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Topological and Geometric Methods in Group Theory

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Abstract. The focus of this meeting was the use of topological and geometric methods to study infinite discrete groups. These methods are increasingly being supplemented by powerful new techniques from analysis. Key topics included group actions on CAT(0) and tree-like spaces, filling invariants, cohomology, K-theory and ℓ^2-cohomology, amenability and Kazhdan's property (T), and deformation spaces, curve complexes and Teichmüller space.

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Introduction by the Organisers

The focus of this meeting was the use of topological and geometric methods to study infinite discrete groups. These methods are increasingly being supplemented by powerful new techniques from analysis. Key topics included group actions on CAT(0) and tree-like spaces, filling invariants, cohomology, K-theory and ℓ^2-cohomology, amenability and Kazhdan's property (T), and deformation spaces, curve complexes and Teichmüller space. More specific information is contained in the abstracts which follow in this volume.

The meeting was organized around a series of 23 lectures each of 50 minutes' duration representing the major recent advances in the area. The first day's lectures were selected from the abstracts and talk proposals which had been submitted in advance of the meeting, and the remainder were decided on Monday and Tuesday morning; in this way were able to take advantage of a full range of abstracts and also to incorporate last-minute information about exciting developments in the field. We had interesting proposals from virtually every participant but lecture slots for fewer than half to speak. We posted all abstracts on the wall of the lecture building and drew attention to them by running a poster event on Tuesday evening, at which every participant took the opportunity to introduce their research and poster. This worked very effectively and was valuable especially to younger people or those visiting the Forschungsinstitut for the first time. It also led to small groups getting together for unofficial lecture sessions in the evening and afternoon breaks on subsequent days.

There were 51 participants from a wide range of countries, including Germany, France, the United States, the United Kingdom, Greece, Russia, Poland, Switzerland and Australia. We are grateful to the European Union, which provided funds to support 6 advanced graduate students, a number of recent Ph. Ds and a few senior researchers.

We feel that the meeting was exciting and highly successful. The quality of the lectures was outstanding, and outside of lectures there was a constant buzz of intense mathematical conversations. One indication of the high degree of current activity and interest in the subject is the fact that four of the participants (including two of the organizers) have been invited to speak at the upcoming International Congress of Mathematicians in Madrid.

Topological and Geometric Methods in Group Theory

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Abstracts

Transitivity properties for group actions on buildings

Kenneth S. Brown
(joint work with Peter Abramenko)

The theory of buildings was created by Tits to provide geometric models for certain classes of groups. The link between buildings and groups is provided classically by a correspondence between strongly-transitive group actions on buildings and groups with a BN-pair. Here one considers a group G acting by type-preserving automorphisms on a building Δ. If $\mathcal{A}$ is a G-invariant system of apartments for Δ, then the action of G on Δ is said to be *strongly transitive* with respect to $\mathcal{A}$ if it is transitive on pairs (Σ, C) with $\Sigma \in \mathcal{A}$ and C a chamber in Σ. The BN-pair is gotten by choosing (Σ, C) as above and taking B to be the stabilizer of C and N to be the stabilizer of Σ.

There is a weaker notion of transitivity, whose definition makes use of the Weyl-group-valued distance function $\delta \colon \mathcal{C} \times \mathcal{C} \to W$, where $\mathcal{C} = \mathcal{C}(\Delta)$ is the set of chambers of Δ and W is the Weyl group of Δ. Namely, we say that the action of G on Δ is *Weyl transitive* if, for each $w \in W$, the action is transitive on the ordered pairs C, C' of chambers such that $\delta(C, C') = w$. As with strong transitivity, there is a group-theoretic formulation of Weyl transitivity. This theory is sketched by Tits in [5], and a full account will appear in [2]. The structure is something like a BN-pair, but one only has the B (sometimes called a *Tits subgroup* of G), and not necessarily the N.

Strong transitivity is very natural if one takes the "old-fashioned" point of view, in which a building is defined as a simplicial complex containing subcomplexes called apartments. Weyl transitivity is equally natural if one takes the "modern" approach to buildings, in which a building is a set $\mathcal{C}$ (whose elements are called chambers), together with a Weyl-distance function $\delta \colon \mathcal{C} \times \mathcal{C} \to W$.

If the building Δ is spherical, then the theory simplifies considerably. First, there is a unique system of apartments, so one can talk about strong transitivity without specifying $\mathcal{A}$. Secondly, strong transitivity turns out to be equivalent to Weyl transitivity. For non-spherical buildings, on the other hand, strong transitivity with respect to some apartment system implies Weyl transitivity, but the converse is false. More precisely, there are Weyl-transitive actions that are not strongly transitive with respect to *any* apartment system. To the best of our knowledge, however, there are no explicit examples of such actions in the literature. All we have found is a general suggestion by Tits [5], where he describes a source of possible examples of Weyl-transitive actions that are not strongly transitive with respect to any apartment system. He does not phrase this in terms of transitivity properties, but rather in group-theoretic terms. In the terminology we introduced above, Tits describes a way to exhibit pairs (G, B) such that B is a Tits subgroup of G that does not come from a BN-pair.

The purpose of the work described here is to fill this foundational gap in the theory of buildings by working out in detail a specific family of examples, in the spirit of Tits's suggestion. A surprising feature of the examples is that strong transitivity holds more often than we had expected.

1. Strong transitivity vs. Weyl transitivity

Before proceeding to the examples, we clarify conceptually the difference between strong transitivity and Weyl transitivity. The following observation is a straightforward exercise in the theory of buildings:

Proposition 1. *The following conditions are equivalent for a type-preserving action of a group G on a building Δ.*

(i) *The G-action on Δ is strongly transitive with respect to some apartment system.*

(ii) *The G-action on Δ is Weyl transitive, and there is an apartment Σ (in the complete system of apartments) such that the stabilizer of Σ acts transitively on $\mathcal{C}(\Sigma)$.*

2. Tits's suggestion

Let G be the group $\mathcal{G}(K)$ of rational points of a simple, simply-connected algebraic group $\mathcal{G}$ over a global field K, and let Δ be the Bruhat–Tits building associated with $\mathcal{G}$ and a non-Archimedean completion $\widehat{K}$ of K. Assume that the K-rank of $\mathcal{G}$ is strictly less than its $\widehat{K}$-rank. (Otherwise Bruhat–Tits theory [3] would imply that the action of G on Δ is strongly transitive with respect to a suitable apartment system.) Then $\widehat{G} := \mathcal{G}(\widehat{K})$ acts strongly transitively on Δ with respect to the complete apartment system. Hence, in particular, the action of $\widehat{G}$ is Weyl transitive.

Standard approximation theorems imply that G is dense in $\widehat{G}$. It follows easily that the restriction to G of the $\widehat{G}$-action on Δ is Weyl transitive. There is no reason to expect, however, that this action should still be strongly-transitive with respect to some apartment system. From the point of view of Proposition 1, there is no reason to expect there to be an apartment whose stabilizer in G is big enough to be transitive on the chambers.

3. The examples

We will take $K = \mathbb{Q}$, $\widehat{K} = \mathbb{Q}_p$ (where p is a prime), and $\mathcal{G}$ equal to the norm 1 group of a quaternion division algebra defined over $\mathbb{Q}$. So we fix nonzero rational numbers α, β, and we let D be the 4-dimensional algebra with basis e_1, e_2, e_3, e_4, where e_1 is the identity element, $e_2^2 = \alpha$, $e_3^2 = \beta$, and $e_2e_3 = -e_3e_2 = e_4$. Here α and β are identified with αe_1 and βe_1. Recall that D is a division algebra if and only if its norm form N is anisotropic. Here the norm of $x = x_1 + x_2e_2 + x_3e_3 + x_4e_4$ is $N(x) := x\bar{x} = x_1^2 - \alpha x_2^2 - \beta x_3^2 + \alpha\beta x_4^2 \in \mathbb{Q}$, where $\bar{x} := x_1 - x_2e_2 - x_3e_3 - x_4e_4$.

Assume that α and β have been chosen so that D is a division algebra. We can assure this, for example, by taking $\alpha, \beta < 0$. For any prime p, let D_p be the quaternion algebra over $\mathbb{Q}_p$ obtained from D by extension of scalars. (It looks just like D, but the coefficients x_i are now allowed to be in $\mathbb{Q}_p$.) Let G (resp. G_p) be the subgroup of D^* (resp. D_p^*) consisting of elements of norm 1. In what follows, we will only be interested in primes p such that D_p splits, which is the case for almost all primes p. This means that D_p is isomorphic to the algebra $M_2(\mathbb{Q}_p)$ of 2×2 matrices, and G_p is isomorphic to $\mathrm{SL}_2(\mathbb{Q}_p)$. The Bruhat–Tits building $\Delta = \Delta_p$ is then the well-known tree associated to $\mathrm{SL}_2(\mathbb{Q}_p)$ (see Serre [4]).

An analysis of the action of G on Δ leads to the following dichotomy:

Theorem 1. *One of the following conditions holds.*

(a) -1 *has a square root in D, and, for almost all primes p, the action of G on Δ_p is strongly transitive with respect to some apartment system.*
(b) -1 *does not have a square root in D, and, for almost all primes p, the action of G on Δ_p is Weyl transitive but is not strongly transitive with respect to any apartment system.*

The proof boils down to deciding whether there is an element $g \in G$ that stabilizes an apartment of Δ (in the complete apartment system) and acts as a reflection on it. This turns out to be equivalent to the existence of an apartment Σ whose stabilizer in G is chamber transitive on Σ. A slight additional argument shows that such a g exists if and only if $-1 \in D^2$. See [1] for details.

4. Final remarks

It is natural to wonder whether the phenomenon in the theorem is special to the case of trees. This is not the case. We have examples in higher rank of Weyl-transitive actions that are not strongly transitive with respect to any apartment system. We also have higher-rank examples where, as in part (a) of the theorem, strong transitivity unexpectedly holds. At this writing, however, we are very far from a complete understanding of when strong transitivity holds in the context of Tits's suggestion.

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On the complexity of fixed infinite words

Gilbert Levitt
(joint work with Arnaud Hilion)

The complexity function $p_X(n)$ of an infinite word X on a finite alphabet associates to an integer n the number of words of length n which actually appear in X. It is at most exponential. If X is not eventually periodic, it is strictly increasing, hence grows at least linearly.

My interest in complexity comes from an amazing theorem by Pansiot, building on earlier work by Ehrenfeucht, Lee, Rozenberg. It is about "substitutions", which for group theorists are positive endomorphisms of a free group F_ℓ.

Consider for instance the map $\alpha : a \mapsto ab$, $b \mapsto a$ (for simplicity, always assume that the image of the first letter a starts with a). Iterating $a \mapsto ab \mapsto aba \mapsto abaab \mapsto \dots$ generates an infinite fixed word $X = abaab\dots$.

Pansiot's theorem [1] states that, up to Lipschitz equivalence, there are only 5 possibilities for the complexity of such a word: bounded (X is eventually periodic), n, $n \log\log n$, $n \log n$, n^2. Here are examples (associated to automorphisms of free groups).

- $a \mapsto ab$, $b \mapsto a$, or $a \mapsto ab$, $b \mapsto a$, $c \mapsto cd$, $d \mapsto c$, generates $X = abaab\dots$ with linear complexity. Note that, under iteration of the map, all letters grow as λ^p, with λ the Perron–Frobenius eigenvalue of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
- For $a \mapsto abc$, $b \mapsto a$, $c \mapsto cd$, $d \mapsto c$, the complexity is $n \log\log n$. Now c and d grow as λ^p, but a and b grow as $p\lambda^p$.
- For $a \mapsto a^2bc$, $b \mapsto a$, $c \mapsto cd$, $d \mapsto c$, the complexity is $n \log n$. Here c and d grow as λ^p, but a and b grow as μ^p, where $\mu > \lambda$ is the eigenvalue of $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$.
- $a \mapsto abc$, $b \mapsto ac$, $c \mapsto c$ and $a \mapsto abc$, $b \mapsto bc$, $c \mapsto c$ are examples with quadratic complexity.

Our project is to generalize Pansiot's theorem to automorphisms of free groups (ideally, to endomorphisms). The first thing to note is that complexity makes sense for any point $X \in \partial F_\ell$ in the boundary of a free group: if one chooses a basis of F_ℓ, then X becomes an infinite word and has a complexity. The complexity function depends on the basis, but its growth doesn't. Geometrically, one represents X by an infinite ray in a graph Γ with $\pi_1(\Gamma) \sim F_\ell$, and one computes the complexity of X by counting the number of subpaths of length n.

As a first step, we show that, *given $\alpha \in Aut(F_\ell)$, any interesting fixed point of α in ∂F_ℓ has complexity at most quadratic.* "Interesting" may be understood as "not belonging to the boundary of the fixed subgroup", or "obtainable as a limit of $\alpha^p(g)$ for some $g \in F_\ell$".

The proof uses the theory of train tracks, developed by Bestvina, Feighn, Handel, as well as combinatorial arguments.

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Free and hyperbolic groups are stable

Zlil Sela

We use the procedure for quantifier elimination over a free group to prove that free and hyperbolic groups are stable (in the sense of Shelah). We do that by showing that Diophantine sets are equational, and minimal rank sets are in the Boolean algebra of equational sets.

To prove stability, we introduce duo limit groups, and prove the existence of a canonical universal collection of duo limit groups that 'cover' all the other duo limit groups that are quotients of a given one.

Given this universal family of duo limit groups, we prove stability using descending chain conditions that essentially follow from equationality of Diophantine sets.

Subdirect products of limit groups

Jim Howie
(joint work with Martin Bridson, Chuck Miller and Hamish Short)

A theorem of Baumslag and Roseblade [2] says that finitely presented subgroups of the direct product of two nonabelian free groups must be of a very special form. In particularly, such a group is, up to finite index, itself a direct product of at most two free groups.

The proof in [2] is rather complicated. Alternative proofs were later found by other authors [12, 18, 7].

In [6], the authors extended this result to subgroups of direct products of (several) surface groups: If $\Gamma_1, \dots, \Gamma_n$ are surface groups, and S is a subgroup of homological type FP_n of their direct product, then S is itself, up to finite index, a direct product of at most n surface groups.

The talk describes the current status of a continuing project to extend this result yet further to a wider class of groups. At present, we are focussing attention on the class of *limit groups*. A limit group is a finitely generated group that is ω-residually-free. This means that any finite subset of the group injects under some homomorphism of the group onto a free group. These groups arise in the work of Sela and others on the first order theory of free groups, and have several other equivalent descriptions. (See for example [14, 15, 16, 10, 11, 8], or [3].)

Conjecture 1. (See [17].) *Let $\Gamma_1, \dots, \Gamma_n$ be limit groups, and let S be a type FP_n subgroup of their direct product. Then S is itself, up to finite index, a direct product of at most n limit groups.*

We make the following simplifying assumptions, which are not essential, in the sense that we can easily reduce the full conjecture to the special case where these conditions hold:

(1) Each Γ_i is non-abelian.
(2) Under each projection map $p_i : \prod_j \Gamma_j \to \Gamma_i$, $p_i(S) = \Gamma_i$.
(3) Each intersection $L_i := S \cap \Gamma_i$ is nontrivial. (Note that, by (2), L_i is normal in Γ_i.)

Here is what we can prove:

Theorem 1. [4] *The conjecture is true in the case where each Γ_i is a subgroup of an elementary free group (that is, a hyperbolic tower group).*

Theorem 2. [5] *The conjecture is true in the case where $n \leq 2$.*

Theorem 3. *The conjecture is true in the case where each quotient group Γ_i/L_i is abelian.*

Theorem 4. *Under the hypotheses of the conjecture, each quotient group Γ_i/L_i is, up to finite index, an extension of a nilpotent group of class at most $n-2$ by a free abelian group of rank at most n.*

Theorem 5. *Under the hypotheses of the conjecture, each Γ_i contains a cyclic subgroup C_i, and a subgroup B_i, such that $L_i \subset B_i$, B_i/L_i is free abelian of rank at most $n-1$, and the number of double cosets $B_i g C_i$, $g \in \Gamma_i$, is finite.*

The last three of these results will appear in a forthcoming paper.

The methods are varied, but include a version for limit groups of the Marshall Hall Theorem for free groups [9] – or, more accurately, of a theorem of Peter Scott's [13] on curves in surfaces. They also include imaginative use of the Lyndon–Hochschild–Serre spectral sequence of a group extension, and of the Bass–Serre theory of groups acting on trees. Each limit group acts in a particularly nice way on a tree – and as a result has a particularly nice kind of classifying space [1, 3]. All these facts, combined with the probable sparcity of non-abelian groups satisfying the conclusions of Theorems 4 and 5, give us great confidence that the conjecture will eventually succumb.

We acknowledge useful conversations with Zlil Sela, Henry Wilton, and (at the Oberwolfach meeting) Emina Alibegović, Karl Gruenberg and Peter Kropholler.

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Hanoi Towers groups

Rostislav Grigorchuk
(joint work with Volodymyr Nekrashevych and Zoran Šunić)

The well known Hanoi Towers Problem has a long history. The original version on 3 pegs was invented by Edouard Lucas in 1883. A version on 4 pegs, known as Reve's Puzzle, was proposed by Dudeney in 1907. The problem was posed for arbitrary number of pegs by Stewart in 1939 in the problem section of American Mathematical Monthly.

Hanoi Towers Problem

Given k pegs, $k \geq 3$, labeled by $0, \ldots, k-1$ and n disks of different size, labeled by $1, \ldots, n$ as they increase in size, a configuration is any placement of the disks on the pegs such that no disk is placed on a smaller disk. In a single move a single disk from the top of one peg can be moved to the top of another peg as long as the new placement of disks is a valid configuration. Initially, all disks are placed on one peg, say 0, and the goal is to move them all to another peg in the smallest possible number of moves.

It is well known that the original version on 3 pegs has a solution in $2^n - 1$ moves. It is also known that the Frame–Stewart algorithm yields a solution in

$\sim 2^{n^{\frac{1}{k-2}}}$ moves (see [Hin89] for more details) and this solution is asymptotically optimal [Sze99]. No optimal solution is known.

Algebraic model by self-similar groups

Configurations on k pegs and n disks can be represented by words of length n over the k-letter alphabet $X = \{0, \dots, k-1\}$. The word $x_1 \dots x_n$ represents the unique configuration in which the disk i is placed on peg x_i. The set X^* of words over X ordered by the prefix relation has the structure of a rooted k-ary tree, denoted $\mathcal{T}$. For $0 \leq i < j \leq k-1$, let a_{ij} be the automorphism of $\mathcal{T}$ defined recursively by

$$a_{ij}(iw) = jw, \qquad a_{ij}(jw) = iw, \qquad a_{ij}(xw) = xa_{ij}(w), \text{ for } x \notin \{i, j\},$$

for $w \in X^*$. The automorphism a_{ij} acts on $\mathcal{T}$ in a such a way that each configuration in the Hanoi Towers Problem on k pegs is mapped to the configuration resulting after a move is performed between peg i and peg j (configurations in which both peg i and peg j are empty do not change under this move).

Definition 1. *The Hanoi Towers group on k pegs is the group $H^{(k)} \leq \mathsf{Aut}(\mathcal{T})$, defined by*

$$H^{(k)} = \langle a_{ij} \mid 0 \leq i < j \leq k-1 \rangle$$

The Schreier graph of the action of $H^{(k)}$ on level n is the graph of all configurations on k pegs and n disks in Hanoi Towers Problem. For example, the Schreier graph for 3 pegs and 3 disks is given in the left half of Figure 1.

Results

Theorem 1. *Let $H = H^{(3)}$ be the Hanoi Towers group on 3 pegs. Then*

(i) H is amenable

(ii) H is not elementary amenable (moreover, H is not subexponentially amenable)

(iii) H is a regular branch group over its commutator subgroup $[H, H]$

(iv) H is conjugate in $\mathsf{Aut}(\mathcal{T})$ to the iterated monodromy group $IMG\left(z^2 - \frac{16}{27z}\right)$, whose self-similar action on the ternary rooted tree is given by

$$\alpha = (01)\ (1, 1, \beta)$$
$$\beta = (02)\ (1, \alpha, 1)$$
$$\gamma = (12)\ (\gamma, 1, 1)$$

(v) The limit space of H and the Julia set of the rational map $z \mapsto z^2 - \frac{16}{27z}$ are homeomorphic to the Sierpiński gasket.

Note that there exists a 4-fold cover of the Sierpiński gasket by the Apollonian gasket (see the right half of Figure 1). This fact is reflected in the next result, which says that there exists a self-similar subgroup A of index 4 in H, whose limit space is the Appolonian gasket.

Theorem 2. *Let* $A = \langle cba, bac, acb \rangle \leq H$. *Then*

(i) A is normal subgroup of index 4 in H, which contains the commutator subgroup $[H,H]$.

(ii) A is a regular branch group over $[H,H]$.

(iii) A is conjugate in $\mathsf{Aut}(\mathcal{T})$ *to the iterated monodromy group* $IMG\left(z^2 + \frac{2}{27z}\right)$, *whose self-similar action on the ternary rooted tree is given by*

$$x = (01)\ (1, y, 1)$$
$$y = (02)\ (x, 1, 1)$$
$$z = (12)\ (1, 1, z)$$

(iv) The limit space of A and the Julia set of the rational map $z \mapsto z^2 + \frac{2}{27z}$ *are homeomorphic to the Apollonian gasket.*

(v) the abelianization $A/[A,A] \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ *is infinite. In terms of x, y and z, the commutator subgroup* $[A,A]$ *consists of all elements in A that can be represented by group words in which the exponent of all three letters x, y and z is 0.*

Note that all proper homomorphic images of a finitely generated branch group are virtually abelian, but it was not known for a while if the image could be infinite. By the above results, the Hanoi Towers group H and the Apollonian group A are examples of finitely generated branch groups that are not just infinite (another example was recently provided by Delzant and the first author).

Theorem 3. *Let* $G = \langle cba, bac \rangle \leq A \leq H$. *Then*

(i) G is regular weakly branch group over its commutator subgroup $[G,G]$.

(ii) G is conjugate in $\mathsf{Aut}(\mathcal{T})$ *to the iterated monodromy group* $IMG\left(z^3 + \frac{3z}{2}\right)$, *whose self-similar action on the ternary rooted tree is given by*

$$x = (01)\ (1, x, 1)$$
$$y = (02)\ (1, 1, y)$$

(iii) The limit space of G and the Julia set of the polynomial map $z \mapsto z^3 + \frac{3z}{2}$ *are homeomorphic and map continuously onto the Apollonian gasket and the Sierpiński gasket.*

(iv) The monoid generated by x and y is free.

In particular, the last result implies that each Hanoi Towers group $H^{(k)}$, $k \geq 3$, has exponential growth.

Further details on Hanoi Towers groups, self-similar groups and iterated monodromy groups can be found in [GŠ06] and [Nek05],

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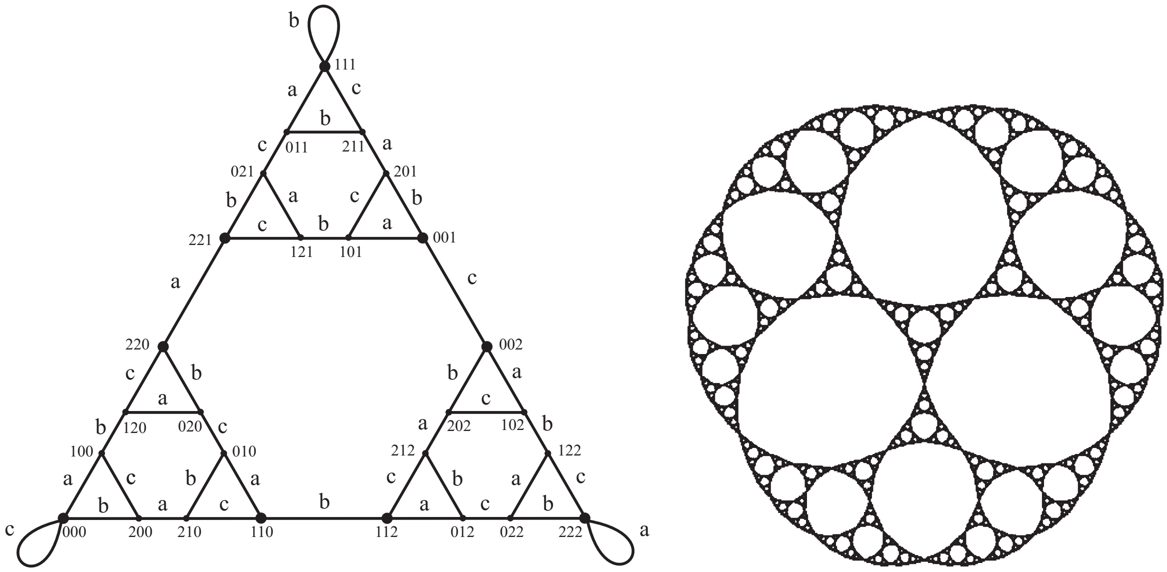

FIGURE 1. Schreier graph of $H^{(3)}$ at level 3 and the limit space of A

[Sze99] Mario Szegedy. In how many steps the k peg version of the Towers of Hanoi game can be solved? In *STACS 99 (Trier)*, volume 1563 of *Lecture Notes in Comput. Sci.*, pages 356–361. Springer, Berlin, 1999.

The algebraic K-theory of the group ring of a word hyperbolic group with arbitrary coefficient ring

WOLFGANG LÜCK
(joint work with Arthur Bartels and Holger Reich)

This is a joint project with Arthur Bartels and Holger Reich. Our main result is:

Theorem: Let R be an (associative) ring (with unit). Let G be a word-hyperbolic group. Then the Farrell–Jones Conjecture for algebraic K-theory with coefficients in R is true for G, i.e. the assembly map

$$H_n(E_{\mathcal{V}Cyc}(G);\mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

The Farrell–Jones Conjecture was formulated by Farrell–Jones in [2]. For a survey about the Farrell–Jones Conjecture and the Baum–Connes Conjecture and their status we refer for instance to [4]. The special case, where G is the fundamental group of a closed Riemannian manifold with strictly negative sectional curvature is already treated in [1].

Theorem: Let $\mathcal{F}$ be the family of subgroups for which the Farrell–Jones Conjecture for algebraic K-theory with coefficients with arbitrary rings R as coefficients is true for G. Then:

(1) If $G \in \mathcal{F}$ and $H \subseteq G$ is a subgroup of G, then $G \in \mathcal{F}$;
(2) If G_1 and G_2 belong to $\mathcal{F}$, then $G_1 \times G_2$ belongs to $\mathcal{F}$;
(3) Word hyperbolic belong to $\mathcal{F}$;
(4) Nilpotent groups belong to $\mathcal{F}$;
(5) Let $\{G_i \mid i \in I\}$ be a directed system of groups $G_i \in \mathcal{F}$. (We do not require the structure maps $G_i \to G_j$ to be injective.) Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{F}$;
(6) Suppose that R is regular with $\mathbb{Q} \subseteq R$. Let $1 \to H \to G \to Q \to 1$ be an extension of groups. Suppose that K is either a word hyperbolic group or an elementary amenable group and that the same is true for Q. Then Farrell–Jones Conjecture for algebraic K-theory with coefficients in R is true for G.

We mention some consequences of the Farrell–Jones Conjecture.

- Let R be a principal ideal domain and $G \in \mathcal{F}$. Then the reduced projective class group $\widetilde{K}_0(RG)$, the Whitehead group $\operatorname{Wh}(G, R)$ and $K_n(RG)$ for $n \leq -1$ all vanish if G is torsionfree.
- Suppose that $G \in \mathcal{F}$. Then the following version of the Bass Conjecture is true:

  Let F be a field of characteristic zero and let G be a group. The Hattori–Stallings homomorphism induces an isomorphism
  $$K_0(FG) \otimes_{\mathbb{Z}} F \to \operatorname{class}_F(G)_f,$$
  where $\operatorname{class}_F(G)_f$ consists of functions $f\colon G \to F$ which vanish on elements of infinite order and are constant on F-conjugacy classes of elements of finite order.
- Suppose that $G \in \mathcal{F}$. Then the following version of the Bass Conjecture is true:

  Let R be a commutative integral domain and let G be a group. Let $g \in G$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R. Then for every finitely generated projective RG-module the value of its Hattori–Stallings rank $\operatorname{HS}_{RG}(P)$ at (g) is trivial.
- Moody's induction result which he proved for virtually poly-cyclic groups in [5] holds for all groups in $\mathcal{F}$, i.e. the canonical map
  $$\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(RH) \to K_0(RG)$$
  is bijective for all regular rings R with $\mathbb{Q} \subseteq R$.
- Higson–Lafforgue–Skandalis [3] give groups G for which the Baum–Connes Conjecture with coefficients is not true. However, as a consequence of our result these groups G belong to $\mathcal{F}$.

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Commensurator of $Out(F_n)$

Michael Handel
(joint work with Benson Farb)

Let F_n denote the free group of rank n and let $\mathrm{Out}(F_n)$ denote the group of outer automorphisms of F_n; that is, $\mathrm{Out}(F_n)$ is the group of all automorphisms of F_n modulo the subgroup of automorphisms of F_n given by conjugation by some element in F_n.

Our main result is the following.

Theorem 1 (Classifying injective homomorphisms)**.** *Let $\Gamma < \mathrm{Out}(F_n)$ be a finite index subgroup and let $\Phi : \Gamma \to \mathrm{Out}(F_n)$ be an injective homomorphism. Then for $n \geq 4$, the map Φ agrees with conjugation by some $g \in \mathrm{Out}(F_n)$.*

Theorem 1 has two main corollaries: a computation of the abstract commensurator of $\mathrm{Out}(F_n)$, and a proof of the co-Hopf property for $\mathrm{Out}(F_n)$ and for its finite index subgroups.

The commensurator. Recall that the *(abstract) commensurator group* $\mathrm{Comm}(\Lambda)$ of a group Λ is defined to be the set of equivalence classes of isomorphisms $\phi : H \to N$ between finite index subgroups H, N of Λ generated by the relation that $\phi_1 : H_1 \to N_1$ is equivalent to $\phi_2 : H_2 \to N_2$ if $\phi_1 = \phi_2$ on $H_1 \cap H_2$. Under this relation, $\mathrm{Comm}(\Lambda)$ becomes a group, which we think of as the group of "hidden automorphisms" of Λ.

This group is in general much larger than $\mathrm{Aut}(\Lambda)$. For example $\mathrm{Aut}(\mathbf{Z}^n) = \mathrm{GL}(n, \mathbf{Z})$ whereas $\mathrm{Comm}(\mathbf{Z}^n) = \mathrm{GL}(n, \mathbf{Q})$. Margulis proved that an irreducible lattice Λ in a semisimple Lie group G is arithmetic if and only if it has infinite index in its commensurator. Mostow–Prasad–Margulis strong rigidity for all lattices Λ in such a $G \neq \mathrm{SL}(2, \mathbb{R})$ can be thought of as proving exactly that the abstract commensurator $\mathrm{Comm}(\Lambda)$ is isomorphic to the commensurator of Λ in G, which in turn is computed concretely by Margulis and Borel–Harish–Chandra; see, e.g., [Ma, Zi].

Outside of lattices in Lie groups there are very few computations of abstract commensurator groups. The group $\mathrm{Comm}(\Lambda)$ was computed for mapping class groups of surfaces by Ivanov [IV], [IV2] and Ivanov–McCarthy [IM], for the Torelli group by Farb–Ivanov [FI] and for word-hyperbolic surface-by-free groups by Farb–Mosher [FM].

Dyer–Formanek [DF] proved that $\mathrm{Out}(\mathrm{Aut}(F_n)) = 1$; Bridson–Vogtmann [BV] proved the related (but different) result that $\mathrm{Aut}(\mathrm{Out}(F_n)) = \mathrm{Out}(F_n)$. These proofs use finite subgroups of $\mathrm{Out}(F_n)$ in an essential way. As $\mathrm{Out}(F_n)$ has a torsion-free subgroup of finite index, these methods do not apply to understanding $\mathrm{Comm}(\mathrm{Out}(F_n))$. While abstract commensurator groups are typically much larger than corresponding automorphism groups, Theorem 1 implies the following:

Corollary 1 (Commensurator Theorem)**.** *For $n \geq 4$, any isomorphism between finite index subgroups of $\mathrm{Out}(F_n)$ agrees with conjugation by some $g \in \mathrm{Out}(F_n)$. In particular the natural injection*

$$\mathrm{Out}(F_n) \to \mathrm{Comm}(\mathrm{Out}(F_n))$$

is an isomorphism.

Corollary 1 answers Question 8 of K. Vogtmann's list (see [Vo]) of open problems about $\mathrm{Out}(F_n)$. Note too that Theorem 1 implies the result of Bridson–Vogtmann that $\mathrm{Aut}(\mathrm{Out}(F_n)) = \mathrm{Out}(F_n)$.

The co-Hopf property. Recall that a group Λ is *co-Hopfian* if every injective endomorphism of Λ is an isomorphism. Unlike the Hopf property, which is true for example for all linear groups, the co-Hopf property holds much less often (consider, for example, any Λ which is free abelian or is a nontrivial free product), and is typically harder to prove. The co-Hopf property was proven for lattices in semisimple Lie groups by Prasad [Pr], for non-elementary, freely-indecomposable, torsion-free word-hyperbolic groups by Sela [Se], and for mapping class groups by Ivanov–McCarthy [IM]. An immediate consequence of Theorem 1 is the following.

Corollary 2 (Co-Hopf property)**.** *For $n \geq 4$, every finite-index subgroup $\Gamma < \mathrm{Out}(F_n)$ is co-Hopfian.*

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Dehn functions for finitely presented groups

Noel Brady
(joint work with Martin Bridson, Max Forester and Krishnan Shankar)

This project is about higher filling invariants for groups G possessing a $K(G,1)$ with finite $(n+1)$-skeleton. Such groups are said to be of type F_{n+1}. The *order n Dehn function* $\delta_G^{(n)}(x)$ of a group of type F_{n+1} gives an upper bound on the number of $(n+1)$-cells needed to fill an n-sphere composed of x n-cells. The idea of using higher filling functions as invariants of groups was introduced by Gromov in [3]. The case $n=1$ is just the ordinary Dehn function of the group G. Precise definitions of the higher order Dehn functions, and initial results were developed in [1].

A basic question in this subject is which functions of the form x^α arise as order n Dehn functions of groups. Since there are only countably many isomorphism classes of finitely presented groups, then for each n there is a countable set of real numbers $\alpha \in [1,\infty)$ such that x^α is the order n Dehn function of some group G of type F_{n+1}. This set is called the order n *isoperimetric spectrum*:

$$IP^{(n)} = \{\alpha \in [1,\infty) \mid f(x) = x^\sigma \text{ is an } n\text{th order Dehn function for some group } G \text{ of type } F_{n+1}\}$$

Fig. 1 shows a picture of our current understanding of the IP spectra.

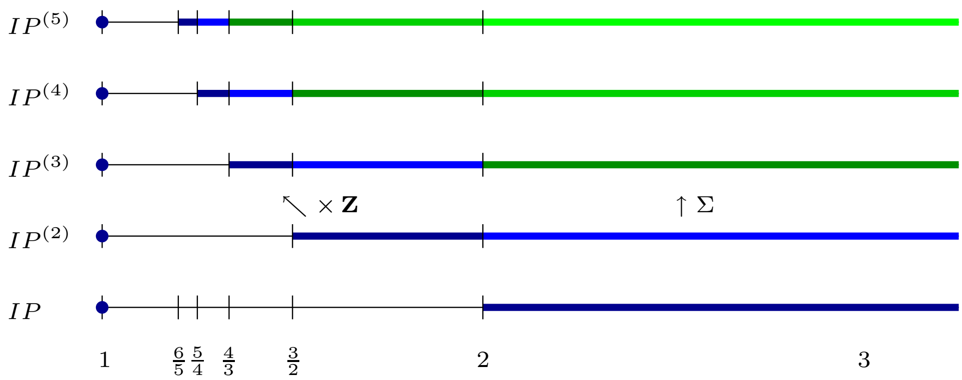

Figure 1. Isoperimetric exponents of $\Sigma^{k-1}G_{r,P} \times \mathbf{Z}^\ell$.

Groups with $\delta^{(1)}(x) \simeq x$ are hyperbolic. A theorem of Gromov states that groups with $\delta^{(1)}(x)$ subquadratic are hyperbolic, and so have $\delta^{(1)}(x) \simeq x$. Thus there is a gap in $IP^{(1)}$ between 1 and 2, and this gap corresponds to an useful criterion for hyperbolicity. The heavy lines are meant to indicate that the IP spectrum is dense in the appropriate interval of the real line. It is entirely possible that the "gaps" in the spectra $\mathrm{IP}^{(n)}$ for $n \geq 2$ will be filled as research on higher Dehn functions progresses.

Here is how we obtain the information in Fig. 1.

(1) First we develop a more flexible version of the snowflake construction used in [2] to obtain 2-dimensional, aspherical groups $G_{r,P}$ (corresponding to a given rational number r and a given irreducible integer matrix P) whose ordinary Dehn functions are of the form $x^{2\log_\lambda(r)}$ where λ is the Perron–Frobenius eigenvalue of the matrix P. The group $G_{r,P}$ is the fundamental group of graphs of groups whose underlying graph has transition matrix equal to P, whose vertex groups are all 2-dimensional right angled Artin groups, and whose edge groups are all $\mathbf{Z}$. Given a positive integer m, the group $G_{r,P}$ admits a monomorphism ϕ_m which maps each generator of the vertex and edge groups to its mth power, and which maps each stable letter to itself.

(2) The map ϕ_r can be used to build up a multiple ascending HNN extension of $G_{r,P}$. This group is denoted by $\Sigma G_{r,P}$. It has a 3-dimensional classifying space. In the case r is an integer, there exist sequences of embedded 3-balls in the universal cover of this classifying space which give a lower bound of $x^{2\log_\lambda(r)}$ for the second order Dehn function of $\Sigma G_{r,P}$.

The fact that r is integral is crucial in our construction of the sequence of embedded balls. This is one key place where the extra flexibility of this new version of the snowflake construction is used. Even though one could define aspherical, 3-dimensional, multiple ascending HNN extensions of the original snowflake groups of [2], there is no obvious way of constructing the sequence of embedded balls giving the appropriate lower bounds for the second order Dehn function.

This process can be iterated inductively to produce groups $\Sigma^k G_{r,P}$ with $(k+2)$-dimensional classifying spaces, and lower bounds of $x^{2\log_\lambda(r)}$ for their order $k+1$ Dehn functions.

(3) An upper bound of $x^{2\log_\lambda(r)}$ is obtained (inductively) for the order $(k+1)$ Dehn function of the group $\Sigma^k G_{r,P}$. The details of this inductive argument necessitate consideration of generalizations of higher order Dehn functions which involve arbitrary manifold fillings of arbitrary closed manifolds. These type of filling invariants were mentioned as worthy of investigation by Gromov in [3]. The upper bound results described here are the first applications of these manifold filling functions. It would be interesting to compute examples of these filling functions, and to compare them with the usual "disk filling sphere" Dehn functions.

(4) The second way we build up higher dimensional groups is to take direct products with $\mathbb{Z}$. This has the effect (in our setting) of taking a k-dimensional group with order $k-1$ Dehn function of the form x^s and producing a $k+1$-dimensional group with order k Dehn function of the

form $x^{2-1/s}$. Note that if $s \in [2,\infty)$ then $2-1/s \in [3/2,2)$. Also, if $s \in [(m+1)/m, m/(m-1))$ then $2-1/s \in [(m+2)/(m+1),(m+1)/m)$ so the more products with $\mathbf{Z}$, the more one can push the IP spectrum toward 1.

As with the suspension procedure, there is a lower bound part which involves embedded representatives, and an upper bound part which involves consideration of arbitrary manifold fillings of arbitrary closed manifolds.

Here are precise statements of the key theorems. The first concerns ordinary Dehn functions of a class of 2-dimensional, aspherical groups.

Theorem 1. *Given an irreducible non-negative integer matrix P with Perron–Frobenius eigenvalue $\lambda > 1$, and a rational number r greater than the largest row-sum of P, there exists a group $G_{r,P}$ with*

$$\delta^{(1)}_{G_{r,P}}(x) \sim x^{2\log_\lambda(r)}.$$

Corollary. *Taking $P = (2^{2q})_{1\times 1}$ and $r = 2^p$ gives $\delta^{(1)}_{G_{r,P}}(x) \sim x^{p/q}$, so $\mathbf{Q}\cap[2,\infty) \subset IP^{(1)}$.*

The next theorem describes the effect of the suspension procedure.

Theorem 2 (Effect of suspensions)**.** *Let P be as above, and r an* integer *greater than the largest row-sum of P. Then*

$$\delta^{(k+1)}_{\Sigma^k G_{r,P}}(x) \sim x^{2\log_\lambda(r)}.$$

Finally, there is a theorem on the effect of taking products with $\mathbb{Z}$.

Theorem 3 (Effect of products with $\mathbf{Z}$)**.** *If G is of the form $\Sigma^k G_{r,P} \times \mathbf{Z}^\ell$ (and so has dimension $2+k+\ell$) with order $(k+\ell+1)$ Dehn function of the form x^s, then $\Sigma^k G_{r,P} \times \mathbf{Z}^{\ell+1}$ has order $(k+\ell+2)$ Dehn function of the form $x^{2-1/s}$.*

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Filling radius and simplicial nonpositive curvature

Jacek Świątkowski
(joint work with Tadeusz Januszkiewicz)

In this note we exhibit a surprising filling property, discovered in a joint work with Tadeusz Januszkiewicz [2], satisfied by spaces and groups with *simplicial nonpositive curvature.* The latter is a purely combinatorial notion introduced in [1] having many consequences similar to metric nonpositive curvature. As an application of these ideas we get, among others, the following.

Theorem 1. *For any positive integer n there is a word-hyperbolic group of cohomological dimension n which contains no subgroup isomorphic to the fundamental group of a nonpositively curved closed riemannian manifold of dimension ≥ 3.*

1. Simplicial nonpositive curvature

Let X be a simplicial complex. Recall that X is *flag* if any set of its vertices that are pairwise connected with edges spans a simplex of X. A *cycle* γ in X is a subcomplex homeomorphic to S^1. The *length* $|\gamma|$ is the number of edges in γ. A *diagonal* in γ is an edge of X that connects some two nonconsecutive vertices of γ.

Definition 1. *Given an integer $k \geq 5$, a simplicial complex L is k-*large *if it is flag and every cycle γ in L of length $3 < |\gamma| < k$ has a diagonal.*

One should think of k-largeness as local condition imposed on links L of a simplicial complex X. It roughly says that not too much distorted cycles in L cannot be short.

Definition 2. *A simplicial complex X is k-*systolic *if it is simply connected and all of its links are k-large. A group is k-*systolic *if it acts properly discontinuously and cocompactly, by simplicial automorphisms, on a k-systolic simplicial complex.*

The above simple combinatorial conditions have consequences similar to metric curvature bounds from above. For example, 6-systolic complexes are contractible and 6-systolic groups are biautomatic (and hence semi-hyperbolic). Moreover, complexes of groups with 6-large local developments are developable. These consequences are analogous to those of metric nonpositive curvature, and hence we call 6-systolicity ‘simplicial nonpositive curvature’.

6-systolicity reduces neither to metric nonpositive curvature, nor to small cancellation conditions, especially in dimensions higher than 2. It does not match well with manifolds of dimension ≥ 3, as there are no triangulations with 6-large links for such manifolds. On the other hand, 6-systolic pseudomanifolds do exist in arbitrary dimension, and there are 6-systolic groups with arbitrary cohomological dimension. There are also such groups that are additionally word-hyperbolic. See [1] for more details and more consequences.

2. Filling radius for spherical cycles

Given a metric space X and a real number $r > 0$, denote by $P_r(X)$ the Rips complex of X, i.e. the simplicial complex with vertex set X in which a finite subset of X spans a simplex iff the elements in this subset are pairwise at distances $\leq r$. We are interested in the following asymptotic property of metric spaces.

Definition 3. *A metric space X has* filling radius for n-spherical cycles constant *(shortly, is S^nFRC) if for any $r > 0$ there is $R \geq r$ such that any simplicial map $f : S^n \to P_r(X)$ (for any triangulation of the sphere S^n) extends to a simplicial map $F : D^{n+1} \to P_R(X)$ in such a way that supp$(F) \subset$ supp(f), where supp is the image in X of the vertex set of S^n or D^{n+1} through f or F, respectively.*

In [2] we show the following facts:

(1) S^nFRC is a coarse invariant; in particular, if a finitely generated group G is S^nFRC then any finitely generated subgroup of G is S^nFRC.
(2) 6-systolic groups are S^nFRC for any $n \geq 2$.
(3) If M is a closed nonpositively curved riemannian manifold then its fundamental group $\pi_1(M)$ is not S^nFRC for all $n \leq \dim M - 1$. In particular, if $\dim M \geq 3$ then $\pi_1(M)$ is not S^2FRC.

3. Consequences

Above properties (1)-(3) yield the following.

Theorem 2. *A 6-systolic group contains no subgroup isomorphic to the fundamental group of a nonpositively curved closed riemannian manifold of dimension ≥ 3.*

Theorem 2, together with the existence of 6-systolic word-hyperbolic groups of arbitrary cohomological dimension, imply Theorem 1.

As another application of filling radius phenomena we obtain in [2] the following.

Theorem 3. *If product $G_1 \times G_2$ of two infinite finitely generated groups is a subgroup in a 6-systolic group then both groups G_i are virtually free. Consequently, product of three infinite finitely generated groups is never a subgroup in a 6-systolic group.*

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Ordinary homology of Bestvina–Brady groups

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(joint work with Müge Saadetoğlu)

Let $\mathbb{T}$ be the circle, or 1-dimensional unitary group, given a CW-structure with one 0-cell and one 1-cell. Suppose also that the identity element of the group is chosen to be the 0-cell. For a set V, let $T(V)$ denote the direct sum $T(V) = \bigoplus_{v \in V} \mathbb{T}$. There is a natural CW-structure on $T(V)$ in which the i-cells are in bijective correspondence with i-element subsets of V.

There is a bijective correspondence between simplicial complexes whose vertex set is contained in V and non-empty subcomplexes of $T(V)$. The empty simplicial complex corresponds to the subcomplex $T_\emptyset$ consisting of just the single 0-cell of $T(V)$, and a non-empty simplicial complex L corresponds to the complex T_L defined by

$$T_L = \bigcup_{\sigma \in L} T(\sigma).$$

The fundamental group of T_L is the right-angled Artin group whose generators are in 1-1 correspondence with the vertices of L, subject only to the relations that the ends of each edge commute. The cohomology ring of T_L is the so-called 'exterior face ring of L' (see [2]), and T_L is aspherical if and only if L is a flag complex.

A point in $T(V)$ is a vector (t_v) of elements of $\mathbb{T}$ indexed by V, such that only finitely many t_v are not the identity element. The group multiplication induces a map $\mu : T(V) \to \mathbb{T}$ which takes the point (t_v) to the product of all of the non-identity t_v's. For each $L \neq \emptyset$ this induces a surjective cellular map $\mu_L : T_L \to \mathbb{T}$. We study the homology and cohomology of the space $\widetilde{T}_L$, the infinite cyclic cover of T_L obtained by pulling back the universal cover of $\mathbb{T}$ via μ_L.

$$\begin{array}{ccc} \widetilde{T}_L & \longrightarrow & \mathbb{R} \\ \big\downarrow & & \big\downarrow \\ T_L & \xrightarrow{\mu_L} & \mathbb{T} \end{array}$$

Let G_L denote the fundamental group of T_L, and let H_L denote the fundamental group of $\widetilde{T}_L$, which can be viewed as a subgroup of G_L. As remarked above, G_L is a right-angled Artin group. The group H_L is the kernel of the homomorphism $\beta_L : G_L \to \mathbb{Z}$ which takes each generator to $1 \in \mathbb{Z}$. The groups H_L are called Bestvina–Brady groups [1]. In the case when L is flag, $\widetilde{T}_L$ is an Eilenberg–Mac Lane space for H_L.

Let Z denote the quotient group G_L/H_L, and let β_L denote the element of $H^1(T_L)$ corresponding to the group homomorphism $\beta_L : G_L \to \mathbb{Z}$ described above. Let R be a non-trivial unital ring, let L be a non-empty simplicial complex, and let $C_*^+(L; R)$ and $C_+^*(L; R)$ denote the augmented cellular chain and cochain complexes for L with coefficients in R. We obtain the following descriptions of the homology and cohomology of $\widetilde{T}_L$. Special cases of these results appeared in the second named author's Ph. D. thesis [4] and in recent work of S. Papadima and A. Suciu [3].

Theorem 1. *There is a short exact sequence of $R[Z]$-modules:*

$$0 \to B^{+}_{n-1}(L;R) \to H_n(\widetilde{T}_L;R) \to \mathbb{Z}[Z] \otimes \overline{H}_{n-1}(L;R) \to 0,$$

where Z acts trivially on $B^{+}_{*}(L;R)$, the boundaries in $C^{+}_{*}(L;R)$.

Theorem 2. *For any L, any commutative ring R, and any $n \geq 0$ there is a short exact sequence of $R[Z]$-modules:*

$$0 \to C^{n-1}_{+}(L;R)/B^{n-1}_{+}(L;R) \to H^n(\widetilde{T}_L;R) \to M \to 0,$$

where Z acts trivially on $C^{*}_{+}(L;R)$. The module M fits in to a short exact sequence:

$$0 \to \overline{H}^{n-1}(L;R) \xrightarrow{\Delta} \prod_{Z} \overline{H}^{n-1}(L;R) \to M \to 0,$$

where Z acts by the 'shift action' on the product and where Δ is the inclusion of the constant sequences.

Theorem 3. *The image of the map $H^*(T_L;R) \to H^*(\widetilde{T}_L;R)$ is equal to the Z-fixed point subring of $H^*(\widetilde{T}_L;R)$ and is isomorphic to the quotient $H^*(T_L;R)/(\beta_L)$. In degree n, the cokernel of this map is isomorphic to an infinite product of copies of $\overline{H}^{n-1}(L;R)$. In particular, the map is a ring isomorphism if and only if L is R-acyclic.*

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The Bieri Neumann Strebel invariant of a Kähler group

Thomas Delzant

A Kähler group is a (finitely presented) group G which can be realized as the fundamental group of a compact Kähler manifold, e.g. a compact nonsingular projective variety.

The Bieri Neumann Strebel invariant of a finitely generated group G is a certain subset of $H^1(G,\mathbb{R})$, whose complement $E^1(G)$ consists of exceptional cohomology classes: a class $w \in H^1(G,\mathbb{R})$ is exceptional if there exists an action of G on some $\mathbb{R}$-tree with no fixed point or line, such that the group G fixes one point at infinity in T, and such that $|w(g)|$ is the translation length of the element g.

The main result is that if G is the fundamental group of a Kähler manifold X, then a class w is exceptional if and only if there exists a hyperbolic 2-orbifold Σ, a non zero class $\omega \in H^1(\Sigma, \mathbb{R})$ and a non constant holomorphic map $F : X \to \Sigma$ with connected fibres, such that $w = F^*\omega$.

Therefore the set $E^1(G)$ is made up as the union of finitely many complex vector subspaces of $H^1(G, \mathbb{R})$.

Using previous results by A. Beauville, and by R. Bieri, W. Neumann and R. Strebel, we deduce that if a Kähler group is solvable, then it is virtually nilpotent.

The main idea is that, in a lot of cases, and due to the work of Gromov–Shoen, Simpson, and Gromov and the author, if a Kähler group acts on a tree without invariant line or fixed point, one can extend this action to a (discrete) action on the unit disk $\{z \in \mathbb{C}, |z| < 1\}$. In the case of the tree obtained by considering an exceptional (in the sense of BNS) class, the disk is obtained by analysing the foliation dual to the closed $(1, 0)$-form whose real part is the harmonic representative of w.

Finitely generated groups acting on $\mathbb{R}$-trees

Vincent Guirardel

Given a finitely generated group G acting on an $\mathbb{R}$-tree T (minimally, without global fix point), what can one say about G? There are several results of this kind.

	Hypothesis on G	Hypothesis on the action	Output
Rips [2, 1]	finitely generated	free	G is a free product of free abelian groups and surface groups
Bestvina–Feighn [1]	finitely presented	stable[1]	G splits over some cyclic extension of subgroup fixing an arc, or T is a line
Guirardel [3]	finitely presented	stable	approximation of T by actions on simplicial trees
Sela [4]	finitely generated	super-stable[2] and trivial tripod stabilizers.	either G splits as a free product, or T has a nice decomposition (implies splitting as in Bestvina–Feighn's result).

[1] Any arc I contains a stable arc J, i.e. an arc such that for any arc $K \subset J$, $\mathrm{Stab}(K) = \mathrm{Stab}(J)$.

[2] Super stability means the stabilizer of each unstable arc is trivial; in other words, if $\mathrm{Stab}(I) \neq \{1\}$, then for any $J \subset I$, $\mathrm{Stab}(I) = \mathrm{Stab}(J)$. Sela's result is actually stated with only the ascending chain condition on arc stabilizers but the proof actually uses super-stability. Moreover we can give a counter-example if this hypothesis is omitted.

Our result generalizes Sela's result by getting rid of the assumption on tripod stabilizers, and by replacing super-stability by a weaker condition. In particular, our result applies for any action of G having small[3] arc stabilizers if small subgroups of G are finitely generated.

Theorem. *Let G be a finitely generated group acting on an $\mathbb{R}$-tree. Assume that we have*

- *Ascending chain condition for arc stabilizers*[4]
- *The stabilizer H of each unstable*[5] *arc is finitely generated and does not contain properly a conjugate of itself.*

Then either G splits over the stabilizer of a tripod or of an unstable arc, or T has a nice decomposition as a graph of actions where each vertex action is a simplicial tree, a line (with a dense action), a tree dual to a measured foliation on a 2-orbifold modulo some kernel.

In particular, G splits over a cyclic extension of a subgroup fixing an arc in T.

One of the tools for the proof is an extended version of Scott's Lemma saying that if a finitely generated group G is a direct limit of groups having non-trivial a free decompositions, then G also has a free decomposition.

Theorem (Extended Scott Lemma)**.** *Let $G_k \circlearrowright S_k$ be a sequence of non-trivial minimal actions of groups on simplicial trees, and $(\phi_k, f_k) : G_k \circlearrowright S_k \to G_{k+1} \circlearrowright S_{k+1}$ be epimorphisms. Assume that*

$$\forall e \in E(S_k), \forall e' \in E(S_{k+1}), \quad e' \subset f_k(e) \Rightarrow G_{k+1}(e') = \phi_k(G_k(e)) \quad (*)$$

Then the inductive limit $G = \lim G_k$ has a non-trivial splitting over the image of an edge stabilizer of some S_k.

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Kleinian groups and curve complexes

Brian Bowditch

My most recent work concerns the interconnections between hyperbolic 3-manifolds, Teichmüller theory and the combinatorics of surfaces. Much of this has been motivated by the work of Minsky, Masur, Brock and Canary towards

[3]A group is *small* if it does not contain a non-abelian free group.

[4]Given any decreasing sequence of arcs $I_1 \subset I_2 \dots$, for i large enough, $\mathrm{Stab}(I_i) = \mathrm{Stab}(I_{i+1})$.

[5]An arc is I unstable if there exists an arc $J \subset I$ with $\mathrm{Stab}(J) \neq \mathrm{Stab}(I)$.

resolving Thurston's ending lamination conjecture [BrCM]. This is closely linked to understanding the large scale geometry of Teichmüller space. An important tool has been the curve complex introduced by Harvey, and a central result, due to Masur and Minsky, shows that this complex is hyperbolic in the sense of Gromov [MaM1]. I have given an alternative proof of this, showing that the hyperbolicity constant is at most logarithmic in the complexity of the surface [Bow1].

There has been a great deal of work on hyperbolic and relatively hyperbolic groups. A significant obstruction to adapting this technology to the curve complex is the fact that this complex is far from being locally finite. Indeed it does not satisfy the more general "fineness" condition (or equivalents) used in much of the work on relatively hyperbolic groups. One possible way of resolving this is via the notion of "tight geodesic" introduced by Masur and Minsky [MaM2]. They show that only finitely many tight geodesics can connect any two points in curve complex. By feeding back ideas from 3-manifold theory it is possible to give a more uniform statement, namely that any slice of the union of tight geodesics between two points has cardinality bounded in terms of the complexity of the surface. From this, and variants of this statement, one can deduce new information about the curve complex: for example that the action of the mapping class group is acylindrical. Recently Bell and Fujiwara [BeF] have shown that it implies that the curve complex has finite asymptotic dimension. Kida [K] has used it to give a proof of property A, as defined by Yu, in connection with Baum–Connes conjecture, for curve complexes. It is also one of the ingredients in the study of the geodesic flow on Teichmüller space by Hamenstädt. It is likely that these kinds of arguments will have further consequences.

More recently I have applied some of these ideas to give an alternative approach to ending lamination conjecture [Bow2], [Bow3]. The overall strategy is related to that of Minsky et al, though the logic is somewhat different. This enables us to use a rather simpler version of the model space that bypasses most of the sophisticated machinery of hierarchies developed by Masur and Minsky. One can also simplify some of the more technical aspects of the proof. One shows that the universal cover of the model is equivariantly quasi-isometric to the hyperbolic manifold in question. As with Minsky et al, in the surface group case, one gets uniform constants. It would be interesting to attempt to construct such a uniform model in general.

It would also be interesting to give a more explicit computational approach to this material. Shackleton [Sh] has shown that the set of tight geodesics between two points is computable, and explores various consequences. However, this does not give uniform constants, and so it would be good to find a method for achieving both simultaneously. At present, most work relating to hyperbolic 3-manifolds in this context makes essential use of some limit, or precompactness, type argument.

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Some analytical tools in group theory

Yehuda Shalom
(joint work with Uri Bader)

The talk focused on the construction of discrete groups which are *just infinite* or *simple*, using methods from rigidity theory. The main theorem, joint with Uri Bader, states:

Theorem. *Let G_1, G_2 be locally compact non-discrete compactly generated groups for which every proper quotient is compact. Let $\Gamma \subset G = G_1 \times G_2$ be a discrete cocompact subgroup which projects densely to both G_i. Then Γ is just infinite, i.e. every non-trivial normal subgroup of Γ has finite index,* unless $G_1 = G_2 = \mathbf{R}$ *and* $\Gamma = \mathbf{Z}^2 \subset \mathbf{R}^2$ *embedded irrationally.*

The method follows the remarkable one introduced by Margulis in his famous normal subgroup theorem for lattices in higher-rank algebraic groups: to show that Γ/N is finite, show that it is both amenable and has property (T). These two independent 'halves' were explained in the talk.

Weighted L^2-cohomology of Coxeter groups

Michael Davis
(joint work with Jan Dymara, Tadeusz Januszkiewicz and Boris Okun)

Suppose (W, S) is a Coxeter system and q is a positive real number. $\mathbf{R}^{(W)}$ denotes the Euclidean space of finitely supported real-valued functions on W and $\{e_w\}_{w\in W}$ its standard basis. Define an inner product $\langle\ ,\ \rangle_q$ by

$$\langle e_v, e_w\rangle_q = \begin{cases} q^{l(w)} & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

$L^2_q(W)$ denotes the Hilbert space completion of $\mathbf{R}^{(W)}$ with respect to this inner product. Deform the multiplication in the group algebra $\mathbf{R}W$ by

$$e_s e_w = \begin{cases} e_{sw} & \text{if } l(sw) > l(w), \\ q e_{sw} + (q-1)e_w & \text{if } l(sw) < l(w). \end{cases}$$

This defines a new multiplication on $\mathbf{R}^{(W)}$ making it into an associative algebra $\mathbf{R}_q W$ called the *Hecke algebra* (see [1, Ex. 22, pp. 56–57]). Define a linear involution $x \to x^*$ of $\mathbf{R}_q W$ by $(e_w)^* := e_{w^{-1}}$. $\mathbf{R}_q W$ acts on L^2_q $(= L^2_q(W))$ by either left or right multiplication. It is easily checked that $*$ is an anti-involution of the algebra $\mathbf{R}_q W$ and that $\langle xy, z\rangle_q = \langle y, x^* z\rangle_q = \langle x, zy^*\rangle_q$. So, $\mathbf{R}_q W$ defines two C^*-algebras of operators (by left or right multiplication); the weak closure of either one is denoted $\mathcal{N}_q$. It is a von Neumann algebra, called the *Hecke - von Neumann algebra*. The *trace* of an element $\phi \in \mathcal{N}_q$ is defined by the usual formula: $\operatorname{tr}\phi := \langle \phi(e_1), e_1\rangle_q$. This extends to definition of a trace for any $\mathcal{N}_q$-endomorphism of a finite direct sum of copies of L^2_q. Hence, if $V \subset \oplus L^2_q$ is any closed $\mathcal{N}_q$-submodule we can define its *von Neumann dimension* by $\dim V := \operatorname{tr} p_V$, where $p_V : \oplus L^2_q \to \oplus L^2_q$ is orthogonal projection onto V.

The von Neumann dimensions of some important $\mathcal{N}_q$-modules are tied to the growth series of W, i.e., to the power series defined by $W(t) := \sum_{w\in W} t^{l(w)}$. Its radius of convergence is denoted by ρ. $W(t)$ is a rational function of t. One way to see this is simply to prove a formula such as

$$\frac{1}{W(t)} = \sum_{T\in\mathcal{S}} \frac{(-1)^{|T|}}{W_T(t^{-1})}, \tag{1}$$

where for any $T \subset S$, $W_T := \langle T\rangle$ is the special subgroup generated by T and $\mathcal{S} := \{T \subset S \mid |W_T| < \infty\}$ is the *poset of spherical subsets.* Formula (1) shows that $W(t)$ is a rational function, since the growth series of any finite group, such as a spherical subgroup W_T, is a polynomial. Hecke algebras (resp. growth series) make sense when q (resp. t) is replaced by a certain I-tuple $\mathbf{q} = (q_i)$ (resp. $\mathbf{t} = (t_i)$), where we are given a function $i : S \to I$ which is constant on conjugacy classes of elements of S. Given $s \in S$, write q_s (resp. t_s) instead of $q_{i(s)}$ (resp. $t_{i(s)}$). Then $q^{l(w)}$ (resp. $t^{l(w)}$) is replaced by $q_w := q_{s_1} \cdots q_{s_n}$ (resp. t_w) where $w = s_1 \cdots s_n$ is any reduced expression for w.

Two important self-adjoint idempotents in $\mathcal{N}_q$ are

$$a_T := \frac{1}{W_T(q)} \sum_{w\in W_T} e_w \quad \text{and} \quad h_T := \frac{1}{W_T(q^{-1})} \sum_{w\in W_T} (-1)^{l(w)} q^{-l(w)} e_w.$$

Right multiplication by a_T (resp. h_T) is a well-defined, bounded linear operator precisely when $q < \rho_T$ (resp. $q > \rho_T^{-1}$), where ρ_T is the radius of convergence of $W_T(t)$. In particular, both are defined when T is spherical. These idempotents define subspaces, $A_T := L^2_q a_T$ and $H_T := L^2_q h_T$ of dimension $1/W_T(q)$ and $1/W_T(q^{-1})$, respectively.

Let Σ be the standard piecewise Euclidean CAT(0) complex on which W acts properly and isometrically (e.g., see [2]). It is defined by pasting together copies of a certain fundamental domain K, one for each element of W. $C^*(\Sigma)$ is its cellular cochain complex, i.e., $C^i(\Sigma) = \{f : \{i\text{-cells}\} \to \mathbf{R}\}$ and $C^i_c(\Sigma) := \{\text{finitely supported f}\}$ is the subcomplex of finitely supported cochains. We can define a weighted inner product on $C^i_c(\Sigma)$ similar to the one discussed above. Given a cell σ of Σ, let $w(\sigma)$ be the element w of shortest length such that $w^{-1}\sigma \subset K$. Assign a weight to the characteristic function e_σ of the cell so that its length is $q^{l(w(\sigma))}$. Its Hilbert space completion is denoted $L^2_qC^i(\Sigma)$. As q varies between 0 and ∞ it interpolates between $C^i(\Sigma)$ and $C^i_c(\Sigma)$. $L^2_qC^i(\Sigma$ inherits the structure of an $\mathcal{N}_q$-module in a straightforward fashion so that the coboundaries are maps of $\mathcal{N}_q$-modules. Its (reduced) cohomology groups are denoted $L^2_qH^*(\Sigma)$. We compute them as $\mathcal{N}_q$-modules for $q \leq \rho$ and for $q > \rho^{-1}$. The von Neumann dimension of $L^2_qH^i(\Sigma)$ is denoted $b^i_q(\Sigma)$ and called the ith *L^2_q-Betti number.* When q is an integer (or when $\mathbf{q}$ is an I-tuple of integers), these numbers are equal to the ordinary L^2-Betti numbers of of any building of type (W, S) and thickness q (or $\mathbf{q}$) with a chamber transitive automorphism group. (Thickness q means that $q+1$ chambers meet along each mirror.) The L^2_q- *Euler characteristic* is the alternating sum of the $b^i_q(\Sigma)$.

Theorem 1. (Dymara [6]). $\chi_q(\Sigma) = 1/W(q)$.

This is proved in the usual fashion by calculating the alternating sum of the dimensions of the spaces of cochains on orbits of cells. There is a cellulation of Σ with one orbit of cells for each $T \in \Sigma$; the dimension of the space of cochains on this orbit is $\dim H_T = 1/W_T(q^{-1})$; so, the theorem follows from (1). The next result of Dymara says that for $q < \rho$, L^2_q-cohomology behaves like ordinary cohomology.

Theorem 2. (Dymara [6]). *For $q < \rho$, $L^2_qH^*(\Sigma)$ is concentrated in dimension* 0 *and is isomorphic to* $\mathbf{R}$ *as a vector space. Its von Neumann dimension is given by* $b^0_q(\Sigma) = 1/W(q)$. *Moreover, for $q > \rho$, $b^0_q(\Sigma) = 0$.*

For each $T \in \mathcal{S}$, let $A_{>T}$ be the $\mathcal{N}_q$-submodule of A_T defined by module defined by $A_{>T} = \sum_{U\in\mathcal{S}_{>T}} A_U$. Put $D_T := A_T/A_{>T}$. Using Theorem 2 we prove the following generalization of a result of L. Solomon [8] when W is finite.

The Decomposition Theorem. ([3]). *For $q > \rho^{-1}$ and any $T \in \mathcal{S}$,*

$$A_T = \overline{\bigoplus_{U\in\mathcal{S}_{>T}} D_U}.$$

This leads directly to the following.

The Main Theorem. ([3]). *For $q > \rho^{-1}$, we have an isomorphism of $\mathcal{N}_q$-modules*

$$L^2_qH^*(\Sigma) \cong \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes D_T.$$

Here the fundamental domain K is contractible complex, it is equipped with a family $\{K_s\}_{s\in S}$ of closed contractible subcomplexes ("mirrors") indexed by S. K^{S-T} denotes the union of those mirrors which are indexed by $S-T$. $H^*(K, K^{S-T})$ is ordinary cohomology.

In [2, 4] we carry out a similar calculation for $H_c^*(\Sigma)$. A consequence is that the inclusion $C_c^*(\Sigma) \hookrightarrow L_q^2C^*(\Sigma)$ induces an injection with dense image, $H_c^*(\Sigma) \to L_q^2H^*(\Sigma)$. So, in this sense weighted L^2-cohomology is like compactly supported cohomology in the range $q > \rho^{-1}$. The big question is what happens in the intermediate range $\rho < q < \rho^{-1}$?

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The Dimension of the Torelli subgroup in $\mathrm{Out}(F_n)$

Kai-Uwe Bux

(joint work with Mladen Bestvina and Dan Margalit)

Abelianizing induces a natural homomorphism from $\mathrm{Out}(F_n)$ to $\mathrm{Gl}_n(\mathbf{Z})$. This homomorphism is surjective. In analogy to the situation in mapping class groups, we call its kernel the *Torelli subgroup* of $\mathrm{Out}(F_n)$, and we denote it by T_n. We show:

Theorem 1. *For $n \geq 3$, the following hold:*

(1) The Torelli subgroup T_n has an Eilenberg–Mac Lane complex of dimension $2n-4$.

(2) Its integral homology in top dimension, $\mathrm{H}_{2n-4}(T_n; \mathbf{Z})$, is not finitely generated. In particular, T_n is not of type FP_{2n-4}.

Our approach is purely geometric: we construct an Eilenberg–Mac Lane space of dimension $2n-3$ as a quotient of the spine of Outer Space. Then, we use combinatorial Morse theory to show that it is homotopy equivalent to a space of dimension $2n-4$. Finally, we exhibit explicitly an infinite family of independent homology classes.

Using the Hochschild–Serre spectral sequence, we obtain:

Corollary 1. *For $n \geq 3$, the following hold:*

(1) $\mathrm{H}_{2n-3}(\mathrm{Aut}(F_n); \mathbf{Z})$ *is not finitely generated.*
(2) Let k be the minimum dimension for which $\mathrm{H}_k(\mathrm{Out}(F_n); \mathbf{Z})$ *is not finitely generated. Then* $\mathrm{H}_k(\mathrm{Aut}(F_n); \mathbf{Z})$ *is not finitely generated, as well.*

We note that our methods also yield a geometric proof of the classical fact (due to Magnus) that T_n is finitely generated.

The paper can be found at the ArXiv: math.GT/0603177

Two boundaries for the mapping class group

Ursula Hamenstädt

The *mapping class group* $\mathcal{M}(S)$ of an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures ($3g-3+m \geq 2$) is the group of isotopy classes of orientation preserving diffeomorphisms of S. This group is well known to be finitely presented. It shares many properties with a lattice Γ in a simple Lie group G of non-compact type and higher rank even though it does not contain any subgroup which is commensurable to such a lattice.

Natural structures determined by the asymptotic properties of a lattice Γ in such a simple Lie group G are the various boundaries for Γ. The best known such boundary is the *Furstenberg boundary* which is defined to be the compact Γ-space G/P where $P < G$ is a minimal parabolic subgroup of G and Γ acts on G/P by left translation. The action of Γ on G/P is *topologically amenable* in the following sense. Let $\mathcal{P}(\Gamma)$ be the space of probability measures on Γ, viewed as a subset of the unit sphere in the (normed) space $\ell^1(\Gamma)$ of summable functions on Γ. There is a sequence ξ_i of weak*-continuous maps $\xi_i : G/P \to \mathcal{P}(\Gamma)$ such that $\|\xi_i(gx) - g\xi_i(x)\| \to 0$ uniformly on compact subsets of $\Gamma \times G/P$. In particular, the point stabilizers of the Γ-action are amenable subgroups of Γ.

Based on earlier work of Yu, Higson [4] established a significant consequence of the existence of a topologically amenable action on a *compact* space. Namely, if a countable group Γ admits such an action, then for every subgroup Γ' of Γ the Novikov conjecture holds.

A *geodesic lamination* for a hyperbolic metric of finite volume on our surface S is a *compact* subset of S foliated into simple geodesics. The space of geodesic laminations on S can be equipped with the *Hausdorff topology* for compact subsets of S. A geodesic lamination λ is called *complete* if every complementary component of λ either is an ideal triangle or a once punctured monogon, and if moreover λ can be approximated in the Hausdorff topology by simple closed geodesics. The space $\mathcal{CL}$ of complete geodesic laminations is a compact $\mathcal{M}(S)$-space. In [2] we show that $\mathcal{CL}$ can be viewed as a Furstenberg boundary for $\mathcal{M}(S)$. In particular, we have.

Theorem 1: *The action of $\mathcal{M}(S)$ on $\mathcal{CL}$ is topologically amenable.*

Corollary: *The Novikov conjecture holds for every subgroup of $\mathcal{M}(S)$.*

The corollary was independently established by Kida [5].

While the Furstenberg boundary of a lattice Γ in a simple Lie group of non-compact type is an essential tool for studying actions of Γ and homomorphisms, it does not give valuable information on the geometry of Γ. Here by the geometry of a finitely generated group Γ we mean the geometry defined by the word metric of a finite generating set. Recall that any two such word metrics are quasi-isometric. Large scale geometric properties of Γ are encoded into topological properties of its *asymptotic cone*.

The *asymptotic cone* of a metric space X with basepoint $x_0 \in X$ and with respect to a non-principal ultrafilter ω is defined as follows. Let $X_\infty = \{(x_i) \in \prod_{i=1}^\infty X \mid d(x_i, x_0)/i \text{ is bounded}\}$; then for $(x_i), (y_i) \in X_\infty$ the *ultralimit* ω-$\lim d(x_i, y_i)/i$ is well defined. These ultralimits define a pseudodistance on X_∞ whose quotient metric space is an asymptotic cone X_ω for X. The asymptotic cone of an euclidean space is euclidean, and the asymptotic cone of a Gromov hyperbolic space is an $\mathbb{R}$-tree. The asymptotic cone of a cocompact lattice in a simple Lie group G of non-compact type coincides with the asymptotic cone of G, and it can be recovered from the *Tits boundary* of G (however, the asymptotic cone of G does not coincide with the cone over its Tits boundary). For the mapping class group $\mathcal{M}(S)$, the asymptotic cone also admits a quite explicit description using a basic building block which can be viewed as a cone over a Tits boundary of the group [3].

Define the *homological dimension* of a topological space X to be the maximal number $n > 0$ such that there are open sets $U \subset V \subset X$ with $H_n(V, U) \neq \emptyset$. For cocompact lattices Γ in simple Lie groups of non-compact type, the homological dimension of the asymptotic cone of Γ coincides with the rank of the group G. The explicit description of the asymptotic cone of $\mathcal{M}(S)$ is used to establish a (modified) version of a result of Behrstock and Minsky [1].

Theorem 2: *The homological dimension of the asymptotic cone of the mapping class group $\mathcal{M}(S)$ equals the maximal rank of an abelian subgroup of $\mathcal{M}(S)$.*

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Cut points of CAT(0) groups

Panos Papasoglu
(joint work with Eric Swenson)

It is known by work of Bowditch that JSJ decompositions of hyperbolic groups are reflected on their boundaries. We show that this holds for CAT(0) groups too under the assumption of no infinite torsion subgroups. This produces a canonical JSJ decomposition. The decomposition we obtain is related to JSJ decompositions constructed by Swarup and Scott.

It is known that the boundaries of CAT(0) groups without infinite torsion subgroups have no cut points. Our work is based on the observation that splittings of such groups correspond to pairs of points $\{a, b\}$ which separate the boundary.

The proof has two parts. In the first part we show that we can associate to a continuum X, with no cut points, an $\mathbb{R}$-tree T encoding the set of cut pairs of X. We call T the JSJ-tree of the continuum. This tree is canonical i.e. any homeomorphism ϕ of X induces a homeomorphism $\bar{\phi} : T \to T$.

In the second part we use the action of G on the JSJ-tree of the boundary of X. In the hyperbolic case one shows that the action has no global fixed point using the fact that G acts on its boundary as a convergence group. In the CAT(0) case we show that the action satisfies a weaker condition, namely it is a π-convergence group action. Using this we infer that the action has no global fixed point.

Amenable covers, minimal volume and L^2-Betti numbers of aspherical manifolds

Roman Sauer

We prove the following claim stated by Gromov in [2, p. 297] (together with an idea that is described below):

Theorem 1. *For every $n \geq 1$ there is a constant $C = C(n) > 0$ with the following property: If M is a closed aspherical n-dimensional Riemannian manifold with* $\mathrm{Ricci}(M) \geq -1$*, then the L^2-Betti numbers satisfy*

$$b_i^{(2)}(\widetilde{M}) \leq C \cdot \mathrm{vol}(M) \text{ for } i \geq 0.$$

It follows that the L^2-Betti numbers of the fundamental group of a closed aspherical manifold bound its minimal volume from below. A particularly interesting aspect of this is that we thus get an orbit equivalence invariant bounding the minimal volume from below.

With related techniques we prove the following general vanishing result.

Theorem 2. *If the closed, aspherical n-dimensional manifold M is covered by open, amenable subsets with multiplicity $\leq n$ then*

$$b_i^{(2)}(\widetilde{M}) = 0 \text{ for } i \geq 0.$$

As a (non-obvious) corollary, for every dimension n there is constant $\epsilon(n) > 0$ such that if the minimal volume of a closed, smooth, aspherical manifold is at most $\epsilon(n)$, then its L^2-Betti numbers vanish. This is consistent with the Atiyah conjecture.

Both theorems have well-known counterparts with *L^2-Betti numbers* replaced by the *simplicial volume*. The simplicial volume (and also bounded cohomology) share lots of properties with L^2-Betti numbers, most notably in the aspherical case. However, we still lack a more direct link. A positive answer to the following conjecture by Gromov would be satisfying in that regard.

Conjecture 1. *If the simplicial volume of a closed, aspherical manifold vanishes, then its L^2-Betti numbers vanish.*

So far this conjecture is wide open.

We briefly describe the framework of the proofs of the two theorems above. We work in the category of $\mathscr{R}$-simplicial complexes (see in [1]) and $\mathscr{R}$-spaces (a topological extension of the latter), where $\mathscr{R}$ is the orbit equivalence relation of some action of the fundamental group $\Gamma = \pi_1(M)$ on a probability space (X, μ).

An (the easiest one) example of an $\mathscr{R}$-space is $X \times \widetilde{M}$ equipped with the diagonal Γ-action. By Gaboriau's theory one can read off the L^2-Betti numbers $b_i^{(2)}(\widetilde{M})$ from the $\mathscr{R}$-simplicial complex $X \times \widetilde{M}$. The rough idea for the estimates is to realize $X \times \widetilde{M}$ as a homotopy retract in the nerves of certain equivariant covers of $X \times \widetilde{M}$. The hypotheses of the theorems enter into the specific constructions of these covers. In the case of the first theorem the idea to use such a homotopy retract argument was already suggested by Gromov. The constructions of the respective covers employ the interplay between geometric/topological properties of M (packing inequalities, covering theory) and ergodic properties of X (mixing actions, Ornstein–Weiss–Rokhlin lemma). Another tool is a new homology theory for the $\mathscr{R}$-quotients of $\mathscr{R}$-spaces modelled on foliated homology.

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Orbit closures and arithmetic of quaternions

Shahar Mozes

(joint work with Manfred Einsiedler)

Let Δ be an affine Bruhat–Tits building, $O \in \Delta$ a fixed vertex, and Γ a group acting freely on Δ. Call $\gamma = \alpha.\beta$ (for $\gamma, \alpha, \beta \in \Gamma$) a *decomposition* if $\alpha O \in \text{cone}(O, \gamma O)$. We shall say that δ *divides* γ ($\delta | \gamma$) when there exist $\alpha, \beta \in \Gamma$ such that $\gamma = (\alpha.\delta).\beta$, with both multiplications being decompositions.

Let $G = \prod_{p \in S} \mathbf{G}(\mathbf{Q}_p)$, where $\mathbf{G}$ is an algebraic group defined over $\mathbf{Q}$ and S is a finite set of places; suppose that $\mathbf{G}$ is simple and $\mathbf{G}(\mathbf{R})$ is compact. Let $\Gamma \subset G$ be a torsion-free lattice commensurable with $\mathbf{G}(\mathbf{Z}[\frac{1}{p}]_{p \in S})$, and $A \subset G$ a maximal split Cartan subgroup, with $\mathrm{rank}(A) \geq 2$. Assume that the only reductive $\mathbf{Q}$-subgroups $H \subset G$ such that $\mathrm{rank}(H) = \mathrm{rank}(G)$ and ΓH is closed supporting a finite H-invariant measure are G or Cartan. Then we show that:

Theorem. *In this setting, for all $\delta_0 \in \Gamma$ and for all $\epsilon > 0$, there exists $w > 0$ such that*

$$e^{\epsilon n} > \left| \left\{ \gamma \in \Gamma \,\middle|\, \mathrm{diam}(\mathrm{cone}(O, \gamma O)) = n,\ \mathrm{width}(\mathrm{cone}(O, \gamma O)) \geq w,\ \delta_0 \not| \gamma \right\} \right|.$$

The main ingredient of the proof is a result of Manfred Einsiedler and Elon Lindenstrauss (following ealier results by Lindenstrauss, Einsiedler–Katok–Lindenstrauss) showing that in the above setting an A-invariant ergodic probability measure on $\Gamma \backslash G$ such that for some regular element $a \in A$ the entropy $\mathsf{h}_\mu(a)$ is positive must be the G-invariant probability measure on $\Gamma \backslash G$.

Foldings, the rank problem and Nielsen equivalence

Richard Weidmann

We discuss how Stallings folds and their generalizations can be used to approximate a group action on a tree and more generally on a hyperbolic space. We then proceed by discussing application of these techniques to the rank problem of Kleinian groups, finiteness of Nielsen equivalence classes of generating tuples and to the Nielsen equivalence problem, i.e. the problem of deciding whether two tuples of group elements are Nielsen equivalent [2].

We are in particular able to prove the following theorem which implies that to decide Nielsen equivalence in a free product it suffices to be able to decide Nielsen equivalence in the factors [5].

Theorem 1. [5] *Let $G = H * K$, $\mathcal{T} = (h_1, \ldots, h_l, k_1, \ldots, k_m)$ and $\mathcal{T}' = (h'_1, \ldots, h'_{l'}, k'_1, \ldots, k'_{m'})$ with $h_i, h'_i \in H$ and $k_i, k'_i \in K$ for all i.*

If $\mathcal{T}$ and $\mathcal{T}'$ are irreducible and Nielsen equivalent then $l = l'$, $m = m'$, $(h_1, \ldots h_l)$ is Nielsen equivalent to $(h'_1, \ldots, h'_l)$ and $(k_1, \ldots k_m)$ is Nielsen equivalent to $(k'_1, \ldots, k'_m)$.

In the case of infinite cyclic factors this is simply Nielsen's theorem: in the case of finite cyclic free factors this result is due to Lustig and Moriah [3]. The above theorem is an algebraic analogue of the Gordon conjecture for Heegaard splittings of connected sums; the Gordon conjecture itself has been recently established independently by Qiu [4] and Bachman [1].

This proves in particular a conjecture of Dunwoody on Zimmermann. In order to prove the theorem we define a graph associated to a tuple of elements of a free product and study basic properties of this graph. This graph seems to be a natural object that should have applications elsewhere.

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Automorphisms of right-angled Artin groups

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(joint work with John Crisp and Karen Vogtmann)

To a finite simplicial graph Γ with vertex set V we associate a *right-angled Artin group* A_Γ defined by the presentation

$$A_\Gamma = \langle V \mid vw = wv \text{ if } v, w \text{ are connected by an edge in } \Gamma\rangle.$$

These groups play a central role in the work of Bestvina–Brady [1], Croke–Kleiner [2], Ghrist [3], and others. At the two extremes of this construction are the case of a graph with n vertices and no edges, in which case A_Γ is a free group of rank n, and that of a complete graph on n vertices, in which case A_Γ is a free abelian group of rank n. In general, right-angled Artin groups can be thought of as interpolating between these two extremes, and thus the automorphism groups $Aut(A_\Gamma)$ interpolate between $Aut(F_n)$ and $GL_n(\mathbb{Z})$. Right-angled Artin groups have been studied by many people, but relatively little is know about their automorphism groups with the exception of the work of Servatius [5] and Lawrence [4] who describe a finite generating set for the automorphism group.

We consider the case in which the defining graph Γ is connected and triangle-free. We show that "maximal joins" in Γ, that is maximal complete bipartite subgraphs, play a key role in the automorphism group of A_Γ. The subgroup A_J generated by such a maximal join J is a direct product of two free groups and these subgroups are preserved up to conjugacy by automorphisms of A_Γ. This gives rise to a homomorphism

$$Out^0(A_\Gamma) \to \prod Out(A_J)$$

whose kernel is a finitely generated free abelian group. (Here, $Out^0(A_\Gamma)$ is a finite index normal subgroup of $Out(A_\Gamma)$ which avoids certain diagram symmetries.)

We then construct an "outer space" $\mathcal{O}(A_\Gamma)$ for $Out^0(A_\Gamma)$. If Γ is a single join J, $\mathcal{O}(A_\Gamma)$ consists of actions of A_J on a product of trees $X_J = T_1 \times T_2$. For a general Γ, a point in outer space is a graph of spaces of the form X_J, with J varying over maximal joins in Γ. We prove that for any connected, triangle-free graph Γ, the space $\mathcal{O}(A_\Gamma)$ is finite dimensional, contractible, and has a proper action of $Out^0(A_\Gamma)$.

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Cut-points in asymptotic cones, and groups acting on tree-graded spaces

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(joint work with Cornelia Druţu)

Observation due to Bestvina and Paulin: if a group has *many* actions on a Gromov-hyperbolic metric space then it acts non-trivially (i.e. without a global fixed point) by isometries on the asymptotic cone of that space which is an $\mathbb{R}$-tree. The word *many* means that the translation numbers of these actions are unbounded.

If the translation number of an action of G on (X, dist) is d then the translation number of the action of G on $X/d = (X, \mathrm{dist}/d)$ is 1. If X is δ-hyperbolic then X/d is δ/d-hyperbolic. If ϕ_n is an action of G on X with translation number d_n, $\lim d_n = \infty$ then G acts with translation number 1 (in particular, without a global fixed point) on the asymptotic cone (= Gromov–Hausdorff limit of X/d_n) of X which is 0-hyperbolic, i.e. an $\mathbb{R}$-tree.

This observation can be applied in each of the following situations:

- An equation (system of equations) $w(\vec{x}) = 1$ has infinitely many non-conjugate solutions in a group G. Then the coordinate group $G[w]$ of this equation has *many* homomorphisms ϕ_n into G, and acts on the Cayley graph of G by left multiplications by $\phi_n(.)$.
- the group $\mathrm{Out}(G)$ is infinite.
- the group G is not Hopfian.
- the group G is not co-Hopfian.

Note that in all these situations, the group acts on the asymptotic cone of G even if G is not hyperbolic. Of course in that case the cone may not be a tree. But it is very often the so-called *tree-graded space* [DS1].

Definition 1. *Let $\mathbb{F}$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets (called* pieces*). Suppose that the following two properties are satisfied:*

(T_1) *Every two different pieces have at most one common point.*

(T_2) *Every simple geodesic triangle (a simple loop composed of three geodesics) in $\mathbb{F}$ is contained in one piece.*

Then we say that the space $\mathbb{F}$ is tree-graded with respect to $\mathcal{P}$.

By [DS1, Proposition 2.17], property (T_2) in this definition can be replaced by each of the following two properties.

(T_2') For every topological arc $\mathfrak{c} : [0, d] \to \mathbb{F}$ and $t \in [0, d]$, let $\mathfrak{c}[t - a, t + b]$ be a maximal sub-arc of $\mathfrak{c}$ containing $\mathfrak{c}(t)$ and contained in one piece. Then every other topological arc with the same endpoints as $\mathfrak{c}$ must contain the points $\mathfrak{c}(t - a)$ and $\mathfrak{c}(t + b)$.

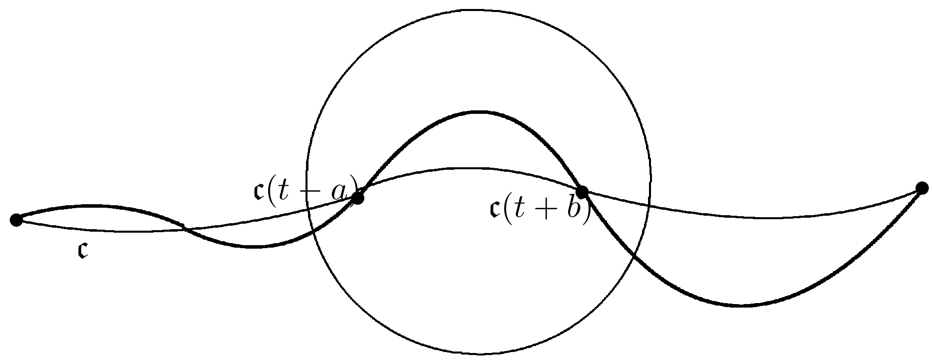

FIGURE 1. Property (T_2').

(T_2'') Every simple loop in $\mathbb{F}$ is contained in one piece.

It is observed in [DS1], that every metric space with global cut-point has a canonical tree-graded structure where pieces are maximal subsets without cut-points.

Although there are wide classes of groups without cut-points in their asymptotic cones (say, all groups satisfying non-trivial law are such), the class of groups with cut-points in their asymptotic cones is also very wide. It includes all relatively hyperbolic groups [DS1], mapping class groups (Behrstock), many right angled Artin groups (Behrstock, Druţu, Mosher), fundamental groups of graph-manifolds, etc.

Conjecture. Every acylindrical amalgamated product $A *_C B$ of two groups has cut-points in its asymptotic cones.

Question. Is there an amenable non-virtually cyclic group with cut-points in its asymptotic cones?

Thus it is important to study actions of groups on tree-graded spaces.

We extend the Rips–Bestvina–Feighn–Levitt–Guirardel theory of groups acting on $\mathbb{R}$-trees to groups acting on tree-graded spaces.

Notation: For every group G acting on a tree-graded space $(\mathbb{F}, \mathcal{P})$,

- $\mathcal{C}_1(G)$ is the set of stabilizers of subsets of $\mathbb{F}$ all of whose finitely generated subgroups stabilize pairs of distinct pieces in $\mathcal{P}$;
- $\mathcal{C}_2(G)$ is the set of stabilizers of pairs of points of $\mathbb{F}$ not from the same piece;
- $\mathcal{C}_3(G)$ is the set of stabilizers of triples of points of $\mathbb{F}$, neither from the same piece nor on the same transversal geodesic.

Here is our main result about groups acting on tree-graded spaces.

Theorem 1 ([DS2])**.** *Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$. Suppose that the following hold:*

(i) *every isometry $g \in G$ permutes the pieces;*
(ii) *no piece in $\mathcal{P}$ is stabilized by the whole group G; likewise no point in $\mathbb{F}$ is fixed by the whole group G.*

Then one of the following four situations occurs:

(**I**) *the group G acts by isometries on an $\mathbb{R}$–tree non-trivially, with stabilizers of non-trivial arcs in $\mathcal{C}_2(G)$, and with stabilizers of non-trivial tripods in $\mathcal{C}_3(G)$;*
(**II**) *there exists a point $x \in \mathbb{F}$ such that for any $g \in G$ any geodesic $[x, g \cdot x]$ is covered by finitely many pieces: in this case the group G acts non-trivially on a simplicial tree with stabilizers of pieces or points of $\mathbb{F}$ as vertex stabilizers, and stabilizers of pairs (a piece, a point inside the piece) as edge stabilizers;*
(**III**) *the group G acts non-trivially on a simplicial tree with edge stabilizers from $\mathcal{C}_1(G)$;*
(**IV**) *the group G acts on a complete $\mathbb{R}$–tree by isometries, non-trivially, such that stabilizers of non-trivial arcs are locally inside $\mathcal{C}_1(G)$-by-Abelian subgroups, and stabilizers of tripods are locally inside subgroups in $\mathcal{C}_1(G)$; moreover if G is finitely presented then the stabilizers of non-trivial arcs are in $\mathcal{C}_1(G)$.*

As applications of our main results we obtain the following generalizations of results of Dahmani, Groves, Delzant–Potyagailo, Belegradek, Szczepański.

Theorem 2 ([DS2])**.** *Suppose that the peripheral subgroups of G are not relatively hyperbolic with respect to proper subgroups. If* $\operatorname{Out}(G)$ *is infinite then one of the followings cases occurs:*

(1) *G splits over a virtually cyclic subgroup;*
(2) *G splits over a parabolic (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup;*
(3) *G can be represented as a non-trivial amalgamated product or HNN extension with one of the vertex groups a maximal parabolic subgroup of G.*

Theorem 3 ([DS2]). *Suppose that a relatively hyperbolic group G is not co-Hopfian. Let ϕ be an injective but not surjective homomorphism $G \to G$. Then one of the following holds:*

- *$\phi^k(G)$ is parabolic for some k.*
- *G splits over a parabolic or virtually cyclic subgroup.*

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