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Real Analysis, Harmonic Analysis and Applications to PDE

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Abstract. There have been important developments in the last few years in the point-of-view and methods of harmonic analysis, and at the same time significant concurrent progress in the application of these to partial differential equations and related subjects.

The conference brought together experts and young scientists working in these two directions, with the objective of furthering these important interactions.

Mathematics Subject Classification (2000): 42xx, 43xx, 44xx, 22xx, 35xx.

Introduction by the Organisers

Major areas and results represented at the workshop are:

I. Methods in harmonic analysis

(a) Multilinear analysis: this is an outgrowth of the method of "tile decomposition" which has been so successful in solving the problems of the bilinear Hilbert transform. Recent progress involves the control of maximal trilinear operators and an extension of the Carleson-Hunt theorem.

(b) Geometry of sets in $\mathbb{R}^d$: This includes recent progress on the interaction of Fourier analysis and geometric combinatorics related to the Falconer distance problem.

(c) Singular integrals: A break trough has been obtained on singular integrals on solvable Iwasawa AN- groups, which require a new type of Calderón-Zygmund decomposition, since the underlying spaces have exponential volume growth. Further significant progress involves the theory of operator-valued Calderón-Zygmund operators and their connection to maximal regularity of evolution equations.

(d) Oscillatory integrals, Fourierintegral operators and Maximal operators: This includes estimates of maximal operators related to polynomial polyhedra and their relations with higher dimensional complex analysis, sharp estimates for maximal operators associated to hypersurfaces in $\mathbb{R}^3$, estimates for degenerate Radon transforms and linear and bilinear estimates for oscillatory integral operators, as well as optimal Sobolev regularity for Fourier integral operators.

II. Applications to P.D.E.

(a) Dispersive linear and non-linear equations: Far reaching new approaches to dispersive estimates for Schrödinger equations via coherent state decompositions and a related new phase space transform adapted to the wave operator were introduced. Further significant progress includes L^p- estimates respectively blow up rates for eigenfunctions and quasimodes of elliptic operators on compact manifolds with and without boundary, and related problems for globally elliptic pseudodifferential operators, and well-posedness of the periodic KP-I equations. All these results are based in part on important ideas in harmonic analysis (such as I(d) above).

(b) Schrödinger operators with rough potentials: This includes quantitative unique continuation theorems and their relations with spectral properties and the study of embedded eigenvalues of Schrödinger operators.

The meeting was attended by 50 participants. The official program consisted of 23 lectures, and left sufficiant room for further activities, such as self-organised sessions and discussions among groups of participants. The organisers made an effort to include young mathematicians, and greatly appreciate the new joint program of the Oberwolfach Institute and the american NSF, which allowed to invite several outstanding young scientists from the United States.

Workshop: Real Analysis, Harmonic Analysis and Applications to PDE

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Abstracts

Quantitative unique continuation theorems

Carlos E. Kenig

We discussed three quantitative unique continuation theorems at infinity. The first theorem, for second order elliptic equations, is joint work with Bourgain, [2], and was a key step in our proof of Anderson localization for the continuous Bernoulli model in higher dimensions, a problem posed by Anderson, [1]. The second theorem,[3], for second order parabolic equations, settles a conjecture of Landis-Oleinik, [6], and extends results of Escauriaza-Seregin-Sverak, [5], which have had applications to the regularity of solutions to the Navier-Stokes equations. This is joint work with Escauriaza, Ponce and Vega. The third theorem,[4], also joint work with the same authors, deals with dispersive equations and can be thought of as an extension to non-linear Schődinger equations of Hardy's uncertainty principle, [7].

References

[1] P. Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev. **109** (1958), 1492–1505.

[2] J. Bourgain and C.E. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimensions*, to appear, Invent. Math.

[3] L. Escauriaza, C.E. Kenig, G. Ponce and L. Vega, *Decay at infinity of caloric functions with characteristic hyperplanes*, preprint

[4] L. Escauriaza, C.E. Kenig, G. Ponce and L. Vega, *On unique continuation of solutions of Schrődinger equations*, preprint

[5] L. Escauriaza, S. Seregin and V.Sverak, $L^{3,\infty}$ *solutions to the Navier-Stokes equations and backward uniqueness*, Russ.Math.Surv. **58:2** (2003), 211–250.

[6] E.M. Landis and O. Oleinik, *Generalized analyticity and some related properties of solutions of elliptic and parabolic equations*, Russ. Math. Serv. **29** (1974), 195–212.

[7] E.M. Stein and R. Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis, Princeton University Press, (2003).

Singular integral of Iwasawa AN groups

Waldemar Hebisch

Let G be a connected, complex semisimple Lie group with Cartan decomposition KAN. We can identify AN with the symmetric space G/K via mapping $x \mapsto x^{-1}K$. In this way we get right invariant Riemannian metric on AN. We use left Haar measure on AN. Let X_i be right invariant vector fields on AN, such that $X_i(e)$ form an orthonormal basis of the tangent space to AN at e. Put

$$L = -\sum X_i^2$$

L is essentially self-adjoint on $C_c^\infty(AN)$.

Then Riesz transforms $X_iL^{-1/2}$ are bounded on L^p, $1 < p \leq 2$ and of weak type $(1, 1)$.

The proof develops ideas which appeared first in [1]. There are two main ingredients. First we show that on all amenable Lie groups we have "good" singular integral theory (called Calderón–Zygmund property in [1]).

Second is an estimate on the gradient of heat kernel. Namely, define heat kernel p_t by the formula $e^{-tL}f = p_t * f$ (where e^{-tL} is the semigroup of operators generated by L). We have

$$\|X_ip_t\|_{L^1} \leq Ct^{-1/2}$$

Note, that only the gradient estimate depend on the specific structure of our groups. Since the singular integral part works on amenable groups, we can get estimates for Riesz transforms on other groups, once we prove that gradient of heat the heat kernel – it is likely the gradient estimate holds on all amenable groups, however we can only prove it in a number of specific cases.

Proof of the gradient estimate uses simple relation between p_t and heat kernel on the symmetric space and explicit formulas available on the symmetric space. The main difficulty here is that on symmetric space simple formulas use radial symmetry, which is destroyed by passage to AN. For derivatives in one direction (corresponding to sum of all positive roots) we get needed estimate computing derivative of basic spherical function ϕ_0 in two different ways: one using Harish-Chandra integral formula, the second using asymptotic expansion of ϕ_0. We extend the estimate to other direction using asymptotic properties of geodetics.

References

[1] W. Hebisch, T. Steger *Multipliers and singular integrals on exponential growth groups*, Math. Zeit. **245** (2003), 37–61.

Optimal Sobolev regularity for Fourier integral operators on $\mathbb{R}^d$

Malabika Pramanik
(joint work with Andreas Seeger)

We investigate the L^p-Sobolev regularity of a class of Fourier integral operators on $\mathbb{R}^d$, $d \geq 3$, that arise from averaging and that have, at worst, one-sided fold singularities. Using a deep Fourier transform estimate of Wolff [26] and Laba-Wolff [11] associated to the light cone, we show that under appropriate "nonvainishing curvature conditions" , an FIO of the above type maps L^p to $L^p_{(d-2)/p}$ for large p. This gain in regularity is optimal, but the range of p is not. We discuss several applications of this result, in particular when the operator under consideration is an X-ray transform associated to a rigid line complex or convolution with respect to a measure along curves in Euclidean space or on the Heisengroup group.

References

[1] J. Bourgain, *Estimates for cone multipliers,* Geometric Aspects of Functional Analysis, Operator theory, Advances and Applications bf 77, ed. by J. Lindenstrauss and V. Milman, Birkhäuser Verlag, 1995.

[2] A. Comech, *Optimal regularity of Fourier integral operators with one-sided folds,* Comm. Partial Differential Equations **24** (1999), 1263–1281.

[3] I.M.Gelfand and M.I. Graev, *Line complexes in the space* $\mathbb{C}^n$*,* Func. Ann. Appl. **2** (1968) 219-229.

[4] A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities,* J. reine ang. Math. **455** (1994), 35–56.

[5] ———, *Oscillatory and Fourier integral operators with degenerate canonical relations,* Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. 2002, Vol. Extra, 93–141.

[6] A. Greenleaf and G. Uhlmann, *Nonlocal inversion formulas for the X-ray transform,* Duke Math. J. **58** (1989), 205–240

[7] ———, *Estimates for singular Radon transforms and pseudo-differential operators with singular symbols,* J. Funct. Anal. **89** (1990) 202–232.

[8] ———, *Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms, I,* Ann. Inst. Fourier (Grenoble) **40** (1990), no. 2, 443–466.

[9] V. Guillemin, Cosmology in $(2+1)$-dimensions, cyclic models, and deformations of $M_{2,1}$, Annals of Mathematics Studies **121**, Princeton University Press, Princeton, NJ, 1989.

[10] L. Hörmander, *Fourier integral operators, I,* Acta Math. **127** (1971), 79–183.

[11] I. Laba and T. Wolff, *A local smoothing estimate in higher dimensions*, J. Anal. Math. **88** (2002), 149–171.

[12] N. Laghi, *Elements of functional calculus and* L^2 *regularity for some classes of Fourier integral operators*, preprint.

[13] D. Oberlin and H. Smith, *A Bessel function multiplier,* Proc. Amer. Math. Soc. **127** (1999), 2911–2915.

[14] D. Oberlin, H. Smith, and C. Sogge, *Averages over curves with torsion,* Math. Res. Lett. 1998, 535–539.

[15] D.H. Phong, E.M. Stein, *Radon transforms and torsion,* Internat. Math. Res. Notices 1991, no. 4, 49–60.

[16] D.H. Phong, *Singular integrals and Fourier integral operators,* Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 286–320, Princeton Math. Ser., 42, Princeton Univ. Press, Princeton, NJ, 1995.

[17] M. Pramanik and A. Seeger, L^p *regularity of averages over curves and bounds for associated maximal operators,* preprint.

[18] M. Pramanik and A. Seeger, L^p *Sobolev regularity of a restricted X-ray transform in* $\mathbb{R}^3$*,* preprint.

[19] A. Seeger, *Some inequalities for singular convolution operators in* L^p*-spaces,* Trans. Amer. Math. Soc., 308 (1988) 259–272.

[20] ———, *Degenerate Fourier integral operators in the plane,* Duke Math. J. 71 (1993), no. 3, 685–745.

[21] ———, *Radon transforms and finite type conditions,* J. Amer. Math. Soc. 11 (1998), no. 4, 869–897.

[22] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, N.J., 1971.

[23] ———, Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton University Press, Princeton, NJ, 1993.

[24] T. Tao and A. Vargas, *A bilinear approach to cone multipliers*, *I. Restriction estimates*, Geom. Funct. Anal. 10 (2000), 185–215; *II, Applications*, Geom. Funct. Anal. 10 (2000), 216–258.

[25] T. Wolff, *A sharp bilinear cone restriction estimate,* Ann. of Math., 153 (2001), 661–698.
[26] ———, *Local smoothing type estimates on L^p for large p,* Geom. Funct. Anal. 10 (2000), no. 5, 1237–1288.

Boundedness problem for maximal operators associated to non-convex hypersurfaces

Isroil A. Ikromov

One of the classical results of real analysis is E.M. Stein's maximal theorem for spherical means on Euclidean space $\mathbb{R}^{n+1}(n \geq 2)$. The 2-dimensional case was later dealt with by J. Bourgain [1]. These results became the starting point for the study of various classes of maximal operators associated to subvariaties, such as maximal operators defined by

$$\mathcal{M}g(x) = \sup_{t>0} \left| \int_S g(x-ty)\psi(y)d\sigma(y) \right|, \tag{1}$$

where S is a smooth hypersurface, ψ is a fixed non-negative function in $C_0^\infty(S)$ and $d\sigma$ is the surface measure on S. For instance, A. Greenleaf [3] proved that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^{n+1})$, if $n \geq 2$ and $p > \frac{n+1}{n}$, provided S has everywhere non-vanishing principal curvature and is star-shaped with respect to the origin. Moreover, he proved that if at every point of the surface there exist at least $k(k \geq 2)$ non-vanishing principal curvatures then the maximal operator is bounded for any $p > \frac{k+1}{k}$. The analogical result was obtained by C.D. Sogge [4] in a more difficult case $k = 1$.

In contrast, the case where the Gaussian curvature vanishes at some points is still widely open, and sharp results for this case are known only for particular classes of surfaces. A result of general nature given by C.D. Sogge and E.M. Stein in [5] shows that if the Gaussian curvature of S does not vanish of infinite order at any point of S then M is bounded on L^p in a certain range $p > p(S)$. However, an estimate for the exponent $p(S)$ in that paper is in general far from being optimal.

It is well-known that the L^p-estimates of the maximal operator (1) are strongly related to the decay of the Fourier transform of measures carried on S, i.e. to oscillatory integrals of the form

$$\int_S e^{i(\xi,x)}\psi(X)d\sigma, \tag{2}$$

where $\psi(X)d\sigma(X)$ is a compactly supported density on S.

But, the decay of the oscillatory integral (2) as $|\xi| \to \infty$ can be rather low. Another important idea, introduced in [5] and applied in several subsequent articles, is to "damp" the oscillatory integral (2), by multiplying the amplitude a by a suitable power of the Gaussian curvature on S, in order to obtain the "optimal" decay of order $|\xi|^{-n/2}$ (as $|\xi| \to \infty$).

We consider the problem in a case of analytic hypersurfaces. Let $S \subset \mathbb{R}^{n+1}$ be an analytic hypersurface. Denote by $A(X)$ the matrix of a second fundamental form defined on the surface S. It is well known that if S is an analytic hypersurface, then $A(X)$ is a symmetric matrix-valued real analytic function, and also one can define a symmetric matrix $A(X) \wedge A(X)$, where " $\wedge$" is an exterior product of the matrices.

We consider two functions defined as follows:

$$\Lambda_1(X) := tr(A(X)^2), \quad \Lambda_2(X) = tr((A(x) \wedge A(X))^2),$$

where "tr" is a trace of the matrix defined by convolution with the first fundamental form of the surface.

Note that if S is an analytic hypersurface then the both $\Lambda_1(X)$ and $\Lambda_2(X)$ are analytic functions on the surface. The damping factors are defined by the following formula:

$$\Lambda(X) := \Lambda_1(X)^{\frac{q_1}{2}} \Lambda_2(X)^{\frac{q_2}{4}},$$

where q_1, q_2 are fixed positive real numbers satisfying the conditions: $q_1 + q_2 = 1$, $q_2 > 0$.

Theorem 1. *Let S be an analytic hypersurface in $\mathbb{R}^{n+1}$ and ψ be a fixed non-negative smooth function with compact support on S and $\mathcal{M}$ be a maximal operator defined by relation (1). If $\Lambda_1^{-\beta} \in L^1_{loc}(S)$, where β is a fixed positive real number, and also $\Lambda_2(X) \not\equiv 0$ then the maximal operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^{n+1})$, whenever $p > 2 + \frac{3}{2\beta}$. In other words, under the conditions the estimate $p(S) \leq p(\beta)$ holds.*

Following a standard approach (see e.g. [5]), Theorem 1 will be shown by embedding $\mathcal{M}$ respectively linearizion of $\mathcal{M}$ into an analytic family of operators.

More precisely, for $z \in C$ with $Re(z) > -\beta$ define a measure $d\sigma_z(X) = \Lambda(X)^z \psi(X) d\sigma(X)$ on S as well as the corresponding maximal operator:

$$\mathcal{M}_z f(x) = \sup_{t>0} \left| \int_S f(x - ty) d\sigma_z(y) \right|.$$

It is easy to see that for any $z \in C$ and $f \in C_0^\infty(\mathbb{R}^{n+1})$ the inequality $\mathcal{M}_z f(x) \leq \mathcal{M}_{Re(z)}|f(x)|$ holds. Since the $\psi(x)d\sigma(x)$ is a positive Borel measure. It easily follows from our assumption on S that if $Re(z) \equiv q > -\beta$ and q_2 is sufficiently small positive then due to Hölder inequality Λ^z is a locally integrable function on S. Thus, $M_q(M_z)$ is bounded on $L^\infty(\mathbb{R}^{n+1})$ for these values of z.

Once, we can show the L^2-boundedness of $\mathcal{M}_q(M_z)$ for $q > \frac{3}{2}$ due to the following Statements and Sobolev's embedding theorem [5].

Theorem 2. *If S is an analytic hypersurface, $(q_1, q_2, q) : q_1 + q_2 = 1$, $q_2 > 0$, $q > \frac{3}{2}$ are fixed real numbers, and $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ then there exists an $\varepsilon > 0$ such that the following inequality*

$$\left| \int_S e^{i(x,\xi)} \Lambda(X)^q \psi(X) d\sigma(X) \right| \leq const \frac{\|\psi\|_{L^1_3(\mathbb{R}^{n+1})}}{|\xi|^{\frac{1}{2}+\varepsilon}}$$

holds.

From Theorem 2 as a simple consequences, we obtain

Corollary 3. *If S is an analytic hypersurface and $(q_1, q_2, q) : q_1 + q_2 = 1$, $q_2 > 0$, $q > \frac{3}{2}$ are fixed real numbers, then the Fourier transform of the measure $d\sigma_\alpha$ satisfies*

$$|\nabla \hat{d\sigma_q}(\xi)| \leq C(1 + |\xi|)^{-(\frac{1}{2}+\varepsilon)},$$

for some $\varepsilon > 0$.

Then Theorem 1 follows from Stein's interpolation theorem for an analytic family of operators $\mathcal{M}_z$.

Some sharp results.

Our sharp results are connected to the three-dimensional case. Let S be a hypersurface in $\mathbb{R}^3$ given as the graph of a smooth function $c + f$ at the origin with $f(0) = 0$ and $\nabla f(0) = 0$. Denote by $h(f)$ the height of the function by Varchenko terminology [7]. We introduce the number

$$P(S, (0,0,c)) = \inf\{p : \exists\, U(0,0,c)\ \forall \psi \in C_0^\infty(U(0,0,c))\, \mathcal{M} \text{ is bounded on } L^p\},$$

where $U(0,0,c)$ is a neighborhood of the point $U(0,0,c)$.

Theorem 4. (I.A. Ikromov, M. Kempe, D. Müller.) *Let $n = 2$ and f be a smooth function with $h \geq 2$ and $c \neq 0$ and S be a hypersurface given as the graph of the function $c + f$. Then $P(S, (0,0,c)) = h(f)$.*

Let's consider some applications of the theorem.

Let $S \subset \mathbb{R}^3$ be a smooth hypersurface. We fix $S_0 \subset S$ a bounded peace of the hypersurface and for $0 \leq \psi \in C_0^\infty(S_0)$ we define a measure by $d\sigma(x) := \psi(x) dS(x)$. Consider the Fourier transform $\hat{d\sigma}(\xi)$. Let's write $\xi = \lambda\omega$, where $\lambda \in \mathbb{R}_+$ and $\omega \in S^2$ where S^2 is the unite sphere centered at the origin. We fix $\omega \in S^2$ and define so-called oscillation index by

$$\beta(\omega) = \inf\{\alpha :\ \forall \psi \in C_0^\infty(S_0)\, \hat{d\sigma}(\lambda\omega) = O(\lambda^\alpha)(\text{as}\, \lambda \to +\infty)\}$$

By the analogy we define an oscillation index at a fixed point $x^0 \in S$

$$\beta(x^0) = \inf\{\alpha : \exists\, U_\alpha(x^0) \neq \emptyset\ \forall \psi \in C_0^\infty(U_\alpha(x^0))\, \hat{d\sigma}(\lambda n_{x^0}) = O(\lambda^\alpha)(\text{as}\, \lambda \to +\infty)\},$$

where n_{x^0} is the unite normal to S at x^0

Theorem 5. (Ikromov I.A., M. Kempe, D. Müller.)*Let $S_0 \subset S \subset \mathbb{R}^3$ be a fixed piece of the smooth hypersurface S. If for any $\omega \in S^2$ the inequality $\beta(\omega) \leq -1/2$ holds then the associated maximal operator is L^p bounded for any $p > 2$.*

Theorem 6. (Ikromov I.A., M. Kempe, D. Müller.)*Let $x^0 \in S$ be a fixed point of the smooth hypersurface in $\mathbb{R}^3$ with property $\beta(x^0) \leq -1/2$ then $p(S, x^0) \leq 2$. Moreover, if $0 \notin T_{x^0}S$ and $\beta(x^0) = -1/2$ then $p(S, x^0) = 2$.*

The Theorems 5 and 6 show that the analogy of A. Greenleaf result [3] holds in the critical case.

References

[1] J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Anal. Math. **47** (1986), 69-85.

[2] M. Cowling, G. Mauceri, *Inequalities for some maximal functions II*, Trans Amer. Math. Soc. **296** (1986), 341–365.

[3] A. Greenleaf, *Principal curvature and harmonic analysis*, Indiana Math. J. **30** (1982), 519–537.
[4] C. D. Sogge, *Averaging operators with one non-vanishing principal curvature*, Preprint (1995).
[5] C. D. Sogge, E. M. Stein, *Averages of functions over hypersurface in* $\mathbb{R}^n$, Invent. Math. **82** (1985), 543–556.
[6] E. M. Stein, *Maximal functions: spherical means*, Proc. Natl. Acad. Sci. USA **73** (1976), 2174–2175.
[7] A. N. Varchenko, *Newton polyhedra and estimations of oscillating integrals*, Funct. anal. and Appl. **10** (1976), 175–196.

Well-posedness theorems for the KP-I initial value problem on $\mathbb{T}\times\mathbb{T}$ and $\mathbb{R}\times\mathbb{T}$

Alexandru Ionescu
(joint work with C. E. Kenig)

I discussed some joint work with C. E. Kenig on local and global well-posedness theorems for the KP-I initial value problem on $\mathbb{T}\times\mathbb{T}$ and $\mathbb{R}\times\mathbb{T}$.

Let $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$. The subject of my talk was the Kadomstev-Petviashvili I initial value problem

$$\begin{cases}\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+u\partial_x u=0;\\ u(0)=\phi,\end{cases} \tag{1}$$

on $\mathbb{T}\times\mathbb{T}$ and $\mathbb{R}\times\mathbb{T}$. KP-I equations, as well as KP-II equations in which the sign of the term $\partial_x^{-1}\partial_y^2 u$ in (1) is $+$ instead of $-$, arise naturally in physical contexts as models for the propagation of dispersive long waves, with weak transverse effects. The KP-II initial value problems are much better understood from the point of view of well-posedness, due mainly to the X_b^s method of J. Bourgain [1]. For instance, the KP-II initial value problem is globally well-posed in L^2, on both $\mathbb{R}\times\mathbb{R}$ and $\mathbb{T}\times\mathbb{T}$ (J. Bourgain [1]).

On the other hand, it has been shown in [4] that KP-I initial value problems are badly behaved with respect to Picard iterative methods in the standard Sobolev spaces, since the flow map fails to be C^2 at the origin in these spaces. Due to this fact, the well-posedness theory of these equations is more limited. For example, global well-posedness of the KP-I initial value problem (1) in the natural energy space $Z^1(\mathbb{R}\times\mathbb{R})$ remains an open problem. It is known, however, that the KP-I initial value problem on $\mathbb{R}\times\mathbb{R}$ is globally well-posed in the "second" energy space $Z^2(\mathbb{R}\times\mathbb{R})$ (C. E. Kenig [3]). On $\mathbb{T}\times\mathbb{T}$, the KP-I initial value problem is known to be globally well-posed in the "third" energy space $Z^3(\mathbb{T}\times\mathbb{T})$ (J. Colliander [2]).

The energy spaces Z^s, $s=0,1,2,\ldots$, are related to the (formal) conservation laws of the KP-I equation. For $g\in L^2(\mathbb{T}\times\mathbb{T})$ or $g\in L^2(\mathbb{R}\times\mathbb{T})$ let $\widehat{g}$ denote its Fourier transform. For $s=0,1,2,\ldots$ we define

$$\begin{aligned}Z^s(\mathbb{T}\times\mathbb{T}) =&\{g:\mathbb{T}\times\mathbb{T}\to\mathbb{R}:\widehat{g}(0,n)=0 \text{ for any } n\in\mathbb{Z}\setminus\{0\} \text{ and}\\ &||g||_{Z^s_{(3)}}=||\widehat{g}(m,n)[1+|m|^s+|n/m|^s]||_{L^2(\mathbb{Z}\times\mathbb{Z})}<\infty\},\end{aligned} \tag{2}$$

and

$$Z^s_{(3)}(\mathbb{R}\times\mathbb{T}) = \{g : \mathbb{R}\times\mathbb{T}\to\mathbb{R} : \quad ||g||_{Z^s_{(3)}} = ||\widehat{g}(\xi,n)[1+|\xi|^s+|n/\xi|^s]||_{L^2(\mathbb{R}\times\mathbb{Z})} < \infty\}. \tag{3}$$

Our main theorem concerns global well-posedness of the KP-I initial value problem in $Z^2(\mathbb{T}\times\mathbb{T})$ and $Z^2(\mathbb{R}\times\mathbb{T})$. In the theorem below assume $\mathbb{S}=\mathbb{T}$ or $\mathbb{S}=\mathbb{R}$, and H^{-1} denotes the standard Sobolev space.

Theorem 1. *Assume that $\phi\in Z^2(\mathbb{S}\times\mathbb{T})$. Then the initial value problem*

$$\begin{cases}\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \text{ on } \mathbb{S}\times\mathbb{T}\times\mathbb{R};\\ u(0)=\phi,\end{cases} \tag{4}$$

admits a unique solution $u\in C(\mathbb{R}:Z^2(\mathbb{S}\times\mathbb{T}))\cap C^1(\mathbb{R}:H^{-1}(\mathbb{S}\times\mathbb{T}))$. In addition, $u\in L^\infty(\mathbb{R}:Z^2(\mathbb{S}\times\mathbb{T}))$, $\partial_x u\in L^1_{\text{loc}}(\mathbb{R}:L^\infty(\mathbb{S}\times\mathbb{T}))$, and the mapping $\phi\to u$ is continuous from $Z^2(\mathbb{S}\times\mathbb{T})$ to $C([-T,T]:Z^2(\mathbb{S}\times\mathbb{T}))$ for any $T\in[0,\infty)$.

In addition, we prove that sufficiently high Sobolev regularity is globally preserved by the flow. We also prove local well-posedness theorems in certain spaces larger than the energy space Z^2.

As in [3], our proofs are based on controlling $||\partial_x u||_{L^1_t L^\infty_{x,y}}$ locally in time, where u is a solution of (4). The main difficulty is that the Strichartz estimates of J.-C. Saut [5] for the free KP-I flow on $\mathbb{R}\times\mathbb{R}$, which are the main tool in [3], fail in periodic settings. We replace these Strichartz estimates with certain time-frequency localized Strichartz estimates, which are still sufficient for our purpose.

References

[1] J. Bourgain, On the Cauchy problem for the Kadomstev–Petviashvili Equation, Geom. Funct. Anal. **3** (1993), 315–341.
[2] J. Colliander, Globalizing estimates for the periodic KPI equation, Illinois J. Math. **40** (1996), 692–698.
[3] C. E. Kenig, On the local and global well-posedness theory for the KP-I equation, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 827–838.
[4] L. Molinet, J.-C. Saut, and N. Tzvetkov, Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation, Duke Math. J. **115** (2002), 353–384.
[5] J.-C. Saut, Remarks on the generalized Kadomstev-Petviashvili equations, Indiana Univ. Math. J. **42** (1993), 1011–1026.

Hardy's Uncertainty Principle

Michael Cowling
(joint work with M. Sundari)

This is an account of joint work with M. Sundari, from IIT Roorkee.

Hardy's Uncertainty Principle states that, if $f \in L^1(\mathbb{R}^n)$,

$$|f(x)| \leq e^{-\alpha|x|^2} \quad \forall x \in \mathbb{R}^n$$
$$|\hat{f}(\xi)| \leq e^{-\beta|\xi|^2} \quad \forall \xi \in \mathbb{R}^n$$

and $\alpha\beta > 1/4$ (here the Fourier transformation involves $e^{\pm ix\cdot\xi}$), then $f = 0$. If $\alpha\beta = 1/4$, then f has to be a Gaussian. This and other related results are described and often proved in the recent survey of this and other Uncertainty Principles by G.B. Folland and A. Sitaram [1].

This result can be generalised in several ways: $\mathbb{R}^n$ may be replaced by other Lie groups (in which case the Fourier transform estimate needs to be appropriately interpreted), or the result may be phrased in terms of operators.

Suppose that $\mathcal{K}$ is a kernel operator on $L^2(\mathbb{R}^n)$, that is, there exists a locally integrable function k on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\mathcal{K}f(x) = \int_{\mathbb{R}^n} k(x,y)f(y)\,dy \quad \forall f \in L^2(\mathbb{R}^n).$$

In particular we may consider the heat operator $\mathcal{P}_t$; the Fourier transform version is

$$(\mathcal{P}_t f)\hat{}\,(\xi) = e^{-t|\xi|^2}\hat{f}(\xi) \quad \forall \xi \in \mathbb{R}^n,$$

and the kernel version is

$$\mathcal{P}_t f(x) = \int_{\mathbb{R}^n} p_t(x,y)f(y)\,dy,$$

where $p_t(x,y) = c(t)e^{-|x-y|^2/4t}$; here $c(t)$ depends on the dimension n and t.

One generalisation of Hardy's Uncertainty Principle is the following.

Theorem 1. *Suppose that $\mathcal{K}$ is a kernel operator on $L^2(\mathbb{R}^n)$, and that*

$$|k(x,y)| \leq p_s(x,y) \quad \forall x, y \in \mathbb{R}^n$$
$$|\mathcal{K}| \leq \mathcal{P}_t$$

in the sense of operators, that is, $\|\mathcal{K}f\|_2 \leq \|\mathcal{P}_t f\|_2$ for all f in $L^2(\mathbb{R}^n)$, and that $s < t$. Then $\mathcal{K} = 0$.

It is interesting to speculate whether this holds in more general contexts, such as subriemannian manifolds. Such a result cannot hold on a compact manifold, but when heat can escape to infinity "fast enough" then it might.

One strategy for dealing with this problem begins with work of A. Hulanicki [2], who studied the algebra of operators generated by a sublaplacian on a Lie group G. Here we may consider the heat operator $\mathcal{P}_t$ and the associated convolution kernel p_t; we let $\Gamma^p(G)$ be the closure in $L^p(G)$ of the linear span of the various p_t, for t

in $\mathbb{R}^+$. Then $\Gamma^1(G)$ is a Banach $*$-algebra, and from the abstract theory of these, there is a Gelfand transform, and a Plancherel Theorem for $\Gamma^2(G)$. Essentially the Gelfand transform maps $\mathcal{P}_t$ to the function $\hat{p}_t : \lambda \mapsto e^{-t\lambda^2}$ on $\mathbb{R}^+$ (or a subset thereof), and the Plancherel Theorem states that

$$\|f\|_2 = \Big(\int_0^\infty |\hat{f}(\lambda)|^2 \, d\mu(\lambda) \Big)^{1/2},$$

for some measure μ. It is possible to identify μ for stratified Lie groups: if G is a stratified group of homogeneous dimension Q, then $d\mu(\lambda) = c\lambda^{Q-1}\, d\lambda$.

The following observation is very familiar to operator algebraists but not to harmonic analysts.

Theorem 2. *The orthogonal projector $\mathcal{E}$ from $L^2(G)$ to $\Gamma^2(G)$ extends to a map from convolution operators on $L^2(G)$ to an appropriate completion of $\Gamma^1(G)$, and this map is a conditional expectation.*

One way of tackling Hardy's Uncertainty Principle on Lie groups is to observe that (in many cases), if $\mathcal{F}$ is a convolution operator with kernel f, and

$$|f(x)| \le p_s(x)$$
$$|\mathcal{F}| \le \mathcal{P}_t$$

where $s < t$, then translates of f satisfy a similar equality, but where p_s is replaced by $p_{s'}$ where $s < s' < t$. Further, $\mathcal{E}f(e) = f(e)$. Thus if one can establish that an Uncertainty Principle holds in $\Gamma^1(G)$, and that $\mathcal{E}$ preserves the estimates for f and $\mathcal{F}$, then one can establish an Uncertainty Principle for G.

We can show the following.

Theorem 3. *Suppose that the Lie group G has a normal subgroup N such that G/N is isomorphic to $\mathbb{R}$. Then Hardy's Uncertainty Principle holds in $\Gamma^1(G)$.*

Further, there are many cases in which we can establish that the conditional expectation $\mathcal{E}$ behaves nicely relative to kernel estimates and operator estimates (these latter are never a problem). So we are able to extend the range of groups for which Hardy's Uncertainty Principle is known to hold.

Interestingly enough, we have trouble with nilpotent groups, such as the Heisenberg group; however work of S. Thangavelu [3] shows that Hardy's Theorem holds in this context too.

References

[1] G.B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey, *J. Fourier Anal. Appl.* **3** (1997), 207–238.

[2] A. Hulanicki, Subalgebra of $L_1(G)$ associated with Laplacian on a Lie group. *Colloq. Math.* **31** (1974), 259–287.

[3] S. Thangavelu, *An introduction to the uncertainty principle.* Progress in Mathematics, 217. Birkhuser Boston, Inc., Boston, MA, 2004.

A phase space transform adapted to the wave equation

Daniel Tataru
(joint work with Dan Geba)

A natural way to study pseudodifferential and Fourier integral operators is by means of phase space transforms. This is easiest to understand within the framework of the S^0_{00} calculus, where the localization occurs on the unit scale both in position and in the frequency. This corresponds precisely to the Bargmann transform,

$$Tu(x,\xi) = c_n \int_{\mathbb{R}^n} e^{i\xi(x-y)} e^{-\frac{(x-y)^2}{2}} u(y)dy$$

The Bargmann transform is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{C}^n)$ so an inverse for it is provided by the adjoint operator. This inverse is not uniquely determined since T is not onto. Precisely, the range of T consists of those functions satisfying a Cauchy-Riemann type equation, $i\partial_\xi v = (\partial_x - i\xi)v$. The connection with the S^0_{00} type calculus is provided by the following simple result:

Theorem 1. *Let $A : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ be a linear operator. Then $A \in OPS^0_{00}$ if and only if the kernel K of TAT^* satisfies*

$$|K(x_1,\xi_1,x_2,\xi_2)| \leq c_N(1 + |x_1 - x_2| + |\xi_1 - \xi_2|)^{-N}$$

This provides an easy way to study the calculus and the L^2 boundedness of OPS^0_{00} pseudodifferential operators. One can also talk about S^0_{00} type Fourier integral operators, etc. For more details and further development of these ideas we refer the reader to [7].

On the other hand, in the study of the wave equation with rough coefficients one is naturally led to consider wave packets, see Smith [6]. These are exact or approximate solutions to the wave equation which are localized in position and frequency on dual scales.

In the initial data space, the wave packets correspond to what is called the second dyadic decomposition. Precisely, we begin with a dyadic decomposition in frequency; then, each dyadic annulus of size pλ is subdivided into sectors of angle $\lambda^{-\frac{1}{2}}$. Thus the Fourier space is partitioned into parallelepipeds which at frequency λ have size $\lambda \times (\lambda^{\frac{1}{2}})^{n-1}$. Then for each such parallelepiped one considers an equipartition of the physical space into rectangles on the dual scale $\lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$.

One can decompose any initial data set for the wave equation into a discrete almost orthogonal superposition of localized initial data on the above scale. Then the wave packets are essentially obtained by transporting those initial data along the corresponding Hamilton flow.

The aim of this talk is to introduce a phase space transform adapted to the scales described above,

$$Tu(x,\xi) = \int_{\mathbb{R}^n} u(y)\overline{\phi_{x,\xi}(y)}dy$$

where for each (x, ξ) the coherent state $\phi_{x,\xi}$ has the phase space localization described above. Via an inversion formula this leads to a continuous (even smooth) counterpart of the discrete second dyadic decomposition for the initial data. In different contexts similar ideas were pursued earlier in Cordoba-Fefferman [2], Folland [3].

Then we consider the associated classes of symbols, and characterize the corresponding pseudodifferential operators using our phase space transform as in Theorem 1 above. Of course, in the kernel bounds the Euclidean distance in the phase space is replaced by the distance with respect to the Riemannian metric associated to the new localization scales. This analysis is not entirely straightforward as it shares some of the features of the $S_{1,1}$ calculus, see Hörmander [4].

Starting with a suitable class of canonical transformations we introduce the Fourier integral operators adapted to this geometry. For these we discuss the calculus and the L^2 boundedness properties.

Finally we consider evolution equations governed by first order operators with real almost homogeneous symbols,

$$(D_t + A(t, x, D))u = 0, \qquad u(0) = u_0$$

We show that the generated evolution operators are in effect Fourier integral operators associated to the canonical transformations generated by the Hamilton flow. In the case of the Bargmann transform and the S^{00} calculus this analysis was carried out in [5]. For related work we refer the reader to Bony [1].

As an application we consider the question of constructing parametrices for half-wave evolutions with rough coefficients. Following the spirit of the paradifferential calculus we regularize the coefficients on a frequency dependent scale to obtain a modified evolution which fits within our setup. On the other hand we show that the original and the modified evolutions are close in the L^2 sense.

References

[1] Jean-Michel Bony. Evolution equations and microlocal analysis. In *Hyperbolic problems and related topics*, Grad. Ser. Anal., pages 17–40. Int. Press, Somerville, MA, 2003.

[2] Antonio Córdoba and Charles Fefferman. Wave packets and Fourier integral operators. *Comm. Partial Differential Equations*, 3(11):979–1005, 1978.

[3] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[4] Lars Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[5] Herbert Koch and Daniel Tataru. Dispersive estimates for principally normal pseudodifferential operators. *Comm. Pure Appl. Math.*, 58(2):217–284, 2005.

[6] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[7] Daniel Tataru. Phase space transforms and microlocal analysis. In *Phase space analysis of partial differential equations*, Pisa 2004, pages 505–524.

Wave packets and boundary value problems

Hart F. Smith
(joint work with Chris D. Sogge)

This abstract reports on the authors' work establishing sharp bounds on the L^p norm of eigenfunctions (and more generally spectral clusters) on two-dimensional manifolds with boundary. It is part of a more general program of applying the Cordoba-Fefferman wave packet transform to the study of L^p bounds on solutions to the wave equation on manifolds with boundary.

The Cordoba-Fefferman transform at frequency scale λ is defined by

$$\begin{aligned}(T_\lambda f)(x,\xi) &= \lambda^{\frac{n}{4}} \int e^{-i\langle \xi, y-x\rangle} g(\lambda^{\frac{1}{2}}(y-x))\, f(y)\, dy \\ &= \int \overline{g_{x,\xi}(y)}\, f(y)\, dy\end{aligned}$$

where we fix a Schwartz functions g with $\|g\|_{L^2} = (2\pi)^{-\frac{n}{2}}$. A direct calculation shows that $T_\lambda^* T_\lambda f = f$, or that

$$f(y) = \int (T_\lambda f)(x,\xi)\, g_{x,\xi}(y)\, dx\, d\xi$$

In or work we take support$(\widehat{g}) \subseteq \{|\eta| \le 1\}$, so that if $\widehat{f}$ is localized in ξ then so is $T_\lambda f$.

This transform has seen applications in the study of dispersive estimates for wave equations, for example [4] and more recently [2], where it is a useful tool since it essentially conjugates a wave operator to differentiation along the Hamiltonian flow. Precisely, if P is a real, first order pseudodifferential operator, then

$$T_\lambda(Pf) = D_P\, T_\lambda f + \tilde{T}_\lambda f$$

where D_P is the Hamiltonian vector field of P, and $\tilde{T}_\lambda$ is a modified Cordoba-Fefferman transform, in that the function g depends in a uniform way on (x,ξ). (We assume here that ξ is localized to $|\xi| \approx \lambda$.)

Suppose now that we are given $A(x,D_x)$ a self-adjoint second order differential operator on a manifold M with boundary, and consider the wave equation

$$D_t^2 u = A(x,D_x)u$$

with Dirichlet or Neumann conditions at ∂M.

We work in geodesic normal coordinates such that $M = \{x_n \ge 0\}$, in which case

$$A(x,D_x) = D_n^2 + \sum_{i,j=1}^{n-1} a^{ij}(x',x_n)\, D_i\, D_j$$

We then extend $a^{ij}(x',x_n)$ evenly across $x_n = 0$, by considering $a^{ij}(x',|x_n|)$, and extend u oddly (Dirichlet) or evenly (Neumann). The extended function u satisfies the extended wave equation, so that we have eliminated the boundary, but at the expense of dealing with a wave equation with coefficients with a Lipschitz singularity. In the model case of the disc $a^{11} = (1 - |x_2|\,)^{-1} \approx 1 + |x_2|$.

After factoring the wave equation as a product of half-wave equations, we are led to considering a first-order hyperbolic equation $D_t - P$, where the elliptic, real symbol $p(x,\xi)$ is even in x_n and smooth on $x_n \geq 0$.

If we conjugate P by T_λ, then error terms arise that are large in a small neighborhood of the set $x_n = 0$. The resulting errors, however, are integrable along geodesics transverse to this set. The result is that, if one microlocalizes to the set where $|\xi_n| \approx \theta\lambda$ (which corresponds to bicharacteristics that meet ∂M at angle between θ and 2θ) then one has good control of the errors on a strip of length θ. Consequently, one can establish L^p bounds on such strips for solutions to the wave equation, such as Strichartz estimates.

Our interest is in establishing L^p bounds on spectral clusters on compact M. We restrict attention to the case of dimension $n = 2$. Suppose given an orthonormal basis of eigenfunctions

$$A(x,D)\phi_j = -\lambda_j^2 \phi_j\,.$$

Then a spectral cluster of frequency λ is of the form $f = \sum_{\lambda_j \in [\lambda,\lambda+1]} c_j \phi_j\,.$

The case $n = 2$ of Sogge [3] says that, for compact M without boundary, then

$$\|f\|_{L^p} \leq \lambda^{2(\frac12 - \frac1p) - \frac12} \|f\|_{L^2}\,, \quad p \geq 6\,.$$

On the other hand, Grieser [1] observed in his thesis that for the disk this estimate fails for $p < 8$; the counter-examples are Bessel function eigenfunctions that concentrate near the boundary $|x| = 1$.

Our recent work shows that the examples of Grieser are worst case.

Theorem. *Let M be a two-dimensional compact manifold with boundary, and $A(x,D_x)$ a second-order elliptic operator which is self-adjoint in some smooth volume form. Then for spectral clusters we have*

$$\|f\|_{L^p} \leq \lambda^{2(\frac12 - \frac1p) - \frac12} \|f\|_{L^2}\,, \quad p \geq 8\,.$$

We establish the estimate on f by establishing $L^p_x L^2_t$ bounds on the solution u to the wave equation with initial data f. If f is a spectral cluster then u is essentially periodic, and one gets similar bounds on u in $L^p_x L^2_t(M \times [0,1])$.

It turns out that to establish this estimate it is sufficient to prove bounds uniformly on strips of length θ in x_1 for the part u_θ of u microlocalized to $\xi_2 \approx \theta\lambda$. For $p > 6$, the localization to small angles leads to a gain over the estimates of Sogge,

$$\|u_\theta\|_{L^p_x L^2_t} \leq C\,\theta^{\frac12 - \frac3p} \lambda^{2(\frac12 - \frac1p) - \frac12} \|f\|_{L^2}\,.$$

In our case the norm on the left must be taken over a set of the form $|x_1 - c| \leq \theta$, and adding this estimate over the $\approx \theta^{-1}$ such strips leads to bounds on M for the piece of u microlocalized to angle θ from tangent:

$$\|u_\theta\|_{L^p_x L^2_t(M \times [0,1])} \leq C\,\theta^{\frac12 - \frac4p} \lambda^{2(\frac12 - \frac1p) - \frac12} \|f\|_{L^2}\,.$$

If $p > 8$, the series is summable over dyadic values of θ (in the proof one need consider only $\theta \geq \lambda^{-\frac{1}{3}}$), and the desired bound follows. For $p = 8$ one needs appropriate orthogonality of the terms u_θ, which involves bounding the leakage of energy from one angle to another. Again, however, it suffices to bound this leakage on strips of length θ in x_1, and consequently the Cordoba-Fefferman transform can be used to yield the desired bound.

References

[1] D. Grieser, *L^p bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries*, Thesis, UCLA, 1992.

[2] H. Smith, *Spectral cluster estimates for $C^{1,1}$ metrics*, Preprint.

[3] C. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–134.

[4] D. Tataru, *Strichartz estimates for operators with nonsmooth coefficients III*, J. Amer. Math. Soc. **15** (2002), 419–442.

Littlewood-Paley theory for matrix weights

Svetlana Roudenko

(joint work with M. Frazier, F. Nazarov)

We study weighted norm inequalities with matrix valued weights on vector-valued generalized function spaces. Namely, if $\mathcal{M}$ is the cone of non-negative-definite $m \times m$ complex-valued matrices, then the *matrix weight* W is a locally integrable map $W : \mathbb{R}^n \to \mathcal{M}$, and the main question we are interested in is under what condition on W and for which indices α, p, q, the following norm equivalence holds:

$$\|\vec{f}\|_{\dot{F}_p^{\alpha q}(W)} \approx \|\vec{s}_Q(f)\|_{\dot{f}_p^{\alpha q}(W)}. \tag{1}$$

Here, $\vec{f} = (f_1, ..., f_m)^T : \mathbb{R}^n \to \mathbb{C}^m$, $\{\vec{s}_Q(f)\}_Q$ - wavelet-type coefficients of $\vec{f}$ and the space $\dot{F}_p^{\alpha q}(W)$ is matrix-weighted (homogeneous[1]) Triebel-Lizorkin space together with its discrete analog (coefficient or sequence space) $\dot{f}_p^{\alpha q}(W)$ (see definitions below).

The original motivation comes from work of Nazarov, Treil and Volberg in [5], [4] and [6], where authors seek the boundedness of the Hilbert transform on $L^p(W)$ and to obtain it, split the problem into two steps: (i) obtaining a 'good' coefficient space (with Haar coefficients) for $L^p(W)$ (the sequence Triebel-Lizorkin space f_p^{02}) and (ii) showing that the Hilbert transform has a 'good' representation (almost diagonal) in that coefficient space. The condition on the matrix weight they require for (i) is $W \in A_p$, which is

$$\|W\|_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q \|W^{1/p}(t) W^{-1/p}(y)\|^p \, dt \right)^{p'/p} dy \right)^{p/p'} < \infty, \tag{2}$$

[1] all results hold similarly for the inhomogeneous space $F_p^{\alpha q}$, for brevity of definitions we use homogeneous spaces in this note

where the supremum is over all cubes $Q \subseteq \mathbb{R}^n$, $p' = p/(p-1)$ is the conjugate index, and the norm inside the integral is the matrix operator norm. (This is an equivalent formulation of the matrix A_p condition, see [3]; here, it is easy to see that for a scalar weight (2) reduces to the well-known Muckenhoupt A_p condition.) Since $L^p \approx F_p^{02}$ (Littlewood-Paley square function representation of an L^p function), and $F_p^{\alpha q} \approx f_p^{\alpha q}$, see [1], step (i) raises the question whether the previous equivalence holds with matrix weights, i.e. whether (1) holds.

Another motivation comes from the fact that the scale of Tribel-Lizorkin spaces includes not only Lebesque spaces ($L^p \approx F_p^{02}$) but also Sobolev ($W^{k,p} \approx F_p^{k2}$), Hardy ($H^p \approx F_p^{02}$, $p < 1$), and several other important spaces. Therefore, by obtaining (1), one can get the matrix-weighted characterization of sequence spaces of Sobolev, Hardy and other spaces (and as a consequence, almost diagonal operators will be bounded on these matrix-weighted spaces).

Now we turn to definitions. The *Triebel - Lizorkin space* $\dot{F}_p^{\alpha q}(W)$ is the collection of all vector-valued distributions $\vec{f} = (f_1, ..., f_m)^T$ with $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$[2], $1 \le i \le m$, such that

$$\| \vec{f} \|_{\dot{F}_p^{\alpha q}(W)} = \left\| \left(\sum_{\nu \in \mathbb{Z}} \left(2^{\nu\alpha} \| W^{1/p} (\varphi_\nu * \vec{f}) \| \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\varphi_\nu(x) = 2^{\nu n} \varphi(2^\nu x)$ for $\nu \in \mathbb{Z}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} \hat{\varphi} \subseteq \{ \frac{1}{2} \le |\xi| \le 2 \}$ and $|\hat{\varphi}(\xi)| \ge c > 0$ for $\frac{3}{5} \le |\xi| \le \frac{5}{3}$.

The *discrete Triebel-Lizorkin space* $\dot{f}_p^{\alpha q}(W)$ is the collection of all sequences $\vec{s} = \{\vec{s}_Q\}_Q$, where each $\vec{s}_Q = ((s_Q)_1, (s_Q)_2, \ldots, (s_Q)_m)^T$, such that

$$\| \vec{s} \|_{\dot{f}_p^{\alpha q}(W)} = \left\| \left(\sum_{Q} \left(|Q|^{-\alpha/n - 1/2} \| W^{1/p} \vec{s}_Q \| \chi_Q \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

(In the case when $q = \infty$, the ℓ^q (quasi-)norm is replaced with the supremum over $\nu \in \mathbb{Z}$.)

Before we state the main result, we need to define the reverse Hölder property for matrix weights and extend the definition of the A_p class for $p < 1$.

A matrix weight W satisfies the *reverse Hölder condition of order* p, $W \in (RH)_p$, if there exist $c, \delta > 0$ such that

$$(3) \qquad \left(\frac{1}{|Q|} \int_Q \| W^{1/p}(t) \vec{x} \|^{p(1+\delta)} \, dt \right)^{1/(1+\delta)} \le c \, \frac{1}{|Q|} \int_Q \| W^{1/p}(t) \vec{x} \|^p \, dt$$

for all $\vec{x} \in \mathbb{C}^m$ and all cubes Q. In other words, $W \in (RH)_p$ if the scalar weights $w_{\vec{x}}(t) = \| W^{1/p}(t) \vec{x} \|^p$ satisfy a uniform reverse Hölder condition.

[2] modulo polynomials

Let $0 < p \le 1$. Then $W \in A_p$, if

$$\|W\|_{A_p} = \sup_{Q} \operatorname*{ess\,sup}_{y\in Q} \frac{1}{|Q|} \int_Q \left\| W^{1/p}(t) W^{-1/p}(y) \right\|^p dt < \infty, \tag{4}$$

where the supremum is over all cubes $Q \subseteq \mathbb{R}^n$. Observe that in the scalar case (4) reduces to the A_1 condition (independent of p).

Theorem 1. *Suppose $0 < p < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$. Then the equivalence (1) holds if $W \in (RH)_p$ for $1 < p < \infty$ and if $W \in A_p$ for $p \le 1$.*

Now we make some comments about the proof. The scalar result of this theorem was obtained in [1] under the doubling condition on the weight (i.e., $w(2Q) \le c\, w(Q)$, where Q is dyadic and $w(Q) = \int_Q w(t)\, dt$). It is not surprising that in the matrix case stronger conditions than doubling are required (i.e. $(RH)_p$ and even more strong A_p). For example, in the case of matrix-weighted Besov spaces we are able to construct a counterexample that for small enough p[3] a norm equivalence similar to (1) but with matrix-weighted Besov spaces fails for non-A_p doubling matrix weight (see [2]). (The matrix weight is *doubling of order p*, if $w_{\vec{x}}(2Q) \le c\, w_{\vec{x}}(Q)$ uniformly on $\vec{x}$, where $w_{\vec{x}}(t) = \|W^{1/p}(t)\, \vec{x}\|^p$.) The proof is different from the scalar case, since many scalar techniques simply fail in the matrix case (e.g., see introduction in [6]). The crucial idea in the proof comes from discretizing the matrix weight W into the *reducing operators* A_Q and then proving a similar to (1) equivalence with reducing operators instead of a matrix weight:

$$\|\vec{f}\|_{\dot{F}^{\alpha q}_p(A_Q)} \approx \|\vec{s}_Q(f)\|_{\dot{f}^{\alpha q}_p(A_Q)}. \tag{5}$$

In fact, this equivalence is practically the scalar case and holds just under the doubling condition on matrix weight W. (For any finite dimensional matrix $W(t)$, there exists a sequence $\{A_Q\}$ of reducing operators such that $\|A_Q \vec{x}\| \approx \left(\frac{1}{|Q|}\int_Q \|W^{1/p}(t)\vec{x}\,\|\, dt\right)^{1/p}$ with equivalence constants independent of $\vec{x}$ and Q, see [6] for $p > 1$ and [2] for $0 < p \le 1$.) For the rest of the proof we establish the relation between the matrix-weighted spaces and spaces with reducing operators. This part also answers the question of Volberg in [6] where he asks for a conditions on W under which the equivalence $\dot{f}^{02}_p(W) \approx \dot{f}^{02}_p(A_Q)$ holds. The following chain of equivalences holds:

$$\dot{F}^{\alpha q}_p(W) \approx \dot{F}^{\alpha q}_p(A_Q) \approx \dot{f}^{\alpha q}_p(A_Q) \approx \dot{f}^{\alpha q}_p(W),$$

where the first equivalence holds if $W \in (RH)_p$ for $1 < p < \infty$ and $W \in A_p$ for $0 < p \le 1$, the second if W is a doubling matrix weight (of order p), and the third holds for any W if $p = q$, for $W \in (RH)_p$ if $1 < p < \infty$ and $p \ne q$, and $W \in A_p$ for $0 < p \le 1$ and $p \ne q$.

[3] less than the doubling exponent of the matrix weight

References

[1] M. Frazier and B. Jawerth, *A Discrete Transform and Decompositions of Distribution Spaces*, J. Funct. Anal. **93** (1990), 34–170.
[2] M. Frazier and S. Roudenko, *Matrix-Weighted Besov Spaces and Conditions of A_p Type for* $0 < p \leq 1$, Indiana Univ. Math. J. **53** (2004), 1225–1254.
[3] S. Roudenko, *Matrix-weighted Besov spaces*, Trans. AMS **355** (2003), 273–314.
[4] F. Nazarov and S. Treil, *The Hunt for a Bellman Function: Applications to Estimates for Singular Integral Operators and to Other Classical Problems of Harmonic Analysis*, Algebra i Analiz (in Russian) **8**, no. 5 (1996), 32–162.
[5] S. Treil and A. Volberg, *Wavelets and the Angle between Past and Future*, J. Funct. Anal. **143** (1997), 269–308.
[6] A. Volberg, *Matrix A_p Weights via S-functions*, J. Amer. Math. Soc., **10**, no.2 (1997), 445–466.

Two problems related to polynomial polyhedra

Alexander Nagel
(joint work with Malabika Pramanik)

We study two kinds of problems whose solutions seem to rely on an understanding of the geometric properties of sets defined by real or complex polynomials. We establish the L^p-boundedness of maximal averages over certain families of monomial polyhedra. We also obtain estimates for the Bergman kernel on the diagonal in certain model domains, including the so-called *cross of iron* domain.

1. Maximal Averages

Let $\mathcal{O}_n = \Big\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,\Big|\, x_j > 0, \quad 1 \leq j \leq n\Big\}$ denote the positive octant in $\mathbb{R}^n$. If $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$, the monomial $F_{\mathbf{p}} : \mathcal{O}_n \to (0, \infty)$ is defined by $F_{\mathbf{p}}(x) = x_1^{p_1} \cdots x_n^{p_n}$. We study maximal averages over a family of polyhedra generated by a finite number of such monomials.

Thus let $\mathcal{R} = \{\mathbf{q}_1, \dots, \mathbf{q}_d\} \subset \mathbb{R}^n$, let $\bar{\delta} = (\delta_1, \dots, \delta_d) \in \mathbb{R}^d_+$, and put

$$B_{\mathcal{R}}(x; \bar{\delta}) = B_{\mathcal{R}}(x; \delta_1, \dots, \delta_d) = \Big\{y \in \mathcal{O}_n \,\Big|\, F_{\mathbf{q}_j}(y) - F_{\mathbf{q}_j}(x) < \delta_j, \quad 1 \leq j \leq d\Big\}.$$

For any $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{R}^n$ and any measurable set $E \subset \mathcal{O}_n$, put

$$m_{\mathbf{e}}(E) = \int_E y_1^{e_1} \cdots y_n^{e_n} \,\frac{dy_1 \cdots dy_n}{y_1 \cdots y_n}.$$

Note that if $\bar{1} = (1, \dots, 1)$, then $m_{\bar{1}}(E) = |E|$, the Lebesgue measure of E. If the vectors $\{\mathbf{q}_1, \dots, \mathbf{q}_d\}$ do not lie in any closed half-space[1] of $\mathbb{R}^n$, then the set $B(x; \bar{\delta})$ is a relatively compact subset of $\mathcal{O}_n$, and so $m_{\mathbf{e}}(B_{\mathcal{R}}(x; \bar{\delta})$ is finite. For $f \in L^1_{loc}(\mathcal{O}_n)$, put

$$\mathcal{M}_{\mathcal{R},\mathbf{e}}[f](x) = \sup_{\bar{\delta} \in \mathbb{R}^n_+} \frac{1}{m_{\mathbf{e}}\big(B_{\mathcal{R}}(x; \bar{\delta})\big)} \int_{B_{\mathcal{R}}(x;\bar{\delta})} |f(y)|\, y_1^{e_1} \cdots y_n^{e_n} \,\frac{dy_1 \cdots dy_n}{y_1 \cdots y_n}.$$

[1]This means that for all $0 \neq \eta \in \mathbb{R}^n$ there exists j so that the inner product $\langle \eta, \mathbf{q}_j \rangle < 0$.

Theorem 1: *Assume that the vectors $\mathcal{R} = \{\boldsymbol{q}_1, \dots, \boldsymbol{q}_d\} \subset \mathbb{R}^n$ do not lie in any closed half-space of $\mathbb{R}^n$. Also assume that the vector $\boldsymbol{e}$ does not belong to the linear span of any subset of $(n-1)$ of the vectors $\{\boldsymbol{q}_1, \dots, \boldsymbol{q}_d\} \subset \mathbb{R}^n$. Then the maximal operator $\mathcal{M}_{\mathcal{R},\mathbf{e}}$ is a bounded operator on $L^p(\mathcal{O}_n)$ for $1 < p \leq \infty$.*

Recall that the strong maximal function on $\mathbb{R}^n$ is defined by

$$\mathcal{M}_{st}[f](x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is take over the set of all rectangles Q with sides parallel to the axes which contain x. Jessen, Marcinkiewics, and Zygmund [2] showed that $\mathcal{M}_{st}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. Such rectangles are clearly special cases of monomial polyhedra, so in general, if $\mathbf{e} = \bar{1}$, the maximal operator $\mathcal{M}_{\mathcal{R},\mathbf{e}}$ dominates the strong maximal function.

The proof of Theorem 1 depends showing that there is a constant $\epsilon > 0$ so that if $B(x;\bar{\delta})$ is a monomial polyhedron as defined above, then under the hypotheses of the theorem, there is an n-tuple $\{\mathbf{q}_{i_1}, \dots, \mathbf{q}_{i_n}\} \subset \mathcal{R}$ and constants $0 < \alpha_k < \beta_k$ such that

$$B_{\mathcal{R}}(x;\delta) \subset \Big\{ y \in \mathcal{O}_n \,\Big|\, 0 < \alpha_k < F_{\mathbf{q}_{i_k}}(y) < \beta_k, \quad 1 \leq k \leq n \Big\}$$

and

$$m_{\mathbf{e}}\big(B_{\mathcal{R}}(x;\delta)\big) \geq \epsilon\, m_{\mathbf{e}}\Big(\Big\{ y \in \mathcal{O}_n \,\Big|\, 0 < \alpha_k < F_{\mathbf{q}_{i_k}}(y) < \beta_k, \quad 1 \leq k \leq n \Big\} \Big).$$

With this geometric information, one can dominate $\mathcal{M}_{\mathcal{R},\mathbf{e}}$ by a sum of a finite number of strong maximal functions, and the L^p-estimates of the theorem then follow from [2].

2. Estimates for the Bergman kernel

If $\Omega \subset \mathbb{C}^n$, the Bergman projection is the orthogonal projection $P_\Omega : L^2(\Omega) \to A^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$ from $L^2(\Omega)$ onto the closed subspace of square integrable holomorphic functions. This operator is given by integration against the Bergman kernel $B_\Omega : \Omega \times \Omega \to \mathbb{C}$:

$$P_\Omega[f](z) = \int_\Omega f(w)\, B_\Omega(z,w)\, dw.$$

Many of the elementary properties of the Bergman kernel can be found in [3], and there is considerable interest in obtaining estimates for $|B(z,w)|$. In particular, the value of the Bergman kernel on the diagonal of $\Omega \times \Omega$ is given by the solution of an extremal problem. Thus if $z \in \Omega$, then

$$B_\Omega(z,z) = \sup \Big\{ |f(z)|^2 \,\Big|\, f \in A^2(\Omega) \text{ and } ||f||_{A^2(\Omega)} \leq 1 \Big\}.$$

For model domains of the form

$$\Omega = \Big\{ z \in \mathbb{C}^{n+1} \,\Big|\, \Re e[z_{n+1}] > \sum_{j=1}^{d} |P_j(z_1, \dots, z_n)|^2 \Big\}$$

where $\{P_1, \dots, P_d\}$ are holomorphic monomials, one wants estimates for $|B_\Omega(z_\delta, z_\delta)|$ where $z_\delta = (a_1, \dots, a_n, \delta + \sum_{j=1}^{d} |P_j(a)|^2)$. The objective is to obtain sharp estimates which are uniform in the base point $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and in δ, which is the distance to the boundary.

To understand the possibile difficulties that arise in finding uniform estimates, consider the special case of the *cross of iron* domain $\Omega^\dagger = \{z \in \mathbb{C}^3 \,|\, \Re e[z_3] > Q(z)\}$, where $Q(z) = |z_1|^6 + |z_2|^6 + |z_1 z_2|^2$. Herbort [1] showed that for points $z_\delta = (0, 0, \delta)$ directly above the origin one has the estimate

$$B_{\Omega^\dagger}\big((0,0,\delta);(0,0,\delta)\big) \approx \frac{1}{\delta^3 \log\big(\delta^{-1}\big)}.$$

On the other hand, if $(a_1, a_2) \neq (0, 0)$, the boundary point $(a_1, a_2, Q(a))$ is strictly pseudoconvex, and so for small δ it is known that

$$B_{\Omega^\dagger}\big((a_1, a_2, \delta + Q(a));(a_1, a_2, \delta + Q(a))\big) \approx \frac{1}{\delta^4}.$$

The problem is to reconcile these two estimates, since the point (a_1, a_2) can be arbitrarily close to the origin $(0, 0)$.

Some results in this direction have been obtained by Tiao [5] and by McNeal [4]. We obtain the following uniform estimates.

THEOREM 2: *Let $a_\delta = \big(a_1, a_2, \delta + Q(a)\big)$. Then*

$$B_{\Omega^\dagger}(a_\delta, a_\delta) \approx \begin{cases} \frac{(|a_1|^2+|a_2|^2)^3}{\delta^4} & \text{if } \delta^{\frac{1}{3}} \lesssim |a_1|^2 + |a_2|^2 \\[2ex] \frac{1}{\delta^3}\left(\frac{|a_1|^2+|a_2|^2}{\delta^{\frac{1}{3}}} + \left[\log^+\left(\frac{\delta^{1/3}}{|a_1 a_2|}\right)^{-1}\right]\right) & \text{if } \begin{cases} |a_1|^2 + |a_2|^2 \lesssim \delta^{\frac{1}{3}} \\ \delta^{\frac{1}{2}} \lesssim |a_1 a_2| \end{cases} \\[4ex] \frac{1}{\delta^3}\left(\frac{|a_1|^2+|a_2|^2}{\delta^{1/3}} + \left[\log^+\left(\frac{1}{\delta}\right)^{-1}\right]\right) & \text{if } \begin{cases} |a_1|^2 + |a_2|^2 \lesssim \delta^{\frac{1}{3}} \\ |a_1 a_2| \lesssim \delta^{\frac{1}{2}} \end{cases} \end{cases}$$

The proof of Theorem 2 uses a scaling argument and an analysis of the Bergman kernel evaluated at the point $(1, 1)$ of the monomial polyhedron

$$\Omega(\delta) = \left\{(w_1, w_2) \in \mathbb{C}^2 \,\middle|\, |w_1^3 - 1| < \delta_1, \quad |w_2^3 - 1| < \delta_2, \quad |w_1 w_2 - 1| < \delta_3\right\}.$$

For some values of the parameters $\{\delta_1, \delta_2, \delta_3\}$, $\Omega(\delta)$ can be approximated by a polydisk, but for other values, in appropriate coordinates the domain $\Omega(\delta)$ is essentially the product of a disk with an annulus, and this accounts for some occurrences of the logarithm term.

References

[1] G. Herbort, *Logarithmic growth of the Bergman kernel for weakly pseudoconvex domains in* $\mathbb{C}^3$ *of finite type*, Manuscripta Math. **45** (1983), 69-76.

[2] B. Jessen, J. Marcinkiewics, A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math **25** (1935), 217-234.

[3] Steven G. Krantz, *Function Theory of Several Complex Variables*, Second Edition, Wadsworth & Brooks/Cole, (1992)

[4] J.D. McNeal, L^2 *harmonic forms on some complete Kähler manifolds*, Math. Ann. **323** (2002), 319-349.

[5] C.-H. Tiao, *The Bergman kernel on monomial polyhedra*, Michigan Math. J. **46** (1999), 53-69.

Energy functionals and flows in Kähler geometry

D. H. Phong

(joint work with Jacob Sturm)

1 Main results

In this lecture[1], we focus on the problem of the convergence of the Kähler-Ricci flow. Let X be a compact complex manifold of dimension n with complex structure J and Kähler form $\omega_0 = \frac{\sqrt{-1}}{2}\sum_{j,k=1}^n g^0_{\bar{k}j} dz^j \wedge d\bar{z}^k$. The Kähler-Ricci flow is the non-linear parabolic flow for the Kähler form $\omega = \frac{\sqrt{-1}}{2}\sum_{j,k=1}^n g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ given by

$$\dot{g}_{\bar{k}j} = -(R_{\bar{k}j} - \mu g_{\bar{k}j}), \qquad g_{\bar{k}j}(0) = g^0_{\bar{k}j}. \tag{1}$$

Here $V = \int_X \omega^n$ and $\mu = \frac{1}{Vn}\int_X R\omega^n$, where $R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log \omega^n$, $R = g^{j\bar{k}} R_{\bar{k}j}$ are respectively the Ricci and the scalar curvature of the metric $g_{\bar{k}j}$. The initial Kähler form ω_0 is assumed to satisfy the cohomological condition $\mu\omega_0 \in c_1(X)$. Since $R_{\bar{k}j}$ is always in $c_1(X)$, it is readily seen that the Kähler-Ricci flow preserves the Kähler class of the metric. Thus

$$g_{\bar{k}j} = g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \phi, \qquad R_{\bar{k}j} = \mu g_{\bar{k}j} + \partial_j \partial_{\bar{k}} h, \tag{2}$$

where $\phi = \phi(t)$, $h = h(t)$ are scalar functions. With ϕ suitably normalized, the Kähler-Ricci flow can then be rewritten as the following parabolic Monge-Ampère equation

$$\dot{\phi} = \log \frac{\det(g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \phi)}{\det g^0_{\bar{k}j}} + \mu\phi - h(0). \tag{3}$$

The maximum principle implies that $\sup_{0 \le t \le T} |\phi| \le C_T$, so general estimates for complex Monge-Ampère equations [12] imply that the Kähler-Ricci flow exists for all time [1]. In this aspect, it is better behaved than the most general Ricci flow of

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R. Hamilton. The main question about the Kähler-Ricci flow concerns the long-time behavior of the flow, and in particular when it converges. The cases $\mu < 0$ and $\mu = 0$ have been completely resolved in [1], so henceforth we assume that $\mu > 0$.

The stationary points of the Kähler-Ricci flow are Kähler-Einstein metrics, $R_{\bar{k}j} = \mu g_{\bar{k}j}$, and the flow is expected to converge if and only if X admits a Kähler-Einstein metric. According to a well-known conjecture of Yau [13], the existence of Kähler-Einstein metrics should be equivalent to the stability of X in the sense of geometric invariant theory. It is a major open problem in Kähler geometry to relate such notions of stability to the convergence of the Kähler-Ricci flow. In fact, when μ is positive, the Kähler-Ricci flow has been shown to converge only when $X = \mathbf{CP}^1$ [7, 4], and when the metric $g^0_{\bar{k}j}$ has positive biholomorphic sectional curvature [5, 2]. This last assumption is extremely strong, since it implies that $X = \mathbf{CP}^n$. In this work, we establish the following theorem. Let (A) and (B) be the following conditions.

(A) The Mabuchi energy functional $\nu_{\omega_0}(\phi)$ is bounded below;
(B) There is no complex structure J' in the closure of the diffeomorphism group orbit of J which admits more independent holomorphic vector fields than J.

Theorem 1 *Assume that the Riemann curvature tensor is bounded along the flow. Then*
- *If* (A) *holds, then* $\lim_{t\to\infty} ||R_{\bar{k}j} - \mu g_{\bar{k}j}||_{(s)} = 0$*, where* $||\cdot||_{(s)}$ *denotes the Sobolev norm of order* s *with respect to the metric* $g_{\bar{k}j}(t)$*;*
- *If both* (A) *and* (B) *hold, and if the diameter of* X *remains uniformly bounded along the flow, then the flow converges exponentially fast in* C^∞ *to a Kähler-Einstein metric.*

The condition (A) is clearly a type of stability condition, since the notion of K-stability is defined by the asymptotic behavior of the Mabuchi functional along certain degenerations of X [11, 6]. The condition (B) is a different type of stability condition, which is certainly necessary if the moduli of admissible complex structures for X is to be Hausdorff. It makes here its first appearance in relation to the convergence of the Kähler-Ricci flow.

In complex dimension 2, the assumption of uniformly bounded curvature can be relaxed:

Theorem 2 *Assume that* $\dim X = 2$*, that* (A) *and* (B) *are satisfied. Assume that the scalar curvature* R *and the diameter of* X *remain bounded along the flow. Assume further that the condition* (C) *below holds:*

(C) *The initial metric* $g^0_{\bar{k}j}$ *has positive Ricci curvature and its traceless curvature operator is 2-nonnegative.*

Then the flow converges exponentially fast in C^∞ *to a Kähler-Einstein metric.*

The condition (C) was introduced in [9] where it was shown to be preserved by the Kähler-Ricci flow in dimension 2. The 2-nonnegativity of the traceless curvature operator is the Kähler analogue of a condition introduced in the Riemannian setting by H. Chen [3]. The uniform boundedness of the scalar curvature and of the diameter have been established by Perelman in as yet unpublished work. Under no stability conditions, the Kähler-Ricci flow can exhibit solitonic behavior. For this, see N. Sesum [10] and references therein.

2 Key estimates and stability

The proof of Theorems 1 and 2 can be found in [9]. Here we stress only two estimates where stability plays a major role. Let h be the function defined in (2), and set $Y(t) = ||\nabla h||^2_{(0)}$. Then the first key estimate is

$$\begin{aligned}\dot{Y} \leq -2\,\lambda_t Y + 2\lambda_t Fut(\pi_t(\nabla h)) - \int_X |\nabla h|^2 (R - \mu n)\omega^n \\ - \int_X \nabla^j h \nabla^{\bar{k}} h (R_{\bar{k}j} - \mu g_{\bar{k}j})\omega^n,\end{aligned} \tag{4}$$

where λ_t is the lowest strictly positive eigenvalue of the Laplacian $\nabla^{\bar{k}}\nabla_{\bar{k}}$ on $T^{1,0}(X)$, π_t is the orthogonal projection with respect to $g_{\bar{k}j}(t)$ on holomorphic vector fields, and $Fut(V) = \int_X (Wh)\,\omega^n$ is the Futaki invariant defined on $W \in H^0(X, T^{1,0})$. This estimate shows that crucial to the exponential decay of $Y(t)$ is a stricly positive lower bound for λ_t along the Kähler-Ricci flow. This is provided in turn by the following estimate. Assume that J satisfies condition (B). Fix $V > 0$, $D > 0$, $\delta > 0$, C_k. The second key estimate asserts that there exists $C > 0$ so that

$$||\bar{\partial} W||^2 \geq C||W||^2, \qquad W \perp H^0(X, T^{1,0}), \tag{5}$$

for all Kähler metrics with volume $\leq V$, diameter $\leq D$, injectivity radius $\geq \delta$, and $\sup_{|\alpha|\leq k}|D^\alpha Riem| \leq C_k$. This estimate follows in turn from a Kähler version of the Cheeger-Gromov-Hamilton compactness theorem for C^∞ bounded geometries (see Theorem 4 in [9]). The above bounds insure the exponential decay of $Y(t)$, and hence, the equivalence of all metrics $g_{\bar{k}j}(t)$ for t large enough. This equivalence is crucial to deriving the convergence of $g_{\bar{k}j}(t)$ from the convergence of $||R_{\bar{k}j} - \mu g_{\bar{k}j}(t)||_{(s)}$.

References

[1] Cao, H.D., *Deformation of Kähler metrics to Kähler-Einstein metrics*, Inventiones Math. **81** (1985), 359–372.

[2] Cao, H.D., B.L. Chen, and X.P. Zhu, *Ricci flow on compact Kähler manifolds of positive bisectional curvature*, math.DG / 0302087.

[3] Chen, H., *Pointwise quarter-pinched 4-manifolds*, Ann. Global Anal. Geom. **9** (1991), 161–176.

[4] Chow, B., *The Ricci flow on the 2-sphere*, J. Diff. Geometry **33** (1991), 325–334.

[5] Chen, X.X. and G. Tian, *Ricci flow on Kähler-Einstein surfaces*, Inventiones Math. **147** (2002) 487-544; Ricci flow on Kähler manifolds, arXiv math.DG / 0108179.

[6] Donaldson, S.K., *Scalar curvature and stability of toric varieties*, J. Diff. Geometry **62** (2002), 289–349.
[7] Hamilton, R., *The Ricci flow on surfaces*, Contemp. Math. **71** (1988), 237–261.
[8] Phong, D.H. and J. Sturm, *On the Kähler-Ricci flow on complex surfaces*, arXiv math.DG / 0407232.
[9] Phong, D.H. and J. Sturm, *On stability and the convergence of the Kähler-Ricci flow*, arXiv: math.DG / 0412185.
[10] Sesum, N., *Convergence of a Kähler-Ricci flow*, arXiv math.DG / 0402238.
[11] Tian, G., *Kähler-Einstein metrics with positive scalar curvature*, Inventiones Math. **130** (1997), 1–37.
[12] Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I*, Comm. Pure Appl. Math. **31** (1978), 339–411.
[13] Yau, S.T., *Open problems in geometry*, Proc. Symp. Pure Math. **54** (1993), 1–28.

Endpoint estimates for some degenerate Radon transforms in the plane

Jong-Guk Bak

The purpose of this talk is to give a brief survey on the problem of endpoint $L^p - L^q$ estimates for the Radon transform in the plane associated to some degenerate phase functions.

Phong and Stein [9] studied the $L^p - L^q$ mapping properties of the Radon transform defined by

$$(Rf)(x,t) = \int_{-\infty}^{\infty} f(y, t + S(x,y))\psi(x,t,y)dy$$

where $S(x,y) = \sum_{k=1}^{n-1} a_k x^k y^{n-k}$, $n \geq 2$, and ψ is a smooth cutoff function supported near the origin in $\mathbb{R}^3$.

Assume $a_1 \neq 0$ and $a_{n-1} \neq 0$. Let Δ_n be the trapezoid (a triangle when $n = 2$) in the plane with vertices $O = (0,0)$, $O' = (1,1)$, $A = (2/(n+1), 1/(n+1)$, $A' = (n/(n+1), (n-1)/(n+1))$. A homogeneity argument shows that for R to be bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$, it is necessary that $(1/p, 1/q)$ lies in Δ_n. In the nondegenerate case, i.e. when $n = 2$, it was known that R is of strong type (p,q) if and only if $(1/p, 1/q)$ is in Δ_2. Also, in the translation-invariant case, i.e. when $S(x,y) = (x-y)^n$, it was known that R is of strong type (p,q) if and only if $(1/p, 1/q)$ is in Δ_n (this result was implicit in [5]).

When $n \geq 3$, Phong and Stein [9] proved that R is of strong type (p,q) if $(1/p, 1/q)$ is on the open segment (A, A'). On the other hand Seeger [10] proved that R is of strong type (p,q) if $(1/p, 1/q)$ is on the open segment (O, A). Since the estimate is trivial when $p = q$, by duality and interpolation it follows that R is of strong type (p,q) if $(1/p, 1/q)$ is in $\Delta_n \setminus \{A, A'\}$. Thus the only remaining question is the boundedness of R at the point A for $n \geq 3$. All of these positive results were obtained based on an approach involving certain oscillatory integral estimates (and analytic interpolation), which also yield optimal Sobolev space estimates for R.

In [1] the remaining question at A was answered in the affirmative. Namely, it was proved that R is of strong type (p,q) at the critical point $(1/p,1/q) = A = (2/(n+1), 1/(n+1)$. The proof was an adaptation of a multilinear proof by Oberlin [8] of the optimal $L^{3/2} - L^3$ estimate for the convolution operator associated to the circle, namely the one whose kernel is the arc length measure on the unit circle in the plane. The proof is reduced to certain multilinear estimate for characteristic functions and hinges on the so-called "multilinear trick" of Christ [4], which is a multilinear version of a bilinear real interpolation theorem of Lions and Peetre (see [3]). It also crucially depends on the fact that the index q is a positive integer and that R is a positive operator. (In particular, this method does not seem to imply any result on Sobolev estimates.)

This result was generalized in [2] to the so-called non-semi-translation-invariant case. This means that in the definition of R the phase function $t + S(x,y)$ is replaced by a smooth function $G(x,t,y)$. The optimal result was proved under the hypotheses of the so-called left finite type n and right finite type m, together with some auxiliary hypotheses. (For a precise statement of this result see [2].)

Lee [6] considered the problem of extending the result in [1] to the case of real-analytic phase functions $S(x,y)$. Write

$$S(x,y) = \sum_{\alpha,\beta>0} a_{\alpha,\beta}\, x^\alpha y^\beta$$

and let Δ be the closed convex hull of the set

$$\{A_{\alpha,\beta} :\ a_{\alpha,\beta} \neq 0\} \cup \{O, O'\}$$

where $A_{\alpha,\beta} = ((\alpha+1)/(\alpha+\beta+1), \alpha/(\alpha+\beta+1))$. (The set Δ is closely related to the Newton polygon for $S(x,y)$.) Lee proved that if ψ has a sufficiently small support near the origin, R is of strong type (p,q) if and only if $(1/p,1/q)$ is in Δ. He succeeded in obtaining this general result by a clever argument combining the results of [2] and certain other $L^p - L^2$ estimate via analytic interpolation. (See [7] for an even more general result of his for smooth phase functions.) Seeger [10] had earlier proved the boundedness of R in the interior of Δ, and Yang [11], independently of Lee's work, obtained the result at all points except the vertices of Δ.

References

[1] J.-G. Bak, *An $L^p - L^q$ estimate for Radon transforms associated to polynomials*, Duke Math. J. **101** (2000), 259–269.

[2] J.-G. Bak, D. Oberlin, A. Seeger, *Two endpoint bounds for generalized Radon transforms in the plane*, Rev. Mat. Iberoamericana **18** (2002), 231–247.

[3] J. Bergh and J. Löfström, *Interpolation spaces*, Grund. der math. Wiss. **223** (1976), Springer-Verlag.

[4] M. Christ, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case*, Trans. Amer. Math. Soc. **287** (1985), 223–238.

[5] M. Christ, *Endpoint bounds for singular fractional integral operators*, preprint (1988), unpublished.

[6] S. Lee, *Endpoint $L^p - L^q$ estimates for degenerate Radon transforms in $\mathbb{R}^2$ associated with real-analytic functions*, Math. Z. **243** (2003), 817–841.

[7] S. Lee, *Endpoint $L^p - L^q$ estimates for some classes of degenerate Radon transforms in* $\mathbb{R}^2$, Math. Res. Lett. **11** (2004), 85–101.
[8] D. Oberlin, *Multilinear proofs for two theorems on circular averages*, Colloq. Math. **63** (1992), 187–190.
[9] D. Phong and E. M. Stein, *Models of degenerate Fourier integral operators and Radon transforms*, Ann. of Math. **140** (1994), 703–722.
[10] A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), 685–745.
[11] C. W. Yang, *L^p improving estimates for some classes of Radon transforms*, Trans. Amer. Math. Soc. **357** (2005), 3887–3903.

A maximal trilinear operator

Christoph Thiele
(joint work with Ciprian Demeter, Terence Tao)

We discuss a hierarchy of maximal operators, starting with the one sided Hardy Littlewood maximal operator in one dimension

$$Mf(x) = \sup_{\epsilon>0} \frac{1}{\epsilon} \int_0^\epsilon |f(x+t)|\, dt$$

This operator satisfies the trivial bound $\|Mf\|_\infty \leq \|f\|_\infty$. A refined geometric tool, the Vitali covering lemma, allows one to lower the exponent and obtain $\|Mf\|_p \leq \|f\|_p$ for $1 < p \leq \infty$. For our hierarchy of maximal operators, the general goal will be to lower exponents of L^p boundedness as far as possible. An application of the Hardy Littlewood maximal operator is to prove the Lebesgue differentiation theorem for $f \in L^p$, namely that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^\epsilon f(x+t)\, dt$$

exists for almost every x and is equal to $f(x)$. This is trivial on the dense subclass of continuous functions, and then can be extended to all of L^p using the corresponding bound on the Hardy Littlewood maximal operator.

Considering large ϵ, we also study the discrete averages

$$Mf(n) = \sup_{N\geq 1} \frac{1}{N} \sum_{n=1}^{N} |f(n+m)|$$

By a simple transference argument, the L^p bounds for the discrete object are the same as for the continuous object. There is however no immediate sense in asking for limits of averages as $N \to \infty$, as these limits either are trivially zero for $f \in l^p$ with $p < \infty$ or need not exist if $f \in l^\infty$. However, there is a meaningful variant in ergodic theory. Let X, μ be a probability space and T a measure preserving transformation. Then there is a maximal operator

$$Mf(x) = \sup_{N\geq 1} \frac{1}{N} \sum_{n=1}^{N} |f(T^n(x))|$$

which again by easy transference satisfies the same L^p bounds as the previous maximal operators. Also one has a theory of limits of averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n(x))$$

as $N \to \infty$, which exist almost everywhere for $f \in L^p$ with $p \geq 1$ by Birkhoff's ergodic theorem. Again one proves this first in a dense subclass and then extends this to all of L^p using bounds on the maximal operator. There is in general no nice dense subclass such as continuous functions on the real line, but for the above averages one has a nice Hilbert space proof for functions in L^2 which can be used as the dense subclass. In L^2, the limit of the averages equals the projection onto the space of functions invariant under T.

Next, we consider the maximal operator

$$M(f,g)(x) = \sup_{\epsilon} \frac{1}{\epsilon}\int_0^{\epsilon} |f(x+t)g(x+2t)|\, dt$$

The parameter 2 in the second argument is quite arbitrary, any other number different from 1 and 0 would give a similar object. In the discrete analogue, the parameter will have to be an integer. Trivially, this operator is pointwise bounded by $M(f)\|g\|_\infty$ and possibly a constant times $\|f\|_\infty M(g)$, which by interpolation immediately gives bounds

$$\|M(f,g)\|_r \leq C\|f\|_p\|g\|_q$$

with $1/r = 1/p + 1/q$ and $1 < p, q \leq \infty$ and $1 < r \leq \infty$. A deep theorem by M. Lacey [9] states that one can relax the last condition to $2/3 < r$. Somewhat surprisingly, despite the positivity of the operator, one uses non-positive tools such as time frequency analysis as in the closely related proof of boundedness of the bilinear Hilbert transform [10] and a Fourier theoretic lemma of Borgain [2]. No geometric proof of Lacey's theorem is known. One has convergence of the corresponding bilinear averages for $\epsilon \to 0$ by extending the result from the case of continous functions. In ergodic theory, by the same transference principle as above, one has the same estimates as in Lacey's theorem for the maximal operator

$$M(f,g)(x) = \sup_N \frac{1}{N}\sum_{n=1}^{N} |f(T^n(x))g(T^{2n}(x))|$$

Convergence almost everywhere of the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n(x))g(T^{2n}(x))$$

then follows in the region of exponents in Lacey's theorem from extension of the result for $f, g \in L^\infty$, which is a deep Fourier analytic result by Bourgain [3]. Averages such as this one and multilinear analogues

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T^n(x))f_2(T^{2n}(x))\ldots f_m(T^{mn}(x))$$

appear in Furstenberg's theory of return time theorems and his proof [6] of Szemeredi's theorem [11] of existence of long arithmetic progressions in sets of integers of positive density. Recently, Green and Tao have used a weighted version of this result to prove existence of long arithmetic progressions in the set of primes, which form a set of positive density in the almost primes.

In joint work with Ciprian Demeter and Terence Tao we study the maximal operator

$$M(f,g,h)(x) = \sup_{\epsilon} \frac{1}{\epsilon^2} \int_0^{\epsilon} \int_0^{\epsilon} f(x+t)g(x+u)h(x+t+u)\, dt\, du$$

and obvious variants of higher order. This operator is pointwise bounded by $CM(f)M(g)\|h\|$ and symmetric estimates, so it trivially satisfies the bounds

$$\|M(f,g,h)\|_s \leq C\|f\|_p\|g\|_q\|h\|_r$$

whenever

$$1/s = 1/p + 1/q + 1/r$$

and $1 < p,q,r, \leq \infty$ and $1/2 < s \leq \infty$. Using the approach of Lacey, we are able [4] to relax the last condition to $2/5 < s$. Using a purely geometric approach [5] , we can give a different proof of this result for $1/2 - \epsilon < s < \infty$. This new approach uses subsets of integers of low complexity, such as those having a small sum set or a small difference set. By the Balog-Szemeredi-Gowers theorem, we use the version in [7], sets with small sum set are closely related to sets with small difference sets, which is a crucial ingredient in the proof.

Again, there is a related theorem for a maximal operator in ergodic theory, and a theorem by Assani [1] assures convergence almost everywhere of the corresponding ergodic averages in a dense subclass.

References

[1] I. Assani *Pointwise convergence of ergodic averages along cubes.* preprint

[2] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein.* Inst. Hautes Études Sci. Publ. Math. No. 69 (1989), 5–45.

[3] J. Bourgain, *Double recurrence and almost sure convergence.* J. Reine Angew. Math. **404** (1990), 140–161.

[4] C. Demeter, T. Tao, C. Thiele, *Maximal multilinear operators.* preprint

[5] C. Demeter, T. Tao, C. Thiele, *A trilinear maximal function estimate via arithmetic combinatorics.* preprint

[6] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions.* J. Analyse Math. **31** (1977), 204–256.

[7] W. T. Gowers, *A new proof of Szemerédi's theorem for arithmetic progressions of length four.* Geom. Funct. Anal. **8** (1998), no. 3, 529–551.

[8] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions* to appear Ann. of Math.

[9] M. T. Lacey, *The bilinear maximal functions map into L^p for $2/3 < p \leq 1$.* Ann. of Math. (2) **151** (2000), no. 1, 35–57.

[10] M. Lacey and C. Thiele, *O*n Calderón's conjecture. Ann. of Math. (2) **149** (1999), no. 2, 475–496.

[11] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression.* Acta Arith. **27** (1975), 199–245.

A generalization of the Carleson-Hunt theorem

Camil Muscalu
(joint work with Xiaochun Li)

The goal of my lecture was to describe the main result obtained jointly with Xiaochun Li in [14]. In this short abstract we will briefly present this theorem.

First of all, let us recall that the maximal Carleson operator is the sub-linear operator defined by

$$Cf(x) := \sup_{N\in\mathbb{R}} \left| \int_{\xi<N} \widehat{f}(\xi)e^{2\pi i x\xi} d\xi \right|, \tag{1}$$

where f is a Schwartz function on $\mathbb{R}$ and the Fourier transform is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} dx. \tag{2}$$

The following result of Carleson and Hunt [1], [7] is a classical theorem in Fourier analysis.

Theorem 1. *The operator C maps $L^p \to L^p$ boundedly, for every $1 < p < \infty$.*

This statement, in the particular weak type $L^2 \to L^{2,\infty}$ special case, was the main ingredient in the proof of Carleson's fameous theorem which states that the Fourier series of a function in $L^2(\mathbb{R}/\mathbb{Z})$ converges pointwise almost everywhere.

For $n \geq 1$, let now consider $m(= m(\xi))$ in $L^\infty(\mathbb{R}^n)$ a bounded function, smooth away from the origin and satisfying

$$|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}} \tag{3}$$

for sufficiently many multi-indices α. Denote by T_m the n-linear operator defined by

$$T_m(f_1, ..., f_n)(x) := \int_{\mathbb{R}^n} m(\xi)\widehat{f_1}(\xi_1)...\widehat{f_n}(\xi_n)e^{2\pi i x(\xi_1+...+\xi_n)} d\xi \tag{4}$$

where $f_1, ..., f_n$ are Schwartz functions on the real line $\mathbb{R}$. The following statement of Coifman and Meyer [2] is also a classical theorem in analysis.

Theorem 2. *T_m maps $L^{p_1} \times ... \times L^{p_n} \to L^p$ boundedly, as long as $1 < p_1, ..., p_n \leq \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_n} = \frac{1}{p}$ and $0 < p < \infty$.*

Now, for $N \in \mathbb{R}^n$ and m as before satisfying (3), denote by $\tau_N m(\xi) := m(\xi - N)$ the translated symbol and by C_m the maximal operator defined by

$$(5) \qquad C_m(f_1, ..., f_n)(x) := \sup_{N \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \tau_N m(\xi) \widehat{f_1}(\xi_1) ... \widehat{f_n}(\xi_n) e^{2\pi i x (\xi_1 + ... + \xi_n)} d\xi \right|$$

where as before $f_1, ..., f_n$ are Schwartz functions on $\mathbb{R}$.

The purpose of the paper [14] was to study the L^p boundedness properties of this Carleson type operator C_m. A simpler version of this operator appeared recently in the study of the bi-Carleson operator in [21] and [22]. The main theorem obtained in [14] is the following.

Theorem 3. *C_m maps $L^{p_1} \times ... \times L^{p_n} \to L^p$ boundedly, as long as $1 < p_1, ..., p_n \leq \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_n} = \frac{1}{p}$ and $0 < p < \infty$.*

Clearly, Theorem 3 contains both Coifman-Meyer theorem and Carleson-Hunt theorem as special cases.

References

[1] Carleson, L., *On convergence and growth of partial sums of Fourier series*, Acta Math., vol 116, 135-157, [1966].

[2] Coifman R. R., Meyer, Y., *Opérateurs multilinéaire*, Hermann, Paris, [1991].

[3] Fefferman C., *Pointwise convergence of Fourier series.*, Ann. Math., vol. 98, pp. 551-571, [1973].

[4] Gilbert J., Nahmod A., *Boundedness of bilinear operators with non-smooth symbols*, Math. Res. Lett., vol. 7, pp. 767-778, [2000].

[5] Grafakos L., Li X, *Uniform bounds for the bilinear Hilbert transform I*, Ann. of Math., vol. 159, 889-933, [2004].

[6] Grafakos L., Li X, *The disc as a bilinear multiplier*, to appear in Amer. J. Math., [2005]

[7] Hunt, R., *On the convergence of Fourier series*, Proc. Conf. Edwardsville, 235-255, [1967].

[8] Lacey M., *The bilinear maximal function maps into L^p for $2/3 < p \leq 1$*, Ann. of Math., vol 151, pp. 35-57, [2000].

[9] Lacey M., Li X., *On the Hilbert transform and $C^{1+\epsilon}$ families of lines*, preprint, [2004]

[10] Lacey M., Thiele C., *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$*, Ann. of Math., vol. 146, pp. 693-724, [1997].

[11] Lacey M., Thiele C., *On Calderon's conjecture*, Ann. of Math., vol. 149, pp. 475-496, [1999].

[12] Lacey M., Thiele C., *A proof of boundedness of the Carleson operator*, Math. Res. Lett., vol. 7, pp. 361-370, [2000].

[13] Li X, *Uniform bounds for the bilinear Hilbert transform II*, to appear in Revista Mat. Iberoamericana, [2005].

[14] Li X, Muscalu C. *Generalizations of the Carleson-Hunt theorem I. The classical singularity case*, preprint, [2005].

[15] Muscalu C. *L^p estimates for multilinear multipliers given by singular symbols.*, PhD Thesis, Brown University, [2000].

[16] Muscalu C., Tao T., Thiele C., *Multilinear operators given by singular symbols*, J. Amer. Math. Soc., vol. 15, 469-496, [2002].

[17] Muscalu C., Tao T., Thiele C., *L^p estimates for the biest I. The Walsh case*, Math. Ann., vol. 329, 401-426, [2004].

[18] Muscalu C., Tao T., Thiele C., *L^p estimates for the biest II. The Fourier case*, Math. Ann., vol. 329, 427-461, [2004].

[19] Muscalu C., Tao T., Thiele C., *A counterexample to a multilinear endpoint question of Christ and Kiselev* Math. Res. Lett., vol. 10, 237-246, [2003].
[20] Muscalu C., Tao T., Thiele C., *A Carleson type theorem for a Cantor group model of the scattering transform*, Nonlinearity vol. 19, 219-246, [2003].
[21] Muscalu C., Tao T., Thiele C., *A discrete model for the bi-Carleson operator*, GAFA, vol. 12, 1324-1364, [2002].
[22] Muscalu C., Tao T., Thiele C., *The bi-Carleson operator*, to appear in GAFA, [2005].
[23] Stein E., *Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press, [1993].
[24] Thiele C., Ph. D. Thesis, Yale University, [1995].
[25] Thiele C., *On the Bilinear Hilbert transform*, Universität Kiel, Habilitationsschrift [1998].
[26] Thiele C., *A uniform estimate*, Ann. of Math., vol. 157, pp. 1-45, [2002].

Fourier zeros of boolean functions

Mihail N. Kolountzakis
(joint work with Evangelos Markakis, Aranyak Mehta)

Let G be the discrete group $\{0,1\}^k$ and $f : G \to \{0,1\}$. Such a function is usually called a *boolean* function and such functions are often studied in the context of computational complexity [1, 2, 3, 4].

The Fourier Transform of such a function is defined by

$$f(\xi) = \frac{1}{|G|} \sum_{x \in G} (-1)^{\xi \cdot x} f(x), \quad (\xi \in G).$$

It is easy to see that $\hat{f}(0)$ equals the average value of f. In certain approaches to *learning* such functions (see, for instance, [3, 4]) it is significant to know what is the the first Fourier non-zero according to the natural *weight* $|\xi| = \sum_{k=1}^k \xi_j$, i.e. the quantity

$$m(f) = \min\{|\xi| : \ \xi \neq 0, \hat{f}(\xi) \neq 0]\}.$$

For the particular case of symmetric boolean functions f, i.e. functions satisfying $f(\pi(x)) = f(x)$ for all vectors $x \in G$ and permutations π of $\{1, \ldots, n\}$, it was known [3] that if f is not constant and not a parity function ($f(x) = x_1 + \cdots + x_k \bmod 2$ or $f(x) = x_1 + \cdots + x_k + 1 \bmod 2$) then $m(f) \leq \frac{3}{31}k$.

In this work we prove that, under the same assumptions, (f not constant and not parity) we have that

$$m(f) \leq Ck/\log k.$$

Our proof relies on a characterization [3] of functions f with a large value of $m(f)$. If f_i is the value f takes for inputs of weight equal to i, the $0-1$ sequence $f_0, f_1, \ldots, f_k$ determines f and if $m(f) = N$ then it must satisfy the recurrence relation

$$\sum_j \binom{N}{j} f_{j+\nu} = 2^N \hat{f}(0), \quad (\nu = 0, 1, \ldots, k - N - 1).$$

On this we apply repeatedly Lucas' theorem for well chosen primes and also use a probabilistic interpretation of the fact $M(f) = N$.

References

[1] A. Blum, Open problems, COLT 2003.
[2] A. Blum and P. Langley, *Selection of relevant features and examples in machine learning*, Artificial Intelligence **97** (1997), 245–271.
[3] R. Lipton and E. Markakis and A. Mehta and N. Vishnoi, *On the Fourier Spectrum of Symmetric Boolean Functions with Applications to Learning Symmetric Juntas*, IEEE Conference on Computational Complexity, 2005.
[4] E. Mossel and R. O'Donnel and R. Servedio, *Learning juntas*, STOC (2003), 206–212.

Traces of Sobolev Spaces with Variable Exponents

Lars Diening
(joint work with Peter Hästö)

From the point of boundary value problems it is important to study the trace spaces of the natural energy space. Indeed, a partial differential equation is in many cases solvable if and only if the boundary values are in the corresponding trace space. In the case of electrorheological fluids [6] the energy space is a Sobolev space with variable exponent, namely $W^{1,p(\cdot)}$. These spaces are defined as follows: For an open set $\Omega \subset \mathbb{R}^m$ let $p\colon \Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponent on Ω with $p^+ := \operatorname{esssup} p(x) < \infty$. Further, let $p^- := \operatorname{essinf} p(x)$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $f\colon \Omega \to \mathbb{R}^m$ for which the modular

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx$$

is finite. Then $\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0\colon \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \le 1\big\}$ defines a norm on $L^{p(\cdot)}(\Omega)$. The space $W^{1,p(\cdot)}(\Omega)$ is the subspace of $L^{p(\cdot)}(\Omega)$ such that $|\nabla f| \in L^{p(\cdot)}(\Omega)$. The norm $\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. For basic properties of $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ we refer to [3, 4].

We are interested in domains Ω with Lipschitz boundary but for sake of simplicity we assume that Ω is just the halfspace $\mathbb{H} = \mathbb{R}^n \times (0,\infty)$. The corresponding results for Lipschitz domains can than be achieved via flattening of the boundary by local Bi-Lipschitz mappings. We simply write $\mathbb{R}^n$ instead of $\mathbb{R}^n \times \{0\}$ for the boundary of $\mathbb{H}$. The trace space of $W^{1,p(\cdot)}(\mathbb{H})$ is naturally defined to be the quotient space of the traces of functions from $W^{1,p(\cdot)}(\mathbb{H})$, i.e.

$$\|f\|_{\operatorname{tr} W^{1,p(\cdot)}(\mathbb{H})} = \inf\big\{\|F\|_{W^{1,p(\cdot)}(\mathbb{H})}\colon F \in W^{1,p(\cdot)}(\mathbb{H}) \text{ and } \operatorname{tr} F = f\big\}.$$

Note that the trace $\operatorname{tr} F$ is well defined, since every $F \in W^{1,p(\cdot)}(\mathbb{H})$ is in $W^{1,1}_{\text{loc}}(\overline{\mathbb{H}})$.

Although this definition is very natural, it depends on the exponent p in the interior of the domain. We will show however that if p is globally log-Hölder continuous (see below) then the definition of trace space depends only on the values of p on the boundary.

We say that the exponent p is globally log-*Hölder continuous* if there exist constants $c > 0$ and $p_\infty \in (1, \infty)$ such that for all points $|x - y| < \frac{1}{2}$ and all points z holds

$$|p(x) - p(y)| \leq \frac{c}{\log(1/|x-y|)}, \qquad |p(z) - p_\infty| \leq \frac{c}{\log(e + |z|)}.$$

Let us denote by $\mathcal{P}(\Omega)$ the class of globally log-Hölder continuous variable exponents p on $\Omega \subset \mathbb{R}^m$ with $1 < p^- \leq p^+ < \infty$. This condition appears quite naturally in the context of variable exponent spaces: For example by [1, 2] we know that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$ if $p \in \mathcal{P}(\Omega)$. Global log-Hölder continuity is the best possible modulus of continuity to imply the boundedness of the maximal operator, see [1, 5]. If the maximal operator is bounded, then it possible to use the technique of mollifiers. Our result is the following:

Theorem 1. *Let $p_1, p_2 \in \mathcal{P}(\mathbb{H})$ with $p_1|_{\mathbb{R}^n} = p_2|_{\mathbb{R}^n}$. Then with equivalence of norms we have* $\mathrm{tr}W^{1,p_1(\cdot)}(\mathbb{H}) = \mathrm{tr}W^{1,p_2(\cdot)}(\mathbb{H})$.

Especially, the definition of the trace space for regular p does only depend on the values of p on the boundary. The theorem is proved by use of the following extension theorem:

Theorem 2. *Let $p \in \mathcal{P}(\mathbb{R}^{n+1})$. Then there exists a bounded, linear extension operator $\mathcal{E} : W^{1,p(\cdot)}(\mathbb{H}) \to W^{1,p(\cdot)}(\mathbb{R}^{n+1})$.*

Although the definition of the trace space above is very natural it is not so useful for deciding if a function is a $W^{1,p(\cdot)}$-trace. For this purpose it is better to have an intrinsic norm. In the classical case the trace space of $W^{1,q}(\mathbb{H})$ is given by the fractional Sobolev space $W^{1-\frac{1}{q},q}(\mathbb{R}^n)$ with intrinsic norm

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q-1}}\, dy\, dx < \infty.$$

It is our aim to generalize this result to variable p. However, the usual approach by means of the semi group of translations does not work in our case: First, translations are continuous on $L^{p(\cdot)}$ if and only if p is constant. Second, $p(x,t)$ depends on $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, so it is not possible to find a fixed space X such that $T(t) : X \to X$ for a semi group T. Therefore, our approach is rather based on mollifiers and oscillations than of translations and differences. Our result is the following:

Theorem 3. *Let $p \in \mathcal{P}(\mathbb{R}^{n+1})$. Then the function f belongs to the trace space* $\mathrm{tr}W^{1,q(\cdot)}(\mathbb{H})$ *if and only if*

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx + \int_0^1 \int_{\mathbb{R}^n} \left(\tfrac{1}{r} M^\sharp_{B^n(x,r)} f\right)^{p(x)} dx\, dr < \infty,$$

where $M^\sharp_{B^n(x,r)} f$ denotes the mean oscillation of f over the ball $B^n(x,r) \subset \mathbb{R}^n$.

References

[1] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238; **29** (2004), 247–249.

[2] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–254.

[3] X.-L. Fan and D. Zhao, *On the spaces spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424–446.

[4] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J. **41(116)** (1991), 592–618.

[5] L. Pick and M. Růžička: *An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), 369–371.

[6] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.

L^p-estimates for the wave equation associated to the Grušin operator

Ralf Meyer

Let L be a positive essential selfadjoint differential operator of second order on a smooth manifold M of dimension d. Consider the following Cauchy problem for the wave equation associated to L on M.

$$\frac{\partial^2 v}{\partial t^2} + Lv = 0, \quad v|_{t=0} = f, \quad \frac{\partial v}{\partial t}|_{t=0} = g$$

The solution to this problem is formally given by

$$v(t,x) := \cos(t\sqrt{L})f(x) + \frac{\sin(t\sqrt{L})}{\sqrt{L}}g(x), \quad (x,t) \in M \times \mathbb{R}.$$

The functions of L are defined by the spectral theorem and the above expression for v makes sense at least for $f, g \in L^2(\mathbb{R}^d)$.

The smoothness properties of the solution v for fixed time t can be measured in terms of Sobolev norms $\|f\|_{L^p_\alpha} := \|(1+L)^{\alpha/2} f\|_{L^p}$ addepted to L. Especially we are interested in estimates of the following kind

$$\|\cos(t\sqrt{L})f\|_{L^p_{-\alpha}} \leq C_{p,t}\|f\|_p \tag{1}$$

holds if $\alpha \geq \alpha(d,p)$.

$$\left\|\frac{\sin(t\sqrt{L})}{\sqrt{L}}g\right\|_{L^p_{-\alpha}} \leq C_{p,t}\|g\|_p \tag{2}$$

holds if $\alpha + 1 \geq \alpha(d,p)$.

For $L = -\Delta$ and $M = \mathbb{R}^d$ we have the usual Cauchy problem on the Euclidean space. For this case estimates have been established by S. Sjöstrand [10], A. Miyachi [6] and J. Peral [8]. J. Peral and A. Miyachi showed 1980 that the estimates 1, 2 hold with $\alpha(d,p) := (d-1)|1/p - 1/2|$.

It is well known (see J. J. Duistermaat [2]) that if the wave equation for a general L is strictly hyperbolic one can express the solutions in terms of elliptic

Fourier integral operators. A. Seeger, C. D. Sogge and E. M. Stein [9] showed that local analogues of the estimates from Miyachi and Peral hold true for a wide class of Fourier integral operators.

Let $\mathbb{H}_m$ denote the $2m+1$-dimensional Heisenberg group. The vector fields $X_j := \partial_{x_j} - \frac{1}{2}y_j\partial_u,\ Y_j := \partial_{y_j} + \frac{1}{2}x_j\partial_u,\ U := \partial_u$ form a natural basis for the Lie algebra of left-invariant vector fields. The *sub-Laplacian*

$$L_{\mathbb{H}} := -\sum_{j=1}^{m}(X_j^2 + Y_j^2)$$

is non-elliptic. Nevertheless is $L_{\mathbb{H}}$ an hypoelliptic operator, since it is an Hörmander type operator [5]. The problem in studying the associated wave equation to this operator is the lack of strict hyperbolicty and so smoothness properties of the solutions cannot be obtained by using Fourier integral operator methods. In 1999 D. Müller and E. M. Stein showed that the estimates (1), (2) hold true with $\alpha(d,p) > (d-1)|1/p - 1/2|$ and $d := 2m+1$ the Euclidean dimension of $\mathbb{H}_m$.

On $\mathbb{R}^2$ the *Grušin operator* G is defined by

$$G := -(\partial_x^2 + |x|^2\partial_u^2).$$

V. V. Grušin studied this and similar operators in 1970 [4]. As in the Heisenberg case the associated wave equation is not strictly hyperbolic, since G is degenerate-elliptic. In contrast to the sub-Laplacian $L_{\mathbb{H}}$ the Grušin operator is not translation invariant. Since G is homogenous of degree 2 with respect to the automorphic dilations $(x,u) \mapsto (rx, r^2u),\ r > 0$ we can restrict to $t = 1$. To prove the estimates (1), (2) for G and $\alpha(p,d) > |1/p - 1/2|$ it suffices to prove the case $p = 1$. In this case we have to show uniform L^1-estimates for the integral kernels of the operators $\cos(\sqrt{L})/(1+L)^{\alpha/2}$, $\sin(\sqrt{(}L))/(\sqrt{L}(1+L)^{\alpha/2})$. Instead of these operators we can also study the operator

$$\frac{\exp(i\sqrt{G})}{(1+G)^{\alpha/2}} =: m^\alpha(G)$$

with integral kernel $K_{m^\alpha(G)}$. We have to show that

$$\|K_{m^\alpha(G)}(x_1,u_1,x,u)\|_{L^1(x,u)}$$

is uniformly bounded in x_1 and u_1.

Since G is translation invariant with respect to u we only have to consider $u_1 = 0$. The speed of propagation of the wave is finite and so if x_1 is large enough the support of the wave propagator of G lies in a set where G is an elliptic Operator and the wave equation is strictly hyperbolic. In this case we can use the theorem of A. Seeger, C. D. Sogge and E. M. Stein together with some scaling arguments.

The case where x_1 is smaller than a constant C is left. Here we have the following theorem.

Theorem 1. *Let $C > 0$.*

$$\sup_{x_1 \leq C} \|K_{m^\alpha(G)}(x_1, 0, x, u)\|_{L^1(x,u)} < \infty$$

holds if $\alpha > 1/2$.

Observe that on the polarized Heisenberg group $\tilde{\mathbb{H}}_m$, which is isomorphic to the Heisenberg group $\mathbb{H}_m$ the sub-Laplacian is $\tilde{L}_{\mathbb{H}} = -\sum_{j=1}^m (\partial_{p_j}^2 + (\partial_{q_j} + p_j \partial_u)^2)$. So we can get the integral kernel of $m^\alpha(G)$ by integrating the kernel of $m^\alpha(\tilde{L}_{\mathbb{H}})$ with respect to the q variable. One can try to prove the theorem by this fact. The kernel of $m^\alpha(\tilde{L}_{\mathbb{H}})$ can be given as a infinite sum over one dimensional oscillatory integrals and so for G one ends up with two dimensional oscillatory integrals which unfortunetly turn out to be very hard to understand.

For the proof of the theorem we therefore derive a formula for the kernels that involve only one dimensional integrals. Here we use the fact that $G = (iU)(-iGU^{-1})$ with $U := \partial_u$. The operator iU is easy to understand. The operator $-iGU^{-1}$ can be written as a sum over so called spectral projection operators to rays. Here we follow an idea by R. S. Strichartz [11], who as carried out these calculations for the sub-Laplacian on the Heisenberg group.

Further information about the Grušin operator can be found in [3]. Since multiplier theorems are connected with wave equation estimates one should also mention a result from M. Cowling and A. Sikora for the sub-Laplacian on the $SU(2)$ [1].

References

[1] M. Cowling, A. Sikora, *A spectral multiplier theorem for a sub-Laplacian on $SU(2)$*, Mathematische Zeitschrift **238**, 2001, 1–36

[2] J. J. Duistermaat, *Fourier Integral Operators*, Courant Institute of Mathematical Sciences, New York University, New York, 1973

[3] P. C. Greiner, D. Holcman, Y. Kannai, *Wave kernels related to second-order operators*, Duke mathematical journal **114[2]**, 2002, 329–386

[4] V. V. Grušin, *On a class of hypoelliptic operators*, Math. USSR Sbornik **12[3]**, 1970, 458–476

[5] L. Hörmander, *Hypoelliptic second-order differential equations*, Acta Mathematica **119**, 1967, 147–171

[6] A. Miyachi, *On some estimates für the wave equation in L^p and H^p*, Journal of the faculty of science, University of Tokyo **27**, 1980, 331–354

[7] D. Müller, E. M. Stein, *L^p-estimates for the wave equation on the Heisenberg group*, Revista Matematica Iberoamericana **15[2]**, 1999, 297–334

[8] J. Peral, *L^p estimates for the wave equation*, Journal of functional analysis **36**, 1980, 114–145

[9] A. Seeger, C. Sogge, E. M. Stein, *Regularity properties of Fourier integral operators*, Annals of Mathematics **134**, 1991, 231–251

[10] S. Sjöstrand, *On the Riesz means of the solutions of the Schrödinger equation*, Ann. Scuola Norm. Sup. Pisa **24**, 1970, 331–348

[11] R. S. Strichartz, *L^p Harmonic Analysis and Radon Transforms on the Heisenberg Group*, Journal of functional analysis **96**, 1991, 350–406

Analytic Embedding in the unit ball of $\mathbb{C}^n$

Stefanie Petermichl
(joint work with Sergei Treil, Brett Wick)

The Carleson Embedding Theorem is an important result in function theory on the unit disk. This result says that:

Theorem Let μ be a non-negative measure in $\mathbb{D}$. Then the following are equivalent:

(i) The embedding operator

$$\mathcal{J} : L^2(\mathbb{T}) \to L^2(\mathbb{D}, \mu),$$

with $\mathcal{J}(f)(z) = f(z)$ denoting the harmonic extension, is bounded.

(ii)

$$C(\mu)^2 := \sup_{z\in\mathbb{D}} \|\mathcal{J} k_z\|^2_{L^2(\mu)} = \sup_{z\in\mathbb{D}} \|\mathcal{P}_z\|_{L^1(\mu)} < \infty,$$

where $k_z(\xi) = \frac{(1-|z|^2)^{1/2}}{1-\xi\overline{z}}$, the reproducing kernel for the Hardy space $H^2(\mathbb{D})$.

(iii)

$$I(\mu) := \sup\left\{\frac{1}{r}\mu(\mathbb{D}\cap Q(\xi, r)) : r > 0, \xi \in \mathbb{T}\right\} < \infty,$$

where $Q(\xi, r) := \{z \in \mathbb{C} : |z - \xi| < r\}$.

Moreover, the following expressions are equivalent $C(\mu) \approx I(\mu) \approx \|\mathcal{J}\|$.

Property (iii) is typically taken as the definition of a Carleson measure on $\mathbb{D}$.

It will be more convenient for us to work with the second definition of a Carleson measure, i.e., μ is a Carleson measure if and only if

$$\sup_{\lambda\in\mathbb{D}} \int_{\mathbb{D}} |k_\lambda(z)|^2 d\mu(z) < +\infty,$$

or equivalently

$$\sup_{\lambda\in\mathbb{D}} \int_{\mathbb{D}} \mathcal{P}_\lambda(z) d\mu(z) < +\infty.$$

with the obvious change when considering the ball $\mathbb{B}$. Here $k_\lambda(z)$ is the (normalized) reproducing kernel for the Hilbert space $H^2(\mathbb{S})$ and $\mathcal{P}_\lambda(z)$ is the Poisson kernel. We also want to emphasize that it is sufficient to only check this supremum over the points that are in the supp μ.

We present a simple proof of the following theorem:

Theorem If either of the following conditions are satisfied,

1. $\sup_{\lambda\in\mathbb{D}} \int_{\mathbb{D}} |k_\lambda(z)|^2 d\mu(z) \le C$ or
2. $\sup_{\lambda\in\text{supp }\mu} \int_{\mathbb{D}} |k_\lambda(z)|^2 d\mu(z) \le C'$,

then

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq 2eC' \|f\|^2_{H^2(\mathbb{D})},$$

for all analytic f.

It is clear the second condition on the measure μ implies the first on μ.

We also present a simple proof of the n-dimensional analogue of the theorem on the disk.

Theorem If either of the following conditions are satisfied,

1. $\sup_{\lambda\in\mathbb{B}} \int_{\mathbb{B}} |k_\lambda(z)|^2 d\mu(z) \leq C$ or
2. $\sup_{\lambda\in \text{supp } \mu} \int_{\mathbb{B}} |k_\lambda(z)|^2 d\mu(z) \leq C'$,

then

$$\int_{\mathbb{B}} |f(z)|^2 d\mu(z) \leq e\frac{(2n)!}{(n!)^2} C' \|f\|^2_{H^2(\mathbb{B})},$$

for all analytic f.

The outline of the proof for the theorem on the disk is as follows, (the outline for the ball is almost identical):
Note that the function

$$\varphi(\lambda) = -\int_{\mathbb{D}} \mathcal{P}_\lambda(z) d\mu(z)$$

is bounded if and only if the measure μ is Carleson. But, in fact more can be said, this function is sub-harmonic, i.e.

$$\Delta\varphi(\lambda) \geq 0.$$

We will use it to construct a smoother measure

$$d\nu := \Delta_\lambda \varphi(\lambda) \ln \frac{1}{|\lambda} |dA(\lambda)$$

whose Carleson norm is easy to bound:

$$\|\nu\|_{\mathcal{C}} \leq e\|\varphi\|_\infty = e\|\mu\|_{\mathcal{C}}.$$

We will then proceed to prove the inequalities as follows

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq c \int_{\mathbb{D}} |f(\lambda)|^2 d\nu(\lambda) \leq e\|\varphi\|_\infty \|f\|_2^2 = e\|\mu\|_{\mathcal{C}} \|f\|_2^2$$

Thus, the discussion in this paper provides a new simple proof of part of the analytic Carleson Embedding Theorem for the unit disk in $\mathbb{C}$ and obtains the best known constant. The previous best constant was obtained by N. Nikolskii using the Schur test to estimate the norm of the embedding operator.

With this new proof of the Carleson Embedding Theorem, it is also possible to obtain better estimates in the interpolation of analytic $H^2(\mathbb{D})$ functions. Some of the previous estimates for the norm of the interpolation operators depended upon the estimates obtained by Nikolskii for the norm of the embedding operator. One can consult the texts [4] or [5] for the connections between the Carleson Embedding Theorem and the solution to the interpolation problem for analytic functions.

The proof on the disk also indicates how one can prove an analogous theorem on the unit ball in $\mathbb{C}^n$. One needs to replace the Laplacian with the invariant Laplacian and the Poisson kernel with the invariant Poisson kernel.

References

[1] A. B. Aleksandrov, "Function Theory in the Unit Ball", Several Complex Variables II, Springer-Verlag, Berlin, 1991.

[2] J. Garnett, "Bounded Analytic Functions", Academic Press, New York, 1981.

[3] S. Krantz, "Several Complex Variables", AMS Chelsea Publishing, Providence, 1992.

[4] N. Nikolskii, "Operators, Functions and Systems, I" AMS Mathematical Surveys and Monographs, Vol. 92, AMS, Providence, 2002.

[5] N. Nikolskii, "Treatise on the Shift Operator", Springer-Verlag, New York, 1986.

[6] W. Rudin, "Function Theory in the Unit Ball of $\mathbb{C}^n$", Springer-Verlag, New York, 1980.

[7] M. Stoll, "Invariant Potential Theory in the Unit Ball of $\mathbb{C}^n$", London Mathematical Society Lecture Note Series **199**, Cambridge University Press, Cambridge, 1994.

[8] K. Stroethoff, Compact Toeplitz Operators on Bergman Space, *Math. Proc. Camb. Phil. Soc.*, **124** (1998), pp. 151-160.

[9] D. Zheng, Toeplitz Operators and Hankel Operators on the Hardy Space of the Unit Sphere, *Journal of Func. Analysis*, **149** (1997), pp. 1-24.

Generalized Gaussian Estimates and Applications to Calderon-Zygmund Theory

Peer Christian Kunstmann

We present a survey on results that use generalized Gaussian estimates of the following form

$$\|1_{B(x,t^{\frac{1}{2m}})}e^{-tA}1_{B(y,t^{\frac{1}{2m}})}\|_{p\to q} \le C\,t^{-\frac{n}{2m}(\frac{1}{p}-\frac{1}{q})}e^{-b(\frac{|x-y|^{2m}}{t})^{\frac{1}{2m-1}}} \tag{1}$$

where $(e^{-tA})_{t>0}$ is an analytic semigroup in $L^2(\mathbb{R}^n)$ and $1\le p<q\le\infty$. In particular, we address boundedness of H^∞–calculi and boundedness of Riesz transforms. Most of the talk is based on joint work with Sönke Blunck. For $p=1$ and $q=\infty$, (1) reduces to a usual Gaussian estimate of order $2m$: the operators e^{-tA} have kernels $k(t,x,y)$ that satisfy

$$|k(t,x,y)| \le C\,t^{-\frac{n}{2m}(\frac{1}{p}-\frac{1}{q})}e^{-b(\frac{|x-y|^{2m}}{t})^{\frac{1}{2m-1}}}. \tag{2}$$

The use of (2) (and more general Poisson type estimates) in establishing weak type $(1,1)$ results has its origins in [7, 6]. There are several classes of operators A on $\mathbb{R}^n$ for which (2) fails in general but (1) still holds for suitable p and q.

Example 1. Schrödinger operators $A=-\Delta-V$ where $V\ge 0$ belongs to a pseudo-Kato class and p and $q=p'$ depend on the form bounds of the potential (cf. [10]), here $m=1$.

Example 2. Uniformly elliptic operators of order $2m$ in divergence form with bounded measurable coefficients (cf. [5]). In case $2m<n$ we have $p=\frac{2n}{n+2m}$, $q=p'=\frac{2n}{n-2m}$, and those values are known to be optimal for $n\ge 5$.

Example 3. Uniformly elliptic second order operators with singular lower order terms (cf. [9] and the literature cited there). Here $m = 1$ and p and q depend, e.g., on certain form bounds for the lower order terms.

The first application of (1) in the context of singular integrals is in [2]. At the center of our talk is the following weak type (p, p) result, the first one ever in Calderon-Zygmund theory which is not restricted to $p = 1$.

In (4) and (5), B denotes an arbitrary closed ball with radius r, and kB denotes the ball with same center as B but radius kr.

Theorem 1 (cf. [3]). *Let (Ω, d, μ) a space of homogeneous type and dimension n. Let $1 \leq p < 2 < q \leq \infty$ and let $T : L^2(\Omega) \to L^2(\Omega)$ be bounded. Suppose that there is an approximation of the identity $(S_r)_{r>0}$ satisfying*

$$\|S_r f\|_2 \leq C\|f\|_2 \quad (r > 0) \tag{3}$$

$$\|1_{(k+1)B\setminus kB} S_r 1_B\|_{p\to q} \leq \mu(B)^{\frac{1}{q}-\frac{1}{p}} g_r(k) \quad k = 0, 1, 2, \ldots \tag{4}$$

$$\|1_{(k+1)B\setminus kB} T(I - S_r)1_B\|_{p\to 2} \leq \mu(B)^{\frac{1}{2}-\frac{1}{p}} g_r(k) \quad k = 4, 5, \ldots \tag{5}$$

where the g_r are functions satisfying $\sup_{r>0} \sum_k k^{n-1} g_r(k) < \infty$. Then T is of weak type (p, p).

Observe that (4) is an on– and off–diagonal estimate whereas (5), which corresponds to the weakened form of the Hörmander condition given in [6], is an off–diagonal estimate.

Sketch of Proof.

One first observes that (4) and (5) imply the pointwise estimates

$$N_{p',r}(S_r^* f) \leq CM_{q'} f \quad \text{and} \quad N_{p',r}((T(I - S_r))^* f) \leq CM_2 f, \tag{6}$$

where $N_{s,r}g(x) := \mu(B(x, r))^{-\frac{1}{s}} \|g1_{B(x,r)}\|_s$ and $M_s g(x) := \sup_{r>0} N_{s,r}g(x) = (M(|g|^s)(x))^{1/s}$. Then let $f \in L^p$ and consider an L^p-Calderon-Zygmund decomposition at height $\alpha > 0$: $f = g + \sum_j b_j$ with b_j supported in balls $B_j = B(x_j, r_j)$ with the finite intersection property. We have

$$Tf = Tg + \sum_j TS_{r_j} b_j + \sum_j T(I - S_{r_j})b_j = Tg + h_1 + h_2$$

where Tg is treated as usual. If we show

$$\|\sum_j S_{r_j} b_j\|_2^2 \leq C\alpha^2 \|\sum_j 1_{B_j}\|_2^2, \tag{7}$$

then, by L^2-boundedness of T and properties of the decomposition,

$$\mu(\{x : |h_1(x)| > \alpha\}) \leq C\alpha^{-2}\|\sum_j S_{r_j} b_j\|_2^2 \leq C \sum_j \mu(B_j) \leq C\alpha^{-p}\|f\|_p^p.$$

For the proof of (7) we take $\phi \in L^2$, $\|\phi\|_2 = 1$, and write (essentially)

$$
\begin{aligned}
|\langle \phi, S_{r_j} b_j \rangle| &= |\langle S_{r_j}^* \phi, b_j \rangle| \le C N_{p', r_j}(S_{r_j}^* \phi)(x_j) \alpha \mu(B_j) \\
&\le C\alpha \int_{B_j} N_{p', r_j}(S_{r_j}^* \phi) \le C\alpha \int_{B_j} M_{q'} \phi.
\end{aligned}
$$

Thus, by boundedness of $M_{q'}$ on L^2 and finite intersection,

$$
|\langle \phi, \sum_j S_{r_j} b_j \rangle| \le C\alpha \int (M_{q'}\phi)(\sum_j 1_{B_j}) \le C\alpha \| \sum_j 1_{B_j} \|_2, \tag{8}
$$

and (7) follows. The estimation for h_2 is similar. □

Pascal Auscher [1] showed that, using Kolmogorov's inequality in the last step of (8), the assertion of Theorem 1 remains true for $q = 2$. We thus obtain the following version of a result in [3].

Theorem 2. *Let A satisfy (1) where $1 \le p \le 2 \le q \le \infty$ and assume that A has a bounded H^∞–calculus in $L^2(\mathbb{R}^n)$. Then A has a bounded H^∞–calculus in $L^r(\mathbb{R}^n)$ for any $r \in (p, q)$.*

This generalizes the case $p = 1$, $q = \infty$ proved in [7]. In [3], Theorem 1 was applied to prove the following result on Riesz transform type operators.

Theorem 3. *Let A satisfy (1) where $1 \le p < 2 \le q \le \infty$. Let $\alpha \in (0, 1)$ and suppose that $B : D(A) \to L^2(\mathbb{R}^n)$ is a linear operator such that (1) holds for $t^\alpha B e^{-e^{\pm i\theta} tA}$ in place of e^{-tA} where $\theta \in (0, \pi/2)$. If $BA^{-\alpha}$ is bounded on $L^2(\mathbb{R}^n)$ then $BA^{-\alpha}$ is of weak type (p, p).*

This generalizes the case $p = 1$, $q = \infty$ proved in [6]. At the end we indicate how (1) may be obtained for elliptic non-divergence operators A in with bounded measurable coefficients of the form

$$
Au(x) = \sum_{|\alpha| \le 2m} a_\alpha(x) \partial^\alpha u(x). \tag{9}
$$

Also for this class of operators it is known that (2) fails in general. Assume that A is of the form (9) and that, for some $p_0 \in (1, \infty)$, $-A$ with domain $W^{2m}_{p_0}$ generates an analytic semigroup in $L^{p_0}(\mathbb{R}^n)$. Denote by p_+ the supremum over all $r \in (1, \infty)$ such that $-A_r$ with domain W^{2m}_r generates an analytic semigroup in L^r, and let $q_+ := \infty$ if $n \le 2mp_+$, $q_+ := \frac{np_+}{n - 2mp_+}$ otherwise. Denote by p_- the infimum over all $r \in (1, \infty)$ such that the semigroup e^{-tA} (originally defined on L^{p_0}) acts boundedly in L^r. For this situation, the following is proved in [8].

Theorem 4. *The operator A satisfies (1) for all $p_- < p < q < q_+$. Moreover, $-A$ with domain W^{2m}_p generates an analytic semigroup in $L^p(\mathbb{R}^n)$ for any $p \in (p_-, p_+)$.*

This is based on perturbation arguments and an extension of Theorem 3 to the case $\alpha = 1$. Similar results hold, e.g., for Dirichlet systems on bounded domains $\Omega \subseteq \mathbb{R}^n$ (cf. [8]).

References

[1] P. Auscher, *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to ellipic operators on $\mathbb{R}^n$ and related estimates*, Preprint 2004.

[2] S. Blunck, P.C. Kunstmann, *Weighted norm estimates and maximal regularity*, Adv. Differential Equations **7**, no. 12, 1513-1532 (2002).

[3] S. Blunck, P.C. Kunstmann, *Calderón-Zygmund theory for non-integral operators and the H^∞-functional calculus*, Rev. Mat. Iberoam. **19**, no. 3, 919–942 (2003).

[4] S. Blunck, P.C. Kunstmann, *Weak type (p,p) estimates for Riesz transforms*, Math. Z. **247**, 137–148 (2004).

[5] E.B. Davies, *Uniformly elliptic operators with measurable coefficients*, J. Funct. Anal. **132**, 141–169 (1995).

[6] X.T. Duong, A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Rev. Mat. Iberoamericana **15**, no. 2, 233–265 (1999).

[7] X.T. Duong, D.W. Robinson, *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*, J. Funct. Anal. **142**, 89–128 (1996).

[8] P.C. Kunstmann, *On maximal regularity of type L^p-L^q for non-divergence form operators under minimal assumptions*, in preparation.

[9] V. Liskevich, Z. Sobol, H. Vogt, *On L_p-theory of C_0-semigroups associated with second order elliptic operators II*, J. Funct. Anal. **193**, 55–76 (2002).

[10] G. Schreieck, J. Voigt, *Stability of the L_p-spectrum of generalized Schrödinger operators with form small negative part of the potential*, in "Functional Analysis" Proc. Essen 1991, Bierstedt et al. (eds.) Marcel Dekker 1994, 95–105.

Blowup rates of eigenfunctions and quasimodes

Christopher Sogge

(joint work with John Toth and Steve Zelditch)

We consider boundaryless Riemannian manifolds (M,g) of dimension $n \geq 2$. To understand analysis in this setting important objects are:

- Geodesic flow (classical system)
- The behavior of eigenfunctions at high energies (quantum system).

Recall that if $\Delta = \Delta_g$ is the associated Laplace-Beltrami operator the spectrum is discrete, $-\Delta e_j(x) = \lambda_j^2 e_j(x)$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. We assume, in what follows, that the eigenfunctions are L^2 normalized.

One expects that the "geometry" (M,g) should be encoded in both of these objects and our goal is to understand to what extent the correspondence principle is true. This says that long term properties of geodesic flow should influence the behavior of eigenfunctions at high energies, and that the behavior of high energy modes should dictate certain properties of geodesics.

One issue is to see what sort of geometries allow and preclude blowup of L^p-norms of eigenfunctions. Recall that one has the sharp estimates [So1], [So2]

$$\|\chi_\lambda f\|_p \leq C\lambda^{\max\{n(1/2-1/p)-1/2,\, \frac{n-1}{2}(1/2-1/p)\}}\|f\|_p, \quad p \geq 2. \tag{1}$$

if $\chi_\lambda f = \sum_{|\lambda-\lambda_j|\le 1} e_j(f)$, with $e_j(f)$ begin the projection of f onto the eigenspace of Δ with eigenvalue λ_j^2. The L^p-estimates for finite p were obtained in [So2], while the L^∞ bounds are much older and go back to Avakumovič [Av] and Levitan [Le].

Since χ_λ reproduces eigenfunctions $e_j(x)$ when $|\lambda-\lambda_j|\le 1$, (1) of course yields

$$\|e_j\|_p \le C\lambda^{\max\{n(1/2-1/p)-1/2,\, \frac{1}{2}(1/2-1/p)\}}. \tag{2}$$

While these estimates are sharp for the standard sphere [So1], unlike the bounds in (1), one does not expect (2) to be sharp in general.

Note that for $p \ge 2$, $n(1/2-1/p)-1/2 > \frac{n-1}{2}(1/2-1/p)$ if and only if $p > 2(n+1)/(n-1)$, while the two exponents agree for this value of p. Therefore, to study this problem there are three cases to consider:

- $p > 2(n+1)/(n-1)$
- $2 < p < 2(n+1)/(n-1)$
- $p = 2(n+1)/(n-1)$

Presumably, in order to get improved bounds for the second range, one would have to assume that there are no stable closed geodesics. For the first range (cf. [DG]) one would expect that one would have to assume that through any point there can only be a set of measure zero of closed geodesics. The 3rd case seems very difficult, and, in order to improve (2) for this exponent, one would expect that both of these conditions are necessary.

Zelditch and the author were able to prove results for the first case in [SZ]. To describe these results, we need to make a couple of definitions. First, given $x \in M$ we let Λ_x be the set of unit directions $\xi \in S_x^*M$ for which the geodesic passing through x returns to x in finite time. We say that x is a recurrent point for geodesic flow if $|\Lambda_x| > 0$.

A main result in [SZ] then is that if x is not a recurrent point, then $|e_j(x)| = o(\lambda_j^{(n-1)/2})$, which is an improvement over the $O(\lambda_j^{(n-1)/2})$ sup-norm estimates in (2). It was also shown that M has no recurrent points then $\|e_j\|_\infty = o(\lambda_j^{(n-1)/2})$. By interpolation with the $L^{2(n+1)/(n-1)}$-estimate in (2), one also gets that $\|e_j\|_p = o(\lambda_j^{n(1/2-1/p)-1/2})$, $p > 2(n+1)/(n-1)$.

In [SZ] this result was used to show that for a generic class of metrics on any manifold M one has $\|e_j\|_2 = o(\lambda_j^{(n-1)/2})$. Zelditch and the author also were able to classify real analytic manifolds with the maximum blowup rate, $\|e_j\|_\infty = \Omega(\lambda_j^{(n-1)/2})$.

One could ask whether one always has the maximum blowup rate $e_j(x) = \Omega(\lambda_j^{(n-1)/2})$ at recurrent points. This turns out to be false. It was shown in [SZ] that there are manifolds M with recurrent points x so that $e_j(x) = o(\lambda_j^{(n-1)/2})$.

Thus, in order to get converse results, one must consider other types of functions. One natural generalization of eigenfunctions, of course, is quasimodes.

It was shown in [STZ] that the $o(\lambda_j^{(n-1)/2})$ bounds for non-recurrent points and the $o(\lambda_j^{(n-1)/2})$ sup-norm bounds when there are no recurrent points also hold when $e_j(x)$ is replaced by a quasimode. There are several definitions of quasimodes in the literature, but the most natural one, (which works for our results) seems to be the following. We say that $\{\psi_{j_k}\}$ is a sequence of quasimodes corresponding to the eigenvalues λ_{j_k} if $\|\psi_{j_k}\|_2 = 1$ and for dimensions $n = 2, 3$

$$\|(\Delta + \lambda_{j_k}^2)\psi_{j_k}\|_2 \leq C,$$

for some uniform constant C. In higher dimensions, one also has to assume (because of the Sobolev embedding theorem) that $\|\Pi_{\lambda_{j_k}}\psi_{j_k}\|_\infty \leq C$, if $\Pi_{\lambda_{j_k}}$ is the projection onto frequencies $[2\lambda_{j_k}, \infty)$ for $\sqrt{-\Delta}$. Note that when $n = 2$ or 3 $\|Pi_{\lambda_{j_k}} f\|_\infty \leq C\|(\Delta + \lambda_{j_k}^2)f\|2$, so, in this case, the 2nd condition is superfluous.

We also have results in the other direction. For instance we are able to obtain results for certain types of blowdown points for geodesic flow. Recall that a point $x \in M$ is a blowdown point if $\Lambda_x = S_x^*M$ and if there is a time ℓ so that every geodesic starting at x returns to x in time ℓ. This gives rise to a map $G_x^\ell : S_x^*M \to S_x^*M$ where, if ξ is the initial direction of the geodesic, $\eta = G_x^\ell(\xi)$ is the direction of the resulting geodesic at time ℓ. In the simplest case where G_x^ℓ is the identity map, Toth, Zelditch and the author show that there is a sequence of quasimodes with $\|\psi_{\lambda_{j_k}}\|_\infty = \Omega(\lambda_{j_k}^{(n-1)/2})$, which is the maximum blowup rate. Other results of this type hold for manifolds like the triaxial ellipsoid that have a finite number of fixed point for G_x^ℓ.

References

[A] V.I. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics **60**, Springer-Verlag, New York, (1998).

[Av] V. I. Avakumovič, *Über die Eigenfunktionen auf geshchlossenen Riemannschen Mannigfaltigkeiten*, Math. Z. **65** (1956), 327–344.

[Besse] A. L. Besse, *Manifolds all of whose geodesics are closed. With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas] **93**, Springer-Verlag, Berlin-New York, 1978.

[B] J. Bourgain, *Eigenfunction bounds for compact manifolds with integrable geodesic flow*, IHES preprint.

[DG] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.

[Ho III] L. Hörmander, The Analysis of Linear Partial Differential Operators, Volumes III, IV, Springer-Verlag Berlin Heidelberg, 1983.

[Iv1] V. Ivriĭ, *The second term of the spectral asymptotics for a Laplace Beltrami operator on manifolds with boundary (Russian)*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34.

[Le] B. M. Levitan, *On the asymptoptic behavior of the spectral function of a self-adjoint differential equation of second order*, Isv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 325–352.

[Saf] Y. Safarov, Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition. (Russian) Funktsional. Anal. i Prilozhen. **22** (1988), no. 3,53–65, 96 translation in Funct. Anal. Appl. 22 (1988), no. 3, 213–223.

[So1] C. D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), 43–65.

[So2] C. D. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–134.

[So3] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Univ. Press, Cambridge, 1993.
[SZ] C. D. Sogge and S. Zelditch, *Riemannian manifolds with maximal eigenfunction growth*, Duke Math. J. **114** (2002), no. 3, 387–437.
[STZ] C. D. Sogge, J. Toth and S. Zelditch, to appear.
[T1] J.A.Toth, *Eigenfunction localization in the quantized rigid body*, J.Diff.Geom. **43** (1996), no. 4, 844–858.
[T2] J.A.Toth, *On the quantum expected values of integrable metric forms*, J.Diff.Geom. **52** (1999), no. 2, 327–374.
[TZ] J. A. Toth and S. Zelditch, *Riemannian manifolds with uniformly bounded eigenfunctions*, Duke Math. J. **111** (2002), no. 1, 97–132.
[TZ2] J. A. Toth and S. Zelditch, Norms of modes and quasi-modes revisited. *Harmonic analysis at Mount Holyoke* (South Hadley, MA, 2001), 435–458, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003.
[TZ3] J. A. Toth and S. Zelditch, *L^p-norms of eigenfunctions in the completely integrable case*, Annales Henri Poincaré 4 (2003), 343–368.
[Ta] M. Taylor, *Eigenfunction expansions and the Pinsky phenomenon on compact manifolds*, J. Fourier Anal. Appl. **7** (2001), no. 5, 507–522.
[Z] S. Zelditch, *Kuznecov sum formulae and Szego limit formulae on manifolds*, Comm. PDE **17** (1992), 221–260.
[Z2] S. Zelditch, *The inverse spectral problem for surfaces of revolution*, JDG **49** (1998), 207–264.
[Z3] S. Zelditch, *On the generic spectrum of a Riemannian cover*, Annales de l'Institut Fourier **40** (1990), 407–42.
[Z4] S. Zelditch, *Fine structure of Zoll spectra*, J. Funct. Anal. **143** (1997), 415–460.
[Z5] S. Zelditch, *Wave invariants for non-degenerate closed geodesics*, GAFA **8** (1998), 179–217.

Dispersive estimates and absence of positive eigenvalues

H. Koch
(joint work with D. Tataru)

Let W be a potential which decays at infinity. Then the Schrödinger operator

$$-\Delta_{\mathbb{R}^n} - W$$

has continuous spectrum $[0,\infty)$. In addition its spectrum may contain eigenvalues which could be positive, negative of zero. It is well known that under weak assumptions like

$$\lim_{|x|\to\infty} |x||W(x)| = 0 \tag{1}$$

there are no positive eigenvalues. The argument is based on Carleman type estimates. In this work we derive dispersive Carleman inequalities and use them to prove nonexistence of positive eigenvalues under L^p conditions on the potential.

More precisely we study

$$(-\Delta - V)u = Wu \tag{2}$$

assuming

Assumption 1, the long range potential. The following inequalities hold.

$$|V| + |x||DV| + |x|^2|D^2V| \leq c, \tag{3}$$

$$\liminf_{|x|\to\infty} V > 0, \tag{4}$$

and

$$\tau_0 := -\liminf_{|x|\to\infty} \frac{x\cdot\nabla V}{4V} < 1/2. \tag{5}$$

Assumption 2, the short range potential. The potential W can be decomposed as $W = W_1 + W_2$ where

$$W_1 \in L^{n/2} + L^{(n+1)/2} \tag{6}$$

$$\limsup_{|x|\to\infty} |x||W_2(x)| < \delta. \tag{7}$$

Theorem Assume that V and W satisfy Assumptions 1 and 2, let $\tau_1 > \tau_0$ and assume that δ is sufficiently small. Let $u \in H^1_{loc}(\mathbb{R}^n)$ satisfy (2) and $(1+|x|^2)^{\tau_1-\frac{1}{2}}u \in L^2$. Then $u \equiv 0$.

This extends previous results by Ionescu and Jerison [1]. The exponents in (6) are sharp: below $n/2$ there are compactly supported eigenfunctions (see [2]) and above $(n+1)/2$ there is a very different counter example by Ionescu and Jerision [1].

Given a measurable function f and the Sobolev space $W^{s,q}$ we define the norm

$$\|f\|_{l^pW^{s,q}} = \left(\sum_{j=1}^{\infty} \|f\|^p_{W^{s,q}(\{2^{j-1}\leq|x|\leq 2^{j+1}\})}\right)^{1/p}$$

with the obvious modification for $p=\infty$. The key estimate of the proof of the Theorem is contained in the following inequality for $\tau >> 1$ and $\varepsilon << 1$.

$$\begin{aligned} &\|e^{h(\ln(|x|))}v\|_{l^2W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} + \|e^{h(\ln(|x|))}\rho v\|_{L^2} \leq \\ &\qquad c\inf_{f_1+f_2=(-\Delta-V)v} \|e^{h(\ln(|x|))}\rho^{-1}f_1\|_{L^2} + \|e^{h(\ln(|x|))}f_2\|_{l^2W^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}} \end{aligned} \tag{8}$$

where ρ is given by

$$\rho = \left(\frac{h'(\ln(|x|))}{|x|^2} + \frac{h'(\ln(|x|))^2h''_+(\ln(|x|))}{|x|^4}\right)^{\frac{1}{4}} \tag{9}$$

and where h satisfies

$$h'_\epsilon(t) = \tau_1 + (\tau e^{\frac{t}{2}} - \tau_1)\frac{\tau^2}{\tau^2+\epsilon e^t}. \tag{10}$$

This inequality contains two parts: An L^2 part, which follows by several pointwise estimates and integration by parts. This part depends on convexity properties of the weight. Then there is an L^p part, which essentially is an application of the dispersive estimates of [3]. Both parts can be generalized to suitable operators with variable coefficients and gradient potentials.

References

[1] A. D. Ionescu and D. Jerison, *On the absence of positive eigenvalues of Schrödinger operators with rough potentials*, Geom. Funct. Anal., **15** (2003),1029–1081.
[2] H. Koch and D. Tataru *Sharp counterexamples in unique continuation for second order elliptic equations*, J. Reine Angew. Math.**542**, (2002), 133–146.
[3] H. Koch and D. Tataru *Dispersive estimates for principally normal pseudodifferential operators*, Comm. Pure Appl. Math. **58**(2):217–284, 2005.

On the Mixed Boundary Value Problem for Laplace's Equation in Planar Lipschitz Domains

Loredana Lanzani
(joint work with R. Brown, L. Capogna)

1. Introduction

We let Ω denote a Lipschitz graph domain in the plane, that is

$$\Omega = \{(x_1, x_2) : x_2 > \phi(x_1)\}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with $\phi(0) = 0$, and consider the following simple partition of the boundary, namely $\partial\Omega = N \cup D$, where we have defined:

$$D := \{(x_1, \phi(x_1)) : x_1 < 0\}, \qquad N := \{(x_1, \phi(x_1)) : x_1 \geq 0\}. \tag{1.1}$$

By the *mixed problem for $L^p(d\sigma)$*, we mean the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(\partial\Omega, d\sigma) & \end{cases} \tag{1.2}$$

where we assume that the Neumann data f_N is in $L^p(N, d\sigma)$, and the Dirichlet data has one derivative in L^p that is, $df_D/d\sigma \in L^p(D, d\sigma)$. Here, $(\nabla u)^*$ is the non-tangential maximal function of the gradient and $d\sigma$ denotes arc-length.

Even though, as it should be expected, the mixed problem has deep connections with the *Neumann problem*:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = F_N, & \text{on } \partial\Omega \\ (\nabla u)^* \in L^p(\partial\Omega, d\sigma) & \end{cases} \tag{1.3}$$

and the *regularity problem*:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = F_D, & \text{on } \partial\Omega \\ (\nabla u)^* \in L^p(\partial\Omega,\, d\sigma) & \end{cases} \tag{1.4}$$

it is well known that the mixed problem is far from being a mere superimposition of (1.3) and (1.4). Indeed, whereas (1.3) and (1.4) are known to be solvable in $L^p(d\sigma)$ for $1 < p < 2 + \epsilon(M)$ with $\epsilon(M) > 0$, see Jerison and Kenig [5], Dahlberg and Kenig [4] and Verchota [7] (here, M denotes the Lipschitz constant of the domain, that is $M = ||\phi'||_{L^\infty(\mathbb{R})}$), simple examples show that the Mixed problem is, in general, not solvable in $L^2(d\sigma)$, not even in the smooth domain category, see [3].

The mixed problem has many applications to Physics, see Sneddon [6], and its study in PDEs has a rich history, see Azzam and Kreyszig [1] and Brown, Capogna and Lanzani [3] for a review of some of the most relevant results. Here we recall the following theorem of Brown [2], which is most directly related to our purposes.

Theorem 1. (Brown) *Suppose $\Omega \subset \mathbb{R}^2$ is a Lipschitz graph domain satisfying a crease condition, that is $\Omega = \{(x_1, x_2) : x_2 > \phi(x_1)\}$ and, for D and N as in (1.1), there exist two constants $\delta_D \geq 0$, $\delta_N \geq 0$ such that*

$$\delta_D + \delta_N > 0\,; \quad \phi'(x_1) \geq \delta_N \ a.e.\ N; \quad \phi'(x_1) \leq -\delta_D \ a.e.\ D. \tag{1.5}$$

Then, the mixed problem (1.2) is uniquely solvable in $L^2(d\sigma)$ and, moreover, we have

$$\|(\nabla u)^*\|_{L^2(\partial\Omega, d\sigma)} \leq C(M, \delta_D, \delta_N) \left(\|f_N\|_{L^2(N,d\sigma)} + \left\| \frac{df_D}{d\sigma} \right\|_{L^2(D,d\sigma)} \right), \tag{1.6}$$

where M denotes the Lipschitz constant of the domain.

As is easily seen, the crease condition imposes: (1) a positive lower bound on the size of the Lipschitz constant (in particular, smooth domains do not satisfy the crease condition) and (2) geometric convexity of the domain (e.g. a concave wedge does not satisfy the crease condition and in fact the mixed problem for a concave wedge with N and D as in (1.1) is not solvable in $L^2(d\sigma)$, see Brown [2])). Our goal is to remove the crease condition. To this end, we have the following partial result [3]

Theorem 2. (Brown, Capogna, Lanzani) *Let Ω be a Lipschitz graph domain with N and D as in (1.1). Assume Ω has Lipschitz constant M less than 1.*

Then, there exists a constant $p_0 = p_0(M)$, $1 < p_0 < 2$, so that for $1 < p < p_0$, if $f_N \in L^p(N, d\sigma)$ and $df_D/d\sigma \in L^p(D, d\sigma)$, the mixed problem (1.2) for $L^p(d\sigma)$ has a unique solution. The solution satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega, d\sigma)} \leq C(p, M) \left(\|f_N\|_{L^p(N,d\sigma)} + \left\| \frac{df_D}{d\sigma} \right\|_{L^p(D,d\sigma)} \right). \tag{1.7}$$

2. Comments on proofs

The proof of Theorem 2 is obtained by interpolating a *weighted* mixed problem in L^2 where the weight is given by $|X|^\epsilon$ for $\epsilon > 0$, and a weighted mixed problem in the atomic Hardy space H^1 with weight given by $|X|^{\epsilon'}$ for $\epsilon' < 0$. (In both cases, X stands for points in the boundary of Ω). In order to solve the weighted

L^2 and H^1 endpoint problems, we observe that there is a holomorphic vector field α, namely:

$$\alpha(z) \approx z^{\epsilon}, \quad \text{where} \quad z := x_1 + ix_2$$

such that the following Rellich-type identity and boundary estimates:

$$\int_{\partial\Omega} |\nabla u(X)|^2 \alpha(X) \cdot \nu(X) - 2\alpha(X) \cdot \nabla u(X) \frac{\partial u}{\partial \nu}(X)\, d\sigma(X) = 0; \tag{2.1}$$

$$\alpha(X) \cdot \nu(X) \approx -|X|^{\epsilon}, \quad X \in N; \tag{2.2}$$

$$\alpha(X) \cdot \nu(X) \approx |X|^{\epsilon}, \quad X \in D \tag{2.3}$$

hold for the outer unit normal vector ν and for any harmonic function u such that $(\nabla u)^* \in L^2(\partial\Omega, |X|^{\epsilon} d\sigma)$. (The restriction on the size of the Lipschitz constant: $M < 1$ is needed in order to obtain (2.2) and (2.3)). Once (2.1)-(2.3) have been established, the proof follows by combining conformal map methods with techniques that were essentially developed in Brown [2] and Dahlberg-Kenig [4]. We refer to [3] for the complete proof and a list of related open questions.

References

[1] A. Azzam and E. Kreyszig *On Solutions of Elliptic Equations satisfying Mixed boundary Conditions*, SIAM J. Math. Analysis **13(2)** (1982), 254–262.

[2] R. Brown *The Mixed Problem for Laplace's Equation in a Class of Lipschitz Domains*, Comm. PDEs **19 (7&8)** (1994), 1217–1233.

[3] R. Brown, L. Capogna and L. Lanzani *The mixed problem in L^p for some two-dimensional Lipschitz domains*, preprint 2005.

[4] B.E.J. Dahlberg and C.E. Kenig *Hardy Spaces and the Neumann Problem in L^p for Laplace's Equation in Lipschitz Domains*, Ann. of Math. **125** (1987), 437–466.

[5] D.S. Jerison and C.E. Kenig *The Neumann Problem on Lipschitz Domains*, Bull. Amer. Math. Soc. **4** (1982), 203–207.

[6] I.N. Sneddon *Mixed Boundary Value Problems in Potential Theory*, Wiley (1966).

[7] G.C. Verchota *Layer Potentials and Regularity for the Dirichlet problem for Laplace's equation on Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611.

Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces

Sanghyuk Lee

We are concerned with the oscillatory integral operator defined by

$$T_\lambda f(z) = \int e^{i\lambda\phi(z,y)} a(z,y) f(y) dy,\ (z,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n,\ n \geq 1,$$

where $a \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\phi \in C^\infty$ on the support of a. The operator T_λ can be thought of as variable coefficient generalization of restriction operator in the sense that the optimal decay estimates

$$\|T_\lambda f\|_q \leq C\lambda^{-\frac{n+1}{q}} \|f\|_p \tag{1}$$

implies the corresponding L^p-L^q boundedness of the adjoint of Fourier restriction to $\{\nabla_z \phi(z_0, y) : y \in B(y_0, \epsilon)\}$ for a small $\epsilon >$ as long as $a(z_0, y_0) \neq 0$ (see [9, 12]).

In connection with Bochner-Riesz conjecture, it was asked in [9] whether it is possible to obtain (1) for $q > (2n+2)/n$, $(n+2)/q \leq n(1-1/p)$ under the assumption that on the support of a $\mathrm{rank}(\partial^2_{zy}\phi) = n$ and if $\theta \in S^n$ is the unique direction for which $\nabla_y \langle \partial_z \phi, \theta \rangle = 0$, then

$$\det(\partial_y^2 \langle \partial_z \phi, \theta \rangle) \neq 0. \tag{2}$$

When $n = 1$, this was proven by Hörmander [9] generalizing the earlier result due to Carleson and Sjölin [6]. In higher dimensions ($n \geq 2$), Stein [17] proved it for $q \geq (2n+4)/n$. Later, it was shown by Bourgain [3, 5] that there are phase functions for which it is impossible to obtain (1) for $q < (2n+4)/n$ when $n \geq 2$ is even. In $\mathbb{R}^3$ he further showed that (1) fails generically for $q < s$ for some $3 < s < 4$ and he obtained a positive result beyond the Stein's result with simpler phases. Recently, Wisewell [25] obtained more concrete range of failure in all dimension bigger than 2 using quadratic phases.

We extend the estimate (1) by imposing an elliptic type condition which was used in the study of restriction problem [23]. Precisely,

Theorem 1. *Suppose ϕ satisfies rank$(\partial^2_{zy}\phi) = n$ and (2), and $\partial_y^2 \langle \partial_z \phi(z_0, y), \theta \rangle$ has eigenvalues of the same sign. Then, for $q \geq 2(n+3)/(n+1)$, $(n+2)/q \leq n(1-1/p)$ and any $\epsilon > 0$,*

$$\|T_\lambda f\|_q \leq C \lambda^{-\frac{n+1}{q}+\epsilon} \|f\|_p. \tag{3}$$

It is also important to consider operator defined by homogeneous phase functions of degree one. This kind of operators naturally appear in the study of Fourier integral operators related to wave equations. In view of restriction to conic surfaces, a natural generalization of (2) to homogeneous is that if $\theta \in S^n$ is the unique direction for which $\nabla_y \langle \partial_z \phi, \theta \rangle = 0$, then

$$\mathrm{rank}\ \partial_y^2 \langle \partial_z \phi, \theta \rangle = n - 1. \tag{4}$$

Under this condition, the optimal L^2-$L^{(2n+2)/(n-1)}$ estimate was obtained in [13].

Theorem 2. *Let $n \geq 2$. If ϕ satisfies rank$(\partial^2_{zy}\phi) = n$, (4) and all eigenvalues of $\partial_y^2 \langle \partial_z \phi \theta \rangle$ have the same sign, then for $q \geq 4$ if $n = 2$ or $q \geq 2(n+3)/(n+1)$ if $n \geq 3$, and $(n+1)/q \leq (n-1)(1-1/p)$, (3) holds.*

When $n = 2$, the elliptic condition is actually unnecessary. For $n = 3$, up to ϵ-loss it establishes best possible estimates corresponding to the restriction to the cone, which is due to Wolff [26].

Both Theorem 1 and Theorem 2 are consequences of bilinear estimates (Theorem 3 and Theorem 4 below) which are variable coefficient generalizations of bilinear L^2-restriction estimates for hypersurfaces [11, 19, 24, 26]. For related subjects and applications of bilinear estimates readers are referred to [4, 7, 8, 10, 11, 20, 22] and references contained therein. For $i = 1, 2$, define

$$T_i f(x,t) = \int e^{i\lambda\phi_i(x,t,\xi)} a_i(x,t,\xi) f(\xi) d\xi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where a_i is compactly supported smooth function and ϕ_i is a smooth function on the support of a_i satisfying that

$$\partial_t \phi_i(x,t,\xi) = q_i(x,t,\partial_x \phi_i(x,t,\xi)). \tag{5}$$

We assume ϕ_i is defined only on the support of a_i.

Theorem 3. *Suppose ϕ_i satisfies* (5), *rank $\partial^2_{x\xi}\phi_i = n$ and $\det \partial^2_{\xi\xi} q_i \neq 0$, and suppose*

$$\langle \partial^2_{x\xi}\phi_i(\partial_\xi q_1 - \partial_\xi q_2), [\partial^2_{x\xi}\phi_i]^{-1}[\partial^2_{\xi\xi} q_i]^{-1}(\partial_\xi q_1 - \partial_\xi q_2)\rangle \neq 0$$

for $i = 1, 2$ (each function may has different variable in case the domains of functions do not coincide). Then for any $\epsilon > 0$ and $q \geq (n+3)/(n+1)$,

$$\|T_1 f T_2 g\|_q \leq C\lambda^{-\frac{n+1}{q}+\epsilon}\|f\|_2\|g\|_2. \tag{6}$$

We also consider operators defined by homogeneous phases and generalize the bilinear restriction estimates for conic surfaces [11, 18, 26] to a variable coefficient version.

Theorem 4. *Let $n \geq 2$. Suppose ϕ_i is a homogeneous function of degree 1 in ξ satisfying* (5), *rank $\partial^2_{x\xi}\phi_i = n$ and the Hessian matrix $\partial^2_{\xi\xi} q_i$ has maximal rank $n-1$, and suppose*

$$\langle \eta_i/|\eta_i|, \partial_\xi q_1(x_1,t_1,\eta_1) - \partial_\xi q_2(x_2,t_2,\eta_2)\rangle \neq 0.$$

for $i = 1, 2$. Then for any $\epsilon > 0$ and $q \geq (n+3)/(n+1)$, (6) *holds.*

Theorem 3 and 4 can serve as bilinear substitutes in variable coefficient settings as bilinear restrictions to hypersurfaces do. Hence these may be applied to various situations arising by variable coefficient generalization. Especially it is possible to obtain improved regularity properties of a class of Fourier integral operators studied in [13].

References

[1] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), 147-187.

[2] ______, *On the restriction and multiplier problems in $\mathbb{R}^3$*, in *Geometric aspects of functional analysis-seminar* 1989-90, Lecture Notes in Math. **1469**, Springer-Berlin, (1991), 179-191.

[3] ______, *L^p-estimates for oscillatory integrals in several variables*, Geom. Funct. Anal. **1** (1991), 321-374.

[4] ______, *Estimates for cone multipliers,* Operator Theory: Advances and Applications **77** (1995), 41-60.

[5] ______, *Some new estimates on oscillatory integrals. Essays on Fourier analysis in honor of Elias M. Stein*, 83-112, Princeton Univ. Press, 1995.

[6] L. Carleson and P. Sjolin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287-299.

[7] B. Erdogan, *On Falconer's distance set conjecture*, to appear in Rev. Mat. Ibero.

[8] ———, *A bilinear Fourier extension theorem and applications to the distance set problem*, to appear in IMRN.
[9] L. Hörmander, *Oscillatory integrals and multipliers on FL^p*, Ark. Mat. **11** (1973), 1-11.
[10] S. Lee, *Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators*, Duke Math. Journ. **122** (2004), 105-235.
[11] ———, *Bilinear restriction estimates for surfaces with curvatures of different signs*, to appear, Trans. AMS.
[12] G. Mockenhaupt, *Bounds in Lebesgue spaces of oscillatory integral operators*, Habilitationsschrift, Universität Siegen (1996).
[13] G. Mockenhaupt, A. Seeger and C. Sogge, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*, J. Amer. Math. Soc. **6** (1993), 60-135.
[14] W. Schlag and C.D. Sogge, *Local smoothing estimates related to the circular maximal theorem*, Math. Res. Lett. **4** (1997), 1-15.
[15] C.D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, **105**, Cambridge University Press, Cambridge, 1993.
[16] ———, *Propagation of singularities and maximal functions in the plane*, Invent. Math. **104** (1991), 349-376.
[17] E. Stein, *Oscillatory integrals in Fourier analysis*, in *Beijing lectures in harmonic analysis*, Ann. of Math. Study # 112, Princeton Univ. Press, Princeton, (1986), 307-355.
[18] T. Tao, *Endpoint bilinear restriction theorems for the cone and some sharp null from estimates*, Math. Z. **238** (2001), 215-268.
[19] ———, *A Sharp bilinear restriction estimate for paraboloids*, Geom. Funct. Anal. **13** (2003), 1359-1384.
[20] ———, *Recent progress on the restriction conjecture*, preprint.
[21] T. Tao and A. Vargas, *A bilinear approach to cone multipliers I. Resriction estimates*, Geom. Funct. Anal. **10** (2000), 185-215.
[22] ———, *A bilinear approach to cone multipliers. II. Application*, Geom. Funct. Anal. **10** (2000), 216-258.
[23] T. Tao, A. Vargas and L. Vega, *A bilinear approach to the restriction and Kakeya conjecture*, J. Amer. Math. Soc. **11** (1998), 967-1000.
[24] A. Vargas, *Restriction theorems for a surface with negative curvature*, Math. Z. **249** (2005), 97-111.
[25] L. Wisewell, *Kakeya sets of curves*, to appear in GAFA.
[26] T. Wolff, *A sharp cone restriction estimate*, Annals of Math. **153** (2001), 661-698.

Calderon-Zygmund operators with operator-valued kernels and evolution equations

Lutz Weis

For a Calderon-Zygmund operator

$$Tf(u) = \int k(u,v)f(v)\,dv \qquad u \notin \mathrm{supp} f$$

on a Bochner space $L_p(\mathbb{R}^n, X)$, $1 < p < \infty$, with a kernel of bounded linear operators $k(u,v)$ on the Banach space X, the usual Calderon-Zygmund theory still ensures that T is bounded on the whole $L_p(\mathbb{R}^n, X)$ scale for $1 < p < \infty$, if T is bounded on $L_q(\mathbb{R}^n, X)$ for a single q ([BCP]). However, if X is not a Hilbert space there is nothing special about the space $L_2(X)$ and it is just as hard to prove boundedness on $L_2(X)$ as it is to prove boundedness on $L_p(X)$. This difficulty was overcome by J. Bourgain ([Bou]) and T. Figiel ([Fig]) for scalar-valued kernels

$k(u,v)$ by developing a Paley-Littlewood decomposition of $L_p(\mathbb{R}^n, X)$ for UMD-spaces X. A Banach space has the UMD-property if the Hilbert transform extends to a bounded operator on $L_2(\mathbb{R}, X)$. This holds for all subspaces and quotient spaces of a $L_q(\Omega)$-space, $1 < q < \infty$, in particular for the usual reflexive Hardy and Sobolev spaces.
Motivated by applications to evolution equations (e.g. maximal regularity [W], sums of closed operators [KW], H^∞-functional calculus for sectorial operators [KW], [KKW]) a theory for translation invariant operators T with operator-valued kernels was developed in recent years ([W], [StW], [AB], [GW], [HW1]). The crucial difference to the scalar case treated in [Bou] and [Fig] is, that boundedness assumptions on Fourier-multiplier functions and kernels cannot be expressed in the operator norm (this was observed by G. Pisier (cf. [AB]), but have to be replaced by R-boundedness. If $X = L_q(\Omega)$, $q < \infty$, a sequence of operators T_j on X is R-bounded if there is a constant $C < \infty$ such that for all $x_j \in X$

$$\left\| \left(\sum_j |T_j x_j|^2 \right)^{\frac{1}{2}} \right\|_{L_q(\Omega)} \leq C \left\| \left(\sum_j |x_j|^2 \right)^{\frac{1}{2}} \right\|_{L_q(\Omega)}$$

For general Banach spaces X this classical square function estimate can be expressed in terms of Rademacher averages which are an important tool in the geometry of Banach spaces. The resulting inequality

$$\mathbb{E} \left\| \sum_j r_j T_j x_j \right\| \leq C \mathbb{E} \left\| \sum_j r_j x_j \right\|$$

may be seen as an extension of Kahane's contraction principle for random series. By now it is well established, that such R-boundedness assumptions are fulfilled in the context of large classes of elliptic and parabolic partial differential operators [DHP], [DDHPV], [KuW], [KKW] and Stokes operators ([NS], [KKW]) and they have proven to be very useful e.g. in establishing maximal regularity.
In the talk we gave a short survey on the interplay of boundedness results for Calderon-Zygmund operators with problems in evolution equations. Then we presented some joint work with T. Hytönen ([HW2]) and C. Kaiser ([KaW]) on the $T1$-theorem for Calderon-Zygmund operators with operator-valued kernels and its application to wavelet decompositions of $L_p(\mathbb{R}^n, X)$.
Suppose that the kernel $k(u,v)$ defines a bilinear form $K : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to B(X)$ such that for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ with $\mathrm{supp}\phi \cap \mathrm{supp}\psi = \emptyset$ we have

$$K(\phi, \psi) = \int\int \psi(u) k(u,v) \phi(v) \, dudv.$$

Furthermore, we replace the standard boundedness assumptions on the kernel k by the R-boundedness e.g. of the following sets of operators in $B(X)$:

$$\{|u-v|^n \, k(u,v) : u \neq v\}$$
$$\left\{ \frac{|u-v|^{n+\gamma}}{|v-v_0|^{\gamma}} (k(u,v) - k(u,v_0)) : |u-v| > 2\,|v-v_0| > 0 \right\}$$
$$\left\{ \frac{|u-v|^{n+\gamma}}{|u-u_0|^{\gamma}} (k(u,v) - k(u_0,v)) : |u-v| > 2\,|u-u_0| > 0 \right\}$$

for some $\gamma \in (0,1]$ and all u_0, v_0. In place of "weak boundedness", we assume now that

$$\{R^{-n} K(\phi(\frac{\cdot - u_0}{R}), \psi(\frac{\cdot - u_0}{R})) : R > 0, u_0 \in \mathbb{R}^n\}$$

is R-bounded in $B(X)$ for all bump functions ϕ and ψ in the unit ball. Then a special $T1$ theorem holds: If X has UMD and if $T1 = 0$, $T'1 = 0$ (which can be defined in a similar way as in the scalar case), we obtain a bounded operator T on $L_p(\mathbb{R}^n, X)$, $1 < p < \infty$, $H_1(\mathbb{R}^n, X)$ and $BMO(\mathbb{R}^n, X)$ (cf. [HW2]).
As a consequence we can improve a result of T. Figiel (see [KaW]): if ϕ and ψ are smooth wavelets generating frames $\phi_{j,k}(u) = 2^{-nj/2}\phi(2^{-j}u - k)$, $\phi_{j,k}$ in $L_2(\mathbb{R}^n)$, then for every R-bounded sequence $M_{j,k} \in B(X)$ the wavelet decomposition

$$Lf = \sum_j \sum_k M_{j,k} \langle \psi_{j,k}, f \rangle \phi_{j,k}$$

converges unconditionally in $L_p(\mathbb{R}^n, X)$ with $1 < p < \infty$ and in $H_1(\mathbb{R}^n, X)$.

We also have some interpretations of "$T1 \in BMO$" which are sufficient to obtain a bounded operator T in $L_p(\mathbb{R}^n, X)$ for $1 < p < \infty$. In general, they are somewhat technical but they reduce to a quite natural understanding of "$T1 \in BMO(\mathcal{A})$" if the operators involved belong to a subspace $\mathcal{A}$ of $B(X)$ which also has the UMD property (e.g. a Schatten class S_p with $1 < p < \infty$ or the Hille-Tamarkin operators). Necessary and sufficient conditions for the boundedness of T are apparently unknown, even in the Hilbert space case. The problem is of course the boundedness of paraproducts (see also [NPTV]).
Tb-theorems of a similar nature are presented in [H]. For boundedness results for pseudo-differential operators with operator-valued symbols which are also based on R-boundedness and which have applications to non-autonomous evolution equations, see [PS].

References

[AB] W. Arendt and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. **240** (2002), 311–343.

[BCP] A. Benedikt, A. Calderon and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. USA, **48**, (1962), 356–363.

[Bou] J. Bourgain, *Vector-valued singular integrals and the $H^\infty - BMO$ duality, In: J. A. Chao, W. Woyczynski (eds.) Probability theory and Harmonic Analysis*, Marcel Dekker, New York, (1986), 1–19.

[DHP] R. Denk, M. Hieber and J. Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., vol. 166 , Amer. Math. Soc, Providence, R.I., 2003.

[DDHPV] R. Denk, G. Dore, M. Hieber, J. Prüß and A. Venni, *New thoughts on old results of R.T. Seeley*, Math. Ann. 328 (2004), 545-583.

[Fig] T. Figiel, *Singular integral operators: a martingale approach. In: P.F.X. Müller, W. Schachenmayer (eds.) Geometry of Banach spaces*, London Math. Soc. Lecture Note Ser. **158**, Cambridge University Press, (1990).

[GW] M. Girardi and L. Weis, *Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces*, J. Funct. Analysis 204 (2003), 320–354.

[H] T. Hytoenen, *An operator-valued Tb-theorem*, preprint.

[HW1] T. Hytoenen and L. Weis, *Singular convolution integrals with operator-valued kernels*, to appear in Math. Zeit.

[HW2] T. Hytoenen and L. Weis, *A $T(1)$ theorem for integral transformations with operator valued kernels*, submitted.

[KaW] C. Kaiser and L. Weis, *Wavelet transform for functions with values in UMD spaces*, submitted.

[KW] N. J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann., **321** (2001), no. 2, 319–345.

[KKW] N. J. Kalton, P. Kunstmann and L. Weis, *Perburbation and interpolation results for the H^∞ calculus with applications to partial differential equations*, to appear in Math. Ann.

[KuW] P. Kunstmann and L. Weis: Maximal L_p Regularity for parabolic equations, Fourier multiplier theorems and H^∞ functional calculus, in : Functional Analytic Methods for Evolution Equations (Levico), 65–312 Lecture Notes in Mathematics 1855.

[NPTV] F. Nazarov, G. Pisier, S. Treil and A. Volberg, *Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts*, J. Reine Angew. Math. 542 (2002), 147–171.

[NS] A. Noll and J. Saal, *H^∞-for the Stokes operator on L_p-spaces*, Math. Z. **244**, 651–688.

[PS] P. Portal and Ž. Štrkalj, *L_p-boundedness of pseudo-differential operators with operator-valued symbols and applications*, submitted.

[StW] Ž. Štrkalj and L. Weis, *On operator-valued Fourier multiplier theorems*, to appear in Transactions AMS.

[W] L. Weis *Operator-valued Fourier Multiplier theorems and maximal L_p-regularity*, Math. Ann., **319**, (2001), 735-758.

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