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Mini-Workshop: Gerbes, Twisted K-Theory and Conformal Field Theory

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August 14th – August 20th, 2005

Abstract. The aim of the mini-workshop was to bring together people whose work in mathematics and mathematical physics centers around geometry and topology applicable in modern string theory. More specifically, whose interests are in one or another way connected to gerbes and twisted K-theory and their applications in topological field theory, conformal field theory and M-theory.

Mathematics Subject Classification (2000): 18xx, 19xx, 55xx, 57xx, 81xx.

Introduction by the Organisers

The Mini-Workshop *Gerbes, Twisted K-Theory and Conformal Field Theory* organized by Branislav Jurčo (München), Jouko Mickelsson (Helsinki) and Christoph Schweigert (Hamburg) was held August 14th – August 20th, 2005. The idea of this mini-workshop was to bring together people working on different aspects of gerbes (and related topics) to exchange different points of view, make the interactions more intense and eventually establish new collaborations. Because of different backgrounds of participants, the organizers asked Jarah Evslin and Hisham Sati to prepare some expository lectures on M-theory, supergravity and superstrings oriented towards a more mathematical audience. Also Christoph Schweigert prepared an introduction on the use of gerbes and gerbe modules in the description of D-branes in Wess-Zumino-Witten models and Dale Husemöller reported on the point of view of stacks and gerbes in the form used in algebraic geometry following Deligne's exposition of nonabelian cohomology in LN 900.

As equivalence classes of line bundles are geometric realizations of the second integral cohomology classes, stably equivalence classes of bundle gerbes are geometric realizations of the third integral cohomology classes. Similarly to line bundle that can be equipped with connection, gerbe can be equipped with connection and curving (B-field). Taking this extra differential geometric structures into account the relevant cohomology is the Deligne cohomology. Gerbes can be equipped with modules, which are vector bundles twisted with the third integral cohomology class in questions. These are the structures important in string theory when we want to describe D-branes in topologically nontrivial backgrounds. D-brane charges are conjecturally classified (at least in some situations) by the corresponding twisted K-theory. This links very nicely K-theory to conformal filed theory. Similarly higher versions of gerbes and their nonabelian generalizations are expected to be useful for description of higher rank forms gauge theories and the conjectural M-theory. As usually, this interaction between geometry, topology and theoretical physics makes the subject attractive to researches with different backgrounds.

Gerbe and gerbe modules turn out to be a crucial input in the description of the Wess-Zumino term in the presence of boundaries. The quantization of the position of D-branes in string backgrounds with three-form fluxes actually arises as a consequence of integrality properties of these modules. This was discussed in talks of Christoph Schweigert and Krzyzstof Gawedzki who explained how gerbes help to solve WZW conformal sigma models and how they provide an uniform and effective treatment of classical and quantum theories without and with boundaries. Closely related to WZW theory is the Chern-Simons gauge theory. The precise relation between these two theories in the language of multiplicative bundle gerbes, 2-bundle gerbes and the corresponding Deligne cohomology classes was described in the talk of Michael Murray.

In the same way as vector bundles give rise to class in the topological K-theory of the manifold, gerbe modules give rise to a class in twisted K-theory. Twisted K-theory has been proposed as the recipient of D-brane. Equivariant twisted K-theory ${}^{\tau}K_G(G)$ on a compact Lie group G can be endowed with the structure of a ring. Due to a result of Freed Hopkins and Teleman this ring turns out to be the Verlinde algebra, familiar from conformal field theory. More generally equivariant twisted K-theory ${}^{\tau}K_H(G)$ in the case when G/H is hermitian symmetric can be related to the $\mathcal{N} = 2$ chiral ring. This extension was discussed in the talk of Sakura Schäfer-Nameki. Volker Braun in his talk showed some explicit computations in twisted K-theory relevant for $N = 1$ supersymmetric Wess-Zumino-Witten models. Explicit construction of twisted equivariant K-classes using Dirac operators related to supersymmetric WZW models was given in talk of Jouko Mickelsson. A nice application of multiplicative bundle gerbes was given by Alan Carey. The multiplicative structure gives rise to the fusion product on bundle gerbe D-branes. Quantization functor relates these to the twisted equivariant K-classes.

Another aspect is the action of dualities on gerbes and gerbe modules and the related twisted K-classes. In the case of circle and torus bundle, there has been a lot of activity recently. In their talks Jarah Evslin and Hisham Sati introduced the concept of duality how it is emerging in M-theory, supergravity and superstring theory. They explained how anomaly cancellation and consistency of quantum field theories under different dualities lead to the description of filed configurations in terms of generalized cohomology theories (e.g. K-theory and elliptic cohomology). Talk of Varghese Mathai was devoted to global aspects of T-duality and relations to noncommutative geometry. T-duality for orbispaces from the point of view of of stacks and gerbes introduced in talk of Dale Husemoller was presented by Ulrich Bunke in his lecture. Peter Bouwknegt gave an overview of the topic of generalized geometry and its relation to gerbes, T-duality and mirror symmetry.

Nonabelian gerbes, their relation to nonabelian cohomology and their application to quantum theory was another topic presented in some lectures. Nonabelian generalizations of bundle gerbes, their modules and differential geometry along with an application to anomaly of five branes were presented by Paolo Aschieri followed by the talk of Branislav Jurčo about simplicial description of nonabelian bundle gerbes with connection and curving. Simplicial description leads naturally to classification of bundle gerbes related to crossed modules. Closely related talk was given by Danny Stevenson who described the string gerbe based on the recent simplicial construction of the string group. He as well as Urs Schreiber in his talk on nonabelian holonomy used the groupoid point of view to gerbes. A nonabelian bundle gerbe can be thought as an example of a 2-bundle which comes together with the concept of higher algebraic structures and categorification, subjects of the lecture of Hendryk Pfeiffer.

We are very glad that there was a lot of interactions across the various groups of researchers. Physicists and mathematicians enjoyed both the talks and the lively after talks discussion. Also mini-workshop benefited a lot from the activity and curiosity of young participants.

Mini-Workshop: Gerbes, Twisted K-Theory and Conformal Field Theory

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Abstracts

M-Theory, Type II String Theory, and (Refinements of) Twisted K-Theory

Jarah Evslin and Hisham Sati

Evslin Sati

This is a series of four lectures delivered by the two authors. The first two were expository aimed mostly at introducing M-theory, supergravity and superstring theories to the mathematics audience. The second two were research-oriented and aimed at reporting the work of the authors on the subject of the workshop.

The different fields that usually occur in physics are generically described in the language of vector bundles or principal bundles as sections or connections, or as cohomology elements. Each field has a corresponding Lagrangian and an equation of motion (EoM). One useful way of classifying the fields is via the spin, which is integral for bosons and half-integral for fermions. Among the important examples are the spinors, with the Dirac operators as the operators in question. A supersymmetric theory is a theory of bosons and fermions, which enjoy, in addition to the symmetries of the separated system, a symmetry that mixes the two kinds called supersymmetry. A (non-)gravitational theory can have rigid or global (local) supersymmetry.

Supergravity (=sugra) is basically built from combining a bosonic Lagrangian, the Einstein-Hilbert Lagrangian, and a fermionic one, the Rarita-Schwinger Lagrangian. In addition to the symmetries that these Lagrangians might separately have, one also introduces an extra symmetry – the supersymmetry– that varies the fermions with respect to the bosons and vice versa, involving a local spinor parameter, i.e. one that depends on coordinates of spacetime. One important requirement of supersymmetry is the equality of the number of degrees of freedom of the bosons and the fermions. For simple supergravity theory in four dimensions, this condition is already satisfied by the Einstein-Hilbert and the Rarita-Schwinger system. However, in higher dimensions, this is no longer the case as the dimension of the spinors grows exponentially, whereas the dimension of the bosons is usually polynomial. The mismatch in bosonic degrees of freedom is supplied by differential forms. The most important such example is eleven-dimensional supergravity [1], where the above form has degree three.

The field content of eleven-dimensional supergravity is (g, C_3, ψ_1), where g is the metric tensor, C_3 is the antisymmetric tensor field, usually called the C-field, and ψ_1 is the Rarita-Schwinger field, i.e. a fermion with spin 3/2. The Lagrangian of this supergravity theory is rather simple compared to other higher-dimensional supergravity theories. It is made of kinetic terms for the three fields involved, and in addition contains an important piece dictated by supersymmetry, which the Chern-Simons term, a topological term independent of the metric. If one includes quantum effects of anomalies then one also adds a one-loop term made of C_3 and some eight-dimensional polynomial in the Pontrjagin classes of the tangent bundle of the eleven manifold (Y^{11}, g).

Varying the action of 11d sugra with respect to each of its field leads to a set of three EoM's: the Einstein equation for g, the Maxwell-like equation for C_3, and the Rarita-Schwinger equation for ψ_1. Solving such differential equations in general is a very difficult task; one usually does so only for a particular ansatz. Among the interesting solutions are the membrane M2 and the fivebrane M5. These are characterized by being BPS. This means that they are stable against perturbations and thus do not receive quantum corrections. This essentially implies that such solutions can be trusted in the quantum theory, i.e. at strong coupling.

M-theory is a quantum theory in eleven dimensions whose weak coupling limit is classical eleven-dimensional supergravity. There is no intrinsic formulation of the theory without using its limits. There are proposals for such definitions, e.g. the matrix model, but they have their limitations, especially as far as topology goes. M-theory also connects the various string theories through a web of dualities, namely perturbative target space T-duality, and nonperturbative strong-weak coupling S-duality. So one can try to study M-theory from its low energy limit, i.e. eleven-dimensional supergravity, or from its connection to the duality web. In the first approach, one uses the above BPS solutions as objects in M-theory itself.

The degree three field C_3 is responsible for the nontrivial topology in M-theory. In analogy to electromagnetism where the one-form potential (=connection) couples to the worldline of the electron, and its dual couples to the worldline of the monopole, one has an analogous situation where C_3, viewed as an electric potential, couples to the M2 worldvolume and the the dual potential C_6, viewed as a magnetic potential, couples to the worldvolume of the M5-brane. This is due to the eleven-dimensional Hodge duality between $G_4 = dC_3$ and $*G_4 = dC_6 + \cdots$, and follows directly from the EoM for C_3.

In supergravity, the non-gravitational fields are usually taken to be differential forms. However, upon taking anomalies into account and looking at the quantum picture, such fields are expected to form classes in integral cohomology. However, the situation is usually more subtle. For the case of G_4 one gets a shifted quantization condition $G_4 - \frac{\lambda}{2} \in H^4(Y^{11}; \mathbb{Z})$, where λ is half the Pontrjagin class of TY^{11}.

Motivated by the $E_8 \times E_8$ heterotic string theory on the boundary of Y^{11}, Witten [2] showed that G_4 can be interpreted as the class of an E_8 bundle in eleven dimensions. In [3] the question of supersymmetry of such a theory is analyzed and an approximate construction of the 11d gravitino as a condensate of the gauge theory fields was given.

The Kaluza-Klein dimensional reduction of eleven-dimensional supergravity leads to type IIA supergravity theory, whose bosonic field content includes the Ramond-Ramond (=RR) fields F_{2p}, $(p = 1, \cdots, 5)$, and the Neveu-Schwarz (=NSNS) field H_3. The global dimensional reduction was performed in [4]. In type II string theory on X^{10} one analogously has refinements of the cohomology description, except that in this case one is led instead to K-theory, $K^0(X^{10})$ for type IIA and $K^1(X^{10})$ for type IIB. In the presence of H_3, the corresponding K-theories are twisted.

The reduction of the E_8 above to ten dimensions leads to an LE_8 bundle. This was proposed first in [5] and interpreted in terms of gerbes in [4], where H_3 serves as the obstruction to lifting the loop group bundle to its central extension. This bundle picture can be considered to be somewhat complementary to the twisted K-theory view above. Adding a cosmological constant F_0 leads necessarily to a H_3 which is trivial in cohomology, i.e. to $H_3 = dB_2$ [4].

Instead of looking at the fields, one can look at the story from a complementary point of view, namely via D-branes. The latter are in homology, and one can go back and forth between cohomology and homology using Poincaré duality. There is an analogy with electromagnetism in this case and the branes act as sources of charges that appear as delta-function violations of the the corresponding Bianchi identities. This can be thought of as higher degree analog of Dirac charge quantization for monopoles (and dyons). The Atiyah-Hirzebruch Spectral Sequence (AHSS) serves as a tool to detect the difference between cohomology and K-theory and is thus a powerful tool in the D-brane realization [6]. The classification of D-branes and solitons, especially in type IIB string theory in this context has been considered in [7] [8] [9] where further physical realizations are made and a modification of the AHSS is proposed in order to describe the group of conserved RR and NSNS charges.

The correspondence between M-theory and type IIA string theory holds at the quantum level in the path integral formulation, i.e. at the level of the partition functions [10]. The former is governed by the E_8 gauge theory and the latter by K-theory. The corresponding match for twisted K-theory was started in [4], where a nontrivial M-theory circle bundle is considered, the NSNS field H_3 nontrivial in cohomology is added, and the corresponding vector bundles are taken not to be lifted from the base. The construction of the K-theory torus is done as in the untwisted case. Eta differential forms – the higher degree generalizations of the eta invariant – make a very interesting appearance, perhaps for the first time in string theory.

The partition function in [10] has an anomaly given the seventh integral Stiefel-Whitney class W_7. In [11] the vanishing of this anomaly was shown to be equivalent to orientability of spacetime with respect to (complex-oriented) elliptic cohomology E. Motivated by this, an elliptic cohomology correction to the IIA partition function was defined. The generators of E were proposed as corresponding to M2 and M5-branes in the M-theory limit.

Other aspects of string theory also point further towards elliptic cohomology. In the presence of background NSNS flux, the description of the RR fields of type IIB string theory using twisted K-theory is not compatible with S-duality. In [12] it was shown that other possible variants of twisted K-theory would still not resolve this issue and a possible solution was proposed using elliptic cohomology. Another evidence for elliptic cohomology is modularity in type IIB, where there is an elliptic curve that lifts the theory to twelve-dimensional F-theory. In [13] an interpretation is given for this elliptic curve in the context of elliptic cohomology.

The above elliptic cohomology description in type II string theory can, at least mathematically, be continued to M-theory via a Kunneth formula for E. One can ask whether E will ultimately be the theory describing the fields of M-theory. Related to this is trying to understand the nature of G_4. For this purpose, the Chern-Simons and the one-loop terms in the M-theory action were written in [14] in terms of new characters involving the M-theory four-form and the string classes. The latter are defined as analogs to the usual string class of rank four, i.e. $\lambda_i = p_i/2$. This suggests the existence of a theory of higher characteristic classes where the Chern classes and the Chern characters are replaced by those new classes and characters. In [15] this formalism is used to give a unified expression for the class of G_4 and its dual (called the Θ class in [16]) in analogy to the K-theoretic quantization of the RR fields.

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Introduction to the WZW Theory and Bundle Gerbes

Christoph Schweigert
(joint work with Konrad Waldorf)

In this talk, I gave an introduction to the application of bundle gerbes and gerbe modules in WZW theories. The goal was to show why bundle gerbes, gerbe modules and their holonomy are the natural and unavoidable language to describe the Wess-Zumino (WZ) term in the action functional of these models.

The classical WZW model is defined by an action for maps from a two-dimensional conformal manifold Σ to a Lie group G. In the present talk, only the case of compact connected simple Lie groups G and oriented surfaces Σ was considered. Conformal invariance of the *quantized* WZW model requires the addition of a WZ term. In Witten's original paper [8], this term was described as follows: fix a three-dimensional manifold B with boundary Σ; extend the map $g : \Sigma \to G$ to a map $\tilde{g} : B \to G$. Then

$$S_{\mathrm{WZ}} = k \int_B \tilde{g}^* H \quad ,$$

where H is a closed biinvariant three-form on G. For this approach, however, the condition $H_2(G) = 0$ is needed for the existence of the extension $\tilde{g}$.

This condition is not fulfilled for the non simply-connected Lie group $G = \mathrm{Spin}(4n)/\mathbb{Z}_2 \times \mathbb{Z}_2$. Simple current techniques [7] in the algebraic approach show, however, that there should be two different conformal field theories that differ "by a choice of discrete torsion". This puzzle is naturally resolved by considering the WZ term as the holonomy of an (equivariant hermitian bundle) gerbe on G. (For the first formulation of this idea in terms of Deligne cohomology see [6].) For this formulation, there is no obstruction in $H_2(G)$, and moreover the possible connections on a gerbe with given Dixmier-Douady class form a torsor over $H^2(G, U(1) \cong \mathbb{Z}_2$ which explains the existence of two different models. Hence, already for the WZW models on closed surfaces Σ, gerbes and holonomies are essential.

For the case of surfaces with boundaries, boundary conditions have to be specified. The naive approach would be to fix a submanifold $\iota : Q \to G$ and a two-form ω on Q such that the pair (H, ω) is closed in the relative de Rham complex for (G, Q). Again, one would have to choose extensions, leading to the obstruction that $g(\Sigma)$ should vanish in the relative homology $H_2(G, Q)$.

In view of the situation for closed surfaces, one immediately wonders whether these obstructions are really needed. Moreover, algebraic results about boundary conditions in conformal field theories with simple current modular invariants [1, 3] show that there are cases (e.g. the conjugacy class of the Lie group SO(3) that is isomorphic to $\mathbb{R}P^2$) where two different boundary conditions have the same submanifold Q and two-form ω. Moreover, algebraic calculations in Gepner models have shown [2] that a simple boundary condition, without Chan-Paton multiplicities, can give rise to non-abelian gauge symmetries. The computation of charges gives further hints that the idea to describe boundary conditions – also called D-branes – in terms of submanifolds with a two-form, possibly refined to a $U(1)$-gauge theory on Q fails.

The solution is provided by gerbe modules [4, 5]. The obstructions can be avoided, the situation for SO(3) is explained by the fact that gerbe modules with fixed curvature form a torsor over the group of isomorphism classes of flat line bundles on Q, which is isomorphic to $\mathbb{Z}_2$ for $\mathbb{R}P^2$. Finally, it was shown by Gawędzki in [5] that irreducible gerbe modules of higher rank, leading to non-abelian gauge symmetries, can occur once the fundamental group is not cyclic.

The talk might therefore be summarized by saying that for the Lagrangian description of the WZW model on oriented surfaces, both with and without boundaries, gerbes, their holonomies and gerbe modules are crucial.

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Holonomy and Aspects of Deligne-Cohomology for Nonabelian Gerbes

Urs Schreiber

Gerbes play a role in string theory mostly as gerbes *with connection*, namely as structures that admit 'parallel transport' of strings and possibly of membranes. This allows physicists to write down globally well-defined ('anomaly free') action functionals for these objects.

According to an argument formalized by Aschieri and Jurčo [1] the endstrings of certain membranes are in particular expected to couple to a *non*abelian gerbe with connection. While the concept of parallel transport as well as that of Deligne classification is well-known for abelian gerbes with connection, its generalization to the nonabelian case has only more recently emerged [2] within the context of '2-bundles' [3], and has further been developed in [4].

In order to motivate this approach first reconsider ordinary G-bundles with connection from the point of view of parallel transport. The most immediate definition of an ordinary G-bundle with connection over a space M in this sense is in terms of its holonomy-functor

$$\mathrm{hol} : \mathcal{P}_1(M) \to G\text{–Tor}$$

which maps paths in M (really morphisms of the groupoid $\mathcal{P}_1(M)$ of *thin* homotopy equivalence classes of paths) to morphisms of G-torsors in a suitable smooth way.

Locally, on contractible open subsets $U_i \subset M$, such a functor is naturally isomorphic to a smooth *local holonomy functor*

$$\mathrm{hol}_i : \mathcal{P}_1(U_i) \to G$$

from paths to group elements. As is well known, these functors are specified by local connection 1-forms

$$A_i \in \Omega^1(U_i, \mathrm{Lie}(G)) .$$

On double intersections $U_{ij} = U_i \cap U_j$ such functors are found to be related by natural isomorphisms

$$\mathrm{hol}_i \stackrel{g_{ij}}{\to} \mathrm{hol}_j$$

which induce the familar cocycle conditions on A_i and g_{ij}. Gauge transformations, coming from different choices of trivializations, correspond to natural isomorphisms of these fuctors.

This means that with respect to a good covering $\mathcal{U} = \{U_i\}_{i\in I}$ of M, the global holonomy functor $\mathrm{hol} : \mathcal{P}_1(M) \to G$–Tor defines an isomorphism class of functors from the Čech-groupoid of $\mathcal{U}$ (whose objects are open sets and whose morphisms are double intersections) to the category of local holonomy functors, which is best thought of, equivalently, as a homotopy class of simplicial maps between the *nerve* of the covering and that of the holonomy functor-category [5].

Now categorify this situation. Fix a smooth category G_2 with strict group structure, called a strict 2-group [6], and consider 2-holonomy 2-functors:

$$\mathrm{hol} : \mathcal{P}_2(M) \to G_2\text{–2Tor}$$

that assign 2-morphisms of G_2-2-torsors to surfaces in M. Given a piece of worldsheet Σ of a 'nonabelian string', $\mathrm{hol}(\Sigma)$ is supposed to be the parallel transport of a string across Σ.

What is the information encoded in the specification of such a 2-functor hol? A combination of abstract diagrammatic reasoning together with some path space analysis shows that such 2-functors specify nonabelian gerbes with connection and curving whose *fake* curvature vanishes and which have a notion of surface holonomy given by hol.

More in detail, the 2-functor hol is locally isomorphic to 2-functors

$$\mathrm{hol}_i : \mathcal{P}_2(U_i) \to G_2$$

that are specified by pairs $A_i \in \Omega^1(U_i, \mathrm{Lie}(G))$ and $B_i \in \Omega^2(U_i, \mathrm{Lie}(H))$ satisfying the *fake* flatness condition [7]

$$F_{A_i} + dt(B_i) = 0 .$$

(Here $H \stackrel{t}{\to} G$ is a crossed module of groups associated with G_2.)

These 2-functors are related on double overlaps by pseudonatural transformations

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$$

which themselves are related on triple overlaps by modifications of pseudonatural transformations

$$g_{ik} \xrightarrow{f_{ijk}} g_{ij} \circ g_{jk} \,.$$

These modifications finally satify a tetrahedron coherence law on quadruple overlaps.

In terms of the pairs (A_i, B_i) all these transformations and coherence laws translate precisely into the cocycle conditions for a fake-flat nonabelian gerbe with connection and curving (as displayed in [8], [9]).

As before, we can think of this situation as a 2-functor from the Čech 2-groupoid of the covering $\mathcal{U}$ to the 2-category of local 2-holonomy 2-functors. One checks that natural isomorphism of such an assignment correspond to gauge transformations of the above cocycle data. Again [5], it is useful to think of this, equivalently, as a homotopy class of simplicial maps between the respective nerves (now using the notion of nerves of 2-categories as in [10]).

This establishes the notion of nonabelian gerbes/2-bundles with connection *and* with holonomy. A little exercise in diagram-gluing produces an explicit formula for computing nonabelian surface holonomy from local 2-forms $\{B_i\}$ which generalizes a similar formula well-known for abelian gerbes [11], [12].

While the classification of these objects in terms of classes of maps between the Čech 2-groupoid and a 2-functor 2-category is available, it does not seem to lend itself to computations. An efficient nonabelian generalization of the cocycle description of abelian gerbes with connection in terms Deligne hypercohomology would therefore be desireable.

It turns out that, at least at a 'linearized' level, such a description is obtainable by suitably 'differentiating' all elements of the above discussion. In order to do so the 2-path 2-groupoids $\mathcal{P}_2(U_i)$ should be replaced by '2-path 2-algebroids' $\mathfrak{p}_2(U_i)$, the structure 2-group G_2 similarly by a Lie 2-algebra $\mathfrak{p}_2(U_i)$, the 2-holonomy 2-functor by a morphism

$$\mathrm{con}_i : \mathfrak{p}_2(U_i) \to \mathfrak{g}_2$$

of 2-algebroids, the transition transformation g_{ij} by a respective 2-algebroid 2-morphism $\mathfrak{g}_{ij}$ and so on.

One technical complication for this program is that p-algebroids have yet to be formulated in a convenient category-theoretic framework that would allow to extract them by mere application of some p-functor from the above disucssion. But for all semistrict Lie p-algebras as well as for 1-algebroids and certain 2-algebroids it is known that they have a *dual* description in terms of p-term differential graded algebras [13], [14]. These again fit naturally in p-categories whose 1-morphsims are given by chain maps, 2-morphism by chain homotopies, and so on.

Using this dictionary, the entire above discussion translates into the study of simplicial maps from Čech-simplices to categories of morphisms $\mathrm{con}_i : \mathfrak{p}_2(U_i) \to \mathfrak{g}_2$ of dg-algebras. Using the differentials of $\mathfrak{p}_2(U_i)$ and $\mathfrak{g}_2$ one naturally obtains a nilpotent operator Q which makes the sheaves L^n of dg-algebra n-morphisms into a complex of sheaves

$$\cdots \to L^n \stackrel{Q}{\to} L^{n-1} \stackrel{Q}{\to} L^{n-2} \to \dots .$$

One finds that a simplicial map from the Čech-groupoid to the category of algebroid morphisms is equivalent to a cochain in the hypercohomology complex

$$\cdots \to \mathcal{H}^0(\mathcal{U}, L^\bullet) \stackrel{D=\delta\pm Q}{\to} \mathcal{H}^1(\mathcal{U}, L^\bullet) \to \dots ,$$

where δ is the Čech coboundary operator. Homotopies of such maps correspond to shifts by D-exact terms.

One checks that in the abelian case this reduces to ordinary Deligne hypercohomology

$$\mathcal{H}^\bullet(\mathcal{U}, L^\bullet) \to \mathcal{H}^\bullet(\mathcal{U}, \Omega^\bullet)$$

classifying abelian (p-)gerbes with $(p+1)$connection. In the nonabelian case the equation $D\omega = 0$ provides an efficient tool for computing the linearized cocycle conditions of these objects. It is however unclear if this generalized Deligne cohomology fully captures the classification of nonabelian gerbes, or if the linearization involved looses information.

On the other hand, the algebroid formalism allows to handle objects with (weak) structure Lie p-algebras that are not integrable to Lie p-groups and hence have no proper p-gerbe analog. An interesting example for this is the semistrict Lie-2-algebra $\mathfrak{so}_k(n)$ which is related to the String-group [15].

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Stacks and Gerbes

Dale Husemöller

My lecture was to be a report on Deligne's exposition of the subject of nonabelian cohomolgy in LN 900. In the process I used also the book of Gérard Lamoun and Laurent Moret-Bailly on Champs algébraiques. A gerbe in the conference usually ment a bundle gerbe, and this lecture was to report on the point of view of stacks and gerbes in the form used in algebraic geometry. This point of view was present in the lecture of U. Bunke.

The definition of fibre category $\pi : \mathcal{E} \to \mathcal{B}$ was introduced and the special case of a groupoid $\mathcal{E}$ over a category considered. Examples were taken from bundles and from principal G-bundles over (top) $= \mathcal{B}$ the category of topological spaces.

In discussions with U. Bunke we could give an equivalent formalism in terms of psedudofunctors (or 2-functors) $\mathcal{B} \to$ (cat) for fibre categories $\pi : \mathcal{E} \to \mathcal{B}$. Both the fibre category and the pseudofunctor approaches have their merits for examples from topology which are currently under consideration.

Grothendieck topologies on a category $\mathcal{T}$ were introduced, and the related categories $\mathcal{T}^\vee$ of presheaves of sets and Sh$\mathcal{T}$ of sheaves of sets were introduced. The adjunction property of the inclusion Sh$\mathcal{T} \to \mathcal{T}^\vee$ was also considered, and its relation to morphisms $\mathcal{T}' \to \mathcal{T}''$ of topologies explained.

The definition of a stack $\pi : \mathcal{E} \to \mathcal{T}$ over a topology $\mathcal{T}$ as a fibre category which is equivalent to the category of descent data relative to the topology on the base was introduced. gerbes over $\mathcal{T}$ were defined as certain stacks of groupoids.

Examples were principal G-bundles over (top), and for a sheaf $\underline{G}$ of groups the category of principal $\underline{G}$-sheaves P (or torseurs) were considered as basic examples of stacks.

Global Aspects of T-duality in String Theory and Twisted K-theory

Varghese Mathai

(joint work with P. Bouwknegt, A. Carey, J. Evslin, K. Hannabuss, M. Murray, J. Rosenberg, D. Stevenson)

String theory is arguably the most exciting research area in modern mathematical physics. It is known to the general public as the "Theory Of Everything", thanks to its great success in unifying Relativity and Quantum Field Theory, yielding Quantum Gravity theory. The impact of string theory is not just felt in physics, but it also has profound interactions with a broad spectrum of modern mathematics, including noncommutative geometry, K-theory and index theory. In this talk, I will give a brief survey of my joint papers on the global aspects of T-duality in string theory and Twisted K-theory.

The theory of D-branes forms an important part of string theory. It arises as the T-dual of open strings on a circle bundle, where the open strings in the dual theory are no longer free to move everywhere in space, but are endowed with Dirichlet boundary conditions so that the endpoints are free to move only on a submanifold known as a D-brane. For a link describing the mathematics behind D-branes, cf. superstrings. Such D-branes come with (Chan-Paton) vector bundles, and therefore their charge determines an element of K-theory, as was argued by Minasian-Moore. In the presence of a nontrivial B-field but whose Dixmier-Douady class is a torsion element of $H^3(M, \mathbb{Z})$, Witten argued that D-branes no longer carry honest vector bundles, but they have a twisted or gauge bundle. In the presence of a nontrivial B-field whose Dixmier-Douady class is a general element of $H^3(M, \mathbb{Z})$, it was proposed in [12] that D-brane charges in type IIB string theories are measured by the twisted K-theory that was described earlier by Rosenberg, and the twisted bundles on the D-brane world-volumes were elements in this twisted K-theory. In [11], using bundle gerbes and their modules, a geometric interpretation of elements of twisted K-theory was obtained, and the the Chern-Weil representatives of the Chern character was studied. This was further generalized to the equivariant and the holomorphic cases in [10]. The relevance of the equivariant case to conformal field theory was highlighted by the remarkable result of Freed, Hopkins and Teleman that the twisted G-equivariant K-theory of a compact connected Lie group G (with mild hypotheses) is graded isomorphic to the Verlinde algebra of G, with a shift given by the dual Coxeter number and the curvature of the B-field, where we recall that Verlinde algebra of a compact connected Lie group G is defined in terms of positive energy representations of the loop group of G, and arises naturally in physics in Chern-Simons theory which is defined using quantum groups and conformal field theory.

Type I D-branes in the presence of an H-flux are studied in [8], where a geometric interpretation of $H^2(M, \mathbb{Z}_2)$ is given in terms of stable isomorphisms of real bundle gerbes, and the twisted KO theory is interpreted geometrically in terms of real projective vector bundles.

One development is the novel discovery in [9, 6] of T-duality isomorphisms in twisted K-theory and twisted cohomology and the character formulae relating these. Briefly, T-duality defines an isomorphism between the twisted K-theory of the total space of a circle bundle, to the twisted K-theory of the total space of a "T-dual" circle bundle with "T-dual" twist, and with a change of parity. Similar statements hold for twisted cohomology. One interesting consequence is that we can construct fusion products in twisted K-theory and twisted cohomology, whenever the twist is a non-trivial decomposable cohomology class. Another interesting consequence of our work is that it gives convincing evidence that a type IIA string theory A on a circle bundle of radius R in the presence of an H-flux, and a type IIB string theory B on a "T-dual" circle bundle of radius 1/R in the presence of a "T-dual" H-flux, are equivalent in the sense that the string states of string theory A are in canonical one to one correspondence with the string states of string theory B. This is a fundamental property of type II string theories that was predicted only in special cases earlier.

[7] studies the more general case of T-duality for principal torus bundles. The new phenomenon that occurs here is that not all H-fluxes are T-dualizable, and

this paper works out the precise class of T-dualizable H-fluxes. The isomophisms in twisted K-theory and twisted cohomology also follow in this case.

In [5], we give a complete characterization of T-duality on principal 2-torus-bundles with H-flux. As noticed in [7] for instance, principal torus bundles with H-flux do not necessarily have a T-dual which is a torus bundle. A big puzzle has been to explain these mysterious "missing T-duals." Here we show that this problem is resolved using noncommutative topology. It turns out that every principal 2-torus-bundle with H-flux does indeed have a T-dual, but in the missing cases (which we characterize), the T-dual is non-classical and is a bundle of noncommutative tori. This suggests an unexpected link between classical string theories and the "noncommutative" ones, obtained by "compactifying" along noncommutative tori.

In [4, 1], we give a complete characterization of T-duality for general principal torus-bundles with H-flux, generalizing the results in [5] to higher rank torus bundles. The striking new feature in the case when the rank of the torus bundle is greater than or equal to 3 is that not every such torus bundle has a T-dual, either classical or nonclassical. The simplest example is the rank 3 torus over a point. We also define the action of the T-duality group $GO(n, n, \mathbb{Z})$ on T-dual pairs of principal torus bundles, where n is the rank of torus bundle, where $GO(n, n, \mathbb{Z})$ is the subgroup of $GL(2n, \mathbb{Z})$ that preserves the bilinear pairing upto sign. All of T-dual pairs in a given orbit of $GO(n, n, \mathbb{Z})$ define physically equivalent type II string theories.

In [3, 2], we initiate the study of C^*-algebras endowed with a twisted action of a locally compact Abelian Lie group, and we construct a twisted crossed product, which is in general a nonassociative, noncommutative, algebra. The properties of this twisted crossed product algebra are studied in detail, and are applied to T-duality in Type II string theory to obtain the T-dual of a general principal torus bundle with general H-flux, which we will argue to be a bundle of noncommutative, nonassociative tori. We also show that this construction of the T-dual includes all of the special cases that were previously analysed.

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T-Duality for Orbispaces

Ulrich Bunke
(joint work with Thomas Schick)

The concept of T-duality has its origin in string theory. It is a relation between one type of string theory on a certain target with another type of string theory on a T-dual target. T-duality is related to various mathematical concepts like mirror symmetry for Calabi-Yau manifolds, Fourier-Mukai transformations or Pontrjagin-Takai-duality for crossed products.

In an ongoing project we study the topological aspects of T-duality in the presence of an H-flux for spaces with torus actions. In the present talk we restrict to $U(1)$-spaces. Let B be a space.

A pair (E,h) over B consists of a $U(1)$-principal bundle $E \to B$ and a class $h \in H^3(E;\mathbb{Z})$.

We study a T-duality relation $(E,h) \stackrel{T}{\sim} (\hat{E},\hat{h})$ between such pairs and the construction of canonical isomorphism classes of T-dual pairs $[\hat{E},\hat{h}] := T(E,h)$. These investigations started with [1], and we refer to [2] for precise mathematical definitions and results. For the case of higher-dimensional torus actions we refer to [3] and the review of the literature therein.

The total space E of the $U(1)$-principal bundle is a free $U(1)$-space. The main goal of the present talk is to show that the definitions and results extend to the cases of non-free $U(1)$-actions with finite stabilizers essentially by a twist in the language. Our results are documented in [4].

Analyzing the case $E = U(1)/(\mathbb{Z}/n\mathbb{Z})$ we came to the conclusion that the first component of the T-dual $(\hat{E},\hat{h}) := T(E,0)$ has to be considered as a topological stack locally modeled by quotients of spaces by finite groups. These stacks will be called orbispaces. In the example we get $\hat{E} \cong U(1) \times [*/\mathbb{Z}/n\mathbb{Z}]$. In [4] we show that the definition of the T-duality relation and the results about canonical T-duals obtained in [2] extend to the case where B is an orbispace, $E \to B$ is a $U(1)$-principal bundle in the category of orbispaces, and $h \in H^3(E;\mathbb{Z})$ is a class in the natural extension of integral cohomology to orbispaces.

Another aspect to T-duality (and a design criterion for the definition of the T-duality relation) is the T-duality transformation $T : h(E,h) \to h(\hat{E},\hat{h})$ in twisted cohomology theories, in particular in twisted real cohomology and K-theory. In [4] we show that any twisted cohomology theory has a Borel extension to orbispaces, and this Borel extension is still T-admissible so that the T-duality transformation is an isomorphism. This is fine for real cohomology, but for K-theory (as in the equivariant case) there is a more natural extension to orbispaces. In is an interesting open problem to decide whether the latter is also T-admissible.

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Nonabelian Bundle Gerbes

Paolo Aschieri

(joint work with Branislav Jurčo)

Abelian bundle gerbes are a higher version of line bundles. Complex line bundles are geometric realizations of the integral 2nd cohomology classes $H^2(M,Z)$ on a manifold, i.e. the first Chern classes. Similarly, abelian (bundle) gerbes are the next level in realizing integral cohomology classes on a manifold, they are geometric realizations of the 3rd cohomology classes $H^3(M,Z)$. One way of thinking about abelian gerbes is in terms of their local transition functions [1]. Local "transition functions" of an abelian gerbe are complex line bundles on double overlaps of open sets satisfying cocycle conditions for tensor products over quadruple overlaps of open sets. The nice notion of abelian bundle gerbe [2] is related to this picture. Abelian gerbes and bundle gerbes can be equipped with additional structures, that of connection 1-form, that of curving (this latter is the 2-form gauge potential that corresponds to the 1-form gauge potential in line bundles) and of curvature (3-form field strength whose de Rham cohomology class is the image in $I\!R$ of the integral third cohomology class of the gerbe).

Following [3], in this talk we have reported on the nonabelian generalization of abelian bundle gerbes and their differential geometry. Nonabelian gerbes arose in the context of nonabelian cohomology [4]. Their differential geometry –from the algebraic geometry point of view– has been recently discussed in [5]. In [3] we study the subject in the context of differential geometry. We show that non-abelian bundle gerbes connections and curvings are very natural concepts in classical differential geometry. We believe that it is primarily in this context that these structures can have mathematical physics applications.

Since local transition functions of an abelian gerbe are complex line bundles (or principal $U(1)$ bundles), nonabelian gerbes should be built gluing appropriate nonabelian principal bundles (called bibundles). Bibundles admit a local description in terms of transition functions. This is the starting point for a local description of nonabelian gerbes and of their differential geometry. While this local viewpoint is usually the most suitable for calculations, only reaching a global description of these geometric structures one can grab their full essence. This is even more the case for the construction of a connection (and curving and curvature) on a nonabelian bundle gerbe. A first achievement of our research is the definition of connection one-form on a bibundle, this is a relaxed version of connection on principal bundles. Nevertheless one can define the exterior covariant derivative and curvature two-form of this connection, and prove a relaxed Cartan structural equation and the Bianchi identity. We then proceed to our main results, the definition of nonabelian bundle gerbes, and expecially of their connections, and the proof that there always exist a connection. Finally the nonabelian curving 2-form and the corresponding curvature 3-form compatible with the nonabelian bundle gerbe connection are defined and their relations studied.

We have also briefly discussed twisted nonabelian bundle gerbes, see [6]. Following the correspondence between line bundles and abelian gerbes, we have that abelian 2-gerbes are geometric realizations of the fourth integral cohomology classes $H^4(M, Z)$. We recall that a twisted nonabelian bundle is a bundle whose cocycle relations hold up to phases. These phases in turn characterize an abelian gerbe. Similarly, twisted nonabelian gerbes are a higher version of twisted bundles. The study of their properties shows that they are associated with abelian 2-gerbes in the same way that twisted bundles are associated with abelian gerbes. This is a new way of looking at (twisted) nonabelian gerbes, namely as modules for abelian 2-gerbes. Using global anomalies cancellation arguments we then see that the geometrical structure underlying a stack of M5-branes is in general indeed that of a twisted nonabelian gerbe. We can also define connections, curvings and curvature for the 2-gerbe and the twisted nonabelian gerbe. It turns out that these structures have an interpretation as M5-branes gauge fields. A prominent role is here played by the E_8 group, indeed up to the 14th-skeleton E_8 is homotopy equivalent to the Eilenerg-MacLane space $K(Z, 3)$, similarly $BE_8 \sim K(Z, 4)$ and therefore, if dim$M \leq 15$, equivalence classes of E_8 bundles on M are one to one with homotopy classes $[M, BE_8] = [M, K(Z, 4)] = H^4(M, Z)$.

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Classification of Nonabelian bundle Gerbes

Branislav Jurčo

This talk was a sequel to that of Paolo Aschieri. We continued the discussion of nonabelian bundle gerbes and their differential geometry. The emphasis was on classification using ideas from simplicial homotopy theory. The subject and some ideas are very closely related to talks given by Hendryk Pfeiffer, Urs Schreiber and Danny Stevenson.

Let us recall that given a crossed module, there is a notion of a crossed module bundle gerbe; the bundle gerbe (equipped with a trivial module with the structure group D) discussed in talk of P. Aschieri [1]. There are two points of view to a crossed module $H \stackrel{\alpha}{\to} D$. It can be viewed either as a 1-category (actually a 1-groupoid) which we will denote as $\mathcal{C}$ or as a 2-category (actually as a 2-groupoid) which we will denote as $\tilde{\mathcal{C}}$. In both cases we can form the corresponding nerves $N\mathcal{C}$ and $N\tilde{\mathcal{C}}$ and their respective geometric realizations $|N\mathcal{C}|$ and $|N\tilde{\mathcal{C}}|$. Both points of view appear to be useful.

Let us start with the simplicial space $N\mathcal{C}$. Because of its origin in a crossed module it is naturally a simplicial group and its geometric realization is a group. If H and D are Lie groups then $N\mathcal{C}$ is a simplicial Lie group and $|N\mathcal{C}|$ is a topological group. It has further interesting interpretations as the homotopy fiber of $BH \stackrel{B\alpha}{\to} BD$ or as the homotopy quotient $D//H = EH \times_\alpha D$ of D by H. In other words it is the classifying space of principal H-bundles with a chosen trivialization after their structure group has been changed from H to D using the crossed module map α. String group of [3], [4] is an example (see also talk of D. Stevenson). As with any simplicial group we can form corresponding the classifying space $\overline{W}N\mathcal{C}$. This brings as to the 2-category point of view. The classifying space $\overline{W}N\mathcal{C}$ appears to be equal equal to $N\tilde{\mathcal{C}}$, the nerve of the 2-category $\tilde{\mathcal{C}}$. This is the origin of the following observation: There is one to one correspondence between stable equivalence classes of $(H \to D)$-crossed module bundle gerbes and principal $|N\mathcal{C}|$-bundles. Hence starting from from the universal $|N\mathcal{C}|$-bundle we get also the universal crossed module gerbe and vice versa.

Locally crossed module bundle gerbes are described as follows. Let us take an open covering $\{O_\alpha\}$ of the manifold X. We can think of a simplicial space $N\{O_\alpha\}$ whose n-th component is formed by disjoint unions of n-fold intersections of O_αs. Then locally a crossed module bundle gerbe is given as a simplicial map $N\{O_\alpha\} \to \overline{W}N\mathcal{C}$, i.e. as a simplicial $N\mathcal{C}$-bundle. This description is useful for the following reason. Namely although (in the case of a Lie crossed module) $N\mathcal{C}$ is a simplicial Lie group $|N\mathcal{C}|$ is not necessarily a Lie group. So there is no differential geometric connection on a principal $|N\mathcal{C}|$-bundle. But we can introduce a notion of a connection on a simplicial principal bundle when the structure group is a simplicial Lie group. This is a collection of connections on the corresponding components of the simplicial bundle which are properly compatible under the face and degeneracy maps. Similarly we can introduce the B-field on a simplicial bundle. In the case of a $N\mathcal{C}$-bundle over $N\{O_\alpha\}$, described above, these reproduce the crossed module bundle gebre connection and curving B-field on the corresponding crossed module gerbe [2], [1].

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Quillen Superconnection, Supersymmetric WZW Model, and Twisted K-Theory

Jouko Mickelsson

Gauge symmetry breaking in quantum field theory is described in terms of families index theory. The Atiyah-Singer index formula gives via the Chern character cohomology classes in the moduli space of gauge connections and of Riemann metrics. In particular, the 2-form part is interpreted as the curvature of the Dirac determinant line bundle, which gives an obstruction to gauge covariant quantization in the path integral formalism. The obstruction depends only on the K-theory class of the family of operators.

In the Hamiltonian quantization odd forms on the moduli space become relevant. The obstruction to gauge covariant quantization comes from the 3-form part of the character. The 3-form is known as the Dixmier-Douady class and is also the (only) characteristic class of a gerbe; this is the higher analogue of the first Chern class (in path integral quantization) classifying complex line bundles.

The next step is to study familes of ' operators' which are only projectivly defined; that is, we have families of hamiltonians which are defined locally in the moduli space but which refuse to patch to a globally defined family of operators. The obstruction is given by the Dixmier-Douady class, an element of integral third cohomology of the moduli space. On the overlaps of open sets the operators are related by a conjugation by a projective unitary transformation. This leads to the definition of twisted K-theory.

In the present talk I will review the basic definitions of both ordinary K-theory and twisted K-theory The construction of twisted (equivariant) K-theory classes on compact Lie groups G is outlined using a supersymmetric model in $1+1$ dimensional quantum field theory. The families of Fredholm operators are acting in a tensor product of a 'bosonic' and of a 'fermionic' Fock space, both carrying a representation of a central extension of the loop group LG. Finally, the Quillen superconnection formula is applied to the projective family of Fredholm operators giving a Chern character alternatively with values in Deligne cohomology on the base G or in global twisted de Rham cocycles. The use of Quillen superconnection has been proposed in general context of twisted K-theory by Daniel Freed but in this talk I will give the details in simple terms using the supersymmetric Wess–Zumino-Witten model.

The case of $G = SU(2)$ is computed (joint work with Juha-Pekka Pellonpää, to be published) explicitly. Although one knows on general grounds that the class in twisted cohomology obtained from the supeconnection is zero, it is interesting to observe that the calculation gives a nonzero cocycle in ordinary cohomology, with a coefficient in front of the 3-form which is given by the dimension of the $SU(2)$ representation on the lowest weight sector in the bosonic sector of the model.

WZW Models with Non Simply Connected Targets via Gerbes and Gerbe Modules

Krzysztof Gawedzki

Bundle gerbes [10] and gerbe-modules [8, 1] find a natural application in analysis of the two-dimensional sigma models of field theory. We shall sketch here, basing on refs. [6, 7, 5], how they help to solve the quantum Wess-Zumino-Witten (WZW) conformal sigma models [11] with non simply connected target groups pointing to the geometric origin of the finite group cohomology that appeared in the algebraic approach to such theories [3]. The geometric approach provides a uniform and effective treatment of the classical as well as quantum theories, in the absence of boundaries and in their presence. It should extend to the supersymmetric WZW models and to the coset conformal field theories.

Gerbes, gerbe modules and branes. Bundle gerbes (of line bundles with unitary connection) on manifold M are geometric objects associated to a closed 3-form H on M, called their curvature. Given H, gerbes $\mathcal{G}$ with curvature H exist if and only if the periods of $\frac{1}{2\pi}H$ are integers. In the latter case, non (stably) isomorphic gerbes are related by twists by elements of $H^2(M, U(1))$.

Gerbe modules $\mathcal{E}$ on M are versions of vector bundles with unitary connection twisted by line bundles entering a gerbe $\mathcal{G}$ on M. Finite rank $\mathcal{G}$-modules exist if and only if the curvature H of $\mathcal{G}$ is an exact form.

$\mathcal{G}$-branes $\mathcal{D}$ in M are pairs $(D, \mathcal{E})$ where D is a submanifold of M and $\mathcal{E}$ is a $\mathcal{G}_D$-module on D of finite rank, where $\mathcal{G}_D$ denotes the pullback of gerbe $\mathcal{G}$ to D. We shall call D the support of the $\mathcal{G}$-brane $\mathcal{D}$ and the rank of $\mathcal{E}$ the rank of $\mathcal{D}$.

By transgression, a gerbe $\mathcal{G}$ on M defines a line bundle $\mathcal{L}_{\mathcal{G}}$ with unitary connection on the loop space LM [4, 6]. Similarly, a gerbe $\mathcal{G}$ on M and a pair of $\mathcal{G}$-branes $\mathcal{D}_i = (D_i, \mathcal{E}_i)$ of ranks N_i, $i = 0, 1$, defines a vector bundle $\mathcal{E}_{\mathcal{D}_0}^{\mathcal{D}_1}$ with unitary connection of rank $N_0 N_1$ on the space of paths $\varphi : [0, \pi] \to M$ such that $\varphi(0) \in D_0$ and $\varphi(\pi) \in D_1$.

Geometric quatization of the WZW sigma models. Let G be a connected compact simple Lie group. Such groups are target manifolds of the WZW sigma models. Let for $k > 0$,

$$H_G \;=\; \tfrac{k}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3\,, \tag{0.1}$$

where $\mathrm{tr}\,XY$ stands for the properly normalized Killing form on the Lie algebra $\mathbf{g}$ of G, be an invariant closed 3-form on G. The bulk WZW model of level k, with group G as the target, is specified by giving a gerbe $\mathcal{G}$ on G with curvature H_G. The space of quantum states of such model is the space of sections of the line bundle $\mathcal{L}_\mathcal{G}$ on the loop group LG:

$$\mathbf{H} \;=\; \Gamma(\mathcal{L}_\mathcal{G})\,. \tag{0.2}$$

This space carries a geometric action of the double current algebra $\hat{\mathbf{g}}\times\hat{\mathbf{g}}$ associated to the Lie algebra $\mathbf{g}$. The representation content of $\mathbf{H}$ may be found by identifying the highest weight sections of $\mathcal{L}_\mathcal{G}$. By the Sugawara construction, also the double Virasoro algebra $Vir \times Vir$ acts in $\mathbf{H}$.

The boundary WZW theory requires, in addition to the choice of a gerbe $\mathcal{G}$ with curvature H_G, a choice of CG-branes specifying the boundary conditions. We shall only consider the so called symmetric $\mathcal{G}$-branes $\mathcal{D} = (D, \mathcal{E})$ such that D is a conjugacy class in G and the curvature form of $\mathcal{E}$ is equal to the scalar 2-form

$$F_D \;=\; \tfrac{k}{8\pi}\,\mathrm{tr}(g^{-1}dg)\,\tfrac{1+Ad_g}{1-Ad_g}(g^{-1}dg)\,. \tag{0.3}$$

The space of quantum states of the WZW theory with the boundary conditions specified by a pair of symmetric $\mathcal{G}$-branes $\mathcal{D}_i$, $i = 0, 1$, is

$$\mathbf{H}^{\mathcal{D}_1}_{\mathcal{D}_0} \;=\; \Gamma(\mathcal{E}^{\mathcal{D}_1}_{\mathcal{D}_0}) \tag{0.4}$$

and it carries a geometric action of a single copy of the current algebra $\hat{\mathbf{g}}$ (due to the restriction to the symmetric $\mathcal{G}$-branes) and an action of the Virasoro algebra Vir. The representation content of $\mathbf{H}^{\mathcal{D}_1}_{\mathcal{D}_0}$ may again be found by identifying the highest weight sections of $\mathcal{E}^{\mathcal{D}_1}_{\mathcal{D}_0}$.

Case of simply connected groups G. For a proper normalization of the Killing form, a gerbe $\mathcal{G}$ on G with curvature H_G exists if and only if the level k is an integer. For each such k, $\mathcal{G}$, explicitly constructed in [9], is unique (up to a stable isomorphism). The space of states of the bulk group G WZW theory decomposes according to

$$\mathbf{H} \;=\; \bigoplus_{\lambda\in P^+_k} \hat{V}_\lambda \otimes \overline{\hat{V}_\lambda} \tag{0.5}$$

where P^+_k is the set of the highest weights of the unitary highest weight modules $\hat{V}_\lambda$ of the current algebra $\hat{\mathbf{g}}$ of level k. These are the weights that lie in the dilated positive Weyl alcove $k\mathcal{A}_W$.

Symmetric $\mathcal{G}$-branes $\mathcal{D}$ in G are (up to an isomorphism) of the form

$$\mathcal{D} \;=\; (\mathcal{C}_\lambda, \mathcal{E}_1 \otimes \boldsymbol{C}^N)\,, \tag{0.6}$$

where $\mathcal{C}_\lambda$ is the conjugacy class of $\mathrm{e}^{2\pi\mathrm{i}\,\lambda/k}$ for $\lambda \in P_k^+$ and $\mathcal{E}_1$ is the unique $\mathcal{G}_{\mathcal{C}_\lambda}$-module of rank 1 with the curvature given by Eq. (0.3). Given a pair of symmetric $\mathcal{G}$-branes $\mathcal{D}_i$ of rank N_i, supported by the conjugacy classes $\mathcal{C}_{\lambda_i}$, $i = 0, 1$, the space of states of the boundary WZW model decomposes according to

$$\mathbf{H}_{\mathcal{D}_0}^{\mathcal{D}_1} \;=\; \bigoplus_{\lambda \in P_k^+} \boldsymbol{C}^{N_0} \otimes \overline{\boldsymbol{C}^{N_1}} \otimes M_{\lambda_0\lambda}^{\lambda_1} \otimes \overline{\hat{V}_\lambda}\,, \tag{0.7}$$

where the spaces of rank 1 multiplicities may be identified [5] with linear subspaces of the highest weight modules V_λ of the Lie algebra $\mathbf{g}$:

$$M_{\lambda_0\lambda}^{\lambda_1} \;=\; \Big\{\, v \in V_\lambda \quad \Big| \quad h\,v = \mathrm{tr}\, h(\lambda_1 - \lambda_0)\, v \quad \text{for} \quad h \in \mathbf{t} \subset \mathbf{g}\,, \quad \mathrm{e}_{\alpha_i}^{1+\mathrm{tr}\,\alpha_i^\vee \lambda_1} v = 0\,, \quad \mathrm{e}_{-\phi}^{k+1-\mathrm{tr}\,\phi^\vee \lambda_1} v = 0 \,\Big\}\,, \tag{0.8}$$

where $\mathbf{t}$ denotes the Cartan subalgebra of $\mathbf{g}$, α_i and $\alpha_i^\vee$ its simple roots and coroots and $\phi = \phi^\vee$ its highest root and the corresponding coroot. The dimensions of the multiplicity spaces $M_{\lambda_0\lambda}^{\lambda_1}$ are equal to the (Verlinde) fusion coefficients $N_{\lambda_0\lambda}^{\lambda_1}$.

Case of the non simply connected groups $G' = G/Z$. There is an obstruction $[u] \in H^3(Z, U(1))$ to the existence of a gerbe $\mathcal{G}'$ on G' with curvature $H_{G'}$ (that has to pull back to the gerbe $\mathcal{G}$ on G with curvature H_G). It may be described explicitly. For $N(T) \subset G$ denoting the normalizer of the Cartan subgroup $T \subset G$, let a map $Z \ni z \mapsto w_z \in N(T)$ be such that

$$z\,\mathrm{e}^{2\pi\mathrm{i}\,\tau/k} \;=\; w_z^{-1} \mathrm{e}^{2\pi\mathrm{i}\,(z\tau)/k} w_z \tag{0.9}$$

for τ and $z\tau$ belonging to the dilated positive Weyl alcove $k\mathcal{A}_W \subset \mathbf{t}$. Let $b_{z,z'} \in \mathbf{t}$ satisfy $w_z w'_z w_{zz'}^{-1} = \mathrm{e}^{\mathrm{i}\,b_{z,z'}}$. One may take [7, 5]

$$u_{z,z',z''} \;=\; \mathrm{e}^{\mathrm{i}\,k\,\mathrm{tr}\,\lambda_z\, b_{z',z''}}\,, \tag{0.10}$$

where λ_z is the simple weight for which $z = \mathrm{e}^{2\pi\mathrm{i}\,\lambda_z}$. Triviality of the cohomology class $[u]$ selects the values of the level k, first found in [2], for which the gerbes $\mathcal{G}'$ on the non simply connected group G' and, consequently, the WZW model with group G' as the target, exist. Different (more precisely, non stably isomorphic) gerbes $\mathcal{G}'$ differ by a twist in $H^2(Z, U(1))$. The latter group is trivial for cyclic Z. For simple compact groups only $Spin(4n)$ has the non cyclic center $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ for which $H^2(Z, U(1)) = \boldsymbol{Z}_2$. There are then two (stably) non isomorphic gerbes $\mathcal{G}'_\pm$ on $Spin(4n)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2) = SO(4n)/\boldsymbol{Z}_2$ and two WZW theories, as already noted in [2]. The ambiguity provides an example of Vafa's discrete torsion. The representation content of the space of states

$$\mathbf{H}' \;=\; \Gamma(\mathcal{L}_{\mathcal{G}'}) \tag{0.11}$$

of the bulk WZW theory for all groups G' was described in ref. [2].

The conjugacy classes in G' are images of the conjugacy classes in G. Only images of the conjugacy classes $\mathcal{C}_\lambda$ may support symmetric $\mathcal{G}'$-branes. The image of $\mathcal{C}_\lambda$ in G' coincides with that of $z\mathcal{C}_\lambda = \mathcal{C}_{z\lambda}$. We shall denote it $\mathcal{C}'_{[\lambda]}$ where $[\lambda]$ stands for the Z-orbit of λ in P_k^+. $\mathcal{C}'_{[\lambda]}$ may be identified with any of the quotients $\mathcal{C}_\lambda/Z_{[\lambda]}$, where the stabilizer subgroup $Z_{[\lambda]}$ is composed of $z\in Z$ such that $\mathcal{C}_\lambda = z\mathcal{C}_\lambda$. There is an obstruction $[c]\in H^2(Z,U(1)^{[\lambda]})\cong H^2(Z_{[\lambda]},U(1))$ to the existence of a rank 1 $\mathcal{G}'_{\mathcal{C}'_{[\lambda]}}$-module $\mathcal{E}'_1$ on $\mathcal{C}'_{[\lambda]}$ that pulls back to rank 1 $\mathcal{G}_{\mathcal{C}_\lambda}$-modules $\mathcal{E}_1$ on $\mathcal{C}_\lambda$. Above, $U(1)^{[\lambda]}$ stands for the Z-module of the $U(1)$-valued functions on the orbit $[\lambda]$. One may take [5]

$$c_{\lambda;z,z'} \;=\; \mathrm{e}^{\mathrm{i}\,\mathrm{tr}\,\lambda\, b_{z,z'}}\, v_{z,z'} \tag{0.12}$$

where $\delta v = u$. If $[c]$ is trivial then the symmetric $\mathcal{G}'$ branes of rank N supported by $\mathcal{C}'_{[\lambda]}$ have the form

$$\mathcal{D}' \;=\; (\mathcal{C}'_{[\lambda]}, \mathcal{E}'_1(1)\oplus\cdots\oplus\mathcal{E}'_1(N))\,, \tag{0.13}$$

where $\mathcal{E}'_1(n)$ are chosen from $|Z|$ non isomorphic rank 1 $\mathcal{G}'_{\mathcal{C}'_{[\lambda]}}$-modules that differ by twists in $H^1(Z,U(1)^{[\lambda]})\cong H^1(Z_{[\lambda]},U(1))$, i.e. by characters of $Z_{[\lambda]}$.

The obstruction class $[c]$ may be non trivial only if the stabilizer subgroup $Z_{[\lambda]}$ is not cyclic. It is indeed non trivial for $G = Spin(4n)$ when $Z = Z_{[\lambda]} = \boldsymbol{Z}_2\times\boldsymbol{Z}_2$ but only for the choice $\mathcal{G}'_-$ of the gerbe on $SO(4n)/\boldsymbol{Z}_2$. In this case, there is no $U(1)$-valued solution to the equation $c=\delta d$, i.e. no numbers $d_{\lambda;z}$ in $U(1)$ for $\lambda\in[\lambda]$ and $z\in Z$ such that

$$c_{\lambda;z,z'} \;=\; d_{z^{-1}\lambda;z'}\, d^{-1}_{\lambda;zz'}\, d_{\lambda;z}\,. \tag{0.14}$$

There are however solutions $d_{\lambda;z}$ with values in unitary matrices of higher rank. Up to a conjugation, they are direct sums of a rank 2 solution expressed by the Pauli matrices [5]. They give rise to the symmetric $\mathcal{G}'_-$-branes

$$\mathcal{D}' \;=\; (\mathcal{C}'_{[\lambda]}, \mathcal{E}_2\otimes\boldsymbol{C}^{N/2}) \tag{0.15}$$

of even rank N. That exhaust the classification of the (isomorphism classes of) symmetric $\mathcal{G}'$-branes in non simply connected simple groups G'.

In order to describe the Hilbert space of states of the boundary G' WZW theory for any choice of the gerbe $\mathcal{G}'$ and of a pair of symmetric $\mathcal{G}'$-branes $\mathcal{D}'_i = (\mathcal{C}'_{[\lambda_i]},\mathcal{E}'_i)$ of ranks N_i, one notes that $\mathcal{D}'_i$ are determined (up to isomorphisms) by solutions $d^i_{\lambda_i,z}$ of Eq. (0.14) with values in rank N_i matrices (of diagonal or (2×2)-block diagonal form). $\mathcal{D}'_i$ pull back to the G-theory $\mathcal{G}$-branes $\mathcal{D}_i = (\mathcal{C}_{\lambda_i},\mathcal{E}_1\otimes\boldsymbol{C}^{N_i})$ for $\lambda_i\in[\lambda_i]$ that we shall call compatible (with the $\mathcal{G}'$-branes $\mathcal{D}'_i$). Consider the corresponding space of all compatible boundary G-theory states

$$\tilde{\mathbf{H}}^{\mathcal{D}'_1}_{\mathcal{D}'_0} \;=\; \bigoplus_{\substack{(\mathcal{D}_0,\mathcal{D}_1)\\ \text{compatible}}} \mathbf{H}^{\mathcal{D}_1}_{\mathcal{D}_0}\,. \tag{0.16}$$

Recall the decomposition (0.7). There is a natural action of group Z in $\tilde{\mathbf{H}}^{\mathcal{D}'_1}_{\mathcal{D}'_0}$ (genuine, not projective) composed of the linear maps

$$\boldsymbol{C}^{N_0}\otimes\overline{\boldsymbol{C}^{N_1}}\otimes M^{\,z^{-1}\lambda_1}_{z^{-1}\lambda_0\,\lambda}\otimes\overline{\hat{V}_\lambda}$$

$$\downarrow \; d^0_{\lambda_0;z} \otimes \overline{d^1_{\lambda_1;z}} \otimes w_z \otimes Id$$

$$\boldsymbol{C}^{N_0} \otimes \overline{\boldsymbol{C}^{N_1}} \otimes M^{\lambda_1}_{\lambda_0\,\lambda} \otimes \overline{\hat{V}_\lambda} \tag{0.17}$$

The space of states $\mathbf{H}^{\mathcal{D}'_1}_{\mathcal{D}'_0}$ of the boundary G'-theory may be canonically identified with the subspace of the Z-invariant states in $\tilde{\mathbf{H}}^{\mathcal{D}'_1}_{\mathcal{D}'_0}$. The situation in the boundary theory is then simpler than in the bulk one where taking Z-invariant states of the G-theory reproduces only the untwisted sector of the G'-theory.

The boundary operator product expansion of the G'-theory may be obtained by projecting the one of the G-theory to the Z-invariant sector [5].

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Twisted equivariant K-theory and D-brane charges in coset models

Sakura Schafer-Nameki

I shall discuss an extension of the result by Freed, Hopkins and Teleman, that twisted G-equivariant K-theory of a simple, compact, simply-connected Lie group G is the Verlinde algebra, to the case of H-equivariant K-theory, for H a connected, maximal rank subgroup of G. In particular, these K-theory groups are shown to agree with the charge lattices of D-branes in $\mathcal{N} = 2$ superconformal coset conformal field theories.

Let G be a simple, compact, simply-connected Lie group and $\tau \in H^3(G, \mathbb{Z}) = \mathbb{Z}$. Consider the action of G on itself via conjugation and let $g^\vee$ be the dual Coxeter number of G. Then Freed, Hopkins and Teleman have proven that the τ-twisted

G-equivariant K-theory of G, ${}^{\tau}K_G(G)$, is isomorphic as a ring to the Verlinde ring $V_k(G) = R_G/I_k(G)$, where R_G the representation ring of G and $I_k(G)$ the Verlinde ideal, of the level $\tau = k + g^\vee$ Wess-Zumino-Witten (WZW) conformal field theory, that is

$$ {}^{\tau}K_G(G) \equiv V_k(G)\,. \tag{0.1} $$

On the other hand it has been conjectured that D-brane charges in string theory backgrounds with non-trivial NSNS three-form flux $\tau \in H^3(M,\mathbb{Z})$, M being the target space of the string theory, are classified by a suitable variant of twisted K-theory. In particular, the K-theories in (0.1) would classify brane charges in the G/G coset model, which in fact is topological. From a string theoretical point of view more interesting are the (non-topological) Kazama-Suzuki (KS) coset models, which preserve $\mathcal{N} = 2$ supersymmetry and are constructed from pairs (G, H) with G satisfying the above-specified conditions and H a connected, maximal rank subgroup.

We have shown in [1, 2] that the associated twisted H-equivariant K-theories are given by

$$ {}^{\tau}K_H(G) \equiv \frac{R_H}{I_k(G)}\,. \tag{0.2} $$

For the comparison to D-brane charges in the KS models, it is vital to taken into account a further equivariance, which in the conformal field theory corresponds to selection rules, which form a group isomorphic to the common centre Z of G and H. The rank of ${}^{\tau}K_{H/Z}(G)$ is then $\frac{d_k(G)}{l_k(G)}\frac{|W_G|}{|W_H|}$, where $d_k(G)$ is the rank of the Verlinde ring of G, W denotes the Weyl groups and $l(Z)$ the length of the orbits of the Z-action on $H//H$. This is in agreement with the known D-brane charge computations in boundary conformal field theory. In particular, this implies also that the rank of the K-theory charge lattice is equal to the rank of the ring of chiral primaries of the G/H Kazama-Suzuki model. However, the ring structure in (0.2) differs. The proof of (0.2) is based on (0.1) and a generalization of the Künneth theorem and can be generalized to equivariant K-theories with respect to not necessarily maximal rank subgroups H.

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Computation of Some K-groups

Volker Braun

1. Introduction

By now a well-established result is that the D-brane charges in string theory are precisely the K-theory group of the space-time, see [1]. Hence, computing certain K-groups has immediate physical interest. For example, cancellation of the total D-brane charge for compact directions places additional restrictions on allowed compactifications, which eliminates some torus orientifold constructions.

In this talk, I will review the computation of the twisted K-theory that is relevant for $N = 1$ supersymmetric Wess-Zumino-Witten models. I solved the case for compact, simple, simply connected Lie groups in [2]. As a non-simply connected example, I will present $SO(3)$ in Section 3. The latter is joint work with Sakura Schäfer-Nameki [3]

2. Twisted K-theory for Lie Groups

In the following, let G always be a compact, simple, simply connected Lie group, together with a gerbe on G with characteristic class

$$t \in H^3\Big(G; \mathbb{Z}\Big)\,. \tag{2.1}$$

The corresponding Grothendieck group of twisted vector bundles on G is the twisted K-theory ${}^tK(G)$. It is a generalized (twisted) cohomology theory. To compute the K-groups, we relate it to equivariant twisted K-theory by rewriting

$${}^tK^*(G) = {}^tK^*_G(G^{\mathrm{Tr}} \times G^{\mathrm{L}}) = {}^tK^*_G(G^{\mathrm{Ad}} \times G^{\mathrm{L}})\,, \tag{2.2}$$

where the superscripts refer to the **T**rivial, **L**eft, and **Ad**joint action of G on itself. The first equality is obvious, the second follows from the G-isomorphism $G^{\mathrm{Tr}} \times G^{\mathrm{L}} = G^{\mathrm{Ad}} \times G^{\mathrm{L}}$ through conjugation. To compute the K-theory of the product, we use a certain equivariant Künneth theorem which follows from [4]:

Theorem 1 (Equivariant Künneth Theorem)**.** *Let G be a compact, simple, simply connected Lie group. Let X be a G-space with twist class, let Y be a G-space. Then there is a spectral sequence*

$$E_2^{-p,*} = Tor^p_{RG}\Big({}^tK^*_G(X),\ K^*_G(Y)\Big) \Rightarrow {}^tK^{p+*}_G(X)\,. \tag{2.3}$$

The point of doing so is that we can now apply the theorem of Freed-Hopkins-Teleman [5], which identifies the twisted equivariant K-theory with the Verlinde algebra at level $k = t - \check{h}$,

$${}^tK^*_G(G^{\mathrm{Ad}}) = RG/I_k\,. \tag{2.4}$$

Hence, it remains to compute

$$Tor^p_{RG}\Big({}^tK^*_G(G^{\mathrm{Ad}}),\ K^*_G(G^{\mathrm{L}})\Big) = Tor^p_{RG}\Big(RG/I_k,\ \mathbb{Z}\Big)\,. \tag{2.5}$$

A widely believed fact is that the Verlinde algebra is a complete intersection, and hence there exists a Koszul resolution. Although not strictly proven, this was checked for a large number of cases in [6]. Henceforth, I assume that there exists a regular sequence $y_1, \ldots, y_n$, $n = rk(G)$. A bit of homological algebra yields

$$(2.6)\quad Tor^p_{RG}\Big(RG/I_k,\ \mathbb{Z}\Big) = Tor^p_{RG}\Big(RG/\langle y_1, \ldots, y_n\rangle,\ \mathbb{Z}\Big) = \bigoplus_{2^{n-1}} \mathbb{Z}_{\gcd(y_1,\ldots,y_n)}\,.$$

Finally, what about higher differentials and extension ambiguities? The dual K-homology spectral sequence is a spectral sequence of algebras under the Pontryagin product. One can use this to show that there are no further differentials, and that all extension ambiguities are trivial. Hence,

$$(2.7)\qquad {}^tK^*(G) = \bigoplus_{2^{n-1}} \mathbb{Z}_{\gcd(y_1,\ldots,y_n)}\,.$$

3. $SO(3)$ Wess-Zumino-Witten Model

As an example of a non-simply connected Lie group, let us consider $SO(3)$. This Wess-Zumino-Witten (WZW) model was treated from the boundary conformal field theory side in [7], where it was found that the D-brane charge groups is either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether $\kappa \overset{\text{def}}{=} k+1$ is odd or even. Interestingly, the charge groups do not grow with the level in this example. This is in contradiction to the usual Atiyah-Hirzebruch spectral sequence, which predicts ${}^kK^*\big(SO(3)\big) = \mathbb{Z}_2 \oplus \mathbb{Z}_k$. Our resolution to this paradox is that D-brane charges in the $SO(3)$ WZW model, that is the bosonic $SO(3)$ supersymmetrized with free fermions, correspond to another twisted K-theory. Recall that the possible twists of K-theory actually contain

$$(3.8)\qquad H^1\Big(SO(3);\, \mathbb{Z}_2\Big) \oplus H^3\Big(SO(3);\, \mathbb{Z}\Big) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}\,.$$

The WZW model of [7] corresponds to the $(-,\kappa)$ twisted K-theory! We can easily estimate the resulting K-groups from a twisted Atiyah-Hirzebruch spectral sequence

$$(3.9)\qquad E_2 = {}^-H^p\Big(SO(3);\, K^q(\text{pt.})\Big) \Rightarrow {}^{(-,\kappa)}K^{p+q}\big(SO(3)\big)\,.$$

to be either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$, depending on an extension ambiguity.

To resolve this ambiguity, we again rewrite the K-groups as certain equivariant K-groups. But since the Künneth theorem fails for non-simply connected groups, we chose to work $SU(2)$ equivariant, and obtain

$$(3.10)\qquad {}^tK^*\big(SO(3)\big) = {}^tK^*_{SU(2)}\Big(SO(3)^{\text{Ad}} \times SU(2)^{\text{L}}\Big)$$

We found the twisted equivariant K-groups ${}^tK^*_{SU(2)}\big(SO(3)^{\text{Ad}}\big)$ by a Mayer-Vietoris sequence for a certain cell decomposition, whose details I am going to skip. The result is that

$$(3.11)\quad \begin{aligned} {}^{(-,\kappa)}K^0_{SU(2)}\Big(SO(3)\Big) &= 0 \\ {}^{(-,\kappa\text{ odd})}K^1_{SU(2)}\Big(SO(3)\Big) &= \mathbb{Z}[\Lambda,\sigma]\Big/\Big\langle {\scriptstyle \Lambda(\sigma-1),\, \sigma^2-1,\, p_\kappa(\Lambda)} \Big\rangle \\ {}^{(-,\kappa\text{ even})}K^1_{SU(2)}\Big(SO(3)\Big) &= \mathbb{Z}[\Lambda,\sigma]\Big/\Big\langle {\scriptstyle \Lambda(\sigma-1),\, \sigma^2-1,\, p_\kappa(\Lambda)+(-1)^{\frac{\kappa}{2}}(1+\sigma)} \Big\rangle \end{aligned}$$

as $RSU(2) = \mathbb{Z}[\Lambda]$ modules, where p_κ are certain degree κ polynomials. A bit of homological algebra then shows that only the Tor^0 in the equivariant Künneth theorem is nonvanishing, and moreover that

$$(3.12)\quad {}^{(-,\kappa)}K^*\Big(SO(3)\Big) = E_2^{0,*} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \kappa \text{ odd} \\ \mathbb{Z}_4 & \kappa \text{ even,} \end{cases}$$

as predicted by the boundary conformal field theory.

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Twisted K-theory and Bundle gerbes

Alan L. Carey

(joint work with Bai-Ling Wang)

This talk was a sequel to that of Michael Murray. The research described was partly motivated by the theorem of Freed-Hopkins-Teleman [7, 8] relating the Verlinde ring to twisted K-theory. In Murray's talk the relationship between Chern-Simons gauge theories and WZW models with target a compact semisimple Lie group is explained based on [3]. This result uncovers the relevance of multiplicative gerbes. A consequence of the multiplicative structure is used in the non-simply connected version of the Freed-Hopkins-Teleman theorem. In the talk I explained an interesting application of multiplicative bundle gerbes.

The results use the notion of a 'generalized rank n bundle gerbe D-brane' for a bundle gerbe $\mathcal{G}$ over simple Lie group G. It is a smooth manifold Q with a smooth map $\mu : Q \to G$ such that the pull-back bundle gerbe $\mu^*(\mathcal{G})$ admits a rank

n bundle gerbe module [4]. There is also a corresponding notion for G-equivariant bundle gerbes $\mathcal{G}$ over a G-manifold M.

When the compact simple Lie group G is simply-connected then we can construct a G-equivariant bundle gerbe $\mathcal{G}_k$ over G whose Dixmier-Douady class is represented by a multiple by a positive integer k of the canonical bi-invariant 3-form on G.

A particularly interesting example of a generalized G-equivariant bundle gerbe D-brane is provided by a quasi-Hamiltonian manifold (M, ω, μ) where M is a G-manifold, ω is an invariant 2-form and $\mu : M \to G$ is a group-valued moment map. Quasi-hamiltonian manifolds are extensively studied by Alekseev-Malkin-Meinrenken in [1]. The correspondence between quasi-Hamiltonian manifolds and Hamiltonian LG-manifolds at level k is illustrated by the following diagram:

(0.1)
$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\mu}} & L\mathfrak{g}^* \\ \pi \downarrow & & \downarrow Hol \\ M & \xrightarrow{\mu} & G, \end{array}$$

where $\hat{\mu} : \hat{M} \to L\mathfrak{g}^*$ is the moment map for the Hamiltonian LG-action at level k and the vertical arrows define ΩG-principal bundles. The quasi-Hamiltonian manifold, when "pre-quantizable", is naturally a generalized rank 1 bundle gerbe D-brane of the bundle gerbe over G.

When G is semisimple and simply connected, any bundle gerbe $\mathcal{G}_k$ is multiplicative and hence related to the Chern-Simons bundle 2-gerbe of [3] over the classifying space BG . We can then introduce the moduli spaces of flat G-connections on Riemann surfaces and the generalized bundle gerbe D-branes they define. Using [11] we define the fusion category $(\mathcal{Q}_{G,k}, \boxtimes)$ of generalized bundle gerbe D-branes of $\mathcal{G}_k$ to be the category of pre-quantizable quasi-Hamiltonian manifolds with fusion product

$$(M_1, \omega_1, \mu_1) \boxtimes (M_2, \omega_2, \mu_2) = (M_1 \times M_2, \omega_1 + \omega_2 + \frac{k}{2} < \mu_1^* \theta, \mu_2^* \bar{\theta} >, \mu_1 \cdot \mu_2),$$

where the G-action on $M_1 \times M_2$ is via the diagonal embedding $G \to G \times G$, θ, $\bar{\theta}$ are the left and right Maurer-Cartan forms on G, and $\mu_1 \cdot \mu_2(x_1, x_2) = \mu_1(x_1) \cdot \mu_2(x_2)$. This fusion product and the corresponding fusion product on Hamiltonian LG-manifolds were studied in [11].

Let $R_k(LG)$ be the free group over $\mathbb{Z}$ generated by the isomorphism classes of positive energy, irreducible, projective representations of LG at level k. The central extension of LG at level k we write as $\widehat{LG}$. The positive energy representation labelled by $\lambda \in \Lambda_k^*$ acts on $\mathcal{H}_\lambda$ and the Kac-Peterson character of $\mathcal{H}_\lambda$ is

(0.2)
$$\chi_{k,\lambda}(\tau) = Tr_{\mathcal{H}_\lambda} e^{2\pi i \tau (L_0 - \frac{c}{24})},$$

where $\tau \in \mathbb{C}$ with $Im(\tau) > 0$, L_0 is the energy operator on $\mathcal{H}_\lambda$ (Cf.[12]), and $c = \dfrac{k dim G}{k + h^\vee}$ is the Virasoro central charge. We mention that $e^{2\pi i \tau (L_0 - \frac{c}{24})}$ is a trace class operator (cf Theorem 6 in [9] and Lemma 2.3 in [6]) for $\tau \in \mathbb{C}$ with $Im(\tau) > 0$. Equipped with the fusion ring structure:

$$\chi_{\lambda,k} * \chi_{\mu,k} = \sum_{\nu \in \Lambda_k^*} N_{\lambda,\mu}^{\nu} \chi_{\nu,k},$$

where $N_{\lambda,\mu}^{\nu}$ is the Verlinde fusion coefficient, we obtain $(R_k(LG), *)$, the Verlinde ring.

Motivated by Guillemin-Sternberg's "quantization commutes with reduction" philosophy, we define a quantization functor on the fusion category of generalized bundle gerbe D-branes of $\mathcal{G}_k$ using $Spin^c$ quantization of the reduced spaces:

$$\chi_{k,G} : \mathcal{Q}_{G,k} \longrightarrow R_k(LG).$$

Note that for a quasi-Hamiltonian manifold M obtained from a pre-quantizable Hamiltonian G-manifold, $\chi_{k,G}(M)$ is the equivariant index of the $Spin^c$ Dirac operator twisted by the pre-quantization line bundle.

Theorem: *The quantization functor* $\chi_{k,G} : (\mathcal{Q}_{G,k}, \boxtimes) \longrightarrow (R_k(LG), *)$ *satisfies*

$$\chi_{k,G}(M_1 \boxtimes M_2) = \chi_{k,G}(M_1) * \chi_{k,G}(M_2),$$

where the product $*$ *on the right hand side denotes the fusion ring structure on the Verlinde ring* $(R_k(G), *)$.

The fusion product on Hamiltonian LG-manifolds at level k involves the moduli space of flat connections on a canonical pre-quantization line bundle over the 'trousers' $\Sigma_{0,3}$ as in [1]. The multiplicative property of the bundle gerbe $\mathcal{G}_k$ over G is essential for this part of the construction.

The difficulty with the above constructions lies in their extension to the case when G is not simply connected. I discussed various subtle issues concerning the non-simply connected case.

Given a compact, connected, non-simply connected simple Lie group $G = \tilde{G}/Z$ for a subgroup Z in the center $Z(\tilde{G})$ of the universal cover $\tilde{G}$, we construct a G-equivariant bundle gerbe $\mathcal{G}_{(k,\chi),G}$ associated to a multiplicative level k and a character $\chi \in Hom(Z, U(1))$. (The level k is defined as the positive integral multiple of the generator of $H^3(G, \mathbb{Z})$ giving the Dixmier-Douady class $\mathcal{G}_{(k,\chi),G}$.) It is multiplicative if it is transgressed from $H^4(BG, \mathbb{Z})$ to $H^3(G, \mathbb{Z})$.

The G-equivariant bundle gerbe $\mathcal{G}_{(k,\chi),G}$ is obtained from the central extension of LG in [13], $1 \to U(1) \longrightarrow \widehat{LG}_\chi \longrightarrow LG \to 1$, associated to (k, χ). One may classify irreducible positive energy representations of $\widehat{LG}_\chi$ following the work of Toledano Laredo in [13].

Let $R_{k,\chi}(LG)$ be the Abelian group generated by the positive energy, irreducible representations of $\widehat{LG}_\chi$. We define the category $\mathcal{Q}_{(k,\chi),G}$ of G-equivariant bundle gerbe modules of $\mathcal{G}_{(k,\chi).G}$ and the quantization functor

$$\chi_{(k,\chi),G} : \mathcal{Q}_{(k,\chi),G} \longrightarrow R_{k,\chi}(LG).$$

When χ is the trivial homomorphism 1 and we are at a multiplicative level, then $\mathcal{Q}_{(k,1),G}$ admits a natural fusion product structure whose resulting category is denoted by $(\mathcal{Q}_{(k,1),G}, \boxtimes)$. Then $\chi_{(k,1),G}$ induces a ring structure on $R_{k,1}(LG)$

although it is not clear how to use our geometric approach in the non-simply connected case to generate an algorithm for constructing the Verlinde coefficients.

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Generalized Geometry, Mirror Symmetry and T-duality

Peter Bouwknegt

In this talk I will review the concept of a *generalized geometry* in the sense of Hitchin, and its application to mirror symmetry and T-duality. The talk is based on the material contained in [1, 2, 3]. My motivation to study generalized geometry mainly derives from the papers [4, 5, 6], but this is by no means a complete set of references for the subject.

Generalized geometry. The basic idea of *generalized geometry* is to replace structures associated with the tangent bundle TM, of a d-dimensional (real) manifold M, by analogous structures on $TM \oplus TM^*$. First of all we observe that we have a symmetric, bilinear form on sections of $TM \oplus TM^*$, i.e. for $X+\xi, Y+\eta \in \Gamma(TM \oplus TM^*)$

$$\langle X+\xi, Y+\eta\rangle = \tfrac{1}{2}(\imath_X \eta + \imath_Y \xi)\,. \tag{0.1}$$

The orthogonal group $O(TM \oplus TM^*) \cong O(d,d)$ associated to $\langle\ ,\ \rangle$ includes, in particular, an element e^B, defined by a 2-form $B \in \Omega^2(M)$, acting on $\Gamma(TM \oplus TM^*)$ as

$$e^B(X+\xi) = X + \xi + \imath_X B\,. \tag{0.2}$$

The role of the Lie bracket on $\Gamma(TM)$ is now played by the *Courant bracket* on $\Gamma(TM \oplus TM^*)$

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(\imath_X\eta - \imath_Y\xi)\,. \tag{0.3}$$

The transformation (0.2), for $B \in \Omega^2(M)$, acts on the Courant bracket as

$$[e^B(X+\xi), e^B(Y+\eta)] = e^B[X+\xi, Y+\eta] + \imath_Y\imath_X dB\,, \tag{0.4}$$

and thus provides an automorphism of the Courant bracket iff $dB = 0$ (in fact, together with diffeomorphisms these are all automorphisms). Thus suggests defining a *twisted Courant bracket* by

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + \imath_Y\imath_X H\,, \tag{0.5}$$

where $H \in \Omega^3(M)$, such that $dH = 0$. This definition is related to the following construction. Given a (bundle) gerbe with connection $(H, B_\alpha, A_{\alpha\beta})$, we can construct a vector bundle W, such that locally $W\big|_{U_\alpha} \cong TM \oplus TM^*\big|_{U_\alpha}$ and such that we have a split short exact sequence

$$0 \longrightarrow TM^* \longrightarrow W \longrightarrow TM \longrightarrow 0\,.$$

The Courant bracket on W, transported to $TM \oplus TM^*$ precisely gives rise to the twisted bracket of (0.5). The (twisted) Courant bracket is skew symmetric, but neither associative nor does it satisfy the Leibnitz rule. Instead we have

$$\begin{aligned} [[A,B],C]_H + \text{cycl} &= \tfrac{1}{3}d\left(\langle [A,B]_H, C\rangle + \text{cycl}\right)\,,\\ [A, fB]_H &= f[A,B]_H + (\rho(A)f)B - \langle A,B\rangle d_H f\,, \end{aligned} \tag{0.6}$$

where $A,B,C \in \Gamma(TM \oplus TM^*)$, $f \in C^\infty(M)$, $\rho : TM \oplus TM^* \to TM$ is the projection, and $d_H = d + H\wedge$ the twisted differential.

Generalized complex and Kähler structures. A *Generalized almost complex structure* is a $\mathcal{J} \in O(TM \oplus TM^*)$, such that $\mathcal{J}^2 = 1$. The eigenspaces

$$\mathrm{Ker}(1 \mp i\mathcal{J}) = \{A \in \Gamma(TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}) \mid \mathcal{J}A = \pm iA\} \subset \Gamma(TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}})\,, \tag{0.7}$$

define maximally isotropic subbundles $E_\pm$ of the complexification of $TM \oplus TM^*$. Examples are the generalized almost complex structures $\mathcal{J}_J$, associated to an almost complex structure J, and $\mathcal{J}_\omega$, associated to a nondegenerate 2-form ω. Explicitly,

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}\,, \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}\,. \tag{0.8}$$

A *(twisted) generalized complex structure* is a generalized almost complex structure $\mathcal{J}$, such that sections of $E_\pm$ are closed under the (twisted) Courant bracket, i.e. $E_\pm$ is involutive with respect to the (twisted) Counrant bracket. The generalized almost complex structures $\mathcal{J}_J$, and $\mathcal{J}_\omega$, define generalized complex structures if J

defines a complex structure and ω is closed (i.e. defines a symplectic structure), respectively. In general, generalized complex structures can be viewed as somehow interpolating between complex and symplectic structures.

A *(twisted) generalized Kähler structure* $(\mathcal{G}, \mathcal{J}_1, \mathcal{J}_2)$ is a pair of commuting (twisted) generalized complex structures, such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ defines a positive definite metric on $TM \oplus TM^*$. Obviously, a Kähler manifold gives rise to a generalized Kähler structure by taking $\mathcal{J}_1 = \mathcal{J}_J,\ \mathcal{J}_2 = \mathcal{J}_\omega$.

Lie algebroids. A *Lie algebroid* is a vector bundle $E \to M$, equipped with a Lie bracket $[\ ,\ ]$ on $\Gamma(E)$, and a smooth map $\rho : E \to TM$ ("anchor"), which is a Lie algebra homomorphism, and satisfies the Leibnitz rule, i.e.

$$\rho([X,Y]) = [\rho(X), \rho(Y)]\,, \qquad [X, fY] = f[X,Y] + (\rho(X)f)Y\,, \tag{0.9}$$

for $X, Y \in \Gamma(E)$, and $f \in C^\infty(M)$. We see from Eqns. (0.6) that isotropic, involutive, subbundles $E \subset TM_\mathbb{C} \oplus TM^*_\mathbb{C}$ give rise to (complex) Lie algebroids where the anchor is the projection $\rho : TM_\mathbb{C} \oplus TM^*_\mathbb{C} \to TM_\mathbb{C}$, restricted to E. In particular, every (twisted) generalized complex structure $\mathcal{J}$ gives rise to a (complex) Lie algebroid E_+. To any Lie algebroid E we can associate 'k-forms' $\Omega^k(E) = \Gamma(\wedge^k E^*)$, on which we can define

- a *differential*: $d_E : \Omega^k \to \Omega^{k+1}$

$$\begin{aligned} d_E\omega(x_0, \ldots, x_k) = &\sum_{i=0}^{k} (-1)^i \rho(x_i)(\omega(x_0, \ldots, \hat{x}_i, \ldots, x_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k)\,. \end{aligned}$$

- a *contraction*: $\imath_x : \Omega^k \to \Omega^{k-1}$, $(\imath_x\omega)(x_1, \ldots, x_{k-1}) = \omega(x, x_1, \ldots, x_{k-1})$.
- a *Lie derivative*: $\mathcal{L}_x : \Omega^k \to \Omega^k$, $\mathcal{L}_x = d_E\imath_x + \imath_x d_E$.

The *Lie algebroid cohomology* H_{d_E}, is the cohomology associated to the complex $(\Omega(E), d_E)$. For example, for $\mathcal{J} = \mathcal{J}_J$, we have $E = E_+ = T^{(0,1)} \oplus T^{*(1,0)}$, such that $\Gamma(\wedge^k E^*) = \bigoplus_{p+q=k} \big((\wedge^q T^{(0,1)}) \oplus (\wedge^p T^{*(1,0)})\big)$, while $d_E = \bar{\partial}$, hence

$$H^k_{d_E} = \bigoplus_{p+q=k} H^p_{\bar{\partial}}(\wedge^q T^{(0,1)})\,. \tag{0.10}$$

Applications. A 2D nonlinear sigma models on a target manifold M, coupled to fields (g, B), possesses $N = (2,2)$ supersymmetry iff the manifold M is equipped with a so-called *bi-hermitean geometry* [4]. It turns out that the notion of a bi-hermitean geometry is equivalent to the existence of a twisted generalized Kähler structure [2]. The sigma model can, as usual, be twisted in two possible ways and leads to the topological A- and B-models, which are related by mirror symmetry. In the absence of the flux $H = dB$, the A-model encaptures the symplectic structure of the manifold M, while the B-model excaptures the complex structure. It has been shown recently that the physical states of the A- and B-model are precisely given by the Lie algebroid cohomology associated to the two complex structures $\mathcal{J}_1$, and $\mathcal{J}_2$, defining the twisted generalized Kähler structure (cf. Eqn. (0.10), which gives the physical states of the B-model in the absence of $H = dB$ flux). It has also been shown [2, 3] that T-duality (for principle circle bundles) fits nicely in the framework of generalized geometry, namely it induces isomorphisms of Clifford algebras and modules, and preserves the Courant bracket structure. Moreover, (non-topological) mirror symmetry fits in nicely as well [6]. Questions that remain are: What are D-branes in the context of a generalized geometry and are they classified by some kind of derived category?

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Chern-Simons and Wess-Zumino-Witten Theories

Michael Murray
(joint work with Alan Carey, Stuart Johnson, Danny Stevenson, Bai-Ling Wang)

In this talk, based on [3], I use bundle gerbes to explain the correspondence between Chern-Simons gauge theories with gauge group a compact, connected, semi-simple group G and Wess-Zumino-Witten models with target the same G. This correspondence takes the form of a map

$$\Psi\colon CS(G) \longrightarrow WZW(G) \tag{0.1}$$

The image of this map are the so-called multiplicative Wess-Zumino-Witten models which correspond to multiplicative bundle gerbes. An interesting application of multiplicative bundle gerbes is given in Carey's talk for which this present talk forms a useful background.

In [4] it is shown that three dimensional Chern-Simons gauge theories with gauge group G can be classified by the integer cohomology group $H^4(BG, \mathbb{Z})$, and conformally invariant sigma models in two dimension with target space a compact Lie group (Wess-Zumino-Witten models) can be classified by $H^3(G, \mathbb{Z})$. It is also established that the correspondence between three dimensional Chern-Simons gauge theories and Wess-Zumino-Witten models is related to the transgression map

$$\tau\colon H^4(BG, \mathbb{Z}) \longrightarrow H^3(G, \mathbb{Z}),$$

which explains the subtleties in this correspondence for non-simply connected Lie groups [6].

In this talk I will define all the elements in the correspondence (0.1) and maps making the diagram

$$\begin{array}{ccc} CS(G) & \xrightarrow{\Psi} & WZW(G) \\ \downarrow & & \downarrow \\ H^4(BG,\mathbb{Z}) & \xrightarrow{\tau} & H^3(G,\mathbb{Z}) \end{array} \tag{0.2}$$

Recall from, for instance [1], the notion of Deligne cohomology. If M is a manifold we denote by $H^p(M, \mathcal{D}^p)$ the pth Deligne cohomology group of M. Our indexing is such that $H^1(M, \mathcal{D}^1)$ corresponds to isomorphism classes of line bundles with connection. We then define:

Definition 1. A Deligne characteristic class d (of degree p) for principal G-bundles with connection is an assignment to any principal G-bundle P with connection A over M of a class $d(P, A) \in H^p(M, \mathcal{D}^p)$ which is functorial in the sense that if $f\colon N \to M$ then

$$d(f^*(P), f^*(A)) = f^*(d(P, A)),$$

where $f^*(P)$ is the pull-back principal G-bundle with the pull-back connection $f^*(A)$.

Denote by $\mathcal{D}_p(G)$ the group of all Deligne characteristic classes of degree p for principal G-bundles. We show in [3] that a Deligne characteristic class d of degree p gives rise to a characteristic class of degree $p + 1$ and hence there is a homomorphism

$$\mathcal{D}_p(G) \to H^{p+1}(BG, \mathbb{Z}).$$

We know have two definitions:

Definition 2. A three dimensional *Chern-Simons gauge theory* with gauge group G is defined to be a Deligne characteristic class of degree 3.

We denote the group of all three dimensional Chern-Simons gauge theories with gauge group G by $CS(G)$, that is $CS(G) = \mathcal{D}_3(G)$.

Definition 3. A *Wess-Zumino-Witten model* on G is defined to be a Deligne class on G of degree 2.

We denote the group of all Wess-Zumino-Witten model models on G by $WZW(G)$, that is $WZW(G) = H^2(G, \mathcal{D}^2)$.

Note that we already have parts of the diagram (0.2) namely:

$$\begin{array}{ccc} CS(G) & & WZW(G) \\ \downarrow & & \downarrow \\ H^4(BG,\mathbb{Z}) & \xrightarrow{\tau} & H^3(G,\mathbb{Z}) \end{array} \tag{0.3}$$

To define the map Ψ in (0.1) we need to define a certain G bundle over $S^1 \times G$. To this end consider first the trivial G bundle $\mathbb{R} \times G \times G$ where the right G action of $k \in G$ is given by $(t, g, h)k = (t, g, hk)$. This is a $\mathbb{Z}$-equivariant bundle for the left $\mathbb{Z}$ action

$$n(t, g, h) = (t + n, g, g^n h)$$

Denote the quotient principal G bundle over $S^1 \times G$ by $\mathcal{P}$. In [3] we give a slightly more involved construction of $\mathcal{P}$ taken from [2] and [9] which is useful to show that it has a canonical connection $\mathbb{A}$. The map Ψ is defined as follows. Given $d \in CS(G)$ we can apply it to $(\mathcal{P}, \mathbb{A})$ to get $d(\mathcal{P}, \mathbb{A}) \in H^3(S^1 \times G, \mathcal{D}^3)$. Using the fact that we can integrate over the fibre with Deligne cohomology [1] we define

$$\Psi(d) = \int_{S^1} d(\mathcal{P}, \mathbb{A}) \in H^2(G, \mathcal{D}^2) = WZW(G).$$

This completes the construction of the diagram (0.2) and we show in [3] that it commutes.

Finally we want to discuss the image of Ψ. Recall from [7] and [8] the basic facts about bundle gerbes. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two bundle gerbes over a manifold M. We can form the product $\mathcal{G}_1 \otimes \mathcal{G}_2$ and the dual $\mathcal{G}^*$. If $f\colon N \to M$ is smooth there is a pull-back bundle gerbe $f^*(\mathcal{G})$ over N. There is a notion of stable isomorphism of bundle gerbes and the group of all stable isomorphism classes of bundle gerbes over M is $H^3(M, \mathbb{Z})$. Bundle gerbes can be endowed with connections and curvings and the group of all stable isomorphism classes of bundle gerbes with connection and curving is $H^2(M, \mathcal{D}^2)$. Finally given a bundle gerbe with connection and curving we can calculate its holonomy over any surface in the underlying manifold.

In [10] Stevenson defines the notion of simplicial bundle gerbe over a simplicial manifold $X^\bullet = \{X^n\}_{n\geq 0}$ with face operators $d_i : X^{n+1} \to X^n$ $(i = 0, 1, \cdots, n+1)$. In such a situation if $\mathcal{G}$ is a bundle gerbe over X^n we can define a bundle gerbe $\delta(\mathcal{G})$ over X^{n+1} by

$$\delta(\mathcal{G}) = d_0^{-1}(\mathcal{G})^* \otimes d_1^{-1}(\mathcal{G}) \otimes d_2^{-1}(\mathcal{G})^* \cdots$$

As the definition of a simplicial line bundle is a little complicated we refer to [10] for details and present a summary here. We start with a bundle gerbe $\mathcal{G}$ over X_1 and a trivialisation of the bundle gerbe $\delta(G)$ over X^2. The latter induces a trivialisation of $\delta(\delta(\mathcal{G}))$ over X^4 but $\delta(\delta(\mathcal{G}))$ has a canonical trivialisation so these two trivialisations differ by the so-called associator line bundle $L \to X^3$. The associator line bundle is required to have a section a and, because $\delta(L) \to X^4$ has a canonical trivialisation, we can, and do, require lastly that $\delta(a) = 1$.

We need two applications of this notion. Firstly consider the simplicial space $X_G^\bullet$ that arises in the construction of BG, see [5]. This has $X_G^n = G^n$ and face maps

$$d_i(g_1,\dots,g_{p+1}) = \begin{cases} (g_2,\dots,g_{p+1}), & i=0,\\ (g_1,\dots,g_{i-1}g_i,g_{i+1},\dots,g_{p+1}), & 1\le i\le p-1,\\ (g_1,\dots,g_p), & i=p. \end{cases}$$

A bundle gerbe over $X^1 = G$ which is a simplicial bundle gerbe for this simplicial space is called a *multiplicative bundle gerbe* over G.

Secondly let $Y \to M$ be a submersion and consider the simplicial space $X_Y^\bullet$. This has $X_Y^{n+1} = Y^{[p]}$ its pth fold fibre product. We define face maps $d_i\colon Y^{[p+1]} \to Y^{[p]}$ by omitting the i-th element. A pair consisting of a submersion $Y \to M$ and a simplicial bundle gerbe for $X_Y^\bullet$ is called a *bundle* 2 *gerbe* over M.

In [3] we show that the image of $\Psi\colon CS(G) \to WZW(G)$ is precisely the Deligne classes in $H^2(G,\mathcal{D}^2)$ which correspond to multiplicative bundle gerbes on G. The Wess-Zumino-Witten action regarded as a function on the space of smooth maps from a closed surface Σ to G exponentiates to the bundle gerbe holonomy of $\mathcal{G}$ and satisfies the following multiplicative property:

$$\exp\big(S_{\mathrm{wzw}}(\sigma_1\cdot\sigma_2)\big) = \exp\big(S_{\mathrm{wzw}}(\sigma_1)\big)\cdot\exp\big(S_{\mathrm{wzw}}(\sigma_2)\big),$$

for a pair of smooth maps σ_1 and σ_2 from Σ to G.

Finally we make a remark on why multiplicative bundle gerbes arise in the image of Ψ. This relates to the important fact that for any Chern-Simons gauge theory the exponential of the Chern-Simons action over a three surface can be shown to arise as the holonomy of a certain universal Chern-Simons bundle 2 gerbe defined in [3]. This arise, in part, as follows. If $Y \to M$ is a principal G bundle then we there is a map $\rho^1\colon Y^{[2]} \to G$ defined by $\rho(y_1,y_2) = g$ where $y_1 g = y_2$. This extends in a natural way to $\rho^p\colon Y^{[p+1]} \to G^p$ and defines a simplicial map $\rho^\bullet\colon X_Y^\bullet \to X_G^\bullet$. It follows immediately that a multiplicative bundle gerbe over G gives rise to a bundle 2 gerbe over M and with some more effort that the latter bundle 2 gerbe has a connection, curving and 2 curving.

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The String Gerbe

Daniel Stevenson

Let G be a compact, simple and simply connected Lie group and let $\pi\colon P \to M$ be a principal G bundle over a smooth manifold M. Let ν denote the universally transgressive generator of $H^3(G; \mathbf{Z}) = \mathbf{Z}$ and let $c \in H^4(M; \mathbf{Z})$ be the transgression of ν. Recall that M is said to be *string*, or admit a *string structure*, if a certain characteristic class in $H^3(LM; \mathbf{Z})$ vanishes (here LM denotes the free loop space of M). This characteristic class is the obstruction to lifting the structure group of the principal LG-bundle $LP \to LM$ to $\widehat{LG}$ — the Kac-Moody group. As has been observed by several authors [1, 5, 4] the obstruction in $H^3(LM; \mathbf{Z})$ is closely related to the characteristic class $c \in H^4(M; \mathbf{Z})$ if M is 2-connected: a lift of the structure group to $\widehat{LG}$ exists precisely when c is zero. As is well known, if $G = \mathrm{Spin}(n)$ then $2c = p_1$. This obstruction problem on LM can be phrased in the language of homotopy theory down on M. Recall that G fits into a short exact sequence of topological groups

$$1 \to K(\mathbf{Z}, 2) \to \hat{G} \to G \to 1$$

where $\hat{G}$ is the 3-connected cover of G. $\hat{G}$ is a topological group which can be defined in a homotopy theoretic manner as the homotopy fibre of the canonical map $G \to K(\mathbf{Z}, 3)$ classifying ν. $\hat{G}$ has vanishing third homotopy group and therefore cannot have the homotopy type of any Lie group. When $G = \mathrm{Spin}(n)$ the group $\hat{G}$ is called $\mathrm{String}(n)$. In their study of elliptic objects Stolz and Teichner [6] give a description of $\mathrm{String}(n)$ in terms of von Neumann algebras. Their description however is awkward from the point of view of classical differential geometry as their model of $\mathrm{String}(n)$ is not a smooth manifold.

In this talk we want to promote the view that there is a useful description of $\mathrm{String}(n)$ as a *group stack* (gr-stack) and that it is possible to do differential geometry in this setting. In recent joint work with Alissa Crans, John Baez and Urs Schreiber we gave a construction of a 2-*group* $\mathcal{P}G$, that is, a category $\mathcal{P}G$ with objects $\mathcal{P}G_0$ and morphisms $\mathcal{P}G_1$ such that $\mathcal{P}G_0$ and $\mathcal{P}G_1$ are (Lie) groups and all structural maps such as source, target and composition are group homomorphisms. One can think of $\mathcal{P}G$ as a groupoid presentation for a gr-stack. As is well known, 2-groups are equivalent to crossed modules; one can describe the corresponding crossed module for $\mathcal{P}G$ via the homomorphism $\widehat{\Omega G} \to P_0G$ where P_0G denotes the group of based paths in G and an action of P_0G on $\widehat{\Omega G}$ lifting the conjugation action of P_0G on ΩG. The main result of the talk will be a construction of a non-abelian gerbe on M — the '*string gerbe*' — which plays the role of a lift of the structure group of P from G to $\hat{G}$. This construction is interesting for two reasons: firstly, it raises the possibility of studying the elliptic objects of Stolz and Teichner within the framework of classical differential geometry, secondly, there are very few examples of non-abelian gerbes — I know of no other examples besides those associated to bundle lifting problems coming from short exact sequences of groups $1 \to K \to E \to G \to 1$. In order to describe this gerbe on M, we need to recast the description of non-abelian gerbes in terms of groupoids due to Breen [3] and more recently [2] in a more convenient language. We will show how gerbes for a crossed module $t\colon H \to G$ can be described as certain groupoids internal to the category of principal bundles. With this description in place we explain how to construct the string gerbe. This can, in a sense, be viewed as an extension of the earlier work of Carey and Murray [1].

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2-Groups, Trialgebras and Their Hopf Categories of Representations

Hendryk Pfeiffer

A strict 2-group is an internal category in the category of groups. Strict 2-groups can also be characterized as 2-categories with one object in which all 1-morphisms and all 2-morphisms have inverses. The notion of a strict 2-group can therefore be viewed as a higher-dimensional generalization of the notion of a group because the set of 2-morphisms of a 2-group has got *two* multiplication operations. Strict 2-groups can be constructed from crossed modules, and so there exist plenty of examples.

Starting from the theory of groups, one can develop the notion of cocommutative Hopf algebras which arise as group algebras, the notion of commutative Hopf algebras which appear as algebras of functions on groups, and the notion of symmetric monoidal categories which arise as the representation categories of groups. Compact topological groups are characterized by their commutative Hopf C^*-algebras of continuous complex-valued functions (Gel'fand representation). Commutative Hopf algebras are characterized by their rigid symmetric monoidal categories of finite-dimensional comodules (Tannaka–Kreĭn reconstruction). This 'commutative' theory forms the basic framework that is required before one can develop the theory of quantum groups.

What is the analogy of all these structures if one systematically replaces the concept of a group by that of a strict 2-group? In particular, what is a good definition of 'group algebra', 'function algebra' and 'representation category' for a strict 2-group? I outline how to define the relevant structures and how to establish generalizations from groups to strict 2-groups of the theorems mentioned above, namely on Gel'fand representation and on Tannaka–Kreĭn reconstruction. The 'group algebra' of a 2-group is a trialgebra, and its 'representation category' a Hopf category [14].

With the step from groups to strict 2-groups, we enter the realm of higher-dimensional algebra. Higher-dimensional algebraic structures have appeared in various areas of mathematics and mathematical physics. A prime example is the higher-dimensional group theory programme of Brown [1], generalizing groups and groupoids to double groupoids and further on, in order to obtain a hierarchy of algebraic structures. The construction of these algebraic structures is motivated by problems in homotopy theory where algebraic structures at a some level of the hierarchy are related to topological features that appear in the corresponding dimension.

Motivated by the construction of topological quantum field theories (TQFTs), Crane has introduced the concept of *categorification*, see, for example [2, 3]. Categorification can be viewed as a systematic replacement of familiar algebraic structures that are modelled on sets by analogues that are rather modelled on categories, 2-categories, and so on. Categorification often serves as a guiding principle in order to find suitable definitions of algebraic structures at some higher level starting from the known definitions at a lower level.

Some examples of higher-dimensional algebraic structures that are relevant in this context, are the following.

- Some three-dimensional TQFTs can be constructed from Hopf algebras [4, 5]. In order to generalize this to four dimensions, Crane and Frenkel [2] have introduced the notion of a *Hopf category*. Roughly speaking, this is a monoidal category with an additional functorial comultiplication.
- Crane and Frenkel [2] also speculate about *trialgebras*, vector spaces with three mutually compatible linear operations: two multiplications and one comultiplication or vice versa.
- Kapranov and Voevodsky [6, 7] have introduced braided monoidal 2-categories and 2-vector spaces, a categorified notion of vector spaces.
- Grosse and Schlesinger have constructed examples of trialgebras [8, 9] in the spirit of Crane–Frenkel.
- Several authors [10, 11, 12, 13, 15] have used 2-groups in order to find generalizations of fibre bundles and of gauge theory. Yetter [16] has used 2-groups in order to construct novel TQFTs, generalizing the TQFTs that

are constructed from the gauge theories of flat connections on a principal G-bundle where G is an (ordinary) group.

All these constructions have a common underlying theme: the procedure of categorification on the algebraic side and an increase in dimension on the topological side. The connection between 2-groups, trialgebras, and Hopf categories shows how these constructions are related.

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