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Combinatorics

Organised by
László Lovász (Redmond)
Hans Jürgen Prömel (Berlin)

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Abstract. This is the report on the Oberwolfach workshop on Combinatorics, held 1–7 January 2006. Combinatorics is a branch of mathematics studying families of mainly, but not exclusively, finite or countable structures – discrete objects. The discrete objects considered in the workshop were *graphs*, *set systems*, *discrete geometries*, and *matrices*. The programme consisted of 15 invited lectures, 18 contributed talks, and a problem session focusing on recent developments in graph theory, coding theory, discrete geometry, extremal combinatorics, Ramsey theory, theoretical computer science, and probabilistic combinatorics.

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Introduction by the Organisers

The workshop *Combinatorics* organised by László Lovász (Redmond) and Hans Jürgen Prömel (Berlin) was held January 1st–January 7th, 2006. This meeting was very well attended with 48 participants from many different countries. The programme consisted of 15 plenary lectures, accompanied by 18 shorter contributions and a vivid problem session led by Vera T. Sós.

The plenary lectures provided a very good overview over the current developments in several areas of combinatorics and discrete mathematics. We were very fortunate that some of the speaker reported on essential progress on longstanding open problems. In particular, Ajtai, Komlós, Simonovits, and Szemerédi solved the Erdős–T. Sós conjecture on trees (for large trees) and Tao and Vu greatly improved the asymptotic upper bound on the probability that a Bernoulli matrix is singular. The shorter contributions ranged over many topics. They were a good platform, especially, for younger researchers to present their results. In the following we include the extended abstracts of all talks in the order they were given.

On behalf of all participants, the organisers would like to thank the staff and the director of the *Mathematisches Forschungsinstitut Oberwolfach* for providing a stimulating and inspiring atmosphere. The organizers also thank the many participants who traveled on New Year's Eve and New Year's Day to arrive on time for the beginning of the workshop.

Workshop: Combinatorics

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Abstracts

New code bounds with noncommutative algebra

Alexander Schrijver

We present a new upper bound on $A(n,d)$, the maximum size of a binary code of word length n and minimum distance at least d. The bound is based on block-diagonalizing the (noncommutative) Terwilliger algebra of the Hamming cube and on semidefinite programming. The bound refines the Delsarte bound [1], which is based on diagonalizing the (commutative) Bose-Mesner algebra of the Hamming cube and on linear programming. The bound is published in [9].

Fix a nonnegative integer n, and let $\mathcal{P}$ be the collection of all subsets of $\{1,\dots,n\}$. We identify code words in $\{0,1\}^n$ with their support. So a code C is a subset of $\mathcal{P}$. The *Hamming distance* of $X,Y\in\mathcal{P}$ is equal to $|X\triangle Y|$. The *minimum distance* of a code C is the minimum Hamming distance of distinct elements of C.

1. The Terwilliger algebra

For nonnegative integers i,j,t, let $M^t_{i,j}$ be the $\mathcal{P}\times\mathcal{P}$ matrix with

$$(1)\qquad (M^t_{i,j})_{X,Y} := \begin{cases} 1 & \text{if } |X|=i, |Y|=j, |X\cap Y|=t,\\ 0 & \text{otherwise,}\end{cases}$$

for $X,Y\in\mathcal{P}$. So $(M^t_{i,j})^{\mathsf{T}} = M^t_{j,i}$. Let $\mathcal{A}_n$ be the set of matrices

$$(2)\qquad \sum_{i,j,t=0}^{n} x^t_{i,j} M^t_{i,j}$$

with $x^t_{i,j}\in\mathbb{C}$. It is easy to check that $\mathcal{A}_n$ is a *C∗-algebra*: it is closed under addition, scalar and matrix multiplication, and taking the adjoint. This algebra is called the *Terwilliger algebra* [10] of the *Hamming cube* $H(n,2)$.

Since $\mathcal{A}_n$ is a C∗-algebra and since $\mathcal{A}_n$ contains the identity matrix, there exists a unitary $\mathcal{P}\times\mathcal{P}$ matrix U (that is, $U^*U=I$) and positive integers $p_0,q_0,\dots,p_m,q_m$ (for some m) such that $U^*\mathcal{A}_nU$ is equal to the collection of all block-diagonal matrices

$$(3)\qquad \begin{pmatrix} C_0 & 0 & \cdots & 0\\ 0 & C_1 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & C_m\end{pmatrix} \text{ where } C_k = \begin{pmatrix} B_k & 0 & \cdots & 0\\ 0 & B_k & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & B_k\end{pmatrix}.$$

Each C_k is a block-diagonal matrix with q_k repeated, identical blocks of order p_k. So $p_0^2 + \cdots + p_m^2 = \dim(\mathcal{A}_n) = \binom{n+3}{3}$ and $p_0q_0 + \cdots + p_mq_m = 2^n$.

Now the positive semidefiniteness of any matrix in $\mathcal{A}_n$ is equivalent to the positive semidefiniteness of the corresponding blocks B_k (which are much smaller).

It turns out that U can be taken real, that $m = \lfloor \frac{n}{2} \rfloor$, and that, for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, there is a block B_k of order $p_k = n - 2k + 1$ and multiplicity $q_k = \binom{n}{k} - \binom{n}{k-1}$.

To describe the blocks of (2), define, for $i, j, k, t \in \{0, \ldots, n\}$:

$$\beta^t_{i,j,k} := \sum_{u=0}^{n} (-1)^{u-t} \binom{u}{t} \binom{n-2k}{u-k} \binom{n-k-u}{i-u} \binom{n-k-u}{j-u}. \tag{4}$$

Then for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, the kth block B_k is the following $(n-2k+1) \times (n-2k+1)$ matrix:

$$\left(\sum_t \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta^t_{i,j,k} x^t_{i,j} \right)_{i,j=k}^{n-k}. \tag{5}$$

2. Application to coding

Let $C \subseteq \mathcal{P}$ be any code m(assuming $\emptyset \neq C \neq \mathcal{P}$). Let Π be the set of (distance-preserving) automorphisms π of $\mathcal{P}$ with $\emptyset \in \pi(C)$, and let Π' be the set of automorphisms π of $\mathcal{P}$ with $\emptyset \notin \pi(C)$. Let $\chi^{\pi(C)}$ denote the incidence vector of $\pi(C)$ in $\{0,1\}^{\mathcal{P}}$ (taken as *column* vector). Define the $\mathcal{P} \times \mathcal{P}$ matrices R and R' by:

$$R := \sum_{\pi \in \Pi} |\Pi|^{-1} \chi^{\pi(C)} (\chi^{\pi(C)})^{\mathsf{T}} \text{ and } R' := \sum_{\pi \in \Pi'} |\Pi'|^{-1} \chi^{\pi(C)} (\chi^{\pi(C)})^{\mathsf{T}}. \tag{6}$$

As R and R' are sums of positive semidefinite matrices, they are positive semidefinite. Moreover, R and R' belong to $\mathcal{A}_n$. To see this, define $x^t_{i,j}$ to be the number of triples $(X, Y, Z) \in C^3$ with $|X \triangle Y| = i$, $|X \triangle Z| = j$, and $|(X \triangle Y) \cap (X \triangle Z)| = t$. divided by $|C| \binom{n}{i-t, j-t, t}$. Then

$$R = \sum_{i,j,t} x^t_{i,j} M^t_{i,j} \text{ and } R' = \frac{|C|}{2^n - |C|} \sum_{i,j,t} (x^0_{i+j-2t,0} - x^t_{i,j}) M^t_{i,j}. \tag{7}$$

The positive semidefiniteness of R and R' is by (5) equivalent to:

(8) for each $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, the matrices

$$\left(\sum_{t=0}^{n} \beta^t_{i,j,k} x^t_{i,j} \right)_{i,j=k}^{n-k} \quad \text{and} \quad \left(\sum_{t=0}^{n} \beta^t_{i,j,k} (x^0_{i+j-2t,0} - x^t_{i,j}) \right)_{i,j=k}^{n-k}$$

are positive semidefinite.

The $x^t_{i,j}$'s moreover satisfy the following constraints, where (iv) holds if C has minimum distance at least d:

(9)
(i) $x^0_{0,0} = 1$,
(ii) $0 \leq x^t_{i,j} \leq x^0_{i,0}$ and $x^0_{i,0} + x^0_{j,0} \leq 1 + x^t_{i,j}$ for all $i, j, t \in \{0, \ldots, n\}$,
(iii) $x^t_{i,j} = x^{t'}_{i',j'}$ if $(i', j', i' + j' - 2t')$ is a permutation of $(i, j, i + j - 2t)$,
(iv) $x^t_{i,j} = 0$ if $\{i, j, i + j - 2t\} \cap \{1, \ldots, d - 1\} \neq \emptyset$.

Moreover, $|C| = \sum_{i=0}^{n} \binom{n}{i} x^0_{i,0}$. Hence we obtain an upper bound on $A(n, d)$ by considering the $x^t_{i,j}$ as variables, and by maximizing $\sum_{i=0}^{n} \binom{n}{i} x^0_{i,0}$ subject to conditions (8) and (9). This is a semidefinite programming problem with $O(n^3)$ variables, and it can be solved in time polynomial in n. The method gives, in the range $n \leq 28$, the new upper bounds on $A(n, d)$ given in the table.

n	d	best lower bound known	new upper bound	best upper bound previously known	Delsarte bound
19	6	1024	1280	1288	1289
23	6	8192	13766	13774	13775
25	6	16384	47998	48148	48148
19	8	128	142	144	145
20	8	256	274	279	290
25	8	4096	5477	5557	6474
27	8	8192	17768	17804	18189
28	8	16384	32151	32204	32206
22	10	64	87	88	95
25	10	192	503	549	551
26	10	384	886	989	1040

The new bound is stronger than the Delsarte bound, which is equal to the maximum value of $\sum_i \binom{n}{i} x^0_{i,0}$ subject to the condition that $x^0_{i,0} \geq 0$ for all i and $x^0_{i,0} = 0$ if $1 \leq i \leq d - 1$, and to the condition that $\sum_{i,j,t} x^0_{i+j-2t,0} M^t_{i,j}$ is positive semidefinite.

3. Concluding remarks

Taking a tensor product of the algebra, this approach also yields a new upper bound on $A(n, d, w)$, the maximum size of a binary code of word length n, minimum distance at least d, and constant weight w. This bound strengthens the Delsarte bound for constant-weight codes.

The present research roots in two basic papers presenting eigenvalue techniques to obtain upper bounds: Delsarte [1], giving a bound on codes based on association schemes, and Lovász [5], giving a bound on the Shannon capacity of a graph. It was shown by McEliece, Rodemich, and Rumsey [7] and Schrijver [8] that the Delsarte bound is a special case of (a close variant of) the Lovász bound. (This is not to say that the Lovász bound supersedes the Delsarte bound: essential in the latter bound is a reduction of the 2^n-vertex graph problem to a linear programming problem of order n.) An extension of the Lovász bound based on 'matrix cuts' was given by Lovász and Schrijver [6] Applying a variant of matrix cuts to the coding problem leads to considering the Terwilliger algebra as above.

Frank Vallentin showed that the block-diagonalization of the Terwilliger algebra may also be derived using the representation theory of the symmetric group. Etienne de Klerk and Dima Pasechnik [3] used the approach to sharpen bounds on the stability number of orthogonality graphs. Monique Laurent [4] extended the above methods and found further improvements on the bounds on codes and the stability number of orthogonality graphs, based on the Lasserre hierarchy of bounds and using the block-diagonalization of the Terwilliger algebra. With Dion Gijswijt and Hajime Tanaka [2] we generalized the bound for (nonconstant-weight) codes to nonbinary codes.

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Disjoint minors in large graphs

Andrew Thomason

The natural measure of the extremal function for a complete minor in a graph was shown by Mader [5] to be

$$c(r) \;=\; \min\{\, c \,:\, e(G) \geq c|G| \text{ implies } G \succ K_r \,\}\,.$$

The function $c(r)$ was estimated by Kostochka [4] (see also [8]) and in [9] it was shown that the asymptotic value is $c(r) = (0.319\ldots + o(1))r\sqrt{\log r}$. Random graphs of appropriately chosen order and density (both $\Theta(r\sqrt{\log r})$) provide examples of extremal graphs, and Myers [6] showed that all extremal graphs are pseudo-random graphs like these, or essentially disjoint unions of such graphs.

The form of these extremal graphs prompts two natural questions about graphs whose order is large with respect to r. First, what is the maximum connectivity if no K_r minor is allowed — is it only linear in r? And second, what is the maximum size if a K_r minor is allowed but many disjoint K_r minors are not?

It turns out in both cases that complete bipartite graphs, with one class small, are blockages. This is illustrated by a recent theorem of Böhme, Kawarabayashi, Maharry and Mohar [1], that if $\kappa(G) \geq 16r$ then G contains s disjoint K_r minors or it contains a subdivision of $K_{r,s}$, provided $|G|$ is large (depending on both r and s). Their theorem (which is actually stronger than stated here) gives an affirmative answer to the first question, though the proof relies on the full graph minor theory of Robertson and Seymour.

As for the second question, an answer was given in principle by Myers and Thomason [7], who showed that graphs G of size more than approximately $sc(r)|G|$ contain s disjoint K_r minors. Random graphs of order approximately $sr\sqrt{\log r}$ provide extremal graphs. However, it is clearly not possible to take disjoint copies of these to provide extremal graphs of large order, and so this result gives no information about the maximum size of graphs of large order having no sK_r minor. We therefore define

$$c_\infty(sK_r) \;=\; \liminf_{n}\{c : |G| \geq n,\; e(G) \geq c|G| \Longrightarrow G \succ sK_r\}\,.$$

The parameter $c_\infty(sK_r)$ displays different behaviour according to whether s is small or large relative to s. In fact, we show that

(a) $c_\infty(r) = (1 + o(1))c(r)$ for fixed s but $r \to \infty$, whereas
(b) $c_\infty(r) = s(r-1) - 1$ for $s \geq 20c(r)$.

The implication of case (a) is that if any fixed number of K_r minors is disallowed then the extremal graphs do not differ much from the extremal graph when just one K_r minor is forbidden. It is a consequence of a more general result: that, given r, s and t, the maximum size of a graph of order n having neither sK_r nor $K_{t,s}$ as a minor is $(c(r) + t - 1)n + o(n)$ as $n \to \infty$. This latter result is best possible, as is seen by taking the sum of K_{t-1} and multiple copies of a graph with no K_r minor.

On the other hand, case (b), for larger s, is a consequence of a more exact result, as follows: let $r \geq 3$, $s \geq 20c(r)$ and $m = s(r-1) - 1$. If G is a graph of order $n > 2^{2rsm}$ with $e(G) \geq e(K_m + \overline{K}_{n-m})$ and $G \not\succ sK_r$, then $G = K_m + \overline{K}_{n-m}$.

This theorem is a direct generalization of a classical result of Erdős and Pósa [3], which is the case $r = 3$.

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Combinatorial and polyhedral surfaces

Günter M. Ziegler

(joint work with Raman Sanyal and Thilo Schröder)

1. What is a surface?

There are several different combinatorial and geometric notions of a "polyhedral surface." *Topologically*, we consider the connected, orientable 2-manifold without boundary of genus $g \geq 0$, denoted M_g.

In the *combinatorial* version, we look at regular cell decompositions of M_g, which may be obtained by drawing graphs on the surfaces M_g, or by combinatorial prescriptions that tell us how to glue the surface from polygons. The "rotation schemes" of Heffter [6], see also Ringel [11], fall into this category. In the following, we will insist on the *intersection condition* to hold, according to which the intersection of any two cells of the surface consists of either one single edge, or one vertex, or is empty.

In the *geometric* version, a polyhedral surface is a complex formed by flat convex polygons, represented without intersections in $\mathbb{R}^3$, or in some $\mathbb{R}^N$.

See [4] and [13] for more detailed discussions of these models.

2. f-Vectors

For any (combinatorial or geometric) surface the *f-vector* (f_0, f_1, f_2) records the number of vertices, edges, and 2-faces. It thus also measures the topological complexity of the surface, whose genus is given by

$$g \quad = \quad 1 + \tfrac{1}{2}(f_1 - f_0 - f_2),$$

and the combinatorial complexity, via the average vertex degree and the average face degree, given by

$$\delta = \tfrac{2f_1}{f_0} \qquad \text{and} \qquad \delta^* = \tfrac{2f_1}{f_2}.$$

A key problem asks to characterize the f-vectors of combinatorial resp. geometric surfaces, and thus distinguish the two models in terms of their combinatorial characteristics.

For $g = 0$, a little lemma of Steinitz [12] characterizes the f-vectors by

$$f_0 - f_1 + f_2 = 2, \qquad 2f_1 \geq 3f_0, \qquad 2f_1 \geq 3f_2$$

for *both* models. Indeed, these are the f-vectors of convex 3-polytopes.

In contrast, for $g > 0$ the inequality $2f_1 \geq 3f_0$ is tight for combinatorial surfaces, while it is strict for geometric surfaces: A geometric surface satisfying $2f_1 = 3f_0$ is necessarily realized in $\mathbb{R}^3$, and convex. Indeed, we have $2f_1 - 3f_0 \geq 6$ for $g > 0$; see Barnette et al. [1].

3. The high genus case

An interesting extremal case to study is when we fix the number $n := f_0$, and ask for surfaces with the maximal genus, or equivalently, for surfaces with the maximal number of edges and 2-faces. For this we may assume that the surface is triangulated, so $2f_1 = 3f_2$, and $g = 1 + \frac{1}{2}(\frac{f_1}{3} - n)$.

In the combinatorial model, the inequality $f_1 \leq \binom{n}{2}$ is tight for infinitely many values of n, for example for $n = 4, 7, 12$ and for $n \equiv 7 \mod 12$, according to Ringel et al., see [11].

On the other hand, geometric surfaces with $f_1 = \binom{n}{2}$ exist in $\mathbb{R}^3$ for $n = 4, 7$, but *not* for $n = 12$, according to Bokowski & Guedes de Oliveira [2] and Schewe (personal communication, 2005). It is an open problem whether the upper bound $f_1 = O(n^2)$ is tight — the best known lower bound is $f_1 = \Omega(n \log n)$ for surfaces, and $f_1 = \Omega(n^{3/2})$ for the weaker model of “almost disjoint triangles” [7].

4. A combinatorial construction

A combinatorial Ansatz for the construction of extremal surfaces traces back to Brehm [3], see also Datta [5]: In a (p, q)-surface all vertex degrees are p and all faces are q-gons. The goal is to construct (p, p)-surfaces with few vertices.

The Ansatz now produces such surfaces on the vertex set $\mathbb{Z}_N$, by taking as the vertex sets of its faces the cyclic translates of a set $0, A_1, A_2, \dots A_{2m}$ with successive differences

$$a_1, a_1, a_2, a_2, \dots, a_m, a_m.$$

This yields a pseudomanifold (and usually a surface) if the a_i are distinct, and the surface will be orientable if the a_i are odd. The key condition to look at is the intersection property, which mandates that the consecutive partial sums of

$$N = A_{2m} \; = \; a_1 + a_1 + a_2 + a_2 + \ldots + a_m + a_m$$

(other than the singleton sums) should be distinct. Datta suggests the choice $a_i = 3^i$, which clearly works, but it is also easy to see that there are choices such as $a_i := m^2 + i - 1$ that yield a sum f_0 of order $O(m^3)$.

The open problem posed in my talk is whether sum of the a_i can be achieved to be $A_{2m} = O(m^2)$, which would clearly be optimal. This asks for a construction of such numbers a_i resp. A_i such that all the differences $A_i - A_j$ are distinct, except for $A_{2k-1} - A_{2k-2} = A_{2k} - A_{2k-1} = a_k$. This asks for a variant of so-called *Sidon sets*. See O'Bryant [10] for a recent survey.

5. A geometric construction

In the last part of the talk, I described a geometric construction that realizes geometric $(p, 2q)$-surfaces in the boundary complex of a polytope of dimension $2 + p(q-1)$, and hence (after a generic projection) in $\mathbb{R}^5$.

For this, we use an *iterated wedge polytope*, a simple $2 + p(q-1)$-polytope $W := \Delta_{q-1} \lhd C_p$ with pq facets. We do not describe this polytope here; it arises from a p-gon by p generalized wedge operations, as described in McMullen [9].

The dual polytope $S := \Delta_{q-1} \wr C_p$ is a simplicial $2 + p(q-1)$-polytope that arises from a p-gon by successively replacing each vertex by q new vertices that span a $(q-1)$-simplex, increasing the dimension by $q-1$, with the given vertex in its barycenter. The *wreath product* polytopes were described by Joswig & Lutz [8].

The vertices of W, and hence the facets of S, may be indexed by arrays

$$(k_1, k_2, \ldots, *, *, \ldots, k_p),$$

with $k_i \in [q]$ and two cyclicly adjacent $*$s. Thus there are $f_0 = p\, q^{p-2}$ vertices. The edges of the surface correspond to the arrays of the form

$$(k_1, k_2, \ldots, *, ., \ldots, k_p),$$

with only one $*$, which yields $f_1 = p\, q^{p-1}$ edges. Finally, the faces of the surfaces are p-gons given by those arrays of the form

$$(k_1, k_2, \ldots, ., ., \ldots, k_p),$$

that additionally satisfy the condition $\sum_{i=1}^{p} k_i \equiv 0$ or $1 \pmod q$. This yields a count of $f_2 = 2\, q^{p-1}$ for the faces of the geometric $(p, 2q)$-surface in question.

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Additive Approximation for Edge-Deletion Problems

Noga Alon
(joint work with Asaf Shapira and Benny Sudakov)

A graph property is *monotone* if it is closed under removal of vertices and edges. In this paper we consider the following algorithmic problem, called the edge-deletion problem; given a monotone property P and a graph G, compute the smallest number of edge deletions that are needed in order to turn G into a graph satisfying P. We denote this quantity by $E'_P(G)$. The first result of this paper states that the edge-deletion problem can be efficiently approximated for any monotone property.

- For any fixed $\epsilon > 0$ and any monotone property P, there is a deterministic algorithm, which given a graph $G = (V, E)$ of size n, approximates $E'_P(G)$ in linear time $O(|V| + |E|)$ to within an additive error of ϵn^2.

Given the above, a natural question is for which monotone properties one can obtain better additive approximations of E'_P. Our second main result essentially resolves this problem by giving a precise characterization of the monotone graph properties for which such approximations exist.

(1) If there is a bipartite graph that does not satisfy P, then there is a $\delta > 0$ for which it is possible to approximate E'_P to within an additive error of $n^{2-\delta}$ in polynomial time.
(2) On the other hand, if all bipartite graphs satisfy P, then for any $\delta > 0$ it is NP-hard to approximate E'_P to within an additive error of $n^{2-\delta}$.

While the proof of (1) is relatively simple, the proof of (2) requires several new ideas and involves tools from Extremal Graph Theory together with spectral techniques. Interestingly, prior to this work it was not even known that computing E'_P *precisely* for the properties in (2) is NP-hard. We thus answer (in a strong form) a question of Yannakakis [1], who asked in 1981 if it is possible to find a large and natural family of graph properties for which computing E'_P is NP-hard.

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On Applications of the Hypergraph Removal Lemma

József Solymosi

A new and powerful tool, an extension of Szemerédi's Regularity Lemma [8] to hypergraphs, has recently been proved by Rödl *et al.* [5]. Similar results with the same consequences have been obtained independently by Gowers [2]. Inspired by the method of [4] and [2], very recently Tao has given another proof of the main results [9]. The following is an important consequence.

Theorem 1 (Removal Lemma [4, 2])**.** *For any $\epsilon > 0$ and $k \geq 2$ integer, there is a $\delta = \delta(\epsilon, k) > 0$, such that if H_k^n contains at least ϵn^k pairwise edge-disjoint cliques then H_k^n contains at least δn^{k+1} cliques.*

The Removal Lemma has many interesting consequences. Some of them are straightforward applications like the following exercise.

Claim 2. *For every $c > 0$ there is a number $n_0 = n_0(c)$, such that if $G(A, B)$ is a bipartite graph, with minimum degree at least four, and $|B| \geq c|A|^3 \geq n_0$, then G contains a $K_{3,3}$ or a cube, Q_3. (Q_3 is equivalent to $K_{4,4} - M$, where M is a perfect matching in $K_{4,4}$.)*

Other applications are more involved. Using hyperplane incidences one can define hypergraphs which lead us to a combinatorial proof of the so called Multidimensional Szemerédi Theorem, which was proved by Furstenberg and Katznelson [1] using ergodic theory.

Theorem 3 ([1])**.** *For every $\epsilon > 0$, and every finite subset S of the d-dimensional integer grid there is a positive integer N such that every subset X of the grid $[N]^d$ of size at least ϵN^d has a subset of the form $x + tS$ for some positive integer t.*

Theorem 4 ([7])**.** *The Removal Lemma (Theorem 1) implies Theorem 3.*

There are still many open questions, conjectures in extremal graph theory, where, in its present form, hypergraph regularity is not enough. An interesting family of problems can be formulated using linear hypergraphs. A hypergraph H is *linear* if for any two edges $e_1, e_2 \in H$ $|e_1 \cap e_2| \leq 1$. The special case, $k = 2$, of the Removal Lemma implies the theorem of Ruzsa and Szemerédi [6].

Theorem 5 ([6])**.** *If a 3-uniform linear hypergraph H on n vertices doesn't contain three distinct edges e_1, e_2, e_3 forming a triangle, $e_1 = \{a, b, *\}$, $e_2 = \{a, c, *\}$, $e_3 = \{b, c, *\}$, then the number of edges of H is $o(n^2)$. (In the positions marked by $*$, any vertex of H can appear.)*

An old conjecture of Frank and Rödl deals with 4-uniform hypergraphs.

Conjecture 6 (Frankl-Rödl, 1985)**.** *If a 4-uniform linear hypergraph H on n vertices doesn't contain four distinct edges e_1, e_2, e_3, e_4, where $e_1 = \{a, b, c, *\}$, $e_2 = \{a, d, *, *\}$, $e_3 = \{b, d, *, *\}$, $e_4 = \{c, d, *, *\}$, then the number of edges of H is $o(n^2)$.*

The next conjecture may be easier to prove and still had interesting consequences.

Conjecture 7. *If a 4-uniform linear hypergraph H on n vertices doesn't contain four distinct edges e_1, e_2, e_3, e_4, where $e_1 = \{a, b, c, *\}$, $e_2 = \{a, d, e, *\}$, $e_3 = \{b, d, *, *\}$, $e_4 = \{c, e, *, *\}$, then the number of edges of H is $o(n^2)$.*

The following is an easy extension of the $k = 2$ case of the Removal Lemma.

Claim 8. *If a 5-uniform linear hypergraph H on n vertices doesn't contain five distinct edges e_1, e_2, e_3, e_4, e_5 where $e_1 = \{a, b, c, *, *\}$, $e_2 = \{a, d, *, *, *\}$, $e_3 = \{b, d, *, *, *\}$, $e_4 = \{b, e, *, *, *\}$, $e_5 = \{c, e, *, *, *\}$, then the number of edges of H is $o(n^2)$.*

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Extremal infinite graph theory? A topological approach.

Reinhard Diestel

Extremal graph theory, understood broadly as the interaction of global graph invariants such as average degree, connectivity or chromatic number with local ones, such as the existence of certain substructures, accounts for much of the research on finite graphs. Interestingly, there is no matching theory for infinite graphs. We look at some of the reasons of why this is so, and how the relevant concepts might be adapted to infinite graphs in such a way as to make an extremal-type theory possible.

One obvious difference between finite and infinite graphs relevant to the interaction of graph invariants as above is that while a large enough minimum degree can force any desired minor in a finite graph, it forces no interesting substructure in an infinite graph; indeed, infinite trees can have arbitrarily large degrees at every vertex. This 'unchecked spread' of a locally finite infinite graph, however, can be prevented: if we compactify it by adding its ends and then prescribe large 'degrees' also at the ends, we can wrap up the graph sufficiently to force at least some structure. For example, if we define the *degree* of an end as the maximum number of disjoint rays converging to it, we have the following result of Stein [6]:

Theorem 1. *Any locally finite graph in which every vertex has degree at least* $6k^2 - 4k + 3$ *and every end has degree at least* $6k^2 - 9k + 5$ *has a* k*-connected subgraph (finite or infinite).*

Another, more subtle, reason why many extremal-type theorems have no infinite counterpart lies in the fact that ordinary paths and cycles in an infinite graph cannot always fulfil the structural role they play in a finite graph. For example, the fact that in a finite planar graph 4-connectedness suffices to imply the existence of a Hamilton cycle (Tutte 1956) says something about the plane; but there are obviously no Hamilton cycles in an infinite graph that might similarly reflect the restrictions placed on it by the assumption of planarity. Likewise, the well-known tree-packing theorem of Nash-Williams and Tutte (1961) that a finite graph contains k edge-disjoint spanning trees if and only if, for every vertex partition into ℓ sets (say), it has at least $k(\ell - 1)$ edges joining different partition sets, fails for infinite graphs. One way to view this is to say that infinite graphs 'do not have enough paths' to form the spanning trees required.

However, if we define paths not combinatorially in a graph G itself but topologically in its Freudenthal compactification $|G|$, we can use the additional paths through the ends to form the cycles and trees required for infinite analogues of those theorems. Call a homeomorphic image of S^1 in $|G|$ a *circle*, and a closed subspace of $|G|$ a *topological spanning tree* if it is path-connected and contains all the vertices of G but no circle. Circles and topological spanning trees can be characterized in various expected ways; see [2], in particular Lemma 8.5.6 and Exercises 65 and 71.

For G finite, our new topological notions default to the usual finite cycles and spanning trees. For infinite G, however, the additional connectivity at infinity makes the following topological version of the tree-packing theorem possible [2]:

Theorem 2. *The following assertions are equivalent for every locally finite multigraph* G *and positive integer* k*:*

- $|G|$ *contains* k *edge-disjoint topological spanning trees.*
- *For every finite vertex partition, into* ℓ *sets say,* G *has at least* $k\,(\ell - 1)$ *edges joining different partition sets.*

Similarly, Tutte's hamiltonicity theorem for 4-connected planar graphs now has a meaningful infinite analogue. Call a circle in $|G|$ a *Hamilton circle* of G if it contains every vertex of G. Bruhn (see [2, 3]) has conjectured the following:

Conjecture 3. *Every 4-connected planar locally finite graph has a Hamilton circle.*

Bruhn and Yu [1] have proved this for 6-connected graphs with finitely many ends.

In a similar vein, Georgakopoulos [5] has announced a proof of the infinite analogue of Fleischner's theorem, conjectured in [3, 4]:

Theorem 4. *The square of any 2-connected locally finite graph has a Hamilton circle.*

Thomassen [7] had previously proved this for 1-ended graphs.

One would hope that these results are only the beginning of what might once become a more fully developed topological analogue for infinite graphs of at least those parts of finite extremal graph theory that deal with structures such as paths, cycles, topological minors etc.

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The angel and the devil in three dimensions

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(joint work with Béla Bollobás)

The game of Angel and Devil appears in Berlekamp, Conway and Guy [1, 2], and has been popularized by Conway (see for example [3]). It is played on the infinite two-dimensional lattice, $\mathbb{Z}^2$. Two players alternate moves, the *Angel* and the *Devil*. Initially, the Angel is at some square. On his turn, he may jump to any square at distance at most c from his current square (it does not really matter what sort of distance we use, but by convention one uses the ℓ_∞ distance – in other words, the Angel may jump to any square at a distance of c or less king's moves from his current square). On the Devil's turn, he kills some square. The Angel loses if he moves to a square that has been killed at some previous time by the Devil, and wins if he can survive forever (without moving to a dead square).

The main question is: if c is large enough, does the Angel escape? Conway offers \$100 for a proof that the Angel escapes for large enough c, and \$1000 for a proof that the Devil always wins, whatever the value of c.

Not much is known about this problem. It is known that for a very small speed, like $c = 1$, the Devil wins. It is also known that if the Angel always moves upwards (in other words, if each move is to a square with greater y-coordinate than the current square) then the Devil wins. For these results, and related background information, see Conway [3]. Most of the complexity of the problem seems to be summarized by the observation that it seems that any particular strategy one tries for the Angel can easily be defeated by the Devil, while any particular strategy one tries for the Devil can easily be defeated by the Angel.

In this talk we shall consider what happens in more than two dimensions. Of course, there is now 'more space' – for example, the set of points at distance n, while only linear in n in $\mathbb{Z}^2$, grows faster in higher dimensions. It is reported in [1, 2] that Körner (unpublished) has shown that, in very high dimension, the Angel wins.

Our aim is to show that, in three dimensions, the Angel wins (if c is large enough). There is a sense in which this result is best possible: it turns out that there is a natural notion of 'Angel and Infinitely Many Devils', for which the Angel still wins in three dimensions but cannot win in two dimensions.

We also make some stronger conjectures about three dimensions, that we have been unable to prove. Interestingly, this work is linked with the two-dimensional case. One conjecture in particular, the 'Time-Bomb conjecture' seems very central.

The fact that the Angel can escape in three dimensions has recently also been proved, independently, by Martin Kutz [4].

For completeness, we state here the Time-Bomb Conjecture. Suppose we consider (in $\mathbb{Z}^3$) the z-axis as a time axis. Then the three-dimensional game, with the Angel moving up by one unit on each move, is exactly the same as the two-dimensional game, in which at each move the Devil does not kill a square but just announces that at one certain time that square will be dead. (The square is dead for just that one unit of time. The Devil may make more that one such announcement concerning any given square.) We call this the Time-Bomb game (in $\mathbb{Z}^2$). To be more precise, on each turn the Angel moves to some square at distance at most c from his current position, and the Devil announces a pair (p, t), where p is a position and t is a positive integer. The Angel loses if, for some t, his tth move is to a position p such that the pair (p, t) has already been named by the Devil. This leads us to the following, which we call the *Time-Bomb conjecture*.

Conjecture 1. *An Angel of speed* 1 *wins the Time-Bomb game in* $\mathbb{Z}^2$.

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Random planar graphs

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(joint work with Colin McDiarmid, Angelika Steger, and Andreas Weißl)

Planar graphs are well-known and well-studied combinatorial objects in graph theory. Roughly speaking, a graph is planar if it can be drawn in the plane in such a way that no two edges cross. A *random planar graph* R_n is a simple planar graph that is drawn uniformly at random from the set $\mathcal{P}(n)$ of all simple planar graphs on the node set $\{1, \ldots, n\}$.

In order to investigate properties of random planar graphs, one has to cope with the difficulty of the dependence of the edges: whether a particular edge can be added, depends on the presence of other edges. This is a fundamental difference to the well-known random graph model $G_{n,p}$, where each edge is present with probability p independently of the presence or absence of all other edges.

The key step in investigating the behaviour of the random planar graph R_n is to estimate the number of planar graphs. McDiarmid, Steger, and Welsh [5] showed that

$$\left(\frac{|\mathcal{P}(n)|}{n!}\right)^{\frac{1}{n}} \to \gamma \quad \text{as } n \to \infty.$$

Giménez and Noy showed using generating functions and singularity analysis that γ satisfies $\gamma \approx 27.2269$ to four decimal places. More recently they give an explicit analytic expression for γ, and show that

$$|\mathcal{P}(n)| \sim g \cdot n^{-\frac{7}{2}} \; \gamma^n \; n! \tag{1}$$

where the constant g has an explicit analytic expression and is about $4.97 \cdot 10^{-6}$ [4]. A corresponding expression for the number of 2-connected planar graphs was given in [1]. This was a major step towards establishing (1).

To deduce certain properties of the random planar graph McDiarmid, Steger, and Welsh [5] only needed that there exists such a *planar graph growth constant* and not its exact value. In particular, they were able to show, among other results, that a random planar graph R_n with high probability (w.h.p., that is, with probability tending to 1 as n tends to infinity) contains linearly many nodes of each given degree, has linearly many faces of each given size in any embedding, and contains linearly many node disjoint copies of any given fixed connected planar graph. Additionally, and perhaps most surprisingly, they showed that the probability that R_n is connected is bounded away from zero and from one by non-zero constants.

In [4] it is shown that the number of edges $|E(R_n)|$ is asymptotically normally distributed, with mean $\sim \kappa n$ and variance $\sim \lambda n$, where the constants κ and λ have explicit analytic expressions and $\kappa \approx 2.213$ and $\lambda \approx 0.4303$. In particular this means that the expected number of edges of a random planar graph on n nodes is approximately $2.213n$, which was an open problem for quite some time. Thus the average degree in R_n is about 4.416 w.h.p.. Lower and upper bounds on the maximum degree are known (see [5]), but the asymptotic behaviour is not known. Furthermore, Giménez and Noy show in [4] additional limit laws for the random planar graph, for instance that the number of 2-connected components in a random connected planar graph and the number of appearances of a fixed connected planar graph in a random planar graph are asymptotically normally distributed.

We are interested in the class $\mathcal{P}(n, m)$ of (simple) labelled planar graphs on n nodes with m edges, and in particular in $\mathcal{P}(n, \lfloor qn \rfloor)$, where the average degree is about $2q$. As was shown in [2], for all $1 < q < 3$ the random planar graph $R_{n,q}$, which is drawn uniformly at random from the set $\mathcal{P}(n, \lfloor qn \rfloor)$, has properties similar to those of a random planar graph R_n. It is known [2] that for $0 \leq q \leq 3$ there is a constant $\gamma(q)$ such that

$$\left(\frac{|\mathcal{P}(n, \lfloor qn \rfloor)|}{n!}\right)^{\frac{1}{n}} \to \gamma(q) \quad \text{as} \quad n \to \infty. \tag{2}$$

(For $q = 3$, we interpret $\mathcal{P}(n, \lfloor qn \rfloor)$ as the set $\mathcal{P}(n, 3n - 6)$ of triangulations.) The limiting result (2) also holds with the same limiting value $\gamma(q)$ if we replace $|\mathcal{P}(n, \lfloor qn \rfloor)|$ by $|\mathcal{P}_c(n, \lfloor qn \rfloor)|$, the set of *connected* graphs in $\mathcal{P}(n, m)$.

Let us first consider properties of the function $\gamma(q)$, and then look more closely at the limiting result. Recall that κ is the parameter for the mean of the number of edges of the random planar graph R_n, and γ is the planar graph growth constant.

The function $\gamma(q)$ on $[0, 3]$ satisfies

(i) $\gamma(q) = 0$ for $0 \leq q < 1$, $\gamma(1) = e$, $\gamma(\kappa) = \gamma$, and $\gamma(3) = 256/27$.
(ii) $\gamma(q)$ is continuous and log-concave on $[1, 3]$, and it is strictly increasing on $[1, \kappa]$ and strictly decreasing on $[\kappa, 3]$.
(iii) $\gamma(q)$ is computable, and analytic on $(1, 3)$.

Now let us look more closely at the limiting result (2), and give two directions in which it can be strengthened, one allowing more freedom in the number of edges and one being far more precise. First, if $1 \leq q \leq 3$ and $m = m(n)$ satisfies $n \leq m \leq 3n - 6$ and $m/n \to q$ as $n \to \infty$, then

$$\left(\frac{|\mathcal{P}(n, m)|}{n!}\right)^{\frac{1}{n}} \to \gamma(q). \tag{3}$$

This result holds also if we replace $\mathcal{P}$ by $\mathcal{P}_c$. These results follow from the proof of Lemma 2.9 in [2], using also the fact that $\gamma(q)$ is continuous on the right at 1.

Secondly, using analytic methods, Giménez and Noy [4] give rather precise asymptotic expressions for $|\mathcal{P}(n, \lfloor qn \rfloor)|$ and $|\mathcal{P}_c(n, \lfloor qn \rfloor)|$, for $q \in (1, 3)$, which we may write as:

$$|\mathcal{P}(n, \lfloor qn \rfloor)| \sim \alpha(q)\ n^{-4} \gamma(q)^n n! \tag{4}$$

and similarly

$$|\mathcal{P}_c(n, \lfloor qn \rfloor)| \sim \alpha_c(q)\ n^{-4} \gamma(q)^n n!, \tag{5}$$

where $\alpha(q)$ and $\alpha_c(q)$ are constants.

As $\gamma(q) = 0$ for $q < 1$ and $\gamma(1) = e$, we know that $\gamma(q)$ is discontinuous at 1 from the left. We can 'explain' this discontinuity as we approach 1 from below, by changing scale appropriately. More precisely, let $\beta \geq 0$ be a constant. If $m = m(n) = n - (\beta + o(1))(n/\ln n)$ then

$$\left(\frac{|\mathcal{P}(n,m)|}{n!}\right)^{\frac{1}{n}} \to e^{1-\beta} \quad \text{as } n \to \infty.$$

The main feature now left open about $\gamma(q)$ is whether the slope stays finite as q approaches 1 from above, and approaches 3 from below. This is not the case. More precisely, since the function $\lambda(q) = \ln \gamma(q)$ is concave and finite on $[1,3]$, its left and right derivatives exist in $(1,3)$ and are finite and non-increasing. Moreover they tend to ∞ as $q \downarrow 1$ and to $-\infty$ as $q \uparrow 3$.

Finally, since we know the approximate number of planar graphs we can deduce the following property of $R_{n,q}$. Let $1 \leq q < 3$ and let H be a fixed connected planar graph, where H must be a tree if $q = 1$, and for a graph G let $f^*_H(G)$ denote the maximum number of pairwise node disjoint copies of H contained in G. Then there exists a constant $\alpha = \alpha(H,q) > 0$ such that

$$\Pr[f^*_H(R_{n,q}) < \alpha n] = e^{-\Omega(n)}.$$

If we let H be a star on the nodes $1,\ldots,k+1$ with centre at node $k+1$ and look a bit more careful at the proof we in fact obtain the following result. Let $1 \leq q < 3$, let k be a positive integer, and for a graph G let $d_k(G)$ denote the number of nodes with degree equal to k. Then there exists a constant $\alpha_k = \alpha_k(q) > 0$ such that

$$\Pr[d_k(R_{n,q}) < \alpha_k n] = e^{-\Omega(n)}.$$

If we let H be a k-cycle on the nodes $1,\ldots,k$ and use some connectivity results [3] we obtain the following consequence. Let $1 < q < 3$, let $k \geq 3$ be an integer, and for a planar graph G let $f_k(G)$ denote the number of faces of size k *minimised* over all plane embeddings of G. Then there exists a constant $\beta_k = \beta_k(q) > 0$ such that

$$\Pr[f_k(R_{n,q}) < \beta_k n] = e^{-\Omega(n)}.$$

We can also consider graph H which are growing with n and we obtain that the maximum degree of a graph $R_{n,q}$ is w.h.p. at most $(1 + o(1)) \log n / \log \log n$. For more details see [3].

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An Exact Result for the Generalized Triangle

Oleg Pikhurko

The *Turán function* $\mathrm{ex}(n, F)$ of a k-graph F is the maximum size of an F-free k-graph H on n vertices. The *Turán density* of F is

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}.$$

Determining $\mathrm{ex}(n, F)$ for hypergraphs F is in general a very difficult problem. For example, the \$1000 prize of Erdős for computing the Turán function of the complete 3-graph of order 4 is still unclaimed. Last decade saw a surge of activity in this field, with the discovery of a variety of new results and methods.

Let us call a k-graph F *stable* if for any $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that any F-free k-graph of order $n \geq n_0$ and size at least $\mathrm{ex}(n, F) - \delta n^k$ can be made into a maximum F-free k-graph by changing (removing or adding) at most εn^k edges. Any 2-graph is stable (Erdős [2], Simonovits [11]). Stability is useful for the hypergraph Turán problem as it often helps in proving exact results (Füredi and Simonovits [5], Keevash and Sudakov [7, 8], and other).

For example, using the stability approach Keevash and Mubayi [6] gave another proof for the exact value of $\mathrm{ex}(n, T_3)$ for $n \geq 33$, where the *generalized triangle* T_k is the k-graph with edges

$$\{1, \ldots, k\},\ \{1, 2, \ldots, k-1, k+1\}, \text{ and } \{k, k+1, \ldots, 2k-1\}.$$

(The asymptotic result for $\mathrm{ex}(n, T_3)$ follows from a paper of Bollobás [1] while the exact result (for $n \geq 3000$) was first obtained by Frankl and Füredi [3].)

Sidorenko [10] determined $\pi(T_4)$ while Frankl and Füredi [4] determined $\pi(T_5)$ and $\pi(T_6)$. Both these papers use the so-call *Lagrange polynomial* of a hypergraph H,

$$\lambda_H(y_1, \ldots, y_m) = \sum_{D\in H} \prod_{i\in D} y_i,$$

and the the *Lagrangian* of H

$$\Lambda_H = \max\{\lambda_H(y_1, \ldots, y_m) \mid y_i \in \mathbb{R},\ y_i \geq 0,\ y_1 + \cdots + y_m = 1\}$$

For example, Sidorenko proved that any T_4-free 4-graph H satisfies

$$\Lambda_H \leq 1/4^4$$

which implies that $\pi(T_4) \leq 4!/4^4$ in view of inequality $|H| \leq n^4 \lambda_H$ where n is the number of vertices of H. Since it is easy to show that $\pi(T_4) \geq 4!/4^4$ (just take the maximum complete 4-partite 4-graph on n vertices), we have $\pi(T_4) = 4!/4^4$.

In [9] we determine $\mathrm{ex}(n, T_4)$ exactly.

Theorem. *There is an n_0 such that for all $n \geq n_0$ we have*

$$\mathrm{ex}(n, T_4) = \left\lfloor \frac{n}{4} \right\rfloor \times \left\lfloor \frac{n+1}{4} \right\rfloor \times \left\lfloor \frac{n+2}{4} \right\rfloor \times \left\lfloor \frac{n+3}{4} \right\rfloor,$$

and, moreover, the complete balanced 4-partite 4-graph is the unique extremal configuration.

Our proof goes by showing that T_4 is stable. This in turn requires to establish the appropriately defined stability property for the problem of maximizing λ_H given that H is T_4-free.

We still do not have the exact result for $\mathrm{ex}(n, T_i)$ for $i \geq 5$. Our method seems promising in attacking the cases $k = 5, 6$, given the results of Frankl and Füredi [4]. One of the difficult steps here is to prove that T_5 and T_6 are stable.

Also, this approach may be useful for other instances of the hypergraph Turán problem where the asymptotic result can be obtained via Lagrangian.

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Erdős-Ko-Rado type theorems proven by analytical methods

Ehud Friedgut

(joint work with Irit Dinur)

By using discrete Fourier analysis one can prove several uniqueness and robustness theorems concerning intersecting families. Here are some selected ones that I mentioned in my talk.

Theorem 1 ([1]). *Let $0 < \zeta$, let $\zeta n < k < (1/2 - \zeta)n$ and let $\mathcal{A} \subset \binom{[n]}{k}$ be an intersecting family. If $|\mathcal{A}| \geq (1-\varepsilon)\binom{n-1}{k-1}$ then there exists a dictatorship $\mathcal{B} \subset \binom{[n]}{k}$ such that*

$$|\mathcal{A} \setminus \mathcal{B}| < c\varepsilon\binom{n-1}{k-1}$$

where $c = c(\zeta)$.

Theorem 2 ([1]). *Let $t \geq 1$ be an integer, let $0 < \zeta$, let $\zeta n < k < (\frac{1}{t+1} - \zeta)n$ and let $\mathcal{A} \subset \binom{[n]}{k}$ be a t-intersecting family.*

If $|\mathcal{A}| \geq (1-\varepsilon)\binom{n-t}{k-t}$. Then there exists a t-umvirate $\mathcal{B} \subset \binom{[n]}{k}$ such that

$$|\mathcal{A} \setminus \mathcal{B}| < c\varepsilon \binom{n-t}{k-t}$$

where $c = c(\zeta)$.

It turns out that when $k = o(n)$ *every* intersecting family is essentially contained in a maximal one.

Theorem 3 ([2]). *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be an intersecting family. Then there exists a dictatorship $\mathcal{B}$ such that $|\mathcal{A} \setminus \mathcal{B}| = O\left(\binom{n}{k-2}\right)$. (Note that in this range $\binom{n}{k-2} = o(\binom{n}{k-2})$.)*

This holds for finer precision too: Let $r \geq 2$ be an integer and let $\mathcal{A} \subseteq \binom{[n]}{k}$ be an intersecting family. Then there exists an intersecting family $\mathcal{B}$ defined by at most $2r-3$ elements such that

$$|\mathcal{A} \setminus \mathcal{B}| = O\left(\binom{n}{k-r}\right).$$

This is easily seen to be tight.

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On a Geometric Generalization of McMullen's Upper Bound Theorem

Uli Wagner

We consider the following generalization of McMullen's Upper Bound Theorem that was proposed by Eckhoff, Linhart, and Welzl [4, 5, 7]:

Conjecture 1. *If $\mathcal{A}$ is an arrangement of n great hemispheres in $\mathbf{S}^d$ and $0 \leq \ell < (n-d)/2$, then the number of vertices of $\mathcal{A}$ at level at most ℓ is at most $v(\ell, n, d)$, the corresponding number for an arrangement that arises by polar duality from the vertex set of a cyclic d-polytope on n vertices. (The* level *of a vertex is the number of hemispheres that it is* not *contained in.)*

This has been verified for $d = 2$ [1], and for $d = 3$ if the intersection of the hemispheres is nonempty [7]. Sharpening previous asymptotic bounds [3], we show that for an arrangement of n affine halfspaces in $\mathbf{R}^d$, the number of vertices at level at most ℓ is at most $2v(\ell, n, d)$. Our bound implies the conjecture up to a factor of 4, and for $\ell < n/(d+1)$, it is tight up to a factor of 2. (The tight bound for halfspaces is known for $d \leq 4$ [5].)

Our proof is based on the h-matrix of a linear program [6], a generalization of the h-vector of a convex polytope: Given a arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ of halfspaces in general position in $\mathbf{R}^d$ and a generic linear functional φ, orient the edges according to increasing φ-value. Out of the $2d$ edges incident to a vertex $v = \bigcap_{i \in B} \partial H_i$ with *basis* $B \in \binom{[n]}{d}$, consider the d edges that span the orthant $\bigcap_{i \in B} H_i$. The out-degree of v is defined with respect to these edges, and we define $h_{j,\ell}(\mathcal{A}, \varphi)$ as the number of vertices at level ℓ with out-degree j. (If the ℓ-level is unbounded, then $h_{j,\ell}$ may depend on φ.)

It suffices to bound the numbers $h_{\le j, \le \ell}$ of vertices of level at most ℓ and out-degree at most j, which reduces to a problem in extremal set theory. Let $v_1, \dots, v_t$ be these vertices, in order of increasing φ-value. With each v_r, we associate the following subsets of $[n]$: Let B_r be the corresponding basis, let A_r be the set of "labels" of outgoing edges at v_r (each of these is determined by dropping one element from B_r), let $C_r = \{i : v_r \notin H_i\}$, and let $D_r = [n] \setminus B_r$. The quadrupels $(A_r, B_r, C_r, D_r)_{r=1}^t$ satisfy:

(1) For $1 \le r \le t$, $|A_r| \le j$, $C_r \le \ell$, $|B_r| = d$, $|D_r| = n-d$, $A_r \subseteq B_r$, $C_r \subseteq D_r$, and $B_r \cap D_r = \emptyset$.
(2) For $1 \le r < s \le t$, $(A_r \cap (D_s \setminus C_s)) \cup ((B_r \setminus A_r) \cap C_s) \ne \emptyset$.

Theorem 2. *Any ordered sequence of quadrupels* $(A_r, B_r, C_r, D_r)_{r=1}^t$ *with properties (1) and (2) satisfies*

$$t \le 2 \sum_{i=1}^{j} \binom{n-d-\ell+j}{i} \binom{d-j+\ell}{d-i} = 2|\mathcal{B}(n,d,j,\ell)|,$$

where $\mathcal{B}(n,d,j,\ell) = \{B \in \binom{[n]}{d} : |B \cap [n-d-\ell+j]| \le j\}$. *Up to the factor of 2, this family provides an extremal example, by defining* $A = B \cap [n-d-\ell+j]$, $D = [n] \setminus B$, *and* $C = D \cap [n-d-\ell+1, \dots, n]$ *and using any linear ordering.*

The proof of this theorem uses exterior algebra and extends an earlier approach by Alon and Kalai [2] for the classic Upper Bound Theorem (the case $\ell = 0$, in which case the bound in Theorem 2, without the factor of 2, is known as the *Skew Bollobás Theorem*.)

Open Questions.

(1) Get rid of the factor of 2, which is an artefact of the proof.
(2) Prove lower bounds for halfspaces for $n/(d+1) \le j < (n-d)/2$. These would follow, for instance, from the existence of neighborly polytopes that are *perfectly balanced* about the origin o, i.e., such that every hyperplane through o has at least $\lfloor (n-d)/2 \rfloor$ vertices on either side.
(3) Define a good generalization of the h-matrix to spherical arrangements.

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Many Hamiltonian cycles

Jeff Kahn
(joint work with Bill Cuckler)

We discuss two theorems giving lower bounds on Hamiltonian cycle (HC) counts, the first in Dirac graphs and the second in regular tournaments.

A graph is said to be *Dirac* if its minimum degree is at least $n/2$. Write $\Psi(G)$ for the number of HC's in a graph G, and $\Psi(n)$ for the minimum of $\Psi(G)$ over n-vertex Dirac graphs G. Nash-Williams [10] proved in 1971 that any Dirac graph has at least $\lfloor \frac{5}{224} n \rfloor$ edge-disjoint Hamiltonian cycles, so in particular this number is a lower bound on $\Psi(n)$. Bondy [4], on page 79 of the Handbook of Combinatorics, and, according to [12], at several conferences, and also Bollobás [3] (p.1260 of the same book), asked for estimates of $\Psi(n)$. Sárközy et al. [12] proved, using the regularity lemma, that $\Psi(n) \geq c^n n!$ for some (small) constant $c > 0$, and conjectured the following stronger statement.

Theorem 1 (B.C.-J.K.). *For any n-vertex Dirac graph G, $\Psi(G) \geq n!/(2+o(1))^n$.*

This is easily seen to be best possible (up to a factor $(1 + o(1))^n$). In fact it is best possible in a strong sense: Brégman's Theorem ([5], formerly the Minc Conjecture) on permanents of $\{0,1\}$-matrices implies that for *any* $(n/2)$-regular G one has $\Psi(G) \leq ((n/2)!)^2 = O(\sqrt{n}2^{-n}n!)$. More generally the following lower bound matches the Brégman upper bound when G is regular of degree at least $n/2$.

Theorem 2 (B.C. and J.K.). *For $d \geq n/2$, any n-vertex G of minimum degree at least d satisfies $\Psi(G) \geq (d/(e + o(1))^n$.*

We again use $\Psi(T)$ for the number of HC's in a tournament T, and recall that a tournament is *regular* if each of its vertices has outdegree $(n-1)/2$ (so n is necessarily odd).

Theorem 3 (B.C.). *For any regular, n-vertex tournament T, $\Psi(T) \geq n!/(2 + o(1))^n$.*

This was proved in response to a question of Friedgut and Kahn [7], who asked for a lower bound of the form $n^{n-o(n)}$, but turns out to also provide a strong answer to a much earlier question of C. Thomassen [14], who gave a lower bound of about $n^{n/3}$ and asked whether one could prove $c^n n!$ for some positive constant c. (The analogous question for Hamiltonian *paths* was suggested in [15], where it was attributed to C. Grinstead.)

By contrast, as more or less observed by Szele [13] (he considered paths rather than cycles), the expected number of HC's in a *random* (not necessarily regular) tournament is $\varphi(n) := 2^{-n}(n-1)!$, and, as observed by Alon [2], Brégman's Theorem gives $\Psi(T) \le O(n^{3/2}\varphi(n))$ for *any* T. This upper bound was improved slightly by Friedgut and Kahn [7], who speculated that the truth might actually be $O(\varphi(n))$ (the analogous and equivalent possibility for paths was suggested by Adler et. al. [1]), and that perhaps $\Omega(\varphi(n))$ is a *lower* bound on $\Psi(T)$ for *regular* T. In fact it could even be that this is true without the "Ω," a possibility somewhat supported by Wormald [16] (see also [1]), who gives asymptotic values for the expected numbers of HC's in several classes of random regular tournaments.

Our approach to Theorems 1-3 involves analysis of an appropriate self-avoiding random walk, say $X = (X_1, \ldots, X_l)$, on the graph or tournament in question. We show that for some $l = n - o(n/\log n)$ the number of possibilities for X is large (at least as large as whatever lower bound we are aiming for) and that most of these possibilities extend to HC's.

For Theorem 3 the walk is just the natural one: the next vertex is chosen uniformly from the as yet unvisited outneighbors of the current vertex. For Theorems (1 and) 2 the walk is taken according to a weighting $\boldsymbol{x} : E \to [0,1]$ (i.e. the next vertex is chosen from the as yet unseen neighbors of the current vertex with probabilities proportional to the edge weights), which is required to be *proper*, meaning $\sum_{e \ni v} \boldsymbol{x}_e = 1$ for each $v \in V$. The key quantity associated with $\boldsymbol{x}$ is its "entropy" (not really entropy since $\sum \boldsymbol{x}_e \neq 1$), $h(\boldsymbol{x}) := \sum_e \boldsymbol{x}_e \log(1/\boldsymbol{x}_e)$. Our main inequality says that, except in some fairly pathological situations, which are handled by more conventional graph-theoretic arguments, the (ordinary) entropy $H(X)$ satisfies

$$H(X) \ge 2h(\boldsymbol{x}) - n\log e - o(n).$$

This gives the desired lower bound on the number of possibilities for X when combined with

Lemma 4. *Any graph of minimum degree at least $d \ge n/2$ admits a proper edge-weighting $\boldsymbol{x}$ with $h(\boldsymbol{x}) \ge (n/2)\log d$.*

Most of our work involves showing, very roughly, that for each k, $\{X_1 \ldots X_k\}$ resembles a random k-subset of V. The proof of this is similar in spirit to the celebrated "nibble" method (e.g. [11] or [9]), and involves, *inter alia*, (a) some (easy) assertions about rapid mixing of the ordinary random walk corresponding to X, and (b) use of Azuma's inequality to show that various quantities associated with X are well predicted by their expectations, where, fairly atypically, these expectations are not fixed in advance, but are themselves functions of the evolving sequence X. (This use of martingales was inspired by [8].)

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Positional games

Michael Krivelevich

A *positional game* is a played on a finite board V, where a family of subsets (a hypergraph) H, whose members are usually called winning sets, is specified. The game is played by two players, taking turns in claiming previously unoccupied elements of V, and ends whenever there are no unoccupied elements. In general, there are two additional parameters, p and q, the first player takes p elements in his turn, while the second one claims q elements.

There are several types of positional games. In the so called Strong Game, a player completing a winning set $A \in H$ first wins, otherwise the game ends in a draw. In a Weak Game, the first player (Maker) wins if he completes a winning set by the end of the game, otherwise the game is won by the second player (Breaker). In the Avoider-Enforcer version, the first player (Avoider) aims to avoid occupying a winning set completely, while the second player (Enforcer) tries to force Avoider to do just so. There are also hybrid versions, where, for example, the first player acts both as Breaker and Avoider.

There is an amazing variety of recreational and mathematical games that can be casted into the above described framework. Examples include Tic-Tac-Toe and its multi-dimensional generalizations, the game of Hex played and studied by John Nash, and various achievement games played on the edges of the complete graph K^n, where for example Maker tries to create a Hamilton cycle, while Breaker aims to prevent Maker from fulfilling his goal.

The mathematical origins of positional games can be traced back to two seminal papers: that of Hales and Jewett from 1963 [5] (who studied multi-dimensional Tic-Tac-Toe), and of Erdős and Selfridge from 1973 [4] (who obtained an extremely useful criterion for Breaker's win in Maker-Breaker games). It was József Beck though whose many papers on the subject, written during the last quarter century, turned this fascinating subject into a mathematical discipline; his book "Combinatorial games", providing a thorough treatment of the subject, is about to be published by Cambridge University Press [3].

In this survey-type talk we will introduce the subject of positional games, and will define and discuss a variety of types of positional games. We will indicate some typical approaches and tools available. Some recent results will be discussed too. We will stress a perhaps surprising yet quite ubiquitous role of probabilistic intuition in analyzing these deterministic games.

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Harmonic Analysis of Boolean Functions

Gil Kalai

Introduction. Harmonic analysis turned out to be a powerful tool in combinatorics and especially so in combinatorial number theory and in extremal and probabilistic combinatorics. To the already classical connections of Harmonic analysis with additive number theory and with discrepancy theory, a surprising connection to the study of Boolean functions was developed in the last two decades. Much of the motivation and application came from questions in theoretical computer science. I will briefly discuss the basic connection in this abstract.

A *Boolean function* is a function $f(x_1, x_2, \dots, x_n)$ where each variable x_i is a Boolean variable, taking the value 0 or 1. The value of f is also 0 or 1. A Boolean function f is monotone if $f(y_1, y_2, \dots, y_n) \geq f(x_1, x_2, \dots, x_n)$ when $y_i \geq x_i$ for every i. Some basic examples of Boolean functions are named after the voting method they describe. For an odd integer n, the *majority* function $M(x_1, x_2, \dots, x_n)$ equals 1 if and only if $x_1 + x_2 + \cdots + x_n > n/2$. The *dictatorship* function is $f(x_1, x_2, \dots, x_n) = x_i$. *Juntas* refer to the class of Boolean functions that depend on a bounded number of variables, namely functions that disregard the value of almost all variables except for a few, whose number is independent of n.

Harmonic analysis of Boolean functions can also be regarded as a fine study of spectral properties of the graph of the discrete cube, and thus is related to spectral graph theory. It is also connected to the field of discrete isoperimetric inequalities, and related notions of concentration of measure (see [19]). Finally, in the last few decades it turned out that very abstract general properties of Boolean functions (and more general objects) are vary useful to study concrete problems in probability and combinatorics. The FKG-inequality and the Shearer Lemma are examples. The subject we describe can be regarded as another (related) example.

The basic definition. Let Ω_n denote the set of 0-1 vectors $(x_1, \dots, x_n)$ of length n. Let $L_2(\Omega_n)$ denote the space of real functions on Ω_n, endowed with the inner product

$$\langle f, g \rangle = \sum_{(x_1, x_2, \dots, x_n) \in \Omega_n} 2^{-n} f(x_1, \dots, x_n) g(x_1, \dots, x_n). \tag{1}$$

The inner product space $L_2(\Omega_n)$ is 2^n-dimensional. The L_2-norm of f is defined by

$$\|f\|_2^2 = \langle f, f \rangle = \sum_{(x_1, x_2, \dots, x_n) \in \Omega_n} 2^{-n} f^2(x_1, x_2, \dots, x_n). \tag{2}$$

Note that if f is a Boolean function, then $f^2(x)$ is either 0 or 1 and therefore $\|f\|_2^2 = \sum_{(x_1, \dots, x_n) \in \Omega_n} 2^{-n} f^2(x)$ is simply the probability $\mu(f)$ that $f = 1$ (with respect to the uniform probability distribution on Ω_n).

For a subset S of $[n]$ consider the function

$$u_S(x_1, x_2, \dots, x_n) = (-1)^{\sum_{i \in S} x_i} . \tag{3}$$

It is not difficult to verify that the 2^n functions u_S for all subsets S form an orthonormal basis for the space of real functions on Ω_n.

For a function $f \in L_2(\Omega_n)$, the Fourier-Walsh coefficient $\hat{f}(S)$ of f is

$$\hat{f}(S) = \langle f, u_S \rangle. \tag{4}$$

Since the functions u_S form an orthogonal basis, we have $\langle f, g \rangle = \sum_{S \subset [n]} \hat{f}(S) \hat{g}(S)$. In particular, $\|f\|_2^2 = \sum_{S \subset [n]} \hat{f}^2(S)$. This relation is called Parseval's formula.

Example 1. Let M_3 represents the majority function on three variables. The Fourier coefficients of M_3 are easy to compute: $\hat{M}_3(\emptyset) = \sum(1/8)M_3(x) = 1/2$. In general, if f is a Boolean function then $\hat{f}(\emptyset)$ is the probability that $f(x) = 1$ and when f is an odd Boolean function, $\hat{f}(\emptyset) = 1/2$. Next, $\hat{M}_3(\{1\}) = 1/8(M_3(0,1,1) - M_3(1,0,1) - M_3(1,1,0) - M_3(1,1,1)) = (1-3)/8$ and thus $\hat{M}_3(\{j\}) = -1/4$, for $j = 1,2,3$. Next, $\hat{M}_3(S) = 0$ when $|S| = 2$ and finally $\hat{M}_3(\{1,2,3\}) = 1/8(M_3(1,1,0) + M_3(1,0,1) + M_3(0,1,1) - f(1,1,1)) = 1/4$.

Connection with edge-expansion. Consider a Boolean function $f(x_1, \ldots, x_n)$ and the associated event $A \subset \Omega_n$, such that $f = \chi_A$, namely that f is the indicator function of A. For $x = (x_1, x_2, \ldots, x_n) \in \Omega_n$ we say that the kth variable is *pivotal* if flipping the value of x_k changes the value of f. Formally, let

$$\sigma_k(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_{k-1}, 1 - x_k, x_{k+1}, \ldots, x_n) \tag{5}$$

and define the kth variable to be pivotal at x if

$$f(\sigma_k(x)) \neq f(x). \tag{6}$$

The *influence* of the kth variable on a Boolean function f, denoted by $I_k(f)$, is the probability that the kth variable is pivotal, i.e.,

$$I_k(f) = \mu(\{x : f(\sigma_k(x)) \neq f(x)\}) \tag{7}$$

The influence of a variable in a Boolean function and more general notions of influences were introduced by Ben-Or and Linial [2] in the context of “collective coin-flipping”.

The total influence $I(f)$ is the sum of the individual influences.

$$I(f) = \sum_{k=1}^{n} I_k(f). \tag{8}$$

The relation between influences and Fourier coefficients is given by the following expressions, whose proof is elementary:

$$I_k(f) = 4 \sum_{S : k \in S} \hat{f}^2(S). \tag{9}$$

$$I(f) = 4 \sum_{S \subset [n]} \hat{f}^2(S)|S|. \tag{10}$$

If f is monotone we also have $I_k(f) = -2\hat{f}(\{k\})$.

Hypercontractivity. It is surprising how far one can get with the simple base-change of the Fourier-Walsh transform and Parseval’s formula. In addition there is a certain hypercontractive (log-Sobolev) inequality by Bonami, Gross and Beckner which played an important role.

For a real function $f : \Omega_n \to \mathbb{R}$, $f = \sum \hat{f}(S)u_S$, define the L_w-norm of a function f to be

$$\|f\|_w = \left(\sum_{x \in \Omega_n} 2^{-n} |f(x)|^w \right)^{1/w} . \tag{11}$$

Note that, due to the normalization coefficient 2^{-n} in the definition, if $1 \leq v < w$ then $\|f\|_v \leq \|f\|_w$.

Next define the operator

$$T_\rho(f) = \sum_{S \subset [n]} \hat{f}(S) \rho^{|S|} u_S, \tag{12}$$

so that $\|T_\rho(f)\|_2^2 = \sum_{S \subset [n]} \hat{f}^2(S) \rho^{2|S|}$. The Bonamie-Gross-Beckner (BGB) inequality asserts that for every real function f on Ω_n,

$$\|T_\rho(f)\|_2 \leq \|f\|_{1+\rho^2}. \tag{13}$$

Because this inequality involves two different norms, it is referred to as "hypercontractive". The inequality can be regarded as an extension of the Khintchine inequality, which states that the different L_w-norms of functions of the form $\sum_k \alpha_k u_{\{k\}}$ differ only by absolute multiplicative constants.

Here is a quick argument giving a flavor of the use of the Bonamie-Gross-Beckner inequality. Note that for a Boolean function f and every $w \geq 1$,

$$\|f\|_w^w = \mu(f). \tag{14}$$

Let $0 < \rho < 1$. Now, if a large portion of the L_2-norm of f is concentrated at "low frequencies" $|S|$, then $\|T_\rho(f)\|_2$ will not be too much smaller than $\|f\|_2$. The BGB inequality implies that in this case, $\|f\|_{1+\rho^2}$ cannot be too much smaller than $\|f\|_2$ either. This fact, however, cannot coexist with relation (14) if $\mu(f)$ is sufficiently small.

More formally, suppose that $\mu(f) = s \leq 1/2$, and we will try to give lower bounds for $I(f)$. Parseval's formula and relation 8 imply that $I(f) \geq 4(s - s^2)$. The classical edge-isoperimetric inequality asserts that $I(f) \geq 2s \log_2(1/s)$. Let us try to understand the appearance of $\log(1/s)$.

Take $\rho = 1/2$ and thus $1 + \rho^2 = 5/4$. The BGB inequality and equation (14) give

$$\sum \frac{\hat{f}^2(S)}{2^{2|S|}} \leq \|f\|_{5/4}^2 = s^{1+3/5}.$$

Noting that $2^{2|S|} < 1/\sqrt(s)$ for $0 < |S| < \log_2(1/s)/4$,

$$\sum_{0 < |S| < \log(1/s)/4} \hat{f}^2(S) \leq \sqrt{s} \cdot s^{3/5} \leq K \sqrt{s(1-s)}$$

for some constant $K < 1$, since $s \leq 1/2$. This implies that a finite fraction of the L_2 norm of f is concentrated at Fourier coefficients $\hat{f}(S)$ where $|S| \geq K' \log(1/s)$. It then follows from relation (8) that $I(f) \geq K''(\mu(f)(1 - \mu(f)) \log(1/\mu(f))$.

Up to a multiplicative constant this gives the fundamental edge-isoperimetric relation, but the information on Fourier coefficients, is even stronger.

Some applications. I will mention briefly a few developments and applications:

(1) A variable with a large influence. One of the earliest applications of these ideas is [16] to show that every balanced Boolean function f on n variables has a variable with influence at least $C \log n/n$. This result is sharp up to a multiplicative constant.

(2) First passage percolation: A similar argument [5] gives an improvement for Kesten's and Talagrand's estimates for the variance and tail behavior of "First Passage Percolation".

(3) Quantitative FKG: A beautiful application found by Talagrand gives a quantitative version of the FKG inequality, see [21]

(4) Threshold phenomena refer to settings in which the probability for an event to occur changes rapidly as some underlying parameter varies. A fundamental result by Russo connects this with influences. Some highlights in developing this line of study are [9, 12, 7], for a recent applications where this is one of several ingredients in the arguments see [6, 14, 1]. See also the surveys [17, 13].

(5) Noise sensitivity: a notion considered in [4] with several applications and nice connections. A recent highlight is [20].

(6) PCP, hardness of approximation and decay. Let me mention the works of Håstad [15], and Dinur and Safra [11] which gives important applications to the area of hardness of approximation for optimization problems. Questions on the decay of Fourier coefficients comes naturally in this area and some recent developments are in [8, 10].

(7) Khot and Vishnoi [18] used results on analysis of Boolean functions to solve an old standing problem on embeddability of metric spaces.

(8) Erdős-Ko-Rado type theorems. A connection to this classical area of combinatorics was found by Friedgut, see his abstract in this collection.

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On the Erdős–T. Sós conjecture on trees

Miklós Simonovits
(joint work with Miklós Ajtai, János Komlós, and Endre Szemerédi)

We proved the famous Erdős–T. Sós conjecture on the extremal number of trees, at least for large trees. The conjecture is motivated by the

Theorem 1 (P. Erdős and T. Gallai, [2], 1959)**.**

$$\mathrm{ext}(n, P_k) \le \frac{1}{2}(k-2)n.$$

Here P_k is a k-vertex path, and $\mathrm{ext}(n, L)$ denotes the maximum number of edges a graph G_n on n vertices can have without containing the "sample graph" L. This is sharp, at least if n is divisible by $k-1$, as shown by the union of $n/(k-1)$ complete graphs K_{k-1}. Observe that for the star S_k (i.e., for a vertex x joined to $k-1$ other vertices) the same estimate holds (is trivial):

$$\mathrm{ext}(n, S_k) \le \frac{(k-2)n}{2}.$$

The tree and the star are two "extreme" trees. This motivates the Erdős–T. Sós conjecture, according to which

$$\mathrm{ext}(n, T_k) \le \frac{(k-2)n}{2}$$

holds for *any* k-vertex tree T_k. Again, if true, this is sharp.

Our main result states that this conjecture holds if $k > k_0$.

Theorem 2 (Main Theorem, Sharp)**.** *There exists an integer k_0 such that if $k > k_0$ and for an arbitrarily fixed T_k, a graph G_n on n vertices contains no T_k then*

$$e(G_n) \le \frac{1}{2}(k-2)n. \tag{1}$$

The proof is rather involved, uses the regularity lemma, [3] a version of Tutte's theorem and a technique to be seen in the paper of Ajtai, Komlós and Szemerédi on the Loebl's conjecture [1] strongly related to the Erdős–Sós conjecture. (That proof is much simpler!) First we prove a weaker theorem.

Theorem 3 (Main Theorem, Approximative)**.** *For every $\eta > 0$ there exists an integer $n_0(\eta)$ such that if $n, k > n_0(\eta)$ and for an arbitrarily fixed tree T_k, a graph G_n on n vertices contains no T_k, then*

$$e(G_n) \le \frac{1}{2}(k-2)n + \eta n. \tag{2}$$

Then we apply stability arguments to get the sharp result. In the proof we distinguish two cases: for a fixed but very large constant Ω either $n \le \Omega k$ (the "bounded case") or $n > \Omega k$ (the "unbounded case").

Using the Regularity Lemma – at least in the "bounded case" – means that we define the "Reduced Graph", check if it contains a 1-factor, or an almost 1-factor. If YES, then we use a "modified greedy" algorithm to embed T_k into G_n. If NOT, then the "reduced graph" has a very special structure, because of the Tutte theorem, or the Gallai-Edmonds theorem, and using this special structure, we distinguish several cases which either reduce to the previous 1-factor case or where we introduce two more "modified greedy algorithms". (To be more precise, when we reduce a non-1-factor case to the previous case, we do not get a 1-factor in the "reduced graph", only a so called "generalized 1-factor" and the proof of the earlier case works also for this case.)

To embed the tree T_k into G_n, we partition T_k into small subtrees and embed these small parts one by one into G_n, mostly into small randomlike bipartite subgraphs of G_n provided by the Regularity Lemma.

In the "unbounded case" we partition the edges into two categories: BLACK and GREY edges. The BLACK edges are covered by small dense random-looking bipartite graphs. Then we classify the vertices of G_n basically in two three classes: $\mathbb{A}, \mathbb{B}, \mathbb{C}$, where $\mathbb{C}$ contains the large degrees, $\mathbb{B}$ the small degrees not covered by our dense pairs, and $\mathbb{A}$ contains the vertices of low degrees, covered by our dense pairs. Depending on, where are many edges in this partition we have to use two further embedding algorithms. In one of the critical cases we have to embed T_k into $G_n[\mathbb{A}]$ (i.e. the subgraph spanned by $\mathbb{A}$) and this case is somewhat similar to the “bounded case”, but much more involved.

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Sparse ε-regular graphs

Angelika Steger

(joint work with Stefanie Gerke, Yoshiharu Kohayakawa, Martin Marciniszyn, and Vojtěch Rödl)

Over the last decades Szemerédi’s regularity lemma [8] has proven to be a very powerful tool in modern graph theory. Unfortunately, in its original setting it only gives nontrivial results for dense graphs, that is graphs with $\Theta(n^2)$ edges. In 1996 Kohayakawa [4] and independently Rödl introduced a variant which holds for sparse graphs, provided they satisfy some additional structural conditions (which essentially mean that the graph does not contain too dense spots). However, using this sparse regularity lemma to prove e.g. extremal and Ramsey type results similar to the known results in the dense case, requires as an additional step the existence of appropriate embedding or counting lemmas. For the sparse case this missing step has been formulated as a conjecture by Kohayakawa, Łuczak and Rödl [5]. For a graph H, let $\mathcal{G}(H,n,m)$ be the family of graphs on vertex set $V=\bigcup_{x\in V(H)} V_x$, where the sets V_x are pairwise disjoint sets of vertices of size n, and edge set $E=\bigcup_{\{x,y\}\in E(H)} E_{xy}$, where $E_{xy}\subseteq V_x\times V_y$ and $|E_{xy}|=m$. Let $\mathcal{G}(H,n,m,\varepsilon)\subseteq\mathcal{G}(H,n,m)$ denote the set of graphs in $\mathcal{G}(H,n,m)$ satisfying that each $(V_x\cup V_y,E_{xy})$ is an (ε)-regular graph.

Conjecture 1 (KŁR-Conjecture [5])**.** *Let H be a fixed graph and*

$$\mathcal{F}(H,n,m)=\{G\in\mathcal{G}(H,n,m): H \text{ is not a subgraph of } G\}.$$

For any $\beta>0$, there exist constants $\varepsilon_0>0$, $C>0$, $n_0>0$ such that for all $m\geq Cn^{2-1/d_2(H)}$, $n\geq n_0$, and $0<\varepsilon\leq\varepsilon_0$,

$$|\mathcal{F}(H,n,m)\cap\mathcal{G}(H,n,m,\varepsilon)|\leq\beta^m\binom{n^2}{m}^{|E(H)|},$$

where $d_2(H)=\max\left\{\frac{|E(F)|-1}{|V(F)|-2}: F\subseteq H, |V(F)|\geq 3\right\}$.

One of the key difficulties in the proof of the KŁR-conjecture is the fact that for $m = o(n^2)$ the size of a neighbourhood of a vertex is on average $o(n)$. The definition of regularity, however, only deals with linear sized subsets and thus regularity seem to be not inherited by subgraphs induced on the neighborhoods of some vertices. In a joint paper [1] with Gerke, Kohayakawa, and Rödl we were recently able to prove that nevertheless in the sparse case a hereditary version holds as well, at least in a probabilistic setting. This result readily implies much shorter and elegant proofs of the results known so far, namely the case of cycles C_k for all $k \geq 3$ and for $H = K_4$ and K_5. In this talk we show that in fact a much stronger property holds. Namely, small sets not only inherit with high probability the regularity property, but they also satisfy with high probability all properties that regular tuples satisfy with high probability. Among other things this allows us to show that the KLR-conjecture holds for all complete graphs for slightly larger number of edges than the conjectured value. In return, we can show the existence of many copies instead of just one copy. That is, we get a so called counting lemma.

Theorem 2 ([2])**.** *For all $\ell \geq 3$, $\delta > 0$, and $\beta > 0$, there exist constants $n_0 \in \mathbb{N}$, $C > 0$, and $\varepsilon > 0$ such that*

$$|\mathcal{F}(K_\ell, n, m, \delta) \cap \mathcal{G}(K_\ell, n, m, \varepsilon)| \leq \beta^m \cdot \binom{n^2}{m}^{\binom{\ell}{2}} \tag{1}$$

provided that $m \geq Cn^{2-1/(\ell-1)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$ and where $\mathcal{F}(K_\ell, n, m, \delta)$ denotes the family of graphs in $\mathcal{G}(K_\ell, n, m)$ that contain less than $(1-\delta)n^\ell(\frac{m}{n^2})^{\binom{\ell}{2}}$ copies of K_ℓ.

In the last part of the talk we also indicate that our results about inheritance of ε-regularity can also be used to analyse online and semi-online Ramsey games on random graphs [6, 7].

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Minors and Bounded expansion

Jaroslav Nešetřil
(joint work with Patrice Ossona de Mendez)

The concept of tree-width [12],[15] is central to the analysis of graphs with forbidden minors of Robertson and Seymour. This concept gained much algorithmic attention thanks to the general complexity result of Courcelle about monadic second-order logic graph properties decidability for graphs with bounded tree-width [2],[3]. It appeared that many NP-complete problems may be solved in polynomial time when restricted to a class with bounded tree-width. However, bounded tree-width is quite a strong restriction, as planar graphs for instance do not have bounded tree-width.

An alternative approach consists in the partition of graphs, such that p parts induce a subgraph of tree-width at most $(p-1)$. Answering a question of Thomas [14], DeVos et al. [4] proved that for any proper minor closed class of graphs $\mathcal{C}$ – that is: any minor closed class $\mathcal{C}$ excluding at least one graph — and any integer p there exists a constant $N(\mathcal{C},p)$ such that the vertex set of any graph $G\in\mathcal{C}$ may be partitioned into at most $N(\mathcal{C},p)$ parts in such a way that any $j\leq p$ parts induce a subgraph of tree-width at most $(j-1)$, what the authors call a *low tree-width partition* of G. This proof, which relies on the Structural Theorem of Robertson and Seymour [13] fails to be effective from a computational point of view.

It appears that low tree-width decomposition may be established in a more general setting for classes with bounded expansion [9][6]. These results are reported here together with the algorithmic analysis. The definition of bounded expansion classes is based on a new graph invariant, the *greatest reduced average degree (grad)* with *rank* r of a graph G, $\nabla_r(G)$. This invariant is defined by $\nabla_r(G)=\max\frac{|E(H)|}{|V(H)|}$, where the maximum is taken over all the minors H of G obtained by contracting a set of vertex-disjoint subgraphs with radius at most r and then deleting any number of edges and vertices. A class of graphs $\mathcal{C}$ has *bounded expansion* if $\sup_{G\in\mathcal{C}}\nabla_r(G)<\infty$ for any integer r. Not only proper minor closed classes of graphs have bounded expansion (as then ∇_r is uniformly bounded independently of r), but so are classes with bounded degree or some usual classes arising from finite element meshes (as skeletons of d-dimensional simplicial complexes with bounded aspect ratio [5]).

Theorem 1. *For any class with bounded expansion $\mathcal{C}$ and any integer p there exists a constant $N(\mathcal{C},p)$ so that the vertex set of any graph $G\in\mathcal{C}$ may be partitioned into at most $N(\mathcal{C},p)$, any $i\leq p$ parts of them induce a subgraph of tree-width at most $(i-1)$ [6] (actually, of* tree-depth [10] *at most i, what is sensibly stronger).*

Such decompositions are central to the resolution of homomorphism problems like *restricted homomorphism dualities* [8] which provided the original motivation for our research.

We gave a simple algorithm to compute such decompositions and prove that if we restrict the input graph to some fixed class $\mathcal{C}$ with bounded expansion, the running time of the algorithm may be bounded by a linear function of the order of the graph (i.e. belongs to $O(F(\mathcal{C}, p)n)$).

This result is applied to get a linear time algorithm for the subgraph isomorphism problem with fixed pattern and input graphs in a fixed class with bounded expansion.

More generally, let ϕ be a first order logic sentence. We prove that graph properties of type $\exists X : (|X| \leq p) \wedge (G[X] \vDash \phi)$ may be decided in linear time for input graphs in a fixed class with bounded expansion.

We also show that classes with sub-exponential expansion (which properly includes proper minor closed classes) have sub-linear vertex separators(see e.g. [1]). This allows classic related approaches to solve reputed hard problems.

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The structure of bull-free graphs

Maria Chudnovsky

The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

Let G be a graph. We say that G is *bull-free* if no induced subgraph of G is isomorphic to the bull. The *complement* of G is the graph with the same vertex set as G, and two vertices are adjacent in the complement of G if and only if they are non-adjacent in G. A *clique* in G is a set of vertices, all pairwise adjacent. A *stable set* in G is a clique in the complement of G. We call a clique of size three a *triangle*, and a stable set of size three a *triad*. Let A, B be two disjoint subsets of $V(G)$. We say that A is *complete* to B if every vertex of A is adjacent to every vertex of B.

An obvious example of a bull free graph is a graph with no triangle, or a graph with no triad; but there are others. Let us call a graph G an *ordered split graph* if there exists an integer n such that the vertex set of G is the union of a clique $\{k_1, \ldots, k_n\}$ and a stable set $\{s_1, \ldots, s_n\}$, and s_i is adjacent to k_j if and only if $i + j \leq n + 1$. It is easy to see that every ordered split graph is bull-free. A large ordered split graph contains a large clique and a large stable set, and therefore the tree classes (triangle-free, triad-free and ordered split graphs) are significantly different.

It turns out, however, that all bull-free graphs can be built starting from graphs that belong to a few basic classes, gluing them together by certain operations. The basic classes we need are triangle-free graphs, triad-free graphs, a certain generalization of the ordered split graphs, and a couple of others, that we will not describe here. Let $\mathcal{B}$ denote the set of all bull-free graphs that belong to one of the basic classes.

In order to state our main result, we need to introduce a few operations, that will allow us to combine two smaller bull-free graphs together, to obtain a new, larger, bull-free graph.

Operation $\mathcal{O}_1$ is the operation of complementation. The input of $\mathcal{O}_1$ is a graph G_1, and the output is the complement of G_1.

Operation $\mathcal{O}_2$ is the operation of taking disjoint union of two graphs. The input of $\mathcal{O}_2$ is a pair of graphs G_1, G_2, and the output is a new graph G_3, with $V(G_3) = V(G_1) \cup V(G_2)$ and $E(G_3) = E(G_1) \cup E(G_2)$.

Operation $\mathcal{O}_3$ is defined as follows. The input of $\mathcal{O}_3$ is a pair of graphs G_1, G_2, and ordered subsets A_1, B_1 of $V(G_1)$ and A_2, B_2 of $V(G_2)$, with the following properties:

- A_1, B_1, A_2, B_2 are stable sets, with $|A_1| = |A_2|$ and $|B_1| = |B_2|$.
- A_1 is complete to B_1, and A_2 to B_2.
- For $i = 1, 2$ let G'_i be the graph obtained from G_i by adding two new vertices a_i, b_i such that $\{a_i\}$ is complete to A_i and $\{b_i\}$ to B_i, and there are no other edges incident with a_i, b_i. Then both G'_1 and G'_2 is bull free.

Under these circumstances, the result of applying $\mathcal{O}_3$ to $G_1, G_2, A_1, B_1, A_2, B_2$ is the graph G_3, obtained from the disjoint union of G_1 and G_2 by identifying the corresponding vertices of A_1 and A_2, and the corresponding vertices of B_1 and B_2.

Operation $\mathcal{O}_4$ is the operation of substitution. The input of $\mathcal{O}_4$ is a pair of graphs G_1, G_2 and a vertex $v \in V(G_1)$. The output is a new graph G_3, with $V(G_3) = V(G_1) \cup V(G_2) \setminus \{v\}$ and $E(G_3) = E(G_1 \setminus \{v\}) \cup E(G_2) \cup \{xy \ : \ x \in V(G_1) \setminus \{v\}, \ y \in V(G_2), \ \textit{and} \ xv \in E(G_1)\}$. Please note that unlike all the previous operations, $\mathcal{O}_4$ is not symmetric between G_1 and G_2.

We remark that if the input graphs of the operations $\mathcal{O}_1, \ldots, \mathcal{O}_4$ are bull free, then so are the outputs.

Let us now state our main result.

Theorem 1. *Let G be a bull-free graph. Then either $G \in \mathcal{B}$, or G can be obtained starting from graphs in $\mathcal{B}$, by repeated applications of operations $\mathcal{O}_1, \ldots, \mathcal{O}_4$. Conversely, every graph obtained in this way is bull-free.*

We now proceed to describe an application of Theorem 1. The Erdős-Hjanal conjecture [1] states that for every graph H, there exists a constant $0 < \delta(H) \leq 1$, such that if a graph G does not contain an induced subgraph isomorphic to H, then G has a stable set or a clique of size $|V(G)|^{\delta(H)}$. In joint work with S.Safra [2], using the structure theorem described above, we were able to settle the Erdős-Hajnal conjecture for the case when H is the bull. We show that:

Theorem 2. *Let G be a bull free graph. Then G contains a stable set or a clique of size $|V(G)|^{\frac{1}{4}}$.*

In order to prove Theorem 2, we prove inductively, using Theorem 1, that every bull-free graph G can be covered by at most $|V(G)|^{\frac{1}{2}}$ induced subgraphs of G, each of which is perfect. It follows that there exists an induced subgraph H of G, containing at least $|V(G)|^{\frac{1}{2}}$ vertices, and such that H is perfect. Consequently, H contains a stable set or a clique of size $|V(H)|^{\frac{1}{2}} \geq |V(G)|^{\frac{1}{4}}$, and Theorem 2 follows.

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Semidefinite programming bounds for codes and coloring

Monique Laurent

We consider the problem of computing the maximum size $A(n, d)$ of a binary code of word length n and minimum distance at least d. The problem can obviously be cast as the problem of finding the stability number $\alpha(G)$ of the graph $G = H(n, \mathcal{D})$ with node set $\{0, 1\}^n$ and with an edge (u, v) if the Hamming distance of $d(u, v)$

lies in the set $\mathcal{D} := \{1,\dots,d-1\}$. (The techniques apply in fact to the case of any subset $\mathcal{D} \subseteq \{1,\dots,n\}$.)

A number of semidefinite hierarchies for $\alpha(G)$ have been introduced in the literature, that find $\alpha(G)$ in $\alpha(G)$ steps; in particular, by Lovaśz and Schrijver [5] and Lasserre [3]. We consider the hierarchy of Lasserre, which is known to be tightest. It is easy to see that, at any fixed stage in the hierarchy, the bound can be computed (to an arbitrary precision) in time polynomial in $|V(G)|$. For Hamming graphs, we show that the bound can be computed in time polynomial in n (while $|V(G)| = 2^n$); this is based on exploiting the large automorphism group of Hamming graphs and applying a result of de Klerk, Pasechnik and Schrijver [1] about the regular $*$-representation for matrix $*$-algebras.

The Delsarte bound for $A(n,d)$ is the first bound in the hierarchy, and the new bound of Schrijver [6] is located between the first and second bounds in the hierarchy. While computing the second bound involves a semidefinite program with $O(n^7)$ variables and thus seems out of reach for interesting values of n, Schrijver's bound can be computed via a semidefinite program of size $O(n^3)$, a result which uses the explicit block-diagonalization of the Terwilliger algebra. We propose some strengthenings of Schrijver's bound with the same computational complexity; in particular, a parameter $\ell(G)$ which gives better upper bounds on $A(n,d)$ on some instances.

Together with Nebojsa Gvozdenović, we also study the problem of approximating the chromatic number $\chi(G)$ of a Hamming graph $G = H(n,\mathcal{D})$. Any upper bound $\beta(G)$ on the stability number $\alpha(G)$ yields the obvious lower bound $\frac{|V(G)|}{\beta(G)}$ for $\chi(G)$. For $\beta(\cdot) = \ell(\cdot)$, this new bound substantially improves the classic lower bound $\vartheta(\bar{G})$ given by the theta number (and the strengthenings of Szegedy and Meurdesoif obtained by adding nonnegativity and triangle inequalities).

We propose a simple construction for yet stronger lower bounds for $\chi(G)$. Namely, we introduce a simple operator Ψ which maps any graph parameter $\beta(\cdot)$, nested between $\alpha(\cdot)$ and $\bar{\chi}(\cdot)$, to a new graph parameter $\Psi_\beta(\cdot)$, nested between $\omega(\cdot)$ and $\chi(\cdot)$, in the following way:

$$\Psi_\beta(G) := \min_{l\in\mathbb{N}} \beta(K_l \Box G),$$

where $K_l \Box G$ is the Cartesian product of G and the clique K_l. Among other properties, $\Psi_\alpha = \chi$, $\Psi_{\bar{\chi}^*} = \Psi_{\bar{\chi}} = \omega$, $\Psi_\vartheta = \lceil\bar{\vartheta}\rceil$, and if $\beta(\cdot)$ is polynomial time computable (resp., given by a semidefinite program) then the same holds for $\Psi_\beta(\cdot)$. As an application, there is *no* polynomial time computable graph parameter nested between $\chi^*(\cdot)$ and $\chi(\cdot)$ unless P=NP. Under some mild assumption, $\Psi_\beta(G) \geq \frac{|V(G)|}{\beta(G)}$. For $\beta(\cdot) = \ell(\cdot)$, $\Psi_\ell(G)$ gives an improved lower bound for some Hamming graphs. To be able to compute $\Psi_\ell(G)$, we apply some symmetry reduction due to the action of the permutation group $\mathrm{Sym}(l)$ on $K_l \Box G$ and we use again the block-diagonalization of the Terwilliger algebra.

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Max Cut in dense graphs

Benny Sudakov

The well-known Max Cut problem asks for the largest bipartite subgraph of a graph G. This problem has been subject of extensive research, both from the algorithmic perspective in computer science and the extremal perspective in combinatorics. Let n be the number of vertices and e be the number edges of G and let $b(G)$ denote the size of the largest bipartite subgraph of G. The extremal part of Max Cut problem asks to estimate $b(G)$ as a function of n and e. This question was first raised almost forty years ago by P. Erdős [7] and attracted a lot of attention since then (see, e.g., [3, 2, 4, 1, 11, 10, 9, 5, 6]).

It is well known that every graph G with e edges can be made bipartite by deleting at most $e/2$ edges, i.e., $b(G) \geq e/2$. To see this just consider a random partition of vertices of G into two parts V_1, V_2 and estimate the expected number of edges in the cut (V_1, V_2). A complete graph K_n on n vertices shows that the constant $1/2$ in the above bound is asymptotically tight. Moreover, this constant can not be improved even if we consider restricted families of graphs, e.g., graphs that contain no copy of a fixed *forbidden* subgraph H. We call such graphs H-free. Indeed, using sparse random graphs one can easily construct a graph G with e edges such that it has no short cycles but can not be made bipartite by deleting less than $e/2 + o(e)$ edges. Such G is clearly H-free for every forbidden graph H which is not a forest. It is a natural question to estimate the error term $b(G) - e/2$ as G ranges over all H-free graph with e edges. We refer interested reader to [3, 2, 1, 11], where such results were obtained for various forbidden subgraphs H.

In this paper we restrict our attention to *dense* ($e = \Omega(n^2)$) H-free graphs for which it is possible to prove stronger bounds for Max Cut. According to a long-standing conjecture of Erdős [8], every triangle-free graph on n vertices can be made bipartite by deleting at most $n^2/25$ edges. This bound, if true, is best possible (consider an appropriate blow-up of a 5-cycle). Erdős, Faudree, Pach and Spencer proved that for triangle-free G of order n it is enough to delete $(1/18 - \epsilon)n^2$ edges to make it bipartite. They also verify the conjecture for all graphs with at least $n^2/5$ edges. Some extensions of their results were further obtained in [10]. Nevertheless this intriguing problem remains open. Erdős also asked similar question for K_4-free graphs. His old conjecture (see e.g., [9]) asserts that it is enough to delete at most $(1 + o(1))n^2/9$ edges to make bipartite any K_4-free graph on n vertices. Here we confirm this in the following strong form.

Theorem 1. *Every K_4-free graph G with n vertices can be made bipartite by deleting at most $n^2/9$ edges. Moreover, the only extremal graph which requires deletion of that many edges is a complete 3-partite graph with parts of size $n/3$.*

This result can be used to prove the following asymptotic generalization.

Corollary 2. *Let H be a fixed graph with chromatic number $\chi(H) = 4$. If G is a graph on n vertices not containing H as a subgraph, then we can delete at most $(1 + o(1))n^2/9$ edges from G to make it bipartite.*

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Pairwise colliding permutations and the capacity of infinite graphs

János Körner
(joint work with Claudia Malvenuto)

We call two permutations of the first n naturals colliding if they map at least one number to consecutive naturals. We give bounds for the exponential asymptotics of the largest cardinality of any set of pairwise colliding permutations of $[n]$. We relate this problem to the determination of the Shannon capacity of an infinite graph and initiate the study of analogous problems for infinite graphs with finite chromatic number.

Let n be an arbitrary natural number and let $[n]$ be the set of all natural numbers from 1 to n. We will say that two permutations of $[n]$ are *colliding* if they map at least one element of $[n]$ into two consecutive numbers, i.e. into numbers differing by 1. It is then natural to ask for the determination of the maximum cardinality $\rho(n)$ of a set of pairwise colliding permutations of $[n]$. One easily sees that this number grows exponentially with n and its asymptotic exponent lies between $\log_2 \frac{1+\sqrt{5}}{2}$ and 1. We will improve the lower bound to $\frac{1}{4}\log 10$. We conjecture the upper bound to be tight.

Certain graphs having as vertex set the permutations of $[n]$ have been introduced before by Cameron and Ku [1] and Larose and Malvenuto [2] These authors considered Kneser–type graphs in which they studied the growth of stable sets describing sets of permutations that are "similar" in some sense, whereas our definition of adjacency corresponds to being "different" and distinguishable in some other, particular sense. In fact, the above Kneser–type problems, unlike ours, have no immediate relation to capacity in the Shannon sense.

We will generalize our introductory problem in several ways. We will consider arbitrary infinite graphs over the natural numbers and introduce various new concepts of capacity. As always, graph capacity measures the exponential growth rate of the largest cliques induced on the Cartesian powers of the vertex set of a graph. In case of an infinite vertex set such as the naturals this is not always interesting, for the graph in itself might have infinite cliques. Then it is reasonable to restrict our attention to particular subsets of the power sets, e. g. those representing permutations. We will present some simple bounds for the value of new capacities so obtained.

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The relative strength of topological graph colouring obstructions

Carsten Schultz

In the proof of Kneser's Conjecture, Lovász has shown that if the neighbourhood complex of a graph G is $(k-1)$-connected, its chromatic number is at least k+2. Later formulations of this theorem replace the neighbourhood complex by the complex $\mathrm{Hom}(K_2, G)$, which is homotopy equivalent to it. One variant of this theorem is the following.

Theorem 1 ([4, 1])**.** *Let G be a graph with at least one edge. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, G) \leq \chi(G) - 2.$$

For graphs H and G, the cell complex $\mathrm{Hom}(H, G)$ has the graph homomorphisms from H to G as 0-cells, while the higher dimensional cells are indexed by multi-homomorphisms, functions which assign to every vertex of H a non-empty set of vertices of G such that every choice of one of these for every vertex of H yields a graph homomorphism. An involution on H which flips an edge makes $\mathrm{Hom}(H, G)$ into a free $\mathbb{Z}_2$-space. The emphasis on the cohomological index of the $\mathbb{Z}_2$-action comes from the work of Babson & Kozlov on the following theorem which proves a conjecture by Lovász.

Theorem 2 ([2, 5])**.** *Let G be a graph with an odd cycle and $r \geq 1$. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(C_{2r+1}, G) \leq \chi(G) - 3.$$

Previous proofs of this theorem can be summarized at follows. One studies the complex $\mathrm{Hom}(C_{2r+1}, K_n)$. This is the hard part. Then the functorality of Hom in the second argument is used to deduce information on $\mathrm{Hom}(C_{2r+1}, G)$ from the existence of an n-colouring of G, i.e. from $\mathrm{Hom}(G, K_n) \neq \emptyset$.

Extending an elegant partial proof of Theorem 2 by Živaljević [6, 7], we present a simpler way of obtaining the desired information on $\mathrm{Hom}(C_{2r+1}, K_n)$, or even $\mathrm{Hom}(C_{2r+1}, G)$ for an arbitrary graph G. This uses the idea that Hom is not only functorial, but that there is a continuous map extending composition of homomorphisms, in this case

$$\mathrm{Hom}(K_2, C_{2r+1}) \times \mathrm{Hom}(C_{2r+1}, G) \to \mathrm{Hom}(K_2, G).$$

Using properties of this map and of the $\mathbb{Z}_2$-actions on $\mathrm{Hom}(K_2, C_{2r+1})$ induced by involutions on K_2 and C_{2r+1} we obtain the following result.

Theorem 3. *Let G be a graph with an odd cycle and $r \geq 1$. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(C_{2r+1}, G) + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, G).$$

This reduces Theorem 2 to Theorem 1. It therefore proves Theorem 2 in a simple way, but also shows that lower bounds on the chromatic number that can be obtained from it can also be obtained from the complex $\mathrm{Hom}(K_2, G)$ originally studied by Lovász.

Theorem 3 can be generalized as follows.

Theorem 4. *Let G, G' be graphs with involutions, the involution on G flipping an edge, and $k \geq 1$. If*

- $\mathrm{coind}_{\mathbb{Z}_2} \mathrm{Hom}(G, {G'}^{\mathbb{Z}_2}) \geq k - 1$,
- *there is a graph homomorphism from G to G' that commutes with the involutions, and*
- $\mathrm{Hom}(G, G')$ *is* $(k-1)$*-connected,*

then

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(G', H) + k \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(G, H)$$

for all graphs H with $\mathrm{Hom}(G', H) \neq \emptyset$.

Here, $G'^{\mathbb{Z}_2}$ is a graph whose vertex set is the set of all orbits of the involution on G'. Its edge set is the largest one such that the map $V(G'^{\mathbb{Z}_2}) \to \mathcal{P}(V(G'))$ assigning to each orbit the orbit itself is a multi-homomorphism.

This theorem can be applied to yield a result analogous to Theorem 2, with circuits of chromatic number 3 replaced by Kneser graphs of chromatic number 4.

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Matching points to hyperplanes

Anders Björner

Let $P \subseteq \mathbb{R}^d$ and $H = \{\text{hyperplanes spanned by } P\}$, and assume that the affine span of P is $\mathbb{R}^d$. We are concerned with the problem:

> *does there exist an injective mapping $f : P \to H$ such that $p \in f(p)$ for all $p \in P$?*

We call such a mapping a *matching.*

Conjecture 1. *For every dimension d and all subsets $P \subseteq \mathbb{R}^d$ there exists a matching $f : P \to H$.*

It follows from a result of matroid theory [2] (based on Hall's marriage theorem) that the conjecture is true for all finite sets P. The case $P = \mathbb{R}^d$ is easy to verify via an explicit construction. The case of general infinite sets is, however, less obvious. Here is what we can prove.

Theorem 2. *Let $P \subseteq \mathbb{R}^d$ and $H = \{$hyperplanes spanned by $P\}$. Suppose that one of the following three conditions is satisfied:*

(i) $d = 2$,

(ii) $d = 3, 4$ *and the cardinal $|P|$ is regular,*

(iii) $|P| < \aleph_\omega$.

Then there exists a matching $f : P \to H$.

From condition (iii) we draw the following conclusion.

Corollary 3. *The Continuum Hypothesis implies Conjecture 1.*

In this connection we would like to point out the existence of results in Euclidean combinatorics that depend on the choice of axioms for set theory. E.g., Shelah and Soifer [4] construct a graph on the real line whose chromatic number is 2 assuming the Axiom of Choice, and is greater than $\aleph_0$ (if it exists) using another consistent axiom system for $\mathbb{R}$.

A proof of the theorem appears in [1]. Part (i) is proved via a direct geometric construction. For the other parts one works in the setting of geometric lattices, which provides a convenient framework for inductive resoning starting from the $d = 2$ case. This part uses the transversal theorem of Milner and Shelah [3] in an essential way.

It is in the course of the induction arguments that one runs into trouble with singular cardinals, a circumstance that necessitates the cardinality restrictions imposed in parts (ii) and (iii). As expressed in the conjecture, we believe that these problems can be overcome. However, examples such as those of Shelah-Soifer [4] are healthy reminders of the conceivable dependence on extra hypotheses, such as CH, for certain combinatorial properties of real space.

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Szemerédi's regularity lemma and compactness

Balázs Szegedy

(joint work with László Lovász)

Szemerédi's regularity lemma is a fundamental tool in graph theory: it has many applications in extremal graph theory, in the area called "Property Testing", combinatorial number theory, etc. Here we present the regularity lemma as a result in analysis. The motivation for this analytic language is comming from the paper [3] where we introduce a convergence notion for graph sequences. Roughly speaking, a graph sequence H_i is convergent if the density of any fixed graph G in the members of the sequence tends to a limit $f(G)$. We prove in [3] that for every convergent graph sequence H_i there is a "natural" limit object from which the limits of the subgraph densities can be read off. These objects are two variable measurable functions $w : [0,1]^2 \mapsto [0,1]$ such that $w(x,y) = w(y,x)$ for every

$x, y \in [0,1]$. Let $\mathcal{W}_0$ denote the set of all such functions and let $\mathcal{W}$ denote the space of all bounded measurable functions $w : [0,1]^2 \mapsto [0,1]$. We consider the following norm on $\mathcal{W}$:

$$\|W\|_\square = \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} W(x,y)\,dx\,dy \right|.$$

For the case of matrices, this norm is called the "cut norm"; various important properties of it were proved by Alon and Naor [2] and by Alon, Fernandez de la Vega, Kannan and Karpinski [1]. We define the density of a graph G in an element $w \in \mathcal{W}_0$ with

$$t(G,w) = \int_{x_1,x_2,\dots,x_n} \prod_{(i,j)\in E(G),i<j} w(x_i,x_j)\ dx_1\ dx_2 \dots dx_n$$

where the vertex set of G is assumed to be $\{1,2,\dots,n\}$. Let $\phi : [0,1] \mapsto [0,1]$ be a measurable function and for $w \in \mathcal{W}$ let w^ϕ denote the function with $w^\phi(x,y) = w(\phi(x),\phi(y))$. It is easy to see that $t(G,w) = t(G,w^\phi)$ if G is a graph, $w \in \mathcal{W}_0$ and ϕ is measure preserving. This motivates the following distance

$$\delta_\square(U,W) = \inf_{\phi,\psi} \|U^\phi - W^\psi\|_\square,$$

where ϕ and ψ range over all measure preserving maps $[0,1] \to [0,1]$. Let $\mathcal{X}_0$ denote the space which is obtained from $\mathcal{W}_0$ by identifying elements whose $\delta_\square$ distance is 0. We prove that

Theorem 1. *The metric space $(\mathcal{X}_0, \delta_\square)$ is compact.*

It turns that this theorem implies the regularity lemma and some of its stronger versions.

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Excluded subposets in the Boolean lattice

Gyula O.H. Katona

Let $[n] = \{1,2,\dots,n\}$ be a finite set, families $\mathcal{F},\mathcal{G}$, etc. of its subsets will be investigated. $\binom{[n]}{k}$ denotes the family of all k-element subsets of $[n]$. Let P be a poset. The goal of the present investigations is to determine the maximum size of a family $\mathcal{F} \subset 2^{[n]}$ which does not contain P as a (non-necessarily induced) subposet. This maximum is denoted by $\mathrm{La}(n,P)$. In some cases two posets, say P_1, P_2 could be excluded. The maximum number of subsets is denoted by $\mathrm{La}(n, P_1, P_2)$ in this case.

The easiest example is the case when P consist of two comparable elements. Then we are actually looking for the largest family without inclusion that is without two distinct members $F, G \in \mathcal{F}$ such that $F \subset G$. The well-known Sperner theorem ([4]) gives the answer, the maximum is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We say that the distinct sets $A, B_1 \dots, B_r$ form an *r-fork* if they satisfy $A \subset B_1, \dots, B_r$. A is called the *handle*, B_is are called the *prongs* of the fork. On the other hand, the distinct sets $A, B_1 \dots, B_r$ form an *r-brush* if they satisfy $B_1, \dots, B_r \subset A$. The r-forks and the r-brush are denoted by $F(r), B(r)$, respectively. An old theorem solves the problem when the 2-fork and the 2-brush are excluded.

Theorem 1 ([3])**.**

$$\mathrm{La}(n, F(2), B(2)) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

The optimal construction is the family

$$\mathcal{F} = \left\{F: \; F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor}\right\} \bigcup \left\{F \cup \{n\}: \; F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor}\right\}.$$

We have proved the following theorem in a paper appearing soon.

Theorem 2 ([2])**.** *Let $n \geq 3$. If the family $\mathcal{F} \subseteq 2^{[n]}$ contains no four distinct sets A, B, C, D such that $A \subset C, A \subset D, B \subset C, B \subset D$, then $|\mathcal{F}|$ cannot exceed the sum of the two largest binomial coefficients of order n, i.e., $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.*

Following the suggestion of J.R. Griggs, such a family could be called a *butterfly-free meadow*. The optimal construction here is obvious, one can take all the subsets of sizes $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor + 1$.

In all of these cases the maximum size of the family is exactly determined. This is not true when the r-fork is excluded. In a paper under preparation A. De Bonis and the present author proved the following theorem.

Theorem 3 ([1])**.**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{r}{n} + O(\frac{1}{n^2})\right) \leq \mathrm{La}(F(r+1)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O(\frac{\log n}{n^{3/2}})\right).$$

A weaker version of the upper bound in this theorem was obtained in [5]: the constant in the second term was larger. There is still a gap between the lower and upper bounds in the second term: a factor 2. This however seems to be a serious difficulty. The best construction (lower bound) contains all sets in one level and a thinned next level.

Let the poset N consist of 4 elements illustrated here with 4 distinct sets satisfying $A \subset B, C \subset B, C \subset D$. We were not able to determine $\mathrm{La}(n, N)$ for a long time. Recently, a new method jointly developed by J.R. Griggs, helped us to prove the following theorem.

Theorem 4.

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+o(\frac{1}{n})\right) \le \mathrm{La}(n,N) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+o(\frac{1}{n})\right).$$

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Extremal Subgraphs of Random Graphs

Graham Brightwell

(joint work with Konstantinos Panagiotou and Angelika Steger)

It is well-known that most large triangle-free graphs are bipartite. Specifically, Osthus, Prömel and Taraz [3], and independently Steger [4] showed that, if $m \ge Cn^{3/2}\sqrt{\log n}$, for C a suitable constant, then the number of triangle-free graphs with n vertices and m edges is asymptotically equal to the number of bipartite graphs with n vertices and m edges.

We consider a problem of a similar flavour. Let $G = G(n,m) = (V,E)$ be a random graph with $|V| = n$ and $|E| = m$. Let $T(G)$ be a largest subset of E containing no triangle, and let $B(G)$ be a largest subset of E such that $(V, B(G))$ is bipartite. We always have $|T(G)| \ge |B(G)|$, but for what range of $m = m(n)$ do we asymptotically almost surely have equality?

For Π a bipartition or *cut* of $V(G)$, let $\Pi(G)$ denote the set of edges going *across* the partition, i.e., joining vertices in opposite parts. Then $|B(G)|$ is the maximum, over bipartitions Π, of $|\Pi(G)|$: this is the size of a maximum cut in G, and is much studied.

The maximum cut in a random graph is normally of more interest if m is smaller, say around cn, for c a constant. As c increases from $\frac{1}{2}$ to ∞, $\mathbb{E}|B(G)|/m$ decreases from 1 to $\frac{1}{2}$. There has been some recent work (see, e.g.,[2]) pinning down this behaviour more precisely. In the range we are considering, $|B(G)|$ exceeds $m/2$ by at most about $C\sqrt{mn}$.

In a 1990 paper, Babai, Simonovits and Spencer [1] proved that there is a positive constant δ such that, for $m \ge (\frac{1}{2} - \delta)\binom{n}{2}$,

$$\Pr\left(|T(G(n,m))| = |B(G(n,m))|\right) \to 1 \quad \text{as } n \to \infty \qquad (*).$$

Perhaps what is most striking about this result is its domain of validity. It seems unlikely that the property $(*)$ has a threshold for m a constant proportion of

$\binom{n}{2}$; indeed, Babai, Simonovits and Spencer asked whether this result could be extended to cover all $m \geq \varepsilon\binom{n}{2}$, for $\varepsilon > 0$ constant.

As far as we know, $(*)$ could hold whenever $m = m(n) \geq n^{3/2+\varepsilon}$, for arbitrary $\varepsilon > 0$. The property does not hold for (say) $m = \frac{1}{10}n^{3/2}\sqrt{\log n}$, as the random graph $G(n,m)$ asymptotically almost surely has an induced 5-cycle H such that no other vertex has more than one neighbour in H: any maximum-size triangle-free subgraph then includes all the edges of H, and is not bipartite.

We prove that property $(*)$ holds whenever $m = m(n) \geq n^{2-\delta}$, for some fixed $\delta > 0$.

Our proof involves a couple of features that may be of interest. The account below gives some of the intuition; there are some technical issues not discussed here.

We begin by using a strong form of the sparse regularity lemma (see the Abstract of Angelika Steger in this collection) to show that the largest triangle-free subgraph of $G = G(n,m)$ asymptotically almost surely differs from some $\Pi(G)$ by at most εm edges.

We then show that we can restrict attention to graphs differing from some $\Pi(G)$ by at most about $(n^2/m)^7$ edges. For any particular Π, we show that the probability that $|\Pi(G)|$ can be improved by such a small perturbation is very small.

However, to apply this we also need to show that there are relatively few cuts Π such that $|\Pi(G)|$ is close to – within about $(n^2/m)^7$ of – $|B(G)|$. We are able to show that this is indeed the case. Our proof method also tells us something about the family of *maximum* cuts in a random graph, namely that there is only a small set U of vertices such that any two maximum cuts differ only on U. For details, we refer to the paper, which is currently still in preparation.

Although our main focus is on the most appealing case of triangle-free graphs, our methods do extend to more general settings.

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Singularity of random matrices

Van H. Vu
(joint work with Terence Tao)

Random matrices is an important area of mathematics, with strong connections to various other fields. One of the main objects in this area is matrices whose entries are i.i.d random variables. We focus on the following basic model

- M_n: n by n matrix whose entries are i.i.d variables with Bernoulli distribution (taking values -1 and 1 with probability half).

A famous problem is to estimate the probability that M_n is singular. Let us denote by p_n this probability. Since M_n is singular if it has two identical rows, it is trivial that $p_n \geq (1/2 + o(1))^n$. A notorious conjecture in the field is that this bound is sharp.

Conjecture 1. *$p_n = (1/2 + o(1))^n$.*

The first result concerning singularity was obtained by Komlós in 1967, who proved $p_n = o(1)$. Later, he improved the bound to $O(n^{-1/2})$. A significant progress was made in 1995, when Kahn, Komlós and Szemerédi proved that $p_n \leq .999^n$ (see [4] and the references therein).

Recently, T. Tao and I made a progress by improving the bound further to further to $(3/4 + o(1))^n$ [6]. We discovered a surprising connection betweeen problems on random matrices and additive combinatorics. In particular, the proof of the new bound uses various ingredients from additive combinatorics (in particular, Freiman's theorem).

The details are somewhat technical, but my feeling is that the optimal bound $(1/2 + o(1))^n$ might be within sight. In fact, I believe that any improvement upon the constant $3/4$ could perhaps lead to the solution of the conjecture. Furthermore, our techniques can be used for other discrete distributions as well and in certain cases we can obtain sharp results.

A closely related question is to estimate the probability that a random symmetric matrix is singular. Let Q_n be the random symmetric n by n matrix whose upper diagonal entries are i.i.d. Bernoulli random variables. Weiss (1980s) conjectured that Q_n is almost surely non-singular. Recently, Costello, Tao and I [1] confirmed this conjecture. Our proof again makes a detour to additive combinatorics, with the main lemma being a quadratic version of the classical Littlewood-Offord-Erdős problem [2].

The relevant papers can be downloaded from my website.

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Perfect packings in graphs and critical chromatic number

Daniela Kühn
(joint work with Oliver Cooley and Deryk Osthus)

Given two graphs H and G, an *H-packing in G* is a collection of vertex-disjoint copies of H in G. H-packings are natural generalizations of graph matchings (which correspond to the case when H consists of a single edge). An H-packing in G is called *perfect* if it covers all vertices of G. If H has a component which contains at least 3 vertices then the question whether G has a perfect H-packing is difficult from both a structural and algorithmic point of view. For example, Tutte's theorem characterizes those graphs which have a perfect H-packing if H is an edge but for other graphs H no such characterization exists. This leads to the search for simple sufficient conditions which ensure the existence of a perfect H-packing. The following theorem of Komlós, Sárközy and Szemerédi [6] is a fundamental result of this kind.

Theorem 1. *For every graph H there exists a constant $C = C(H)$ such that every graph G whose order n is divisible by $|H|$ and whose minimum degree is at least $(1 - 1/\chi(H))n + C$ contains a perfect H-packing.*

This confirmed a conjecture of Alon and Yuster [2]. As observed in [2], there are graphs H for which the above constant C cannot be omitted completely. Thus one might think that this settles the question of which minimum degree guarantees a perfect H-packing.

However, there are graphs H for which the bound on the minimum degree can be improved significantly: Kawarabayashi [4] conjectured that if $H = K_\ell^-$ (i.e. a complete graph with one edge removed) and $\ell \geq 4$ then one can replace the chromatic number with the critical chromatic number in Theorem 1 and take $C = 0$. He [4] proved the case $\ell = 4$. Here the *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as

$$\chi_{cr}(H) := \frac{(\chi(H) - 1)|H|}{|H| - \sigma(H)},$$

where $\sigma(H)$ denotes the minimum size of the smallest colour class in a colouring of H with $\chi(H)$ colours. Note that $\chi_{cr}(H)$ always satisfies $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$. The critical chromatic number was introduced by Komlós [5]. He (and independently Alon and Fischer [1]) observed that for *any* graph H it gives a lower bound on the minimum degree that guarantees a perfect H-packing.

Our main result of [8] is that for any graph H, either its critical chromatic number or its chromatic number is the relevant parameter which governs the existence of perfect packings in graphs of large minimum degree. The exact classification depends on a parameter which we call the highest common factor of H and which is defined as follows. We say that a colouring of H is *optimal* if it uses exactly $\chi(H) =: \ell$ colours. Given an optimal colouring c, let $x_1 \le x_2 \le \cdots \le x_\ell$ denote the sizes of the colour classes of c. Put $\mathcal{D}(c) := \{x_{i+1} - x_i \mid i = 1, \dots, \ell - 1\}$. Let $\mathcal{D}(H)$ denote the union of all the sets $\mathcal{D}(c)$ taken over all optimal colourings c. We denote by $\text{hcf}_\chi(H)$ the highest common factor of all integers in $\mathcal{D}(H)$. (If $\mathcal{D}(H) = \{0\}$ we set $\text{hcf}_\chi(H) := \infty$.) We write $\text{hcf}_c(H)$ for the highest common factor of all the orders of components of H. If $\chi(H) \neq 2$ we say that $\text{hcf}(H) = 1$ if $\text{hcf}_\chi(H) = 1$. If $\chi(H) = 2$ then we say that $\text{hcf}(H) = 1$ if both $\text{hcf}_c(H) = 1$ and $\text{hcf}_\chi(H) \le 2$. So for example K_ℓ^- and $K_{2,5,7}$ both have $\text{hcf} = 1$.

As indicated above, our main result in [8] is that in Theorem 1 one can replace the chromatic number by the critical chromatic number if $\text{hcf}(H) = 1$. It turns out that Theorem 1 is already best possible up to the value of the constant C if $\text{hcf}(H) \neq 1$. Combining both results yields a minimum degree threshold for perfect graph packings which up to an additive constant is best possible for *all* graphs H. Indeed, let

$$\chi^*(H) := \begin{cases} \chi_{cr}(H) & \text{if } \text{hcf}(H) = 1; \\ \chi(H) & \text{otherwise.} \end{cases}$$

Also let $\delta_{Pack}(H, n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \ge k$ contains a perfect H-packing.

Theorem 2. *For every graph H there exists a constant $C = C(H)$ such that*

$$\left(1 - \frac{1}{\chi^*(H)}\right) n - 1 \le \delta_{Pack}(H, n) \le \left(1 - \frac{1}{\chi^*(H)}\right) n + C.$$

Note that while the definition of the parameter χ^* is somewhat complicated, the form of Theorem 2 is similar to that of the Erdős-Stone theorem. Related algorithmic aspects are considered in [7]. In [3] we considered the case when $H = K_\ell^-$ and proved the conjecture of Kawarabayashi for large graphs G.

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An approach to obtain exact results for an extremal hypergraph problem

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(joint work with Oleg Pikhurko)

Let F be a k-uniform hypergraph (k-graph for short). Write $\mathrm{ex}(n, F)$ for the maximum number of edges in an n-vertex k-graph containing no copy of F. When $k > 2$, computing $\mathrm{ex}(n, F)$, in fact even determining $\lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{k}$ (which is known to exist), is usually difficult. We propose a general method for this problem.

For $l \geq k$, let $T_l^k(n)$ be the complete l-partite k-graph with part sizes $\lfloor n/l \rfloor$ or $\lceil n/l \rceil$: every edge of $T_l^k(n)$ has at most one vertex in each of the l parts, and all edges subject to this restriction are present. Let

$$t_l^k(n) = |T_l^k(n)|.$$

(We identify a k-graph with its edge set.)

Suppose that we wish to prove that $\mathrm{ex}(n, F) = t_l^k(n)$ for a given F. Our method has four steps:

Step 1. Define an appropriately chosen family K of k-graphs such that $F \in K$. There is no general recipe for K. A particular property that K should possess is that any F-free k-graph of order n can be made K-free by removing $o(n^k)$ edges. Then $\mathrm{ex}(n, F) = \mathrm{ex}(n, K) + o(n^k)$ but, hopefully, $\mathrm{ex}(n, K)$ is easier to analyze.

Step 2. Prove that K is *stable* with respect to $T_l^k(n)$. Loosely speaking, this means that every K-free k-graph G on n vertices with close to $\mathrm{ex}(n, K)$ edges can be transformed to $T_l^k(n)$ without changing too many edges.

Step 3. From the stability of K, deduce the stability of F. This can be achieved for a large class of examples by using the hypergraph regularity Lemma and its associated counting Lemma (see Gowers [2] and Nagle-Rödl-Schacht-Skokan [6]).

Step 4. Using the stability of F, deduce the exact result $\mathrm{ex}(n, F) = t_l^k(n)$. This technique was first employed by Simonovits [9] to determine $\mathrm{ex}(n, F)$ exactly for color-critical 2-graphs F. Recently, stability has been used to determine exact results for several hypergraph Turán problems [1, 3, 4, 5, 7, 8].

We illustrate this approach via two examples. The first is a possible generalization of Turán's graph theorem to k-graphs, and yields the first infinite family of k-graphs, for each $k > 2$, whose extremal function is exactly determined. The second is a generalization of Mantel's theorem in a different direction.

Example 1. Fix $l, k \geq 2$. Let H_l^k be the k-graph with vertex set $A \dot\cup \dot\bigcup_{S \in \binom{A}{2}} B_S$, where $|A| = l, |B_S| = k-2$ for every S, and edge set $\{S \cup B_S : S \in \binom{A}{2}\}$. Thus H_l^k is the k-graph obtained from the complete graph K_l by enlarging each edge with a set of $k-2$ new vertices. H_l^k has $l + (k-2)\binom{l}{2}$ vertices and $\binom{l}{2}$ edges.

Theorem 2. *Let $l, k \geq 2$ and n be sufficiently large. Then the maximum number of edges in an n-vertex k-graph containing no copy of H_{l+1}^k is $t_l^k(n)$. The only k-graph for which equality holds is $T_l^k(n)$.*

Example 3. Let Fan^k be the k-graph comprising $k+1$ edges $E_1, \dots, E_k, E$, with $E_i \cap E_j = \{x\}$ for all $i \neq j$, where $x \notin E$, and $|E_i \cap E| = 1$ for all i. In other words, k edges share a single common vertex x and the last edge intersects each of the other edges in a single vertex different from x. Note that Fan^2 is simply a triangle, and in this sense Fan^k generalizes the definition of K_3.

Theorem 4. *Let $k \geq 3$. Then, for all sufficiently large n, the maximum number of edges in an n-vertex k-graph containing no copy of* Fan^k *is $t_k^k(n)$. The only k-graph for which equality holds is $T_k^k(n)$.*

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Perfect Matchings and Hamilton cycles in uniform hypergraphs

Deryk Osthus

(joint work with Daniela Kühn)

Matchings in uniform hypergraphs. The so called 'marriage theorem' of Hall provides a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. For hypergraphs there is no analogue of this result—up to now only partial results are known.

A simple corollary of Hall's theorem for graphs states that every bipartite graph with vertex classes A and B of size n whose minimum degree is at least $n/2$ contains a perfect matching. This can also be easily proved directly by considering a matching of maximum size. In [2] we proved an analogue of this result for uniform hypergraphs. Instead of two vertex classes and a set of edges joining them (as in the graph case), we now have r vertex classes and a set of (unordered) r-tuples, each of whose vertices lies in a different vertex class. A natural way to define the minimum degree of an r-uniform r-partite hypergraph $\mathcal{H}$ is the following. Given $r-1$ distinct vertices $x_1, \dots, x_{r-1}$ of $\mathcal{H}$, the *neighbourhood $N_{r-1}(x_1, \dots, x_{r-1})$ of*

$x_1, \dots, x_{r-1}$ in $\mathcal{H}$ is the set of all those vertices x which form a hyperedge together with $x_1, \dots, x_{r-1}$. The *minimum degree* $\delta'_{r-1}(\mathcal{H})$ is defined to be the minimum $|N_{r-1}(x_1, \dots, x_{r-1})|$ over all tuples $x_1, \dots, x_{r-1}$ of vertices lying in different vertex classes of $\mathcal{H}$.

Theorem 1. *Suppose that $\mathcal{H}$ is an r-uniform r-partite hypergraph with vertex classes of size $n \geq 1000$ which satisfies $\delta'_{r-1}(\mathcal{H}) \geq n/2 + \sqrt{2n \log n}$. Then $\mathcal{H}$ has a perfect matching.*

Theorem 1 is best possible up to the error term $\sqrt{2n \log n}$. Surprisingly, a simple argument already shows that a significantly smaller minimum degree guarantees a matching which covers *almost all* vertices of $\mathcal{H}$:

Theorem 2. *Suppose that $\mathcal{H}$ is an r-uniform r-partite hypergraph with vertex classes of size n which satisfies $\delta'_{r-1}(\mathcal{H}) \geq n/r$. Then $\mathcal{H}$ has a matching which covers all but at most $r-2$ vertices in each vertex class of $\mathcal{H}$.*

Again, the bound on the minimum degree in Theorem 2 is essentially best possible.

We used Theorems 1 and 2 to obtain analogues for r-uniform hypergraphs $\mathcal{H}$ which are not necessarily r-partite. The resulting bounds on the minimum degree are best possible up to an error term of $O(\sqrt{n} \log n)$. Recently, Rödl, Ruciński and Szemerédi [6] improved this error bound to $O(\log n)$.

Hamilton cycles in 3-uniform hypergraphs. A classical theorem of Dirac states that every graph on n vertices with minimum degree at least $n/2$ contains a Hamilton cycle. If one seeks an analogue of this result for 3-uniform hypergraphs $\mathcal{H}$, then several alternatives suggest themselves. We define the *minimum degree* $\delta(\mathcal{H})$ of $\mathcal{H}$ to be the minimum $|N(x,y)|$ over all pairs of distinct vertices $x, y \in \mathcal{H}$ (where $N(x,y)$ is defined as in the previous section).

We say that a 3-uniform hypergraph $\mathcal{C}$ is a *cycle of order n* if there a exists a cyclic ordering $v_1, \dots, v_n$ of its vertices such that every consecutive pair $v_i v_{i+1}$ lies in a hyperedge of $\mathcal{C}$ and such that every hyperedge of $\mathcal{C}$ consists of 3 consecutive vertices. A cycle is *tight* if every three consecutive vertices form a hyperedge. A cycle of order n is *loose* if it has the minimum possible number of hyperedges among all cycles on n vertices. Thus if the number n of vertices in a loose cycle $\mathcal{C}$ is even and at least 6, then consecutive hyperedges in $\mathcal{C}$ have exactly one vertex in common and the number of hyperedges in $\mathcal{C}$ is exactly $n/2$. A *Hamilton cycle* of a 3-uniform hypergraph $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$ which is a cycle containing all its vertices. In [4] we proved the following result.

Theorem 3. *For each $\sigma > 0$ there is an integer $n_0 = n_0(\sigma)$ such that every 3-uniform hypergraph $\mathcal{H}$ with $n \geq n_0$ vertices and minimum degree at least $n/4 + \sigma n$ contains a loose Hamilton cycle.*

The bound on the minimum degree in Theorem 3 is best possible up to the error term σn. In fact, if the minimum degree is less than $\lceil n/4 \rceil$, then we cannot even guarantee *any* Hamilton cycle. Recently, Rödl, Ruciński and Szemerédi [5] proved that if the minimum degree is at least $n/2 + \sigma n$ and n is sufficiently large, then one can even guarantee a tight Hamilton cycle. Their bound is best possible up to the error term σn. The proof of Theorem 3 relies on the Regularity Lemma for 3-uniform hypergraphs due to Frankl and Rödl [1]. As a tool, we use a 'blow up' type result: every 'pseudo-random' hypergraph contains a loose Hamilton cycle. This in turn uses a probabilistic argument based on results about random perfect matchings in pseudo-random graphs [3].

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Extremal problems on packing of graphs

Alexandr Kostochka

(joint work with Gexin Yu)

Recall that n-vertex graphs $G_1, G_2, \ldots, G_k$ are said to *pack* if there exist injective mappings of their vertex sets onto $[n] = \{1, \ldots, n\}$ such that the images of the edge sets do not intersect. In particular, two n-vertex graphs G_1 and G_2 pack if G_1 is a subgraph of the complement $\overline{G}_2$ of G_2 (and vice versa).

In terms of packing, some graph theory problems or concepts can be generalized or made more natural. For example, the problem of existence of a hamiltonian cycle in an n-vertex graph G is equivalent to the question whether the n-cycle C_n packs with the complement $\overline{G}$ of G. Another example: For a graph G on n vertices, being equitably k-colorable is equivalent to pack with the n-vertex graph whose components are cliques with $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices.

Study of extremal problems on packings of graphs started in the 1970s by Sauer and Spencer [10] and Bollobás and Eldridge [3].

In particular, Sauer and Spencer [10] proved the following result.

Theorem 1. *Suppose that G_1 and G_2 are graphs of order n such that $2\Delta(G_1)\Delta(G_2) < n$. Then G_1 and G_2 pack.*

One of the main conjectures in the area is the Bollobás–Eldridge-Catlin (BEC) conjecture (see [3]) stating that *if G_1 and G_2 are n-vertex graphs and $(\Delta(G_1) + 1)(\Delta(G_2) + 1) \le n + 1$, then G_1 and G_2 pack.*

If true, the conjecture would be sharp. It is a considerable extension of the Hajnal-Szemerédi Theorem on equitable colorings. The conjecture has been proved in the case $\Delta_1 \leq 2$ by Aigner and Brandt [1] and Alon and Fisher [2], and in the case $\Delta_1 = 3$ and n is huge by Csaba, Shokoufandeh, and Szemerédi [5]. The aim of this talk is to report very recent results on the topic and to state new problems.

The restriction $2\Delta(G_1)\Delta(G_2) < n$ in Theorem 1 is sharp in the sense that if G_2 is a perfect matching and $G_1 = K_{0.5n+1}$, then $2\Delta(G_1)\Delta(G_2) = n$, but G_1 and G_2 do not pack. However, there are not many such examples. Kaul and Kostochka [7] gave a characterization of the pairs (G_1, G_2) of n-vertex graphs with $2\Delta_1\Delta_2 = n$ that do not pack.

Theorem 2. *Let G_1 and G_2 be n-vertex graphs with maximum degrees $\Delta(G_i) = \Delta_i$ for $i = 1, 2$. Let $2\Delta_1\Delta_2 \leq n$. G_1 and G_2 do not pack if and only if one of G_1 and G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.*

Bollobás, Kostochka and Nakprasit [4] proved that when one of the two graphs is sparse, to be precise, d-degenerate for a small d, then much weaker conditions on Δ_1 and Δ_2 imply the existence of a packing.

Theorem 3. *Let $d \geq 2$. Let G_1 be a d-degenerate graph of order n and maximal degree Δ_1 and G_2 a graph of order n and maximal degree at most Δ_2. If*

$$40\Delta_1 \ln \Delta_2 < n \quad \text{and} \quad 40d\Delta_2 < n, \tag{1}$$

then there is a packing of G_1 and G_2.

Both restrictions in (1) are weakest up to a constant factor. Kaul, Kostochka and Yu [8] proved the following weakening of the BEC conjecture that improves the bounds of Theorem 1 for large Δ_1 and Δ_2.

Theorem 4. *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If $\Delta_1, \Delta_2 \geq 300$ and*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq 0.6n + 1, \tag{2}$$

then G_1 and G_2 pack.

This gives a partial answer to Problem 4.4 in [6].

Let $\overline{\sigma}_2(G)$ denote the maximum of $0.5(\deg_G(v) + \deg_G(u))$ over all edges of G. In other words, $\overline{\sigma}_2(G) = 0.5\Delta(L(G)) + 1$, where $L(G)$ is the line graph of G. Then the theorem of Ore [9] refining Dirac's theorem on hamiltonian cycles can be stated in terms of packings as follows.

Theorem 5. *If $n \geq 3$ and G is an n-vertex graph with $\overline{\sigma}_2(G) \leq 0.5n - 1$, then G packs with the cycle C_n.*

We conjecture that the following Ore-type analog of the BEC-conjecture is true.

Conjecture 6. *If G_1 and G_2 are n-vertex graphs and $(\overline{\sigma}_2(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$, then G_1 and G_2 pack.*

Conjecture 6 would imply the Ore-type version of Hajnal-Szemerédi Theorem.

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The random planar graph process

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(joint work with Stefanie Gerke, Dirk Schlatter and Angelika Steger)

The study of the random graph process $(G_{n,t})_{t=0}^{N}$, where one starts with an empty graph on n vertices and adds all $N := \binom{n}{2}$ edges in a random order, was initiated by Erdős and Rényi in a series of papers more than 40 years ago [3]. During the past decades, there has been a wealth of fascinating results in the area, and although some problems still remain unsolved, the model in general seems to be well understood. But comparatively little is known about variants of this process, where extra conditions have to be satisfied when inserting the edges. These conditions distort the randomness in such a way that the methods and tools employed for the original case are of little use.

A restricted random graph process $(P_{n,t})_{t=0}^{N}$ is a random graph process equipped with an additional acceptance test: after we have randomly chosen the edge to be inserted, we check whether the present graph together with this edge preserves a certain property. If so, we take it, otherwise we reject it (and never look at it again).

Special cases which have been considered include the properties triangle-freeness and cycle-freeness. In both cases, the outcome of these random graph processes differ significantly from the corresponding uniform models. Erdős, Suen, and Winkler [4] have shown that with high probability the outcome of the random triangle-free graph process only has close to $n^{3/2}$ edges, whereas a well-known result by Erdős, Kleitman, and Rothschild [2] states that with high probability a uniformly random triangle-free graph is bipartite and has $\theta(n^2)$ edges. Aldous [1] investigated the random cycle-free graph process $(T_{n,t})_{t=0}^{N}$ and showed amongst other results that the number of leaves in the resulting tree is concentrated around approximately

$0.406n$. By noting that the leaves of a tree are precisely those vertices whose label does not appear in a Prüfer code, it is easy to see that the number of leaves in a uniformly random tree is concentrated around n/e.

Here we require the graph to be *planar* and shall be interested in the *evolution* of this restricted random process. There are two ways in which we will parametrize the process. Let P_{n,t_0} denote the random planar graph obtained after t_0 edges have been considered. $P_{n,m=m_0}$, on the other hand, describes the random planar graph after m_0 edges have been accepted. As edges between vertices in different components are always accepted, it is obvious that $T_{n,t} \subseteq P_{n,t} \subseteq G_{n,t}$ for all $t = 0, \ldots, N$. Thus, after the connectivity threshold for $G_{n,t}$, which lies at $t = n \log n/2$, $P_{n,t}$ must have at least $n-1$ edges with high probability. The following theorem may thus seem somewhat surprising.

Theorem 1. *For every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\mathbb{P}\left[e\left(P_{n,\delta n^2}\right) \geq (1+\epsilon)n\right] < e^{-n}.$$

The uniform model of random planar graphs has found considerable attention in the literature over the past decade. Recently, Giménez and Noy [6] gave rather precise asymptotic expressions for both the number of simple, labelled planar graphs with n vertices and dn edges, and the number of those which are connected. These results yield an analytic expression for the probability that a uniformly random planar graph with dn edges is connected. As it turns out, this probability is bounded away from 0 and 1 for every $1 < d < 3$. From Theorem 1, we can immediately infer that this is not true for $P_{n,m=dn}$.

Theorem 2. *For every $1 < d < 3$,*

$$\mathbb{P}\left[P_{n,m=dn} \text{ is connected}\right] \longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

Gerke, McDiarmid, Steger, and Weißl [5] have shown the following result about the containment of a fixed planar graph H in a graph $\hat{P}_{n,m=dn}$ which is chosen uniformly at random from the class of all planar graphs with n vertices and dn edges:

$$\mathbb{P}\left[\hat{P}_{n,m=dn} \text{ contains at most } \alpha n \text{ pairwise vertex–disjoint copies of } H\right] < e^{-\alpha n},$$

for every $1 < d < 3$ and a positive constant $\alpha = \alpha(H,d)$.

In this respect, the two models do agree: the following analogue is our second main result.

Theorem 3. *Let H be a planar graph. For every $1 < d < 3$, there exists $\alpha = \alpha(H,d) > 0$ such that*

$$\mathbb{P}\left[P_{n,m=dn} \text{ contains at most } \alpha n \text{ pairwise vertex–disjoint copies of } H\right] < e^{-\alpha n}.$$

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