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Arbeitsgemeinschaft mit aktuellem Thema: Modern Foundations for Stable Homotopy Theory

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Abstract. In recent years, *spectral algebra* or *stable homotopical algebra* over structured ring spectra has become an important new direction in stable homotopy theory. This workshop provided an introduction to structured ring spectra and applications of spectral algebra, both within homotopy theory and in other areas of mathematics.

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Introduction by the Organisers

Stable homotopy theory started out as the study of generalized cohomology theories for topological spaces, in the incarnation of the stable homotopy category of spectra. In recent years, an important new direction became the *spectral algebra* or *stable homotopical algebra* over structured ring spectra. Homotopy theorists have come up with a whole new world of 'rings' which are invisible to the eyes of algebraists, since they cannot be defined or constructed without the use of topology; indeed, in these 'rings', the laws of associativity, commutativity or distributivity only hold up to an infinite sequence of coherence relations. The initial 'ring' is no longer the ring of integers, but the *sphere spectrum* of algebraic topology; the 'modules' over the sphere spectrum define the stable homotopy category. Although ring spectra go beyond algebra, the classical algebraic world is properly contained in stable homotopical algebra. Indeed, via Eilenberg-Mac Lane spectra, classical algebra embeds into stable homotopy theory, and ordinary rings form a full subcategory of the homotopy category of ring spectra. Topology interpolates algebra in various ways, and when rationalized, stable homotopy theory tends to become purely algebraic, but integrally it contains interesting torsion information.

There are plenty of applications of structured ring spectra and spectral algebra within homotopy theory, and in recent years, these concepts have started to appear in other areas of mathematics (we will sketch connections to algebraic K-theory and arithmetic, and to algebraic geometry via motivic homotopy theory and derived algebraic geometry).

After this general introduction, we now want to give some history and more background and point out some areas where structured ring spectrum methods have been used for calculations, theorems or constructions. Several of the following topics were taken up in the talks of this AG.

Some history. A crucial prerequisite for spectral algebra is an associative and commutative smash product on a good point-set level category of spectra, which lifts the well-known smash product pairing on the *homotopy category.* To illustrate the drastic simplification that occurred in the foundations in the mid-90s, let us draw an analogy with the algebraic context. Let R be a commutative ring and imagine for a moment that the notion of a chain complex (of R-modules) has not been discovered, but nevertheless various complicated constructions of the unbounded derived category $\mathcal{D}(R)$ of the ring R exist. Moreover, constructions of the *derived* tensor product on the *derived* category exist, but they are complicated and the proof that the derived tensor product is associative and commutative occupies 30 pages. In this situation, you could talk about objects A in the derived category together with morphisms $A \otimes^L_R A \to A$, in the derived category, which are associative and unital, and possibly commutative, again in the derived category. This notion may be useful for some purposes, but it suffers from many defects – as one example, the category of modules (under derived tensor product in the derived category), does not in general form a triangulated category.

Now imagine that someone proposes the definition of a chain complex of R-modules and shows that by formally inverting the quasi-isomorphisms, one can construct the derived category. She also defines the tensor product of chain complexes and proves that tensoring with a bounded below (in homological terms) complex of projective modules preserves quasi-isomorphisms. It immediately follows that the tensor product descends to an associative and commutative product on the derived category. What is even better, now one can suddenly consider differential graded algebras, a 'rigidified' version of the crude multiplication 'up-to-chain homotopy'. We would quickly discover that this notion is much more powerful and that differential graded algebras arise all over the place (while chain complexes with a multiplication which is merely associative up to chain homotopy seldom come up in nature).

Fortunately, this is not the historical course of development in homological algebra, but the development in stable homotopy theory was, in several aspects, as indicated above. The first construction of what is now called 'the stable homotopy category', including its symmetric monoidal smash product, is due to Boardman (unpublished); accounts of Boardman's construction appear in [57] and [1, Part III] (Adams has to devote more than 30 pages to the construction and formal properties of the smash product). With this category, one could consider ring spectra 'up to homotopy', which are closely related to multiplicative cohomology theories.

However, the need and usefulness of ring spectra with rigidified multiplications soon became apparent, and topologists developed different ways of dealing with them. One line of approach used operads for the bookkeeping of the homotopies which encode all higher forms of associativity and commutativity, and this led to the notions of A_∞- respectively E_∞-ring spectra. Various notions of point-set level ring spectra had been used (which were only later recognized as the monoids in a symmetric monoidal model category). For example, the orthogonal ring spectra had appeared as *$\mathscr{I}_*$-prefunctors* in [34], the *functors with smash product* were introduced in [6] and symmetric ring spectra appeared as *strictly associative ring spectra* in [18, Def. 6.1] or as *FSPs defined on spheres* in [19, 2.7].

At this point it had become clear that many technicalities could be avoided if one had a smash product on a good point-set category of spectra which was associative and unital *before* passage to the homotopy category. For a long time no such category was known, and there was even evidence that it might not exist [28]. In retrospect, the modern spectra categories could maybe have been found earlier if Quillen's formalism of *model categories* [37, 23] had been taken more seriously; from the model category perspective, one should not expect an intrinsically 'left adjoint' construction like a smash product to have good homotopical behavior in general, and along with the search for a smash product, one should look for a compatible notion of cofibrations.

In the mid-90s, several categories of spectra with nice smash products were discovered, and simultaneously, model categories experienced a major renaissance. Around 1993, Elmendorf, Kriz, Mandell and May introduced the *S-modules* [14] and Jeff Smith gave the first talks about *symmetric spectra*; the details of the model structure were later worked out and written up by Hovey, Shipley and Smith [24]. In 1995, Lydakis [29] independently discovered and studied the smash product for Γ-spaces (in the sense of Segal [51]), and a little later he developed model structures and smash product for *simplicial functors* [30]. Except for the S-modules of Elmendorf, Kriz, Mandell and May, all other known models for spectra with nice smash product have a very similar flavor; they all arise as categories of continuous, space-valued functors from a symmetric monoidal indexing category, and the smash product is a convolution product (defined as a left Kan extension), which had much earlier been studied by category theorist Day [9]. This unifying context was made explicit by Mandell, May, Schwede and Shipley in [33], where another example, the *orthogonal spectra* was first worked out in detail. The different approaches to spectra categories with smash product have been generalized and adapted to equivariant homotopy theory [11, 31, 32] and motivic homotopy theory [12, 25, 26]. In this AG we will present these various setups with an emphasis on symmetric spectra.

Algebra versus homotopy theory. Many constructions and invariants for classical rings have counterparts for structured ring spectra. These ring spectra have well behaved module categories; algebraic K-theory, Hochschild homology and André-Quillen homology admit refinements; classical constructions such as localization, group rings, matrix rings, Morita theory and Galois theory carry over, suitably adapted. This export of concepts from algebra to topology illuminates both fields. For example, Dwyer, Greenlees, and Iyengar [13] have shown that Gorenstein duality, Poincaré duality, and Gross-Hopkins duality become results of a single point of view. Morita theory for ring spectra [4, 50, 48] gives a new perspective at *tilting theory*. The extension of Galois theory to ring spectra [42] has genuinely new kinds of examples given by classical and higher forms of real topological K-theory. Certain algebraic extensions of commutative rings can be lifted to the sphere spectrum; for example, roots of unity can be adjoined to E_∞-ring spectra away from ramification [46]. Some algebraic notions, for example the units or the center of a ring, are more subtle, and their generalizations to ring spectra show richer features than the classical counterparts.

Power operations. Recognizing a multiplicative cohomology theory as an E_∞-ring spectrum leads to additional structure which can be a powerful theoretical and calculational tool. We want to illustrate this by the example of the Adams spectral sequence [1, III.15], [35, II.9], [52, Ch. 19], a tool which has been used extensively for calculations of stable homotopy groups. In its most classical instance, the Adams spectral sequence converges to the p-completed stable homotopy groups of spheres and takes the form

$$E_2^{s,t} \;=\; \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p) \;\Longrightarrow\; (\pi^{\mathrm{stable}}_{t-s} S^0)^\wedge_p \quad .$$

The E_2-term is given by Ext-groups over the Steenrod algebra $\mathcal{A}_p$ of stable mod-p cohomology operations. This spectral sequence neatly separates the problem of calculating homotopy groups into an algebraic and a purely homotopy theoretic part.

The Steenrod algebra has various explicit descriptions and the E_2-term can be calculated mechanically. In fact, computer calculations have been pushed up to the range $t - s \leq 210$ (for $p = 2$) [8, 36]. The 1-line is given by the primitive elements in the dual Steenrod algebra, which for $p = 2$ are classes $h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $i \geq 0$. The first few classes h_0, h_1, h_2 and h_3 are infinite cycles and they detect the Hopf maps $2\iota, \eta, \nu$ and σ; the Hopf maps arise from the division algebra structures on $\mathbb{R}$, $\mathbb{C}$, the quaternions and the Cayley numbers.

On the other hand, identifying differentials and extensions in the Adams spectral sequence is a matter of stable homotopy theory. The known differentials can be derived by exploiting more and more subtle aspects of the homotopy-commutativity of the stable homotopy groups of spheres (or, in our jargon, the E_∞-structure of the sphere spectrum). The first d_2-differential in the mod-2 Adams spectral sequence is a consequence of the graded-commutativity $yx = (-1)^{|x||y|}xy$ in the ring of stable homotopy groups of spheres: like any element of odd dimension, the third Hopf map σ in the stable 7-stem has to satisfy $2\sigma^2 = 0$. The class $h_0 h_3^2$ which represents $2\sigma^2$ in $E_2^{3,17}$ is non-zero, so it has to be in the image of some differential. In these dimensions, the only possible differential is $d_2(h_4) = h_0 h_3^2$, which simultaneously proves that the class $h_4 \in E_2^{1,16}$ is not an infinite cycle and thus excludes the existence of a 16-dimensional real division algebra.

Graded-commutativity of the multiplications is only a faint shadow of the E_∞-structure on the sphere spectrum. A more detailed investigation reveals that an E_∞-structure on a ring spectrum yields *power operations* [7, Ch. I], [40, Sec. 7]. Various kinds of power operations can be constructed on the homotopy and (generalized) homology groups of the E_∞-ring spectrum and on the E_2-term of the Adams spectral sequence. These operations interact in certain ways with the differential, and this interaction is an effective tool for determining such differentials. For example, the operations propagate the first d_2-differential, which proves that the classes h_i for $i = 0, 1, 2$ and 3 are the only infinite cycles on the 1-line and excludes the existence of other real division algebras.

Algebraic K-theory. Structured ring spectra and algebraic K-theory are closely related in many ways, and in both directions. Algebraic K-theory started out as the Grothendieck group of finitely generated projective modules over a ring. Quillen introduced the higher algebraic K-groups as the homotopy groups of a certain topological space, for which he gave two different constructions (the *plus-construction* and the *Q-construction* [38]). Nowadays, algebraic K-theory can accept various sorts of categorical data as input, and produces spectra as output (the spaces Quillen constructed are the underlying infinite loop spaces of these K-theory spectra). For example, the symmetric monoidal category (under direct sum) of finitely generated projective modules leads to the K-theory spectrum of a ring, and the category of finite sets (under disjoint union) produces the sphere spectrum. If the input category has a second symmetric monoidal product which suitably distributes over the 'sum', then algebraic K-theory produces commutative ring spectra as output.

It was Waldhausen who pioneered the algebraic K-theory of structured ring spectra, introduced the *algebraic K-theory of a topological space X* as the algebraic K-theory of the *spherical group ring* $S[\Omega X] = \Sigma^\infty \Omega X_+$ and established close links to geometric topology, see [58]. Waldhausen could deal with such ring spectra, or their modules, in an unstable, less technical fashion before the modern categories of spectra were found: his first definition of the algebraic K-theory of a ring spectrum mimics Quillen's plus-construction applied to the classifying space of the infinite general linear group. Similarly, Bökstedt [6] (in cooperation with Waldhausen) defined *topological Hochschild homology* using so-called *Functors with Smash Product*, which were only later recognized as the monoids with respect to the smash product of simplicial functors [30]. In any event, a good bit of the development of A_∞-ring spectra was motivated by the applications in algebraic K-theory.

Later, Waldhausen developed one of the most flexible and powerful K-theory machines, the *$S_\bullet$-construction* [59] which starts with a *category with cofibrations and weak equivalences* (nowadays often referred to as a *Waldhausen category*). Originally, Waldhausen associated to this data a sequential spectrum and indicated [59, p. 342] how a categorical pairing leads to a smash product pairing on K-theory spectra. The $S_\bullet$-construction can be applied to a category of cofibrant highly structured module spectra. Applying this to Eilenberg-Mac Lane ring spectra gives back Quillen K-theory, and applied to spherical group rings of the form $S[\Omega X]$ gives an interpretation of Waldhausen's $A(X)$.

Hesselholt eventually recognized that the iterated $S_\bullet$-construction gives K-theory as a symmetric spectrum and that it turns good products on the input category into symmetric ring spectra [16, Prop. 6.1.1]. Elmendorf and Mandell [15] have a different way of rigidifying the algebraic K-theory of a bipermutative category into an E_∞-symmetric ring spectrum.

Units of a ring spectrum. One of the more subtle generalizations of a classical construction is that of the units of a ring spectrum. The units of a structured ring spectrum form a loop space, and in the presence of enough commutativity (i.e., for E_∞- or strictly commutative ring spectra), the units even form an infinite loop space. These observations are due to May and are highly relevant to orientation theory and bordism. A recent application of these techniques is the construction by Ando, Hopkins and Rezk, of E_∞-maps $MO\langle 8\rangle \to \mathbf{tmf}$ from the string-cobordism spectrum to the spectrum of topological modular forms which realize the Witten genus on homotopy groups [21, Sec. 6] (this refines earlier work of Ando, Hopkins and Strickland on the $MU\langle 6\rangle$-orientation of elliptic spectra [20, 2, 3]). The details of this are not yet publicly available, but see [21, Sec. 6], [27], [40].

Waldhausen's first definition of algebraic K-theory of a ring spectrum is based on Quillen's plus-construction and uses the units of matrix ring spectra. While for a commutative ring in the classical sense, the units are always a direct summand in the first K-group, Waldhausen showed that the units of the sphere spectrum do not split off its K-theory spectrum, not even on the level of homotopy groups. This phenomenon has been studied systematically by Schlichtkrull [45], producing non-trivial classes in the K-theory of an E_∞-ring spectrum from homotopy classes which are not annihilated by the Hopf map η.

Homotopical algebraic geometry. In this Arbeitsgemeinschaft, we are trying to convey the idea that the foundations of multiplicative stable homotopy theory are now in good shape, and ready to use. In fact, the machinery allows to 'glue' commutative ring spectra into more general algebro-geometric objects, making them the affine pieces of 'schemes' or even 'stacks'. This area is becoming known as *homotopical algebraic geometry*, and one set of foundations has been pioneered by Toën and Vezzosi in a series of papers [53, 54, 55, 56].

Another promising line of research is to investigate small (e.g., Deligne-Mumford) stacks which come equipped with a flat morphism to the moduli stack of formal groups; one might hope to lift the graded structure sheaf of such a stack to a sheaf of ring spectra and capture a snapshot of stable homotopy theory. In this context, one may think of the generalized algebro-geometric objects as an ordinary scheme or stack, together with a sheaf of E_∞-ring spectra, which locally looks like 'Spec' of an E_∞-ring spectrum. The structure sheaf of the underlying ordinary stack can be recovered by taking π_0 of the sheaf of ring spectra. It is essentially by this program that Hopkins and his coworkers produced *topological modular forms* (the 'universal' version of elliptic cohomology) and the recent work of Lurie shows that these ideas can be extended to almost any situation where the Serre-Tate theorem on deformations applies. Here the flow of information goes both ways: the number theory informs homotopy theoretic calculations, but surprising algebraic phenomena, such as the Borcherds congruence in modular forms, have natural homotopy theoretic explanations [21, Thm. 5.10].

While this is a 'hot' area in algebraic topology, the organizers decided not to have talks about it in this AG because of the lack of publicly available literature (but see [5, 17, 20, 21, 22, 27, 39]).

Rigidity theorem. After having discussed various models for the stable homotopy category, the last day of the AG will be devoted to its 'rigidity' property [49]; this says that the stable homotopy category admits essentially only one model. More precisely, any model category whose homotopy category is equivalent, as a triangulated category, to the homotopy category of spectra is already *Quillen equivalent* to the model category of spectra. Loosely speaking, this says that all higher order homotopy theory is determined by the homotopy category, a property which is very special. Examples of triangulated categories which have inequivalent models are given in [47, 2.1, 2.2] or [10, Rem. 6.8] (which is based on [44]).

In algebra, a 'rigidity theorem' for unbounded derived categories of rings is provided by *tilting theory*; it is usually stated in the form that if two rings are derived equivalent, then, under a flatness assumption, there is a complex of bimodules X such that derived tensor product with X is an equivalence of triangulated categories [41] (a reworking of this result in model category terms, which also removes the flatness assumption, can be found in [10, Thm. 4.2]). Rigidity fails for categories of dg-modules over differential graded algebras [10, Rem. 6.8]. Incidentally, neither for derived categories of rings nor for the stable homotopy category is it known whether every derived equivalence lifts to a Quillen equivalence, or equivalently, whether there are exotic self-equivalences of these triangulated categories.

Questions about rigidity and exotic models are related to the problem of whether algebraic K-theory is an invariant of triangulated categories. Quillen equivalent model categories have equivalent K-theory spectra [10, Cor. 3.10], [43], and in special 'rigid' situations, the triangulated category determines the model, and thus the algebraic K-theory.

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Arbeitsgemeinschaft mit aktuellem Thema: Modern Foundations for Stable Homotopy Theory

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Abstracts

Background on Spectra

Dale Husemöller

We study the homotopy category of spectra as some background and motivation leading up to the study of the Bousfield-Friedlander category of spectra.

The suspension functor $\Sigma : (\text{top}/^*) \to (\text{top}/^*)$ is the reduced product given by $\Sigma X = S^1 \wedge X$. The stable maps $\{X, Y\}$ from X to Y are given by

$$\{X, Y\} = \varinjlim_k [\Sigma^k X, \Sigma^k Y].$$

In order to study conditions when $[X, Y] \to [\Sigma X, \Sigma Y]$ and $[X, Y] \to \{X, Y\}$ are bijections, we use the adjoint functor relation $\Sigma \dashv \Omega$ and the adjunction unit $\beta : Y \to \Omega\Sigma(Y)$.

Theorem. Let X be a path-connected space with $H_*(X, k)$ flat as a k-module. Then the morphism of coalgebras

$$\widetilde{H}_*(\beta) : H_*(Y, *; k) \;\to\; H_*(\Omega\Sigma(Y); k)$$

induces a morphism of Hopf algebras $T(\widetilde{H}_*(Y; k)) \to H_*(\Omega\Sigma(Y); k)$ which is an isomorphism from the tensor Hopf algebra on the coalgebra $\widetilde{H}_*(Y; k)$ onto the Hopf algebra $H_*(\Omega\Sigma(Y); k)$.

Theorem. If X is an n-dimensional CW-complex, and if Y is r-connected, then the maps $[X, Y] \to [\Sigma X, \Sigma Y] \to \{X, Y\}$ are bijections for $n < 2r - 2$ and surjective for $n < 2r - 1$.

Definition. A spectrum $X_\bullet$ is a sequence of spaces with a sequence of morphisms $S^1 \wedge X_n \to X_{n+1}$ for all $n \in \mathbb{N}$.

Definition. For a space X and a spectrum $Y_\bullet$ we define

$$\{X, Y_\bullet\} = \varinjlim_k [\Sigma^k X, Y_k]$$

with transition morphisms the composition of Σ and the morphism induced by the spectrum structure morphism

$$[\Sigma^k X, Y_k] \;\to\; [\Sigma^{k+1} X, S^1 \wedge Y_k] \;\to\; [\Sigma^{k+1} X, Y_{k+1}]\,.$$

Example. If $Y_\bullet$ is the suspension spectrum $S_\bullet \wedge Y$ on a space Y, then $\{X, Y\} = \{X, S_\bullet \wedge Y\}$. The sphere $S_\bullet$ is a spectrum with the natural folding morphism $S_\bullet \wedge S_\bullet \to S_\bullet$. The Eilenberg-Mac Lane sequence $K(\pi) = K(\pi, n)_{n\in\mathbb{N}}$ is a spectrum freely generated by $S^1 \wedge K(\pi, n) \to K(\pi, n + 1)$, the adjoint to the homotopy equivalence $K(\pi, n) \to \Omega K(\pi, n + 1)$.

With this notion of stable maps into a spectrum we can define generalized homology and cohomology with values in a spectrum. We introduce a graded version of $\{X, Y_\bullet\}$ which leads to the grading in cohomology and homology.

For a space X and a spectrum $Y_\bullet$ we denote

$$\{X, Y_\bullet\}_n = \varinjlim_k [\Sigma^{k+n} X, Y_k] \quad \text{and} \quad \{X, Y_\bullet\}^n = \varinjlim_k [\Sigma^{k-n} X, Y_k]\,.$$

Since the upper and lower indexing are related as usual by a sign $\{X, Y_\bullet\}_n = \{X, Y_\bullet\}^{-n}$, the graded abelian group $\{X, Y_\bullet\}$ is defined for all $n \in \mathbb{Z}$.

Definition. Let $E_\bullet$ be a spectrum. The (reduced) cohomology and homology of X with values in the spectrum $E_\bullet$ are $H^n(X, E_\bullet) = \{X, E_\bullet\}^n$ and $H_n(X, E_\bullet) = \{S^0, X \wedge E_\bullet\}_n$ where $(X \wedge E_\bullet)_n = X \wedge E_n$.

Remark. The exact sequence property follows from the two exact sequence properties of the graded abelian groups $\{X, Y\}$ and $\{X, Y_\bullet\}$ related to the mapping sequence

$$X \xrightarrow{f} Y \xrightarrow{a} C(f) \xrightarrow{b} \Sigma X.$$

It has the form of an exact triangle

$$\begin{array}{ccc} \{X, Z_\bullet\}^* & \longleftarrow & \{Y, Z_\bullet\}^* \\ & {\scriptstyle \deg+1} \searrow \quad \nearrow & \\ & \{C(f), Z_\bullet\}^* & \end{array}$$

for generalized cohomology. For generalized homology, we use

$$X \wedge E_\bullet \to Y \wedge E_\bullet \to C(f) \wedge E_\bullet$$

giving an exact triangle

$$\begin{array}{ccc} \{S^0, X \wedge E_\bullet\}_* & \longrightarrow & \{S^0, Y \wedge E_\bullet\}_* \\ & {\scriptstyle \deg-1} \nwarrow \quad \swarrow & \\ & \{S^0, C(f) \wedge E_\bullet\}_* & \end{array} \quad .$$

Remark. In the case of $E_\bullet = K(\pi)_\bullet$ we obtain $H^*(X, \pi) = H^*(X, K(\pi)_\bullet)$ and $H_*(W, \pi) = H_*(W, K(\pi)_\bullet)$.

While ordinary cohomology and K-theory are basic generalized cohomology theories which can be studied rather directly, it is bordism homology which made the study of spectra an urgent matter rather early in the study of generalized homology, for imbeddings of manifolds in Euclidean space give homotopy elements in Thom spaces, and the Thom spaces $T(B_r)$ are related by a suspension, so:

Theorem. The bordism group of n-dimensional $(B_\bullet, \phi_\bullet)$-manifolds is isomorphic to $\{S^n; T(B_\bullet)\} = \varinjlim_k \pi_{n+r}(TB_r)$.

Model Categories I

Daniel Müllner

Model categories form the common basis for all kinds of homotopy theories. In fact, the axioms for a model category are quite restrictive so that strong conclusions can be drawn and there is much structure to be discovered. Besides, model categories help to overcome a set-theoretic problem when one wants to localize a category with respect to a collection of morphisms.

A model structure on a category consists of three specified classes of morphisms—fibrations, cofibrations and weak equivalences—which fulfill certain axioms. I presented topological and algebraic examples of model categories. After this, I explained all necessary notions for defining the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$: cofibrant and fibrant objects, cylinder object, path object, left/right homotopy, cofibrant and fibrant replacement functors. One important slogan to be learned from these constructions is that in general one cannot control the morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ between two arbitrary objects well enough, but if X and Y are fibrant and cofibrant then the morphisms and homotopy relations behave conveniently.

There is a functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ which is the identity on objects but sends every map to its cofibrant and fibrant replacement map. The main theorem presented in the talk was the following (see [1, Thm. 4.6], [2, Thm. I.1]):

Theorem. Let $\mathcal{C}$ be a model category and $W \subseteq \mathcal{C}$ the class of weak equivalences. Then the functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is a localization of $\mathcal{C}$ with respect to W. Moreover, there is an equivalence of categories between $\mathrm{Ho}(\mathcal{C})$ and $\pi\mathcal{C}_{cf}$, the category of both cofibrant and fibrant objects of $\mathcal{C}$ and homotopy classes of maps.

In the last part, I explained Quillen functors and Quillen equivalences. Given an adjoint pair of functors between model categories

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

The left adjoint F is called a *left Quillen functor* if it preserves cofibrations and trivial cofibrations (i. e., maps which are both cofibrations and weak equivalences). An equivalent condition is that G preserves fibrations and trivial fibrations. G is then called correspondingly a *right Quillen functor*. Given this, the total left and right derived functors exist (see [1, Thm. 9.7.]):

$$\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbf{R}G$$

If, in addition, a map $f : X \to G(Y)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^\flat : F(X) \to Y$ is a weak equivalence in $\mathcal{D}$, then $\mathbf{L}F$ and $\mathbf{R}G$ are equivalences of categories, inverse to each other. The adjunction between F and G is then called a *Quillen equivalence*.

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Model categories II

Thorsten Eppelmann

This talk was a continuation of the previous talk on model categories. The goal was to learn more examples of model categories and Quillen equivalences.

The first example was a model category structure on the category of simplicial sets $\mathcal{SS}et$ [4, Ch II.3, Thm 3],[3, Ch 3.2]. Therefore, we quickly recalled the definition of simplicial sets and the geometric realization functor $|\cdot|$ to topological spaces $\mathcal{T}op$. The latter is used to define weak equivalences of simplical sets.

$$f : X \to Y \text{ is a weak equivalence } \Leftrightarrow |f| : |X| \to |Y| \text{ is one.}$$

The cofibrations are given by morphisms which are injective in each dimension, and the fibrations by Kan fibrations.

These three classes then define a model category structure on $\mathcal{SS}et$, and the geometric realization functor is part of a Quillen equivalence with its right adjoint being the singular functor which assigns to a topological space the simplicial set of its singular simplices.

Next, we discussed the small object argument, which in many instances gives a useful tool to provide the factorizations required in the axioms of a model category [4, Ch II.3, Lemma 3],[2],[3, Thm 2.1.14].

Given a set of maps $F = \{f_i : A_i \to B_i \mid i \in I\}$ with all A_i sequentially small, any map $p : X \to Y$ can be factored as $P = s \circ t$ such that $s : X' \to Y$ has the right lifting property with respect to all maps f_i.

Using the characterization of cofibrations in $\mathcal{SS}et$ as those maps having the right lifting property with respect to the maps $\partial\triangle^n \hookrightarrow \triangle^n$ and a similar characterization of trivial cofibrations, it was indicated how to construct the required factorizations.

Finally, we constructed a model category structure on the category of sequential spectra introduced in the first talk [1]. We treated the cases of spectra of topological spaces and of simplicial sets simultaneously. A map $f : X \to Y$ of sequential spectra is a weak equivalence if it induces isomorphisms on the stable homotopy groups, it is a cofibration if the induced maps

$$X_0 \to Y_0 \quad \text{and} \quad X_{n+1} \amalg_{S^1 \wedge X_n} S^1 \wedge Y_n \to Y_{n+1}$$

are cofibrations in $\mathcal{T}op$ or $\mathcal{SS}et$ respectively. The fibrations are all maps which have the right lifting property with respect to all trivial cofibrations. We closed with a more explicit description of fibrations.

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Symmetric spectra

Miguel A. Xicoténcatl

A symmetric spectrum X is a sequence $X_0, X_1, \ldots, X_n, \ldots \in \mathcal{S}_*$ of pointed simplicial sets, together with structure maps $\sigma : S^1 \wedge X_n \to X_{n+1}$ and for all $n \geq 0$ a basepoint preserving action of the symmetric group Σ_n on X_n such that the composition

$$\sigma \circ (S^1 \wedge \sigma) \circ \ldots (S^{p-1} \wedge \sigma) \ : \ S^p \wedge X_n \ \longrightarrow \ X_{p+n}$$

is $\Sigma_p \times \Sigma_n$-equivariant. The category $\mathcal{S}p^\Sigma$ of symmetric spectra is bicomplete and simplicial, but the main difference from the category of sequential spectra is the existence of a nice symmetric monoidal smash product $\wedge$.

Every symmetric spectrum has an underlying sequence $X_0, X_1, \ldots$ of pointed simplicial sets with a basepoint preserving action of Σ_n on X_n; these are called symmetric sequences and with the obvious maps they also form a category $\mathcal{S}_*^\Sigma$. For $X, Y \in \mathcal{S}_*^\Sigma$ define their tensor product $X \otimes Y$ by

$$(X \otimes Y)_n := \bigvee_{p+q=n} (\Sigma_n)_+ \ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)$$

This is a symmetric monoidal product in $\mathcal{S}_*^\Sigma$, having as unit the sequence $e = (S^0, *, *, \ldots)$ and an appropriate twist isomorphism $\tau : X \otimes Y \to Y \otimes X$ (see [1]). If $\mathbb{S} = (S^0, S^1, \ldots)$ is the underlying symmetric sequence of the sphere spectrum, then: (i) $\mathbb{S}$ is a commutative monoid on $\mathcal{S}_*^\Sigma$ and (ii) the category $\mathcal{S}p^\Sigma$ is naturally isomorphic to the category of left $\mathbb{S}$-modules. Now, for $X, Y \in \mathcal{S}p^\Sigma$ define $X \wedge Y$ as the coequalizer in symmetric sequences of the natural maps

$$X \otimes \mathbb{S} \otimes Y \quad \underset{m\tau \otimes 1}{\overset{1 \otimes m}{\rightrightarrows}} \quad X \otimes Y$$

The main result here is that $\wedge$ is a closed symmetric monoidal product on the category $\mathcal{S}p^\Sigma$. Finally, a symmetric ring spectrum is a symmetric spectrum R together with morphisms $\eta : \mathbb{S} \to R$ and $\mu : R \wedge R \to R$, called the unit and multiplication map, which satisfy the usual associativity and unit conditions. Some examples of symmetric ring spectra are: the Eilenberg–Mac Lane spectrum $H\mathbb{Z}$, monoid ring spectra, the cobordism spectrum MO (see [3]) and topological KO-theory (see [2]).

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The model structure on symmetric spectra

Philipp Reinhard

We describe here the stable model structure on the category Sp^Σ of symmetric spectra following [1]. There are many model category structures on Sp^Σ, all with their own properties, advantages and disadvantages, but we will only discuss the stable model structure.

Its main properties are the following points:

Theorem. It is a model structure such that:

- Sp^Σ is Quillen-equivalent to the category of Bousfield-Friedlander-spectra, where the right adjoint is given by the forgetful functor. In particular, the homotopy category of Sp^Σ is equivalent to the stable homotopy category, which has been discussed in the first talk.
- All suspension spectra, in particular the sphere spectrum $\mathbb{S}$, are cofibrant.
- There are compatibility properties with the symmetric monoidal smash product $\wedge$ on Sp^Σ, so that for a cofibrant spectrum X the functor $X \wedge -$ respects stable equivalences and the model structure gives rise to model structures on the category of R-modules and R-algebras for R a symmetric ring spectrum, which will be discussed in the next talk.

The exact definition of the stable model structure is given in [1], sections 3.1 and 3.4. The stable equivalences are not the π_*-isomorphisms as in the usual model structure on the category of Bousfield-Friedlander-spectra, but are the maps which induce isomorphisms on cohomology theories (see below for a precise statement), which includes the π_*-isomorphisms. The stable cofibrations are then defined to be the maps with the left lifting property with respect to the maps which are both levelwise Kan-fibrations (called level fibrations) and levelwise weak equivalences (called level weak equivalences), whereas the stable fibrations are defined to be the maps with the right lifting property with respect to the maps that are stable cofibrations and stable equivalences.

Proposition. We have the following characterisations:

- The stably fibrant objects are exactly the Ω-spectra.
- The stable fibrations which are stable equivalences are the maps which are level fibrations and level weak equivalences.
- Denote by $\overline{\mathbb{S}} = \{*, S^1, S^2, \ldots\}$ the spectrum which equals the sphere spectrum in positive degree and is a single point on level zero. A symmetric spectrum X is stably cofibrant if and only if for all n the map $L_nX = (\overline{\mathbb{S}} \wedge X)_n \longrightarrow X_n$ is a monomorphism such that Σ_n acts freely on the simplices not in the image.

- A stable equivalence is a map $f : X \to Y$ so that for any Ω-spectrum E, the map
$$E^0(f) : E^0(Y^{\mathrm{cof}}) \longrightarrow E^0(X^{\mathrm{cof}})$$
is an isomorphism, where $(-)^{\mathrm{cof}}$ is a cofibrant replacement functor and E^0 denotes homotopy classes of morphisms of symmetric spectra. Equivalently, it is a map so that $E^0(Y) \longrightarrow E^0(X)$ is an isomorphism for any injective Ω-spectrum E. Or equivalently it is a map so that the induced map of simplicial sets $\mathrm{Map}_{Sp^\Sigma}(Y, E) \longrightarrow \mathrm{Map}_{Sp^\Sigma}(X, E)$ is an isomorphism for any injective Ω-spectrum E.
- The weak equivalences between Ω-spectra are the π_*-isomorphisms.

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Symmetric ring and module spectra

Nathalie Wahl

Let $\mathcal{C}$ be a monoidal model category satisfying the monoid axiom. Examples of such categories are the category of simplicial sets (SSets, $\times$), the category of chain complexes over a commutative ring $(\mathrm{Ch}, \otimes)$, the category of symmetric spectra $(Sp^\Sigma, \wedge)$.

Given a monoid R in $\mathcal{C}$, model structures for the categories of R-modules and R-algebras (when R is commutative) are directly inherited from the model structure of $\mathcal{C}$ ([3, Cor. 5.4.2, 5.4.3] and [6, Thm. 4.1]). The weak equivalences and fibrations are defined to be the weak equivalences and fibrations of the underlying objects in $\mathcal{C}$ and the cofibrations are then defined to be the maps having the left lifting property with respect to the acyclic fibrations. Note that the category of monoids in $\mathcal{C}$ is the category of S-algebras, where S is the unit of $\mathcal{C}$.

The above procedure does not work to produce a model structure on the category of commutative monoids. Indeed, in the case of symmetric spectra, the existence of a lift of the stable model structure to commutative ring spectra using the stable fibrations and weak equivalences of the underlying spectra would imply that QS^0 is homotopy equivalent to a product of Eilenberg-Mac Lane spaces, and this is known not to be the case. This is a reincarnation in model category terms of Lewis' observation [4].

To obtain a model structure on the category of commutative symmetric ring spectra, we modify the stable model structure on Sp^Σ. The *positive* stable model structure on Sp^Σ has cofibrations the stable cofibrations which are moreover isomorphisms in level 0. The weak equivalences are the stable equivalences and the fibrations are defined via the right lifting property. The fibrant objects in the positive stable structure are the Ω-spectra "from level 1 onwards". (See [5, Sec. 15].)

The positive stable model structure lifts to the category of commutative symmetric ring spectra using the fibrations and weak equivalences of the underlying spectra [5, Thm. 15.1]. More generally, let $\mathcal{P}$ be an operad in the category of simplicial sets and consider the category of $\mathcal{P}$-algebras in Sp^{Σ}, where $\mathcal{P}$ acts on a spectrum by taking the smash product of simplicial sets levelwise [2, Thm. 1.2.3]. As for commutative rings, the positive stable model structure lifts to the category of $\mathcal{P}$-algebras. If $\mathcal{P}$ is an E_∞-operad, or actually any operad such that $\mathcal{P}(n)$ is contractible for each n, then the category of $\mathcal{P}$-algebras in Sp^{Σ} is Quillen equivalent to the category of commutative symmetric ring spectra ([2, Thm. 1.2.4] or [1, Thm. 1.4]). This contrast with the situation for topological spaces is explained by the following fact: for a cofibrant object X in Sp^{Σ}, the action of the symmetric group Σ_n on $X^{\wedge n} = X \wedge \cdots \wedge X$ is free.

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S-modules

Markus Spitzweck

The theory of S-modules developed in [1] is a vast improvement of the theory of coordinate-free spectra used in [3] as a model for stable homotopy theory. Most importantly it has a point-set level symmetric monoidal smash product which can be used to develop a homotopically well behaved theory of modules, algebras and commutative algebras.

Fix a universe U, i. e., a topological real inner product space isomorphic to $\mathbb{R}^\infty$ with the standard scalar product and the union topology. A coordinate-free spectrum E indexed on U in the sense of [3, I. Definition 2.1] is an assignment $V \mapsto EV$, where V runs through all finite dimensional subspaces of U, together with homeomorphisms $\sigma_{V,W} : EV \xrightarrow{\cong} \Omega^{W-V} EW$ for $V \subset W$ which satisfy a transitivity condition for $V \subset W \subset Z$. Here $W - V$ denotes the orthogonal complement of V in W.

The category of spectra $\mathcal{S}$ enjoys many nice properties, in particular it is a topologically enriched model category, with weak equivalences the levelwise weak equivalences, whose homotopy category is the stable homotopy category. It has the disadvantage that there is no point-set level symmetric monoidal smash product which lifts the smash product on the homotopy category.

For $i \in \mathbb{N}$ let $\mathcal{L}(i)$ be the space of isometric embeddings of U^i into U endowed with the function space topology. Together these spaces form a topological operad $\mathcal{L}$. The notion of an $\mathbb{L}$-spectrum uses the *twisted half-smash product* defined in [1, Appendix A.], which assigns to any spectrum E indexed on U^i and map $A \to \mathcal{L}(i)$ a spectrum $A \ltimes E$ indexed on U.

The endofunctor $E \mapsto \mathbb{L}E := \mathcal{L}(1) \ltimes E$ of $\mathcal{S}$ has naturally the structure of a monad.

Definition. [1, I. Definition 4.2 and Definition 5.1]

- An $\mathbb{L}$-*spectrum* is defined to be an $\mathbb{L}$-algebra.
- The smash product of two $\mathbb{L}$-spectra M and N is
$$M \wedge_{\mathcal{L}} N := \mathcal{L}(2) \ltimes_{\mathcal{L}(1)^2} (M \wedge N),$$
where $M \wedge N$ is the external smash product.

The special properties of the linear isometries operad imply that the smash product of $\mathbb{L}$-spectra is commutative and associative [1, I. Proposition 5.2 and Theorem 5.5]. Moreover there is always a natural map $S \wedge_{\mathcal{L}} M \to M$ [1, Proposition 8.3].

Definition. [1, II. Definition 1.1] An S-*module* is an $\mathbb{L}$-spectrum M such that the map $S \wedge_{\mathcal{L}} M \to M$ is an isomorphism.

The smash product $\wedge_{\mathcal{L}}$ defines a closed symmetric monoidal structure on the category of S-modules [1, II. Theorem 1.6].

One of the main results of the theory of S-modules is the following

Theorem. [1, Chapter VII.] The category $\mathcal{M}_S$ of S-modules is a finitely generated symmetric monoidal topological model category with generating cofibrations and trivial cofibrations those induced from the standard ones on spaces (see [2, Definition 2.4.3 and Theorem 2.4.25]) by applying the functors $S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^\infty$ where the right adjoint of the evaluation at V functor is denoted by Σ_V^∞.

For a fixed commutative S-algebra R the model structure on $\mathcal{M}_S$ generates topological model structures on the categories of R-algebras, commutative R-algebras and R-modules, the latter one is symmetric monoidal. For an R-algebra A the category of A-modules is a model category enriched over R-modules. Maps of (commutative) algebras define Quillen adjunctions, and these are Quillen equivalences for weak equivalences.

The theory of S-modules compares nicely to the theory of symmetric spectra by the following result of S. Schwede:

Theorem. [4, Main Theorem] Any cofibrant replacement S_c^{-1} of S^{-1} in $\mathcal{M}_S$ together with a weak equivalence $S_c^{-1} \wedge_{\mathcal{L}} S^1 \to S$ defines a lax symmetric monoidal functor from $\mathcal{M}_S$ to symmetric spectra of spaces whose left adjoint is symmetric monoidal. This adjoint pair of functors is a Quillen equivalence for the positive model structure on symmetric spectra. There are induced Quillen equivalences for model categories of algebras, commutative algebras and modules.

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Other types of spectra

Halvard Fausk

To construct the category of symmetric spectra with values in based simplicial sets one starts out with the diagram category $\mathcal{D}$ of finite sets and all bijections (it is convenient to choose an equivalent small category, for example the one with objects $[n] = \{1, 2, \ldots, n\}$ for $n > 0$ and $\emptyset$ for $n = 0$.) The category $\mathcal{D}$ has a symmetric tensor product given by disjoint union. One then considers $\mathcal{D}$-diagrams in the category of based simplicial sets. The category of such $\mathcal{D}$-spaces is given the levelwise (projective) model structure; the weak equivalences are the levelwise weak equivalences and the fibrations are the levelwise fibrations. The category of $\mathcal{D}$-spaces is a symmetric tensor category [1]. Next one considers a symmetric monoid S in the category of $\mathcal{D}$-spaces given by sending the set $[n]$ to the smash product $(S^1)^n$, where S^1 is the simplicial 1-sphere $\Delta^1/\partial\Delta^1$. Symmetric monoids in the category of $\mathcal{D}$-spaces are equivalent to strict tensor functors from $\mathcal{D}$ to the category of based simplicial sets. The symmetric spectra are the modules over S. One can now form a symmetric closed tensor category of S-modules and give it a model structure via the forgetful right adjoint functor from the category of S-modules to the model category of $\mathcal{D}$-spaces.

Sequential spectra (earlier: prespectra) are defined in a similar way using the subcategory of $\mathcal{D}$ consisting of all identity maps. The functor S is not a strict tensor functor in this case, so the category of sequential spectra is not a symmetric monoidal category. The insight to make use of a strict monoidal sphere functor S to construct a *symmetric* tensor category of spectra is due to Jeff Smith.

So far the framework is formal and can be formulated in a very general setting. This has been done by O. Renaudin [6]. He considers a symmetric tensor indexing category $\mathcal{D}$ enriched over a symmetric closed tensor model category $\mathcal{V}$, together with a $\mathcal{V}$-enriched strict tensor functor S from $\mathcal{D}$ to $\mathcal{V}$. The spectra are $\mathcal{V}$-enriched functors $\mathcal{D} \to \mathcal{V}$ that are S-modules. He also considers the process of localizing the levelwise model category to form a stable model category of diagram spectra. The passage to a stable model category has also been studied by M. Hovey [3]. The process of forming the local model structure is subtle and often done case by case.

There is a stable model structure on symmetric spectra so that the stable fibrations are the levelwise fibrations whose levelwise (homotopy) fibers are Ω-spectra, and so that the cofibrations are the same as the cofibrations in the projective model structure. The weak equivalences in the stable model category are called stable equivalences. The stable homotopy groups of a spectrum X are defined by $\pi_*(colim_n \Omega^n X([n]))$. A map between spectra that induces an isomorphism on the stable homotopy groups is called π_*-isomorphism. All π_*-isomorphisms are stable equivalences. The converse is not true for symmetric spectra, although it is true for sequential spectra.

Symmetric spectra are well suited for generalizations by replacing the category of simplicial sets by some other symmetric closed tensor model category. For example one can form a category of symmetric spectra by using the (Quillen) model category of based topological spaces instead of based simplicial sets. The Quillen equivalence between simplicial sets and topological spaces gives a Quillen equivalence between these two stable model categories of symmetric spectra.

There is a category of space valued diagram spectra, called orthogonal spectra, that is often more convenient than the category of space valued symmetric spectra. Interesting spectra like Thom spectra and K-theory are most naturally given as orthogonal spectra. The diagram category for orthogonal spectra is the category of finite dimensional real inner product spaces and linear isometries. This category has a symmetric tensor product given by direct sum of vector spaces and it is enriched over based topological spaces (add a disjoint base point). For various constructions it is necessary to choose an equivalent small diagram category. This is done by fixing a suitable universe of vector spaces.

There is a full inclusion of the diagram category of symmetric spectra into the diagram category of orthogonal spectra. It is given by sending $[n]$ to the free real inner product space generated by $[n]$. The symmetric group Σ_n is naturally a subgroup of the orthogonal group $O(n)$. The strict tensor functor S from the indexing category to the category of based topological spaces is given by the one-point compactification. One can form a stable model category of orthogonal spectra in the same way as for symmetric spectra. In this case it turns out that the π_*-isomorphisms and the stable equivalences coincide. This is a consequence of the fact that $O(N)/O(n)$ is $(n-1)$-connected for $N \geq n$. This is in contrast to the case of symmetric spectra where Σ_N/Σ_n is a discrete space which is not connected for $N > n$.

Orthogonal spectra are also well suited for equivariant generalizations [4]. Let G be a compact Lie group. The diagram category $\mathcal{D}$ is the category of finite dimensional orthogonal G-representations and orthogonal maps. (Variations are obtained by considering various universes of G-representations). This category is enriched over based G-spaces. The $\mathcal{D}$-spaces are enriched $\mathcal{D}$-diagrams into the model category of based topological G-spaces (with weak equivalences the maps whose H-fixed point maps are weak equivalences for all closed subgroups H of G). The functor S is formed by the one-point compactification. The orthogonal G-spectra are the S-modules in the category of $\mathcal{D}$-spaces. There is a stable model structure on the category of orthogonal G-spectra.

One can also describe other space valued diagram spectra, for example sequential spectra and Segal's Γ-spaces. The Γ-spaces are obtained by letting $\mathcal{D}$ be the category of finite based sets with a symmetric tensor product given by smash product of based sets. These two classical categories of spectra have serious drawbacks: The category of sequential spectra is not a symmetric tensor category. The category of Γ-spaces gives only connective spectra.

Inclusions of diagrams, and restriction of the monoid S, give restriction functors between various categories of (space valued) diagram spectra. There are also 'prolongation' functors that are left adjoint functors to these restriction functors. In particular, the restriction and prolongation functors give a Quillen equivalence between the stable model structures of both symmetric and sequential spectra and the stable model structure of orthogonal spectra. Γ-spaces are compared to connective orthogonal spectra via a category of diagram spectra for a suitable diagram category that contains both the finite based sets and the one-point compactifications of real vector spaces. These comparisons are discussed in [5].

Diagram spectra in motivic homotopy theory have been given by B. Dundas, P.A. Østvær, and O. Röndigs [2].

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Algebraic K-theory

Markus Szymik

Waldhausen's $S.$-construction associates a space $|wS.\mathcal{C}|$ to a category $(\mathcal{C}, w)$ with cofibrations and weak equivalences, the algebraic K-theory space of $(\mathcal{C}, w)$, see [4]. He observed that this construction can be iterated in order to define a sequential spectrum $(|w\mathcal{C}|, |wS.\mathcal{C}|, |wS.S.\mathcal{C}|, \dots)$. In fact, a multi-simplicial version of the $S.$-construction can be used to obtain a symmetric spectrum at one go. This version I have presented, following [3], with details enough and to spare.

A rich source for categories with cofibrations and weak equivalences are model categories. (This is so since the cofibrant objects in a model category automatically satisfy the gluing property.) I have explained that Quillen equivalent model categories have equivalent algebraic K-theory spectra, and gave a proof along the lines of [1]. First one shows that a Quillen pair of functors induces an equivalence between the classifying spaces of the categories with the weak equivalences. This shows that the zeroth spaces of the algebraic K-theory spectra are equivalent. Then one shows that the categories of simplices in the $S.$-construction are equivalent to such categories where the previous argument can be applied. Consequently, one obtains an equivalence of realisations, so that the first spaces of the algebraic K-theory spectra are equivalent. The rest of the equivalence follows either by iteration or by generalisation of these arguments.

As pointed out in [2], if E is a ring spectrum, a suitable category of E-module spectra gives rise to the algebraic K-theory spectrum of E. I have mentioned two special cases. On the one hand, if R is an ordinary ring, the algebraic K-theory spectrum of the Eilenberg-Mac Lane spectrum HR is equivalent to Quillen's algebraic K-theory of R. (This follows from the fact that the model category of HR-module spectra is Quillen equivalent to the model category of R-complexes.) On the other hand, if X is a space, the algebraic K-theory spectrum of the spherical monoid ring $S[\Omega X]$ is equivalent to the algebraic K-theory of X in the sense of Waldhausen. Thus, while Quillen's algebraic K-theory of ordinary rings and Waldhausen's algebraic K-theory of spaces have been around before the advent of modern foundations for categories of module spectra, they now fit into the common framework of algebraic K-theory of ring spectra.

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Power Operations

Niko Naumann

We started by introducing the Adams-Novikov spectral sequence based on a suitable generalized homology theory E

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*Y) \Rightarrow [X,Y]^E_{t-s}$$

and then had a closer look at the E_2-chart for the classical Adams spectral sequence computing the stable stems at the prime 2

$$E_2^{s,t} = \mathrm{Ext}_{A_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*^s \otimes \mathbb{Z}_2, \tag{1}$$

where $A_* = P(\xi_i)$ is the dual mod 2 Steenrod algebra. We then introduced the elements $h_i \in E_2^{1,2^i}$ dual to Sq^{2^i} and discussed the Hopf invariant one problem and its solution, i. e., h_i is a permanent cycle in (1) if and only if $i = 0, 1, 2, 3$.

We then introduced power operations in the homotopy of a commutative S-algebra X following [1]: For $n \geq 1$ put

$$D_n X := (E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n},$$

the extended powers of X, and for $i \geq 0$

$$D_n^i X := (E\Sigma_n)_+^i \wedge_{\Sigma_n} X^{\wedge n}$$

where $(E\Sigma_n)^i$ denotes the i-skeleton. The multiplication of X induces maps

$$\xi : D_n^i X \longrightarrow X.$$

Given $\alpha \in X_N(D_n^i S^n)$ one defines $\alpha^* : \pi_n X \longrightarrow \pi_N X$ as

$$\alpha^*(S^n \xrightarrow{f} X) := (S^N \xrightarrow{\alpha} X \wedge D_n^i S^n \xrightarrow{X \wedge D_n^i f} X \wedge D_n^i X \xrightarrow{\xi} X).$$

Using Milgram's generalization of the Adams spectral sequence, we discussed some of these operations and the relations among them, e.g. for $n \equiv 1\,(4)$ there is an operation

$$h_1 P^{n+1} : \pi_n^s \longrightarrow \pi_{2n+2}^s$$

such that $2(h_1 P^{n+1})(x) = \eta^2 x^2$ for $x \in \pi_n^s$ and with $\eta \in \pi_1^s$ the Hopf map.

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Units, Thom spectra, and orientation theory

Ulrich Bunke

The goal of this contribution to the Arbeitsgemeinschaft was to review the construction of the spectrum of units of a structured ring spectrum and its application to orientation theory. For an illustration we presented the discussion of the J-homomorphism due to [8] combined with more recent results such as the infinite loop Adams conjecture [2], [3].

We now describe the contents of the talk in more detail. In the first part we explained the construction of the units $gl_1(E)$ of an E_∞-spectrum E following [8]. The multiplicative structure of E gives the zeroth space E_0 of E the structure of an algebra over an E_∞-operad. The subspace of invertible components is the zeroth space $GL_1(E) := gl_1(E)_0$ of the units. The whole spectrum $gl_1(E)$ is then obtained by an application of a delooping machine, see e.g. [7].

The next topic of the talk were E-orientations of spherical fibrations. We explained the orientation sequence

$$Gl_1(E) \to B(Gl_1(E); G) \to BG \xrightarrow{w} gl_1(E)_1,$$

where G is a classical group (in the sense of [8]) and $B(Gl_1(E); G)$ classifies E-oriented stable spherical fibrations with reduction of the structure group to G. The universal Stiefel-Whitney class $w \in H^1(BG, gl_1(E))$ is the universal obstruction against E-orientability of a stable spherical fibration with reduction of the structure group to G. We explained how this sequence can be viewed as a fibre sequence of infinite loop spaces. An E-orientation of G is a split map $s : BG \to B(GL_1(E); G)$. It is called good if it is an infinite loop map.

As an illustration we discussed the Atiyah-Bott-Shapiro kO-orientation of $Spin$. In order to see that it is good, we first use results of [4], [5], [6] to construct a kO-orientation of the $Spin$-bordism theory as a map of E_∞-ring spectra $MSpin \to kO$. As explained in [8], this leads to a good kO-orientation of $Spin$.

The last part of the talk was devoted to the J-theory diagram of [8]. The goal was to understand the splitting (as infinite loop spaces)

$$J_{(p)} \xrightarrow{\alpha} sl_1(S)_{(p)} \xrightarrow{\epsilon} J_{(p)}$$

of the units of the sphere spectrum S, where p is an odd prime, $J_{(p)}$ is the fibre of the (lift of the) Adams operation $\psi^r : BO_{(p)} \to BSpin_{(p)}$, and r generates the units of $\mathbb{Z}/p^2\mathbb{Z}$. We explained how the discussion of [8] can be simplified by the knowledge of the infinite-loop version of the Adams conjecture and the goodness of the Atiyah-Bott-Shapiro orientation. The result of the calculation ([1] and follow-ups) of the image of α was stated.

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Complex cobordism, formal groups, and MU-algebras

Jens Hornbostel

In the first part (based on the textbooks of Switzer [7] and Ravenel [5]), we compute the homology groups and then - using the $H\mathbf{F}_p$-based Adams spectral sequence introduced in a previous lecture - the p-completed homotopy groups of the Thom spectrum MU. The E_2-term is given by $\mathrm{Ext}^{s,t}_{A_*}(\mathbf{F}_p, H_*(MU, \mathbf{F}_p))$, where A_* is the dual Steenrod algebra. For $p \neq 2$, there is an isomorphism of graded Hopf algebras $A_* \cong E \otimes C$, where E is an exterior algebra on generators $\tau_0, \tau_1, ...$ with $deg(\tau_i) = 2p^i - 1$ and $C = \mathbf{F}_p[\xi_1, \xi_2, ...]$ with $deg(\xi_i) = 2(p^i - 1)$. Moreover, the ring spectrum MU contains a direct summand BP such that $H_*(BP, \mathbf{F}_p) \cong C$.

The isomorphism of graded A_*-comodules $H_*(MU, \mathbf{F}_p) \cong C \otimes \mathbf{F}_p[u_i | i \neq p^t - 1]$ together with some homological algebra of A_*-comodules computes the E_2-term of the spectral sequence and shows it degenerates, thus yielding the p-completed homotopy groups of MU. By a theorem of Milnor and Novikov, there is even an isomorphism of graded rings $\pi_*(MU) \cong \mathbb{Z}[x_1, x_2, ...]$ with $\deg(x_i) = 2i$.

In the second part, we give the definition of an oriented cohomology theory following Adams [1] and explain how these give rise to one-dimensional commutative formal group laws. Then we state the theorems of Lazard and Quillen which say that the universal fromal group law is defined over the coefficient ring MU_* computed above [1, sections 7 and 8]. We also mention that BP_* carries the universal p-typical formal group law. Then we provide some other examples of orientable theories, including the Johnson-Wilson spectrum $E(n)$ and the Morava-K-theory spectrum $K(n)$ arising in chromatic homotopy theory.

In the last part, we sketch how inverting and killing elements on coefficient rings of strictly associative and commutative ring spectra such as MU lifts to the ring spectrum itself [2, chapter 5]. We conclude by stating some theorems about the question when the so-obtained spectra are still MU-ring-spectra [2, V.3.2], [6, Proposition 2.8] and even strictly associative [3, Corollary 5.10],[4]. We refer to the next lecture for obstruction theories that are used for proving such results.

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Obstruction theories

Kathryn Hess

Let (E, μ) be a homotopy-commutative, homotopy-associative ring spectrum. Let $F(-,-)$ denote the function spectrum functor. The *endomorphism operad* of E is the symmetric sequence of spectra $\mathcal{E}nd(E)$ such that $\mathcal{E}nd(E)(n) = F(E^{\wedge n}, E)$ for all $n \geq 0$, endowed with the obvious composition operations. An E_∞*-structure* on

(E, μ) consists of a morphism of operads $\mathcal{E} \to \mathcal{E}nd(E)$ extending μ, where $\mathcal{E}$ is an E_∞-operad, i.e., a cofibrant replacement of the commutative operad.

There are two well-known obstruction theories, due to Goerss and Hopkins [3] and to Robinson and Whitehouse [9, 10, 11], that can be applied to determining whether (E, μ) admits an E_∞-structure. As established by Basterra and Richter [2], the two theories lead to isomorphic obstruction groups. Due to time limitations, only the theory of Robinson and Whitehouse was treated in this talk. Concerning Goerss-Hopkins theory, we mention here only that the essential idea is to calculate the homotopy type of the moduli space of realizations of E_*E as a commutative E_*-algebra.

Let Λ denote a commutative ring with unit. Let $\mathbb{L}_n$ denote the free Lie algebra on n generators over Λ, with $Lie(n)$ the submodule spanned by Lie monomials in which each generator appears exactly once, which admits an obvious action of Σ_n on the left. Let $Lie(n)^*$ denote the Λ-dual of $Lie(n)$, with its inherited right Σ_n-action.

Let Γ denote the category of finite, based sets and basepoint-preserving maps. Let $\Phi : \Gamma \to {}_\Lambda\mathbf{Mod}$ be any functor, where Λ is a commutative ring. Motivated by the geometry of their obstruction theory, as outlined below, Robinson and Whitehouse define a bicomplex $\Xi(\Phi)$ of Λ-modules, of which the q^{th} line is

$$\Xi(\Phi)_{*,q} = \mathbf{B}(Lie(q+1)^*, \Lambda[\Sigma_{q+1}], \Phi[q+1]),$$

the two-sided bar construction. The horizontal differential is that of the bar construction, while the vertical differential has an intricate definition in terms of certain canonical surjections among finite sets. They then define $\mathrm{H}\Xi_*(\Phi)$ to be the homology of the total complex of $\Xi(\Phi)$, which Robinson [9] and Pirashvili and Richter [6] proved to be isomorphic to the stable homotopy of Φ.

Let R be a commutative, graded algebra. Let Λ be a commutative R-algebra and M a Λ-module. The *Loday functor* $\mathcal{L}(\Lambda|R; M) : \Gamma \to {}_\Lambda\mathbf{Mod}$ is specified on objects by $\mathcal{L}(\Lambda|R; M)[n] := \Lambda^{\otimes n} \otimes M$. The *gamma homology* $\mathrm{H}\Gamma_*(\Lambda|R; M)$ of the pair (Λ, R) with coefficients in M is then $\mathrm{H}\Xi_*\big(\mathcal{L}(\Lambda|R; M)\big)$, while its *gamma cohomology* $\mathrm{H}\Gamma^*(\Lambda|R; M)$ is defined to be the cohomology of the cochain complex $\mathrm{Hom}_\Lambda\big(\mathrm{Tot}\Xi\big(\mathcal{L}(\Lambda|R; \Lambda), M\big)$. Note that gamma homology and cohomology inherit a second, internal grading from the grading on R, Λ and M.

Within the framework of Robinson-Whitehouse obstruction theory, we work with the following E_∞-operad. Let $\mathcal{E}'$ denote the *Barrat-Eccles operad*, i.e., $\mathcal{E}'(n) = E\Sigma_n$, the free, contractible Σ_n-space, which admits a natural, increasing filtration by skeleta .. $\subseteq F^jE\Sigma_n \subseteq F^{j+1}E\Sigma_n \subseteq \ldots$. Let $\mathcal{T}$ denote the *tree operad*, where $\mathcal{T}(n)$ is the space of isomorphism classes of labeled n-trees, which is a contractible, finite, cubical complex. The E_∞-operad we need is then $\mathcal{E} := \mathcal{E}' \times \mathcal{T}$, which admits a *diagonal filtration*, given by $\big(\nabla^n\mathcal{E}\big)(k) := F^{n-k}E\Sigma_k \times \mathcal{T}(k)$. An *$n$-stage* $\mu_{(n)}$ for an E_∞-structure on (E, μ) consists of a set of appropriately compatible maps $\{\mu_k : \big(\nabla^n\mathcal{E}\big)(k) \to \mathcal{E}nd(E)(k) \mid k \leq n\}$, agreeing with μ for $k = 2$.

Let $R = E_*$ and $\Lambda = E_*E$. The ring spectrum (E, μ) satisfies the *perfect universal coefficient formula* if Λ is R-flat, which implies that $E_*(E^{\wedge n}) \cong \Lambda^{\otimes n}$, and if the natural homomorphism $E^*(E^{\wedge n}) \to \mathrm{Hom}_R(\Lambda^{\otimes n}, R)$ is an isomorphism for all n.

Theorem (Robinson-Whitehouse [9, 10, 11])**.** Suppose that (E, μ) satisfies the perfect universal coefficient formula. Let $\mu_{(n)}$ be an n-stage for (E, μ). There is a canonical cocycle $\theta(\mu_{(n)})$ in $\mathrm{Tot}\Xi^{n,2-n}\big(\mathcal{L}(\Lambda|R;R)\big)$ such that

- $\theta(\mu_{(n)}) = 0$ if and only if there is an $(n+1)$-stage extending $\mu_{(n)}$, and
- the cohomology class $[\theta(\mu_{(n)})] = 0$ if and only if there is an $(n+1)$-stage extending the $(n-1)$-stage underlying $\mu_{(n)}$.

Hence, if $\mathrm{H}\Gamma^{n,2-n}(\Lambda|R;R) = 0$ for all $n \geq 3$, then (E, μ) admits at least one E_∞-structure. Furthermore, if $\mathrm{H}\Gamma^{n,1-n}(\Lambda|R;R) = 0$ for all $n \geq 2$, then (E, μ) admits at most one E_∞-structure.

Theorem 1 can be applied to proving that the n^{th} Morava E-theory spectrum, E_n, admits an E_∞-structure, for all $n \geq 1$. We begin by recalling the definition of E_n.

Let k be a perfect field of characteristic $p > 0$, and let G be a formal group law over k. Let G' be a formal group law over a complete, local ring B with maximal ideal $\mathfrak{m}$. Let $\pi : B \to B/\mathfrak{m}$ denote the projection morphism, and let $i : k \to B/\mathfrak{m}$ be a homomorphism. The pair (G', i) is a *deformation* of (k, G) if $i^*G = \pi^*G'$. An isomorphism $f : G_1 \to G_2$ of formal groups laws over B is a *morphism of deformations* from (G_1, i) to (G_2, i) if $\pi^*f = Id_{i^*G}$.

Theorem (Lubin-Tate [5])**.** For any finite-height, formal group law G over k, there exists

- a complete, local ring $E(k, G)$ with maximal ideal $\mathfrak{m}$,
- an isomorphism $i : k \to E(k, G)/\mathfrak{m}$, and
- a formal group law F over $E(k, G)$ that is a *universal deformation* of (k, G), i.e., for every deformation (G', i) over B of (k, G), there is a unique ring homomorphism $\varphi : E(k, G) \to B$ such that there is a (unique) morphism of deformations from (φ^*F, i) to (G', i).

It follows from Theorem 2 that the automorphism group of G acts on $E(k, G)$. In particular, when $G = F_n$, the Honda formal group law of height n over $\mathbb{F}_{p^n}$, then the group S_n of automorphisms of F_n, also known as the n^{th} *Morava stabilizer group*, acts on $E(\mathbb{F}_{p^n}, F_n)$, as does $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

We remark that $E(\mathbb{F}_{p^n}, F_n)$ is isomorphic to $\mathbb{W}\mathbb{F}_{p^m}[[u_1, \ldots, u_{n-1}]]$, where $\mathbb{W}\mathbb{F}_{p^m}$ is the ring of Witt vectors. The ring $E(\mathbb{F}_{p^n}, F_n)$ is thus a complete, local ring, with maximal ideal $(p, u_1, \ldots, u_{n-1})$.

Recall that MU_* and its formal group law F_{MU} classify formal group laws of degree -2 [1]. Given a formal group (k, G), define a functor $(E_{k,G})_*$ from topological spaces to graded abelian groups by $(E_{k,G})_*(X) := (E_{k,G})[u^\pm] \otimes_{MU_*} MU_*(X)$, where the MU_*-module structure on $(E_{k,G})[u^\pm]$ is induced by the map $MU_* \to (E_{k,G})[u^\pm]$ classifying the degree -2 formal group law $\overline{F}(x, y) = u^{-1}F(ux, uy)$, where $\deg u = 2$.

Theorem (Landweber [4]). Let R_* be an MU_*-module. Let v_k be the coefficient of x^{p^k} in the p-series of F_{MU}. If R is *Landweber exact*, i.e., if $(p, v_1, v_2, \dots)$ is a regular sequence in R_* for each prime p, then the functor $X \mapsto R_* \otimes_{MU_*} MU_*(X)$ is a homology theory.

Theorem (Hopkins-Miller-Rezk [7]). The MU_*-module $(E_{k,G})[u^{\pm}]$ is Landweber exact.

Hopkins and Miller showed that $(E_{k,G})_*$ could be realized naturally as a homotopy commutative A_∞-ring spectrum, denoted $E_{k,G}$. In special case of the Honda formal group law, we obtain the n^{th} *Morava E-theory* or the n^{th} *Lubin-Tate spectrum*, $E_n := E_{\mathbb{F}_{p^n}, F_n}$. The following deep results in homological algebra, due to Hopkins and Miller [7], imply that E_n satisfies the perfect coefficient formula and that therefore Robinson-Whitehouse obstruction theory applies.

- If E_* and F_* are Landweber exact MU_*-modules, then E_*F is flat over E_*.
- If M is a flat $(E_{k,G})_*$-module, then $\mathrm{Ext}^s_{(E_{k,G})_*}(M, (E_{k,G})_*) = 0$ for all $s > 0$.
- The natural map $E_n^*(E_n) \to \mathrm{Hom}_{(E_n)_*}((E_n)_*E_n, (E_n)_*)$ is an isomorphism.

As proved by Goerss and Hopkins [3] and, later, by Richter and Robinson [8],

$$\mathrm{H\Gamma}^*\big((E_n)_*(E_n) \mid (E_n)_*; (E_n)_*\big) = 0.$$

Theorem 1 therefore implies that E_n admits a unique E_∞-structure.

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Galois Theory

Holger Reich

The talk reported on Galois theory of structured ring spectra as developed by John Rognes in [5]. Let G be a finite group. In elementary topology one has the notion of a G-covering, or G-principal bundle: let $p\colon X \to Y$ be a map of topological spaces and suppose G acts on X through maps over Y. Then this situation is called a G-covering if

(1) The natural map $X/G \to Y$ is a homeomorphism.
(2) The map $G \times X \to X \times_Y X$, $(g, x) \mapsto (x, gx)$ is a homeomorphism.

If one transports this definition to affine algebraic geometry, or equivalently to commutative ring theory, one is lead to the notion of a G-Galois extension of commutative rings, compare [3] and [2]. Transported further to the framework of commutative $\mathbb{S}$-algebras, that was developed during the Arbeitsgemeinschaft, one obtains the notion of a G-Galois extension of commutative $\mathbb{S}$-algebras:

Let A be an $\mathbb{S}$-algebra, let B be an A-algebra, and suppose G acts on B through A-algebra maps, then this situation is called a G-Galois extension if

(1) The natural map $A \to B^{hG}$ is a weak equivalence.
(2) The map $B \wedge_A B \to F(G_+, B)$, adjoint to the action map of G on the right smash-factor, is a weak equivalence.

The Eilenberg-McLane functor applied to a Galois extension of commutative rings yields a Galois extension of commutative $\mathbb{S}$-algebras.

But new examples arise immediately. Taking $\mathbb{F}_p$-cochains, i.e., applying the contravariant functor $F((-)_+, H\mathbb{F}_p)$ to a G-covering $X \to Y$ of topological spaces, yields a G-Galois extension of $H\mathbb{F}_p$-algebras if the Eilenberg-Moore spectral sequence for the computation of $H^*(X \times_Y X; \mathbb{F}_p)$ converges strongly. If one takes G as a finite p-group and $p\colon X \to Y$ as the natural quotient map $EG \to BG$, then one obtains an example of a non-trivial Galois extension $A \to B$. In this special case $B = F(EG_+, H\mathbb{F}_p) \simeq H\mathbb{F}_p$. Note that for commutative rings it would be completely absurd to have a non-trivial Galois extension $A \to B$ with $B = \mathbb{F}_p$.

The complexification map $KO \to KU$ is a C_2-Galois extension.

There is an analogue of the Main Theorem of Galois Theory in the new context. John Rognes proves that the sphere spectrum $\mathbb{S}$ admits no non-trivial connected Galois extension. This uses and is the analogue of the Minkowsky Theorem, which makes the same statement for $\mathbb{Z}$ in the context of commutative rings.

If one works $K(n)$-locally (here $K(n)$ is Morava K-theory) then the extension $L_{K(n)}\mathbb{S} \to E_n$ from the $K(n)$-local sphere to the Lubin-Tate theory E_n can be interpreted as a profinite Galois extension. The Galois group is the Morava stabilizer group, compare [4], [1]. A slight variant of E_n is conjectured to be a "separable closure" of the $K(n)$-local sphere.

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Rigidity for the stable homotopy category

Constanze Roitzheim

If two model categories $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent, then their homotopy categories $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$ are equivalent. But if $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$ are equivalent categories, can anything be said about the underlying model structures? For the stable homotopy category $Ho(\mathcal{S})$ (i.e., the homotopy category of spectra) there is the following result:

Theorem (Rigidity Theorem [2])**.** Let $\mathcal{C}$ be a stable model category and

$$\Phi : Ho(\mathcal{S}) \longrightarrow Ho(\mathcal{C})$$

be an equivalence of triangulated categories. Then $\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

This means that all higher homotopy information in $\mathcal{S}$ such as algebraic K-theory or mapping spaces is already encoded in the triangulated structure of the stable homotopy category. However, this theorem does not claim that the equivalence Φ is the derived of a Quillen functor.

In this talk we will construct the most important category-theoretic tool used in the proof of the Rigidity Theorem:

Theorem (Universal Property of Spectra [3])**.** Let $\mathcal{C}$ be a stable model category, $X \in \mathcal{C}$ a fibrant and cofibrant object. Then there is a Quillen adjoint functor pair

$$X \wedge \_ : \mathcal{S} \leftrightarrows \mathcal{C} : \mathrm{Hom}(X, \_)$$

with $X \wedge S^0 \simeq X$.

Here $\mathcal{S}$ is the category of sequential spectra with the stable model structure of Bousfield and Friedlander [1, Sec. 2].

The right adjoint $\mathrm{Hom}(X, \_)$ will be constructed in the case that $\mathcal{C}$ is a simplicial model category by using mapping spaces. Then we give the left adjoint and prove that $\mathrm{Hom}(X, \_)$ is a right Quillen functor.

In the set-up of the Rigidity Theorem this Quillen pair will provide the desired Quillen equivalence. More precisely, for $X = \Phi(S^0)$, the composition

$$Ho(\mathcal{S}) \xrightarrow{L(X\wedge\_)} Ho(\mathcal{C}) \xrightarrow{\Phi^{-1}} Ho(\mathcal{S})$$

of the left derived Quillen functor and the given equivalence Φ is an endofunctor of the stable homotopy category sending the sphere to itself. Hence, as the following talks will show, it must be a self-equivalence of $Ho(\mathcal{S})$. Consequently, the derived functor of the Quillen functor $X \wedge \_$ is an equivalence of categories which means that $\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

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Reduction to Adams filtration one

Marc A. Nieper-Wisskirchen

Recall the following theorem (see [2] and the previous talk):

Rigidity Theorem (S. Schwede)**.** Let $\mathcal{C}$ be a stable model category. If the homotopy category of $\mathcal{C}$ and the homotopy category of spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathcal{C}$ and the model category of spectra.

We start by showing (following [2]) that the proof of this theorem can be reduced to the proof of the following proposition by using the "universal property of the category of Bousfield-Friedlander spectra" as given by the uniqueness theorem in [3].

Proposition. Let F be an exact endofunctor of the stable homotopy category of spectra which preserves infinite sums and takes the sphere spectrum $\mathbf{S}^0$ to itself, up to isomorphism. If for every odd prime p the morphism $F(\alpha_1) : F(\mathbf{S}^{2p-3}) \to F(\mathbf{S}^0)$ is non-trivial, then F is a self-equivalence. □

(Here $\alpha_1 \in \pi_{2p-3}(\mathbf{S}^0) \otimes \mathbb{Z}_p$ is a generator of the first non-trivial mod-p stable homotopy group of the sphere spectrum.)

The proposition itself is proven p-locally, one prime at a time, which is possible because F as an exact functor commuting with infinite sums also commutes with p-localisation. The problem is further reduced to the subcategory of finite p-local spectra, i.e., one has to show that F restricts to an equivalence on these spectra.

By a cell induction argument on finite spectra from [1], this can be broken down to the statement that the map

$$F : [\mathbf{S}^0_p, \mathbf{S}^0_p]_* \to [F(\mathbf{S}^0_p), F(\mathbf{S}^0_p)]_*$$

is an isomorphism for all primes p, i.e., we have to check the action of F on all elements of the stable homotopy of the sphere spectrum.

We recall the definition of the mod-p Adams filtration of these stable homotopy groups. It is a leading principle in the study of the stable homotopy groups that they are generated (in terms of compositions, Toda brackets and higher homotopy operations) by the elements of Adams filtration one, which we state in a very precise manner usable for our purposes here, following [1]. The non-trivial elements of Adams filtration one for $p = 2$ are the three Hopf maps η, ν, σ and for odd prime p the "greek letter" map α_1 mentioned above. All of them are generators of the localised stable homotopy groups, in which they live, i.e., it remains to show that $F(\gamma)$ is a generator for γ one of the four elements.

For $\gamma = \alpha_1$, this is the assumption of the proposition above, which is the topic of the next talk. For γ one of the Hopf maps, we give proofs by studying the mod-2 Moore spectrum.

Finally, we hint how the existence of α_1 can be deducted from the Adams spectral sequence and how this element is detected by the mod-p Steenrod reduced power P^1. For purposes of the next talk, we also define the element $\beta_1 \in \pi_{p(2p-2)-2}(\mathbf{S}^0)$ and show how all of its extensions to the mod-p Moore spectrum are detected by P^p.

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Proof of the Rigidity Theorem

Anthony D. Elmendorf

We reduced the proof of the rigidity theorem of [1] to the following proposition.

Proposition. Let $\mathcal{S}$ be the category of spectra, let $\mathcal{C}$ be a stable model category, and let $\Phi : \mathrm{Ho}(\mathcal{S}) \to \mathrm{Ho}(\mathcal{C})$ be an equivalence of triangulated categories. Then for every prime p, the map

$$\alpha_1 \wedge \Phi(\mathbf{S}^0) : S^{2p} \wedge \Phi(\mathbf{S}^0) \to S^3 \wedge \Phi(\mathbf{S}^0)$$

is nontrivial.

Here α_1 is the first p-torsion element in the homotopy of S^3, which also gives, after suspension, the first p-torsion element in the stable homotopy groups of spheres. To prove this proposition, we introduced the idea of a *k-coherent M-action*, where k is a positive integer and M is the Moore space with bottom cell in dimension 2. This is a precise way of allowing the extended powers of M to act on an object of $\mathcal{C}$ with k degrees of homotopy associativity; in particular, M acts on itself via a canonical $(p-1)$-coherent action. The obstruction to extending this action to a p-coherent action is precisely the class α_1. Proceeding by contradiction, we assumed that the map $\alpha_1 \wedge \Phi(\mathbf{S}^0)$ was trivial, and then used this to construct a sequence $E_0, \ldots, E_{p-1}$ of spectra with E_i having cells in precisely dimensions jpq and $jpq+1$ for $0 \leq j \leq i$ and where $q = 2p-2$ as usual. Further, the construction of the E_i showed that the i'th iterate of the Steenrod power operation P^p gave an isomorphism

$$(P^p)^i : H^0(E_i) \to H^{ipq}(E_i),$$

as well as a map $a_i : S^{(i+1)pq-1} \to E_i$ detected by P^p. Consequently, when taking the mapping cone Ca_{p-1} of the last map, $S^{p^2q-1} \to E_{p-1}$, we obtained a spectrum with cells only in dimensions ipq and $ipq+1$ for which the p'th iterate of the Steenrod power P^p gave an isomorphism

$$(P^p)^p : H^0(Ca_{p-1}) \to H^{p^2q}(Ca_{p-1}).$$

However, for dimensional reasons, the Steenrod power P^1 is trivial on this spectrum, but by the Adem relations, $(P^p)^p$ is in the left ideal generated by P^1, so must be 0 on this spectrum as well. This contradiction established the proposition.

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