Mathematisches Forschungsinstitut Oberwolfach

Report No. 2/2006

Mathematical Aspects of General Relativity

Organised by
Piotr Chruściel (Tours)
James Isenberg (Eugene)
Alan D. Rendall (Golm)

January 8th – January 14th, 2006

Abstract. The conference covered mathematical general relativity. The Einstein equations, the key to the subject, can be split into constraints and evolution equations. Many of the talks at the conference concerned the constraints and the concepts of mass and horizons while others dealt with the Einstein evolution equations and related hyperbolic problems. Applications to cosmology were also well represented.

Mathematics Subject Classification (2000): 83-XX.

Introduction by the Organisers

The conference brought together people working in mathematical general relativity, a field which lies at the interface of analysis, differential geometry and physics. There was a mix of established workers in the subject with participants at the start of their careers and researchers from neighbouring fields.

The Einstein equations are at the heart of general relativity theory. They form a system of evolution equations and their solutions are conveniently parametrized by initial data. These initial data are required to satisfy constraint equations and these were the topic of several lectures at the conference. The talks of Corvino and Pollack were concerned with recent progress in methods for constructing solutions of the constraints. An important concept for solutions of the constraints is that of mass, including quasilocal mass. This was a central theme of the talks of Degeratu, Huisken and Shi. Huisken introduced a striking new approach to the definition of mass based on isoperimetric inequalities. Dain presented a variational characterization of the extreme Reissner-Nordström solution, thus relating the study of the constraint equations to the theory of black holes. The talk of Wohlfarth was rather outside the main area of the conference. He described a new concept of geometry motivated by string theory which may come to enrich the circle of ideas within mathematical relativity.

Many of the talks at the conference were on evolution equations related to general relativity. Dafermos and Finster presented results related to the dynamical stability of black holes. Dafermos described new results on the rate of decay of solutions of the wave equation on the Schwarzschild spacetime while Finster explained applications of methods of functional analysis to the wave equation on the Kerr spacetime. Struwe's talk concerned uniqueness for supercritical nonlinear wave equations which from the point of view of the Einstein equations are an important example to compare with. The Maxwell equations are another important comparison system and Bauer showed how they can be used to understand more about radiation formulae. Velázquez gave an introduction to singularity formation in the Keller-Segel model, a parabolic system coming from mathematical biology. This is a possible source of insight for obtaining a rigorous understanding of critical collapse in general relativity. The talks of Andréasson, Choptuik and Lindblom dealt with various aspects of the application of numerical techniques to the study of the Einstein evolution equations. Choptuik showed impressive new simulations of coalescing black holes due to Pretorius which could hardly have been imagined just a year ago.

Cosmology is at present a very active area of research in general relativity. This is in part due to the challenge of understanding the observed accelerated expansion of our universe. Mathematics is beginning to make its mark in this subject and this was reflected by talks of Heinzle, Rendall and Tod.

Mathematical relativity is a meeting point for many ideas and the abstracts which follow give some idea of the variety of the subject. In fact the spectrum of topics discussed by the participants at the conference was much wider than those for which talks could be scheduled. By limiting the number of presentations it was possible to leave plenty of time for people to exchange insights. The lively interactions observed make us hopeful that this conference has given a boost to the development of the subject in the next few years.

Workshop: Mathematical Aspects of General Relativity

Table of Contents

Abstracts

Area geometry and string backgrounds

Mattias N. R. Wohlfarth

1. Introduction

Interest in a geometric understanding of generalized backgrounds in string theory has recently been fueled by Hitchin's proposal of a unified description of the spacetime metric and the Neveu-Schwarz two-form [1]. This approach is of intrinsic mathematical appeal and has proven valuable in compactifications of string theory on generalized complex manifolds [2, 3], and in studying D-branes and mirror symmetry [4, 5, 6, 7]. However, the basic premise of string theory, namely the replacement of point particles by strings, suggests an alternative geometric picture: manifolds M equipped with an area measure G, not with a metric length measure. These present a true generalization of Lorentzian manifolds, which in particular may include a B-field, because only some area measures can be induced from a metric. We aim at the construction of differential geometric structures on manifolds (M, G), with a view towards their characterization and to the construction of gravitational dynamics that could play an important role in understanding the gravitational sector of strings and branes.

2. Area metric manifolds

The affine variety of oriented areas consists of the simple antisymmetric two-tensors, or bivectors, Ω in $\bigwedge^2 TM$ with $\Omega \wedge \Omega = 0$. Any such area can be represented by two vectors as $X \wedge Y$, which expression encodes the $SL(2, \mathbb{R})$ invariance under basis change. We consider smooth d-dimensional manifolds M equipped with an area measure G given by a symmetric metric on the areas' embedding space $\bigwedge^2 TM$. So G is a fourth rank covariant tensor with the symmetries

$$G(X, Y, A, B) = -G(Y, X, A, B) = G(A, B, X, Y)\,. \tag{1}$$

The area measure naturally provides a linear map $G : \bigwedge^2 TM \to \bigwedge^2 T^*M$. In case the inverse of this map exists everywhere on M, we call G an area metric and (M, G) an area metric manifold.

Any metric manifold (M, g) is an area metric manifold (M, G_g), by virtue of the induced area metric $G_g(X, Y, A, B) = g(X, A)g(Y, B) - g(X, B)g(Y, A)$. (Note that $G_g(X, Y, X, Y)$ is the squared area of the parallelogram spanned by X and Y.) However, not every area metric is induced by a single metric, but rather by a finite collection of metrics $\{g^{(1)} \dots g^{(N)}\}$ via a decomposition theorem for algebraic curvature maps due to Gilkey [8]. It follows from this theorem that any area metric can be decomposed as

$$G = F + \sum_{i=1}^{N} \sigma_{(i)}\, G_{g^{(i)}}\ , \tag{2}$$

with signs $\sigma_{(i)} = \pm 1$ and a four-form F. The latter, as well as the $G_{g^{(i)}}$, are irreducible representations under the local frame group $SL(d,\mathbb{R})$.

The above decomposition is not unique. Hence two different points of view on area metric geometry may be taken. The first redefines area metric geometry as multi-metric geometry (by picking a particular decomposition as the basic data on the manifold). This approach is discussed in [9], and allows the construction of curvature invariants in terms of the constituent metrics $g^{(i)}$. The second route, considered in the following, is canonical in not relying on any decomposition of the area measure G, see [10] for more detail.

3. Differential geometry of stationary surfaces

In order to identify geometric structures on area metric manifolds (M,G), we proceed from the equation for surfaces of stationary area, i.e., the equation of motion of the classical bosonic string. This equation is obtained from variation, with respect to x, of the surface area integral

$$\int d^2\sigma \sqrt{G(\Omega,\Omega)}\,, \tag{3}$$

for a surface $x(\sigma)$ embedded in some target space with tangent area $\Omega = \dot{x}\wedge x'$. Fixing the reparametrization invariance by a constant normalization $G(\Omega,\Omega)$, and satisfying suitable boundary conditions, one finds for all vectors Z the condition

$$dG(\Omega,\cdot)(\Omega\wedge Z) = 0\,. \tag{4}$$

The stationary surface equation can be concisely recast in terms of differential geometric structures on area metric manifolds (M,G). However, due to the non-linear nature of area spaces, connections on $\bigwedge^2 TM$ do not play a similarly important role as tangent bundle connections for geodesics, or autoparallels, on metric manifolds. Better adapted is the concept of pre-connections. For any vector X and two sections Ω and Σ of $\bigwedge^2 TM$ we define a symmetric pre-connection $D_X(\Omega,\Sigma)$ in $C^\infty M$ by the properties:

$$D_X(\Omega,\Sigma) = D_X(\Sigma,\Omega)\,, \tag{5}$$

$$D_{X+fY}(\Omega,\Sigma) = D_X(\Omega,\Sigma) + fD_Y(\Omega,\Sigma)\,, \tag{6}$$

$$D_X(\Omega,\Sigma+\Phi) = D_X(\Omega,\Sigma) + D_X(\Omega,\Phi)\,, \tag{7}$$

$$D_X(f\Omega,\Sigma) = fD_X(\Omega,\Sigma) + Xf\,G(\Omega,\Sigma)\,, \tag{8}$$

which are symmetry, $C^\infty M$-linearity in X, and $\mathbb{R}$-linearity together with a certain 'Leibniz' rule for the sections Ω and Σ. We also define an antisymmetric pre-connection $D_X[\Omega,\Sigma]$ in $C^\infty M$ on (M,G), with precisely the same properties as stated above for the symmetric pre-connection, except an additional minus sign appearing in the first line.

It is now simple to show that symmetric and anti-symmetric pre-connection together determine a connection on $\bigwedge^2 TM$: for X and Ω as above we define

$$\nabla_X\Omega = \frac{1}{2}G^{-1}(D_X(\Omega,\cdot) + D_X[\Omega,\cdot],\cdot)\,, \tag{9}$$

which has all the expected properties. Conversely, any $\bigwedge^2 TM$ connection can be uniquely decomposed into symmetric and antisymmetric pre-connections. Due to the symmetry of G, the natural requirement of area metric compatibility for any such connection, $\nabla_X G = 0$, is a condition on the symmetric pre-connection alone, which it fully determines:

$$\nabla_X G = 0 \quad \Rightarrow \quad D_X(\Omega, \Sigma) = XG(\Omega, \Sigma)\,. \tag{10}$$

Area metric compatibility has further geometric implications. Via the covariant constancy of the volume form $\omega(G)$ on area metric manifolds of even dimension $d \geq 4$ one can show that parallel transport along any curve preserves the intersection of area distributions and the simplicity of sections of the bundle of antisymmetric two-tensors.

The symmetric pre-connection enables the construction of a one-form valued derivative action of areas $\Sigma = X \wedge Y$ on sections Ω of $\bigwedge^2 TM$. Starting from the simple expression $D_X(\Omega, Y \wedge Z)$ for an additional vector field Z, and requiring $C^\infty M$-linearity in Z, $SL(2,\mathbb{R})$-invariance for the pair (X, Y), and $C^\infty M$-homogeneity under rescalings of Σ, one is led to the definition of

$$\mathcal{D}_\Sigma \Omega(Z) = D_X(\Omega, Y \wedge Z) - \frac{1}{2} G(\Omega, [X, Y] \wedge Z) + \text{terms cyclic in } X, Y, Z\,. \tag{11}$$

Based on an area metric compatible pre-connection this expression simplifies to $\mathcal{D}_\Sigma \Omega(Z) = dG(\Omega, \cdot)(\Sigma \wedge Z)$. We may hence rewrite the stationary surface equation for any tangent area distribution $\Omega = \dot{x} \wedge x'$ as

$$\mathcal{D}_\Omega \Omega = 0\,. \tag{12}$$

The expression $\mathcal{D}_\Omega \Omega$ thus acquires a neat geometric interpretation as the mean curvature one-form of a two-surface represented by Ω.

The work detailed here, and in [9, 10], only presents a first step in understanding the geometry of area metric manifolds. Of particular importance for further research appears to be the construction of invariants for these manifolds, maybe through the investigation of curvature analogues, which would then allow for the formulation of gravitational dynamics.

References

[1] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford Ser. **54** (2003) 281 [arXiv:math.dg/0209099].

[2] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Supersymmetric backgrounds from generalized Calabi-Yau manifolds*, JHEP **0408** (2004) 046 [arXiv:hep-th/0406137].

[3] M. Grana, J. Louis and D. Waldram, *Hitchin functionals in $N = 2$ supergravity*, arXiv:hep-th/0505264.

[4] M. Gualtieri, *Generalized complex geometry*, Oxford DPhil thesis, [arXiv:math.dg/0401221].

[5] R. Zucchini, *Generalized complex geometry, generalized branes and the Hitchin sigma model*, JHEP **0503** (2005) 022 [arXiv:hep-th/0501062].

[6] P. Koerber, *Stable D-branes, calibrations and generalized Calabi-Yau geometry*, JHEP **0508** (2005) 099 [arXiv:hep-th/0506154].

[7] S. Fidanza, R. Minasian and A. Tomasiello, *Mirror symmetric* SU(3)*-structure manifolds with NS fluxes*, Commun. Math. Phys. **254** (2005) 401 [arXiv:hep-th/0311122].

[8] P. B. Gilkey, *Geometric properties of natural operators defined by the Riemann curvature tensor*, World Scientific, Singapore, 2001.
[9] F. P. Schuller and M. N. R. Wohlfarth, *Geometry of manifolds with area metric*, arXiv:hep-th/0508170.
[10] F. P. Schuller and M. N. R. Wohlfarth, *Canonical differential structure of string backgrounds*, arXiv:hep-th/0511157.

The long-time dynamics of scalar waves in the Kerr geometry

Felix Finster
(joint work with Niky Kamran, Joel Smoller, Shing-Tung Yau)

We consider the scalar wave equation in the Kerr geometry for Cauchy data which is smooth and compactly supported outside the event horizon. We derive an integral representation which expresses the solution as a superposition of solutions of the radial and angular ODEs which arise in the separation of variables. In particular, we prove completeness of the solutions of the separated ODEs. From this integral representation, we deduce decay of the solutions in L^∞_{loc}.

In the talk, the methods of the proof are outlined. The main difficulty is that the classical energy density can be negative inside an annular region around the event horizon, the so-called *ergosphere*. For classical particles, this effect leads to the so-called Penrose process, which allows to extract energy and angular momentum from a rotating black hole. A similar phenomenon occurs for scalar waves and is called *superradiance*. Mathematically, the indefinite energy density leads to the difficulty that the inner product is not positive, and thus we cannot use spectral methods in a Hilbert space.

In order to overcome this difficulty, we use several methods: First, we consider the problem in finite volume, where spectral methods in *Pontrjagin spaces* can be applied. Second, we derive *resolvent estimates*, which make it possible to quantitatively compare the dynamics in finite and in infinite volume. Next, we make use of the separation of variables in the Kerr metric together with estimates for the resulting ordinary differential equations (so-called *WKB estimates* and *invariant region estimates for the complex Riccati equation*). Moreover, we use spectral estimates for the angular equation, which were worked out in collaboration with Harald Schmid [2]. Combining these methods with Whiting's mode stability result [4], we obtain an integral representation of the scalar wave propagator which involves contour integrals in the complex plane [1].

In order to derive decay rates, one needs to move the contours onto the real axis. To this end, we need *asymptotic estimates* for the radial equation, which are obtained using the Jost equation. Then a *causality argument* allows us to rule out poles of the resolvent on the real axis. We thus obtain an integral representation of the propagator as an infinite sum over the angular momentum modes, each of which is an integral of the energy variable ω on the real line [3]. This representation has similarity with a Fourier representation, and we can apply the Riemann-Lebesgue lemma to get the desired decay in L^∞_{loc}.

References

[1] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, *An integral spectral representation of the propagator for the wave equation in the Kerr geometry*, gr-qc/0310024, Commun.Math. Phys. **260** (2005) 257-298

[2] F. Finster, H. Schmid, *Spectral estimates and non-selfadjoint perturbations of spheroidal wave operators*, math-ph/0405010, to appear in Crelle Journal (2006)

[3] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, *Decay of solutions of the wave equation in the Kerr geometry*, gr-qc/0504047, to appear in Commun.Math.Phys. (2006)

[4] B. Whiting, *Mode stability of the Kerr black hole*, J. Math. Phys. **30** (1989) 1301–1305

A numerical investigation of the stability of steady states and critical phenomena for the spherically symmetric Einstein-Vlasov system

Håkan Andréasson

(joint work with Gerhard Rein)

We investigate the stability of steady states for the spherically symmetric, asymptotically flat Einstein-Vlasov system. This system describes a self gravitating collisionless gas in the framework of general relativity. Here the matter is thought of as a large ensemble of particles, which is described by a density function on phase space, and the individual particles move along geodesics; for further information on this system cf. [1]. In astrophysics it is used as a model for compact star clusters and galaxies. In this context the stability question was first studied by Zeldovich et al. in the sixties, [14]. They characterize a steady state by its central redshift and binding energy and conjecture that the binding energy maximum along a steady state sequence signals the onset of instability. Ipser [6] and Shapiro and Teukolsky [13] found numerical support for this conjecture for isotropic steady states. In our investigation we find that this conjecture also holds for non-isotropic steady states where the density on phase space depends on the particle energy and angular momentum. We also investigate the role of the binding energy and find that states with a positive binding energy will be bound and the solution will oscillate in a neighbourhood of the steady state, while a negative binding energy leads to dispersion. Our initial motivation for studying the stability of steady states was its role in critical collapse. This topic started with the work of Choptuik [3] where he studied the Einstein-Scalar-Field system. He took a fixed initial profile for the scalar field and scaled it by an arbitrary constant factor. This gives rise to a family of initial data depending on a real parameter A. It turned out that there exists a critical parameter A_* such that for $A < A_*$ the corresponding solutions disperses, while for $A > A_*$ the corresponding solutions collapse and produce a black hole. This was in accordance with the theoretical results established by Christodoulou, see [5] for a review. The surprising result was that the limit of the mass $M(A)$ of the black hole tends to zero for $A \to A_*$ so that in such a one-parameter family there are black holes with arbitrarily small mass. Choptuik found that this fact is related to the existence of self-similar solutions of the Einstein-scalar-field equations, in particular, the critical solution is self-similar and universal, i.e., independent of the initial profile which determines the one-parameter family. Later on a similar investigation was performed for the Einstein-Yang-Mills equations by Choptuik, Chmaj and Bizoń [4]. Here both cases where $\lim_{A\to A_*} M(A) = 0$ and $\lim_{A\to A_*, A>A_*} M(A) > 0$ were found, called type II and type I respectively. In the latter case there is accordingly a mass gap.

As opposed to the field theoretic matter models mentioned above the Vlasov model is phenomenological, but in contrast to fluid models several global results have been established. For the spherically symmetric and asymptotically flat case it was shown in [8] that sufficiently small initial data launch global, geodesically complete solutions which disperse for large times. It is also known that there do exist initial data, necessarily large, which develop singularities [12]. The proof relies on the Penrose singularity theorem. There are no general results on the behavior of large data solutions yet, except for the following: If data on a hypersurface of constant Schwarzschild time give rise to a solution which develops a singularity after a finite amount of Schwarzschild time, then the first singularity occurs at the center of symmetry [10]. In [2] further results on the global behaviour of solutions for large data can be found. The transition between dispersion and gravitational collapse was numerically investigated by Rein, Rendall, and Schaeffer [11], and it was found that there is a mass gap in the $M(A)$ curve. This result was later confirmed by Olabarrieta and Choptuik [7]. In addition, these authors reported evidence that the mass gap is due to the presence of static solutions, and they conjectured that the critical solution is universal.

In the present investigation we address the role of steady states in critical phenomena for the Einstein-Vlasov system and the question of universality by explicitly exploiting the fact that for this system the existence of an abundance of steady states is well-established [9], and that these steady states can easily be computed numerically. Computing a steady state f_s we consider the family Af_s of initial data. Within every family of steady states given by a specific dependence on particle energy and angular momentum we found unstable steady states which act as critical solutions: If they are perturbed with $A > 1$ they collapse to a black hole, if they are perturbed with $A < 1$ they either disperse or oscillate in a neighbourhood of the steady state, depending on the sign of the binding energy. Due to the abundance of possible such dependences on particle energy and angular momentum there cannot be a universal critical solution in spherically symmetric collapse for the Einstein-Vlasov system.

Our main motivation for this numerical study is that it may lead to conjectures on the behavior of solutions of the Einstein-Vlasov system which may eventually be proven rigorously.

References

[1] H. Andréasson, *The Einstein-Vlasov system/kinetic theory,* Living Rev. Relativity, **8** (2005).

[2] H. Andréasson, *On global existence of the spherically symmetric Einstein-Vlasov system in Schwarzschild coordinates,* Indiana University Math. J. To appear.

[3] M. W. Choptuik, *Universality and scaling in the gravitational collapse of a scalar field,* Phys. Rev. Lett., **70** (1993), 9–12.
[4] M. W.Choptuik, T. Chmaj, P. Bizoń, *Critical behaviour in gravitational collapse of a Yang-Mills field,* Phys. Rev. Lett., **77** (1996), 424–427.
[5] D. Christodoulou, *On the global initial value problem and the issue of singularities,* Classical Quantum Gravity, **16** (1999), A23–A35.
[6] J. R. Ipser, *Relativistic, spherically symmetric star clusters. III. Stability of compact isotropic models,* The Astrophys. J., **158** (1969), 17–43.
[7] I. Olabarrieta, M. W. Choptuik, *Critical phenomena at the threshold of black hole formation for collisionless matter in spherical symmetry,* Phys. Rev. D., **65** (2002), 024007.
[8] G. Rein, A. D. Rendall, *Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data,* Commun. Math. Phys., **150** (1992), 561–583. Erratum: Commun. Math. Phys., **176** (1996), 475–478.
[9] G. Rein, A. D. Rendall, *Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics,* Math. Proc. Camb. Phil. Soc., **128** (2000), 363–380.
[10] G. Rein, A. D. Rendall, J. Schaeffer, *A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system,* Commun. Math. Phys., **168** (1995) 467–478.
[11] G. Rein, A. D. Rendall, J. Schaeffer, *Critical collapse of collisionless matter—a numerical investigation,* Phys. Rev. D, **58** (1998), 044007.
[12] A. D. Rendall, *Cosmic censorship and the Vlasov equation,* Class. Quantum Grav., **9** (1992), L99–L104.
[13] S. L. Shapiro, S. A. Teukolsky, *Relativistic stellar dynamics on the computer. II—Physical Applications,* Astrophysical J., **298** (1995), 58–79.
[14] Ya. B. ZelĎovich, I. D. Novikov, *Relativistic Astrophysics* Vol. **1**, Chicago: Chicago University Press (1971).

Some recent results on the Einstein constraint equations

Justin Corvino

1. On the Existence and Stability of the Penrose Compactification

In the 1960's Penrose [26] proposed a model of isolated gravitational systems based on the conformal compactification of Minkowski space. As the model has had enormous influence on the study of gravitational radiation, one would like to establish stability results which yield new examples through perturbation. Friedrich attacked this problem by rewriting the Einstein equations to emphasize the conformal structure, and he obtained a semi-global stability result for Minkowski space: for hyperboloidal data suitably close to a given hyperboloidal data set in Minkowski space (intersecting future null infinity), the resulting solution of the initial-value problem for Einstein's vacuum equation admits a conformal compactification to the future [18]; see also more recent work of Anderson and Chruściel [1]. Thus, if one could control the asymptotics near infinity on an asymptotically flat initial data set, in such a manner that it will evolve to a spacetime with suitable hyperboloidal slices (to the future and the past), then one could invoke the stability result to evolve the data to a spacetime which possesses a smooth compactification. In fact, Cutler and Wald [16] use this method to produce solutions of the Einstein-Maxwell field equations that admit a smooth compactification.

We state a version of linearization stability of the conformal compactification (in the vacuum case) in terms of the initial data [14]. The proof involves a careful study of the construction in [12] of perturbations of given asymptotically flat, scalar flat metrics to ones which are Schwarzschild near infinity (in the time-symmetric case, the constraint equations reduce to the vanishing of the scalar curvature). Recall that at the flat metric, the linearization L of the scalar curvature operator is given by $L(h) = -\Delta(\operatorname{tr} h) + \operatorname{div}(\operatorname{div}(h))$. The Euclidean metric is a critical point for the ADM mass function, in an appropriate space of solutions to the Einstein constraints ([2], [3], [10]). We say that a solution h of $L(h) = 0$ is *nondegenerate* if the second variation of the mass in the direction of h is positive.

Theorem 1. *Let h be any smooth, compactly supported, symmetric (0,2)-tensor on $\mathbb{R}^3$ with $L(h) = 0$, and for sufficiently small ϵ, let $g_\epsilon = u_\epsilon^4(\delta + \epsilon h)$ be asymptotically flat with zero scalar curvature. If h is nondegenerate, there is an $R_0 > 0$ so that for all ϵ small enough, there is a metric $\bar{g}_\epsilon$ of zero scalar curvature which agrees with g_ϵ in $\{x : |x| \leq R_0\}$ and is exactly Schwarzschild outside $\{x : |x| \geq 2R_0\}$, and so that the maximal Ricci-flat spacetime with the three-geometry $\bar{g}_\epsilon$ as a totally geodesic Cauchy surface admits a smooth conformal compactification. Moreover the path $\bar{g}_\epsilon$ is tangent to h at $\epsilon = 0$.*

We remark that one can approximate any solution h (in an appropriate weighted function space) of the linearized constraint $L(h) = 0$ by a compactly supported solution [14]. Note that a TT-tensor (trace-free and divergence-free) with respect to the flat metric is in the kernel of L and is nondegenerate [9]. It is known that there is an infinite-dimensional space of compactly supported TT-tensors at the flat metric ([4], [17], [14]). We thus have as a corollary that there exists an infinite-dimensional family of solutions of the vacuum constraint equations whose evolution admits a Penrose compactification; this echoes and augments the result of Chruściel and Delay [11], who construct an infinite-dimensional family of such solutions which are parity-symmetric. We note that all of these constructed examples (including the Cutler-Wald examples) are Schwarzschild in a neighborhood of spatial infinity, which is consistent with the known restrictions at space-like infinity for asymptotically simple spacetimes as given by Friedrich [19] and Valiente Kroon [29].

2. Asymptotically Flat and Scalar-Flat Metrics on $\mathbb{R}^3$ with Multiple Horizons

We consider asymptotically flat initial data for the time-symmetric vacuum field equations, given by an asymptotically flat three-manifold (M, g) with zero scalar curvature. From Meeks, Simon and Yau [24], if M has nontrivial topology, then (M, g) has a stable minimal sphere (horizon). The natural question then is how to construct horizons on $(\mathbb{R}^3, g)$, where the topology is trivial. The first existence result was obtained by Beig and Ó Murchadha [5] by conformally rescaling critical sequences of metrics for the conformal Laplacian on $\mathbb{S}^3$. There is related work of Yan [30] which gives metric criteria on $(\mathbb{S}^3, g)$ to guarantee the existence of minimal spheres in the conformal rescaling $G^4 g$ on $\mathbb{S}^3 \setminus \{P\}$, where G is the Green's function at P of the conformal Laplacian. Further existence results have been obtained by Shi and Tam [28], [27]. A construction due to Miao [25] produces examples by first filling in the Schwarzschild metric to produce a metric on $\mathbb{R}^3$ with nonnegative scalar curvature, and then using two types of scalar curvature deformation (one local and one conformal) to deform the metric to zero scalar curvature so that the horizon persists. One may apply Miao's construction to the multi-horizon data constructed by Chruściel-Delay [11] (which has nontrivial topology) to prove the following theorem from [13].

Theorem 2. *There exist asymptotically flat metrics on* $\mathbb{R}^3$ *with zero scalar curvature and multiple minimal spheres.*

The proof uses several methods of deforming the scalar curvature on a manifold: the conformal method, as well as two localized methods, one due to Lohkamp [23], and the other due to us [12].

3. On Isoperimetric Surfaces in General Relativity

One of the major recent developments in mathematical relativity is the resolution of the Riemannian case of the Penrose conjecture, by Huisken-Ilmanen [22] and Bray [7]. Bray had obtained earlier partial results in his thesis [6] by using isoperimetric surface techniques. Bray established that the isoperimetric profile of the time-symmetric Schwarzschild initial data (of positive mass) is given by the radially symmetric spheres (*i.e.* these spheres are the surfaces homologous to the horizon which minimize area for net volume against the horizon), the method of proof of which has been codified in Bray-Morgan [8]. The main idea is that one can deduce the isoperimetric profile of a given metric if one can construct an appropriate map to a model space (for instance Euclidean space or hyperbolic space) in which the profile is known. We use the method to deduce the isoperimetric profile for the time-symmetric Reissner-Nordström and Schwarzschild-Anti-deSitter initial data [15], which in each case is again given by the the radially symmetric spheres. In contrast, in the negative mass Schwarzschild, the radially symmetric spheres are unstable. For recently announced work by Huisken which explores the relation between isoperimetric inequalities and the mass of asymptotically flat metrics, see [20], [21].

References

[1] Anderson, M., Chruściel, P. T.: Asymptotically simple solutions of the vacuum Einstein equations in even dimensions. *Preprint*: ArXiv:gr-qc/0412020 (2004)

[2] Bartnik, R.: The mass of an asymptotically flat manifold. Comm. Pure Appl. Math. **34** pp. 661-693 (1986)

[3] Bartnik, R.: Phase space for the Einstein equation. *Preprint*: ArXiv:gr-qc/0402070 (2004)
[4] Beig, R.: TT tensors and conformally flat structures on three-manifolds. *Mathematics of Gravitation, Part I* (Warsaw, 1997). Banach Ctr. Publ. **41**, Part I, Polish Acad. Sci., pp. 109-118 (1997)
[5] Beig, R., Ó Murchadha, N.: Trapped surfaces due to concentration of gravitational radiation. Phys. Rev. Lett. **66**, no. 19, 2421-2424 (1991)
[6] Bray, H.L.: *The Penrose Inequality in General Relativity and Volume Comparison Theorems Involving Scalar Curvature.* Thesis. Stanford University: 1997
[7] Bray, H.L.: Proof of the Riemannian Penrose Inequality Using the Positive Mass Theorem. J. Diff. Geom. **59**, no. 2, 177-267 (2001)
[8] Bray, H.L., Morgan, F.: An isoperimetric comparison theorem for Schwarzschild space and other manifolds. Proc. Amer. Math. Soc. **130**, no.5, 1467-1472 (2002)
[9] Brill, D., Deser, S.: Variational methods and positive energy in general relativity. Ann. Phys. **50**, 548-570 (1968)
[10] Choquet-Bruhat, Y., Marsden, J.: Solution of the local mass problem in general relativity. Comm. Math. Phys. **51**, 283-296 (1976)
[11] Chruściel, P., Delay, E.: Existence of non-trivial, vacuum, asymptotically simple space-times. Classical Quantum Gravity. **19**, L71-79 (2002)
[12] Corvino, J.: Scalar Curvature Deformation and a Gluing Construction for the Einstein Constraint Equations. Comm. Math. Phys. **214**, 137-189 (2000)
[13] Corvino, J.: A note on asymptotically flat metrics on $\mathbb{R}^3$ which are scalar-flat and admit minimal spheres. Proc. Amer. Math. Soc. **133**, no. 12, 3669-3678 (2005)
[14] Corvino, J.: On the Existence and Stability of the Penrose Compactification. *In Preparation.*
[15] Corvino, J., Gerek, A., Greenberg, M., Krummel, B.: On isoperimetric surfaces in general relativity. *Submitted*
[16] Cutler, C., Wald, R.: Existence of radiating Einstein-Maxwell solutions which are C^∞ on all of $\mathcal{J}^+$ and $\mathcal{J}^-$. Classical Quantum Gravity **6** 453-466 (1989)
[17] Dain, S., Friedrich, H.: Asymptotically flat initial data with prescribed regularity at infinity. Comm. Math. Phys. **222**, 569-609 (2001)
[18] Friedrich, H.: On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. Comm. Math. Phys. **107** 587-609 (1986)
[19] Friedrich, H.: Gravitational fields near space-like and null infinity. J. Geom. Phys. **24**, no. 2, 83-163 (1998)
[20] Huisken, G.: Lecture on isoperimetric inequalities and the ADM mass of asymptotically flat manifolds. Conference on the Einstein Constraint Equations, Isaac Newton Institute, December 2005. http://www.newton.cam.ac.uk/webseminars/pg+ws/2005/gmr/gmrw03/1212/huisken/
[21] Huisken, G.: Lecture on isoperimetric inequalities and the ADM mass of asymptotically flat manifolds. Oberwolfach Meeting, January 2006. This volume, p. 87.
[22] Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose Inequality. J. Diff. Geom. **59**, no. 3, 353-437 (2001)
[23] Lohkamp, J.: Scalar curvature and hammocks. Math. Ann. **313**, no. 3, 385-407 (1999)
[24] Meeks, W. III, Simon, L., Yau, S.-T.: Embedded Minimal Surfaces, Exotic Spheres, and Manifolds with Positive Ricci Curvature. Ann. of Math. **116** 621-659 (1982)
[25] Miao, P.: Asymptotically Flat and Scalar Flat Metrics on $\mathbb{R}^3$ Admitting a Horizon. Proc. Amer. Math. Soc. **132**, no. 1, 217-222 (2004)
[26] Penrose, R.: Asymptotic properties of fields and space-times. Phys. Rev. Lett. **10** 66-68 (1963)
[27] Shi, Y.: Lecture on quasi-local mass and the existence of horizons. Oberwolfach, January 2006. This volume, p. 88.
[28] Shi, Y., Tam, L.-F.: Quasi-local mass and the existence of horizons. *Preprint*: ArXiv: math.DG/0511398

[29] Valiente Kroon, J. A.: A new class of obstructions to the smoothness of null infinity. Comm. Math. Phys. **244**, 133-156 (2004)
[30] Yan, Y.: The existence of horizons in an asymptotically flat 3-manifold. Math. Res. Lett. **12**, no. 2-3, 219-230 (2005)

An isoperimetric concept for mass and quasilocal mass

Gerhard Huisken

For a complete Riemannian 3-manifold (M^3, g) with asymptotically flat end $(\bar{M}, g) \subset (M^3, g)$, $\bar{M} \simeq \mathbb{R}^3 \setminus B_1(0)$ with $g \in C^2(M^3)$, $|g - \delta| \leq c/r$ the classical ADM-mass is a flux integral at infinity

$$m_{\mathrm{ADM}} = \lim_{R \to \infty} \frac{1}{16\pi} \int_{\partial B_R} (g_{ij,i} - g_{ii,j}) d\nu^j$$

which is known to be geometric invariant when assuming appropriate decay assumptions for the first and second derivatives of the metric. The notion of mass is motivated from General Relativity, where (M^3, g) arises as a 3-dimensional spacelike hypersurface of a Lorentzian 4-manifold modelling an isolated gravitating system such as a star, a black hole or a galaxy. In this setting the mass represents the total energy of the isolated system including the contributions of the gravitational field. Einsteins field equations together with a natural energy condition from physics for the matter fields leads to the consideration of metrics g with nonnegative scalar curvature $R \geq 0$.

The classical positive mass theorem first proven by Schoen and Yau then states that for asymptotically flat 3-manifolds (M^3, g) with nonnegative scalar curvature the ADM-mass of each end is nonnegative with equality holding only on Euclidean space.

The current lecture proposes to interpret the mass as an asymptotic isoperimetric defect, namely we define

$$m_{\mathrm{ISO}} = \limsup_{R \to \infty} \frac{2}{|\partial B_R|} \left(\mathrm{Vol}(B_R) - \frac{1}{6\sqrt{\pi}} |\partial B_R|^{\frac{3}{2}} \right)$$

The isoperimetric mass of three dimensional flat space is zero in view of the isoperimetric inequality. Using both mean curvature flow and inverse mean curvature flow we show in a first step that the new concept is consistent with the classical ADM-mass when it is defined and satisfies the positive mass theorem on manifolds with nonnegative scalar curvature: $0 \leq m_{\mathrm{ISO}} \leq m_{\mathrm{ADM}}$.

Since the new concept of isoperimetric mass only needs a C^0-metric and since the condition of nonnegative scalar curvature can also be interpreted in terms of (local) isoperimetric defects for such metrics, we will ultimately prove a C^0-version of the positive mass theorem: A Riemannian 3-manifold with locally nonnegative isoperimetric defect has nonnegative isoperimetric total mass. In this setting the isoperimetric inequality becomes the natural analogue for the mean value inequality satisfied by subharmonic functions in the Newtonian theory.

We also propose a new isoperimetric definition of quasilocal mass for a 2-surface $\Sigma^2 \subset (M^3, g)$ and domains $\Omega \subset (M^3, g)$ based on the isoperimetric profile of the spatial Schwarzschild manifold (M^3, g_m), $g_m = (1 + m/2r)^4\delta$. It turns out that the new definition of quasilocal mass has all desired properties of such a quantity such as monotonicity and positivity properties. In the smooth case the isoperimetric quasilocal mass will be attained on constant mean curvature surfaces giving a natural link to the center of mass definition of Huisken and Yau [2] via cmc-foliations. It should be noted that H.Bray and F. Morgan [1] proved the isoperimetric minimizing property of the concentric cmc-slices in the spatial Schwarzschild metric.

References

[1] H. Bray and F. Morgan, *An isoperimetric comparison theorem for Schwarzschild space and other manifolds*, Proc. Amer. Math. Soc. **130** (2002), 1467–1472.

[2] G. Huisken and S. T Yau, *Definition of the Center of Mass for Isolated Physical Systems and Unique Foliations by Stable Spheres of Constant Mean Curvature*, Invent. Math. **124**, No. 1–3 (1996), 281–311.

[3] G. Huisken and T. Ilmanen, *The Riemannian Penrose Inequality*, Int. Math. Res. Notices **20** (1997), 1045–1058.

Quasi-local mass and the existence of horizons

Yuguang Shi

(joint work with Luen-Fai Tam)

In 1972, Thorne made the following conjecture, which later became known as the hoop conjecture (see [2]):

Conjecture 1. *Black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $\mathcal{C} \leq 4\pi M$.*

The conjecture is loosely formulated . Several concepts such as mass, circumference etc. are not clearly defined. Hence, this conjecture allows many different precise interpretations. In 1983, Schoen and Yau in [5], define a kind of radius of a bounded region, and derive an upper bound for this radius for a region in spacetime without apparent horizons in terms of the lower bound of mass density. By studying Jang equation, they obtained some important results for the existence of horizons in the spirit of the hoop conjecture. However, Schoen-Yau's arguments cannot be used in time symmetric case, since in this case, Jang equation has obvious solution.

In this talk, we will focus on the time symmetric case, in this case, a horizon is a kind of minimal surface. We always assume (Ω, g) is a 3-dimensional compact Riemannian manifold with smooth boundary and nonnegative scalar curvature. Besides this, we also assume the Gaussian curvature and the mean curvature with respect to outward unit norm of the boundary is positive.

Known Result (Meeks-Yau). *Let Ω be as above, and it is not diffeomorphism with $\mathbb{B}^3$, then it contains a minimal surface; in addition, if Ω is simply connected, then the minimal surface is the boundary some subdomain of Ω.*

So, in the following, we assume Ω is diffeomorphism with $\mathbb{B}^3$. We want to use the following quasi-local mass to investigate the problem.

- Hawking mass: $\mathcal{M}_{\mathrm{H}}(\partial\Omega) = \sqrt{\frac{|\partial\Omega|}{16\pi}}(1 - \frac{1}{16\pi}\int_{\partial\Omega} H^2)$, H is the mean curvature of $\partial\Omega$ with respect to outward norm. $|\partial\Omega|$ is area of $\partial\Omega$.
- Brown-York mass: $\mathcal{M}_{\mathrm{BY}}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega}(H_0 - H)d\sigma$, here H_0 is the mean curvature of isometric embedding image of $\partial\Omega$ in $\mathbb{R}^3$.

It is believed that the lower bound estimate of these quasilocal mass relates to the existence of horizon, for instance, see Walter Simon's observation, (see [1]). By the above expression, we see that quasi-local mass only depends the geometry of $\partial\Omega$ and the mean curvature. However, we have:

Proposition 2. *There exist asymptotically flat metrics g_1, g_2 with nonnegative scalar curvature on $\mathbb{R}^3$ such that $g_1 = g_2$ outside some compact set, $(\mathbb{R}^3, g_1)$ contains a stable minimal sphere but $(\mathbb{R}^3, g_2)$ does not contain any compact minimal surfaces.*

By Proposition 2, we may construct two domains with the same boundary geometry and mean curvature, while one contains a horizon the other does not. From the proposition, in order to find a sufficient condition for the existence of compact minimal surfaces, we need to know information in the interior of the domain. This motivates us to introduce the following quantity using Hawking mass of some compact surfaces inside a domain.

Let $\Omega_1 \subset\subset \Omega_2 \subset \Omega$ such that Ω_1 and Ω_2 have smooth boundaries. We need the following lemma from [4]

Lemma 3 (Meeks-Yau). *With the above notations and let d be the distance between Ω_1 and $\partial\Omega_2$. Let ι be the infimum of the injectivity radius of points in $\{x|\ d(x, \partial\Omega_2) > \frac{d}{4}\}$. Let $K > 0$ be the upper bound of the curvature of Ω_2. Suppose N is a minimal surface and $x \in N$ with $d(x, \partial\Omega_2) = \frac{d}{2}$, so that $d(x, \partial N) \geq \frac{d}{2}$, then*

$$|N \cap B_x(r)| \geq CK^{-2}\int_0^r \tau^{-1}(\sin K\tau)^2 d\tau \tag{1}$$

where $r = \min\{\frac{d}{2}, \iota\}$. Here C is a positive absolute constant.

For such Ω_1, Ω_2, let

$$\alpha^2_{\Omega_1;\Omega_2} = \min\left\{\frac{CK^{-2}\int_0^r \tau^{-1}(\sin K\tau)^2 d\tau}{|\partial\Omega_1|}, 1\right\} \tag{2}$$

Let $\mathcal{F}_{\Omega_2}$ be the family of precompact connected minimizing hulls with C^2 boundary in Ω_2, for the definition of minimizing hull please see [3]. Define

$$m(\Omega_1;\Omega_2) = \sup_{E\in\mathcal{F}_{\Omega_2}, E\subset\Omega_1} m_{\mathrm{H}}(E). \tag{3}$$

Define

$$m(\Omega) = \sup \alpha_{\Omega_1;\Omega_2} m(\Omega_1;\Omega_2) \tag{4}$$

where the supremum is taken over all $\Omega_1 \subset\subset \Omega_2 \subset \Omega$ with smooth boundaries. Here and below Ω_i is always assumed to be nonempty.

In general, the Hawking mass of a compact surface may be negative. However, one can prove that $m(\Omega) \geq 0$, and for manifolds with nonnegative scalar curvature $m(\Omega) = 0$ implies local flatness of manifolds.

Now, we are able to give some lower bounds of the Brown-York mass in terms $m(\Omega)$. By this, we give a sufficient condition on existence of horizons in Ω.

Theorem 4. *Let (Ω, g) be a compact three manifold with connected smooth boundary and with nonnegative scalar curvature. Assume that Ω is simply connected and suppose $\partial\Omega$ has positive Gauss curvature and positive mean curvature with respect to the outward normal. Then*

$$m_{\mathrm{BY}}(\partial\Omega) \geq m_{\mathrm{H}}(\partial E). \tag{5}$$

for any connected minimizing hull E in Ω where $E \subset\subset \Omega$ with $C^{1,1}$ boundary. Moreover, equality holds for some minimizing hull E with the above properties if and only if Ω is a standard ball in $\mathbb{R}^3$ and E is a standard ball in Ω. In particular, $m_{\mathrm{BY}}(\partial\Omega) \geq m_{\mathrm{H}}(\partial\Omega)$ and equality holds if and only if Ω is a standard ball in $\mathbb{R}^3$.

In the above theorem, we do not assume that (Ω, g) contains no horizons. In order to obtain a sufficient condition that (Ω, g) contains a horizon, we need another estimate of the Brown-York mass. Let $m(\Omega)$ as defined in (4). We have the following:

Theorem 5. *Let (Ω, g) be a compact manifold with connected smooth boundary and with nonnegative scalar curvature. Assume that Ω is simply connected and suppose $\partial\Omega$ has positive Gauss curvature and positive mean curvature with respect to the outward normal. Suppose (Ω, g) has no horizons. Then $m(\Omega) \leq m_{\mathrm{BY}}(\partial\Omega)$. Equality holds if and only if Ω is a domain in $\mathbb{R}^3$.*

Remark 6. *The similar lower bounded estimate is also true for Bartnik mass.*

Let Ω be as in the theorem. Isometrically embed $\partial\Omega$ in $\mathbb{R}^3$. Let R be the radius of the smallest circumscribed ball of $\partial\Omega$ in $\mathbb{R}^3$. We have the following:

Corollary 7. *Let (Ω, g) be a compact manifold with nonnegative scalar curvature and with connected boundary which has positive mean curvature and positive Gauss curvature. Suppose Ω is simply connected and suppose $m(\Omega) \geq m_{\mathrm{BY}}(\partial\Omega)$, then there is a horizon in Ω unless Ω is a domain in $\mathbb{R}^3$. Hence if $m(\Omega) \geq 2R$, then Ω contains a horizon. In particular, if $m(\Omega) \geq 2\,\mathrm{diam}(\partial\Omega)$ then Ω contains a horizon. Here $\mathrm{diam}(\partial\Omega)$ is the diameter of $\partial\Omega$ with respect to the metric induced by g.*

There are examples so that $m(\Omega) > m_{\mathrm{BY}}(\Omega)$. Consider the metrics $(\mathbb{R}^3, ds_m^2 = u_m^4(d\rho^2 + \rho^2 d\sigma^2))$ defined in the proof of Proposition 2, where $d\rho^2 + \rho^2 d\sigma^2$ is the Euclidean metric. Since as $m \to 0$, ds_m^2 converges uniformly on the Euclidean ball

$B_4 = \{x \in \mathbb{R}^3 | \, |x| < 4\}$ to the standard metric on the unit sphere, we see that for sufficiently small $m > 0$, the boundary of B_τ is mean convex for all $0 < \tau < 2$ with respect to ds_m^2. B_1 is a minimizing hull in B_2. Moreover, there is $\delta > 0$ such that the Hawking mass $m_{\rm H}(\partial B_1)$ of ∂B_1 with respect to ds_m^2 is at least δ, provided $m > 0$ is small enough. Hence for all $m > 0$ small enough, $\alpha_{\Omega_1;\Omega_2} m(B_1; B_2) \geq \delta$. Now consider the domain $\Omega = B_\rho$, where $\rho = 8/m > 2\rho_0$. It is easy to see that $\partial\Omega$ has positive mean curvature and positive Gauss curvature with diameter $d \leq Cm$, here C is a universal constant. Hence

$$m(\Omega) \geq \alpha_{\Omega_1;\Omega_2} m(B_1; B_2) \geq \delta > 2d \geq m_{\rm BY}(\Omega)$$

provided m is small enough.

References

[1] Bartnik, R., *Mass and 3-metrics of non-negative scalar curvature*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 231–240, Higher Ed. Press, Beijing, 2002.

[2] Flanagan, E., *Hoop conjecture for black-hole horizon formation*, Phys. Rev.D 44, 2409(1991).

[3] Huisken, G. and Ilmanen, T., *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), 353–437.

[4] Meeks, W. H., and Yau, S.-T., *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. **112** (1980), 441–484.

[5] R. Schoen and S.-T. Yau *The existence of a black hole due to condensation of matter* Comm. Math. Phys. **90** (1993), 575-579.

[6] Shi, Y.-G., Tam, L.-F., *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*,. J. Differential Geom. **62** (2002), 79–125.

Conformal gauge singularities

Paul Tod

(joint work with Christian Luebbe)

A *conformal gauge singularity* is a space-time singularity at which the metric g_{ab} is singular but the conformal equivalence class $[g_{ab}]$ of the metric is not. Thus the singularity may be seen as arising simply from the choice of representative metric in the conformal class, in other words from the choice of conformal gauge.

The interest in these arises from a wish to study cosmological models in General Relativity with singularities at which the Weyl tensor is finite. This interest in turn is generated by Penrose's *Weyl Curvature Hypothesis* [1], [2]. Penrose argues that initial singularities, notably the Big Bang, are different in character from final singularities, such as those arising in gravitational collapse. This difference is, he says, *'something like: the Weyl tensor C_{abcd} vanishes at any initial singularity'* ([1]).

According to Penrose, this restriction on initial singularities is enforced by physical laws, but without needing to know what these are one can simply seek to study the resulting class of space-times. Two questions present themselves: first, can one produce examples of cosmological models with an initial singularity at which the

Weyl tensor is finite or zero? and second, how may one recognise a space-time singularity at which the Riemann tensor is singular while the Weyl tensor is not, and is the singularity then a conformal gauge singularity?

The first question can be answered positively after the following observation: if the metric is conformally-rescaled according to $\tilde{g}_{ab} = \Omega^2 g_{ab}$ then the Weyl tensor transforms as $\tilde{C}_{abc}{}^{d} = C_{abc}{}^{d}$. Thus suppose we have a manifold with metric (M, g); take a smooth space-like surface Σ in M and a function Ω which vanishes at Σ, not necessarily smoothly, and is positive and smooth on $\tilde{M}$, the part of M to the future of Σ. Now rescale the metric as above, then $(\tilde{M}, \tilde{g})$ is a space-time with a curvature singularity at Σ at which the Weyl tensor is finite (since it is equal to the Weyl tensor in M). In the terminology of the start of this abstract, this is a conformal gauge singularity.

It is known that, for cosmological models whose source is a perfect fluid with a certain equation of state [3] or massless collisionless matter [4], there is a well-posed initial value problem with data at the singularity surface Σ. The proofs rely on an existence theorem of Claudel-Newman [5]. Thus these are cosmologies with conformal gauge singularity.

The second, or converse question is to ask whether, or when, cosmological singularities with finite Weyl tensor are conformal gauge singularities. First we need a definition of *finite Weyl tensor* at a singularity. For this we invoke two conformally-invariant notions: conformal geodesics, and conformal propagation along a conformal geodesic (see e.g. [6]). These depend only on the conformal class of the metric, and we can say that Weyl tensor is finite at a curvature singularity if its components are bounded in a conformally-propagated frame along a conformal geodesic going into the singularity. Next we would like a result connecting the existence of a conformal extension through a space-time singularity with conditions on the Weyl tensor framed in this way.

For this, we first recall the extension theorem of Racz [7]: Given an incomplete causal geodesic γ in a strongly causal space-time, and boundedness conditions on the components of the Riemann tensor in a parallelly-propagated frame along γ, Racz shows the existence of a regular coordinate system on a neighbourhood U of a final segment of γ. Then given bounds on the Riemann curvature and its derivatives (to say order $k+1$) in this coordinate system, he shows that there is a C^k-extension of the metric to an open V containing U. With an incomplete conformal geodesic and conditions on conformal curvatures (in fact the tractor curvatures of [8]) we can parallel this theorem to construct similar coordinate systems. At the time of writing it appears that we can then find a conformal factor so that the conditions of Racz's theorem are satisfied and thus conclude that there is a conformal extension. Thus, with these conditions, the singularity is a conformal gauge singularity.

This is a local extension theorem. To find a global extension theorem, we consider spatially-homogeneous perfect-fluid space-times with an initial singularity. It turns out that the matter flow lines are actually conformal geodesics (as well as metric geodesics). If we impose the conditions

- There is a choice of projective parameter on these conformal geodesics so that the singularity is at a finite distance.
- The conformal factor dictated by this choice makes the rescaled fluid expansion finite.
- The conformal (tractor) curvatures are finite in the conformally-propagated frame.

Then the initial singularity is a conformal gauge singularity.

References

[1] R. Penrose, Singularities and time-asymmetry, in *General Relativity and Gravitation: an Einstein centennial volume* ed. by W. Israel and S.W. Hawking (CUP, Cambridge 1979)

[2] R. Penrose, *The Emperor's New Mind* (OUP, Oxford 1989)

[3] K. Anguige and K.P. Tod, *Isotropic cosmological singularities: 1 polytropic perfect fluid space-times; 2 the Einstein Vlasov system* Ann.Phys.(NY) **276** (1999) 257–293 and 294–320 gr-qc/9903008 and gr-qc/9903009

[4] K. Anguige, *Isotropic cosmological singularities 3: the Cauchy problem for the inhomogeneous conformal Einstein-Vlasov equations* Ann.Phys.NY **282** (2000) 395–419 gr-qc/9903018

[5] C.M.Claudel and K.P.Newman, *The Cauchy problem for quasi-linear hyperbolic evolution equations with a singularity in the time* Proc.Roy.Soc.Lond. **A 454** (1998) 1073–1107

[6] H.E.Friedrich and B.G.Schmidt *Conformal geodesics in general relativity.* Proc. Roy. Soc. London **A 414** (1987) 171–195.

[7] I.Racz, *Space-time extensions. I* J. Math. Phys. **34** (1993) 2448–2464.

[8] T.N.Bailey, M.G.Eastwood and A.R.Gover, *Thomas's structure bundle for conformal, projective and related structures* Rocky Mountain J. Math. **24** (1994) 1191–1217.

A new generalized harmonic evolution system

Lee Lindblom

(joint work with M. A. Scheel, L. E. Kidder, R. Owen, and O. Rinne)

This report describes recent work on finding a formulation of the Einstein equations suitable for constructing stable numerical evolutions. The formulation studied here specifies the coordinate degrees of freedom with a generalized harmonic gauge source function rather than with the usual lapse and shift. This type of formulation appears to have played a critical role in the very impressive binary black hole evolutions performed recently by Pretorius [1, 2]. This report analyzes why this type of formulation is so effective for numerical work, describes a recent extension of the system that makes it possible to construct boundary conditions which prevent the influx of constraint violations, and describes numerical tests that demonstrate the effectiveness of the new equations and boundary conditions.

The gauge source function H_a is defined as the action of the scalar-wave operator on the coordinate functions x^a:

$$H_a(x) \equiv \psi_{ab}\nabla^c\nabla_a x^b = -\psi^{bc}\Gamma_{abc} \equiv -\Gamma_a, \qquad (1)$$

where ψ_{ab} is the spacetime metric and Γ_{abc} is the usual Christoffel symbol. The coordinates are fixed in this approach by requiring that $\Gamma_a = -H_a$, for a prescribed

H_a. The existence of solutions to the inhomogeneous wave Eq. (1) guarantees the existence of such coordinates. Choosing the coordinates in this way has two important consequences. The first is well known: the vacuum Einstein equations,

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \tag{2}$$

where $\mathcal{C}_a = H_a + \Gamma_a$, are manifestly hyperbolic since the principal part of the equations is just $\psi^{cd}\partial_c\partial_d\psi_{ab}$ for any value of the gauge source function [3]. The second consequence is less widely appreciated: The constraints of the system are profoundly transformed. The condition $\mathcal{C}_a = 0$ is the primary constraint of this system, while the standard Hamiltonian and momentum constraints $\mathcal{M}_a = G_{ab}t^b$ (where t^a is the unit normal to a Cauchy surface) are determined by the derivatives of $\mathcal{C}_a$: $\mathcal{M}_a = t^b(\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c)$. This means that the primary constraints depend on the first but not the second derivatives of the metric.

Adding multiples of the constraints to the Einstein equations is known to have a significant effect on the growth rates of constraint-violating solutions [4]. However, multiples of the Hamiltonian and momentum constraints can be added only in very restricted ways consistent with the hyperbolic structure of the equations; this is because the addition of these constraints changes the principal part of the equations. In contrast, arbitrary multiples of the gauge constraint $\mathcal{C}_a$ can be added to the system, Eq. (2), without effecting the hyperbolic structure at all. Pretorius [2], based on the suggestion of Gundlach, et al. [5], used a modified evolution system that included the following additional gauge constraint terms designed to suppress the growth of the constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\big[t_{(a}\mathcal{C}_{\,b)} - \frac{1}{2}\psi_{ab}\,t^c\mathcal{C}_c\big]. \tag{3}$$

The Bianchi identities then imply that $\mathcal{C}_a$ satisfies the damped wave equation,

$$0 \;=\; \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^b\big[\,t_{(b}\mathcal{C}_{a)}\big] + \mathcal{C}^{\,b}\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\gamma_0\,t_a\mathcal{C}^{\,b}\mathcal{C}_b, \tag{4}$$

which exponentially suppresses all small short-wavelength constraint violations when the parameter γ_0 is positive [5]. This constraint-suppressing feature of the modified generalized harmonic system, Eq. (3), contributed significantly to the success of Pretorius' impressive binary black-hole evolutions [2].

We have recently extended the modified generalized harmonic evolution system, Eq. (3), to a first-order symmetric-hyperbolic form. (See Ref. [6] for the details.) This new system is linearly degenerate, so shocks do not form from smooth initial data. This system also includes new constraints that arise when additional fields are added to make the system first order. Appropriate terms (proportional to the constraints times a second constraint-damping parameter γ_2) are added to suppress the growth of these new constraints. Constraint-preserving and physical boundary conditions are also presented, and the well-posedness of the new evolution system with these boundary conditions is analyzed.

We tested the new evolution system by evolving initial data for a Schwarzschild black hole. In these evolutions we "freeze" the values of the incoming characteristic fields on the boundaries. We performed these numerical evolutions using spectral

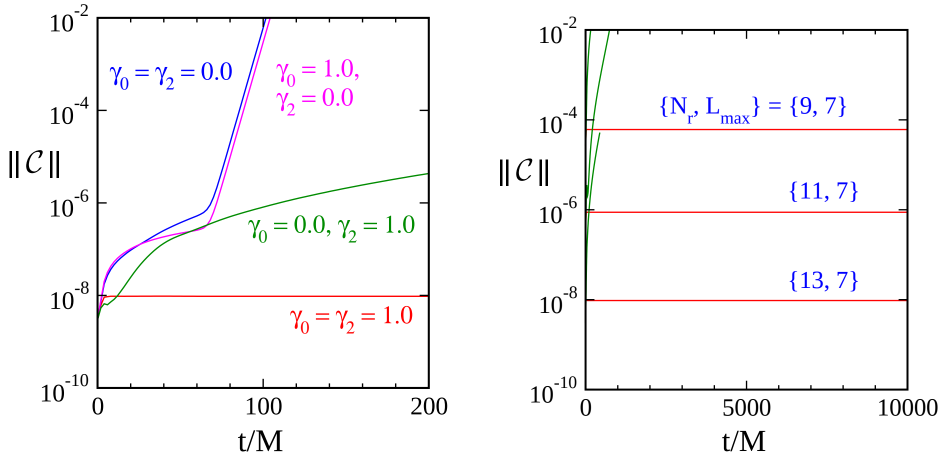

FIGURE 1. Evolution of Schwarzschild initial data.

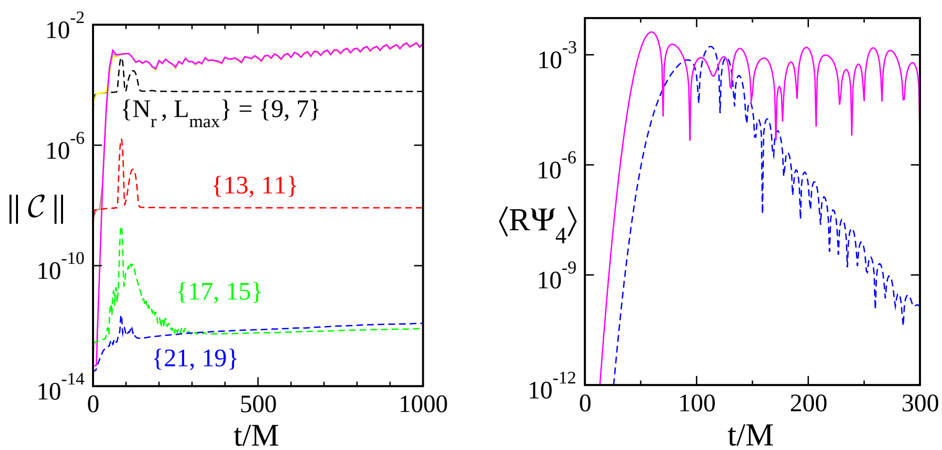

FIGURE 2. Evolution of perturbed Schwarzschild.

methods as described in Ref. [7] for a range of numerical resolutions specified by N_r (the highest order radial basis function) and $L_{\max}$ (the highest order spherical-harmonic). Figure 1 shows the time dependence of the constraint norm $||\mathcal{C}||$ for several values of the constraint-damping parameters γ_0 and γ_2. These tests show that without constraint damping the extended evolution system is extremely unstable, but with constraint damping the evolutions of the Schwarzschild spacetime are completely stable up to $t = 10{,}000M$ (and forever, we presume). These tests also illustrate that both the γ_0 and the γ_2 constraint damping terms are essential.

We also tested our new boundary conditions by evolving a black hole perturbed by an incoming gravitational wave (GW) pulse. We perturb Schwarzschild initial data by injecting a GW pulse through the boundary with time profile $f(t) = \mathcal{A}\, e^{-(t-t_p)^2/w^2}$ and $\mathcal{A} = 10^{-3}$, $t_p = 60M$, and $w = 10M$. Figure 2 shows the evolution of $||\mathcal{C}||$ for both constraint-preserving boundary conditions (dashed curves) and simple boundary conditions that freeze all the incoming characteristic fields (solid curves). These results illustrate that the new boundary conditions indeed prevent the influx of constraint violations. Figure 2 also illustrates the time dependence of the Weyl tensor component Ψ_4 averaged over the outer boundary of the computational domain. The dashed curve (using constraint-preserving boundary conditions) shows black-hole quasi-normal oscillations with the correct complex frequency, while the solid curve (using freezing boundary conditions) is completely unphysical. These results show that proper constraint-preserving boundary conditions are essential if accurate gravitational waveforms are needed.

This work was supported in part by a grant from the Sherman Fairchild Foundation to Caltech and Cornell, by NSF grants PHY-0099568, PHY-0244906 and NASA grants NAG5-10707, NAG5-12834 at Caltech, and by NSF grants PHY-0312072, PHY-0354631, and NASA grant NNG05GG51G at Cornell.

References

[1] F. Pretorius. Numerical relativity using a generalized harmonic decomposition. *Class. Quantum Grav.*, 22:425–452, 2005.
[2] F. Pretorius. Evolution of binary black hole spacetimes. *Phys. Rev. Lett.*, 95:121101, 2005.
[3] H. Friedrich. On the hyperbolicity of Einstein's and other gauge field equations. *Commun. Math. Phys.*, 100:525–543, 1985.
[4] M. A. Scheel, L. E. Kidder, L. Lindblom, H. Pfeiffer, and S. A. Teukolsky, Toward stable 3D numerical evolutions of black-hole spacetimes. *Phys. Rev. D*, 66:124005, 2002.
[5] C. Gundlach, G. Calabrese, I. Hinder, and J. M. Martin-Garcia. Constraint damping in the Z4 formulation and harmonic gauge. *Class. Quantum Grav.*, 22:3767–3774, 2005.
[6] L. Lindblom, M. A. Scheel, L. E. Kidder, R. Owen, and O. Rinne, A New Generalized Harmonic Evolution System. *Class. Quantum Grav.*, submitted, 2006; gr-qc/0512093.
[7] L. E. Kidder, L. Lindblom, M. A. Scheel, L. T. Buchman, and H. P. Pfeiffer. Boundary conditions for the Einstein evolution system. *Phys. Rev. D*, 71:064020, 2005.

The redshift effect and decay rates for the wave equation on a Schwarzschild exterior

Mihalis Dafermos
(joint work with Igor Rodnianski)

We consider the following problem: Let $(\mathcal{M}, g)$ denote the maximally extended Schwarzschild spacetime [7] with parameter $M > 0$. Let $\mathcal{S}$ denote a complete Cauchy surface, and consider locally C^6 solutions of the wave equation

$$\Box_g \phi = 0 \tag{1}$$

on $\mathcal{M}$, such that $\phi|_{\mathcal{S}}$ and $\nabla\phi|_{\mathcal{S}}$ decay sufficiently rapidly at spatial infinity. (We do *not* assume ϕ vanishes at the sphere of bifurcation of the event horizon.) The main result presented in this talk is a set of decay rates for ϕ and its energy flux in the closure of the domain of outer communications $\overline{J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)} \subset \mathcal{M}$. In particular, the decay rates apply along the event horizon $\mathcal{H}^+$.

To state precisely the result, let us introduce some notation: By u and v, we mean standard Eddington-Finkelstein retarded and advanced time coordinates on

$J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)$.[1] The origin of these coordinates can be chosen arbitrarily. Let us set $v_+ = \max\{v, 1\}$, $u_+ = \max\{u, 1\}$. If $\mathcal{S}' \subset \overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)}$ then set $v_+(\mathcal{S}') = \max\{\inf_{\mathcal{S}'} v, 1\}, u_+(\mathcal{S}') = \max\{\inf_{\mathcal{S}'} u, 1\}$. Finally, let $\mathrm{Flux}(\phi, \mathcal{S}')$ denote the flux of the energy of ϕ as measured by the standard static Killing field.

Theorem 1. *Let ϕ be a solution of* (1) *as described above. For any achronal hypersurface $\mathcal{S}' \subset \overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)}$, we have*

$$\mathrm{Flux}(\phi, \mathcal{S}') \leq C((v_+(\mathcal{S}'))^{-2} + (u_+(\mathcal{S}'))^{-2}).$$

(We also allow $\mathcal{S}' \subset \mathcal{I}^+$, interpreted in the obvious limiting sense.)

In addition, we have the pointwise decay rates

$$|\phi| \leq C v_+^{-1}$$

in $\overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)}$, and

$$|r\phi| \leq C_{\hat{R}}(1 + |u|)^{-\frac{1}{2}}$$

in $\{r \geq \hat{R} > 2M\} \cap J^+(\mathcal{S})$.

The result of the above theorem was proven for the 0-th spherical harmonic of ϕ in [3], as a special case of a much more general result concerning the collapse of a self-gravitating scalar field. Partial results in the direction of Theorem 1 are obtained independently by [1]. The uniform boundedness of ϕ is a classical result of Kay and Wald [8].[2] Polynomial decay rates of the form proven here were first heuristically obtained in [9, 6].

The motivation for studying uniform decay rates as in Theorem 1 is discussed at length in [3, 4, 5]. Briefly, these decay rates are related to astrophysical observations of black holes, their internal structure, in particular the nature of apparent horizons and singularities (see [2]), and finally, the possibility of proving non-linear stability for these spacetimes (see the remarks in [3]).

The proof of Theorem 1 is contained in the preprint [5].

References

[1] P. Blue and J. Sterbenz, preprint

[2] M. Dafermos *The interior of charged black holes and the problem of uniqueness in general relativity* Comm. Pure Appl. Math. **58** (2005) 0445–0504

[3] M. Dafermos and I. Rodnianski *A proof of Price's law for the collapse of a self-gravitating scalar field*, Invent. Math. **162** (2005) 381–457

[4] M. Dafermos and I. Rodnianski *Small amplitude non-linear waves on a black hole background*, J. Math. Pures Appl. **84** (2005) 1147–1172

[5] M. Dafermos and I. Rodnianski *The red-shift effect and radiation decay on black hole spacetimes*, gr-qc/0512119

[6] C. Gundlach, R. H. Price, and J. Pullin *Late-time behavior of stellar collapse and explosions. I. Linearized perturbations* Phys. Rev. D **49** (1994), 883–889

[1] Here, $\mathcal{I}^+$, etc., denotes a connected component of *null infinity*, i.e. we are restricting to one of the asymptotically flat ends.

[2] Theorem 1 gives in fact an independent proof of the boundedness theorem of Kay and Wald that does not use inversion of the Laplacian or the discrete isometries of spacetime.

[7] S. Hawking and G. F. R. Ellis *The large scale structure of space-time* Cambridge Monographs on Mathematical Physics, No. 1 Cambridge University Press, London, 1973
[8] B. Kay and R. Wald *Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere* Class. Quantum Grav. **4** (1987) 893–989
[9] R. Price *Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations* Phys. Rev. D (3) **5** (1972), 2419-2438

Singular solutions for the Keller-Segel model of chemotaxis

Juan J. L. Velázquez

The term chemotaxis denotes the tendency of some biological organisms to move in the direction of increasing gradients of a chemical substance. A system of equations that describes such phenomenon by means of a simple system of partial differential equations is the so-called Keller-Segel model. This system describes the concentration of organisms by means of two variables, namely the concentration of the organism n and the concentration of chemoattractant substance c. In the derivation of the Keller-Segel model it is assumed that both the organism and the chemical substance diffuse in a classical fickian manner. It is also assumed that the velocity of the organism towards the region having larger concentrations of chemical is proportional to the gradient of the substance. Under these assumptions the Keller-Segel system becomes, in suitable dimensionless units to (cf. [4]):

$$n_t = \Delta n - \chi \nabla (n \nabla c) \quad , \quad x \in \Omega \subset IR^2 , t > 0 \tag{1}$$

$$c_t = D \Delta c + n - \alpha c \quad , \quad x \in \Omega \subset IR^2 , t > 0 \tag{2}$$

On the other hand, if the diffusivity of the chemical is much larger than the diffusivity of the microorganism, the Keller-Segel model reduces to the following elliptic-parabolic system (cf. [5]):

$$n_t = \Delta n - \chi \nabla (n \nabla c) \quad , \quad x \in \Omega \subset IR^2 , t > 0 \tag{3}$$

$$0 = \Delta c + n - \bar{n} \quad , \quad x \in \Omega \subset IR^2 , t > 0 \tag{4}$$

where $\bar{n}$ is the mean value of n in the domain Ω.

The first rigorous proof of the existence of singularities for chemotaxis models was obtained in [5]. On the other hand, a detailed construction of solutions of the systems (1), (2) or (3), (4) yielding Dirac mass formation in finite time might be found in the references [1], [2], [3].

In my talk I review the singularity formation results contained in these articles. I will describe also some recent results derived using matched asymptotic expansions that show how to continue the solutions of the system (3), (4) beyond the time of formation of the singularity. Such arguments, indicate that the solutions of (3), (4) can be extended as some singular solutions including a regular, bounded part $n_{\text{reg}}(x,t)$ plus a set of Dirac masses. The regular part of the solution and the Dirac mass part solve the following system of equations (cf. [6], [7]):

(5) $$\frac{\partial n_{\text{reg}}}{\partial t} = \Delta n_{\text{reg}} + \sum_{j=1}^{N} \frac{M_j(t)}{2\pi} \frac{(x - x_j(t))}{|x - x_j(t)|^2} \cdot \nabla n_{\text{reg}} - \nabla (n_{\text{reg}} \nabla c_{\text{reg}})$$

(6) $$c_{\text{reg}} = -\frac{1}{2\pi} \int_{IR^2} \log(|x-y|)\, n_{\text{reg}}(y,t)\, dy$$

(7) $$\dot{x}_i(t) = \Gamma(M_i(t))\, A_i(t) \quad , \quad i = 1, ..., N$$

(8) $$A_i(t) = -\sum_{j=1}^{N} \frac{M_j(t)}{2\pi} \frac{(x - x_j(t))}{|x - x_j(t)|^2} + \nabla c_{\text{reg}}(x_i(t), t)$$

(9) $$\frac{dM_i(t)}{dt} = c_{\text{reg}}(x_i(t), t)\, M_i(t) \quad , \quad i = 1, ..., N$$

where

$$n(x,t) = \sum_{i=1}^{N} M_i(t)\, \delta(x - x_i(t)) + n_{\text{reg}}(x,t)$$

The well-posedness of the system (5)-(9) in Hölder spaces has been obtained in [8].

References

[1] M. A. Herrero and J. J. L. Velázquez, Singularity patterns in a chemotaxis model, Math. Ann. 306, 583-623, 1996.

[2] M. A. Herrero and J. J. L. Velázquez, Chemotactic collapse for the Keller-Segel model, J. Math. Biol. 35, 177-194, 1996.

[3] M. A. Herrero and J. J. L. Velázquez, A blow-up mechanism for a chemotaxis model, Annali Scuola Norm. Sup. Pisa, Serie IV, 633-683, 1997.

[4] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol. 26, 399-415, 1970.

[5] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. AMS, 329, 819-824, 1992.

[6] J. J. L. Velázquez, Point dynamics for a singular limit of the Keller-Segel model I: Motion of the concentration regions. SIAM J. Appl. Math. 64, 4, 1198-1223, 2004.

[7] J. J. L. Velázquez, Point dynamics for a singular limit of the Keller-Segel model II: Formation of the concentration regions. SIAM J. Appl. Math. 64, 4, 1224-1248, 2004.

[8] J. J. L. Velázquez, Well posedness of a model of point dynamics for a limit of the Keller-Segel problem. J. Diff. Equations 206, 2, 315-352, 2004.

The Einstein-scalar field constraint equations on compact manifolds

Daniel Pollack

(joint work with Yvonne Choquet-Bruhat and James Isenberg)

This focus of this talk is on the constraint equations for the Einstein-scalar field system on compact manifolds. Using the conformal method we reformulate these equations as a determined system of nonlinear partial differential equations. We contrast the difficulties encountered between analyzing this system and systems obtained from either the vacuum constraint equations of other matter/field models, e.g. the Einstein-Maxwell system. By introducing a new conformal invariant, which is sensitive to the presence of the initial data for the scalar field, we are able to divide the set of free conformal data into subclasses depending on the possible signs for the coefficients of terms in the resulting Einstein-scalar field Lichnerowicz equation. For most of these subclasses we determine whether or not a solution exists. We consider this system in such generality so as to include the vacuum constraint equations with an arbitrary cosmological constant, the Yamabe equation and even (all cases of) the prescribed scalar curvature problem as special cases.

The field variables for an Einstein-scalar field theory consist of a spacetime metric g and a real-valued scalar field Ψ, both specified on an $(n+1)$-dimensional spacetime manifold M. The coupling of a scalar field to the Einstein gravitational field theory does not add any new constraint equations to the theory. We have the usual Hamiltonian and momentum constraints, but with added scalar field source terms. Writing these out in terms of the $n+1$ decomposition fields on an n-dimensional spacelike hypersurface Σ $\{\bar{\gamma}$ (the spatial metric), $\bar{K}$ (the second fundamental form, or extrinsic curvature), $\bar{\psi}$ (the scalar field restricted to Σ), $\bar{\pi}$ (the normalized time derivative of Ψ restricted to Σ)$\}$ we have

$$R(\bar{\gamma}) - |\bar{K}|^2_{\bar{\gamma}} + (\operatorname{tr}\bar{K})^2 = \bar{\pi}^2 + |\nabla\bar{\psi}|^2_{\bar{\gamma}} + 2V(\bar{\psi}) \tag{1}$$

$$\operatorname{div}_{\bar{\gamma}}\bar{K} - \nabla(\operatorname{tr}\bar{K}) = -\bar{\pi}\nabla\bar{\psi}, \tag{2}$$

where all derivatives and norms are taken with respect to the metric $\bar{\gamma}$ on Σ. Here $V(\cdot)$ is the potential of our scalar field, a priori this is an arbitrary smooth function. These constraints are to be solved for the Cauchy data $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$ on a chosen n-dimensional manifold Σ.

We use the conformal method is to recast the constraint equations (1)-(2) into a form which is more amenable to analysis, by splitting the Cauchy data into (i) the "conformal data", which one can choose freely, and (ii) the "determined data", which is determined by solving the recast constraints. For the gravitational data, one achieves an optimal form via the decomposition of the covariant 2-tensors

$$\bar{\gamma} = \phi^{\frac{4}{n-2}}\gamma \qquad \text{and} \qquad \bar{K} = \phi^{-2}(\sigma + \mathcal{D}W) + \frac{\tau}{n}\phi^{\frac{4}{n-2}}\gamma$$

where the conformal data consists of a Riemannian metric $\gamma = \gamma_{ab}$, a symmetric tensor $\sigma = \sigma_{ab}$ which is divergence-free and trace-free with respect to γ (so that σ is what is commonly referred to as a TT-tensor) and a scalar τ representing the mean curvature of the Cauchy surface Σ in the resulting spacetime; while the determined data consists of the positive function ϕ and the vector field $W = W^a$. Here the operator $\mathcal{D}$ is the conformal Killing operator relative to γ, defined by $(\mathcal{D}W)_{ab} := \nabla_a W_b + \nabla_b W_a - \frac{2}{n}\gamma_{ab}\nabla_m W^m$, where ∇ is the covariant derivative for the metric γ. The kernel of $\mathcal{D}$ consists of conformal Killing fields. We decompose the scalar field initial data $(\bar{\psi}, \bar{\pi})$ as follows:

$$\bar{\psi} = \psi \qquad \text{and} \qquad \bar{\pi} = \phi^{-\frac{2n}{n-2}}\pi.$$

Combining these decompositions of the gravitational and the scalar field data, we write out the conformal form of the constraint equations as follows:

$$\begin{aligned}\Delta_\gamma\phi - \tfrac{n-2}{4(n-1)}\left(R(\gamma) - |\nabla\psi|^2_\gamma\right)\phi \quad &+ \quad \tfrac{n-2}{4(n-1)}\left(|\sigma + \mathcal{D}W|^2_\gamma + \pi^2\right)\phi^{-\frac{3n-2}{n-2}} \\ &- \quad \tfrac{n-2}{4(n-1)}\left(\tfrac{n-1}{n}\tau^2 - 4V(\psi)\right)\phi^{\frac{n+2}{n-2}} = 0.\end{aligned} \tag{3}$$

$$\operatorname{div}_\gamma(\mathcal{D}W) = \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\nabla\tau - \pi\nabla\psi. \tag{4}$$

The operator $\operatorname{div}_\gamma \circ \mathcal{D}$ appearing in (4) is a second order, self-adjoint, linear, elliptic operator whose kernel consists of the space of conformal Killing vector fields on (Σ, γ). It follows that for a given set of functions (ϕ, τ, ψ, π) we may solve (4) provided $\frac{n-1}{n}\phi^{\frac{2n}{n-2}}\nabla\tau - \pi\nabla\psi$ is orthogonal to this space. The resulting solution is unique if and only if the space of conformal Killing vector fields on (Σ, γ) is empty.

To facilitate our subsequent discussion of (3) we make the following definitions. We set $c_n = \frac{n-2}{4(n-1)}$ and let

$$\mathcal{R}_{\gamma,\psi} = c_n\left(R(\gamma) - |\nabla\psi|^2_\gamma\right), \qquad \mathcal{A}_{\gamma,W,\pi} = c_n\left(|\sigma + \mathcal{D}W|^2_\gamma + \pi^2\right)$$

and

$$\mathcal{B}_{\tau,\psi} = c_n\left(\frac{n-1}{n}\tau^2 - 4V(\psi)\right).$$

We may then rewrite the Lichnerowicz equation (3) for the Einstein-scalar conformal data $(\gamma, \sigma, \tau, \psi, \pi)$ with a given vector field W satisfying (4) as

$$\Delta_\gamma\phi - \mathcal{R}_{\gamma,\psi}\,\phi + \mathcal{A}_{\gamma,W,\pi}\,\phi^{-\frac{3n-2}{n-2}} - \mathcal{B}_{\tau,\psi}\,\phi^{\frac{n+2}{n-2}} = 0. \tag{5}$$

We denote the conformal class of the metric γ by $[\gamma]$. The Yamabe-scalar field conformal invariant is then defined by

$$\mathcal{Y}_\psi([\gamma]) = \inf_{u\in H^1(\Sigma)} Q_{\gamma,\psi}(u) = \inf_{u\in H^1(\Sigma)} \frac{c_n^{-1}\int_\Sigma [|\nabla u|^2_\gamma + c_n\left(R(\gamma) - |\nabla\psi|^2_\gamma\right)u^2]\,d\eta_\gamma}{\left(\int_\Sigma u^{\frac{2n}{n-2}}\,d\eta_\gamma\right)^{\frac{n-2}{n}}}.$$

$\mathcal{Y}_\psi([\gamma])$ is independent of the choice of background metric in the conformal class used to define it, and is therefore an invariant of the conformal class. Using Hölder's inequality we observe that $|\int_\Sigma (R(\gamma) - |\nabla\psi|^2_\gamma)u^2]\,d\eta_\gamma| \le c\|u\|^2_{p_n}$ for some constant c independent of u. This shows that $\mathcal{Y}_\psi([\gamma]) > -\infty$. The following proposition may be proved in an analogous way to the well known result for the usual Yamabe conformal invariant (when $\psi \equiv 0$).

Proposition 1. *The following conditions are equivalent:*

(i) $\mathcal{Y}_\psi([\gamma]) > 0$ *(respectively* $= 0, < 0$*).*

(ii) *There exists a metric* $\tilde\gamma \in [\gamma]$ *which satisfies* $(R(\tilde\gamma) - |\tilde\nabla\psi|^2_{\tilde\gamma}) > 0$ *everywhere on* Σ *(respectively* $= 0, < 0$*).*

(iii) *For any metric* $\tilde\gamma \in [\gamma]$*, the first eigenvalue,* λ_1*, of the self-adjoint, elliptic operator* $-L_{\tilde\gamma,\psi}$ *is positive (respectively zero, negative).*

Our main focus is determining which sets of vacuum CMC conformal data permit the vacuum Lichnerowicz equation to be solved and which do not. For the Einstein-scalar case, the classification of the data is complicated, primarily because there are more relevant possibilities for the signs of the coefficients in (3). We collect the results of our analysis, in the cases where $\mathcal{B}$ does not change sign, in the following two tables, where "Y" indicates that the Lichnerowicz equation can be solved for that class of conformal data, "N" indicates that the corresponding Lichnerowicz equation has no positive solution, "PR" indicates that we have partial results and "NR" indicates that for this class of initial data we have no results indicating existence or non-existence.

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	PR	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	Y	N	N
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	PR	N	N	N

Table 1: Results for $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	PR	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	N	Y	Y
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	NR	Y	Y	Y

Table 2: Results for $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

References

[1] Y. Choquet-Bruhat, J. Isenberg and D. Pollack *The Einstein–scalar field constraints on asymptotically Euclidean manifolds*, ArXiv: gr-qc/0506101, Chinese Annals of Mathematics, Vol **26**, ser. B

[2] Y. Choquet-Bruhat, J. Isenberg and D. Pollack *The constraint equations for the Einstein–scalar field system on compact manifolds*, In preparation.

[3] J. Isenberg, D. Maxwell and D. Pollack, *A gluing constructions for non-vacuum solutions of the Einstein constraint equations,* ArXiv: gr-qc/0501083, Adv. Theor. Math. Phys., Vol. **9** No. 1, 129–172.

Angular momentum–mass inequality for axisymmetric black holes

Sergio Dain

In this talk I prove that extreme Kerr is a strict absolute minimum of the total mass in an appropriately defined Banach space. As a consequence, we obtain that any vacuum, maximal, asymptotically flat, axisymmetric initial data for Einstein equations in this space satisfy the inequality

$$\sqrt{|J|} \leq m,$$

where m and J are the total mass and angular momentum of the data.

Multipole radiation in systems of collisionless gases

Sebastian Bauer
(joint work with Markus Kunze, Gerhard Rein, Alan D. Rendall)

In this talk I report on some recent results obtained in [4]. That contribution is concerned with the mathematical properties of certain models for the interaction of matter, described by a kinetic equation, with radiation, described by hyperbolic equations. The first model, the relativistic Vlasov-Maxwell system, plays an important role in plasma physics. The motivation for studying the second model, the Vlasov-Nordström system, comes from the theory of gravitation. On a mathematical level the Vlasov-Maxwell system can also give insights into gravity.

The most precise existing theory of gravitation, general relativity, predicts that certain astrophysical systems, such as colliding black holes or neutron stars, will give rise to gravitational radiation. There is a major international effort under way to detect these gravitational waves [6]. In order to relate the general theory to predictions of what the detectors will see it is necessary to use approximation methods - the exact theory is too complicated. The mathematical status of these approximations remains unclear although partial results exist. This paper is intended as a contribution to understanding the mathematical structures involved.

Since the solutions of the equations of general relativity are so difficult to analyze rigorously it is useful to start with model problems. One possibility is the scalar theory of gravitation considered here, the Vlasov-Nordström theory [5]. It has already been used as a model problem for numerical relativity in [9].

Among the approximation methods used to study gravitational radiation those which are most accessible mathematically are the post-Newtonian approximations. Some information on these has been obtained in [7] and [8]. Results which are analogous to these but go much further have been obtained for the Vlasov-Maxwell and Vlasov-Nordström systems in [3], [1] and [2] respectively. Only the last of these results include radiation explicitly. Here we take a another step in doing so. For the case of finite particle systems interacting with their self-induced fields there are several rigorous results concerning radiation; see [10] for an up-to-date review.

The main results [4, Theorem 1.4 and Theorem 1.9] are relations between the motion of matter and the radiation flux at infinity for the Vlasov-Maxwell and Vlasov-Nordström systems respectively. They are analogues of the Einstein quadrupole formula [11, (4.5.13)] which is a basic tool in computing the flux of gravitational waves from a given source. In the case of the Einstein and Maxwell equations a spherically symmetric system does not radiate. For the Vlasov-Nordström system a spherical system can radiate and the specialization of the general formula to that case is computed. In [9] a difference between the spherically symmetric and the general case was claimed but we have not succeeded in connecting this to our results. The main theorems are obtained under plausible assumptions on the behavior of global solutions of the relevant system ([4, Assumption 1.1 and Assumption 1.6]). The former can be proved to hold in the case of small data.

For the systems we are going to consider the (scalar) energy density e and the (vector) momentum density $\mathcal{P}$ are related by the conservation law

$$\partial_t e + \nabla \cdot \mathcal{P} = 0.$$

Defining the local energy in the ball of radius $r > 0$ as

$$\mathcal{E}_r(t) = \int_{|x|\le r} e(t,x)\,dx,$$

this conservation law and the divergence theorem imply that

$$\frac{d}{dt}\,\mathcal{E}_r(t) = \int_{|x|\le r} \partial_t e(t,x)\,dx = -\int_{|x|\le r} \nabla\cdot\mathcal{P}(t,x)\,dx = -\int_{|x|=r} \bar{x}\cdot\mathcal{P}(t,x)\,d\sigma(x),$$

where $\bar{x} = \frac{x}{|x|}$ denotes the outer unit normal. More specifically, for the relativistic Vlasov-Maxwell system with two particle species,

$$\begin{aligned} e_{\mathrm{RVM}}(t,x) &= c^2 \int \sqrt{1+c^{-2}p^2}\,(f^+ + f^-)(t,x,p)\,dp \\ &\qquad + \frac{1}{8\pi}\big(|E(t,x)|^2 + |B(t,x)|^2\big), \\ \mathcal{P}_{\mathrm{RVM}}(t,x) &= c^2 \int p(f^+ + f^-)(t,x,p)\,dp + \frac{c}{4\pi}\,E(t,x)\times B(t,x), \end{aligned}$$

whereas for the Vlasov-Nordström system,

$$\begin{aligned} e_{\mathrm{VN}}(t,x) &= c^2 \int \sqrt{1+c^{-2}p^2}\,f(t,x,p)\,dp + \frac{c^2}{8\pi}\big((\partial_t\phi(t,x))^2 + c^2|\nabla\phi(t,x)|^2\big), \\ \mathcal{P}_{\mathrm{VN}}(t,x) &= c^2 \int pf(t,x,p)\,dp - \frac{c^4}{4\pi}\,\partial_t\phi(t,x)\,\nabla\phi(t,x). \end{aligned}$$

Here f^+ and f^- are the phase space distributions of positive and negative charges respectively, E and B are the electromagnetic fields; f is the phase space distribution of the gravitating matter and ϕ is the scalar gravitational potential. The assumptions on the support of the distribution function are such that the contributions of $\int p(f^+ + f^-)\,dp$ to $\mathcal{P}_{\mathrm{RVM}}$ and $\int pf\,dp$ to $\mathcal{P}_{\mathrm{VN}}$ vanish for $|x| = r$ large. Hence

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{RVM}}(t) = \frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t,x)\,d\sigma(x)$$

for the relativistic Vlasov-Maxwell system, and

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{VN}}(t) = \frac{c^4}{4\pi}\int_{|x|=r} \bar{x}\cdot(\partial_t\phi\nabla\phi)(t,x)\,d\sigma(x)$$

for the Vlasov-Nordström system.

The main results of [4] are concerned with the expansion of these energy fluxes for $r, c \to \infty$ and $|t - c^{-1}r| \leq$ const. Under suitable assumptions it is proved that, to leading order,

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{RVM}}(t) \sim -\frac{2}{3c^3}\,|\partial_t^2 \mathcal{D}(u)|^2,$$

where $u = t - c^{-1}r$ denotes the retarded time and $\mathcal{D}(u) = \int x\,\rho_0(u,x)\,dx$ is the dipole moment associated to the Newtonian limit of the relativistic Vlasov-Maxwell system. Similarly,

$$\frac{d}{dt}\,\mathcal{E}_r^{\mathrm{VN}}(t) \sim -\frac{1}{4\pi c^5}\int_{|\omega|=1} \big(\partial_t \mathcal{R}(\omega,u)\big)^2\,d\sigma(\omega),$$

with a more complicated radiation term $\mathcal{R}$ associated to the Newtonian limit of the Vlasov-Nordström system. In the spherically symmetric case, $\partial_t \mathcal{R}(\omega, u)$ is found to be proportional to $\partial_t \mathcal{E}_{\mathrm{kin}}(u)$, the change of kinetic energy of the Newtonian system. The exact statements are contained in [4, Theorems 1.4 and 1.6].

References

[1] Bauer S.: Post-Newtonian approximation of the Vlasov-Nordström system, to appear in *Comm. Partial Differential Equations*

[2] Bauer S.: Post-Newtonian Dynamics at Order 1.5 in the Vlasov-Maxwell System, preprint `arxiv:math=ph/0602031`

[3] Bauer S. & Kunze M.: The Darwin approximation of the relativistic Vlasov-Maxwell system, *Ann. H. Poincaré* **6**, 283-308 (2005)

[4] Bauer S., Kunze M., Rein G. & Rendall A.D.: Multipole Radiation in a Collisionless Gas Coupled to Electromagnetism or Scalar Gravitation, to appear in *Commun. Math. Phys.*, preprint `arxiv:math-ph/0508057`

[5] Calogero S.: Spherically symmetric steady states of galactic dynamics in scalar gravity, *Classical Quantum Gravity* **20**, 1729-1742 (2003)

[6] Bradaschia, C. (ed.): Proceedings of the 5th Edoardo Amaldi Conference on Gravitational Waves, in *Classical Quantum Gravity* **21**, S377-S1263 (2004)

[7] Rendall A.D.: On the definition of post-Newtonian approximations, *Proc. Roy. Soc. London Ser. A* **438**, 341-360 (1992)

[8] Rendall A.D.: The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system, *Comm. Math. Phys.* **163**, 89-112 (1994)

[9] Shapiro S.L. & Teukolsky S.A.: Scalar gravitation: A laboratory for numerical relativity, *Phys. Rev. D* **47**, 1529-1540 (1993)

[10] Spohn H.: *Dynamics of Charged Particles and Their Radiation Field*, Cambridge University Press, Cambridge-New York 2004

[11] Straumann N.: *General Relativity and Relativistic Astrophysics*, Texts and Monographs in Physics, Springer, Berlin-New York 1984

On the Positive Mass Conjecture in higher dimensions

Anda Degeratu
(joint work with Mark Stern)

In the context of general relativity, it has been proved [4, 5, 3] that the total mass of an isolated system is never negative, provided that the sources of the gravitational field consist of matter with positive mass density moving no faster than light and that spacetime is asymptotically flat.

In 1961 Arnowitt, Desser and Misner introduced the mass of an asymptotically flat hypersurface in spacetime [1]; their definition extends to higher dimension. A non-compact Riemannian manifold (M^n, g) is *asymptotically flat* if, outside a compact set, the metric asymptotically approaches the Euclidean metric on $\mathbb{R}^n$. This means that at infinity $g_{ij} = \delta_{ij} + \mathcal{O}(r^{-n+2})$, with appropriate decay for the derivatives of g. The mass of M is defined to be

$$m(M,g) = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) d\Omega^i, \tag{1}$$

where S_r denotes the sphere of radius r in the coordinate system at infinity.

Positive Mass Conjecture. *If (M^n, g) is an asymptotically flat manifold of dimension $n \geq 3$ and the scalar curvature is positive, then the mass is positive. Moreover, the mass vanishes if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \mathrm{eucl})$.*

In the case $n = 3$ this conjecture was proved by Schoen and Yau in 1979 using minimal surfaces techniques. Their proof generalizes inductively up to dimension $n \leq 7$. The reason for which it cannot be pushed further is that for manifolds of dimension 8, minimal representatives of classes in $H_{n-1}(M)$ have singularities in codimension 7. Recently Lohkamp announced a strategy to deal with this kind of singularities.

In 1981 Witten gave another proof of the conjecture in the case $n = 3$ using a spinorial approach, [5]. A rigorous mathematical interpretation was given by Parker and Taubes [3]. This proof generalizes to all asymptotically flat manifolds with positive scalar curvature [2].

In this talk I reported on a possible approach towards solving the conjecture in higher dimension, based on Witten's spinorial proof. Unlike the case of 3-manifolds, a higher dimensional manifold need not be spin. The obstruction to having a spin structure on an oriented Riemannian manifold is given by the second Stiefel-Whitney class. Our idea is to cut the manifold M, replace it with an incomplete manifold $M \setminus V$ with a spin structure, and consider the corresponding Dirac operator. The challenge is to find the right way to do the analytical manipulations near V so that the positivity of the mass be obtained through a similar argument as in Witten's proof.

References

[1] R. Arnowitt, S. Desser and C. Misner, *Coordinate invariance and energy expressions in General Relativity*, Phys. Rev. **122** (1961), 997–1006.

[2] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.
[3] T. Parker and C. Taubes, *On Witten's proof of the positive energy theorem*, Comm. Math. Phys. **84** (1982), no. 2, 223–238.
[4] R. Schoen and S. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.
[5] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402.

On uniqueness and stability for supercritical nonlinear wave and Schrödinger equations

Michael Struwe

The results we present below are described in detail in the forthcoming paper [11]. They extend previous results from [10].

Consider the Cauchy problem for the equation

$$u_{tt} - \Delta u + mu + f(u) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^n \tag{1}$$

with data

$$(u, u_t)_{|t=0} = (u_0, u_1) \in C_0^\infty \times C_0^\infty(\mathbb{R}^n), \tag{2}$$

where $m \geq 0$ and where $f = F'$ for some C^2-function $F \colon \mathbb{R} \to \mathbb{R}$ satisfying

$$0 \leq uf(u) \leq CF(u) \tag{3}$$

with some uniform constant C. Moreover, we request polynomial type behavior in the sense that for all $R > 0$ there exist numbers $\varepsilon = \varepsilon(R) > 0$, $C = C(R)$ such that the conditions

$$F(u+w) - F(u) - f(u)w \geq \varepsilon F(w) - C|w|^2 \tag{4}$$

as well as

$$F(u+w) - F(u) - f(u)w \leq CF(w) + C|w|^2 \tag{5}$$

and

$$|f(u+w) - f(u) - f'(u)w| \leq CF(w) + C|w|^2 \tag{6}$$

hold true for all w whenever $|u| \leq R$. Clearly, we may assume that $\varepsilon \leq 1$. The conditions (3)-(6) are satisfied for $F(u) = |u|^p$ for any $p \geq 2$; however, they fail to hold for example when $F(u) = e^{u^2} - 1$.

For classical solutions u of (1), upon multiplying equation (1) by u_t and integrating over any time-slice $[0, t] \times \mathbb{R}^n$, one easily finds the energy identity

$$E(u(t)) = \int_{\{t\} \times \mathbb{R}^n} \left(\frac{|Du|^2 + mu^2}{2} + F(u) \right) dx = E(u(0)), \tag{7}$$

where $Du = (u_t, \nabla u)$ is the space-time gradient. Moreover, these solutions have spatially compact support and therefore together with their derivatives are uniformly bounded on any such time-slice.

In the case of nonlinearities with "supercritical" growth the Cauchy problem (1), (2) is not known to admit global smooth solutions. However, by results of Segal [6], Lions [5], and Strauss [8], assuming (3) we always can obtain weak solutions to this equation with $Du, mu \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^n))$ and with $F(u) \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^n))$, satisfying the equation (1) in the sense of distributions and such that the energy inequality

$$E(u(t)) \le E(u(0)) \tag{8}$$

holds for all t; see for instance [7], Chapter 6.2, or [9], Theorem 3.1. We call such solutions "of energy class". Note that the map $t \mapsto Du(t) \in L^2(\mathbb{R}^n)$ is weakly continuous for energy class solutions u of (1). The condition (8) and strict convexity of the L^2-norm then imply that the initial data are continuously attained in the H^1-norm.

We now have the following stability result for smooth solutions – whenever they exist – within the *a priori* much larger class of distribution solutions of (1), (2) satisfying (8).

Theorem 1. *Suppose $u \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ is a classical solution to problem* (1), (2), *where f satisfies* (3) - (6). *For $(v_0, v_1) \in H^1 \times L^2(\mathbb{R}^n)$ with $F(v_0) \in L^1(\mathbb{R}^n)$ let v be an energy class solution to* (1) *with Cauchy data $(v, v_t)_{|t=0} = (v_0, v_1)$ and satisfying the energy inequality*

$$E(v(t)) \le E(v(0)) \text{ for all } t. \tag{9}$$

Then, letting $w = v - u$, if $m > 0$ and if u in addition is uniformly C^1-bounded in space-time, for all $t \ge 0$ with constants $C_i = C_i(u)$ we have the estimate

$$E(w(t)) \le C_1 e^{C_2 t} E(w(0)). \tag{10}$$

If $m = 0$ or if u fails to be uniformly bounded, for any time $T > 0$ and any $0 \le t \le T$ with a constant $C = C(u, T)$ there holds

$$E(w(t)) + ||w(t)||^2_{L^2} \le C(E(w(0)) + ||w(0)||^2_{L^2}). \tag{11}$$

In particular, we obtain the following uniqueness result.

Theorem 2. *Suppose $u \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ is a classical solution to equation* (1) *with Cauchy data $(u, u_t)_{|t=0} = (u_0, u_1) \in C_0^\infty \times C_0^\infty(\mathbb{R}^n)$, where f satisfies* (3) - (6). *Also let v be an energy class solution to* (1), (2), *satisfying* (8). *Then $u \equiv v$.*

Theorem 2 is similar to the uniqueness result of Ladyzenskaya for the Navier-Stokes equations [4]. Moreover, Theorems 1, 2 are related to results of Dafermos [1] and DiPerna [3] for hyperbolic systems of conservation laws; see also Dafermos [2], Chapter 5.3. Note, however, that in contrast to [2] we do not require the energy to be non-increasing in forward time. Thus, whenever the Cauchy problem (1), (2) admits a smooth solution, Theorem 2 also can be used to establish convergence in the energy norm of standard approximation schemes for equation (1), which often yield weak solutions of energy class; see [11] for details.

The proofs of Theorems 1 and 2 are straightforward and, hopefully, can be carried over to other settings.

References

[1] C. M. Dafermos: *The second law of thermodynamics and stability*, Arch. Rational Mech. Anal. 70 (1979), 167-179.
[2] C. Dafermos: *Hyperbolic conservation laws in continuum physics*, Grundlehren 325, 2nd edition, Springer (2005).
[3] R. J. DiPerna: *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. 28 (1979), 137-188.
[4] O. A. Ladyženskaya: *The mathematical theory of viscous Incompressible flow*, 2nd edition, Gordon and Breach, New York-London-Paris, 1969.
[5] J. L. Lions: *Une remarque sur les problèmes d'evolution non linéaires dans des domaines non cylindriques*, Revue Roumaine Math. Pur. Appl. 9 (1964), 11-18.
[6] I. E. Segal.: *The global Cauchy problem for a relativistic scalar field with power interaction*, Bull. Soc. math. France 91 (1963), 129-135.
[7] J. Shatah, M. Struwe: *Geometric wave equations*, Courant Lect. Notes 2, 2nd edition, Amer. Math. Soc. (2000).
[8] W. A. Strauss: *On weak solutions of semi-linear hyperbolic equations*, Anais Acad. Brasil. Cienc. 42 (1970), 645-651.
[9] W. A. Strauss: *Nonlinear wave equations*, CBMS Lect. Notes 73, Amer. Math. Soc. (1989).
[10] M. Struwe: *Uniqueness for critical nonlinear wave equations and wave maps via the energy inequality*, Comm. Pure Applied Math. 52 (1999), 1179-1188.
[11] M. Struwe: *On uniqueness and stability for supercritical nonlinear wave and Schrödinger equations*, preprint (2005), to appear in IMRN.

Accelerated expansion from pure gravity in higher dimensions

J. Mark Heinzle
(joint work with Lars Andersson)

A simple way of modeling accelerated expansion in cosmology is to consider gravity coupled to matter described by nonlinear scalar fields (inflaton, quintessence, k-essence, ...). Nonlinear scalar fields arise rather naturally from the dimensional reduction of supergravity models which are based on string theory or M-theory. It is thus natural to consider the dimensional (Kaluza-Klein) reduction of a D-dimensional spacetime (where typically $D = 10$ or $D = 11$) and to investigate whether the dimensionally reduced spacetime (a four-dimensional spacetime with nonlinear scalar field) exhibits phases of accelerated expansion.

Townsend and Wohlfarth [1] have shown that the reduction of a D-dimensional vacuum spacetime, given by the warped product of a four-dimensional flat Friedmann model and a hyperbolic internal space, leads to a transient phase of acceleration. In [2] it has been shown that it is possible to obtain late time accelerated expansion, if the four-dimensional model is a $\kappa = -1$ Friedmann model. In subsequent papers the dependence of the results on the dimension D was noted and the considerations were extended to multiply warped product spacetimes [3, 4, 5, 6].

Here we restrict our attention to the simple case of D-dimensional vacuum doubly warped spacetimes. The method of scale invariant dynamics enables us to discuss the problem in a systematic way: we can give a comprehensive description of the global dynamics of the D-dimensional spacetimes and their dimensional reduction. In particular, we are able to prove that there exists a unique dimensionally reduced model with eternal acceleration. Generic models exhibit a transient phase of acceleration, late-time acceleration, or no accelerated expansion at all.

Let (M,g) and (N,h) be Einstein manifolds of dimension m and n with

$$\text{Ric}[g] = k_g(m+n-1)g\,, \qquad \text{Ric}[h] = k_h(m+n-1)h\,; \tag{1}$$

k_g and k_h are constants; we set $k_g = k_h = -1$. On the $D = 1+m+n$ dimensional spacetime $\mathbb{R}\times M\times N$ consider a metric of the form of a doubly warped product

$$-dt^2 + a^2(t)\,g + b^2(t)\,h\,; \tag{2}$$

$a > 0$, $b > 0$. We introduce scale invariant variables according to

$$P = -\frac{\dot a}{aH}\,, \qquad Q = -\frac{\dot b}{bH}\,, \qquad A = -\frac{1}{aH}\,, \qquad B = -\frac{1}{bH}\,, \tag{3}$$

where H is the mean curvature, $H = -m\,\dot a/a - n\,\dot b/b$. The D-dimensional vacuum equations lead to the evolution equations,

$$A' = A\,[P - (mP^2+nQ^2)]\,, \qquad B' = B\,[Q-(mP^2+nQ^2)] \tag{4a}$$

$$P' = P\,[1-(mP^2+nQ^2)] + (m+n-1)k_gA^2\,, \tag{4b}$$

$$Q' = Q\,[1-(mP^2+nQ^2)] + (m+n-1)k_hB^2\,, \tag{4c}$$

which are supplemented by two constraint equations, $C_1 = mP+nQ-1 = 0$,

$$C_2 = (mP^2+nQ^2) - (m+n-1)(k_gmA^2+k_hnB^2) - 1 = 0\,. \tag{5}$$

A prime denotes differentiation w.r.t. the time τ, given by $\partial_\tau = H^{-1}\partial_t$. Note that with our conventions $H < 0$, hence by introducing τ we have the singularity to the future. The equation for H decouples from (4), $H' = H(mP^2+nQ^2)$.

The state space $\{(A,B,P,Q)\mid (C_1=0)\wedge(C_2=0)\wedge(A>0)\wedge(B>0)\}$ has compact closure. Due to its regularity the system (4) can be smoothly extended to also include $A = 0$ and $B = 0$. The equations on $A = 0$ (respectively $B = 0$) can be interpreted as the system of equations that arises when the first factor (respectively the second factor) of the metric (2) is Ricci flat. This is because setting $A = 0$ in (4) and (5) corresponds to setting $k_g = 0$ and discarding the decoupled equation for A. The method of scale invariant dynamics thus allows for the simultaneous treatment of all cases $k_g \le 0$, $k_h \le 0$.

The system (4) can be analyzed with dynamical systems methods. The attracting set consists of five fixed points: (F_1) and (F_2), which are local sinks, (F_A) and (F_B), which are saddles, and (F_*), which is the global source. Orbits converging to $(\mathrm{F}_{1,2})$ as $\tau\to\infty$ represent solutions that are asymptotically Kasner, i.e., $a\propto t^p$, $b\propto t^q$ as $t\to 0$, where $mp+nq = 1$, $mp^2+nq^2 = 1$. Orbits converging to (F_A) are associated with solutions $a\to\text{const}$, $b\propto t$; orbits converging to (F_*) as $\tau\to-\infty$ lead to Friedmann type solutions, $a\propto b\propto t$ as $t\to\infty$. A schematic depiction of the global dynamics is given in Fig. 1(a).

Dimensional (Kaluza-Klein) reduction transforms a $D = 1+m+n$ dimensional vacuum spacetime $(\mathbb{R}\times M\times N)$, cf. (2), to a $(1+m)$-dimensional spacetime $\mathbb{R}\times M$

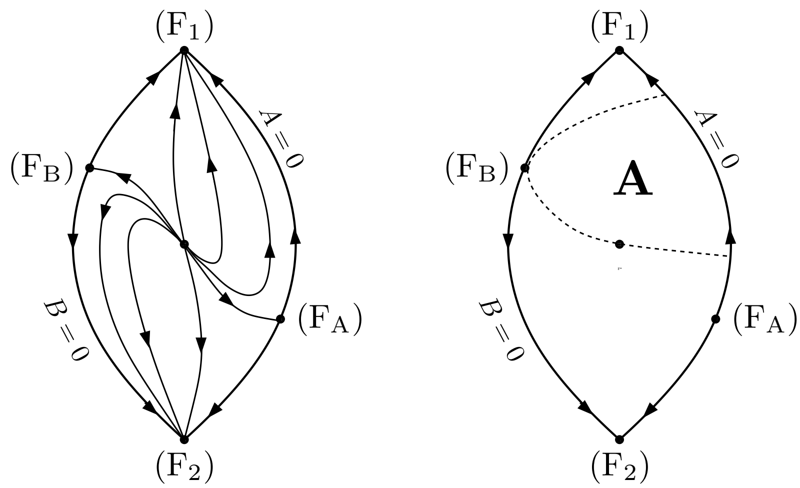

FIGURE 1. Schematic of the flow on the state space and the domain of acceleration; $D = 1 + m + n \geq 10$.

with metric γ and nonlinear scalar field φ. We find that

$$(6) \qquad \gamma = b^{2n/(m-1)}(-dt^2 + a^2 g) = -d\bar{t}^2 + \left(ab^{n/(m-1)}\right)^2 g = -d\bar{t}^2 + \bar{a}^2\, g$$

with $\bar{a} = ab^{n/(m-1)}$, and $\varphi = (8\pi)^{-1/2}\,(m+n-1)\,(m-1)^{-1}\log b$; the energy-momentum tensor of φ is that of a scalar field with the nonlinear potential $V(\varphi)$, $V(\varphi) \propto \exp(-2\sqrt{8\pi}\, n^{1/2}(m-1)^{-1/2}(m+n-1)^{1/2}\varphi)$. The spacetime $(\mathbb{R}\times M, \gamma, \varphi)$ is a solution of the Einstein nonlinear scalar field equations.

By construction, every orbit in Fig. 1(a) corresponds to a solution $(\mathbb{R}\times M, \gamma, \varphi)$, where $\bar{a} = ab^{n/(m-1)}$ is determined via (3). A simple calculation shows that $d\bar{a}/d\bar{t} = (m-1)^{-1}[1-P]/A > 0$, since $P < 1$ from (5). Further differentiation leads to

$$(7) \quad \frac{d^2\bar{a}}{d\bar{t}^2} = \frac{ab^{-n/(m-1)}}{m-1} H^2 \Big[-(1-P)^2 - (m+n-1)\left(k_g(m-1)A^2 + k_h n B^2\right)\Big] .$$

There exists a domain **A** in the state space such that

$$(8) \qquad \frac{d^2\bar{a}}{d\bar{t}^2} > 0 \quad \text{for all } (A,B,P,Q) \in \mathbf{A} .$$

The domain **A** is the *domain of acceleration*, depicted in Fig. 1(b): whenever an orbit passes through this domain, the cosmological model $(\mathbb{R}\times M, \gamma)$ it represents undergoes accelerated expansion. In particular we can prove the following

Theorem 1 (Existence and uniqueness of eternal acceleration)**.** *Let $D \geq 10$ (with $(m,n) \neq (2,7)$). Then there exists a unique solution $\gamma = -d\bar{t}^2 + \bar{a}^2 g$ of the $(1+m)$-dimensional Einstein equations (with nonlinear scalar field φ) arising from the dimensional reduction of a $(1+m+n)$-dimensional vacuum solution, such that $d^2\bar{a}/d\bar{t}^2 > 0$ for all $\bar{t}$.*

For the proof of the theorem we show that the orbit connecting (F_B) with (F_*) lies entirely in **A**. Since (F_B) is a saddle it follows immediately that there cannot exist any other orbit with the same property.

Superimposing Figs. 1(a) and 1(b) we can read off if a generic orbit leads to a cosmological model exhibiting accelerated expansion. For instance, the orbit (F_1)–(F_A) on the boundary $A = 0$ leads to a (Ricci flat) model with a transient phase of acceleration; this model is the one originally described in [1]. It would be of interest to quantify how much accelerated expansion one can actually obtain, e.g., to compute the number of e-foldings for the models. Furthermore, a generalization of the formalism to the case of multiply warped product metrics suggests itself. Whether the resulting dynamical system can be treated by simply generalizing the present methods (especially in view of the global dynamics and the methods used in proof of the theorem) remains to be seen.

References

[1] P.K. Townsend and M.N.R. Wohlfarth, *Accelerating Cosmologies from Compactification*, Phys. Rev. Lett. **91** (2003), 061302.

[2] C.-M. Chen,P.-M. Ho, I.P. Neupane, N. Ohta, and J.E. Wang, *Hyperbolic Space Cosmologies*, J. High Energy Phys. **0310** (2003), 058.

[3] C.-M. Chen, P.-M. Ho, I.P. Neupane, and J.E. Wang, *A Note on Acceleration from Product Space Compactification*, J. High Energy Phys. **0307** (2003), 017.

[4] M.N.R. Wohlfarth, *Inflationary Cosmologies from Compactification?*, Phys. Rev. D **69** (2004), 066002.

[5] P.K. Townsend and M.N.R. Wohlfarth, *Cosmology as Geodesic Motion*, Class. Quantum Grav. **21** (2004), 5375.

[6] I.P. Neupane and D.L. Wiltshire, *Cosmic acceleration from M theory on twisted spaces*, Phys. Rev. D **72** (2005), 083509.

A brief history of the 2–body problem in numerical relativity

Matthew W. Choptuik

Following a very brief synopsis of numerical work on the collision of two black holes (∼1970 – 2005) I focused on Pretorius' new "Generalized Harmonic Code" and results obtained over the last year or so with that code. These results include the late stages of inspiral (currently, up to four-and-a-half orbits), followed by merger and ringdown to a final Kerr hole, as well as axisymmetric collisions between a black hole and a relativistic (but not especially compact) (mini-)boson star (i.e. the complex boson field ϕ satisfies $\Box\phi = m^2\phi$, with m the bosonic [particle] mass).

Accelerated cosmological expansion and k-essence

Alan D. Rendall

An important recent development in cosmology is the realization that the expansion of our universe is accelerating. The cause of this is not known and many mechanisms have been proposed in the literature. The simplest of these arises for the Einstein equations with normal matter in the presence of a positive cosmological constant Λ. There are several mathematical results on this situation. In particular there is a nice theorem due to Wald [6] in the homogeneous case. It applies to solutions which are forever expanding. This includes most homogeneous models, more specifically the Bianchi models of types I-VIII which are expanding at some time. Any kind of matter which satisfies the dominant and strong energy conditions is allowed. Global solutions isotropize and their spatial curvature becomes negligible, as does the energy density of normal matter. For interesting classes of matter such as perfect fluids and collisionless matter this result can be extended. Global existence can be proved and details of the late-time behaviour obtained. In these spacetimes the expansion is exponential.

Among more general models leading to accelerated expansion the simplest and most frequently studied are based on a nonlinear scalar field ϕ with potential V. These are sometimes known under the name quintessence. Whether or not accelerated expansion is obtained depends on the form of V. The only result on existence of models without symmetry which have certain asymptotics is that of [2]. It applies to certain exponential potentials. In the homogeneous case much more is known. One important class of models is that where the potential has a strictly positive lower bound. In that case the direct analogue of Wald's theorem holds and the late-time expansion is exponential. The value of the potential at the minimum determines the exponent. Another class which frequently occurs is where the potential is positive but tends to zero at infinity. If it goes to zero slower than any exponential then an analogue of Wald's theorem holds, with the expansion being faster than any power and no faster than exponential. Detailed asymptotics can be obtained using the slow-roll approximation [3].

The standard Lagrangian for a nonlinear scalar field is $L = -V(\phi) + X$, where $X = -\nabla^\alpha\phi\nabla_\alpha\phi/2$. The generalization where $L = L(\phi, X)$ for some nonlinear function L of ϕ and X is known as k-essence [1]. The k in the name stands for kinetic. The equation of motion is

$$(1)\qquad \left(\frac{\partial L}{\partial X}g^{\alpha\beta} - \frac{\partial^2 L}{\partial X^2}\nabla^\alpha\phi\nabla^\beta\phi\right)\nabla_\alpha\nabla_\beta\phi + \frac{\partial^2 L}{\partial\phi\partial X}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi = -\frac{\partial L}{\partial\phi}$$

The mathematical properties of solutions of the Einstein equations coupled to k-essence and ordinary matter were recently examined in [4]. A basic problem, which has already been solved, is what conditions on L are necessary for the equation of motion of ϕ to be hyperbolic. In that case the Einstein equations coupled to k-essence are also hyperbolic. It is also known for which choices of L superluminal propagation is possible. The dominant, strong and weak energy conditions have been analysed for this class of matter models.

In the homogeneous case statements can be obtained about the late-time dynamics. The first goal is, as in the simpler models, to obtain analogues of Wald's theorem. This has been done for class of models which generalize quintessence models with a positive lower bound for the potential. It should also be possible to obtain results for a situation like that of quintessence with a potential which tends to zero at infinity. These are not, however, the most interesting classes of k-essence models. These models were originally introduced as a way of getting solutions with qualitative properties different from those of previously known cases. Here it typically happens that X does not go to zero as $t \to \infty$. Results on solutions with this kind of behaviour were obtained in [4]. Unfortunately they do not apply to the case where $L(\phi, X) = \phi^{-2}\tilde{L}(X)$ which are of particular interest in applications and here more work is required.

One reason why models in the restricted case just mentioned are difficult to handle is that they are expected to exhibit power-law expansion. It is known from the study of quintessence, where power-law expansion corresponds to an exponential potential, that this is more difficult to treat than faster (e.g. exponential) expansion. Even the case of quintessence with a potential which is asymptotically exponential but not exactly exponential had not been treated until very recently. A result of this kind was obtained as a by-product of the work on k-essence in [4]. For an exactly exponential potential $V(\phi) = V_0 e^{-k\phi}$ accelerated expansion is obtained if the positive exponent k is less than $4\sqrt{\pi}$. In [4] it was shown that the same holds true for potentials which are asymptotically exponential in a suitable sense under the stronger restriction that that the asymptotic exponent is less than $4\sqrt{\pi/3}$.

In [4] it was assumed for simplicity that all models considered satisfied the dominant energy condition. After this work was completed the author discovered a paper of Vikman [5] which explores what happens in the absence of this assumption. It is easy to see that there are Lagrangians L for which the dominant energy condition is not always satisfied. An interesting question which is motivated by the observational data is whether a solution can evolve from a regime in which the dominant energy condition is satisfied to one in which it is not. The results of [5] indicate that under reasonable physical assumptions a transition of this kind is not possible. Depending on the choice of L there may be no solutions at all exhibiting a transition of this kind or those which do may be unstable with respect to homogeneous or inhomogeneous perturbations.

The results discussed above seem like only small forays into the unknown territory of k-essence models. A comprehensive overview of the terrain is not yet available. It is a challenge for the future to obtain one. There are also many more models of accelerated expansion which remain to be understood mathematically. For instance the k-essence scalar field can be replaced by several coupled scalar fields. Other kinds of fields can be coupled to the Einstein equations and theories of gravity other than Einstein's can be investigated. Clearly many things remain to be explored.

References

[1] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, *A dynamical solution to the problem of a small cosmological constant and late-time cosmological acceleration*, Phys. Rev. Lett. **85** (2000), 4438–4441.

[2] J. M. Heinzle and A. D. Rendall *Power-law inflation in spacetimes without symmetry.* Preprint gr-qc/0506134.

[3] A. D. Rendall *Intermediate inflation and the slow-roll approximation.* Class. Quantum Grav. **22** (2005) 1655–1666.

[4] A. D. Rendall *Dynamics of k-essence.* Preprint gr-qc/0511158.

[5] A. Vikman *Can dark energy evolve to the phantom?* Phys. Rev. D **71** (2005), 023515.

[6] R. M. Wald *Asymptotic behaviour of homogeneous cosmological models with cosmological constant.* Phys. Rev. D **28** (1983), 2118–2120.

Reporter: Roger Bieli

Participants

Prof. Dr. Lars Andersson
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Hakan Andreasson
Department of Mathematics
Chalmers University of Technology
S-412 96 Göteborg

Dr. Sebastian Bauer
Fachbereich 4 - Mathematik
Universität Duisburg-Essen
45117 Essen

Prof. Dr. Robert Beig
Institut für Theoretische Physik
Universität Wien
Boltzmanngasse 5
A-1090 Wien

Roger Bieli
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Lydia Bieri
Departement Mathematik
E.T.H. Zürich
HG G 53
Rämistr. 101
CH-8092 Zürich

Dr. Simone Calogero
Departamento de Matematica
para a Ciencia e Tecnologia
Campus de Azurem da Universidade
do Minho
Guimares 4800-058
Portugal

Prof. Dr. Matthew W. Choptuik
Department of Physics and Astronomy
University of British Columbia
6224 Agricultural Road
Vancouver BC V6T 1Z1
Canada

Prof. Dr. Yvonne Choquet-Bruhat
Academie des Sciences
23, Quai Conti
F-75006 Paris

Prof. Dr. Piotr T. Chrusciel
Departement de Mathematiques
Faculte des Sciences
Universite de Tours
Parc de Grandmont
F-37200 Tours

Prof. Dr. Justin Corvino
Department of Mathematics
Lafayette College
Easton, PA 18042-1781
USA

Dr. Mihalis Dafermos
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Dr. Sergio Dain
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Anda Degeratu
Dept. of Mathematics
Duke University
P.O.Box 90320
Durham, NC 27708-0320
USA

Prof. Dr. Erwann Delay
Laboratoire d'Analyse non lineaire
et Geometrie
Universite d'Avignon
33, rue Louis Pasteur
F-84000 Avignon

Prof. Dr. Felix Finster
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Dr. Michel Grueneberg
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Carsten Gundlach
School of Mathematics
University of Southampton
Highfield
GB-Southampton, SO17 1BJ

Prof. Dr. Dietrich Häfner
Institut de Mathematiques
Universite de Bordeaux I
351 Cours de la Liberation
F-33405 Talence

Dr. Mark Heinzle
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Stefan Hollands
Institut für Theoretische Physik
Universität Göttingen
Friedrich-Hund-Platz 1
37077 Göttingen

Prof. Dr. Gerhard Huisken
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. James Isenberg
Department of Mathematics
University of Oregon
Eugene, OR 97403-5203
USA

Prof. Dr. Markus Kunze
Fachbereich 4 - Mathematik
Universität Duisburg-Essen
45117 Essen

Dr. Hayoung Lee
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Philippe G. LeFloch
Laboratoire Jacques-Louis Lions
Universite Paris 6
4, Place Jussieu
F-75252 Paris Cedex 05

Dr. Lee Lindblom
Div. of Physics, Mathematics and
Astronomy
CALTECH 130-33
Pasadena, CA 91125
USA

Prof. Dr. Joachim Lohkamp
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Marc Mars
Facultad de Ciencias
Universidad de Salamanca
E-37008 Salamanca

Prof. Dr. Lionel J. Mason
St. Peter's College
University of Oxford
GB-Oxford OX1 2DL

Prof. Dr. Daniel Pollack
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
USA

Prof. Dr. Gerhard Rein
Mathematisches Institut
Universität Bayreuth
95440 Bayreuth

Prof. Dr. Alan Rendall
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Hans Ringström
Dept. of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
S-100 44 Stockholm

Dr. Bernd Schmidt
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Yuguang Shi
School of Mathematical Sciences
Peking University
Beijing 100871
P.R. CHINA

Prof. Dr. Michael Struwe
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Dr. Blaise Tchapnda
Albert-Einstein-Institut
MPI für Gravitationsphysik
Am Mühlenberg 1
14476 Golm

Prof. Dr. Paul Tod
Mathematical Institute
Oxford University
24-29, St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Claes Uggla
Department of Mathematics & Physics
University of Karlstad
S-65188 Karlstad

Prof. Dr. Juan J. L. Velazquez
Matematica Aplicada
Universidad Complutense
Plaza de Ciencias 3
E-28040 Madrid

Dr. Mattias Wohlfarth
II. Inst. für Theoretische Physik
Universität Hamburg
Luruper Chaussee 149
22761 Hamburg