes [4] showed $\mu(S) \leq 7.6$. Even more recently, Navarra proved $\mu(S) \leq 6.33$ [7]. In this talk, we sketch a proof of $\mu(S) \leq 6$ [1]. This matches the lower bound given in [3] and [9].

Our proof is influenced by the method used in [5, 9, 3]. It works as follows. The cost of each edge e of the MST is represented by a geometric shape. In the case of [5] and [9], so-called diamonds where used for this purpose. Diamonds consist of two isosceles triangles with an angle of 120°. The area of a diamond for an edge e with length $|e|$ is $\lambda \cdot |e|^2$, with $\lambda = \sqrt{3}/6$. Therefore, $\mu(S)$ can be expressed as $1/\lambda$ times the total area generated by the diamonds. Diamonds are considered being open sets. It can be shown that the diamonds do not intersect if one puts them along the edges of an MST with one triangle on each side of the edges. Using this property, one can show that the largest area that can be covered by the diamonds is 12.15λ. Therefore one can conclude $\mu(S) \leq 12.15\lambda/\lambda = 12.15$.

Among the shapes that do not intersect, diamonds seem to be the best possible geometric shape for this kind of analysis. For a better bound, we need to use larger shapes and we need to deal with the intersections of these shapes accurately. The shapes used for our new bound are pairs of equilateral triangles, one on each side of the edge as depicted in Figure 1 on the left. The equilateral triangles intersect heavily, and therefore the analysis becomes much more involved.

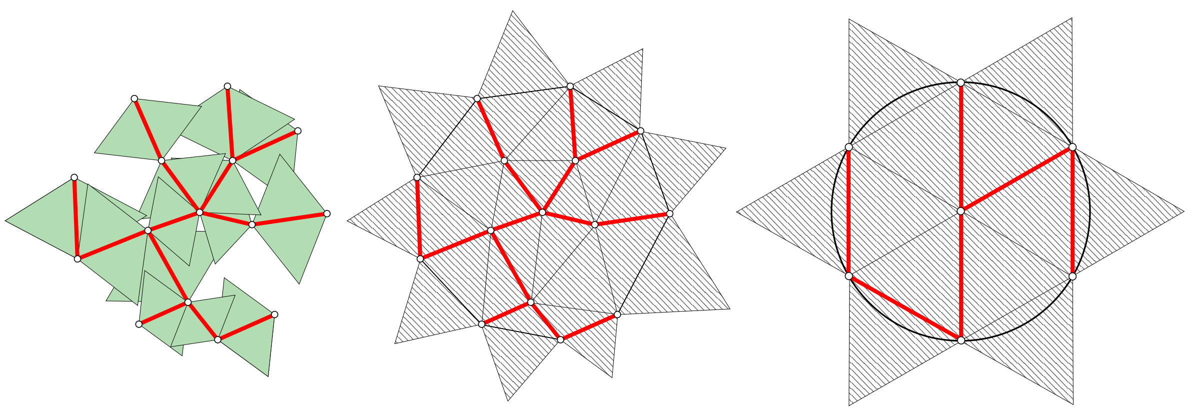

Figure 1. The total area of the equilateral triangles on the left is bounded by the hatched area in the middle. The point set that maximizes the hatched area is shown on the right.

A high level description of the proof of our bound is the following. Consider a point set S with n points. Hence, the MST will have $n-1$ edges and therefore, there will be $2(n-1)$ equilateral triangles representing the cost of the MST. Let M be the total area generated by these triangles.

In order to obtain an upper bound on M, let c be the number of edges of the convex hull of S. By triangulating S, we obtain a planar graph G with $2(n-1)-c$ triangles. Hence, if we add c equilateral triangles along the convex hull of S as depicted in the center of Figure 1, the number of triangles becomes $2(n-1)$, which is equal to the number of triangles involved in M. Let A be the total area of the triangles within the convex hull of S plus the c additional triangles along the convex hull, as depicted in the center of Figure 1.

It can be shown that $M \leq A$. To get an intuitive understanding of this fact, consider a point set S obtained from the triangular grid for which all edges of the triangulation of its convex hull have the same length. In this case, all triangles that are involved in M and A are congruent. Furthermore, since their number is equal, it holds $M = A$. Intuitively, if the edges of the triangulation have different lengths, M will be smaller compared to A since the MST will be composed mainly of small edges.

Having $M \leq A$ at hand, one can then show that A is maximized by the point set shown on the right of Figure 1, which directly leads to the bound $\mu(S) \leq 6$.

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Unbounded number of geometric permutations for families of translates in $\mathbb{R}^3$.

Andrei Asinowski

(joint work with Meir Katchalski)

Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a finite family of n pairwise disjoint convex sets in $\mathbb{R}^d$. A line l is a *transversal* of $\mathcal{F}$ if it intersects all the members of $\mathcal{F}$. Each non-directed transversal intersects the members of $\mathcal{F}$ in an order which can be described by a pair of permutations of $\{1, 2, \ldots, n\}$ which are reverses of each other. Such a pair is called a *geometric permutation*.

There are several results concerning the maximal number of geometric permutations for families of n disjoint convex sets in $\mathbb{R}^d$. Some results deal with families with the restriction that the members of the family are disjoint translates of a convex set. Katchalski, Lewis and Liu proved [3, 4] that for such families in $\mathbb{R}^2$, the maximal number of geometric permutations is 3. They also conjectured [4] that for each natural d, there is a *constant* upper bound on the number of geometric permutations for such families in $\mathbb{R}^d$ (the conjectured upper bound was $\frac{(d+1)!}{2}$). However, the only known upper bound in $\mathbb{R}^d$ is $O(n^{d-1})$ (follows from [6]). A constant upper bound is known in a special case: for families of congruent balls in $\mathbb{R}^d$ [5] (improved in [1]; the bound is 2 when $n \geqslant 9$).

We refute the mentioned above conjecture, showing the following:

For each $n \in \mathbb{N}$, $n > 1$, there exists a convex set $X = X(n)$ in $\mathbb{R}^3$ and a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of X that admits at least $n+1$ geometric permutations.

The proof is by construction of an example of such a family. The construction uses the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$. The idea is to take first a family of disjoint sets that have, or nearly have, the desired transversal properties, but are not translates of each other, and then to append them one to another in order to obtain translates, preserving their disjointness and transversal properties (the same idea was used in a construction due to Holmsen and Matoušek [2]).

A brief description of our construction follows.

Points, lines and the set X

Denote by Σ the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$. For each $i \in \{0, 1, \ldots, n\}$, let λ_i be the plane $y = i$, and let l_i be the line $\lambda_i \cap \Sigma = \{(x, y, z) : y = i, z = xi\}$.

For each $m \in \{1, 2, \ldots, n\}$, define four points on Σ as follows:

$$P_{m,1} = (2mn^2, m-1, 2mn^2 \cdot (m-1)),$$
$$P_{m,2} = (2mn^2+1, m, (2mn^2+1) \cdot m);$$
$$Q_{m,1} = (2mn^2, m, 2mn^2 \cdot m),$$
$$Q_{m,2} = (2mn^2+1, m-1, (2mn^2+1) \cdot (m-1)).$$

Let a_m be the segment that contains $P_{m,1}$ and $P_{m,2}$ with endpoints in the planes λ_0 and λ_n, and let b_m be the segment that contains $Q_{m,1}$ and $Q_{m,2}$ with endpoints in the planes λ_0 and λ_n. Figure 1 shows a_i's and b_i's for $n = 3$ (In this figure, the solid parts of the segments are above Σ, and the dashed are below it. Note that the figure is not drawn to scale: in fact, the segments are much further apart).

Now define two sets X^L and X^U. Each of them is a polygonal line: $X^L = \tilde{a}_1 \cup \tilde{a}_2 \cup \cdots \cup \tilde{a}_n$, $X^U = \tilde{b}_1 \cup \tilde{b}_2 \cup \cdots \cup \tilde{b}_n$, where each $\tilde{a}_m$ is a translate of a_m, and each $\tilde{b}_i$ is a translate of b_i, so that:

- the lowest point of $\tilde{a}_1$ is $(0, 0, 0)$, and for each $m \in \{2, 3, \ldots n\}$ the lowest point of $\tilde{a}_m$ coincides with the highest point of $\tilde{a}_{m-1}$;
- the highest point of $\tilde{b}_1$ is $(0, n^2, H_L + H + H_U)$ (where H_L and H_U are the z-heights of X^L and X^U respectively, and H is a large positive number),

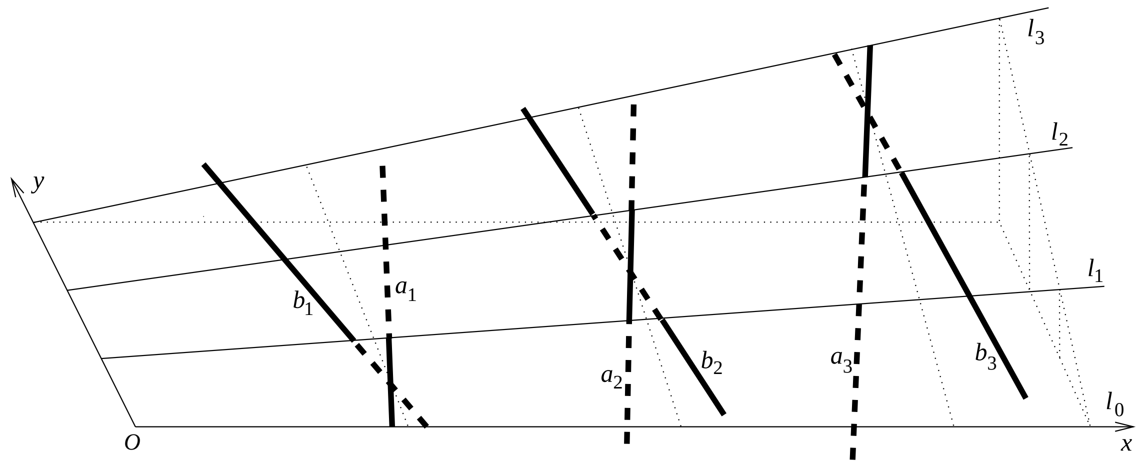

FIGURE 1. The segments a_i and b_i, for $n = 3$.

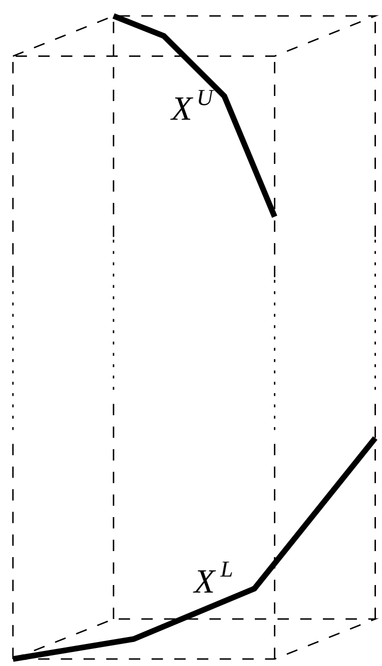

FIGURE 2. The set X for $n = 3$: the bold polygonal lines are X^L and X^U; X is their convex hull.

and for each $m \in \{2, 3, \ldots n\}$ the highest point of $\tilde{b}_m$ coincides with the lowest point of $\tilde{b}_{m-1}$.

Let $X = \operatorname{conv}(X^L \cup X^U)$ (see Figure 2; X^U is situated high above X^L).

The family $\mathcal{F}$ of disjoint translates of X

For each $m \in \{1, 2, \ldots, n\}$, define A_m to be a translate of X with $\tilde{a}_m$ translated to a_m, and B_m to be a translate of X with $\tilde{b}_m$ translated to b_m.

Define $\mathcal{F} = \{A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n\}$. It can be checked that the members of $\mathcal{F}$ are pairwise disjoint, that the lines $l_0, l_1, \ldots, l_n$ are transversals of $\mathcal{F}$, and that these lines induce the following geometric permutations on $\mathcal{F}$:

$$\begin{aligned}
&l_0 : (A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n)\\
&l_1 : (B_1, A_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n)\\
&l_2 : (B_1, A_1, B_2, A_2, A_3, B_3, \ldots, A_n, B_n)\\
&l_3 : (B_1, A_1, B_2, A_2, B_3, A_3, \ldots, A_n, B_n)\\
&\quad \ldots\\
&l_n : (B_1, A_1, B_2, A_2, B_3, A_3, \ldots, B_n, A_n).
\end{aligned}$$

Thus $\mathcal{F}$ is a family of $2n$ disjoint translates of the convex set X that has the $n+1$ geometric permutations listed above.

To summarize, the maximal number of geometric permutations for families of n disjoint translates of a convex set in $\mathbb{R}^3$ is $O(n^2)$ (by [6]) and $\Omega(n)$ (by our construction). The problem of narrowing this gap remains open.

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Extremal problems concerning convex lattice polygons

Imre Bárány

(joint work with Maria Prodromou)

In this survey-type talk, four extremal problems on convex lattice polygons were discussed.

(1) which convex lattice n-gon has the smallest area?

(2) which convex lattice n-gon has the smallest lattice width?

(3) given a norm in $\mathbb{R}^2$, which convex lattice n-gon has the shortest perimeter?

The answer to (1) is that the minimal area is about $c \cdot n^3$, where the value of the constant c is about 0.0186067. It is also nown the minimiser has very oblong shape. Details can be found in [2].

The answer to (2) is that the minimiser has lattice width $\lfloor n/2 \rfloor$. The proof of this is very simple and is left to the reader.

The solution to the third problem is based on the fact that for even n the minimiser is almost unique: if B is the unit ball of the given norm, we let rB be smallest blown-up copy B that contains n primitive vectors. These n vectors come in pairs $+p, -p$ so their sum is zero. Consequently there is a convex lattice n-gon having exactly these primitive vector as edges. Extending this construction to the n is odd case causes some difficulties. Details can be found in a forthcoming paper by M. Prodromou [4].

In this extended abstract I describe the fourth extremal problem more thoroughly. Let $K \subset \mathbb{R}^2$ be a convex body and let $\mathbb{Z}_t = \frac{1}{t}\mathbb{Z}^2$ be a shrunken copy of the usual integer lattice, t is large. Write $\mathcal{P}(K,t)$ for the set of all convex $\mathbb{Z}_t$-lattice polygons that are contained in K. The size of $\mathcal{P}(K,t)$ is known asymptotically (cf. [1] or [6] or [5]):

$$\log|\mathcal{P}(K,t)| = 3\sqrt[3]{\frac{\zeta(3)}{4\zeta(2)}}A(K)t^{2/3}(1+o(1)),$$

where $A(K)$ is the supremum of $AP(S)$, the affine perimeter of $S \subset K$, with the supremum taken over all convex subsets of K. It is also known (cf. [1]) that there is a unique convex $K_0 \subset K$ such that $AP(K_0) = A(K)$. Moreover, the overwhelming majority of the elements of $\mathcal{P}(K,t)$ are very close to K_0 as $t \to \infty$. In other words, $\mathcal{P}(K,t)$ has a "limit shape" as $t \to \infty$.

The fourth extremal problem is the following: Determine

$$m(K,t) = \max\{n : \mathcal{P}(K,t) \text{ contains an } n-\text{gon}\}.$$

The answer is as follows:

Theorem 1. *With the above notation*

$$\lim_{t\to\infty} t^{-2/3}m(K,t) = \frac{3}{(2\pi)^{2/3}}A(K).$$

Moreover, we also proved that the maximisers have a limit shape, actually the same limit shape as $\mathcal{P}(K,t)$. Namely, if $Q_t \in \mathcal{P}(K,t)$ is a maximiser for $m(K,t)$, then the Hausdorff distance of Q_t and K_0 tends to zero as t tends to infinity. The proof is based on a combination of arguments from number theory, or rather geometry of numbers, and convex geometry. Details can be found in [3].

An interesting by-product of the proof is a novel characterisation of the mapping $K \to K_0$. Write $\mathcal{K}$ for the set of all convex bodies in $\mathbb{R}^2$, and $\mathcal{C}$ for those $K \in \mathcal{K}$ whose centre of gravity coincides with the origin. Also, define $F : \mathcal{K} \to \mathcal{K}$ by $F(K) = K_0$.

Let $\rho(u) = \rho_C(u)$ be the radial function of C in direction u where $u \in S^1$ is a unit vector, and for $K \in \mathcal{K}$, let $R(u) = R_K(u)$ be the radius of curvature at the point on the boundary of K where the outer unit normal is u.

The condition that $C \in \mathcal{C}$ has its centre of gravity at the origin is equivalent to

$$\int_{S^1} \rho_C^3(u)\underline{du} = \underline{0}$$

where underlining means vector integration. With this condition in mind, Minkowski's classical theorem states that for every $C \in \mathcal{C}$ there is a convex body, $C^* \in \mathcal{K}$ say, such that for every unit vector u

$$R_{C^*}(u) = \rho^3_C(u),$$

and this convex body C^* is unique, apart from translations. Using this one can prove the following result:

Theorem 2. *For every $K \in \mathcal{K}$ there is a unique $C \in \mathcal{C}$ such that $F(K)$ is a translated copy of C^*. Moreover, every $C \in \mathcal{C}$ satisfies $F(C^*) = C^*$.*

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Ball-Polytopes

Karoly Bezdek

A ball-polytope is an intersection of finitely many unit balls in Euclidean space. The talk surveyed several of their basic combinatorial and metric properties including a version of the Caratheodory theorem and the corresponding Euler-Poincare formula. As it turns out the underlying so-called 1-convexity and unit-circle are ,,distance" (which is in fact, not a metric in a well-controlled way) play important rofes. Based on this then we discussed the following rather well-known problems for the class of ball-polytopes:

- isoperimetric inequalities for circle-polygons;
- illumination problem (it was shown that every ball-polyhedron with generating unit spheres whose center-center distance are less than I can be illuminated by 6 point-sources);
- ball-polyhedra with symmetric sections (here we proved within the class of ball-polyhedra the still unsolved/open conjecture of the speaker (1997) according to which a convex body in $\mathbb{R}^3$ is a solid of revolution or an ellipsoid if and only if all planar sections of i are axially symmetric);
- in connection with the Helly-numbers of unit spheres in $\mathbb{R}^n$ we disproved a conjecture of Maehara (1989) for all $n \geqslant 4$.

A lower bound for Lebesgue's universal cover problem

Peter Brass
(joint work with Mehrbod Sharifi)

The universal cover problem was first stated 1914 in a personal communication by Lebesgue to Pál [1]; Lebesgue asked for the minimum area of a convex set U in the plane such that for each set C of diameter 1 there is a congruent copy C' contained in U. So U is a universal cover for the family of sets of diameter 1, under congruence, and we wish to determine the minimum area of a convex set with that property.

This problem became a prototype for many similar universal cover problems, where possible parameters include the family of sets to be covered (e.g., in Moser's worm problem, the curves of length one), the allowed transformations (congruence or translation), the size measure to be minimized (area, perimeter, diameter, mean width), and whether the cover is assumed to be convex (see [11] section 11.4 for a survey). In this talk, we stick to Lebesgue's original version.

An easy example of a universal cover for sets of diameter 1 is the circle of radius $\frac{1}{\sqrt{3}}$; Jung [2] proved that the smallest ball that contains all sets of diameter 1 is the ball circumscribed to the equilateral simplex of diameter 1 (a different proof for the planar case was also given by Jung [3]). This circle has area $\frac{\pi}{3} \approx 1.047$. The unit square is a smaller universal cover, and it is also a universal cover even under translation.

Pál constructed a sequence of better and better universal covers in his paper [1], culminating in his truncated hexagon, a regular hexagon circumscribed to the unit circle, with two corners cut off; this universal cover has the area 0.8454. Further universal covers were constructed by Sprague [4], Duff [5] (nonconvex), and Hansen [6, 7, 8]; also Eggleston [9] observed that the set obtained as union of a Reuleaux triangle of diameter 1 and a circle of diameter 1, when the triangle vertices are antipodal points of the circle, is a universal cover. But all progress was small, and after Sprague [4] almost infinitesimal, the smallest currently known universal cover has area 0.844.

As lower bound, Pál [1] observed that any set that contains congruent copies of all sets of diameter 1 must contain at least congruent copies of the circle and equilateral triangle of diameter 1; if the set is additionally convex, the area is at least the minimum area of the convex hull of a circle and a triangle of diameter 1. Pál shows that this minimum is reached when circle and triangle are concentric; that set has area $\frac{\pi}{8} + \frac{\sqrt{3}}{4} \approx 0.8257$. This lower bound could be improved if one could add further sets of diameter 1 to this family, for which the area of the convex hull is minimized. This was already observed by Pál, but he found unsurmountable difficulties in extending his method from two sets (disc and triangle) to three sets. This step was finally taken by Elekes [10], more than seventy years later, when Elekes showed that the smallest convex hull of a circle, and all regular 3^i-gons, all of diameter 1, is reached if all these sets are concentric and equally aligned; this raised the lower bound to ≈ 0.8271.

The improvement was comparatively small since the next set included in this sample, the regular 9-gon, is already very near a circle, and the improvement decreases fast with the number of vertices. It would have been much more efficient if one could have taken circle, equilateral triangle, and regular fivegon, of diameter 1; but the analytic methods do not extend to this situation. It is the result of this talk to use instead computational methods to bound the minimum area of the convex hull of a circle, triangle, and fivegon, as a lower bound for the minimum area of a universal cover for sets of diameter 1.

Theorem: A convex set in the plane that contains a congruent copy of each set of diameter one has area at least 0.832.

The placement of triangle, fivegon and circle that gives the smallest convex hull we know of appears quite irregular, certainly the three sets are not concentric, which was crucial for the proofs by Pál [1] and Elekes [10]. This suggests that the analytic methods for finding the minimizing position are not applicable anymore.

Our method is in principle quite standard, we provide an initial bound for the space of possible placements, and then subdivide it in cells. For each cell, we compute a lower bound for the area, and subdivide the cell if the lower bound is not good enough, until we have checked our claimed lower bound for all cells. But the lower bound must be a quite strong lower bound, since the search space of possible placements of the three sets is five-dimensional (the circle is fixed, the triangle might be rotated to be axis-aligned, only the five-gon has three degrees of freedom), and near the minimum we need an error less than 0.1%. At that resolution, five-dimensional space is already enormously large, and adding another set would raise the dimension of the search space to eight and make our approach again infeasible.

The same method could of course be used for all similar universal cover problems, but for each different problem we need a new local bound, and this method is of course not suitable to find the exact value. So it is only reasonable in situations when we do not have a conjecture for the optimal arrangement.

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Universally optimal configurations of points on spheres

Henry Cohn
(joint work with Abhinav Kumar)

What is the best way to distribute N points on the unit sphere S^{n-1} in $\mathbb{R}^n$? One way to make this notion precise is *spherical codes*: how large can the minimal distance between the points be? A more general approach is potential energy minimization. Given a potential function $f : (0, 4] \to \mathbb{R}$, define the energy $E_f(\mathcal{C})$ of a configuration $\mathcal{C} \subset S^{n-1}$ by

$$E_f(\mathcal{C}) = \sum_{x,y \in \mathcal{C},\ x \neq y} f\big(|x-y|^2\big).$$

One seeks the configuration with $|\mathcal{C}| = N$ that minimizes this quantity. Maximizing the minimal angle occurs as a degenerate case by taking $f(r) = 1/r^s$ with $s \to \infty$ (the asymptotics of $E_f(\mathcal{C})$ are determined by the minimal angle).

In most cases the optimal configuration depends on the choice of potential function f, but not always. For example, it is not hard to convince oneself intuitively that when $N = n + 1$ the regular simplex should be optimal for all reasonable choices of f.

Definition 1. *A configuration $\mathcal{C} \subset S^{n-1}$ is universally optimal if $E_f(\mathcal{C}') \geq E_f(\mathcal{C})$ whenever $\mathcal{C}' \subset S^{n-1}$ satisfies $|\mathcal{C}'| = |\mathcal{C}|$ and f is completely monotonic (in other words, $f \in C^\infty((0,4])$ and $(-1)^k f^{(k)}(x) \geq 0$ for each $k \geq 0$ and $x \in (0,4]$).*

This definition is more natural than it may appear: the condition for $k = 1$ simply means the force law is repulsive (as makes sense for energy minimization), the convexity condition with $k = 2$ is very plausible, and the conditions with $k \geq 3$ are the natural continuation.

We prove that all the configurations listed in Table 1 are universally optimal. The first five configurations are the vertices of certain regular polytopes (specifically, those regular polytopes whose faces are simplices; no other regular polytopes are universally optimal [CCEK]). The next seven cases are derived from the E_8 root lattice in $\mathbb{R}^8$ and the Leech lattice in $\mathbb{R}^{24}$. The 240-point and 196560-point configurations are the minimal (nonzero) vectors in those lattices. In sphere packing terms, these are the *kissing configurations*, the points of tangency in the corresponding sphere packings. Each arrangement with the label "kissing" is the kissing configuration of the arrangement above it: each configuration yields a sphere packing in spherical geometry by centering an identical spherical cap at

n	N	M	Inner Products	Name
2	N	$N-1$	$\cos(2\pi j/N)$ for $1 \le j \le \lfloor \frac{N}{2} \rfloor$	N-gon
n	$n+1$	2	$-1/n$	simplex
n	$2n$	3	$-1, 0$	cross polytope
3	12	5	$-1, \pm 1/\sqrt{5}$	icosahedron
4	120	11	$-1, \pm 1/2, 0, (\pm 1 \pm \sqrt{5})/4$	600-cell
8	240	7	$-1, \pm 1/2, 0$	E_8 roots
7	56	5	$-1, \pm 1/3$	kissing
6	27	4	$-1/2, 1/4$	kissing/Schläfli
5	16	3	$-3/5, 1/5$	kissing
24	196560	11	$-1, \pm 1/2, \pm 1/4, 0$	Leech lattice
23	4600	7	$-1, \pm 1/3, 0$	kissing
22	891	5	$-1/2, -1/8, 1/4$	kissing
23	552	5	$-1, \pm 1/5$	[DGS], Ex. 8.3
22	275	4	$-1/4, 1/6$	[DGS], Ex. 8.3
21	162	3	$-2/7, 1/7$	[DGS], Ex. 9.2
22	100	3	$-4/11, 1/11$	[DGS], Ex. 9.2
$q\frac{q^3+1}{q+1}$	$(q+1)(q^3+1)$	3	$-1/q, 1/q^2$	[CGS]

Table 1. The known sharp configurations, together with the 600-cell.

each point, with radius as large as possible without making their interiors overlap. The points of tangency on a given cap form a spherical code in a space of one fewer dimension. (In general different points in a packing can have different kissing configurations, but that does not occur here. See Chapter 14 of [CS] for the details of these configurations.)

The last line of the table describes a remarkable family of sharp configurations from [CGS]. The parameter q must be a prime power. When $q = 2$ this arrangement is the 27-point configuration in $\mathbb{R}^6$, but for $q > 2$ it is different from all the other entries in the table.

More generally we can prove universal optimality for any *sharp configuration* (those listed in Table 1 are all sharp, except for the 600-cell):

Definition 2. *A finite subset of the unit sphere is a sharp configuration if it is a spherical M-design, there are m inner products that occur between distinct points in it, and $M \ge 2m - 1 - \delta$, where $\delta = 1$ if the configuration is antipodal and $\delta = 0$ otherwise.*

Recall that a *spherical M-design* is a finite subset of the sphere such that every polynomial on $\mathbb{R}^n$ of total degree at most M has the same average over the design as over the entire sphere.

Theorem 3. *Let $\mathcal{C}$ be a sharp configuration or the vertices of a regular* 600*-cell. Then $\mathcal{C}$ is universally optimal.*

This theorem generalizes a theorem of Levenshtein [L1], which says that all sharp configurations are optimal spherical codes. It is proved using linear programming bounds for potential energy minimization, which were introduced by Yudin [Y] and developed by Kolushov and Yudin [KY1, KY2] and Andreev [An1, An2]

Table 1 coincides with the list of known cases in which the linear programming bounds for spherical codes are sharp; the list comes from [L2, p. 621], except for the 600-cell, which was dealt with in [An3] and [E]. We conjecture that our techniques apply to a configuration if and only if the linear programming bounds for spherical codes prove a sharp bound for it.

One can set up analogous linear programming bounds for potential energy minimization in Euclidean space (along the lines of [CE]). We conjecture that they prove universal optimality for the hexagonal lattice in $\mathbb{R}^2$, the E_8 root lattice in $\mathbb{R}^8$, and the Leech lattice in $\mathbb{R}^{24}$. See [CK] for details.

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The realizability of graphs

Robert Connelly
(joint work with Maria Belk)

A graph G is *d-realizable* if, for each configuration of its vertices in $\mathbb{E}^N$, there exists another corresponding configuration in $\mathbb{E}^d$ with the same edge lengths. A graph G is 1-realizable if and only if it is a forest. A graph G is 2-realizable if and only if it is a partial 2-tree, i.e. a subgraph of the 2-sum of triangles. The k-sum of two graphs is obtained by taking disjoint copies of each and identifying them along corresponding copies of the complete graph K_k. The main result of our work is to show that a graph is 3-realizable if and only if it does not have K_5 or the edge graph of the octahedron as a minor. A graph H is a *minor* of the graph G if H can be obtained by a sequence of operations identifying an edge (with its endpoints) to a point or deleting an edge.

On application of this work is related to the 'molecule problem". This is the problem, where one is given positive scalar weights on the edges of a graph G, and one looks for a configuration in $\mathbb{E}^d$ which has edge lengths equal to weights. The class of 3-realizable graphs, identified above, is such that this molecule problem can be solved with a polynomial time implimentation, essentially. It is possible to find a configuration numerically in some, possibly high dimensional Euclidean space $\mathbb{E}^N$, and then use the algorithms of our work to 'push down" the realization into $\mathbb{E}^3$ keeping edge lengths of G fixed. If G does not have K_5 or the edge graph of the octahedron as a minor, then there is a realization in at least $\mathbb{E}^4$, or higher, that cannot be re-realized in $\mathbb{E}^3$.

The 3-realizable graphs are obtained as the 3-sum, or lower, of partial 3-trees and two other graphs V_8 and $C_5 \times C_2$. The graph V_8 is obtained by taking the cycle of length 8, and connecting opposite vertices with edges. The graph $C_5 \times C_2$ is obtained by taking two cycles of length 4 and connecting corresponding vertices by edges. For partial 3-trees the idea is to fold each of the summands down into the lower dimensional space. However, for V_8 and $C_5 \times C_2$ the task is more complicated. Roughly the idea is to strech certain pairs of vertices until the rest of the graph is forced to lie in $\mathbb{E}^3$, and is the work of M. Belk.

Polytopes, Lie Algebras, Computing

Jesús A. De Loera
(joint work with Tyrrell B. McAllister)

Given highest weights λ, μ, and ν for a finite dimensional complex semisimple Lie algebra, we denote by $C^{\nu}_{\lambda\mu}$ the multiplicity of the irreducible representation V_ν in the tensor product of V_λ and V_μ; that is, we write

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} C^{\nu}_{\lambda\mu} V_\nu. \tag{1}$$

In general, the numbers $C^{\nu}_{\lambda\mu}$ are called *Clebsch–Gordan coefficients*. In the specific case of type A_r Lie algebras, the values $C^{\nu}_{\lambda\mu}$ defined in equation (1) are called *Littlewood–Richardson coefficients*. When we are specifically discussing the type A_r case, we will adhere to convention and write $c^{\nu}_{\lambda\mu}$ for $C^{\nu}_{\lambda\mu}$.

The concrete computation of Clebsch–Gordan coefficients (sometimes known as the *Clebsch–Gordan problem* [5]) has attracted a lot of attention from not only representation theorists, but also from physicists, who employ them in the study of quantum mechanics (e.g. [3, 12]). The importance of these coefficients is also evidenced by their widespread appearance in other fields of mathematics besides representation theory. For example, the Littlewood–Richardson coefficients appear in combinatorics via symmetric functions and in enumerative algebraic geometry via Schubert varieties and Grassmannians (see for instance [10, 6]). More recently, Clebsch–Gordan coefficients are playing an important role on the study of P vs. NP (see [8]). Very recently, Narayanan has proved that the computations of Clebsch–Gordan coefficients is in general $\#P$-complete [9]. Here are our contributions:

(1) We combine the lattice point enumeration algorithm of Barvinok [1] with polyhedral tools due to Knutson and Tao [4] and Berenstein and Zelevinsky [2] in the polyhedral realization of Clebsch–Gordan coefficients to produce a new algorithm for computing these coefficients. We can prove:

Theorem 1. *For fixed rank r, if $\mathfrak{g}$ is a complex semisimple Lie algebra of rank r, then one can compute Clebsch–Gordan coefficients for $\mathfrak{g}$ in time polynomial in the input size of the defining weights.*

Moreover, as a consequence of the polynomiality of linear programming and the proof of the saturation property of Lie Algebras of type A_r, deciding whether $c^{\nu}_{\lambda\mu} = 0$ can be done in polynomial time, even when the rank is not fixed.

This settles a conjecture of Mulmuley and Sohoni [8]

(2) We implemented the algorithm for types A_r, B_r, C_r, and D_r (the so-called "classical" Lie Algebras) using the software packages `LattE` and `Maple`. In many instances, our implementation performs faster than standard methods, such as those implemented in the software `LiE`. Our software is freely available at `http://math.ucdavis.edu`

(3) Via computer experiments, we explored general properties satisfied by the Clebsch–Gordan coefficients for the classical Lie algebras under the operation of *stretching of multiplicities*: By stretching of multiplicities we refer to the function $e\colon \mathbb{Z}_{>0} \to Z_{\geqslant 0}$ defined by $e(n) = C^{n\nu}_{n\lambda,n\mu}$.

It follows from the definitions of the BZ-polytopes that, given any highest weights λ, μ, ν of a semisimple Lie algebra, $C^{n\nu}_{n\lambda,n\mu} = e(n)$ is a quasi-polynomial in n. Indeed, $e(n)$ is, in polyhedral language, the *Ehrhart quasi-polynomial* of the corresponding BZ-polytope.

We can prove the following theorem:

Theorem 2. *(Stretching Conjecture) Given highest weights λ, μ, ν of a Lie algebra of type A_r, B_r, C_r, or D_r, then*

$$C_{n\lambda,n\mu}^{n\nu} = \begin{cases} f_0(n) & \text{if } n \equiv 0 \quad mod 2, \\ \vdots & \\ f_1(n) & \text{if } n \equiv 1 \quad mod 2 \end{cases}$$

Moreover we conjecture, supported on our experiments:

Conjecture 3. *The coefficients of each polynomial f_i are all nonnegative.*

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On incidences in Euclidean spaces

György Elekes

The talk gave a short survey of certain incidence questions with emphasis on incidence bounds in three and higher dimensions [1], [2]. Relations to planar incidence problems (involving parabolas or circles, see [3], [4]) were also mentioned.

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Orthogonal Surfaces

Stefan Felsner
(joint work with Sarah Kappes)

Let $\mathbb{R}^d$ be equipped with the dominance order:

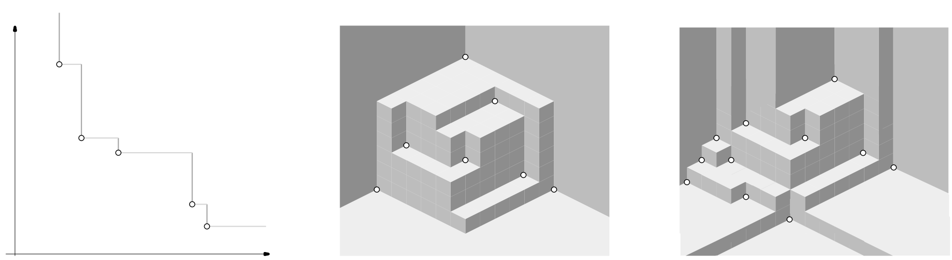
$$x \leqslant y \iff x_i \leqslant y_i \quad \text{for} \quad i = 1, .., d$$

Let $V \subset \mathbb{R}^d$ be a finite antichain in the dominance order. The orthogonal surface $\mathcal{S}_V$ generated by V is the boundary of the filter

$$I_V^{\geqslant} = \{ y \in \mathbb{R}^d \ : \ \exists x \in V \text{ with } y \geqslant x \}$$

generated by V.

Example.



- The left figure shows an orthogonal 1-surface, i.e., an orthogonal surface in two dimensions.
- The middle figure shows a *suspended* and *generic* orthogonal 2-surface in three dimensions. Suspended means that there are special suspension vectors

$$s_i = (0, .., 0, M, 0, ..0) \in V$$

  such that $0 < x_j^i < M$ for all the other elements of V. Generic means that the non-suspension vectors in V have pairwise different coordinates.
- The right figure shows a 2-surface which shows all kinds of 'unfriendly' features.

Orthogonal surfaces are related to various mathematical fields:

- Study of discrete production sets in mathematical economics, (Scarf).

- Resolutions of monomial ideals, (Miller, Sturmfels)
- Connections with order dimension.
- Planar graphs and Schnyder woods.

The most remarkable result in the theory of orthogonal surfaces goes back to Scarf [5].

Theorem. *Generic suspended orthogonal surfaces in* $\mathbb{R}^d$ *induce simplicial complexes which are face complexes of simplicial* d*-polytopes (minus one facet).*

One question motivated by this result is to find criteria to distinguish between simplicial d-polytopes which are induced by an orthogonal surface and those which are not.

The strongest availabe tools to answer this kind of question are of order theoretical nature. Investigations of order dimension imply that a neighbourly 4-polytope with more than 13 vertices is not realizable by an orthogonal 3-surface. Moreover, the number of edges of 4-polytopes with n vertices which are realizable by an orthogonal 3-surface is known to be $\frac{3}{8}n^2 + o(n)$, see [1]. We prove a counting criterion for realizability. This *edge-facet* criterion allows to classify 2344 of the 2957 non-realizable triangulations of a 3-ball (i.e., 4-sphere with a removed facet) on 9 vertices as non-realizable. (Thanks to Frank Lutz, who provided us with the data set of triangulated 4-spheres).

We also address the question of generalizations of Scarf's theorem to non-generic surfaces. In 3-dimensions such a generalization is known, see [4] and [3]. For higher dimensions we are in search for a set of conditions which is less restrictive than genericity and but still allows such a generalization.

A point p on the surface $\mathcal{S}_V$ is called a *generated point* if there is a $G \subset V$ such that $p = \bigvee_{v \in G} v$. A surface is called *non-degenerate* if all minimal generating sets of every generated point have the same size.

The *characteristic points* of a surface S_V are the points where d-different 'flats' of the surface meet. In the non-degenerate case, we can give a more combinatorial definition for characteristic points: Point p is characteristic iff it is generated and for every generating set G_p of p and every $v \in G_p$, there is a minimal generating set $G'_p \subseteq G_p$ that contains v. It turns out that in this case the minimal generating sets for characteristic points are the bases of a matroid.

A non-degenerate surface S_V is *rigid* if any two characteristic points with minimal generating sets of the same size are incomparable in the dominance order.

In the 3-dimensional case the dominance order on the characteristic points of a rigid surface is the face lattice of a 3-polytope (minus one facet).

In higher dimensions the dominance order on the characteristic points of a rigid surface has some features in common with the face lattice of a polytope. However, there are examples which show that it need not even be a lattice.

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Few-distance sets in d-dimensional normed spaces

Zoltán Füredi

Let N be a normed space, $N := (R^d, \|.\|)$. Frequently we use upper index to indicate dimension so we write N^d. E^d denotes the d-dimensional Eucledian space, ℓ_∞^d the maximum norm, as usual.

A set $P \subset R^d$ in the normed space N is called a k-distance set if

$$|\{\|\mathbf{x}-\mathbf{y}\|_N : \mathbf{x},\mathbf{y} \in P, \mathbf{x} \neq \mathbf{y}\}| \leqslant k.$$

In the case $k = 1$ the set P is equidistant. Let $f(N^d, k) :=$ max size of a k-distance set in the d-dimensional normed space N. Considering the lattice points $\{0, 1, 2, \ldots, k\}^d$ one gets $f(\ell_\infty^d, 1) \geqslant (k+1)^d$. On the other hand, it is known that $f(N^d, k) \leqslant 2^{kd}$, independently of the norm. (Petty 1971 for $k = 1$, 5^d for $k = 2$ by Einhorn and Schoenberg 1966, and Swanepoel 1999 for all k). It is conjectured that the upper bound can be decreased to $(k+1)^d$, and it was proved for $d = 2$ and 3.

Unit distance sets. Groemer showed $f(N^d, 1) \leqslant 2^d - 1$ unless $N = \ell_\infty^d$. It is conjectured that this can be further improved for smooth, strictly convex norms. However, in [2] such a norm is constructed with $f(N^d) > 1.05^d$ (This was improved by Talata to $2^{d/2}$, see in Böröczky's book 2004).

It is easy to see that $f(E^d, 1) = d + 1$, and it is conjectured that $f(\ell_1^d) = 2d$ (Kusner 1983). The unitvectors $\pm\mathbf{e}_i$ give the extremal configuration. This was proved for for $d = 3$ (Bandelt, Chepoi and Laurent 1998) and for $d = 4$ (Koolen, Laurent and Schrijver 2000). The best upper bound is due to Alon and Pudlák (2003): if $p \geqslant 1$ odd integer then $\exists c_p > 0$ with $f(\ell_p^d) \leqslant c_p d \log d$. In general thy showed that $f(\ell_p^d) \leqslant c_p d^{(2d+2)/(2d-1)}$ for all $p \geqslant 1$.

Lower bounds. It is conjectured that $f(N^d, 1) \geqslant d+1$ for all N. Dexter (2000) showed that this is true if N is "almost" Euclidean, and Brass (1999) gave a lower bound tending to infinity: $f(N^d) \geqslant c\left(\frac{\log d}{\log\log d}\right)^{1/3}$. For $1 < p < 2$ one has $f(\ell_p^d) > d + 1$ for $d > d_0(p)$ (Swanepoel 2004, also see: C. Smyth 2005).

Euclidean plane and space. There are many excellent results on $f(E^2, k)$ and $f(E^3, k)$. Erdős (1946) conjectures that the $\sqrt{n} \times \sqrt{n}$ lattice gives the minimum number of distinct distances for n-sets in plane, i.e., $f(E^2, k) = O(k\sqrt{\log k})$, and similarly, the $n^{1/3} \times n^{1/3} \times n^{1/3}$ lattice minimizes $f(E^3, k)$, i.e., $f(E^3, k) = O(k^{3/2} \log\log k)$.

The best upper bound due to Katz and Tardos 2004, $f(E^2, k) = O(k^{1.157\ldots})$, and Aronov, Pach, Sharir and Tardos 2004: $f(E^3, k) \leqslant k^{141/77+\varepsilon}$.

Euclidean k-distance sets. Obviously, $f(E^2, 2) = 5$ (regular pentagon), and $f(E^3, 2) = 6$ (octahedron by Croft 1962). Blokhuis (1984) slightly improving a result of Larman, Rogers and Seidel (1977) showed $f(E^d, 2) \leqslant \frac{1}{2}(d+1)(d+2)$. The 0-1 vectors of weight two in R^{d+1} give a lower bound $f(E^d, 2) \geqslant \frac{1}{2}d(d+1)$.

In general the best known upper bound, $f(E^d, k) \leqslant \binom{d+k}{k}$, is due to Blokhuis (1981) and independently Bannai, Bannai and Stanton (1983).

k-dependent sets

A set $P \subset R^d$ in the normed space N is called a k-dependent if $\forall X \subset P$,

$$|X| = k \text{ determines less than } \binom{k}{2} \text{ distances.}$$

For example, for $k = 3$, every triangle is isosceles. Let $g(N^d, k)$ denote the maximum size of such a set. $\{1, \ldots, \binom{k}{2}\}^d$ shows

$$g(\ell_\infty^d, k) \geqslant \binom{k}{2}^d. \tag{1}$$

Our aim is to prove that $\exists$ c dependent only on d, k such that $g(N^d, k) \leqslant c$, independently on the norm. We show,

$$g(N^d, k) \leqslant \left(\frac{1}{2}(k-1)(k^2 - 2k + 6)\right)^{c3^d d \log d}. \tag{2}$$

also (in the case $k = 3$) If every triple from $P \subset R^d$ form an isosceles triangle in the normed space N^d, then

$$|P| = g(N^d, 3) \leqslant 3^{c3^d d \log d}. \tag{3}$$

Sidon sets, the case $d = 1$. There is only one norm for dimension 1.

A set of reals X is called a *Sidon set* (or B_2-*sequence*) if

$$a + b = c + d$$

has only trivial solutions in X (i.e., $\{a, b\} = \{c, d\}$). In other words, for distinct elements $x_1, x_2, x_3, x_4 \in X$, we have $x_1 - x_2 \neq x_3 - x_4$ and $x_1 - x_2 \neq x_2 - x_3$. Let $s(k) := \max\{m : |P| = m$, reals, and $\forall$ Sidon subset $X \subset P$ one has $|X| < k\}$. (I.e., $s(k) = g(N^1, k)$). By (1) we have $s(k) \geqslant \binom{k}{2}$. It was proved by Komlós, Sulyok and Szemerédi (1975) that there exists a $c > 0$ such that $s(k) < ck^2$.

Thin cones. To prove (2) we need a notion introduced in [4]. A cone $C \subset R^d$ in the normed space N is called **thin** if

$$\mathbf{x}, \mathbf{y} \in C \setminus \{\mathbf{0}\} \text{ implies } \|\mathbf{x}+\mathbf{y}\| > \|\mathbf{x}\|, \|\mathbf{y}\|.$$

It is easy to see [4] that C is thin if

$$\operatorname{diam}(C \cap \partial B(\mathbf{0},1)) < 1, \tag{4}$$

where B is the unit ball of the normed space.

Following the latest randomized proof of the Erdős-Rogers (1962) covering theorem, i.e., using Lovász Local Lemma as in Füredi and Kang (2003 one can show that

Theorem 1. For any N^d and $r < 1$ the unit ball B can be covered by m open balls of radius r such that no point of the space is covered by more than $cd \log d$ times and

$$m < c' \left(1 + \frac{1}{r}\right)^d d \log d,$$

where c, c' are positive constants independent from d, r and N. □

Substituting $r = 1/2$ this implies that one can cover the whole space by at most

$$cd \log d 3^d \tag{5}$$

thin cones. Let $\Theta(N^d)$ denote the minimum size of a thin cone covering, i.e., $\Theta(N^d)$ denote the minimum number m such that $\exists C_1, \ldots, C_m$ thin convex cones with $\cup C_i = R^d$.

Increasing sets. An ordered set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{t}\}$ in N^d form an *increasing* point set, if

$$\|\mathbf{x}_\beta - \mathbf{x}_\gamma\| \leqslant \|\mathbf{x}_\alpha - \mathbf{x}_\delta\|$$

holds for any $1 \leqslant \alpha \leqslant \beta < \gamma \leqslant \delta \leqslant t$. Moreover, equality can hold only if $\alpha = \beta$ and $\gamma = \delta$.

For example, if the vectors $\mathbf{p}_i - \mathbf{p}_j$ $(i > j)$ all belong to the same thin cone C.

One can see that an increasing set of size

$$\frac{1}{2}(k-1)(k^2 - 2k + 6) \tag{6}$$

cannot be k-dependent. Let $t := t(N^d, k)$ be the largest integer such that the normed space N^d contains an increasing t-set without k independent points.

Main result. We need the following classical theorem on graph coloring. Any directed graph which contains no directed (simple) path of length ℓ, $\ell \geqslant 1$, is ℓ-colorable (proved independently Gallai 1968, Hasse 1965, Roy 1967, Vitaver 1962). Using this one obtains that

Theorem 2. $g(N^d, k) \leqslant t(N^d, k)^{\Theta(N)}$.

Finally, (5) and (6) yield (2). □

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Introducing New Software Tools in Polyhedral Computation

Komei Fukuda

(joint work with Anders Jensen and Christophe Weibel)

There are quite a number of mathematical computations can be formulated in one way or another as problems on certain convex polyhedra in some vector space $\mathbb{R}^d$. For example, computation of the Voronoi diagram and the Delaunay triangulation for a given set of points in $\mathbb{R}^d$ can be reduced to the representation conversion (between V-representation and H-representation) for convex polyhedra in $\mathbb{R}^{d+1}$ and various software packages for the conversion are available, see [2].

On the other hand, there are many fundamental problems in polyhedral computation that are not yet sufficiently understood in terms of their (worst-case, average) complexities, the existence of efficient algorithms, relations to other problems, etc. These problems include (variants of) the problems of computing the Minkowski addition of several convex polytopes, the volume of a convex polytope, projections of a convex polytope, and recognizing the convexity of the union of several convex polytopes. Note that the complexity of a problem depends very much on how input and output are specified (e.g. V-representation or H-representation) and thus there are many variants. Also, some problems can be efficiently solved if we restrict our attention to a special class of polytopes.

Although we are very far from having a "complete" set of polyhedral computation software tools, there are more and more software tools available. We are pleased to announce the availability of two new software packages with unique functionalities that reflect some recent progresses in polyhedral computation.

(1) `Minksum` [6] is a program to compute the V-representation (i.e. the set of vertices) of the Minkowski addition of several convex polytopes given by their V-representation in $\mathbb{R}^d$. It is an implementation in C++ language of the reverse search algorithm [1] whose time complexity is polynomially bounded by the sizes of input and output.
(2) `Gfan` [5] is a program to list all reduced Gröbner bases of a general polynomial ideal given by a set of generating polynomials in n-variables. It is an implementation in C++ language of the reverse search algorithm [4].

Both packages use `GMP` (GNU multi-precision library) and the exact LP solver of `cddlib` [3]. They are both licensed under GPL (GNU Public License).

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Double-permutation sequences and geometric transversals

Jacob E. Goodman

(joint work with Richard Pollack)

Approximately twenty-five years ago [5], the authors introduced a combinatorial encoding of planar point configurations designed to open problems on configurations to purely combinatorial investigation. This encoding, which assigned to each planar configuration of n points a circular sequence of permutations of $1, \dots, n$, has been used in a number of papers since then, in dual form (as an encoding of line arrangements) as well as in primal form; because the same object encodes pseudoline arrangements as well, it has also been used to derive results on pseudoline arrangements. For a survey of results obtained by this technique, see, e.g., [4]. (Recent applications include [9] and [10].)

In recent work [6], we have extended the encoding of point configurations by circular sequences of permutations to an encoding of planar families of disjoint compact convex sets by circular sequences of what we call "double permutations." It turns out that this new encoding applies as well to a more general class of objects: families of compact connected sets in the plane with a specified arrangement of pairwise tangent and pairwise separating pseudolines, and thereby permits us to extend results known for convex sets to these more general objects. In particular, we use the double-permutation sequence encoding to prove the theorem of Edelsbrunner and Sharir [3] that a collection of n mutually disjoint compact convex sets in the plane has no more than $2n-2$ "geometric permutations," and thereby establish this result in greater generality as well.

In brief, the new encoding works as follows. Given a planar family $\mathcal{C} = \{C_1, \dots, C_n\}$ of mutually disjoint compact convex sets, we project the sets onto a directed line L and denote the endpoints of each projected set C_i by i, i', according to their order on L. This gives, in general, a permutation of the $2n$ indices

$1, \ldots, n, 1', \ldots, n'$. We then rotate L counterclockwise, and record the circular sequence consisting of all the "double permutations" of $1, \ldots, n$ that arise in this way. (Notice that if the convex sets are points, this encoding reduces to the encoding by circular sequences of permutations.)

It turns out that this simple-minded encoding is strong enough to capture all of the features of the family $\mathcal{C}$ that are essential in determining the (partial and complete) transversals that $\mathcal{C}$ possesses, and that it extends in a natural way to the case where the sets C_i are merely connected, provided we specify — for each pair of sets — four pairwise tangent pseudolines and a separating pseudoline that together are compatible in a single arrangement (every two meet just once and cross there).

Our main application of this new encoding is

Theorem. *Suppose $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a family of compact connected sets in the affine plane, each pair C_i, C_j separated by a pseudoline L_{ij} and provided with a set of pairwise tangent pseudolines, two internal and two external. Suppose further that these $5\binom{n}{2}$ pseudolines form an arrangement $\mathcal{A} = \mathcal{A}_T \cup \mathcal{A}_S$ ($\mathcal{A}_T$ being the tangents, $\mathcal{A}_S$ the separators). Then $(\mathcal{C}, \mathcal{A})$ has no more than $2n - 2$ geometric permutations.*

Here, by a *geometric permutation* of $(\mathcal{C}, \mathcal{A})$, we mean an ordering of the sets $C_1, \ldots, C_n$ such that there exists a pseudoline L compatible with $\mathcal{A}$ meeting them in that order (and its reverse).

This result generalizes the Edelsbrunner-Sharir theorem in several ways. For one thing, we replace convex sets with connected sets, and their pairwise tangent lines with pairwise tangent pseudolines. Moreover, we don't have to assume that all of the pseudolines inducing different geometric permutations are together compatible in a single arrangement (with this assumption the result would follow by paraphrasing the original Edelsbrunner-Sharir proof in a "topological plane" containing the entire pseudoline arrangement, whose existence is guaranteed by a result of [8]), but merely that each one *separately* is compatible with the arrangement of tangents and separators.

The proof proceeds by rephrasing a number of geometric properties of our family of connected sets and pseudolinear tangents in combinatorial terms, thereby establishing combinatorial properties of our double-permutation sequences, and then using a purely combinatorial argument on the sequences themselves.

We also define the notion of an "allowable sequence of double permutations," discuss some of its combinatorial properties, and pose several questions concerning these sequences.

For recent surveys in geometric transversal theory, see [2, 7, 11, 12]; for recent work on pseudoline arrangements, see [1, Chap. 6] and [4].

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On Norm Maximization in Geometric Clustering

Peter Gritzmann

(joint work with Andreas Brieden and Christoph Metzger)

In geometric clustering points of some finite point set in some Minkowski space have to be grouped together according to some balancing constraints so as to optimize some objective function. The prime example of a real-world clustering problem that motivated our study is that of a lend-lease initiative for the consolidation of farmland. In fact, in many regions farmers cultivate a number of small lots that are distributed over a wider area. This leads to high overhead costs and economically prohibits use of high tech machinery hence results in a non-favorable cost-structure of production. The classical form of land consolidation is typically too expensive and too rigid, whence consolidation based on lend-lease agreements has been suggested. Of course, the underlying mathematical clustering problem is NP-hard even in the most simple cases.

We give and analyze a new norm maximization model for geometric clustering where in effect the centers of gravity of the clusters are pushed apart. With respect to the effective intrinsic dimension this model compares favourably with other possible formulations of the task.

The most efficient problem adjusted approach for solving the underlying convex maximization problem is based on the use of Minkowski spaces whose unit balls stems from the dual of a cartesian products of permutahedra. It is shown how these unit balls themselves can be tightly approximated by Hardamard matrix based polytopes with only linearly many facets. The facet normals are then used as objective function vectors for a polynomial-time linear programming approximation algorithm. We derive a worst case bound for the approximation error but also report on the practical performance of this method for land consolidation in some typical regions in Northern Bavaria, Germany.

On a separation problem by Tverberg

Andreas Holmsen

It was shown by Tverberg [3] that for every positive integer k there exists a minimal positive integer $s(k)$ such that for any collection of at least $s(k)$ nonempty convex sets in the plane with pairwise disjoint relative interiors, there is a $(1,k)$-separation, i.e. a closed half-plane that contains at least 1 of the sets, while the complementary closed half-plane contains at least k of the remaining sets. Hope and Katchalski [2] gave the bounds $3k-1 \leqslant s(k) \leqslant 12(k-1)$. We discuss this problem and the recent extension of allowable sequences by Goodman and Pollack [1].

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On geometric graph Ramsey numbers

Gyula Károlyi
(joint work with Vera Rosta)

For any finite sequence $G_1, G_2, \dots, G_t$ of simple graphs, $R(G_1, G_2, \dots, G_t)$ denotes the smallest integer r with the property that whenever the *edges* of a complete graph on at least r vertices are partitioned into t colour classes $C_1, C_2, \dots, C_t$, there is an integer $1 \le i \le t$ such that C_i contains a subgraph isomorphic to G_i. Such a subgraph will be referred to as a monochromatic subgraph in the ith colour.

In the special case, when each G_i is a complete graph on some k_i vertices, we will simply write $R(k_1, k_2, \dots, k_t)$ for $R(G_1, G_2, \dots, G_t)$. In general, if G_i has k_i vertices, then the existence of $R(G_1, G_2, \dots, G_t)$ follows directly from that of $R(k_1, k_2, \dots, k_t)$, the latter was first observed and applied to formal logic by Ramsey [9]. For more on Ramsey theory in general, we refer to the monograph [3] and the collection of survey articles [8].

A *geometric graph* is a graph drawn in the plane so that every vertex corresponds to a point, and every edge is a closed straight-line segment connecting two vertices but not passing through a third. The $\binom{n}{2}$ segments determined by n points in the plane, no three of which are collinear, form a *complete* geometric graph with n vertices. A geometric graph is *convex* if its vertices correspond to those of a convex polygon. Further, we say that a subgraph of a geometric graph is *non-crossing*, if no two of its edges have an interior point in common.

For a sequence of graphs $G_1, G_2, \dots, G_t$, the *geometric* Ramsey number that we denote by $R_g(G_1, G_2, \dots, G_t)$ is defined as the smallest integer r with the property that whenever the edges of a complete geometric graph on at least r vertices are partitioned into t colour classes, the ith colour class contains a non-crossing copy of G_i, for some $1 \le i \le t$. The number $R_c(G_1, G_2, \dots, G_t)$ denotes the corresponding number if we restrict our attention to *convex* geometric graphs only. These concepts have been introduced by Károlyi, Pach and Tóth in [5] and further explored in [6] and [7].

These numbers exist if and only if each graph G_i is *outerplanar*, that is, can be obtained as a subgraph of a triangulated cycle (convex n-gon triangulated by non-crossing diagonals). The necessity of the condition is obvious, whereas the 'if part' is implied by the following theorem, based on a result of Gritzmann et al. [4].

Theorem 1. *Let, for each $1 \le i \le t$, G_i denote an outerplanar graph on k_i vertices. Then*

$$R(G_1, \dots, G_t) \le R_c(G_1, \dots, G_t) \le R_g(G_1, \dots, G_t) \le R(k_1, \dots, k_t).$$

We denote by P_k a path of k vertices, by C_k a cycle of k vertices and by D_k a cycle of k vertices triangulated from a point. In addition, $M_{2k} = kP_2$ will stand for any perfect matching on $2k$ vertices.
Results on geometric Ramsey numbers for paths and cycles were found by Károlyi et al. [6] and they were extended in [7]. For example, if $k \ge 3$, then

$$2k - 3 = R_c(P_k, P_k) \le R_g(P_k, P_k) = O(k^{3/2}).$$

Moreover, if k and l are integers larger than 2, then

$$\begin{aligned}(k-1)(l-1)+1 &= R_c(C_k, P_l) \le R_g(C_k, P_l) \le R_g(D_k, D_l) \\ &\le (k-1)(l-2)+(k-2)(l-1)+2.\end{aligned}$$

Here we prove that in the case $k = 3$ the upper bound is sharp:

Theorem 2. *For any integer $l \ge 3$,*

$$R_c(C_3, C_l) = R_g(C_3, C_l) = R_g(D_3, D_l) = 3l - 3.$$

It is proved in [2] that

$$R(k_1P_2, k_2P_2, \dots, k_tP_2) = \sum_{i=1}^{t} k_i + \max_{1 \le i \le t} k_i - t + 1.$$

This results, in the case $t = 2$ has been extended to geometric graphs as follows

Theorem 3 ([5]).

$$R_c(M_{2k}, M_{2\ell}) = R_g(M_{2k}, M_{2\ell}) = R(M_{2k}, M_{2\ell}) = k + \ell + \max\{k, \ell\} - 1.$$

It implies the following general upper bound for

$$R_g^{(t)}(M_{2k}) = R_g(\underbrace{M_{2k}, \dots, M_{2k}}_{t \text{ times}}).$$

Theorem 4.

$$R_g^{(t)}(M_{2k}) \leq \begin{cases} \frac{3t}{2}k - \frac{3t}{2} + 2 & \text{for } t \text{ even,} \\ \frac{3t+1}{2}k - \frac{3t+1}{2} + 2 & \text{for } t \text{ odd.} \end{cases} \tag{1}$$

In particular, $R_g(M_{2k}, M_{2k}, M_{2k}) \leq 5k-3$ and $R_g(M_{2k}, M_{2k}, M_{2k}, M_{2k}) \leq 6k-4$. We can prove that the latter bound is sharp:

Theorem 5. $R_c(M_{2k}, M_{2k}, M_{2k}, M_{2k}) = R_g(M_{2k}, M_{2k}, M_{2k}, M_{2k}) = 6k - 4.$

Combining this result with the obvious inequality

$$R_c^{(t+1)}(M_{2k}) \geq R_c^{(t)}(M_{2k}) + (k-1)$$

we obtain the following general lower bound:

Theorem 6. *If $t \geq 4$ and k are positive integers, then $R_c^{(t)}(M_{2k}) \geq (t+2)k - t$.*

In [1], Araujo et al. studied the chromatic number of some geometric Kneser graphs. For example, given n points in convex position in the plane, let G_n be the graph whose vertices are the $\binom{n}{2}$ line segments determined by the points, two such vertices connected by an edge in G_n if and only if the corresponding line segments are disjoint.

Theorem 7 ([1]).

$$2\left\lfloor \frac{n+1}{3} \right\rfloor - 1 \leq \chi(G_n) \leq \min\left\{n-2, n - \frac{\log n}{2}\right\}.$$

In this result, the lower bound is derived as a consequence of Theorem 3. Note that any improvement upon the upper bound in Theorem 4, even when R_g is replaced by R_c, would have an impact on the lower bound in the above theorem.

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Abstract regular polytopes: recent developments

Peter McMullen

This talk reported on some developments on abstract regular polytopes since the publication of the monograph [4]; for the most part, papers in the bibliography of [4] are not cited individually here. Since these developments have largely been concerned with realizations of polytopes ([4, Chapter 5] gives the background here), for the purposes of the talk, a geometric viewpoint was adopted. A regular polytope P is to be identified with its *symmetry group* $\mathcal{G} = \mathcal{G}(P)$, which is of the form $\mathcal{G} = \langle R_0, \ldots, R_{n-1} \rangle$, where the R_j are *reflexions* (isometries of period 2) of the ambient space E, which (initially) is taken to be spherical or euclidean; these satisfy $R_j R_k = R_k R_j$ if $|j-k| \geqslant 2$ and, further, the intersection property

$$\langle R_i \mid i \in \mathsf{I} \rangle \cap \langle R_i \mid i \in \mathsf{J} \rangle = \langle R_i \mid i \in \mathsf{I} \cap \mathsf{J} \rangle$$

for all $\mathsf{I}, \mathsf{J} \subseteq \{0, \ldots, n-1\}$. It is convenient to identify R_j with its *mirror* of fixed points $\{x \in E \mid xR_j = x\}$. For geometric reasons, the period p_j of $R_{j-1}R_j$ is at least 3 for each $j = 1, \ldots, n-1$; then $\{p_1, \ldots, p_{n-1}\}$ is the Schläfli type of P. This number n is called the *rank* of P, and is denoted by $\operatorname{rank} P$; then P is referred to as an *n-polytope*.

The geometric structure of P is as follows: the initial vertex is a point $v \in R_1 \cap \cdots \cap R_{n-1}$, the *Wythoff space* (and $v \notin R_0$ to avoid degeneracy); recursively, the initial j-face is $F_j := \{F_{j-1}G \mid G \in \langle R_0, \ldots, R_{j-1} \rangle\}$, and the general faces are the images of the initial faces under $\mathcal{G}$. These form a poset under iterated membership. It is assumed that the vertex-set $V := v\mathcal{G}$ spans E (as a sphere or euclidean space), in which case the *dimension* of P is defined to be $\dim P := \dim E$; moreover, if V is finite, then E is a sphere, while if V is infinite, then E is a euclidean space and V is discrete.

In a space of a fixed dimension, the classification proceeds in two stages. First, the possible *dimension vectors* $(\dim R_0, \ldots, \dim R_{n-1})$ are determined. Second, within each class (of dimension vectors) the individual polytopes are identified. There are various restrictions on these and on $\dim P$.

- $\dim P \geqslant \operatorname{rank} P - 1$ $(= n-1)$;
- $\dim R_j \geqslant j$ for $j = 0, \ldots, n-2$ and $\dim R_{n-1} \geqslant n-2$.

If $\dim P = \operatorname{rank} P - 1$, then P is said to be *of full rank*. In this case

- $\dim R_j = j$ or $n-2$ for $j = 0, \ldots, n-2$ and $\dim R_{n-1} = n-2$.

When P is finite, it is customary to think of its ambient sphere as embedded in the euclidean space of one higher dimension; in this case, all the dimensions occurring above should be raised by one. Moreover, infinite polytopes are called *apeirotopes*.

The first case considered was that of the regular polytopes and apeirotopes of full rank; for fuller details, see [2]. The following table gives their numbers in each dimension.

dimension	polytopes	apeirotopes
0	1	-
1	1	1
2	∞	6
3	18	8
4	34	18
$\geqslant 5$	6	8

A few comments on the constructions are appropriate; too much detail would not be. If X is a finite set in a euclidean space E, let $\mathcal{R}(X)$ denote the group generated by the reflexions (inversions) in the points of X. Then $\mathcal{R}(X)$ is discrete if and only if X is *rational*, meaning that the points of X have rational coordinates with respect to some affine basis. A polytope P is rational if its vertices are; if $\mathcal{G}(P) = \langle R_1, \ldots, R_n \rangle$, then the apeirotope $\operatorname{apeir} P$ has $\mathcal{G}(\operatorname{apeir} P) = \langle R_0, R_1, \ldots, R_n \rangle$, where $R_0 = \{v\}$, with v the initial vertex of P (so that $v \in (R_2 \cap \cdots \cap R_n) \setminus R_1$).

Again in the finite case, by assumption, $R_0 \cap \cdots \cap R_{n-1} = \{o\}$, so that, if the reflexion R_0 is a line reflexion), then $-R_0 = (-I)R_0$ is a hyperplane reflexion; this replacement yields another finite group. More generally, if $K_k := R_k \cap \cdots \cap R_{n-1}$, then the operation κ_{jk} is defined by

$$(R_0, \ldots, R_{n-1}) \mapsto (R_0, \ldots, R_{j-1}, R_j K_k, R_{j+1}, \ldots, R_{n-1}).$$

The case $j = k$ is most important, and leads to the interchange of the two possible dimensions of R_k; however, a finite group need not result. The operation $\pi := \kappa_{n-3,n-1}$ generalizes the usual Petrie operation; it cannot be applied if p_{n-3} is odd.

It turns out that each of the regular polytopes and apeirotopes of full rank (except some of those obtained by the "apeir" construction) can be derived from a classical one (whose group is a hyperplane reflexion group) by means of these various operations. A striking feature is that all three Petrie-Coxeter apeirohedra $\{4,6\,|\,4\}$, $\{6,4\,|\,4\}$ and $\{6,6\,|\,3\}$ of [1] occur as 3-faces of 5-apeirotopes in $\mathbb{E}^4$. Indeed, the second occurs twice; the first is also a 3-face of an $(n+1)$-apeirotope in $\mathbb{E}^n$ for each $n \geqslant 3$.

Next considered are the 4-dimensional finite regular polyhedra; here, [3] is followed. The possible dimension vectors are:

- $(1,3,3)$, $(2,3,3)$;
- $(3,2,3)$, $(2,2,3)$, $(1,2,3)$;
- $(2,3,2)$;
- $(2,2,2)$.

The groupings indicate polyhedra related by κ_0 and π. The first family consists of those polyhedra which are – roughly speaking – liftings of ordinary polyhedra in $\mathbb{E}^3$. The second (and its relatives) and third are obtained by applying automorphisms (at least one of which must be outer) to hyperplane reflexion groups represented by diagrams – possibly with fractional marks – which admit suitable symmetries. Those in the class $(3, 2, 3)$ include ones described by Coxeter in [1]. Curiously, the only groups to occur in the second and third groups are extensions (by involutory outer automorphisms) of products $D_r \times D_r$ of dihedral groups, and the group $[3, 4, 3]$ of the 24-cell.

The class $(2, 2, 2)$ is anomalous. A group of this kind is a subgroup of the rotation group SO_4, and polyhedra with such groups occur in two enantiomorphic (mirror image) forms. The approach here is through quaternions; indeed, some of the groups are not related to reflexion (Coxeter) groups either as subgroups or as extensions.

Something briefly should be said about chiral polytopes. Roughly speaking, such an n-polytope P has only rotational symmetries: its group is of the form $\mathcal{G}(P) = \langle S_1, \ldots, S_{n-1} \rangle$, where $S_j S_{j+1} \cdots S_k$ has period 2 for $j < k$, and the obvious analogue of the intersection property holds. It was shown in [2] that a (strictly) chiral polytope cannot be of full rank. Thus the first non-trivial case is that of chiral apeirohedra in $\mathbb{E}^3$. These have been classified by Schulte in [5, 6], and fall into two families, one with finite and the other with infinite 2-faces.

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Rigidity and the lower bound theorem for doubly Cohen-Macaulay complexes

Eran Nevo

We prove that for $d \geqslant 3$, the 1-skeleton of any $(d-1)$-dimensional doubly Cohen Macaulay (abbreviated 2-CM) complex is generically d-rigid. This implies the following two corollaries (see Kalai [6] and Lee [7] respectively): Barnette's lower bound inequalities for boundary complexes of simplicial polytopes ([3], [2]) hold for every 2-CM complex (of dimension $\geqslant 2$). Moreover, the initial part (g_o, g_1, g_2) of the g-vector of a 2-CM complex (of dimension $\geqslant 3$) is an M-sequence. It was conjectured by Björner and Swartz [10] that the entire g-vector of a 2-CM complex is an M-sequence.

The g-theorem gives a complete characterization of the f-vectors of boundary complexes of simplicial polytopes. It was conjectured by McMullen in 1970 and proved by Billera and Lee [4] (sufficiency) and by Stanley [9] (necessity) in 1980. A major open problem in f-vector theory is the g-conjecture, which asserts that this characterization holds for all homology spheres. The open part of this conjecture is to show that the g-vector of every homology sphere is an M-sequence, i.e. it is the f-vector of some order ideal of monomials.

Definition 1. *A simplicial complex K is $2-CM$ if it is Cohen-Macaulay and for every vertex $v \in K$, $K - v$ is Cohen-Macaulay of the same dimension as K (over a fixed field k).*

$K - v$ is the simplicial complex $\{T \in K : v \notin T\}$. By a theorem of Reisner [8], a simplicial complex L is Cohen-Macaulay iff for every face $T \in L$ (including the empty set) and every $i < dim(Lk_L(T))$, $\tilde{H}_i(Lk_L(T); k) = 0$ where $Lk_L(T) = \{S \in L : T \cap S = \emptyset, T \cup S \in L\}$ and $\tilde{H}_i(M; k)$ is the reduced i-th homology of M over k. This is the formulation which we use in the proof of Theorem 7.
Based on the fact that homology spheres are 2-CM and that the g-vector of some other classes of 2-CM complexes is known to be an M-sequence (e.g. [10]), Björner and Swartz recently suspected that

Conjecture 2. *([10], a weakening of Problem 4.2.) The g-vector of any 2-CM complex is an M-sequence.*

We prove a first step in this direction, namely:

Theorem 3. *Let K be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geqslant 4$. Then $(g_o(K), g_1(K), g_2(K))$ is an M-sequence.*

This theorem follows from the following theorem, combined with an interpretation of rigidity in terms of the face ring (Stanley-Reisner ring), due (implicitly) to Lee [7].

Theorem 4. *Let K be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geqslant 3$. Then K has a generically d-rigid 1-skeleton.*

Kalai [6] was the first to notice that if a $(d-1)$-dimensional simplicial complex ($d \geqslant 3$) has a generically d-rigid 1-skeleton then it satisfies Barnette's lower bound inequalities. Applying this observation to Theorem 4 implies

Corollary 5. *Let K be a $(d-1)$-dimensional 2-CM simplicial complex where $d \geqslant 3$. Then for all $0 \leqslant i \leqslant d-1$ $f_i(K) \geqslant f_i(n,d)$ where $f_i(n,d)$ is the number of i-faces in a (equivalently every) stacked d-polytope on n vertices. (Explicitly, $f_{d-1}(n,d) = (d-1)n-(d+1)(d-2)$ and $f_i(n,d) = \binom{d}{k}n - \binom{d+1}{k+1}k$ for $1 \leqslant i \leqslant d-2$.)*
□

Theorem 4 is proved by decomposing K into a union of minimal $(d-1)$-cycle complexes (Fogelsanger's notion [5]). Each of these pieces has a generically d-rigid

1-skeleton ([5]), and the decomposition is such that gluing the pieces together results in a complex with a generically d-rigid 1-skeleton. The gluing lemma that we use is due to Asimov and Roth [1]. The decomposition is detailed in Theorem 7.

Let us recall the concept of minimal cycles. Fix a field k (or more generally, an abelian group) and consider the formal chain complex on a ground set $[n]$, $C = (\oplus\{kT : T \subseteq [n]\}, \partial)$, where $\partial(1T) = \sum_{t \in T} sign(t,T)T \setminus \{t\}$ and $sign(t,T) = (-1)^{|\{s \in T : s<t\}|}$. Define *subchain*, *minimal* $d-$*cycle* and *minimal* $d-$*cycle complex* as follows: $c' = \sum\{b_T T : T \subseteq [n], |T| = d+1\}$ is a *subchain* of a d-chain $c = \sum\{a_T T : T \subseteq [n], |T| = d+1\}$ iff for every such T $b_T = a_T$ or $b_T = 0$. A d-chain c is a $d-cycle$ if $\partial(c) = 0$, and is a *minimal* $d-cycle$ if its only subchains which are cycles are c and 0. A simplicial complex K which is spanned by the support of a *minimal* $d-cycle$ is called a *minimal* $d-cycle$ *complex* (over k), i.e. $K = \{S : \exists T\ S \subseteq T, a_T \neq 0\}$ for some minimal d-cycle c as above. For example, triangulations of connected manifolds without boundary are minimal cycle complexes - fix $k = \mathbb{Z}_2$ and let the cycle be the sum of all facets.

Theorem 6. *(Fogelsanger [5]) For $d \geqslant 3$, every minimal $(d-1)$-cycle complex has a generically d-rigid 1-skeleton.*

We are now ready to decompose a 2-CM simplicial complex:

Theorem 7. *Let K be a d-dimensional 2-CM simplicial complex over a field k $(d \geqslant 1)$. Then there exists a decomposition $K = \cup_{i=1}^m S_i$ such that each S_i is a minimal d-cycle complex over k and for every $i > 1$ $S_i \cap (\cup_{j<i} S_j)$ contains a d-face.*

Moreover, for each $i_0 \in [m]$ the S_i's can be reordered by a permutation $\sigma : [m] \to [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$ $S_{\sigma^{-1}(i)} \cap (\cup_{j<i} S_{\sigma^{-1}(j)})$ contains a d-face.

This theorem is proved by induction on the dimension of K. Note that a graph is 2-CM iff it is 2-connected. The base of the induction, where K is a graph, easily follows from a theorem of Whitney [11] which asserts that a graph is 2-connected iff it has an open ear decomposition.

Problem 8. *Can the S_i's in Theorem 7 be taken to be homology spheres?*

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Forbidden patterns

János Pach
(joint work with Gábor Tardos)

At most how many edges (hyperedges, nonzero entries, characters) can a graph (hypergraph, zero-one matrix, string) have if it does not contain a fixed forbidden pattern? Turán-type extremal graph theory, Erdős–Ko–Rado-type extremal set theory, Ramsey theory, the theory of Davenport–Schinzel sequences, etc. have been developed to address questions of this kind. They produced a number of results that found important applications in discrete and computational geometry.

In the present talk, we discuss an extension of extremal graph theory to ordered graphs, i.e., to graphs whose vertex set is linearly ordered. In the most interesting cases, the forbidden ordered graphs are bipartite, and the basic problem can be reformulated as an extremal problem for zero-one matrices avoiding a certain submatrix P. We disprove a general conjecture of Füredi and Hajnal [3] related to the latter problem, by exhibiting a forbidden submatrix P, for which the maximum number of ones in an $n \times n$ zero-one matrix avoiding P is much larger than the solution of the same problem with the difference that the order of the rows and columns can be arbitrarily changed. However, we conjecture that these functions must be close to each other for adjacency matrices P of acyclic graphs. We verify this conjecture in a few special cases.

We call a sequence $C = (p_0, p_1, \dots, p_{2k})$ of positions in a matrix P an *orthogonal cycle* if $p_0 = p_{2k}$ and the positions p_{2i} and p_{2i+1} belong to the same row, while the positions p_{2i+1} and p_{2i+2} belong to the same column, for every $0 \leq i < k$. If the entry of P in positions p_i is 1 for all $0 \leq i \leq 2k$, then C is said to be an *orthogonal cycle of P*. Given a position $p = (i, j)$ of the matrix P and an orthogonal cycle $C = (p_0, p_1, \dots, p_{2k})$, define $C(i, j)$ to be the number of times that the possibly self-intersecting polygon $p_0 p_1 \dots p_{2k}$ encircles (in the counter-clockwise direction) a point $p' = (i + 1/2, j + 1/2)$ of the plane. (Here we interpret the position (i, j) in a matrix as the point (i, j) or the Euclidean plane.) An orthogonal cycle is said to be *positive* if $C(i, j) \geq 0$ for every pair (i, j) and $C(i, j)$ is strictly positive for at least one such pair. We prove that the maximum number of ones in an $n \times n$ zero-one matrix containing no positive orthogonal cycle is $O(n^{4/3})$. The order of magnitude of this bound cannot be improved.

Our results lead to a new proof of the celebrated theorem of Spencer, Szemerédi, and Trotter [5] stating that the number of times that the unit distance can occur among n points in the plane is $O(n^{4/3})$. This is the first proof that does not use any tool other than a forbidden pattern argument. We present another geometric application, where the forbidden pattern P is the adjacency matrix of an acyclic graph. A *hippodrome* is a $c \times d$ rectangle with two semicircles of diameter d attached to its sides of length d. Improving a result of Efrat and Sharir [2], we show that the number of "free" placements of a convex n-gon in general position in a hippodrome H such that simultaneously three vertices of the polygon lie on the boundary of H, is $O(n)$. This result is related to the Planar Segment-Center Problem.

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Bounding the volume of facet-empty lattice tetrahedra

Julian Pfeifle

(joint work with Han Duong, Christian Haase, Bruce Reznick)

A lattice polytope is *empty* if it contains no lattice points except for its vertices. Already in 1957, Reeve [10] noticed that empty three-dimensional lattice simplices may have unbounded volume. In 1982, Zaks, Perles & Wills [12] constructed a family of d-dimensional lattice simplices, each member of which contains k lattice points in total and has the rather large volume

$$\frac{k+1}{d!}\, 2^{2^{d-1}-1}\,.$$

In the following year, Hensley [4] proved that the volume of any d-dimensional lattice polytope containing $k \geq 1$ lattice points in its interior is bounded by a constant that depends only on d and k. By sharpening Hensley's basic diophantine approximation lemma, Lagarias & Ziegler [6] in 1991 improved his bound and showed that the maximal volume $V(d,k)$ of a d-dimensional lattice polytope with k interior lattice points is bounded by

$$V(d,k) \;\leq\; k\big(7(k+1)\big)^{d2^{d+1}},$$

which for $d=3$ reads

$$V(3,k) \;\leq\; k\big(7(k+1)\big)^{48}.$$

This bound was further sharpened by Pikhurko [8], who was able to prove an upper bound with only a linear dependence on k:

$$\begin{aligned} V(d,k) &\leq (8d)^d \cdot 15^{d\cdot 2^{2d+1}} \cdot k\,, \\ V(3,k) &\leq 24^3 \cdot 15^{384}\, k\,. \end{aligned}$$

A *facet-empty* or *clean* lattice polytope is a lattice polytope whose only lattice points on the boundary are its vertices. In our talk we focused on the special class of *facet-empty k-point lattice tetrahedra*, which contain exactly $k+4$ lattice points, k of them in the relative interior. It is known [10], [11] that via unimodular transformations any facet-empty lattice tetrahedron may be brought into the normal form

$$T_{a,b,n} \;=\; \operatorname{conv}\big\{(0,0,0),(1,0,0),(0,1,0),(a,b,n)\big\},$$

where $(a,b,n)=(0,0,1)$, or $n\geq 2$, $0\leq a,b\leq n-1$, and $\gcd(a,n)=\gcd(b,n)=\gcd(1-a-b,n)=1$. Note that $\operatorname{vol} T_{a,b,n}=n$.

We sketched a proof of the following theorem, which significantly improves Pikhurko's bound for this special family of 3-dimensional lattice polytopes:

Theorem 1. *The maximal (normalized) volume of a clean lattice tetrahedron Δ with $k\geq 1$ interior lattice points is*

$$\operatorname{vol}\Delta \;\leq\; 12k+8\,.$$

This bound is attained by the family of clean k-point lattice tetrahedra

$$\big\{T_{3,6k+1,12k+8} : k\geq 1\big\}.$$

The overall structure of the proof is as follows. We first show that the number k of interior lattice points of Δ equals the number of times that the sum

$$f(z) \;=\; \left\lfloor \frac{(a+b-1)z}{n} \right\rfloor - \left\lfloor \frac{az}{n} \right\rfloor - \left\lfloor \frac{bz}{n} \right\rfloor$$

equals 1, as z takes on integer values between 1 and n.

Next, we use that $f(z)\in\{0,\pm 1\}$ and $f(n-z)=-f(z)$ for $1\leq z\leq n-1$ to express k as half the second moment of the sequence $\big(f(z) : 1\leq z\leq n-1\big)$. This second moment is then expressed using *Dedekind sums* $s(a,n)$:

Proposition 2. *Set $c = 1-a-b \bmod n$ and let $aa' = bb' = cc' = 1 \bmod n$. Then*

$$\frac{1}{2}\sum_{z=1}^{n-1} f(z)^2 = \frac{n-3}{6} + \frac{1}{3n} - s(c,n) - s(a,n) - s(b,n) + s(a'b,n) + s(a'c,n) + s(b'c,n)\,.$$

Remark 3. *After obtaining this expression, we realized that we should have expected the appearance of Dedekind sums in this expression, because they appear in a formula of Pommersheim* [9] *for the Ehrhart polynomial of lattice tetrahedra. In fact, our derivation of Proposition 2 yields an elementary proof of this formula in the case of facet-empty tetrahedra; in particular, we do not need to evaluate the Todd class of the associated toric variety.*

There are now at least two ways to complete the proof of the theorem. First, we can express each Dedekind sum $s(a,n)$ as essentially the sum of digits of the *negative-regular continued fraction expansion* of $n/(n-a)$; see [1], [5], [7], [9].

Proposition 4. *Let* $n/(n-a) = b_1 - 1/(b_2 - 1/(\cdots - 1/b_r))$ *be the* negative-regular continued fraction expansion *of* $n/(n-a)$, *where* $0 \le a < n$ *are coprime and we require* $b_i \ge 2$ *for* $i = 1, 2, \ldots, r$. *Moreover, define* $a' \in \mathbb{N}$ *by* $aa' = 1 \bmod n$ *and* $0 \le a' < n$. *Then*

$$s(a,n) \;=\; \frac{1}{12}\left(\sum_{i=1}^{r}(3-b_i) + \frac{a+a'}{n} - 2\right).$$

We then use a detailed analysis of the behavior of digit sums of negative-regular continued fraction expansions to bound k from below. This approach requires a fairly substantial amount of case distinctions.

The other way of proving the theorem is not by passing to continued fractions, but instead by bounding the individual Dedekind sums directly and controlling the interaction between the six summands in Proposition 2. This alternative proof of the theorem is still work in progress; we have reason to hope that it requires a substantially smaller number of case distinctions.

It has been observed several times that $s(a,n)$ changes drastically if a is close to numbers of the form $n \cdot c/d$. The behaviour of Dedekind sums in the neighborhood of such values was studied by Girstmair [2], [3], who introduced the notion of "F-neighbors". First, define a *Farey point* to be a real number of the form $n \cdot c/d$, where d is "small"; more precisely, $1 \le d \le \sqrt{n}$, $0 \le c \le d$ and $\gcd(c,d) = 1$. (Note that this c is different from the one used before.) The denominator d is called the *order* of the Farey point. The *F-neighbors of order d* are all real numbers x such that $0 \le x \le n$ and $|x - n \cdot c/d| \le \sqrt{n}/d^2$, for some $0 \le c \le d$ with $\gcd(c,n) = 1$. An integer $x \in [0,n]$ that is not an F-neighbor of order $1 \le d \le \sqrt{n}$ is called an *ordinary integer.*

Proposition 5. [3, Theorem 1 and Section 3]

(a) *If* $n \ge 15$ *and* $x \in [0,n]$ *is an ordinary integer, then* $|s(x,n)| \le \frac{1}{4}\sqrt{n} + \frac{5}{12}$.

(b) *Let* x *be a* F*-neighbor of order* d, *let* $n \cdot c/d$ *be the corresponding* F*-point, and put* $q = xd - cn$, *so that* $|q| \le \sqrt{n}/d$. *Then*

$$s(x,n) \;=\; \frac{n}{12dq} + \frac{1}{12}\,E(d + |q| + 4),$$

where E *denotes an error term such that* $|E(z)| \le z$.

The next proposition is crucial for analyzing the sum from Proposition 2.

Proposition 6. *Let $x = (cn + q)/d$ with $\gcd(c, d) = \gcd(x, n) = 1$ be a Farey neighbor of order d to cn/d. Then for any $0 \leq c' < q$ relatively prime to q, there exists a parametrization $n = \alpha s + \beta$ with $\alpha, \beta \in \mathbb{Z}$ and $s \in \mathbb{N}$ such that the inverse x' of x modulo n has the form $x' = (c'n + d)/q$.*

In particular, inverting x leaves the product dq invariant.

It turns out that we may assume all of the six first arguments $\{a, b, c, a'b, b'c, c'a\}$ of the Dedekind sums in Proposition 2 to be F-neighbors. Moreover, by the estimates from Proposition 5 we only need to consider those 20 sets of arguments such that the associated values $d_i q_i$ satisfy $\sum_{i=1}^{6} 1/(d_i q_i) > 1$. We leave the completion of this argument for further study.

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On Geometric Graphs with no Pair of Parallel Edges

Rom Pinchasi

A *geometric graph* is a graph drawn in the plane with its vertices as points and its edges as straight line segments connecting corresponding points. A topological graph is defined similarly except that its edges are simple Jordan arcs connecting corresponding points. Two edges in a geometric graph are said to be *parallel*, if they are two opposite edges of a convex quadrilateral.

In [2, 3] Katchalski, Last, and Valtr prove a conjecture of Kupits and obtain the following result:

Theorem 1. *A geometric graph on n vertices with no pair of parallel edges has at most $2n - 2$ edges.*

In this talk we give two very simple proofs for Theorem 1. We also give a strengthening of this result in the case where G does not contain a cycle of length 4. In the latter case we show that G has at most $\frac{3}{2}(n-1)$ edges.

In the first proof we show that any geometric graph with no pair of parallel edges can be redrawn as a *generalized thrackle*, that is, a topological graph every pair of whose edges meet an odd number of times. We then use results of Cairns and Nikolayevsky ([1]) concerning such graphs. The second proof is self-contained and relies on some purely combinatorial lemma.

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Cinderella.2 — an ϵ before release

Jürgen Richter-Gebert

(joint work with Ulrich Kortenkamp)

The interactive geometry software Cinderella [2] is an environment for doing constructions of elementary geometry on a computer. After a construction has been generated by mouse interactions, the base elements can be moved and the dependent elements follow accordingly. At first sight the task to implement such a programm seems to be almost trivial from a mathematical point of view. However, a closer look shows that the demand of *contineous behavior* of dependend objects requires strategies from complex function theory that navigate on Riemann surfaces that are implicitely definded by a geometric construction (see [1]). Also projective geometry is necessary to deal with elements at infinity appropriately.

The version of Cinderella that is available now since about 5 years was build on these principles. Currently a new version of Cinderella is close to its final release. This new release significantly broadens the scope of the programm. Besides the original focus on elementary (euclidean and non-euclidean) geometry also the following topics are covered:

- advanced geometric primitive operations,
- transformation geometry,
- transformation groups,
- iterated function systems,
- simulation of scenarios in physics,
- internal and external scripting of the programm,
- interfaces to other programming languages,
- handdrawing recognition,
- advanced screen recording capabilities.

Perspective view of an iterated function system generated by two similarities

In the following sections some of these new features are mentioned and sample images are provided.

1. Transformations and transformation groups

Transformations can be defined by declaring preimages and images of a collection of points under the transformation. For instance a projective transformation is defined by specifying four original points and four image points. It is possible to define translations, similarities, affine transformations, projective transformations and Moebius transformations. Transformations can be combined to form transformation groups. It is also possible to display the iterated function system generated by a couple of transformations. All parameters of the transformations can be varied continuously so that it is for instance possible to interactively explore iterated function systems.

2. Physics Simulations

Cinderella.2 has the build-in functionality to simulate scenarios with point-masses and forces. The physics environment is completely compatible with the geometry part of the programm so that one can easily incorporate a geometric analysis of a simulation. Forces can be generated by force fields, by springs or from short range particle interactions. The simulation engine is based on a DormandPrince45-integrator that allows for fast and still reasonably exact calculations.

3. Scripting

A feature of particular interest is the possibility to control the behavior of a construction by a script. The scripting language *CindyScript* is designed to have a mimimum of syntactical overhead. It is a functional language and does not require explicit typing. For input and output the facilities of Cinderella are used. By this, one can create very complex functionality by only a few lines of code. The application range of the scripting language ranges from interactive enhancements of geometric constructions via calculus and discrete geometry, to the automatic analysis of physics simulations. The example on this page shows the complete script to produce a Lindenmayer-System that generates a plant like structure on the screen.

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The number of spanning trees in a planar graph

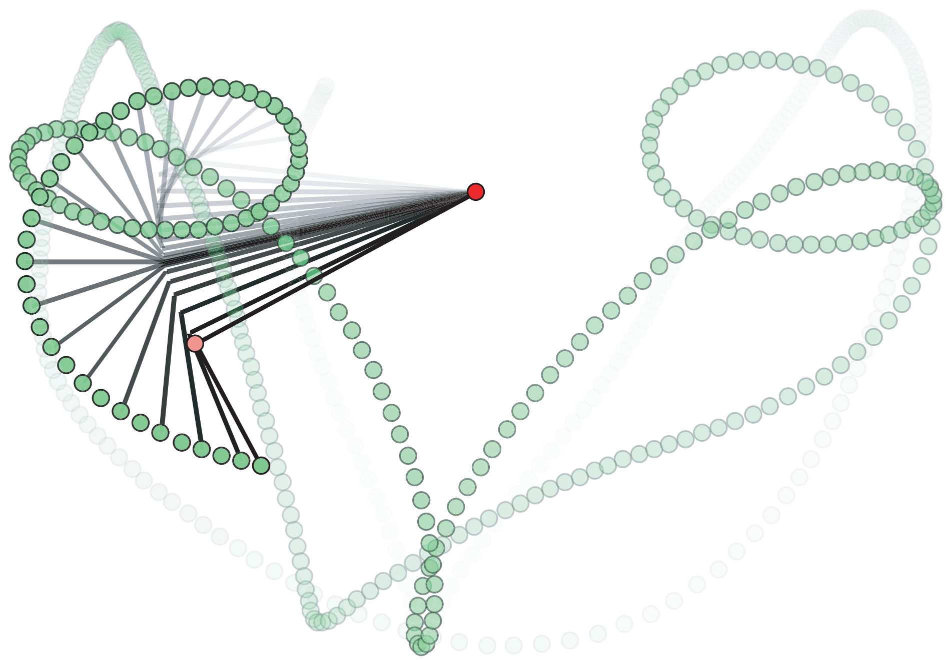
Günter Rote

(joint work with Ares Ribó, Xuerong Yong)

Theorem 1. *(1) A planar graph with n vertices has at most $5.33333333\ldots^n$ spanning trees.*

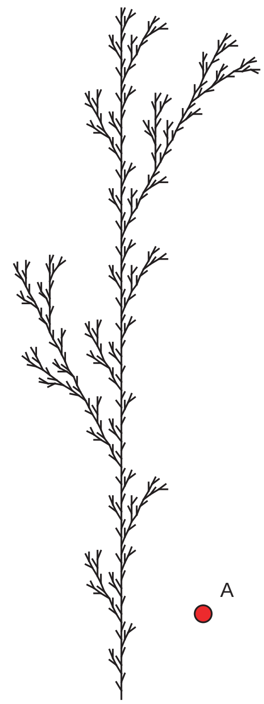
(2) A planar graph with n vertices and without a triangle has at most $(\frac{4}{\sqrt[8]{e}})^n < 3.529988^n$ spanning trees.

(3) A three-connected planar graph with n vertices and without a face cycle of length three or four has at most $(\sqrt[3]{36}/e^{4/27})^n < 2.847263^n$ spanning trees.



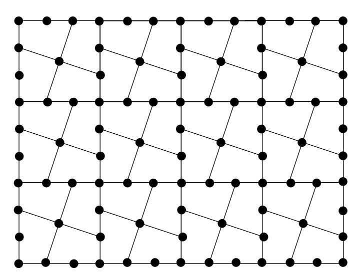
Trace of a chaotic pendulum

```
list(x):=(
  gsave();
  repeat(length(x),turtle(x_#));
  grestore();
);

turtle(x):=(
  (if(x=="F",foreward));
  (if(x=="+",left));
  (if(x=="-",right));
  (if(x=="[",open));
  (if(x=="]",close));
);
linecolor((0,0,0));
foreward:=(draw((0,0),(1,0));translate((1,0)));
left:=rotate(angle);
right:=rotate(-angle);
open:=gsave();
close:=grestore();

l=0.2;
angle=A.x/4;

n=4;
s="F";
repeat(n,s=replace(s,"F","F[+F]F[-F]F"));
rotate(pi/2);
list(s)
```



Linenmayer-System generated by a script

Lower bounds that complement parts (1) and (2) come from the triangular and square grids [7], which have asymptotically $\approx 5.029545^n$ and $\approx 3.209912^n$ spanning trees, respectively, see also [6, (2.17–2.19)]. (The exact values are $\exp\big(\frac{3\sqrt{3}}{\pi}(1-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{11^2}+\frac{1}{13^2}-\cdots)\big)$ and $\exp\big(\frac{4}{\pi}(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\cdots)\big)$.) A large graph with pentagonal faces with the regular structure shown on the right is a candidate for the best construction in case (3). The asymptotic number of trees of this example can be calculated by the technique of Shrock and Wu [6], but we haven't done this.



Our motivation for studying this problem comes from the task of realizing 3-dimensional polytopes with (small) integral vertex coordinates. The combinatorial structure of a 3-polytope is specified by a three-connected planar graph. Such a graph always contains at least a triangular, a quadrilateral, or a pentagonal face; this is the reason why we did not continue after part (3) of Theorem 1.

To construct a 3-polytope with a given combinatorial structure, we follow the approach described in Richter-Gebert [5, Part IV]: we construct a planar equilibrium embedding for a specified self-stress and lift it to a polyhedral surface via the Maxwell-Cremona correspondence. The analysis of the determinant of the linear system of equations which is used to define the equilibrium embedding leads directly to the number of spanning trees of the graph, via the Matrix-Tree theorem.

With the improved bounds of Theorem 1 and some additional technique for graphs containing a quadrilateral face, we can improve the results of Richter-Gebert as follows:

Theorem 2. *(1) A 3-polytope P with n vertices can be realized with integral coordinates of absolute value less than 2^{12n^2} (or more precisely, $n^{10n}\, 2^{10n^2}$).*
(2) If P contains a quadrilateral face, the bound is reduced to 156^n.
(3) If the graph of P contains a triangle, the bound is reduced to 29^n. □

In this abstract, we will only sketch the techniques for proving Theorem 1. Full details can be found in [4]. The proof of part (1) is rather simple: we add edges until we obtain a triangulated supergraph G; its dual graph G^* is 3-regular and has $2n-4$ vertices. Applying the upper bound for regular (not necessarily planar) graphs of McKay [3], and of Chung and Yau [1] yields our bound.

For parts (2) and (3) of Theorem 1, we introduce the *Outgoing Arc Approach*. We choose an arbitrary root vertex r. In the directed graph obtained by replacing every edge by two opposite directed arcs, we form a subset R of arcs by selecting one outgoing arc uniformly at random for each vertex different from the root.

If R does not contain cycles, it forms a spanning tree. Each tree is generated in exactly one way by this process. Multiplying the number of possibilities, which is the product of the vertex degrees $\prod_{v\in V-\{r\}} d_v$, by the "success probability" yields the following expression for the number T of spanning trees:

Lemma 3.

$$T = \prod_{v\in V-\{r\}} d_v \cdot \text{Prob}(R \text{ does not contain a cycle}) \qquad \square$$

From the product of vertex degrees $\prod d_v$ and the arithmetic-geometric mean inequality we already get an easy upper bound of 6^n for the number of spanning trees of planar graphs. We improve this by estimating the probability that some cycle appears. The probability that a particular cycle c appears can be easily calculated as the reciprocal of the product of the degrees. However, cycles do not appear independently. Cycles are independent if they have disjoint vertex sets, and hence we expect that "most" short cycles will be independent of each other. We use Suen's inequality for this case of controlled dependence. Suen's inequality uses the concept of a dependency graph. Let $\{X_i\}_{i\in\mathcal{I}}$ be a family of random variables. A *dependency graph* is a graph L with node set $\mathcal{I}$ such that if A and B are two disjoint subsets of $\mathcal{I}$ with no edge between A and B, then the families $\{X_i\}_{i\in A}$ and $\{X_i\}_{i\in B}$ are mutually independent. In particular, two variables X_i and X_j are independent unless there is an edge in L between i and j. If there exists such an edge, we write $i \sim j$. Suen's inequality is useful in cases in which there exists a sparse dependency graph. The expected value of a random variable X is denoted by $\mathbb{E}X$. The following theorem is a special case of Suen's inequality, see [2]:

Theorem 4. *Let I_i, $i \in \mathcal{I}$, be a finite family of Bernoulli random variables with success probability p_i, having a dependency graph L. Let $X = \sum_i I_i$ and $\lambda = \mathbb{E}X = \sum_i p_i$. Moreover, let $\Delta = \frac{1}{2}\sum_i \sum_{j:i\sim j} \mathbb{E}(I_i I_j)$ and $\zeta = \max_i \sum_{k\sim i} p_k$. Then*

$$\mathrm{Prob}(X = 0) \leqslant \exp(-\lambda + \Delta e^{2\zeta}).$$

In our case, the nodes of the dependency graph are all directed cycles in the graph that avoid r. We connect two cycles by an edge if they share some vertex. The independent choice of an outgoing arc for each vertex in R the ensures that this dependency graph is valid for our model.

Two directed cycles c and c' that share a vertex can never occur together in R, because every vertex has only one outgoing arc in R. Hence, $i \sim j$ implies that $\mathbb{E}(I_i I_j) = 0$, which means that $\Delta = 0$ in Theorem 4. Therefore, we have

$$\mathrm{Prob}(R \text{ does not contain a cycle}) = \mathrm{Prob}(X = 0) \leqslant \exp(-\lambda),$$

where λ is the sum of probabilities for all directed cycles c that can appear in R:

$$(1) \quad \lambda = \sum_c (1/\prod_{v\in c} d_v) = \sum_{(i,j)\in\mathcal{C}_2} \frac{1}{d_i d_j} + \sum_{(i,j,k)\in\mathcal{C}_3} \frac{2}{d_i d_j d_k} + \sum_{(i,j,k,l)\in\mathcal{C}_4} \frac{2}{d_i d_j d_k d_l} + \cdots$$

Here $\mathcal{C}_b$ denotes the set of undirected cycles of length b that don't contain r. To prove an upper bound on $\prod d_v \cdot e^{-\lambda}$ we truncate the sum (1) after $\mathcal{C}_2$. We let the variable f_{ij}, with $i \leqslant j$, stand for the number of edges connecting a vertex of degree i and a vertex of degree j. The logarithm of $\prod d_v \cdot e^{-\lambda}$ can then be written as a linear function in the variables f_{ij}:

$$Z = \sum_{v\in V} \ln d_v - \sum_{(i,j)\in E} \frac{1}{d_i d_j} = \sum_{i\leqslant j} f_{ij} \left(\frac{\ln i}{i} + \frac{\ln j}{j} - \frac{1}{ij} \right)$$

We maximize Z under constraints that reflect the total number n of vertices and the total number of edges in a planar graph (at most $3n$):

$$(2) \qquad \sum_{i\leqslant j} f_{ij} \left(\frac{1}{i} + \frac{1}{j} \right) = n, \text{ and } \sum_{i\leqslant j} f_{ij} \leqslant 3n$$

The optimum $Z = \ln 6 - \frac{1}{12}$ with $e^Z \approx 5.5203$ is achieved when $f_{66} = 3n$ and all other $f_{ij} = 0$. However, this bound for part (1) is not as strong as the easy bound that comes from the dual graph. If we replace the edge bound $3n$ in (2) by $2n$ and $5n/3$, respectively, we obtain parts (2) and (3) of Theorem 1. The corresponding optimal solutions are $f_{44} = 2n$ (corresponding to the square grid), and $f_{33} = n/3$, $f_{34} = 4n/3$ (corresponding to the grid graph with pentagonal faces shown on the first page). Planarity enters this proof only via the bound on the number of edges.

As a next step, one can include in the sum (1) larger cycles $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_5$. If we consider only *face cycles* and introduce corresponding variables f_{ijk}, f_{ijkl}, f_{ijklm} for the number of faces with vertices of degree i, j, k, l, m, calculations indicate that this would reduce the bound in part (2) of Theorem 1 to 3.5026. (In this case, no cycles of length 3 appear.) However, this appears quite complicated to prove. Also, it appears that one cannot beat the current bound for part (1) with this technique, even if longer and longer cycles are included.

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Cubical Polytopes and Spheres

Thilo Schröder

(joint work with Michael Joswig)

We derive a non-recursive combinatorial description of the cubical spheres constructed by Babson, Billera, and Chan [2]. This enables us to deduce many interesting properties of these spheres, for example neighborliness and (non)-polytopality. (For detailed study and references we refer the reader to [4].) The ingredients needed for this construction are the mirror complex, the cubical fissure and BBC sequences.

1. Ingredients

The *mirror complex* was first described by Coxeter [3] in terms of reflection groups. It is a cubical complex that may be constructed for any simplicial complex. It reflects some of the properties of the simplicial complex, for example:

- if the simplicial complex is k-neighborly, then the mirror complex is $(k+1)$-neighborly, and
- if the simplicial complex is a $(d-1)$-sphere, then the mirror complex is a cubical d-manifold.

The *cubical fissure* is a technique to construct new cubical complex from a given one by choosing a subcomplex an then introducing a fissure along (prism over) its boundary.

We introduce BBC sequences, special sequences of simplicial balls. These arise naturally from vertex orderings of simplicial polytopes, as used by Babson, Billera, and Chan, and are closely related to spheres that are directly obtainable in the sense of Altshuler [1].

2. Cubical Spheres

Using these ingredients, Babson, Billera, and Chan inductively constructed cubical d-spheres from BBC sequences of simplicial $(d-1)$-balls. Since neighborliness is one of the properties preserved by mirroring, they were able to prove the existence of neighborly cubical spheres.

2.1. Combinatorial Description.

Following the inductive construction of Babson, Billera, and Chan, we derive a purely combinatorial description of the cubical spheres constructed from a BBC sequence of simplicial balls. The cubical sphere is explicitly given as a subcomplex of a cube depending only on the faces of the BBC sequence. This representation is close to the Cubical Gale Evenness Condition of Joswig and Ziegler [5]. We show that the boundary of the neighborly cubical polytopes constructed by Joswig and Ziegler [5] are isomorphic to neighborly cubical spheres constructed from particular vertex orderings of cyclic polytopes.

With this combinatorial description we prove that if the cubical sphere constructed from a BBC sequence is polytopal, then all the boundary spheres of the BBC sequence must be polytopal as well. This allows us to construct an explicit example of a non-polytopal neighborly cubical 5-sphere. It can also be shown, using entirely different techniques, that if the BBC sequence arises from a vertex ordering of a simplicial polytope, then the cubical sphere constructed from this sequence is polytopal [9].

In dimension less than five, all the cubical spheres constructed from BBC sequences are polytopes, since they are built from simplicial balls of dimension less or equal two. These simplicial balls are pulling triangulations of directly obtainable 1- or 2-spheres, which are polytopes according to Altshuler.

2.2. Polyhedral Surfaces.

The above results yield a new proof for the realizability of some of the equivelar surface of type $\mathcal{M}_{4,q}$ realized by McMullen, Schulz, and Wills [6, 7]. Surfaces of type $\mathcal{M}_{p,q}$ consist of p-gons, where each vertex has degree q. The quad-surfaces of type $\mathcal{M}_{4,q}$ with $n = 2^k$ vertices ($k \geqslant 3$) were already mentioned by Coxeter [3] and Ringel [8] and are particularly interesting because of their 'unusually high genus' $\mathcal{O}(n \log n)$ as pointed out by McMullen, Schulz, and Wills.

The mirror complex of the boundary of a q-gon Q is a quad surface S with vertex degree q, since the link of a vertex of the mirror complex corresponds to the simplicial complex Q. To realize this surface, we consider a BBC sequence $\mathcal{T}$ constructed from a $(q-1)$-gon. With our combinatorial description we show that the mirror complex of a pulling triangulation of a q-gon is contained in the cubical sphere constructed from $\mathcal{T}$. The dimension of the sphere is three and it is isomorphic to the boundary of a 4-dimensional neighborly cubical polytope of Joswig and Ziegler. The Schlegel diagram of this polytope is a 3-dimensional cubical complex which contains the cubical surface S.

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On lattice sphere packings and coverings

Achill Schürmann

(joint work with Mathieu Dutour and Frank Vallentin)

Classical problems in the geometry of numbers are the determination of most economical lattice sphere packings and coverings in Euclidean d-space $\mathbb{R}^d$. We report on some recent progress in the study of covering and (simultaneous) packing–covering lattices: We found some new best known low dimensional lattices and have promising new approaches using symmetry. Moreover, we report on the successful application of a new technique to prove the local optimality of lattices, as for example the Leech lattice.

A lattice L is a discrete subgroup of $\mathbb{R}^d$. We may assume that L has full rank d and is spanned by a basis $A \in \mathrm{GL}_d(\mathbb{R})$, that is, $L = A\mathbb{Z}^d$. The determinant $\det(L) = |\det(A)|$ of L is at the same time the volume of the Dirichlet-Voronoi polytope (DV-cell) $\mathrm{DV}(L) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \leqslant |\boldsymbol{x} - \boldsymbol{y}| \text{ for all } \boldsymbol{y} \in L \right\}$. The inradius of $\mathrm{DV}(L)$ is called the packing radius of L, denoted by $\lambda(L)$, and its circumradius is the covering radius of L, denoted by $\mu(L)$. The *covering density* of L is

$$\Theta(L) = \frac{\mu(L)^d}{\det(L)} \cdot \kappa_d$$

where κ_d denotes the volume of the d-dimensional unit ball. The (simultaneous) *packing-covering constant* of L is defined by

$$\gamma(L) = \frac{\mu(L)}{\lambda(L)}.$$

Problem (lattice covering problem). For $d \geqslant 1$ determine $\Theta_d = \min_L \Theta(L)$ and lattices L of rank d attaining it.

Problem (lattice packing-covering problem). For $d \geqslant 1$ determine $\gamma_d = \min_L \gamma(L)$ and lattices L of rank d attaining it.

Both problems have been solved for dimensions $d \leqslant 5$ only. We refer to [7] for a detailed overview about the two problems.

Both problems can be reduced to finitely many convex optimization problems. The main mathematical tool needed for an algorithmic solution is a reduction theory developed by Voronoi (see [11]), in which lattices are categorized according to their DV-cell: Assume a basis A of L and an associated positive definite quadratic form (PQF) $Q[\boldsymbol{x}] = |A\boldsymbol{x}|^2 = \boldsymbol{x}^t A^t A \boldsymbol{x}$ are given. Then the DV-cell of Q is the polytope

$$\mathrm{DV}(Q) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : Q[\boldsymbol{x}] \leqslant Q[\boldsymbol{x} - \boldsymbol{y}] \text{ for all } \boldsymbol{y} \in \mathbb{Z}^d \right\}.$$

Each vertex $\boldsymbol{c}$ of $\mathrm{DV}(Q)$ corresponds to a Delone polytope $P = \mathrm{conv}\{\boldsymbol{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ with $\boldsymbol{v}_i \in \mathbb{Z}^d$ and $Q[\boldsymbol{c} - \boldsymbol{v}]$ minimal among integral points $\boldsymbol{v}$, if and only if $\boldsymbol{v}$ is one of the vertices of P. Two PQFs (and corresponding lattices) are said to be of the same L-type, if the star of Delone polytopes with vertex at the origin $\boldsymbol{0}$ is the same. The Delone star defines a Delone subdivision of $\mathbb{R}^d$, by translating it by all vectors of $\mathbb{Z}^d$.

Voronoi's theory allows in principle an enumeration of all L-types. Due to a combinatorial explosion this is currently practically impossible in dimensions $d \geqslant 6$. Nevertheless, given a fixed L-type it follows from theorems of Barnes and Dickson [2] and Ryshkov [6] that among lattices of the given L-type there exists a unique lattice (up to orthogonal transformations) minimizing the covering density and a unique minimum of the packing-covering constant. In the first case the problem can be formulated as a determinant maximization problem (see [10]), in the second case we have to solve a semidefinite program. Using the software MAXDET by Wu, Vanderberghe and Boyd (`http://www.stanford.edu/~boyd/MAXDET.html`) we were not only able to verify all of the known results in dimension $d \leqslant 5$, we heuristically found new best known lattices for the lattice covering and the lattice packing-covering problem in dimension 6. The interpretation of these results helped us to discover new best known lattice coverings in dimension 7 and 8 as well.

In dimensions above 8 not only the number of L-types explodes, but also the size of the input and the dimension of the convex optimization problems become intractable. With a new equivariant approach we introduce in [4] we hope to overcome these difficulties and to find new record breaking lattices in dimensions up to 8 and beyond. The key idea is to fix a finite matrix group G and to restrict the search to a Bravais space

$$B(G) = \left\{ Q \in \mathcal{S}^d_{>0} : g^t Q g = Q \text{ for all } g \in G \right\}$$

of G within the space of PQFs $\mathcal{S}^d_{>0}$. Here we have a natural extension of Voronoi's reduction theory with respect to the action of the normalizer

$$N_{\mathbb{Z}}(G) = \left\{h \in \mathrm{GL}_d(\mathbb{Z}) : h^{-1}Gh = G\right\}$$

on $B(G)$. With the new theory we hope to find new best known covering lattices whose automorphism group has a low dimensional Bravais space. A similar generalization of Voronois reduction theory by perfect forms has been described by Martinet, Bergé and Sigrist in [5].

Inspired by the experimental success, we have developed a technique (see [8], [9]) that allows to compute local lower bounds for the covering density and the packing covering constant. With it, we were able to prove the following theorem (see [8]) on the local optimality of the highly symmetric 24-dimensional Leech lattice (see [3]). "Locally optimal" in this context means that choosing a fixed base, sufficiently small perturbations of it (unless they are dilations or orthogonal transformations) yield lattices with a larger covering density or packing-covering constant respectively.

Theorem 1. *The Leech lattice is a locally optimal covering lattice and a locally optimal packing covering lattice.*

Note that the Leech lattice gives a first known example of a locally optimal covering lattice whose Delone subdivision is not a triangulation, which was an open question before. It may surprise at first that a similar bound is not attained by the root lattice E_8. In fact, the E_8 lattice is even far from being a local minimum.

Theorem 2. *Let Q_{E_8} be a PQF associated to the root lattice E_8 and let Q be a quadratic form such that the Delone subdivision of $Q_\lambda = Q_{E_8} + \lambda Q$ is a triangulation for all sufficiently small $\lambda > 0$. Then the covering density of E_8 is strictly larger than the covering density of a lattice associated to Q_λ, for all sufficiently small $\lambda > 0$.*

A generic Q satisfies the condition of the Theorem. Thus instead of a local optimum we may speak of a local pessimum as suggested by Peter McMullen during the talk. Similar is true for the root lattice D_4 and we would like to know if it is true for the Coxeter-Todd lattice in dimension 12 and the Barnes-Wall lattice in dimension 16. For the packing-covering problem it currently remains open whether or not E_8 gives a local or even global minimum as conjectured by Zong in [12].

Concluding we must say that the lattice covering problem is far from being understood. This is even more the case for the sphere covering problem in general without the restriction to lattices. The following are major open challenges:

- Prove that the bcc-lattice (A_3^*) gives a least dense covering in $\mathbb{R}^3$.
- Prove that the Leech lattice gives a least dense (lattice) covering in $\mathbb{R}^{24}$.
- Find a dimension d and a non-lattice covering that is less dense than any lattice covering in its dimension.

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Counting Crossing-Free Configurations in the Plane

Micha Sharir

(joint work with Emo Welzl)

Let P be a set of n points in the plane in general position. We consider various classes of crossing-free geometric graphs on P (i.e., they have P as their vertex set, and edges are drawn as crossing-free straight segments), such as matchings, spanning cycles, crossing-free partitions (these are partitions of the set, so that the convex hulls of the individual parts are disjoint), and spanning trees. We obtain improved upper bounds for the maximum possible number of graphs in each of these families, as a function of n. Specifically, we show that the maximum number of prefect matchings is at most 10.52^n; the maximum number of partial matchings is at most 10.92^n; the maximum number of crossing-free partitions is at most 12.92^n; the maximum number of spanning cycles is at most 78.2^n; and the maximum number of spanning trees is at most 296^n. Unlike previous upper bounds, all of which essentially depend on bounds for the number of triangulations of P, our approach directly establishes an upper bound on the number of matchings, independently of any assumed triangulation, and then uses this bound to derive all other mentioned upper bounds.

Geometry in Additive Combinatorics

József Solymosi

One of the central problems in additive combinatorics is to describe the structure of sets with small sumsets. The sumset of a set A is denoted by $A+A$, where

$$A + A = \{a + b | a, b \in A\}.$$

If the sumset is very small, $|A + A| \leqslant C|A|$, where C is a constant, then the structure of A is similar to an arithmetic progression. Freiman's theorem [3] says that A is a $c-$dense subset of a $d-$dimensional generalized arithmetic progression, where d and c depend on C only. (For the notations and details about Freiman's theorem we refer to Nathanson's book [4].) In particular, one can show that A contains long arithmetic progressions [1]. For larger but not very large sumsets, when

$$|A| \ll |A + A| \leqslant |A|^{1+\epsilon},$$

Freiman's theorems gives no valuable information and it is not true in general, that A would even contain an arithmetic progression of length three. On the other hand one can show that A contains a special subset with a "nice" additive structure.

Definition 1. We say that B_d is an affine cube of reals with dimension d, if there are real numbers $x_0, x_1, \ldots, x_d$, such that

$$B_d = \left\{ x_0 + \sum_{i \in I} x_i \middle| I \subset [1, 2, \ldots, d] \right\}.$$

Affine cubes of integers were introduced by Hilbert [2], who proved that for any partition of the integers into finitely many classes, one class will always contain arbitrary large affine cubes of integers. His result was extended to various directions, the most famous are Schur's theorem [5] and van der Waerden's Theorem [6]. Here we prove the following.

Theorem 1. *For every d there is a $\delta > 0$ and a threshold $n_0 = n_0(d)$ such that for any set of reals, A, if $|A + A| \leqslant |A|^{1+\delta}$ and $|A| \geqslant n_0$, then A contains an affine cube with dimension d.*

In the second part of the talk we will see more examples, how geometry can help to visualize problems from number theory.

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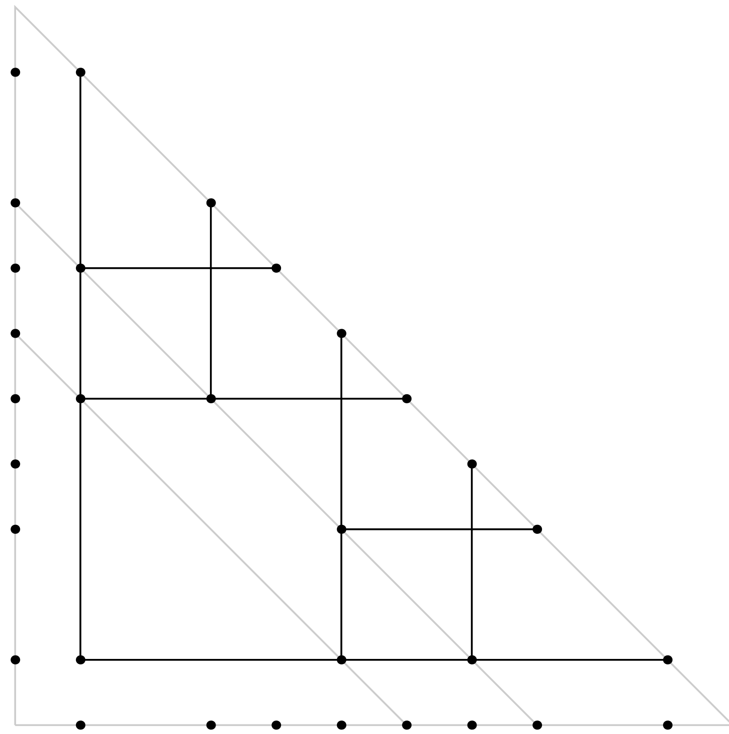

FIGURE 1. If the sumset is small, then the Cartesian product, $A \times A$, can be covered by using only a few lines of slope -1. Select one, which covers the most points of the Cartesian product. Consider the points lying on this line and the smaller Cartesian product defined by these points. Repeat the process, until the most popular line with slope -1 contains only one point of the last Cartesian product. Going back from this point along the nested Cartesian products we get a binary tree. It is easy to check that the projection of the tree onto the horizontal (or vertical) line determines an affine cube in A.

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Rigid Components: Geometric Problems, Combinatorial Solutions

ILEANA STREINU

The rigidity of bar-and-joint frameworks in dimension 2 and body-and-bar, body-and-hinge frameworks in arbitrary dimensions is characterized generically by graphs satisfying a certain sparsity condition (cf. well-known theorems of Laman

[4], Lovasz and Yemini [5] and Recski [7], resp. Tay [11]). When such structures are under-constrained, we are interested in detecting their maximal rigid substructures (*rigid components*) efficiently. Applications include understanding flexibility of large molecules (such as proteins near the native state), computing rigid components in pseudo-triangulation mechanisms and deciding when $3D$ polyhedra constructed from flat rigid faces connected by *some* hinges are rigid or flexible. In this last case, hinges are along the edges of the polyhedron, but some of them may be *broken*.

The class of (multi)-graphs capturing the generic rigidity of these frameworks can be extended in the most general way to a matroidal structure by the concept of tight (k, l)-sparse graphs. A multi-graph (possibly, with loops) G on n vertices is (k, l)-sparse, for fixed k and l, $0 \leqslant l < 2k$, if every subset of $n' \leqslant n$ vertices spans at most $kn' - l$ edges. G is *tight* if, in addition, has exactly $kn - l$ edges. Special cases include k-arborescences (edge-disjoint unions of k spanning trees), which are the tight (k, k)-sparse graphs by a theorem of Tutte [12] and Nash-Williams [6], and their generalization to (k, l)-arborescences by Haas [2]: graphs which become k-arborescences by the addition of *any* l edges (these are tight and $(k, k + l)$-sparse).

In [8], we characterize (k, l)-sparse graphs via a family of simple, elegant and efficient algorithms called the (k, l)-pebble games. In [9], we give a full analysis of their $O(n^2)$ running time (including the necessary data structures). A (k, l)-pebble game induces a special orientation of a tight (k, l)-sparse graph: every vertex has out-degree at most k, and exactly l out-edges are missing from the total of kn edges, which would be the count if all vertices had k-out-degree.

As additional applications, we use the pebble games for computing *components* (maximal tight subgraphs) in sparse graphs, to obtain inductive (Henneberg) constructions, and, when $l = k$, edge-disjoint tree decompositions. Finally, we derive an $O(n^2)$-time pebble-game-based algorithm for computing a Henneberg sequence of a Laman graph, which doesn't rely on tree decompositions. This provides an alternative to a recent algorithm of Bereg [1] for the same problem, and improves from $O(n^3)$ to $O(n^2)$ an algorithm of Haas et al. [3] for embedding planar Laman graphs as pseudo-triangulations.

A special case of the problem of finding rigid components has been considered in [10]. In this case, the special geometry of the embedding makes possible a linear time algorithm.

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Variants of the Crossing Number Problem

László A. Székely

1. Biplanar crossing numbers

We assume that the Reader is familiar with drawings of graphs and with crossing numbers. We will denote the crossing number of the graph G by $\mathrm{cr}(G)$. The survey papers [6], [8] or [12] contain the definitions not shown here.

Owens [5] introduced the *biplanar crossing number* of a graph G. A *biplanar drawing* of a graph G means drawings of two subgraphs, G_1 and G_2, of G, on two disjoint planes under the usual rules for drawings for crossing numbers, such that $G_1 \cup G_2 = G$. The *biplanar crossing number* $\mathrm{cr}_2(G)$ is the minimum of $\mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ over all biplanar drawings of G. Owens had motivation from electrical engineering. One can naturally extend the definition to k-planar drawings and k-planar crossing numbers of graphs. Czabarka, Sýkora, Székely and Vrťo in [2] started the systematic study of the biplanar crossing number. They determined $\mathrm{cr}_2(K_{5,q})$ and made conjectures for the biplanar crossing number of several families of complete bipartite graphs. However, there is no conjecture yet for the biplanar crossing number of all complete bipartite graphs, i.e. a "biplanar Zarankiewicz conjecture". A surprising and so far unexplained phenomenon is that the exact or conjectured biplanar crossing number of complete bipartite graphs with even number of edges seems to be always realizable with *isomorphic* G_1 and G_2. (In such cases *one* drawing suffices to produce to biplanar drawing, just labeling is needed twice.)

Unfortunately, for the biplanar crossing number most of the powerful lower bound techniques for crossing number, like bisection width and graph embedding, fail. The counting method and lower bounds based on the number of edges, of course, work. However, we are not aware of any "structural" lower bound improving on them. The only step in this direction is Spencer's [10] result on the biplanar crossing number of random graphs.

In a sequel to [2], [11], we show using a randomized algorithm, that for all graphs G,

$$\mathrm{cr}_2(G) \leqslant \frac{3}{8}\mathrm{cr}(G). \tag{1}$$

We also point out that one cannot give a good upper bound for $\mathrm{cr}(G)$ in terms of $\mathrm{cr}_2(G)$, since there are graphs G of order n and size m, with crossing number $\mathrm{cr}(G) = \Theta(m^2)$ (i.e. as large as possible) and biplanar crossing number $\mathrm{cr}_2(G) = \Theta(m^3/n^2)$ (i.e. as small as possible), for any $m = m(n)$, where m/n exceeds a certain absolute constant. We do not know what is the smallest constant (in place of 3/8), with which (1) is still true.

In [11] we use (1) to estimate the thickness of a graph G from above (roughly speaking) by $\mathrm{cr}_2(G)^{\cdot 406}$.

2. An optimality criterion

I have found the following optimality criterion in [12] and [13]:

Let us be given a touching-free drawing D of the simple graph G, in which any two edges cross at most once, and adjacent edges (edges with an endpoint in common) do not cross. Let us associate with every edge $e = \{x, y\} \in E(G)$ an arbitrary vertex set $A_e \subseteq V(G) \setminus \{x, y\}$. If the edges e and f are non-adjacent, then we define the *parity* of this edge pair as 0 or 1 according to

$$par(e, f) = |e \cap A_f| + |f \cap A_e| \quad \text{modulo} \;\; 2. \tag{2}$$

If non-adjacent edges e, f cross in D, then we write $e \times_D f$, otherwise write $e||_D f$.

Theorem. *Using the notation above, the condition that for all choices of the sets A_e the inequality*

$$\sum_{\substack{par(e,f)=1\\ e\times_D f}} 1 \leqslant \sum_{\substack{par(e,f)=1\\ e||_D f}} 1 \tag{3}$$

holds, implies that D realizes $CR(G)$. Checking the condition with brute force requires exponential time. Are there graphs for which such a criterion can be verified with more theoretical tools? The Zarankiewicz' drawing of the complete bipartite graph may be a candidate. (Sergiu Norine has found an optimality criterion for complete bipartite graphs, which seems to be different.)

3. Outerplanar crossing numbers

The concept of the outerplanar (in alternative terminology, convex or one-page) crossing number was pioneered by Paul Kainen. The convex crossing number problem requires the placement of the vertices on a circle and edges are drawn in straight line segments. Shahrokhi, Sýkora, Székely, and Vrťo [9] showed that for the outerplanar crossing number $CR^*(G)$ we have

$$CR^*(G) = O([CR(G) + \sum_v d_v^2] \log n). \tag{4}$$

Since the crossing number is less or equal the rectilinear crossing number, and the rectilinear crossing number is less or equal the outerplanar crossing number, we have that notwithstanding the examples of Bienstock and Dean [1], in "non-degenerate" cases there is at most a $\log n$ times multiplicative gap between the crossing number and the rectilinear crossing number. It is worth noting that the estimate in (4) is tight for an $n \times n$ grid [9]. This example is kind of degenerate since the error term dominates the RHS. However, many examples of dense graphs have been found where $CR^*(G) = \Theta(CR(G) \log n)$ in a follow-up paper of Czabarka, Sýkora, Székely and Vrťo, [3].

This result is interesting for the following reason. When estimating variants of the crossing number, sometimes logarithmic factors creep in and we do not know if those factors must be there or just results of some technical inconvenience. A good example of it is the bisection width lower bound for the pair crossing number, Kolman and Matoušek [4]. We exhibited in [3] a logarithmic factor which *is* needed.

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The unit-distance problem for convex sets

Pavel Valtr

In 1946 Erdős asked what is the minimum number, $u(n)$, of unit segments (unit distances) determined by a set of n points in the euclidean plane. The problem motivated a lot of research, e.g. see [1] for a survey. Currently the best known bounds on $u(n)$ are:

$$\Omega(n^{1+c/\log\log n}) \le u(n) \le O(n^{4/3}).$$

The lower bound was proved already in the original paper of Erdős [2]. It is attained by a properly scaled square lattice $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$ and it is conjectured to be essentially best possible. The original upper bound $O(n^{3/2})$ of Erdős was improved several times (e.g. see [1] for a survey). There are several proofs of the current upper bound, the simplest and most elegant one is due to Székely [4].

In the talk we consider the unit-distance problem for translates of a convex body. The distance of two non-empty closed sets $C, D \subset R^d$ is defined by

$$d(C,D) = \min_{\substack{c\in C\\ d\in D}} ||c-d||.$$

Given a family of closed sets $C_1,\ldots,C_n \subset R^d$, let $h(C_1,\ldots,C_n)$ denote the number of pairs $\{C_i, C_j\}$ with unit mutual distance, i.e. with $d(C_i,C_j)=1$. Let $t_d(n)$ be the maximum $h(C_1,\ldots,C_n)$, where we maximize over all families $C_1,\ldots,C_n \subset R^d$ of n pairwise disjoint translates of the same convex compact set in R^d. As noted in [3],

$$u_d(n) \le t_d(n),$$

where $u_d(n)$ is the d-dimensional version of the above unit-distance function $u(n)$.

Theorem 1 (Erdős and Pach [3])**.** $t_2(n) = O(n^{4/3})$.

Conjecture 1 (Erdős and Pach [3])**.** There is an $\varepsilon > 0$ such that $t_2(n) = \Omega(n^{1+\varepsilon})$.

In the talk we outline a construction verifying Conjecture 1. In fact, the asymptotic bound in Theorem 1 is best possible:

Theorem 2.

$$t_2(n) = \Theta(n^{4/3}).$$

Moreover, the translates in our construction are centrally symmetric. Thus,

Theorem 3.

$$t_2^{\text{symm}}(n) = \Theta(n^{4/3}),$$

where $t_d^{\text{symm}}(n)$ is defined similarly as $t_d(n)$, except that now the translates are also required to be centrally symmetric.

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Projected polytopes, Gale diagrams, and polyhedral surfaces

Günter M. Ziegler

(joint work with Raman Sanyal and Thilo Schröder)

We report about a new construction scheme that yields interesting 4-dimensional polytopes and polyhedral surfaces. The basic pattern is as follows:

1. Fix a combinatorial type of a high-dimensional simple polytope whose 2-skeleton contains a high-genus surface that uses all the vertices.
2. Construct explicit matrices for a special "deformed" realization of the polytope.
3. Project it to $\mathbb{R}^4$ such that all the faces of the surface realized in the boundary complex "survive" the projection.
4. Determine the combinatorics (in particular, the f-vector) of the resulting 4-polytope, in terms of Gale transforms.
5. Construct a Schlegel diagram to obtain a polyhedral surface realized in $\mathbb{R}^3$.

In the lecture, we outlined two instances of this program. The first one concerns the "projected products of polygons" presented in [7], whose construction has been simplified and further analyzed by the current authors:

- For $n \geq 4$ even and $r \geq 2$, the product of n-gons $(C_n)^r \subset \mathbb{R}^{2r}$ is a simple $2r$-dimensional polytope with n^r vertices and nr facets.
- A "deformed product realization" P_n^{2r} of $(C_n)^r$ is constructed in terms of explicit, lower block-triangular matrices $A_n^{2r} \in \mathbb{R}^{nr \times 2r}$. (The polytopes P_n^{2r} may be seen as iterated rank 2 deformed products; our construction goes beyond the "rank 1 deformed products" as discussed by Amenta & Ziegler [1].)
- Projection of P_n^{2r} to the last four coordinates yields 4-dimensional polytopes $\pi_4(P_n^{2r})$. All the vertices and edges of P_n^{2r}, as well as all the n-gon 2-faces, are "strictly preserved" by the projection.
- The construction of suitable matrices A_n^{2r}, as well as the combinatorial description of the resulting 4-polytopes $\pi_4(P_n^{2r})$, is achieved in terms of Gale diagrams: The rows of matrices A_n^{2r} are obtained by perturbation of the rows of a reduced matrix $\bar{A}_n^{2r} \in \mathbb{R}^{2r \times 2r}$. Deletion of the last four columns of $\bar{A}^{2r}$

results in a matrix $\bar{\bar{A}}_n^{2r} \in \mathbb{R}^{2r\times(2r-4)}$ that is the Gale diagram of a pyramid over an $(2r-1)$-gon, given by a matrix $G \in \mathbb{R}^{2r\times 3}$. The rows of A_n^{2r} and their positive dependencies, and thus the faces of $\pi_4(P_n^{2r})$, can be analyzed in terms of lexicographic triangulations of this pyramid.

- The 4-polytopes $\pi_4(P_n^{2r})$ have unusual f-vectors: For $n, r \to \infty$ the *fatness* of these polytopes approaches 9. (This is the largest value currently known. See [5] [6] for "fatness" and its role in the f-vector problem for 4-polytopes.)
- For $n = 4$, $\pi_4(P_n^{2r})$ is a *neighborly cubical polytope*, a cubical 4-polytope with the graph of the $2r$-cube. (These were first obtained by Joswig & Ziegler [2].)
- The Schlegel diagrams of the polytopes $\pi_4(P_n^{2r})$ yield geometric realizations of *equivelar* polyhedral surfaces of type $(4, 2r)$ in $\mathbb{R}^3$, all of whose faces are quadrilaterals and all of whose vertices have degree $2r$. Thus we have a new construction for a class of surfaces of "unusually high genus" $g \sim N \log N$ on $N = n^r$ vertices, as first obtained by McMullen, Schulz & Wills [4].

A second interesting instance for our construction scheme is as follows:

- For $n \geq 3$, the *totally wedged polytope* W^{n+2} is obtained from an n-gon by forming n successive wedges over facets that correspond to the edges of the original polygon. This is a simple $(n+2)$-dimensional polytope with $2n$ facets. (It is dual to a *wreath product* $I \wr C_n$, as described by Joswig & Lutz [3].)
- The boundary complex of W^{n+2} contains an equivelar surface of type $(n, 4)$ consisting of n-gons, where each vertex has degree 4. For $n = 2r$ this surface is combinatorially dual to the $(4, 2r)$-surfaces discussed above.
- We describe a special "deformed" realization Q^{n+2} of W^{n+2} in $\mathbb{R}^{n+2}$ in terms of explicit matrices, designed such that all vertices and all the n-gon 2-faces "survive" the projection $\pi_4 : \mathbb{R}^{n+2} \to \mathbb{R}^4$ to the last four coordinates.
- The combinatorial structure of $\pi_4(Q^{n+2})$ is again analyzed in terms of Gale transforms.
- Construction of a Schlegel diagram for $\pi_4(Q^{n+2})$ yields equivelar polyhedral surfaces of type $(n, 4)$ in $\mathbb{R}^3$. (This is another family of surfaces of high genus, first constructed by McMullen, Schulz & Wills [4].)

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What is known about unit cubes

Chuanming Zong

Unit cubes, from any point of view, are among the simplest and the most important objects in n-dimensional Euclidean space. In fact, they are not simple at all. On the one hand, the known results about them have been achieved by employing complicated machineries from Number Theory, Group Theory, Probability Theory, Matrix Theory, Hyperbolic Geometry, Combinatorics and etc.; On the other hand, the answers for many basic problems about them are still missing. In addition, the geometry of unit cubes does serve as a meeting point for several applied subjects such as Design Theory, Coding Theory and etc. The purpose of this talk is to figure out what is known about the unit cubes.

For convenience, let E^n denote the n-dimensional Euclidean space, let I^n denote the unit cube $\{\mathbf{x} \in E^n : |x_i| \leq \frac{1}{2}\}$ and let $\overline{I^n}$ denote the unit cube $\{\mathbf{x} \in E^n : 0 \leq x_i \leq 1\}$.

In this talk, we will discuss three problems: *What is the maximum* (*or minimum*) *area of a k-dimensional cross section of I^n? What is the maximum* (*or minimum*) *area of a k-dimensional projection of I^n? What is the maximum volume $\gamma(n,k)$ of a k-dimensional simplex inscribed in an n-dimensional unit cube?* and three conjectures: Minkowski's conjecture, Furtwängler's conjecture, and Keller's conjecture. Especially, we will emphasize the machineries from Probability Measures, Brascamp-Lieb Inequality, Convex Geometry, Hyperbolic Geometry, Group Theory, and Graph Theory used to attack the problems and conjectures.

As examples, let us mention a couple of the main results:

Theorem 1. *For any k-dimensional hyperplane H^k which contains the origin we have*

$$1 \leq v_k(I^n \cap H^k) \leq \min\left\{ \left(\tfrac{n}{k}\right)^{\frac{k}{2}}, 2^{\frac{n-k}{2}} \right\}.$$

Theorem 2.

$$\gamma(n,k) \leq \begin{cases} \frac{1}{k!2^k}\sqrt{\frac{(k+1)^{k+1}n^k}{k^k}} & \text{if } k \text{ is odd,} \\ \frac{1}{k!2^k}\sqrt{\frac{(k+2)^k n^k}{(k+1)^{k-1}}} & \text{if } k \text{ is even.} \end{cases}$$

Theorem 3. *There is a k-fold lattice tiling $I^n + \Lambda$ of E^n which has no twin if and only if*

1. *$n = 4$ and k is a multiple of a square of an odd prime.*
2. *$n = 5$ and $k = 3$ or $k \geq 5$.*
3. *$n \geq 6$ and $k \geq 2$.*

Theorem 4. *When $n \leq 6$, Keller's conjecture is right; When $n \geq 8$, Keller's conjecture is false.*

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