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The Rigorous Renormalization Group

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Abstract. The renormalization group is a mathematical tool used in the analysis of systems with infinitely many degrees of freedom associated with length scales. The workshop presents recent work in this domain concerning probability theory, statistical and solid state physics and field theory as well as the methodology of the renormalization group.

Mathematics Subject Classification (2000): 1311,1312,1313,1314.

Introduction by the Organisers

The workshop on *The Rigorous Renormalization Group*, was attended by more than 40 participants coming mainly from Western Europe and from America. The official programme consisted in 19 lectures of 60 minutes each (plus discussion). Four of them were devoted to noncommutative field theory, three of them presented methods used for and results on the construction of a non-gaussian fixed point in a statistical mechanics/quantum field theory model, and two lectures concerned, respectively, nonlinear σ-models, the functional renormalization group, and quantum electrodynamics. The remaining six lectures were on the Brockett-Wegner version of the renormalization group, on random walks, on Fermi liquids, on anomalies in quantum field theory, on renormalization in curved spaces and on functional integrals for many boson systems. The scientific programme, the atmosphere and the Oberwolfach style of the meeting, leaving much room for informal discussions and joint work, were generally highly appreciated. The abstracts of the lectures are presented in chronological order.

Workshop: The Rigorous Renormalization Group

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Abstracts

Noncommutative Quantum Field Theories and Renormalization

Harald Grosse

Noncommutative Quantum Field Theory shows some unexpected features called IR/UV mixing. They arise upon quantization and renormalizability of Noncommutative Quantum Field Theory is spoiled. The planar contributions show the standard singularities which can be handled by a renormalization procedure. The nonplanar loop contributions are finite for generic momenta, however they become singular at any order at exceptional momenta. Without imposing a special structure such as supersymmetry, the renormalizability seems lost. Progress was made, when we were able together with Raimar Wulkenhaar to solve this problem for the special case of a scalar four dimensional theory defined on the deformed Moyal space (CMP 256(2005)305). The IR/UV mixing contributions were taken into account through a modification of the free Lagrangian by adding an oscillator term, which modifies the spectrum of the free Hamiltonian. The harmonic oscillator term was obtained as a result of the renormalization proof. The model fulfills then the Langmann-Szabo duality relating short distance and long distance behavior. Our proof followed ideas of Polchinski.
It turned out, that the IR/UV mixing occurs also on other Fuzzy spaces like noncommutative Tori as well as on the kappa Poincare deformed space (work together with Michael Wohlgenannt hep-th/0507030).
There are indications that a constructive procedure might be possible and will give nontrivial interacting models, which are currently under investigation. In the self-dual case we obtain special matrix models, which we are studying in two and four dimensions together with H. Steinacker (hep-th/0512203 and hep-th/0603052). The Φ^3 model is of special interests, since it can be solved in closed form using the solution of the Kontsevich model. We managed to renormalize the genus zero contribution of the model and proved existence of correlation functions. In four dimensions both a mass renormalization as well as a tadpole renormalization was necessary.
On the other hand in work with M. Wohlgenannt (to be published) we coupled a scalar field to a gauge field and calculated the one loop divergent contributions using a heat kernel approach. We obtained this way gauge models for two and four dimensions. We completed this calculation for the special values of parameters for which the model is self-dual. Away from the duality point the calculation is more involved and will be finished next. The resulting models will hopefully be renormalizable and might become the starting point of a renormalizable deformed Standard model, which will be the final goal.

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A combinatorial generalisation of Cramer's Rule

D. C. Brydges
(joint work with A. Abdesselam)

1. Introduction

The result described here is a Corollary of a bijection discovered by Xavier Viennot, [3, Proposition 6.3]. It is a combinatorial generalisation of Cramer's formula for the inverse of a matrix and it is closely related to the Mayer Expansion.

For motivation we begin with a combinatorial interpretation of Cramer's formula. Let $A = (A_{xy},\ x, y \in \mathcal{S})$ be a matrix. A *path* ω from a to b is any finite sequence

$$\big(\omega_1, \omega_2, \cdots, \omega_n\big) \in \bigcup_{n\in\mathbb{N}^*} \mathcal{S}^n$$

with $\omega_1 = a$ and $\omega_n = b$. The sites $\omega_1, \omega_2, \cdots, \omega_n$ need not be distinct; if they are we say that the path is *self-avoiding*. The set of distinct sites in the sequence ω is called the *support of* ω. If $a = b$ then the only self-avoiding path is (a).

The resolvent expansion in powers of A represents $(I - A)^{-1}$ by the formal power series

$$(I - A)^{-1}_{ab} = \sum_{\omega: a\to b} A^\omega$$

where

$$A^\omega = \begin{cases} \prod_{i=1}^{n-1} A_{\omega_i\omega_{i+1}} & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}$$

We call $c \subset \mathcal{S} \times \mathcal{S}$ a *self-avoiding loop* if for some $n \in \mathbb{N}^*$,

$$c = \begin{cases} \{(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1)\} & \text{if } n \geq 2 \\ \{(x_1, x_1)\} & \text{if } n = 1, \end{cases}$$

where $x_1, \ldots, x_n$ are *distinct*. The *support* of a loop is the set $\{x_1, \ldots, x_n\} \subset \mathcal{S}$.

By writing $\det(I - A)$ in terms of permutations and decomposing the permutations into cycles, one has

$$\det(I - A) = \sum_{r=0}^{\infty} \sum_{\{c_1,\ldots,c_r\}} (-A^{c_1}) \cdots (-A^{c_r}) \tag{1}$$

where the $r = 0$ term equals 1 by definition and $c_1, \ldots, c_r$ are self-avoiding loops with disjoint supports and

$$A^c = \prod_{(x,y)\in c} A_{xy}.$$

There is a similar expansion for the ab cofactor of A,

$$\det A^{(b,a)} = \sum_{r=0}^{\infty} \sum_{\bar\omega,\{c_1,\dots,c_r\}} A^{\bar\omega}(-A^{c_1})\cdots(-A^{c_r}), \tag{2}$$

where $\bar\omega, c_1, .., c_r$ have disjoint supports, $c_1, .., c_r$ are cycles and $\bar\omega$ is a self avoiding path from a to b. If $a = b$ then $\bar\omega = (a)$ and $A^{\bar\omega} = 1$.

By Cramer's formula

$$\frac{\det(I-A)^{(b,a)}}{\det(I-A)} = (I-A)^{-1}_{ab}, \tag{3}$$

so, as elements in the ring of power series in A with rational coefficients,

$$\frac{\sum_{r=0}^{\infty}\sum_{\bar\omega,\{c_1,\dots,c_r\}} A^{\bar\omega}(-A^{c_1})\cdots(-A^{c_r})}{\sum_{r=0}^{\infty}\sum_{\{c_1,\dots,c_r\}}(-A^{c_1})\cdots(-A^{c_r})} = \sum_{\omega:a\to b} A^{\omega}. \tag{4}$$

The result of Viennot generalises this formula to the case where the weights on the self-avoiding loops and the weight on the self-avoiding path are arbitrary. Let $\mathcal{C}$ be the finite set of all self-avoiding loops in $\mathcal{S}$. For each $c \in \mathcal{C}$ we require a formal variable λ_c and let $\lambda = \{\lambda_c : \ c \in \mathcal{C}\}$. Likewise let $\mathcal{C}(a,b)$ be the set of all self-avoiding paths from a to b. For each $\bar\omega \in \mathcal{C}(a,b)$ there is a formal variable $\alpha_{\bar\omega}$ and $\alpha = \{\alpha_{\bar\omega} : \ \bar\omega \in \mathcal{C}(a,b)\}$. Let $\mathcal{R}$ be the ring of power series with rational coefficients in λ, α. Define an element of this ring by

$$\langle a,b\rangle = \frac{\sum_{r=0}^{\infty}\sum_{\bar\omega,\{c_1,\dots,c_r\}} \alpha_{\bar\omega}\lambda_{c_1}\cdots\lambda_{c_r}}{\sum_{r=0}^{\infty}\sum_{\{c_1,\dots,c_r\}} \lambda_{c_1}\cdots\lambda_{c_r}}. \tag{5}$$

By traveling along the path ω starting at a and ending at b, recursively erasing self-avoiding loops in the order in which they appear, one obtains a possibly empty list $\mathcal{E}(\omega)$ of erased self-avoiding loops and a self-avoiding path $\bar\omega(\omega)$ from a to b.

$$\mathcal{E}(\omega) = \begin{cases} (c_1, c_2, \dots, c_r) & \text{if } r \geq 1 \\ () & \text{if } r = 0 \end{cases}$$

where $c_1, c_2, \dots, c_r$ are self-avoiding loops. In the case where $a = b$, $\bar\omega(\omega) = (a)$.

Theorem 1.1. [3]. *For $a, b \in \mathcal{S}$, as an identity in the ring of power series $\mathcal{R}$,*

$$\langle a,b\rangle = \sum_{\omega:a\to b} (-\lambda)^{\mathcal{E}(\omega)}\alpha_{\bar\omega(\omega)}$$

where

$$(-\lambda)^{\mathcal{E}(\omega)} = \prod_{i=1}^{r}(-\lambda_{c_i}).$$

Cramer's formula is the particular case where the formal variables are specialised according to

$$\lambda_c = -A^c, \quad \alpha_{\bar\omega} = A^{\bar\omega}.$$

The proof of the bijection discovered by Viennot is elegant. It is based on the theory of heaps. Our proof uses the Mayer expansion [6, 2] to perform the division

in (5). By using the theory of combinatorial species [5, 4, 1] the Mayer expansion is proved to be equal to a sum over a species of (loop ensemble and one self-avoiding path) connected by edges of a tree graph. The edges of the tree graph taken in the right order describe how to insert all the loops into the self-avoiding path to obtain a species consisting of a single path. Loop erasure is the inverse surgery, which applied to this path recreates the ensemble of loops, the self-avoiding path and the tree.

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On the Brockett-Wegner Diagonalizing Flow

Volker Bach

(joint work with Jean-Bernard Bru)

In 1991, R. Brockett found a method to diagonalize a self-adjoint complex $N \times N$ matrix $H \in \mathcal{M}_{N\times N}(\mathbb{C})$ by a family $\{U(t,s)\}_{0\leq s\leq t} \subseteq U(N)$ of unitary transformations. With H as initial value, its diagonalized form H_∞ is obtained in the limit $H_\infty = \lim_{t\to\infty} H_t$, where $H_t := U(t,0)HU(t,0)^*$. A similar idea was independently developed by F. Wegner in 1994.

Both derivations are formal, however, in the sense that the question of global existence of H_t, which results from a nonlinear evolution equation, is not addressed. J.-B. Bru and V. Bach have specified sufficient conditions insuring global existence which are presented in the seminar.

Moreover, both Brockett's and Wegner's methods are restricted in applicability to trace-class operators, and it is not clear, how to extend it to more general classes of operators. Together with M. Walser, J.-B. Bru and V. Bach show that the method is at least transferable to second quantizations $d\Gamma(h)$ of operators h, whose off-diagonal part is Hilbert-Schmidt. This is possible because the identity used by Brockett, which results from the cyclicity of the trace, can be pulled back from the second quantization $d\Gamma(h)$ of the operator h to the operator h itself, where it yields finite values for the trace.

Exit distribution for random walks in random environments

Erwin Bolthausen
(joint work with Ofer Zeitouni)

We consider the standard random walks in random environments (RWRE for short) on $\mathbb{Z}^d$, $d \geq 3$. Let $\mathcal{P}$ be the set of probability distributions on $\mathbb{Z}^d$, charging only neighbors of 0. If $\varepsilon \in (0, 1/2d)$, we set

$$\mathcal{P}_\varepsilon \stackrel{\text{def}}{=} \left\{ q \in \mathcal{P} : \left| q(\pm e_i) - \frac{1}{2d} \right| \leq \varepsilon,\ \forall i \right\},$$

where the e_i are the standard base vectors in $\mathbb{R}^d$. We call an element $\omega \in \Omega \stackrel{\text{def}}{=} \mathcal{P}^{\mathbb{Z}^d}$ a *random environment*. For $\omega \in \Omega$, and $x \in \mathbb{Z}^d$, we consider the transition probabilities $p_\omega(x, y) \stackrel{\text{def}}{=} \omega_x(y - x)$, if $|x - y| = 1$, and $p_\omega(x, y) = 0$ otherwise.

The random walk $\{S_n\}_{n \geq 0}$ with initial position $x \in \mathbb{Z}^d$ is, given the environment ω, the Markov chain with $S_0 = x$ and transition probabilities

$$P_{\omega,x}(S_{n+1} = y | S_n = z) = \omega_z(y - z).$$

We are interested in the case of a *random* ω. Given a probability measure μ on $\mathcal{P}$, we therefore consider the product measure $\mathbb{P}_\mu \stackrel{\text{def}}{=} \mu^{\otimes \mathbb{Z}^d}$ on Ω. Our basic assumption is

Condition 1. *μ is invariant under lattice isometries, i.e. $\mu f^{-1} = \mu$ for any orthogonal mapping f which leaves $\mathbb{Z}^d$ invariant, and $\mu(\mathcal{P}_\varepsilon) = 1$ for some (small) $\varepsilon \in (0, 1/2d)$.*

This model of RWRE has been studied extensively. We refer to [6] for recent surveys. A major open problem is the determination, for $d > 1$, of laws of large numbers and central limit theorems in full generality (the latter, both under the *quenched* measure, i.e. for $\mathbb{P}_\mu$-almost every ω, and under the *annealed* measure $\mathbb{P}_\mu \otimes P_{x,\omega}$). Although much progress has been achieved in recent years ([3, 4, 1]), a full understanding of the model has not yet been achieved.

We consider here exit distributions of the random walk from balls

$$V_L \stackrel{\text{def}}{=} \left\{ x \in \mathbb{Z}^d : |x| \leq L \right\},$$

where $|\cdot|$ denotes the Euclidean distance. For $x \in V_L$, we write $\Pi_{L,\omega}(x, \cdot)$ for the exit distribution of the RWRE from V_L i.e.

$$\Pi_{L,\omega}(x, z) \stackrel{\text{def}}{=} P_{\omega,x}\left(S_{\tau_{V_L}} = z \right),$$

where τ_{V_L} is the first exit time of the walk from V_L. We also write $\pi_L(x, \cdot)$ for the exit distribution of ordinary random walk. The main result is that $\Pi_{L,\omega}(x, \cdot)$ with probability close to 1 approaches $\pi_L(x, \cdot)$ if L is large. The distance between the two distributions can however not be measured in terms of the total variation distance as this cannot approach 0, due to the random environment close to the boundary. However, only some slight smoothing is necessary.

Let $f : \mathbb{R}^d \to [0, \infty)$ be a smooth rotational symmetric probability density. For $r > 0$ (large), we set

$$f_r(x) \stackrel{\text{def}}{=} \frac{f(x/r)}{\sum_y f(y/r)},\ x \in \mathbb{Z}^d.$$

Our main result is

Theorem 2. *There exists $\varepsilon > 0$ (depending only on the dimension $d \geq 3$) such that if Condition 1, then for any $\eta > 0$ there exists $r(\eta)$ with*

$$\lim_{L\to\infty} \mathbb{P}_\mu \left(\left\{ \omega : \left\| \sum_y \left[\Pi_{L,\omega}(0,y) - \pi_L(0,y) \right] f_r(\cdot - y) \right\|_1 \geq \eta \right\} \right) = 0.$$

If r increases with $L : r_L \nearrow \infty$, then

$$\left\| \sum_y \left[\Pi_{L,\omega}(0,y) - \pi_L(0,y) \right] f_{r_L}(\cdot - y) \right\|_1 \to 0$$

in $\mathbb{P}_\mu$-probability.

In a recent paper [5], the authors obtain for a model of a diffusion in random environment, essentially under similar conditions, a functional central limit theorem. Such a CLT is not immediate from the above theorem. What is lacking is essentially some control of the holding times. On the other hand, the local control we get of the exit distribution is more precise than what is obtained in [5] for the diffusion case.

For an earlier work on the same models, see [2].

The above result is proved using a multiscale analysis, showing that if the exit distributions of the RWRE are close to the ones of ordinary random walk, then on a bigger scale, they are even closer. This is achieved by representing the exit distribution on the larger scale through the smaller ones, using the well known perturbation expansion of the Green's function. The main delicacy is coming from the above mentioned fact that there can be no such contraction in total variation norm. One therefore has to adapt the representation using an appropriate smoothing. However, it turns out that the representation of the smoothed exit distributions on the large scale is not possible through the smoothed ones on the smaller scale alone. Fortunately, the representation can be adapted in such a way that "non-smoothed" exit distributions are needed only in somewhat minor parts of the perturbation expansion for which one needs only somewhat crude bounds.

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Infrared finite algorithms in QED

Alessandro Pizzo
(joint work with Volker Bach, Jürg Fröhlich)

We consider a nonrelativistic electron moving in the Coulomb field of a single nucleus of unit charge and interacting with the soft modes of the quantized electromagnetic field. Our main concern is how to rigorously control the higher order radiative corrections to the scattering amplitudes in the low energy regime (Rayleigh scattering). In fact, Taylor formula is ill-defined when no infrared regularization is adopted. We develop a proper perturbation theory and we provide an asymptotic expansion, up to any order in the coupling constant, for the scattering amplitudes, which represents a first important step towards a rigorous analysis of metastable states. At this stage (scattering amplitudes), the asymptotic expansion of the groundstate vector of the system is the main technical issue. Concerning this expansion, we use a scaling analysis based on the iterated analytic perturbation theory.

Renormalisation scalar quantum field theory on 4D-Moyal plane

Raimar Wulkenhaar
(joint work with Harald Grosse; Vincent Rivasseau, Fabien Vignes-Tourneret)

Quantum field theories on the Moyal plane characterised by the $\star$-product (in D dimensions)

$$(1) \qquad (a\star b)(x) := \int d^D y \frac{d^D k}{(2\pi)^D}\, a(x+\tfrac{1}{2}\theta\cdot k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}ky}\ , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}$$

became fashionable after their appearance in string theory [1] and the discovery of the UV/IR-mixing problem [2]. The UV/IR-mixing contains a clear message: If we make the world noncommutative at very short distances, we must at the same time modify the physics at large distances. The required modification is, to the best of our knowledge, unique: It is given by an harmonic oscillator potential for the free field action. In fact, we can prove the following

Theorem 3. *The quantum field theory associated with the action*

$$(2) \quad S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\ ,$$

for $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}\, x^\nu$, ϕ-real, Euclidean metric, is perturbatively renormalisable to all orders in λ.

There are by now three different proofs of the theorem.

(1) by exact renormalisation group equations in a matrix base of the Moyal algebra [3],
(2) by multi-scale analysis again in a matrix base of the Moyal algebra [4]
(3) by multi-scale analysis directly in position space [5].

First proof [H. Grosse and R. Wulkenhaar] The $\star$-product (1) leads in momentum space to oscillating phase factors which result for some non-planar Feynman graphs in convergent but not absolutely convergent integrals. Our starting point was the conjecture that the selection of a certain order of integration by the renormalisation scheme is at the origin of the UV/IR-mixing problem. Thus, our idea was to invent a regularisation where the model with cut-off is well-defined and no ambiguity in the order of integration appears.

We selected the renormalisation group approach as the right strategy. In order to make use of the simplicity of renormalisation proofs based on exact renormalisation group equations, it was necessary to have amplitudes for vertices and the propagator which are manifestly positive, not oscillating. This lead us to the use of the matrix base of the Moyal algebra where, with respect to that base (given by Laguerre polynomials), the $\star$-product becomes a product of infinite matrices. The price for this achievement was a rather complicated kinetic matrix of the ϕ^4-action. We eventually succeeded in computing the propagator in the matrix base by identifying the eigenvectors of the kinetic matrix as Meixner polynomials. At the end, the matrix propagator was expressed as a finite sum over hypergeometric functions. See [3].

The adaptation of the renormalisation group equation framework to the action (2) requires two steps:

- Prove bounds for the cut-off propagator.
- Compute the amplitude of a graph as a function of these bounds.

We started with the second step. In [6] we proved a power counting theorem for general dynamical matrix models, characterised by a scaling behaviour of the propagator, in the exact renormalisation group approach. The proof is by induction in the number of vertices and loops. As Feynman graphs for matrix models are ribbon graphs characterised by their topology, i.e. genus and number of holes of the Riemann surface, the difficulty was:

- to guess the power counting theorem in terms of the topology,
- to prove that the scaling behaviour (in terms of the topology) is independent of the history in which a graph of given topology arises from smaller graphs with their topology.

The second step alone goes over 20 pages!

It turned out numerically [3] that the propagator obtained for $\Omega = 0$ in (2) has scaling properties which make the perturbative renormalisation impossible. Of course, this is a manifestation of UV/IR-mixing. With inclusion of the harmonic oscillator piece, also motivated by a duality argument [7], the scaling behaviour

together with the power-counting theorem [6] implied that all non-planar graphs and all graphs with more than 4 external legs are irrelevant. See [3].

As a result, only planar graphs with two or four external legs can be relevant or marginal. However, these graphs are labelled by an infinite number of matrix indices. Here, we invented a discrete Taylor expansion in the matrix indices of the external legs which decomposes the (infinite number of) planar two- and four-leg graphs into a linear combination of four relevant or marginal base functions and an irrelevant remainder. The explicite realisation of the propagator in terms of hypergeometric functions was essential in this estimation. These four universal base functions have the same index dependence as the original action in matrix formulation, which implies the renormalisability of the model [3]. A summary of the main ideas and techniques can be found in [8].

We have also computed in [9] the one-loop β-functions of the model which describe the dependence of the bare coupling constant and the bare oscillator frequency on the cut-off matrix size. It turned out that $\frac{\lambda}{\Omega^2}$ remains constant under the renormalisation flow. As $|\Omega|$ is bounded by 1, the running coupling constant can be kept arbitrarily small over all scales for a sufficiently small renormalised coupling constant. This is a sign that a constructive renormalisation of the non-commutative ϕ_4^4-model is possible.

Second Proof [V. Rivasseau, F. Vignes-Tourneret, R. Wulkenhaar] In [3] the asymptotic properties of the propagator are only numerically determined. This shortcoming was cured in [4] where we proved these bounds rigorously, for Ω large enough. The idea was to use the Schwinger representation of the matrix propagator and to cut it into slices $M^{-i} \leq \alpha \leq M^{-i+1}$. We proved bounds in the matrix indices as a function of the scale index i.

These bounds confirmed the previous numerically estimation, but also gave rise to a different renormalisation proof. For given attribution of scale indices to each propagator, we were able to sum all independent matrix indices of the graph, thus giving the amplitude in terms of the scale attribution. In order to determine the independent matrix summation indices, the $SO(2) \times SO(2)$ symmetry of the model (2) was used, which is most conveniently realised in the dual of the graph. Then, the lines of the dual graph are distinguished into tree lines (chosen according to the scale attribution) and loop lines. The bounds implied that summation over the loop angular momenta do not cost anything so that the power-counting degree of divergence boils down to twice the number of completely inner vertices of the dual graph minus the number of propagators. This is precisely the topological degree of divergence found in [3]!

Third Proof [R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret] Renormalisation should be basis independent. In particular, the model (2) should also be renormalisable in position space (or, which by duality [7] is the same, momentum space). This was indeed confirmed in [5]. The advantage of position space is that the propagator is simple: it is given by the Mehler kernel. The price to pay are the oscillating phase factors in the vertex. The problem is elegantly circumvented by first proving that non-orientable graphs (which are always non-planar)

are irrelevant even if one bounds the oscillating phases by 1. And for orientable graphs (which can be planar or non-planar) the phases can be globally handled.

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Asymptotically safe renormalization flow in sigma-models with warped product target spaces

Max Ralph Niedermaier

Research context: The renormalization problem for the quantized gravitational field remains unsolved. Truncated variants of the problem allow one to gain insight into obstructions and prospects. A fruitful truncation is one where the functional integral over geometries 'weighed' with the Einstein-Hilbert action is restricted to those allowing for two commuting Killing vectors. This truncation captures the 'spin two' aspect of the gravitational field and keeps infinitely many local dynamical degrees of freedom. Suitably formulated the core part of the resulting renormalization problem consists in defining the quantum theorie of two-dimensional nonlinear sigma-models whose target space is a warped product of the form

$$G/K \,{}_h\!\times\, \mathbb{R}^{1,1}\,,$$

where G/K is a noncompact symmetric space (with a non-amenable Lie symmetry group G) of curvature $\zeta < 0$, $\mathbb{R}^{1,1}$ is isometric to two-dimensional Minkowski space, and the 'warp function' h is a function of ρ, the area radius of the two Killing vectors. The warped product then has scalar curvature $\zeta \dim(G/K)/h(\rho)$. Without the warp factor noncompact sigma-models of this type are of independent interest, see the contribution by E. Seiler. The results described below can be put into the context of the asymptotic safety scenario for quantum gravity [4] and provide nontrivial evidence for it.

Results [1, 2]: The systems are perturbatively non-renormalizable with a finite number of couplings, reflecting the corresponding property of the untruncated gravity theory. Using the covariant background field technique and dimensional regularization the following results are obtained to all loop orders: (i) strict cutoff independence can be achieved by allowing for infinitely many couplings. They can be combined into a generating function of one real variable, the above 'warp function' h. (ii) The renormalization flow of h is govered by a beta functional which can be expressed in closed form in terms of the (one coupling) beta function of the G/K sigma-model. (iii) The h-flow has a degenerate Gaussian and a non-Gaussian fixed point function h_*. (iv) The warping (i.e. the coupling to gravity) reverses the signs of the flow. While the single coupling of the G/K models is infrared free, all the couplings contained in h are asymptotically safe. That is, arbitrary linear perturbations $h_* + \delta h$ are to all loop orders driven back to h_* in the ultraviolet. (v) the trace anomaly vanishes at the non-Gaussian fixed point and the gravitational constraints can be defined as composite operators.

The results suggest that the warped-product sigma-models exist as quantum field theories beyond perturbation theory. An 'exact' bootstrap type construction has been proposed for $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ [3] but contact to the functional integral is missing. The investigation of nonperturbative aspects in the functional integral formulation is an important open problem.

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Fermi liquid behavior in the weak coupling 2D Hubbard model at exponentially small temperatures

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(joint work with Giuseppe Benfatto, Vieri Mastropietro)

We consider the two dimensional Hubbard model:

$$H = \sum_{\vec{x}\in\Lambda} \sum_{\sigma=\uparrow\downarrow} a^+_{\vec{x},\sigma} \big(-\frac{\Delta}{2} - \mu \big) a^-_{\vec{x},\sigma} + U \sum_{\vec{x}\in\Lambda} a^+_{\vec{x},\uparrow} a^-_{\vec{x},\uparrow} a^+_{\vec{x},\downarrow} a^-_{\vec{x},\downarrow} \tag{1}$$

where:

- $\Lambda \subset \mathbb{Z}^2$ is a square sublattice of $\mathbb{Z}^2$ with side L;
- $a^\pm_{\vec{x},\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma = \uparrow\downarrow$ and site index $\vec{x} \in \Lambda$, satisfying periodic boundary conditions in $\vec{x}$;

- Δ is the discrete Laplacian on $\mathbb{Z}^2$;
- $\mu > 0$ is the chemical potential, whose value fixes the average density of particles;
- U is the strength of the on–site density–density interaction; it can be either positive or negative.

We study the (imaginary time) two point correlation function

$$S(\vec{x},t) = \lim_{L\to\infty} \frac{\mathrm{Tr}\left\{e^{-\beta H}\mathbf{T}\, a^-_{(\vec{x},t),\sigma} a^+_{(\vec{0},0),\sigma}\right\}}{\mathrm{Tr}\, e^{-\beta H}} \tag{2}$$

where $a^\pm_{\vec{x},t} = e^{tH} a^\pm_{\vec{x}} e^{tH}$ and $\mathbf{T}$ is the fermionic time ordering, i.e. $\mathbf{T}\, a^-_{\vec{x},t} a^+_{\vec{0},0}$ equals $a^-_{\vec{x},t} a^+_{\vec{0},0}$ $(-a^+_{\vec{0},0} a^-_{\vec{x},t})$ if $t > 0$ $(t \le 0)$. By fermionic multiscale analysis we devise an iterative resummation scheme to compute $S(\vec{x},t)$ in terms of an expansion convergent in the region of weak coupling and up to exponentially small temperatures. The result is that two point correlation function at small U and $\beta^{-1} \ge \exp\{-const/|U|\}$, can be written in the same qualitative form as the correlation fucntion of a free Fermi gas, up to a (finite) renormalization of the Fermi surface and of the wave functon renormalization constant. The result can be interpreted as evidence of Fermi liquid behavior for small coupling and up to exponentially small temperatures. The precise statement of our result is the following:

Theorem [2] *Let us consider the 2D Hubbard model with* $0 < \mu < \frac{2-\sqrt{2}}{2}$ *and* $\beta^{-1} \ge e^{-\frac{a}{|U|}}$ *where* $a > 0$ *is a suitable constant. There exists a constant* $U_0 > 0$ *such that, if* $|U| \le U_0$*, the Fourier transform* $\hat{S}(\mathbf{k})$*,* $\mathbf{k} = (\vec{k}, k_0)$*, of the two point correlation function can be written as*

$$\hat{S}(\mathbf{k}) = \frac{1}{Z(\theta)} \frac{1}{-ik_0 + \vec{v}_F(\theta)\cdot\big(\vec{k} - \vec{p}_F(\theta)\big) + R(\mathbf{k})} \tag{3}$$

with $Z(\theta), \vec{v}_F(\theta)$ *and* $\vec{p}_F(\theta)$ *real and*

$$Z(\theta) = 1 + a(\theta)U^2 + O(U^3) \tag{4}$$

$$\vec{v}_F(\theta) = \vec{v}_F^{(0)}(\theta) + \vec{b}(\theta)U^2 + O(U^3) \tag{5}$$

$$\vec{p}_F(\theta) = \vec{p}_F^{(0)}(\theta) + \vec{c}(\theta)U + O(U^2) \tag{6}$$

where $a(\theta), |\vec{b}(\theta)|, |\vec{c}(\theta)|$ *are bounded above and below by positive* $O(1)$ *constants uniformly in the region* $\beta^{-1} \ge e^{-\frac{a}{|U|}}$*. Moreover*

$$|R(\mathbf{k})| \le C\big[|\vec{k} - \vec{p}_F(\theta)|^2 + k_0^2 + |\vec{k} - \vec{p}_F(\theta)||k_0|\big] \tag{7}$$

for some constant $C > 0$*, uniformly in the considered range of parameters.*

Remarks.
a) In the free case ($U = 0$) the two point correlation function has an expression similar to (3) with $Z(\theta)$ replaced by $Z_0 = 1$, $\vec{p}_F(\theta)$ replaced by $\vec{p}_F^{(0)}(\theta)$ (the parametric equation of the free Fermi surface $\cos k_1 + \cos k_2 = 2 - \mu$) and $\vec{v}_F(\theta)$ replaced

by $\vec{v}_F^{(0)}(\theta) = (\sin k_1, \sin k_2)|_{\vec{k}=\vec{p}_F(\theta)}$.
b) The condition $0 < \mu < \frac{2-\sqrt{2}}{2}$ physically corresponds to the condition that umklapp processes do not contribute to the scattering of $n \leq 4$ particles.
c) The condition $\beta^{-1} \geq e^{-\frac{a}{|U|}}$ physically corresponds to the condition the temperature is larger than the temperature where possible superconducting instabilities are present.

The above theorem says that the 2D Hubbard model is a *Fermi liquid* up to exponentially small temperatures and far from the half filled band case, in the following sense. Comparing the representation (3) of the interacting two–point Schwinger function with the free one, given in Remark (a), we see that they are apparently similar but the parameters $Z(\theta)$, $\vec{v}_F(\theta)$ and $\vec{p}_F(\theta)$, differently from Z_0, $\vec{v}_F^{(0)}(\theta)$ and $\vec{p}_F^{(0)}(\theta)$, are functions of the temperature β^{-1}, for $\beta^{-1} \geq e^{-a/|U|}$. However such dependence can be stronger or weaker and the different sensitivity to a variation of the temperature has important physical consequences. In the case of the 2D Hubbard model with $\mu < \frac{2-\sqrt{2}}{2}$ we prove that $Z(\theta)$, $\vec{v}_F(\theta)$ and $\vec{p}_F(\theta)$ are slowly depending on β for $\beta^{-1} \geq e^{-a/|U|}$, that is they are essentially constant in β above an exponentially small temperature. This means that, in the considered range of parameters, the interacting two–point correlation is essentially identical to the free one, *up to a renormalization of the parameters essentially independent on the temperature*; in this sense we say that the system shows a *Fermi liquid behavior* for temperatures larger than an exponentially small one. This notion of Fermi liquid is the natural mathematical interpretation of the notion of Fermi liquid often used in the theoretical physics literature, and it is essentially the same as the one adopted, for instance, in [3, 4].

Of course the property to be a Fermi liquid (in the above sense) is not trivial at all and it is not verified in many cases. For instance, in the 1D Hubbard model, the wave function renormalization Z depends logarithmically on β, that is $c_1 U^2 \log \beta \leq |Z - 1| \leq c_2 U^2 \log \beta$, with c_1, c_2 two positive constants, for temperatures above an exponentially small temperature; so, with our definition, the 1D Hubbard model is not a Fermi liquid in such range of temperatures. In the 2D Hubbard model at half–filling (i.e. at $\mu = 2$) it has been recently proved [5, 6] that, for temperatures above an exponentially small temperature, $c_1 U^2 \log^2 \beta \leq |Z - 1| \leq c_2 U^2 \log^2 \beta$, so that the system is not a Fermi liquid at half–filling in that range of temperatures. On the contrary, an example of Fermi liquid in the above sense is provided by the continuum approximation of model (1) in $d = 2$, the so–called *jellium model*, for which [4] showed that, in a range of temperatures above an exponentially small temperature, $c_1 U^2 \leq |Z - 1| \leq c_2 U^2$, and the system is a Fermi liquid. Note that in the jellium model, due to rotation invariance, the interacting and the free Fermi surfaces have exactly the same shape, that is a circle, and the effect of the interaction essentially consists just in changing its radius.

The proof of the Theorem is achieved by determinantal bounds for the fermionic expectations and by an iterative resummation of the original power series expansion for the two–point correlation function and by a renormalization of the free "measure", which takes into account, in particular, the modification of Fermi surface. This allows us to reexpress iteratively the original power series in U as an expansion in an increasing number of parameters (they are indeed functions), called the "effective couplings" and physically describing the effective interaction at different momentum scales, denoted by $(\lambda_1 = U, \lambda_0, \lambda_{-1}, \lambda_{-2}, \ldots)$; moreover, the coefficients of the new series are themselves depending on U through the renormalized single scale propagators. The new series will be well–defined whenever $\bar{U} = \max_{h \le 1} |\lambda_h|$ will be smaller than U_0, where U_0 is a constant *independent of the temperature*. From the physical point of view, this means that the temperature dependence at all orders in the expansion for $\hat{S}(\mathbf{k})$ is essentially all included in the effective couplings, whose size in turn will depend strongly on the temperature. We stress that the possibility of resumming the series into a new series admitting this kind of "uniform bounds" is specific of $d = 2$ far from half–filling; for instance in $d = 1$ the coefficient at order $2n \ge 2$ of the resummed expansion for $Z(\theta)$ behaves like $(\bar{U}/U_0)^{2n} (\log \beta)^n$, instead of $(\bar{U}/U_0)^{2n}$, *even assuming that the effective interactions are bounded*. This is not the case in $d = 2$ far from half–filling; in this case the breaking of Fermi liquid behavior can be due only to some instability occurring in the effective interactions.

Our result should be compared to [4], in which a proof of Fermi liquid behavior was given for the jellium model. We have taken from such papers two crucial technical ingredients: the idea of using anisotropic sectors (and the relative *sector lemma* of [7]) for the bounds and the idea of further decomposing some sector into isotropic sectors in order to improve the bounds for the self energy; note however that the technical implementation of such ideas in the proofs is rather different with respect to [4], mainly for the heavy use of *trees* for reorganizing the perturbative series and for the fact that we do not need neither a "1PI analysis" to extract our power counting improvements needed to prove the Theorem. Moreover, the presence of a non circular Fermi surface causes many new technical problems with respect to the case in [4]. The most important one is that, while in the Jellium case the interacting Fermi surface is fixed *a priori* to be a *circle* as consequence of rotational symmetry, here on the contrary the shape and the regularity or convexity properties of the interacting Fermi surface are completely unknown: in fact there is no a priori evidence of the fact that the interacting Fermi surface is regular and convex uniformly in β in the considered range of temperatures. Hence we cannot in our case *fix* the interacting Fermi surface by properly tuning the chemical potential, as it is done in [1, 4]; on the contrary, we proceed in a way similar to that used in [8], by inserting at each integration step all the quadratic part of the interaction in the free fermionic measure. In this way to each fermionic integration at a certain momentum scale corresponds a different Fermi surface, and one has to check that the geometrical conditions for defining sectors and to apply the sector lemma are verified at each scale.

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Renormalization of Non Commutative Field Theory in Direct Space

Vincent Rivasseau

(joint work with R. Gurau, J. Magnen and F. Vignes-Tourneret)

In joint work with R. Gurau, J. Magnen and F. Vignes-Tourneret [1], using direct space methods, we provided recently a new proof that the Grosse-Wulkenhaar scalar Φ^4_4 theory on the Moyal space $\mathbb{R}^4$ is renormalizable to all orders in perturbation theory.

The Grosse-Wulkenhaar breakthrough [2][3] was to realize that the right propagator in non-commutative field theory is not the ordinary commutative propagator, but has to be modified to obey Langmann-Szabo duality [4],[3].

Our method builds upon previous work of Filk and Chepelev-Roiban [5][6]. These works however remained inconclusive [7], since these authors used the right interaction but not the right propagator, hence the problem of ultraviolet/infrared mixing prevented them from obtaining a finite renormalized perturbation series.

We also extended the Grosse-Wulkenhaar results to more general models with covariant derivatives in a fixed magnetic field [8]. Our proof relies solely on a multiscale analysis analogous to [9] but in direct x configuration space.

Non-commutative field theories (for a general review see [10]) deserve a thorough and systematic investigation. Indeed they may be relevant for physics beyond the standard model. They are certainly effective models for certain limits of string theory [11]-[12]. What is often less emphasized is that they can also describe effective physics in our ordinary standard world but with non-local interactions.

In this case there is an interesting almost complete reversal of the initial Grosse-Wulkenhaar problematic. In the Φ^4_4 theory on the Moyal space $\mathbb{R}^4$, the vertex is sort of God-given by the Moyal structure, and it is LS invariant. The challenge was to overcome uv/ir mixing and to find the right propagator which makes the theory

renormalizable. This propagator turned out to have LS duality. The harmonic potential introduced by Grosse and Wulkenhaar can be interpreted as a piece of the covariant derivatives in a constant magnetic field.

Now to explain the (fractional) quantum Hall effect, which is a bulk effect whose understanding requires electron interactions, we can almost invert this logic. The propagator is known since it corresponds to non relativistic electrons in two dimensions in a constant magnetic field. It has LS duality. But the interaction is unclear, and cannot be local since at strong magnetic field the spins should align with the magnetic field, hence by Pauli principle the local interaction should vanish.

We can argue that among all possible non-local interactions, a few renormalization group steps should select the only ones which form a renormalizable theory with the corresponding propagator. In the commutative case (i.e. zero magnetic field) local interactions such as those of the Hubbard model are just renormalizable in any dimension because of the extended nature of the Fermi-surface singularity. Since the non-commutative electron propagator (i.e. in non zero magnetic field) looks very similar to the Grosse-Wulkenhaar propagator (it is in fact a generalization of the Langmann-Szabo-Zarembo propagator) we can conjecture that the renormalizable interaction corresponding to this propagator should be given by a Moyal product. That's why we hope that non commutative field theory is the correct framework for a microscopic *ab initio* understanding of the fractional quantum Hall effect which is currently lacking.

Even for regular commutative field theory such as non-Abelian gauge theory, the strong coupling or non perturbative regimes may be studied fruitfully through their non commutative (i.e. non local) counterparts. This point of view is forcefully suggested in [12], where a mapping is proposed between ordinary and non commutative gauge fields which do not preserve the gauge groups but preserve the gauge equivalent classes. We can at least remark that the effective physics of confinement should be governed by a non-local interaction, as is the case in effective strings or bags models.

In other words we propose to base physics upon the renormalizability principle. Renormalizability means genericity; only renormalizable interactions survive a few RG steps, hence only them should be used to describe generic physics. This search for renormalizabilty could be the powerful principle on which to orient ourselves in the jungle of all possible non-local interactions.

Fermionic theories such as as the two dimensional Gross-Neveu model can be shown renormalizable to all orders in their Langmann-Szabo covariant versions, using either the matrix basis [13] or the direct space version developed here [14]. However the x-space version seems the most promising for a complete non perturbative construction, using Pauli's principle to controll the apparent (fake) divergences of perturbation theory. They are treated in the talk of F. Vignes-Tourneret.

In the case of ϕ_4^4, recall that although the commutative version is until now fatally flawed due to the famous Landau ghost, there is some hope that the non-commutative field theory treated at the perturbative level in this paper may also

exist at the constructive level [15][16]. Again the x-space version of renormalization is probably better than the matrix basis for a rigorous investigation of this question.

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A non trivial fixed point in a three dimensional quantum field theory

P.K. Mitter

Exact Renormalization Group methods give a constructive tool to analyze critical behaviour of classical statistical mechanical systems and the related problem of constructing a continuum limit (ultraviolet cutoff removal) for euclidean quantum field theory models. In this context an open challenging problem is to construct a massless ϕ^4 field theory in $\mathbf{R}^3$. It is expected that such a massless field theory would be in the universality class of the critical Ising model in $\mathbf{Z}^3$. This expectation

is supported by the renormalization group calculations of Wilson, Fisher, and many others following them in various approximation schemes. But to date there is no mathematical solution of this problem.

Brydges, Mitter and Scoppola in [BMS 2003] considered a variant of this model baptized the $\phi^4_{3,\epsilon}$ model in $\mathbf{R}^3$. This corresponds to a perturbation by a ϕ^4 interaction of a Gaussian random field with a covariance depending on a real parameter ϵ in the range $0 \leq \epsilon \leq 1$. The covariance is the Green's function of a stable Lévy process in $\mathbf{R}^3$ and is both pointwise and Osterwalder-Schrader positive. For $\epsilon = 1$ one recovers the Newtonian potential (the covariance of the standard massless free scalar field) whereas for $\epsilon = 0$ the ϕ^4 interaction is marginal. The infinite volume critical theory with a fixed ultraviolet cutoff at unit length scale was considered. It was proved that for $\epsilon > 0$ and held sufficiently small there exists a nontrivial hyperbolic fixed point of the renormalization group iterations. The stable manifold was constructed in a small neighborhood of this fixed point. It is an open problem to extend the stable manifold beyond this region and to prove that it connects the (unstable) Gaussian fixed point with the nontrivial (stable) fixed point. A.Abdesselam has made progress on the construction of a renormalization group trajectory connecting the two fixed points and he reports on this issue in the present workshop, [A 2006].

In order to prove the above result some new methods of renormalization group analysis were introduced in [BMS]. These build on earlier methods due to Brydges and Yau in [BY 1990] and Brydges, Dimock and Hurd in [BDH 1998] but simplify them considerably. A crucial ingredient is the use of multiscale expansions of the underlying Gaussian random field as the sum of fluctation fields whose covariances have finite range. This is accomplished in the continuum context above by adopting an appropriate ultraviolet cutoff scheme. But such finite range multiscale expansions are by no means special to a particular choice of ultraviolet cutoff. For example it was shown by Brydges, Mitter and Guadagni in in [BGM 2004] that a large class of Gaussian random fields on a lattice $\mathbf{Z}^d$ have finite range expansions. This has been generalized considerably by Brydges and Talarczyk in [BT 2006]. The upshot is that the analysis of fluctuation integrations get considerably simplified and cluster expansions and analyticity norms can be dispensed with. At a generic step of the discrete renormalization group flow the partition function density is represented through a polymer gas representation where the polymer activities $K(X, \phi)$ are supported on closed connected disjoint sets X (polymers) which are unions of closed cubes in $\mathbf{R}^3$. In the complement of the union of these sets there sits a local potential V (local functional of fields). The couple (V, K) gives coordinates for the RG trajectory which make sense in infinite volume (in contrast to the partition function itself). The discrete RG flow can be now considered as the iteration of a single RG map. Moreover one profits from the non uniqueness of this representation so that all expanding (relevant) variables under the RG map: $(V, K) \to (V', K')$ are collected in the local part V so that the polymer activities K represent contracting (irrelevant) directions. The local part $V(Y, \phi)$ in a region Y is the integral of a field polynomial with coefficients

$\underline{\lambda} = (\xi, g, \mu, ..)$ that may be generically called coupling constants which evolve under the RG map. The space of functionals $(\underline{\lambda}, K)$ can be realized as a Banach space. The nonlinear part of the RG map $(\underline{\lambda}, K) \to (\underline{\lambda}', K')$ satisfies a Lipshitz property and the discrete flow is then analyzed by methods of stable manifold theory in a Banach space context leading to the claimed result.
The results in [BMS 2003] together with those reported in [A 2006] realise part of what would be the Wilson program in the context of this model. An important task is to study the scaling limit for correlation functions and in the process construct elementary and composite scaling fields. Prelimnary investigations lead us to expect that the simplest composite fields have anomalous scaling dimensions.

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Functional renormalization group approach to interacting electron systems

W. Metzner

Interacting electron systems usually exhibit very different behavior on different energy scales. Collective phenomena and composite objects emerge at scales far below the bare energy scales of the microscopic Hamiltonian. For example, in high-temperature superconductors one bridges three orders of magnitude from the highest scale, the bare Coulomb interaction, via the intermediate scale of short-range magnetic correlations, down to the lowest scale of d-wave superconductivity and other ordering phenomena.

This diversity of scales is a major obstacle to a straightforward numerical solution of microscopic models, since the most interesting phenomena emerge only at low temperatures and in systems with a very large size. It is also hard to deal with by conventional many-body methods, if one tries to treat all scales at once and within the same approximation, for example by summing a subclass of Feynman diagrams. Perturbative approaches which do not separate different scales are plagued by infrared divergences, and are therefore often inapplicable even at weak coupling, especially in low dimensions.

It is thus natural to treat degrees of freedom with different energy scales successively, descending step by step from the highest scale present in the microscopic system. This is the main idea behind all renormalization group (RG) schemes. Using a functional integral representation this idea can be implemented by integrating out degrees of freedom (bosonic or fermionic fields) successively, following a suitable hierarchy of energy scales. This generates a one-parameter family of effective actions which interpolates smoothly between the bare action of the system, as given by the microscopic Hamiltonian, and the final effective action from which all physical properties can be extracted. The Green or vertex functions corresponding to the effective action at scale Λ obey a hierarchy of differential flow equations [1, 2, 3, 4]. This hierarchy is exact and involves the flow of functions of generally continuous variables. For these reasons it is frequently referred to as *"exact"* or *"functional"* RG.

The exact hierarchy can be solved exactly only in special cases, where the underlying model can also be solved exactly (and more easily) by other means. However, the functional RG is a valuable source for devising approximation schemes, which can be obtained by truncating the hierarchy and/or by a simplified parametrization of the Green or vertex functions. These approximations have several distinctive advantages: i) they have a renormalization group structure built in, that is, scales are handled successively and infrared singularities are thus treated properly; ii) they can be applied directly to microscopic models, not only to effective field theories which capture only some asymptotic behavior; iii) they are physically transparent, for example one can see directly how and why new correlations form upon lowering the scale; iv) one can use different approximations at different scales. Small steps from a scale Λ to a slightly smaller scale Λ' are much easier to control than an integration over all degrees of freedom in one shot.

Truncations of the functional RG hierarchy of flow equation have recently led to a number of powerful new computation schemes for interacting electron systems [5].

Applications of truncated functional RG equations to the *two-dimensional Hubbard model* have greatly improved our knowledge of its leading instabilities [6, 7, 8]. In particular, the existence of d-wave superconductivity in that model was "conclusively" (though not rigorously) established for weak repulsive interactions.

The complex behavior of isolated static *impurities* in weakly interacting one-dimensional Fermi systems *(Luttinger liquids)* is captured already by a first order truncation of the functional RG hierarchy of flow equations [9, 10]. Universal asymptotic power-laws as well as non-universal behavior and crossover phenomena at higher energy scales are obtained within the same scheme.

The functional RG can be expected to lead to further substantial progress in a variety of problems in interacting Fermi systems which are characterized by many energy scales and complicated infrared singularities. Promising candidates under investigation are the computation of ordered (symmetry-broken) phases and quantum critical behavior in itinerant electron systems.

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Functional Renormalization for Disordered Systems

Kay Jörg Wiese

(joint work with Pierre Le Doussal)

The statistical mechanics of even well-understood physical systems subjected to quenched disorder still poses major challenges. For a large class of these systems, as e.g. random-field models or elastic manifolds in quenched disorder, an apparent simplification appears: Supposing that all moments of the disorder are finite, one can show that all correlation functions in the disordered model, in the limit of zero temperature, are equivalent to those of the pure system at finite temperature in two space-dimensions less, at a temperature proportional to the second moment of the quenched disorder [1]. This phenomenon is called dimensional reduction (DR). However, one also knows that dimensional reduction gives the wrong result at large scales, more precisely at scales larger than the Larkin length. The latter is obtained from an Imry-Ma type argument, balancing elastic energy and disorder, as we detail below. For a d-dimensional elastic manifold in quenched disorder, the elastic and disorder energy are

$$E_{\text{elastic}}[u] = \int \mathrm{d}^d x \, \frac{1}{2} (\nabla u(x))^2 \ , \qquad E_{\text{DO}}[u] = \int \mathrm{d}^d x \, V(x, u(x)) \ . \tag{1}$$

For $d = 1$, these are polymers, for $d = 2$ membranes; and for $d = 3$ elastic crystals, as e.g. charge density waves. For simplicity we consider disorder which at the

microscopic scale is Gaussian and short-ranged with second moment

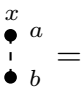
$$\overline{V(x,u)V(x',u')} = \delta^d(x-x')R(u-u') \ . \tag{2}$$

The most important observable is the roughness exponent ζ, which describes the scaling of the 2-point function $\overline{[u(x)-u(x')]^2} \sim |x-x'|^{2\zeta}$. The Imry-Ma argument compares elastic energy $E_{\rm el} \sim L^{d-2}$ and disorder energy $E_{\rm DO} \sim L^{d/2}$ to conclude that in dimensions smaller than four, disorder always wins at large scales, leading to an RG-flow to strong coupling (in a way to be made more precise below). This suggests that the dimensional reduction result $\zeta_{\rm DR} = \frac{4-d}{2}$ will become incorrect below four dimensions.

Functional RG was first introduced in [2, 3], and pioneered for the problem at hand in [4, 5]. Important improvements have been obtained by several authors, see [8] for a more detailed introduction and review.

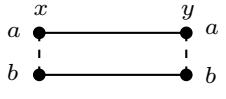
Having identified four as the upper critical dimension, one would like to expand in $\epsilon = 4-d$. Taking the dimensional reduction result $\zeta = (4-d)/2$ in $d = 4$ dimensions tells us that the field u is dimensionless. Thus, the width $\sigma = -R''(0)$ of the disorder is not the only relevant coupling at small ϵ, but any function of u has the same scaling dimension in the limit of $\epsilon = 0$, and might thus equivalently contribute. The natural consequence is that one has to follow the full function $R(u)$ under renormalization, instead of just its second moment $R''(0)$. Such an RG-treatment is most easily implemented in the replica approach: The n times replicated partition function becomes after averaging over disorder $\overline{\exp\bigl(-\frac{1}{T}\sum_{a=1}^n E_{\rm el}[u_a] - \frac{1}{T}\sum_{a=1}^n E_{\rm DO}[u_a]\bigr)} =$
$\exp\Bigl(-\frac{1}{T}\sum_{a=1}^n E_{\rm el}[u_a] + \frac{1}{2T^2}\sum_{a,b=1}^n \int \mathrm{d}^d x\, R\Bigl(u_a(x)-u_b(x)\Bigr)\Bigr)$. Perturbation theory is constructed along the following lines: The bare correlation function is graphically depicted as a solid line, with momentum k flowing through and replicas a and b at its end; equivalently the disorder vertex is represented as a dashed line:

(3)

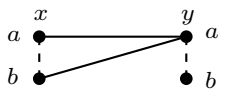
$= \dfrac{T\delta_{ab}}{k^2}$, 

$= \displaystyle\int_x \sum_{a,b} R\Bigl(u_a(x)-u_b(x)\Bigr)$.

The rules of the game are to find all contributions which correct R, and which survive in the limit of $T = 0$. At leading order, i.e. order R^2, counting of factors of T shows that only the terms with one or two correlators contribute. On the other hand, $\sum_{a,b} R(u_a - u_b)$ has two independent sums over replicas. Thus at order R^2 four independent sums over replicas appear, and in order to reduce them to two, one needs at least two correlators (each contributing a δ_{ab}). Thus, at leading order, only diagrams with two correlators survive. These are the following (noting $C(x-y)$ the Fourier transform of $1/k^2$):

(4) 

$= \displaystyle\int_x R''(u_a(x)-u_b(x))R''(u_a(y)-u_a(y))C(x-y)^2$

(5) 

$= -\displaystyle\int_x R''(u_a(x)-u_a(x))R''(u_a(y)-u_a(y))C(x-y)^2$.

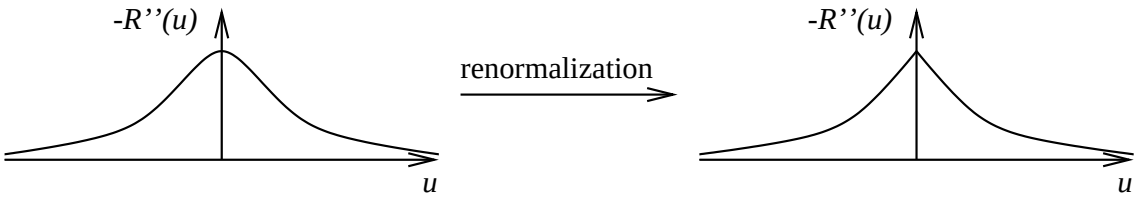


FIGURE 1. Change of $-R''(u)$ under renormalization and formation of the cusp.

In a renormalization program, we are looking for the divergences of these diagrams, for $\epsilon \to 0$. These divergences are localized at $x = y$, which allows to approximate $R''(u_a(y) - u_b(y))$ by $R''(u_a(x) - u_b(x))$. The integral $\int_{x-y} C(x-y)^2 = \int_k \frac{1}{(k^2+m^2)^2} = \frac{m^{-\epsilon}}{\epsilon}$ (using the most convenient normalization for $\int_k$), is the standard 1-loop diagram, which we have chosen to regulate in the infrared by a mass, i.e. physically by a harmonic well which is seen by the manifold. Note that the following diagram also contains two correlators (correct counting in powers of temperature), but is not a 2-replica but a 3-replica sum: 

.

Taking into account the combinatorial factors, and a rescaling of R (which remember has dimension ϵ for a dimensionless field u) as well as of the field u (its dimension being the roughness exponent ζ), we arrive at [5]

$$-m\frac{\partial}{\partial m}R(u) = (\epsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0) \ . \tag{6}$$

Note that the field u itself does not get renormalized due to the statistical tilt symmetry $u(x) \to u(x) + \alpha x$.

The important observation is that starting with smooth microscopic disorder, integration of the RG-equation leads to a cusp in the second derivative of the renormalized disorder at the Larkin-length, as depicted on figure 1. This can easily be seen from the flow-equation of the fourth derivative (supposing analyticity), which from (6) is obtained as

$$-m\frac{\partial}{\partial m}R''''(0) = \epsilon R''''(0) + 3R''''(0)^2 \ . \tag{7}$$

(Note that this explains also the appearance of the combination $\epsilon - 4\zeta$ in (6)). This equation has a singularity after finite renormalization time, equivalent to the appearance of the cusp, as depicted on figure 1. After that dimensional reduction is no longer valid.

Let us sketch how to proceed:

Different microscopic disorder leads to different RG fixed points. The latter are solutions of equation (6), generalized to 2 loop [6, 7], with $-m\frac{\partial}{\partial m}R(u) = 0$; it is important to note that given a microscopic disorder, the exponent ζ, solution of (6) is unique. For random-bond disorder (short-ranged potential-potential correlation function) the result is $\zeta = 0.20829804\epsilon + 0.006858\epsilon^2$ [5, 6, 7]. In the case of random

field disorder (short-ranged force-force correlations) $\zeta = \frac{\epsilon}{3}$. Both results compare well with numerical simulations.

Different physical situations can then be adressed. For lack of space, we refer to [8] for a more detailed introduction, and to `http://www.lpt.ens.fr/~wiese` for up-to date references.

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Nonperturbative Anomalies in QFT by Rigorous RG

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(joint work with G. Benfatto, P. Falco)

A central role in QFT is played by chiral *Ward Identities* (WI) and the relative *Anomalies*, which are corrections to the *Naive Ward Identities* found by formal manipulations of field equations [1]. WI provide in QED or in Electroweack theory relations between the bare parameters which are crucial even to prove the *renormalizability* of the theories. There is some agreement on the validity of a property, the Adler-Bardeen Theorem, see [1, 3], stating that the anomaly is given *exactly* by its lower order non vanishing contribution; on this property is based the proof of renormalizability of Electroweak theory. In general all the issues about WI and anomalies, and their role in the construction of a QFT, are quite subtle and delicate, so that it is convenient to consider models at lower dimension, which can be considered a laboratory to test general ideas.

We consider the Thirring model, describing a system of Dirac fermions with a local current-current interaction (but our techniques can be applied to a larger class of models), whose generating function $W_{\kappa,K}(J,\phi)$ is given by the log of the following *Grassmann integral*

$$\int P(\psi) e^{-\frac{\lambda_K}{2} V(\sqrt{Z_K}\psi) + \int d\mathbf{x}[\bar\phi_{\mathbf{x}}\psi_{\mathbf{x}} + \bar\psi_{\mathbf{x}}\phi_{\mathbf{x}} + Z_K^{(2)} J_{\mathbf{x}}\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}]}\} \tag{1}$$

where $\psi_{\mathbf{x}}, \bar{\psi}_{\mathbf{x}}$ are Grassmann spinor variables, $\mathbf{x} = (x_0, x)$, $V = \int d\mathbf{x} \bar{\psi}_{\mathbf{x}} \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{x}} \psi_{\mathbf{x}}$ and λ_K, Z_K, $Z_K^{(2)}$ are respectively the bare coupling, wave function and density renormalization. We find it convenient to introduce Weyl Grassmann variables $\psi^{\pm}_{\omega,\mathbf{x}}$ with $\omega = \pm$ such that $\psi_{\mathbf{x}} = (\psi^-_{\mathbf{x},+}, \psi^-_{\mathbf{x},-})$ and $\bar{\psi} = \psi^+ \gamma_0$; analogous definitions are applied to the external fields $J_{\mathbf{x}}$ and $\phi_{\mathbf{x}}$. $P(\psi)$ is a fermionic measure with propagator

$$g^{\kappa,K}_{\omega,\omega'}(\mathbf{k}) = \frac{1}{Z_K} \chi_{\kappa,K}(\mathbf{k}) \left[\begin{pmatrix} -ik_0 + k & i\mu_K \\ -i\mu_K & -ik_0 - k \end{pmatrix}^{-1} \right]_{\omega,\omega'}$$

where $\chi_{\kappa,K}(\mathbf{k})$ is a *smooth cutoff function* selecting momenta $\kappa \le |\mathbf{k}| \le K$, with $\kappa < 1$, $K > 1$ and μ_K is the bare mass. In the massless limit $\mu_K = 0$ the propagator is simply given by $g^{\kappa,K}_{\omega}(\mathbf{k}) = \frac{\chi_{\kappa,K}(\mathbf{k})}{Z_K D_\omega(\mathbf{k})}$ with $D_\omega(\mathbf{k}) = -ik_0 + \omega k$. By taking functional derivatives of $W_{\kappa,K}$ with respect to ϕ we get the Schwinger functions, while taking also derivatives with respect to J we get the vertex functions. In particular we consider $G^{2,1}_{\kappa,K,\omega,\omega'}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega'} \partial \phi^-_{\mathbf{z},\omega'}} W(\phi, J)|_{0,0}$ and $G^2_{\kappa,K,\omega}(\mathbf{y},\mathbf{z}) = \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega} \partial \phi^-_{\mathbf{z},\omega}} W(\phi, J)|_{0,0}$. The presence of the ultraviolet cutoff K and the infrared cutoff κ makes the functional integral (1) well defined, but singularities are found when cutoffs are removed for generic values of the bare parameters; to carry out the renormalization program at non-perturbative level we have to prove that there exist bare parameters (only depending on K) such that, in the limit $\kappa \to 0, K \to \infty$, the Schwinger functions verify the OS axioms [21]. The choice of regularizing the functional integral (1) via the cutoff function $\chi_{\kappa,K}(\mathbf{k})$ is particularly convenient but many others are possible; indeed we can prove that we get the same theory for a large class of regularizations, including a *lattice regularization*. On the other hand all the regularizations we can use break the *local gauge invariance* $\psi^{\pm}_{\mathbf{x},\omega} \to e^{\pm \alpha_{\omega,\mathbf{x}}} \psi^{\pm}_{\mathbf{x},\omega}$, which is valid at a classical level. Note finally that, despite the enormous literature devoted to the Thirring model, a complete nonperturbative construction of it starting from a regularized functional integral like (1) is lacking.

Non-perturbative Renormalization. Our basic result consists in the proof that cutoffs can be removed in (1) with a proper choice of the bare parameter.

THEOREM *Given λ and μ small enough, there exist bare parameters $\lambda_K = \lambda + O(\lambda^2)$, $Z_K^{(2)} Z_K^{-1} \equiv \xi_K = 1 + O(\lambda)$,*

$$Z_K = K^{-\eta}\big(1 + \mathrm{O}(\lambda^2)\big) \qquad \mu_K = \mu K^{-\bar{\eta}}\big(1 + \mathrm{O}(\lambda)\big) \tag{2}$$

with η and $\bar{\eta}$ independent of μ and such that $\eta = a\lambda^2 + \mathrm{O}(\lambda^3)$, $\bar{\eta}_\lambda = -b\lambda + \mathrm{O}(\lambda^2)$, $a, b > 0$, such that the Schwinger functions at non-coinciding points exist in the limit $\kappa \to 0, K \to \infty$ and verify OS axioms. In particular, if $\lim_{\kappa^{-1},K\to\infty} G^2_{\kappa,K,\omega} \equiv G^2_\omega$, there are positive constants C and c, such that

$$\big|G^2_\omega(\mathbf{x},\mathbf{y})\big| \le \frac{C}{|\mathbf{x}-\mathbf{y}|^{1+\eta}} e^{-c\sqrt{\mu^{1+\hat{\eta}}|\mathbf{x}-\mathbf{y}|}} \tag{3}$$

with $\hat{\eta} = -b_1\lambda + O(\lambda^2)$, $b_1 > 0$. *Moreover,* $G^2_\omega(\mathbf{x},\mathbf{y})$ *is singular for* $\mathbf{x} \to \mathbf{y}$ *and diverges as* $|\mathbf{x}-\mathbf{y}|^{-1-\eta}$.

The above results are *uniform* in the mass, and the massless limit is reached smoothly.

Anomaly and Anomaly Renormalization. We can consider the WI in the massless limit $\mu_K = 0$; the presence of the cutoff function $\chi_{h,N}$ breaks the chiral invariance, so that the WI has additional terms

$$D_\omega(\mathbf{p})G^{2,1}_{\kappa,K,\omega,\omega'}(\mathbf{p},\mathbf{k}) = \delta_{\omega,\omega'}[G^2_{\kappa,K,\omega}(\mathbf{k}-\mathbf{p}) - G^2_{\kappa,K,\omega}(\mathbf{k})] + \Delta^{2,1}_{\kappa,K,\omega,\omega'}(\mathbf{p},\mathbf{k}) \tag{4}$$

where $\Delta^{2,1}_{\kappa,K,\omega,\omega'} = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega'}\partial\phi^-_{\mathbf{z},\omega'}} H_{\kappa,K}(\phi,J)|_{0,0}$ and $H_{\kappa,K}(\phi,J)$ is given by the log of a functional integral similar to (1), with the difference that $Z^{(2)}_N \int d\mathbf{x} J(\mathbf{x})\bar{\psi}_{\mathbf{x}}\psi_{\mathbf{x}}$ in the exponent is replaced by $Z_N \int d\mathbf{k}d\mathbf{p} C^{\kappa,K}_\omega(\mathbf{k},\mathbf{k}-\mathbf{p})\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},\omega}$ where $C^{\kappa,k}_\omega(\mathbf{k},\mathbf{k}+\mathbf{p})$ is given by $([\chi_{\kappa,K}(\mathbf{k})]^{-1}-1)D_\omega(\mathbf{k}) - ([\chi_{\kappa,K}(\mathbf{k}+\mathbf{p})]^{-1}-1)D_\omega(\mathbf{k}+\mathbf{p})$. Note that $\Delta^{2,1}_{\kappa,K,\omega,\omega'}$ is a rather complex functional integral depending on the details of the cutoff function; however, , it can be written in a remarkably simple form; we prove in fact that $\Delta^{2,1}_{\kappa,K,\omega,\omega'}$ *can be written as*

$$\nu_+ D_\omega(\mathbf{p})G^{2,1}_{\kappa,K,\omega,\omega'} + \nu_- D_{-\omega}(\mathbf{p})G^{2,1}_{\kappa,K,-\omega,\omega'} + H^{2,1}_{\kappa,K,\omega,\omega'} \tag{5}$$

where ν_+ *and* ν_- *are suitable functions of* λ *such that*

$$|\nu_+ - a_+\lambda^2| \le c\lambda^3 \quad |\nu_- - a_-\lambda| \le C\lambda^2 \tag{6}$$

with $a_- = \frac{1}{4\pi}$ *and* a_+ *strictly negative; moreover, for fixed non-zero* $\mathbf{k},\mathbf{p}$,

$$\lim_{\kappa^{-1},K\to\infty} H^{2,1}_{\kappa,K,\omega,\omega'}(\mathbf{p},\mathbf{k}) = 0 \tag{7}$$

Eqs. (4), (5) and (7) imply that, in the limit $\kappa^{-1}, K \to \infty$, the model (1) really verifies an anomalous WI of the form assumed in [17]; however the values of the anomaly coefficients are *different* with respect to the ones in [17] and in particular the Anomaly Non-renormalization property *does not* hold.

New anomaly in the Schwinger-Dyson equation. The two point function $G^2_{\kappa,K,\omega}$ verifies a *Schwinger-Dyson* (SD) equation

$${\xi_K}^{-1} G^2_{\kappa,K,\omega}(\mathbf{k}) = g^{\kappa,K}_\omega(\mathbf{k})[Z_K^{-1} - \lambda_K \int d\mathbf{p} G^{2,1}_{\kappa,K;-\omega,\omega}(\mathbf{p},\mathbf{k})] \tag{8}$$

By using the WI (4) and (5), we get, if $\alpha = (1-\nu_+-\nu_-)^{-1}$, $\bar{\alpha} = (1-\nu_++\nu_-)^{-1}$ and $A_\omega = (\alpha + \omega\bar{\alpha})/2$

$$G^2_{\omega,\kappa,K}(\mathbf{k}) = \frac{g^{\kappa,K}_\omega(\mathbf{k})}{Z_K} - \lambda_K A_- g^{\kappa,K}_\omega(\mathbf{k}) \int d\mathbf{p} \frac{G^2_{\omega,\kappa,K}(\mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})} - \sum_{\omega'} \lambda_K A_{\omega'} g^{\kappa,K}_\omega(\mathbf{k}) \int d\mathbf{p} \frac{H^{2,1}_{K,\kappa,\omega,\omega'}(\mathbf{p},\mathbf{k})}{D_{-\omega}(\mathbf{p})} \tag{9}$$

If the last term in (9) were vanishing in the limit $\kappa^{-1}, K \to \infty$, one would get a closed equation for G^2_ω, which is identical to the closed equation obtained in [17], inserting the WI in the SD equation. *However this is not what happens*; $H^{2,1}_{\kappa,K,\omega,\omega'}$

is vanishing in the limit $\kappa^{-1}, K \to \infty$ at $\mathbf{k}, \mathbf{p}$ fixed, but the same is *not* true for its integral over $\mathbf{p}$. Intuitively this can be understood by noting that the integral involves momenta close to the u.v. cutoff scale K, where $H^{2,1}_{\kappa,K,\omega,\omega'}$ is not small at all. In other words: *even if the WI and the SD equation are true in the model (1), in the limit $\kappa^{-1}, K \to \infty$, the closed equation obtained combining the two identities is not verified; this is a new anomaly* which is hard to see in a purely perturbative approach and in fact it was never noticed before. The fact that the last term in (9) is not vanishing in the limit $\kappa^{-1}, K \to \infty$ could imply that there is no closed equation for G^2_ω; however we can prove that $\lambda_K g^{\kappa,K}_\omega(\mathbf{k}) \int d\mathbf{p} \frac{H^{2,1}_{\kappa,K,\omega,\varepsilon\omega}}{D_{-\omega}(\mathbf{p})}$ *can be written as*

$$\alpha_{\varepsilon,\kappa,K} \frac{g^{\kappa,K}_\omega(\mathbf{k})}{Z_K} + \rho_\varepsilon G^2_{\omega,\kappa,K}(\mathbf{k}) + R^{4,\kappa,K}_\varepsilon(\mathbf{k}) \tag{10}$$

with $\lim_{\kappa^{-1},K\to\infty} R^{4,\kappa,K}_\varepsilon(\mathbf{k}) = 0$ *and* $\rho_\pm$ *suitable functions of* λ *such that* $\rho_+ = c_2\lambda^2 + O(\lambda^2)$, $\rho_- = c_4\lambda + O(\lambda)$. By inserting (10) in (9) we get then, in the limit $\kappa^{-1}, K \to \infty$, a closed equation which is however *different* from the one assumed in [17]; in particular, the relation between the critical index η of the two point Schwinger function and the anomaly coefficients $\nu_\pm$ is given by

$$\eta = \frac{\lambda_\infty}{2\pi} \frac{\nu_-}{(1-\nu_+)^2 - \nu_-^2} \frac{1}{1 + \sum_\varepsilon A_\varepsilon \rho_\varepsilon} \tag{11}$$

As $\sum_\varepsilon A_\varepsilon \rho_\varepsilon = c_0\lambda + O(\lambda^2)$, $c_0 > 0$, we see that the new anomaly produces a different relation with respect to the ones found in [17]. Note that the combination of WI and SD equations is a rather general technique in QFT; it is used, for instance, in QED by [19, 11] and in condensed matter physics in [20], hence this new anomaly could be relevant in a more general setting.

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Renormalization of the Non-commutative Gross-Neveu Model

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From the rebirth of non-commutative quantum field theories [1, 2, 3], people were faced to a major difficulty. A new (with respect to the usual commutative theories) kind of divergences appeared in non-commutative field theory [4, 5]. This UV/IR mixing incited people to declare such theories non-renormalizable. Nevertheless H. Grosse and R. Wulkenhhar found recently the way to overcome such a problem by modifying the propagator. Such a modification will be now called "vulcanization". They proved the perturbative renormalizability, to all orders, of the non-commutative Φ^4 theory (1) on the four-dimensionnal Moyal space [6, 7].

$$S = \int dx \, \frac{1}{2}\phi \left(-\Delta + \Omega^2 \tilde{x}^2 + m^2\right) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \tag{1}$$

Their proof is written in the matrix basis. This is a basis for the Schwartz class functions where the Moyal product becomes a simple matrix product [8, 9]. A Moyal based interaction has a non-local oscillating kernel. The main advantage of the matrix basis is that the interaction is then of the type $\mathrm{Tr}\Phi^4$. This form is much easier to use to get useful bounds. The main drawback is the very complicated propagator (see [10] for a complete study of the Gross-Neveu propagator in the matrix basis). This is one of the reasons which lead collaborators and myself to recover in a simplified manner the renormalizability of the non-commutative Φ^4 theory in x-space [11]. The direct space has several advantages. First of all, the propagator may be computed exactly (and used). It has a Mehler-like form in the Φ^4, LSZ and Gross-Neveu theories [10, 11, 12]. The x-space allows to compare the behaviour of commutative and non-commutative theories. It seems to allow a simpler handling of symmetries like parity of integrals. This point is very useful for the renormalization of the Gross-Neveu model. We also plan to extend renormalizability proofs into the non-perturbative domain thanks to constructive techniques developped in x-space. Finally, at the end, when Physics enter into the game, we would like to have some experience with our physical space. Of course x-space has also drawbacks. It forces to deal with non absolutely convergent integrals. We have to take care of oscillations. Until now it is much more difficult to get the exact topological power-counting of the known non-commutative field theories in direct space than in the matrix basis. The non-commutative parametric representation would certainely provide an other way to get the full power-counting [13].

Apart from the Φ^4_4, the modified bosonic LSZ model [11] and supersymmetric theories, we now know several renormalizable non-commutative field theories. Nevertheless they either are super-renormalizable (Φ^4_2 [9]) or (and) studied at a special point in the parameter space where they are solvable (Φ^3_2, Φ^3_4 [14, 15], the LSZ models [16, 17, 18]). Although only logarithmically divergent for parity reasons, the non-commutative Gross-Neveu model is a just renormalizable quantum field theory as Φ^4_4. Its main interesting feature is that it is completely equivalent to a non-local fermionic field theory in a constant magnetic background. Then apart from strengthening the "vulcanization" procedure to get renormalizable non-commutative field theories, the Gross-Neveu model may also be useful for the study of the quantum Hall effect. It is also a good first candidate for a constructive study [19] of a non-commutative field theory as fermionic models are usually easier to construct. Finally its commutative counterpart being asymptotically free and exhibiting dynamical mass generation [20, 21, 22], a study of the physics of this model would be interesting.

In [23], I prove the perturbative renormalizability of the *orientable* non-commutative Gross-Neveu model defined by the following action functionnal

$$S[\bar{\psi},\psi] = \int dx \left(\bar{\psi}\left(-\imath\gamma^\mu\partial_\mu + \Omega\gamma^\mu\widetilde{x}_\mu + m\right)\psi + V_{\rm o}(\bar{\psi},\psi)\right)(x) \tag{2}$$

where $V_{\rm o}(\bar{\psi},\psi) = \frac{1}{4}\sum_{a,b}\lambda_1\bar{\psi}_a\star\psi_a\star\bar{\psi}_b\star\psi_b + \lambda_2\bar{\psi}_a\star\psi_b\star\bar{\psi}_b\star\psi_a + \lambda_3\bar{\psi}_a\star\psi_b\star\bar{\psi}_a\star\psi_b$. The main difficulty in the proof is due to the propagator. By multiscale analysis [19], the Φ^4 propagator in a slice i behaves like

$$C^i_\Phi(x,y) \sim M^{2i}\, e^{-M^{2i}(x-y)^2 - M^{-2i}(x+y)^2}. \tag{3}$$

The term $\exp -M^{-2i}(x+y)^2$ was called "masslet". In the Gross-Neveu model, the masslet is replaced by an oscillation. The propagator behaves like

$$C^i_{\rm GN}(x,y) \sim M^i\, e^{-M^{2i}(x-y)^2 + 2\imath\Omega x\Theta y}. \tag{4}$$

Whereas, in the Φ^4 case, the (vertex) oscillations were only useful to prove the convergence of the non-planar graphs, here the total oscillation (vertex + propagator) is needed to get the power-counting even for planar graphs. This requires a carefull exploitation of those oscillations.

Moreover the Gross-Neveu model exhibits two new features (with repsect to Φ^4). At first, there is a remaining UV/IR mixing. It concerns the planar four-point graphs with two broken faces ($N = 4, g = 0, B = 2$, see [6, 24]). These graphs are logarithmically divergent if the two points in the second broken face are joined by a single line at a lower scale. They converge if it is not the case. Whereas those connected components are not renormalizable by a vertex-like counterterm of the initial Lagrangian, they are nevertheless renormalizable by the corresponding two-point function. The second feature is that the vacuum graphs recover translation invariance.

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Towards a complete renormalization group trajectory between two fixed points

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This is a report on work in progress [1] aiming at constructing a complete RG trajectory, with rigorous control across all scales, joining a Gaussian ultraviolet fixed point to a nontrivial infrared fixed point. Note that since this is ongoing work, some statements and definitions might be altered in future accounts. The model we consider is the $\phi^4_{3,\epsilon}$ Euclidean quantum field theory studied by Brydges, Mitter and Scoppola [3]. It is a three-dimensional ϕ^4 model in infinite volume with a modified free propagator depending on a bifurcation parameter ϵ. When $\epsilon = 1$ this is the usual ϕ^4_3 model; whereas at $\epsilon = 0$ the coupling constant becomes marginal as in the ϕ^4_4 model. Here we consider the situation where $\epsilon > 0$ is very small.

More precisely, let $v : \mathbb{R}^3 \to \mathbb{R}$ be a pointwise nonnegative C^∞ and rotation invariant function which vanishes when $|x|_\infty \geq \frac{1}{2}$ where $|x|_\infty = \max_{1\leq i\leq 3}(x_i)$, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let $u = v * v$ be the convolution of v with itself. The u function is positive both in direct and momentum spaces; it is also strictly short-ranged in direct space. We define the smooth UV-regularized covariance

$$C(x,y) = \int_1^{+\infty} \frac{dl}{l} l^{-(\frac{3-\epsilon}{2})} u\left(\frac{x-y}{l}\right)$$

as well as the corresponding Gaussian measure $d\mu_C(\phi)$. The model we consider is that of functional integrals $Z = \int d\mu_C(\phi) \ldots e^{-V(\phi)}$ with an interaction potential $V(\phi) = g \int d^3x : \phi^4 :_C (x) + \mu \int d^3x : \phi^2 :_C (x)$ with Wick ordering with respect to the covariance C. The RG transformation we use in order to study this model is that of [3], in the formal infinite volume limit. It is given by a recursion

$$\left\{ \begin{array}{lcl} g_{n+1} & = & L^\epsilon g_n - L^{2\epsilon} a(L,\epsilon) g_n^2 + \xi_{g,n}(g_n, \mu_n, R_n) \\ \mu_{n+1} & = & L^{\frac{3+\epsilon}{2}} \mu_n + \xi_{\mu,n}(g_n, \mu_n, R_n) \\ R_{n+1} & = & \mathcal{L}_n(R_n) + \xi_{R,n}(g_n, \mu_n, R_n) \end{array} \right.$$

where the integer $L > 1$ is the scale ratio of a single RG step; $a(L,\epsilon) = \mathcal{O}(\log L)$ is the, second order in perturbation theory, contribution given by the bubble graph to the renormalization of the coupling g; and the R_n live in a family of Banach spaces $\mathcal{B}_{R,n}$ of polymer activities in the sence of [4]. The map $\mathcal{L}_n$ is a linear operator $\mathcal{B}_{R,n} \to \mathcal{B}_{R,n+1}$ which incorporates the integration over the fluctuation field, rescaling, volume effect (summing small cubes inside big ones), and the substraction of dangerous relevant parts. The map actually depends on the other two dynamical variables g_n and μ_n, but we suppressed this dependence in what follows for ease of exposition.

Our aim is to construct a double-sided sequence $s = (g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ which solves this recursion and such that $\lim_{n\to-\infty}(g_n, \mu_n, R_n) = (0,0,0)$ the Gaussian ultraviolet fixed point, and $\lim_{n\to+\infty}(g_n, \mu_n, R_n) = (g_*, \mu_*, R_*)$ the BMS nontrivial infrared fixed point [3]. As sets and even as topological vector spaces the $\mathcal{B}_{R,n}$

are the same; what changes is the norm. Indeed, $\mathcal{B}_{R,n}$ is equipped with a norm $||\cdot||_{R,n}$ which incorporates a measurment of the typical size of a low-momentum field $|\phi(x)| \sim h_n \sim g_n^{-\frac{1}{4}}$. This is an avatar of the large-field problem and the domination procedure addressed for instance in [6, 5, 7, 2]. In addition to adapting the estimates of [3] to the present context (already a rather time consuming process), the main new difficulty we had to overcome is due to this variation of the norms with the discrete coordinate n along the trajectory.

We begin by considering the approximate sequence $\bar{s} = (\bar{g}_n, 0, 0)_{n\in\mathbb{Z}}$ where $\bar{g}_{n+1} = f(\bar{g}_n)$ with $f(x) = L^\epsilon x - L^{2\epsilon} a(L,\epsilon) x^2$; and construct the deviation $\delta s = s - \bar{s}$ by a contraction mapping argument in a Banach space of sequences with a weighted norm

$$||s|| \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}} \left(\max\left\{|g_n|\bar{g}_n^{-1}, |\mu_n|\bar{g}_n^{-\lambda_\mu}, ||R_n||_{R,n}\bar{g}_n^{-\lambda_R}\right\}\right) .$$

The only free parameter is $g_0 = \bar{g}_0$ the coupling at the "anthropic" (or perhaps "antropic") scale. One uses the approximate sequence $\bar{g}_n$ in order to define the polymer activity norms $||\cdot||_{R,n}$. One also needs λ_μ slightly smaller than 2, and λ_R slightly smaller than 3, for the construction to work. In terms of the deviation $\delta s = (\delta g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ the RG map is

$$\left\{\begin{array}{lcl} \delta g_{n+1} &=& f'(\bar{g}_n)\delta g_n + \left[-L^{2\epsilon} a\ \delta g_n^2\ + \xi_{g,n}(\bar{g}_n + \delta g_n, \mu_n, R_n)\right] \\ \mu_{n+1} &=& L^{\frac{3+\epsilon}{2}}\mu_n + \xi_{\mu,n}(\bar{g}_n + \delta g_n, \mu_n, R_n) \\ R_{n+1} &=& \mathcal{L}_n(R_n) + \xi_{R,n}(\bar{g}_n + \delta g_n, \mu_n, R_n) \ . \end{array}\right.$$

When the latter is iterated, using both forward and backward discrete integral equations, we obtain our contraction mapping $s \mapsto s'$, or rather $\delta s \mapsto \delta s'$, as

$$\delta g'_0 \stackrel{\text{def}}{=} 0 \ ,$$

$\forall n > 0$

$$\delta g'_n = \sum_{0\le p<n} \left(\prod_{p<j<n} f'(\bar{g}_j)\right) \left[-L^{2\epsilon} a\ \delta g_p^2\ + \xi_{g,p}(\bar{g}_p + \delta g_p, \mu_p, R_p)\right] \ ,$$

$\forall n < 0$

$$\delta g'_n = -\sum_{n\le p<0} \left(\prod_{n\le j\le p} \frac{1}{f'(\bar{g}_j)}\right) \left[-L^{2\epsilon} a\ \delta g_p^2\ + \xi_{g,p}(\bar{g}_p + \delta g_p, \mu_p, R_p)\right] \ ,$$

$\forall n \in \mathbb{Z}$

$$\mu'_n = -\sum_{p\ge n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)}\ \xi_{\mu,p}(\bar{g}_p + \delta g_p, \mu_p, R_p) \ ,$$

$\forall n \in \mathbb{Z}$

$$R'_n = \sum_{p<n} \mathcal{L}_{n-1} \circ \mathcal{L}_{n-2} \circ \cdots \circ \mathcal{L}_{p+1} \left(\xi_{R,p}(\bar{g}_p + \delta g_p, \mu_p, R_p)\right) \ .$$

The crux of the construction relies on choosing the RG map, and in particular the so-called extraction step, appropriately so that the linear operators $\mathcal{L}_n$ are uniformly contractive. This is a translation of the fact the R's are irrelevant terms.

Finally, this work which is a follow up on [3], is only the second step in a vast program aiming at a rigorous mathematical investigation of the finer features of K. G. Wilson's theory [8]. In the abscence of a nonperturbative definition of dimensional regularization, the $\phi^4_{3,\epsilon}$ model probably provides the best available testing ground for this enterprise. Future work should address the true infinite volume and scaling limits, as well as the construction of correlation functions, the investigation of the presence or abscence of anomalous dimensions for primary or composite fields, and eventually the operator product expansion. Also note that an interesting feature of our framework is that by varying the g_0 one should obtain a sequence of interval patches along a conjectural one-dimensional invariant manifold joining the two fixed points. One could try to see if one can glue together these patches into a curve parametrized by the logarithmic scale. This could possibly open the door to the investigation of the Stückelberg–Peterman/Gell-Mann–Low Prewilsonian continuous RG equation which has so far remained elusive in Bosonic constructive field theory. As one can surmise, this program might very well provide twelve tasks each for twelve clones of Hercules.

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Limiting Absorption Principle and Local Decay for the Standard Model of the Non-relativistic QED

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(joint work with Jürg Fröhlich, Marcel Griesemer)

The mathematical framework of the theory of non-relativistic matter interacting with the quantized electro-magnetic field (non-relativistic quantum electrodynamics) is well established. It is given in terms of the standard quantum Hamiltonian

$$H_g^{SM} = \sum_{j=1}^{n} \frac{1}{2m_j}(i\nabla_{x_j} + gA(x_j))^2 + V(x) + H_f \tag{1}$$

acting on the Hilbert space $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}$, the tensor product of the state spaces of the particle system and the quantized electromagnetic field. Here SM stands for 'standard model' and $g := \alpha^{3/2}$ where $\alpha \approx \frac{1}{137}$ is the fine-structure constant. (The notation above and units we use are explained below.) This model describes, in particular, the problem of radiation — emission and absorption of radiation by systems of matter, such as atoms and molecules — as well as other processes of interaction of quantum radiation with matter (e.g. photoeffect).

For reasonable potentials $V(x)$ the operator H_g^{SM} is self-adjoint and its spectral and resonance structure - and therefore dynamics for long but finite time-intervals - is well understood. However, we still know little about its asymptotic dynamics. In particular the scattering theory for this operator does not, at present, exist.

At the foundat ion of the construction of the modern quantum scattering theory lies the property of local decay (or limiting absorption principle) which states that the system under consideration is either in a bound state, or, as time goes to infinity, it breaks apart in the sense that the probability to occupy any bounded region of the physical space tends to zero (average distance between the particles goes to infinity).

So far the local decay for the Hamiltonian H_g^{SM} is proven only for the energies away from an $O(g^2)$ neighborhood of the ground state energy. This situation is not satisfactory, since starting from any energy, the system eventually winds up in this neighborhood. Indeed, while the total energy is conserved, the photons carry away the energy from regions of the space where matter is concentrated. In this presentation we report on a proof of local decay of states in the spectral interval for H_g^{SM} in the $(e_1 - e_0)/12$-neighborhood of the ground state energy. Here e_0 and e_1 are the ground state and the first excited state energies of the particle system.

However, the main achievement of this paper, we feel, lies in its method, namely, in application of the spectral renormalization group, developed in [2, 3, 1], to analysis unitary quantum dynamics - to proving the local decay property mentioned above.

In this presentation we use the dimensionless units in which the Planck constant divided by 2π and speed of light are equal to 1, $\hbar = 1$ and $c = 1$, and the electron charge is equal to $-\sqrt{\alpha}$ ($e = -\sqrt{\alpha}$). In these units the distance is measured

in the units of the Bohr radius $r_{bohr} = (m\alpha)^{-1}$, and the energy, in the units of $m\alpha^2 = 2(Rydberg)$.

Next, we explain the notation employed in (1). Our particle system consists of n particles of masses m_j (the ratio of the mass of the j-th particle to the mass of an electron) and positions x_j, where $j = 1, ..., n$. We write $x = (x_1, \ldots, x_n)$. The total potential of the particle system is denoted by $V(x)$. The Hilbert space of the particle system, denoted by $\mathcal{H}_{el}$, is either $L^2(\mathbb{R}^{3n})$ or a subspace of this space determined by a symmetry group of the particle system.

The quantized electromagnetic field is described by the quantized vector potential

$$A(y) = \int (e^{iky} a(k) + e^{-iky} a^*(k)) \frac{\chi(k) d^3k}{(2\pi)^3 \sqrt{2|k|}}, \tag{2}$$

where χ is an ultraviolet cut-off: $\chi(k) = 1$ in a neighborhood of $k = 0$ and it vanishes sufficiently fast at infinity, and its dynamics, by the quantum Hamiltonian

$$H_f \;=\; \int d^3k \; a^*(k) \; \omega(k) \; a(k) \; , \tag{3}$$

both acting on the Fock space $\mathcal{F}$. Above, $\omega(k) = |k|$ is the dispersion law connecting the energy, $\omega(k)$, of the field quantum with wave vector k, $a^*(k)$ and $a(k)$ denote the creation and annihilation operators on $\mathcal{F}$ and the right side can be understood as a weak integral.

One verifies that H_f defines a positive, self-adjoint operator on $\mathcal{F}$ with purely absolutely continuous spectrum, except for a simple eigenvalue 0 corresponding to the eigenvector Ω (the vacuum vector).

To obtain expression (1) we first assume that $V(x)$ is the total Coulomb potential of the particle system and rescale the original Hamiltonian appropriately (see [4]); but then we relax this restriction on $V(x)$ and consider rather general Kato potentials $V(x)$. In order not to deal with the problem of center-of-mass we assume that either some of the particles (nuclei) are infinitely heavy or the system is placed in an external potential field.

In this presentation we will be also consider the Nelson model. We will not describe the Hamiltonian H_g^N for this model here but mention only that here, as before, g is a positive parameter - a coupling constant - which we assume to be small, and the coupling function is taken to be $\frac{\kappa(k)}{|k|^{1/2}} \, e^{-ikx}$. Here, $\kappa = \kappa(k)$ is a real function with the property that

$$\|\kappa\|_\mu \;:=\; \Big(\int \frac{d^3k}{|k|^{3+2\mu}} \, |\kappa(k)|^2 \Big)^{1/2} \;<\; \infty \, , \tag{4}$$

where $d \geq 1$ is the dimension of the physical space and $\mu > 0$, and $g \geq 0$ is a coupling constant.

In this presentation we describe results of [7] on the limiting absorption principle implies the local decay property for the Hamiltonians H_g^{SM} and H_g^N. (Note that our approach can also handle the perturbations quadratic in creation and annihilation operators, a and a^*.) The limiting absorption principle states that

the resolvent sandwiched by appropriate weights has Hölder continuous limit on the spectrum. The local decay property which we establish below says roughly that in evolving states with energies away from the eigenvalues the Hamiltonian the photons (or phonons) move out of any bounded region of the physical space. This result, important in its own right, serves also as the first step in constructing modern scattering theory for quantum particles and massless fields.

Let B denote the self-adjoint generator of dilatations on the Fock space $\mathcal{F}$. It can be expressed in terms of creation- and annihilation operators as

$$B \;=\; \frac{i}{2}\int d^3k\; a^*(k)\left\{k\cdot\nabla_k+\nabla_k\cdot k\right\}a(k)\,. \tag{5}$$

We further extend it to the Hilbert space $\mathcal{H}=\mathcal{H}_{el}\otimes\mathcal{F}$. Let $\langle B\rangle := (\mathbf{1}_{\{+\}}B^2)^{1/2}$. Let furthermore e_0 and e_1 are the ground state and first excited state energies of H_{el}. Henceforth the restriction $g \ll 1$ will mean that there is a positive number g_0 depending on the norms like (4) and $g \le g_0$ and similarly for other parameters. In [7] we prove the following

Theorem 4. *Let H be either H_g^{SM} or H_g^N, the two Hamiltonians defined above, and let $g \ll 1$. Let $\Delta \subset (E_g, E_g+\frac{1}{2}(e_1-e_0))$, where E_g is the ground state energy of H. If $s>1/2$, then*

$$\langle B\rangle^{-s}(H-\lambda-i0)^{-1}\langle B\rangle^{-s} \in C^\alpha(\Delta). \tag{6}$$

for $0<\alpha<s-\frac{1}{2}$.

The above theorem has the following consequence.

Corollary 5. *For Δ as above and for any function $f(\lambda)$ with* supp$g \subseteq \Delta$ *and for $\alpha < s-\frac{1}{2}$, we have*

$$\|\langle B\rangle^{-s}e^{-iHt}f(H)\langle B\rangle^{-s}\| \le Ct^{-\alpha}. \tag{7}$$

The statement follows, in a standard way, from (6) and the formula

$$\langle B\rangle^{-s}e^{-iHt}f(H)\langle B\rangle^{-s} = \int_{-\infty}^{\infty} d\lambda f(\lambda)e^{-i\lambda t}\mathrm{Im}\langle B\rangle^{-s}(H-\lambda-i0)^{-1}\langle B\rangle^{-s}.$$

Remark 6. . *We conjecture that one can extend the method of this paper to the more general energy interval $\sigma(H)\setminus\sigma_{\rm pp}(H)$ for the Nelson model and $\big(\sigma(H)\setminus\sigma_{\rm pp}(H)\big)\bigcap(-\infty,\Sigma_{el})$, where $\Sigma_{el}:=\inf\sigma(H_{el})$.*

Previously the limiting absorption principle and local decay estimates were proven in [5] for the standard model of non-relativisitic QED and for the Nelson model away from neighborhoods of the ground state energy and ionization threshold and in [8, 9], under rather stringent assumptions, including that on the infra-red behavior of the coupling functions, but for all values of the coupling constant, for the Nelson model (see also [2, 3, 4, 10, 12, 6, 11] for earlier works).

We prove the theorem by combining the Mourre estimate with the renormalization group (RG) approach of [2, 3, 1]. We do this in three steps. In the first step we apply a specially designed renormalization map, $\mathcal{R}^0_{\rho_0}$, to the family $H-\lambda\mathbf{1}$,

where H is either H_g^{PF} or H_g^N. We show that the result of this application, $H^{(0)}(\lambda) := \mathcal{R}^0_{\rho_0}(H - \lambda\mathbf{1})$, is in a polidisc in certain Banach space. On this Banach space we construct a renormalization group transformation, $\mathcal{R}_\rho$, which we apply iteratively to the family of operators $H^{(0)}(\lambda)$. After sufficiently many iterations we obtain an operator, $H^{(n)}(\lambda) := \mathcal{R}^n_\rho(H^{(0)}(\lambda))$, which is close to the operator wH_f for some $w \in \mathbb{R}^+$. We then analyze $H^{(n)}(\lambda)$ using the Mourre theory.

Since the operators $H^{(n)}(\lambda), n \geq 0$ are bounded most of the Mourre theory (the virial theorem, the second commutator estimate) become rather simple. Also, a specific form of the interaction and the coupling functions becomes irrelevant. In particular, as was mentioned above, our approach can handle the perturbations quadratic in creation and annihilation operators, a and a^*. Also, with little more work one can establish an explicit restriction (in terms of the particle energy difference $e_1 - e_2$) on the coupling constant g.

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The strange world of nonlinear sigma models with non-amenable target space

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(joint work with Anthony Duncan and Max Niedermaier)

Nonlinear sigma models with noncompact target space and non-amenable symmetry group were introduced in 1979 by F. Wegner in the study of disordered electron systems. They also occur in dimensionally reduced quantum gravity.

Duncan, Niedermaier and the author [1, 2] investigated these models and found that in any dimension, even one and two, spontaneous symmetry breaking necessarily occurs as a consequence of the non-amenability of their symmetry group. Superficially this might seem to be in conflict with the Mermin-Wagner theorem, but on closer inspection it is not. There are large fluctuations of 'divergent variance' in spite of the symmetry breaking; nevertheless the fluctuations are insufficient to restore the symmetry. A result which is in some sense complementary has been obtained recently by Spencer and Zirnbauer [3]: they showed that in dimension 3 and higher these models show conventional spontaneous symmetry breaking with normal fluctuations.

The one-dimensional model is analyzed to a large extent by explicit computation, whereas for the two-dimensional one we employ numerical as well as analytic methods (such as the limit of infinitely many dimensions in target space).

In addition to the symmetry breaking we find that the low-dimensional models show other peculiarities, such as dependence on boundary conditions even for invariant observables and after the thermodynamic limit has been taken. The Osterwalder-Schrader reconstruction yields a nonseparable Hilbert space and in addition to a unitary, continuous representation on the ground state space, also discontiuous representations occur. In one dimension this might be considered a pathology of this reconstruction, since the corresponding quantum mechanical model can be defined directly on a separable Hilbert space carrying a continuous unitary representation; in two dimensions, however, this reconstruction seems to be the only possible path for obtaining a quantum mechanical interpretation (or possibly a quantum field theory) corresponding to these models; so this nonseparability has to be taken seriously.

In [4] Niedermaier and the author analyze the generalized ground state space from the point of view of harmonic analysis of the global symmetry group. This is done in any dimension, but on a spatially finite lattice. The problem requires solving a problem analogous to the separation of the center of mass motion in n-particle qunatum mechanics, but this time not in euclidean space but rather on a symmetric Riemnannian space of constant negative curvature. Under rather general conditions it turns out that the ground state space carries a distinguished unitary irreducible representation of the symmetry group, namely the limit of the spherical principal series.

Perturbative Renormalization Group analysis predicts for 2 dimensions asymptotic freedom in the infrared (as in the Φ^4_4 model) [5]. Whether this is confirmed by

a rigorous nonperturbative analysis or turns out to be an artefact of perturbation theory remains a challenging question. Likewise rigourous control of the thermodynamic limit in more than one dimension remains a challenging open problem.

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Renormalization on Riemannian Manifolds

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(joint work with Christoph Kopper)

Since the pioneering work of Polchinski [1] perturbative renormalization based on Wilson's differential flow equation has attracted much attention, leading to extensions in various directions. Technically, the translation symmetry of Minkowski space and of its Euclidean descendant proved very efficacious, it allows in particular to treat the perturbative flow equations in momentum space. On a given curved spacetime, however, renormalization via flow equations has to be approached in the position space. In our work [2] such an approach is being developed, albeit only on a Euclidean version of curved spacetime, i.e. on a Riemannian manifold. In contrast to flat space, there is no Wick rotation on curved spacetime, in general. Nevertheless, Bros, Epstein and Moschella [3], and, with a different method, Birke and Fröhlich [4] have shown that quantum field theories defined on particular curved spacetimes can be analytically continued to a corresponding Euclidean version, and that furthermore this Euclidean formulation allows Osterwalder-Schrader reconstruction.

Using functional integration, we consider the quantum field theory of a self-interacting real scalar field ϕ with mass m on a four-dimensional simply-connected (noncompact) Riemannian manifold $(\mathcal{M}, g)$ without boundary, which is assumed to be geodesically complete and whose sectional curvatures are bounded between two constants $-k^2$ and κ^2 . Our aim is to demonstrate perturbative renormalizability of this theory by way of Wilson's differential renormalization group equation. In the regularized free propagator considered

$$C^{\varepsilon,t}(x,y) \;=\; \int_{\varepsilon}^{t} dt'\, e^{-m^2 t'} K(t',x,y), \qquad K(t,x,y) := \text{kernel}\, e^{\,t\Delta}$$

with $0 < \varepsilon \leq t < \infty$ and Δ the Laplace-Beltrami operator associated to the Riemannian manifold, the short distance cutoff ε is kept fixed, to be removed finally, whereas t acts as the Wilsonian flow parameter. The free propagator $(-\Delta + m^2)^{-1}$ emerges in the limit $t \to \infty$, $\varepsilon \to 0$. Our treatment relies heavily on rather sharp pointwise bounds on the heat kernel $K(t, x, y)$ and its derivatives found in the mathematical literature, see e.g.[5], [6], [7]. From Wilson's differential equation follows the system of flow equations relating the connected free-propagator-amputated Schwinger functions $\mathcal{L}_{n,\,l}^{\varepsilon,\,t}(x_1, \cdots, x_n)$, $n \in 2\mathbf{N}$, after a formal loop expansion, $l \in \mathbf{N}_0$. Estimating bounds, the distributional character of these amputated Schwinger functions is accounted for by smearing $n-1$ arguments of an n-point function with a test function appropriately chosen. To extract the relevant parts in the 2- and 4-point function a covariant Taylor expansion with remainder term of the respective test functions is performed,

$$\begin{aligned}\mathcal{L}_{2,l}^{\varepsilon,t}(x_1, \vec{p}) \; &= a_l^{\varepsilon,t}(x_1)\; \vec{p}(x_1) \;-\; f_l^{\mu,\varepsilon,t}(x_1)\; (\nabla_\mu \vec{p})(x_1) \;-\; b_l^{\mu\nu,\varepsilon,t}(x_1)(\nabla_\mu \nabla_\nu \vec{p})(x_1) \\ &\quad +\; \ell_{2,l}^{\varepsilon,t}\; (x_1, \vec{p})\;, \\ \mathcal{L}_{4,l}^{\varepsilon,t}(x_1, \vec{p}) &= c_l^{\varepsilon,t}(x_1)\; \vec{p}(x_1, x_1, x_1) \;+\; \ell_{4,l}^{\varepsilon,t}(x_1, \vec{p})\,,\end{aligned}$$

with the relevant terms

$$a_l^{\varepsilon,t}(x_1) = \int_{x_2} \mathcal{L}_{2,l}^{\varepsilon,t}(x_1, x_2)\;, \quad f_l^{\mu,\varepsilon,t}(x_1) = \int_{x_2} \sigma(x_2, x_1)^\mu\; \mathcal{L}_{2,l}^{\varepsilon,t}(x_1, x_2)\;,$$

$$b_l^{\mu\nu,\varepsilon,t}(x_1) = -\,\frac{1}{2}\int_{x_2} \sigma(x_2, x_1)^\mu\; \sigma(x_2, x_1)^\nu \mathcal{L}_{2,l}^{\varepsilon,t}(x_1, x_2)\,,$$

and $c_l^{\varepsilon,t}(x_1)$ similarly. This procedure entails to treat in parallel a system of flow equations for Schwinger functions which involve insertions $\sigma(x_i, x_1)^\mu$ resulting from this expansion. The renormalization problem, primarily related to the behaviour of the heat kernel at small values of t, is essentially solved if we can integrate the flow equations up to some finite value t_R of t. Therefore, in a first stage renormalization conditions are posed at $t_R = 1$, fixing there the relevant terms specified above as a given smooth scalar-, vector-, symmetric tensor- and scalar-field, respectively. Via the flow equations we then inductively establish bounds which guarantee the existence of finite Schwinger functions $\lim_{\varepsilon \to 0} \mathcal{L}_{n,\,l}^{\varepsilon,\,t}(x_1, \cdots, x_n)$ upon removing the short distance regularization.

In a second stage, presently still under investigation, we aim at demonstrating that a particular choice of the renormalization conditions exists which leads at $t = \varepsilon$ to a 'minimal form' of the bare interaction with $a_l^{\varepsilon,\,\varepsilon}(x) = a_l^\varepsilon + R(x)\,\xi_l^\varepsilon$, $f_l^{\mu,\varepsilon,\,\varepsilon}(x) \equiv 0$, $b_l^{\mu\nu,\,\varepsilon,\,\varepsilon}(x) = g^{\mu\nu}(x)\, b_l^\varepsilon$, $c_l^{\varepsilon,\,\varepsilon}(x) = c_l^\varepsilon$, where $R(x)$ denotes the scalar curvature of the manifold and a_l^ε, ξ_l^ε, b_l^ε, c_l^ε do not depend on position.

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A Functional Integral Representation for Many Boson Systems

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(joint work with T.Balaban, J.Feldman, E.Trubowitz)

We are developing a set of tools and techniques for analyzing the large distance/infrared behaviour of many boson systems as the temperature tends to zero. The first tool is a functional integral representation for the grand canonical partition function and correlation functions of many bosons moving in a space X with a finite number of points. This tool provides a rigorous version of the formal functional integral representation used by physicists [2, §2.2].

Informally, the statement of one of our main results is

$$\mathrm{Tr}\, e^{-\beta(\mathrm{H}-\mu N)} = \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} d\mu_{R(\frac{\beta}{p})}(\phi^*_\tau, \phi_\tau)\ \ e^{\mathcal{F}(\frac{\beta}{p}, \phi^*, \phi)} \tag{1}$$

Here, H is the Hamiltonian and N is the number operator of our system of identical bosons. As usual, μ is the chemical potential and the temperature is $\frac{1}{k\beta} > 0$. For each natural number p, the discrete time interval $\mathcal{T}_p$ is given by

$$\mathcal{T}_p = \left\{\ \tau = q\frac{\beta}{p}\ \middle|\ q = 0\,,\,\cdots\,,\,p-1\ \right\}$$

For each point $(\mathbf{x},\tau)$ in the discrete space–time $X\times\mathcal{T}_p$, we introduce the complex variable $\phi(\tau,\mathbf{x}) = \phi_\tau(\mathbf{x})$ and the measure $\frac{d\phi^*_\tau(\mathbf{x})\wedge d\phi_\tau(\mathbf{x})}{2\pi\imath} = \frac{1}{\pi}\, d\Re\phi_\tau(\mathbf{x})\, d\Im\phi_\tau(\mathbf{x})$ on the complex plane $\mathbb{C}$. For each $r > 0$, we set

$$d\mu_r(\,\phi^*,\,\phi) = \prod_{\mathbf{x}\in X}\left[\frac{d\phi^*(\mathbf{x})\wedge d\phi(\mathbf{x})}{2\pi\imath}\ \chi_r(|\phi(\mathbf{x})|)\right]$$

where, χ_r is the characteristic function of the closed interval $[0,r]$. The sequence $R(\frac{\beta}{p}) > 0$ in (1) tends to infinity at an appropriate rate as $p\to\infty$. The "action" $\mathcal{F}(\varepsilon,\phi^*,\phi)$ is given by

$$\begin{aligned}\mathcal{F}(\varepsilon,\phi^*,\phi) = &\iint d\tau d\mathbf{x}\ \phi^*_\tau(\mathbf{x})(\partial^\varepsilon\phi_\tau)(\mathbf{x}) - \iint d\tau d\mathbf{x}\ \phi^*_\tau(\mathbf{x})(\mathrm{h}\phi_\tau)(\mathbf{x})\\ &+\mu\iint d\tau d\mathbf{x}\ \phi_\tau(\mathbf{x})^*\phi_\tau(\mathbf{x}) - \iiint d\tau d\mathbf{x}d\mathbf{y}\ \phi_\tau(\mathbf{x})^*\phi_\tau(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, \phi_\tau(\mathbf{y})^*\phi_\tau(\mathbf{y})\end{aligned}$$

where h is the single particle operator (for example discrete Laplacian), v the two body interaction

$$\begin{aligned}\iint d\tau d\mathbf{x}\ \psi(\tau,\mathbf{x}) &= \varepsilon \sum_{\tau\,\in\,\mathcal{T}_p}\ \sum_{\mathbf{x}\in X} \psi(\tau,\mathbf{x})\\ \iiint d\tau d\mathbf{x} d\mathbf{y}\ \psi(\tau,\mathbf{x},\mathbf{y}) &= \varepsilon \sum_{\tau\,\in\,\mathcal{T}_p}\ \sum_{\mathbf{x}\in X}\ \sum_{\mathbf{y}\in X} \psi(\tau,\mathbf{x},\mathbf{y})\end{aligned}$$

and the difference operator ∂^ε acts by

$$\partial^\varepsilon\,\phi(\tau,\mathbf{x}) = \varepsilon^{-1}\big(\phi(\tau+\varepsilon,\mathbf{x}) - \phi(\tau,\mathbf{x})\big)$$

In (1), ϕ_β is determined by the periodic boundary condition $\phi_\beta = \phi_0$.

During the course of the proof we derive a rigorous version of the formal resolution of the identity

$$\mathbb{1} = \int \prod_{\mathbf{x}\in X} \Big[\tfrac{d\phi^*(\mathbf{x})d\phi(\mathbf{x})}{2\pi\imath}\Big]\ e^{-\int d\mathbf{y}\ |\phi(\mathbf{y})|^2}\ \ |\,\phi\,\rangle\,\langle\,\phi\,|$$

We also show that the trace formula

$$\mathrm{Tr}\,B = \int d\mu(\phi^*,\,\phi)\ e^{-\int d\mathbf{y}\ |\phi(\mathbf{y})|^2}\ \ \langle\,\phi\,|\,B\,|\,\phi\,\rangle$$

applies rigorously to a certain class of operators B. These are then used to prove a rigorous variant of the formal integral representation

$$\begin{aligned}\mathrm{Tr}\ e^{-\beta(H-\mu N)} &= \lim_{p\to\infty} \int \prod_{\substack{\tau\in\mathcal{T}_p\\ \mathbf{x}\in X}} \Big[\tfrac{d\phi^*_\tau(\mathbf{x})d\phi_\tau(\mathbf{x})}{2\pi\imath}\ e^{-\int d\mathbf{y}\ |\phi_\tau(\mathbf{y})|^2}\Big]\\ &\qquad \prod_{\tau\in\mathcal{T}_p} \Big\langle\ \phi_\tau\ \Big|\ e^{-\varepsilon\,(\mathrm{H}-\mu N)}\ \Big|\ \phi_{\tau+\frac{\beta}{p}}\ \Big\rangle\end{aligned}$$

Following an analysis of $\big\langle\ \alpha\ \big|\ e^{-\varepsilon(\mathrm{H}-\mu N)}\ \big|\ \phi\ \big\rangle$, we prove the functional integral representation (1).

The next step towards controlling the thermodynamic limit of a many boson system in d space dimensions, is to express the temporal, ultraviolet limit $p\to\infty$ in (1) in a form suitable for an infrared renormalization group analysis. Principally, to extract an effective potential that exhibits the mechanism for symmetry breaking. This is quite subtle because the exponential of $\langle\phi^*,\partial^\varepsilon\phi\rangle_\varepsilon$ is highly oscillatory. The analogous step in the analysis of a many fermion system is given in [1].

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