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Teichmüller Space (Classical and Quantum)

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Abstract. This is a short report on the conference "Teichmüller Space (Classical and Quantum) " held in Oberwolfach from May 28th to June 3rd, 2006.

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Introduction by the Organisers

In a broad sense, the subject of Teichmüller theory is the study of moduli for geometric structures on surfaces. The progenitor of the subject is usually considered to be G. F. B. Riemann, who in a famous paper on Abelian functions, studied the moduli space of algebraic curves and stated that the space of deformations of equivalence classes of conformal structures on a closed orientable surface of genus $\mathfrak{g} \geq 2$ is of complex dimension $3\mathfrak{g} - 3$. This was explicated by O. Teichmüller who laid the foundations of the theory in a series of famous papers (during a remarkably brief period). Many prominent mathematicians including L. Ahlfors and L. Bers continued developing the theory over several decades. In the 1970s, W. Thurston introduced techniques of hyperbolic geometry in the study of Teichmüller space and its asymptotic geometry. In the 1980s, new combinatorial treatments of Teichmüller and moduli spaces evolved with a concurrent interplay of ideas from string theory in high-energy physics.

Teichmüller theory is one of those precious subjects in mathematics which have the advantage of bringing together, at an equally important level, fundamental ideas coming from different fields. Among the fields associated to Teichmüller theory, one can surely mention complex analysis, hyperbolic geometry, discrete group theory, algebraic geometry, low-dimensional topology, Lie groups, symplectic geometry, dynamical systems, topological quantum field theory, string theory, and many others. Teichmüller theory is growing at a fantastic rate, and the fact that it involves all these areas is probably a consequence of the fact that Teichmüller space itself carries a diversity of rich structures. As a matter of fact, this space can be seen from at least three points of view: as a space of equivalence classes of hyperbolic metrics, as a space of equivalence classes of conformal structures, and as a space of equivalence classes of representations of the fundamental group of a surface into a Lie group. Each of these points of view endows Teichmüller space with various structures, including several interesting metrics, a natural complex structure, a symplectic structure, a real analytic structure, an algebraic structure, cellular structures, various boundary structures, a natural discrete action by the mapping class group, interesting geodesic and horocyclic flows on the quotient Riemann moduli space, a quantization theory of its Poisson structure, and the list goes on and on.

The quantization of Teichmüller space was developed in the last few years by L. Chekhov and V. Fock and independently in work by R. Kashaev. This theory produces noncommutative families of deformations of the Poisson or symplectic structure of Teichmüller space in the form of $*$-algebras, with an action of the mapping class group of the surface as an outer automorphism group. In particular, quantization of Teichmüller space leads to new invariants of hyperbolic three-manifolds.

The conference brought together people in almost all of the active areas of Teichmüller theory. The fact that Teichmüller theory is a living and rich subject connecting several areas of mathematics was reflected in the richness of the talks that were presented, and in the variety of the new perspectives that were discussed at the problem session, on which we report separately.

We note that many other attendees were ready to give interesting talks than time permitted. As a general rule, younger researchers were given the opportinity to present their own work. In this short report, we have divided the talks that were delivered in five groups:

1) Metric theory. U. Hametsädt reported on her recent work on the behaviour in moduli space of images of certain closed geodesics for the Teichmüller metric, namely, for every compact set K, one can find such images which do not intersect K. G. Schmithüsen reported on Teichmüller disks, which are embeddings of the Poincaré disks which are isometric with respect to the Teichmüller metric. G. Théret gave a talk on Thurston's asymmetric metric on Teichmüller space and presented results on the convergence of certain geodesics to points on Thurston's boundary.

2) Mapping class groups and the associated simplicial complexes. V. Markovich gave a review of several realization problems for the mapping class group and he reported on his result stating that for any closed surface S of genus ≥ 6, the natural projection from the space of homeomorphisms to the mapping class group has no section. This result answered a famous open problem. D. Kotschick gave a survey of his work on quasi-homomorphisms with applications to the mapping class group. J. McCarthy gave a talk in which he described the automorphism group of a recently introduced simplicial complex, the complex of domains of a surface (joint work with A. Papadopoulos). E. Irmak described recent results on superinjective simplicial maps of the curve complex, on the automorphism group of the complex of nonseparating curves, and on the Hatcher-Thurston complex of cut systems of curves (some of this work is joint with J. McCarthy and with M. Korkmaz). N. Wahl described a stability theorem for the homology of the mapping class group of non-orientable surfaces which is analogous to Harer's theorem for orientable surfaces. K. Fujiwara spoke on the geometry of the curve graph showing that the asymptotic dimension of this graph is finite, and that for surfaces of genus ≥ 2 with one bundary component, the dimension is at least two. A description of symplectic structures on Lefschetz fibrations using algebraic properties of the mapping class group was given by M. Korkmaz. At a more algebraic level, N. Kawazumi described recent work on characterictic classes in the mapping class group, in which he constructs higher analogues of the period matrix in order to obtain "canonical" differential forms that represent all the Morita-Mumford classes and their higher relations. R. Cohen gave a talk on joint work with I. Madsen on a generalized Mumford conjecture on the stable cohomology of the mapping class group and a general version of homology stability for that group in the setting of spaces of Riemann surfaces with appropriate boundary conditions in a simply connected target manifold. G. Mondello reported on his work relating the tautological classes to cycles of Witten and Kontsevich, which are constructed combinatorially (using fatgraphs).

3) Quantum theory. R. Kashaev described a new and elegant quantization of a homology bundle over Teichmüller space related to his earlier work. L. Chekhov reported (on his joint work with Penner) quantizing Thurston's projective lamination space for the once-punctured torus. V. Fock discussed an example from cluster algebras giving an explicit relationship between Teichmüller geometry and representation theory which is related to his recent work with A. Goncharov on higher Teichmüller spaces. Y. Gerber described a new construction of a collection of surface mapping classes with computable quantum invariants leading to new invariants for fibered knots. F. Bonsante gave a report on his recent work with R. Benedetti on constant curvature Lorentzian structures on manifolds that are topologically the product of a hyperbolic surface with the real line.

4) Dynamics. M. Möller reported on joint work with I. Bouw on billiards in relation to Veech surfaces, projective affine groups and Teichmüller curves in the moduli space of curves characterizing these curves by properties of the variation of Hodge structures. The talk by U. Hamenstädt, mentioned in *1)* above, involved the Teichmüller geodesic flow on quotient of the space of quadratic differentials by the action of the mapping class group. M. Mirzakhani studied the ergodic properties of natural flows on moduli space in relation to the asymptotic behaviour of simple closed geodesics on hyperbolic surfaces.

4) Complex geometry. Y. Imayoshi gave a talk on joint work with T. Nogi on the complex analytic structure of moduli space together with its Deligne-Mumford compactification. Continuing ideas that originate in work of Kodaira, he described a cut-and-paste construction which produces holomorphic families of closed Riemann surfaces of genus two over a four-punctured torus, to which they associate two holomorphic sections.

5) Higher Teichmüller theory A. Wienhard gave a talk on her recent work with M. Burger and A. Iozzi on representations of the fundamental group of the surface into simisimple Lie groups of Hermitian type. G. McShane described geometric identities for surfaces that are related to Hitchin's a component of the representation variety of the fundamental group of a compact surface into $SL(n, R)$. D. Dumas and S. Kojima spoke on complex projective structures on surfaces, the space $\mathcal{P}(S)$ of which (equivalence calsses) can be considered as a higher-analog of Teichmüller space. $\mathcal{P}(S)$ is a fibre bundle over Teichmüller space, and like Teichmüller space itself, can be studied from different points of view: complex analysis (via the Schwarzian derivative) and hyperbolic geometry(via Thurston's *grafting* map). Some of the most interesting questions in the theory of projective structures relate the two points of view, and the talk by Dumas focused on this relation. Kojima described a geometric parametrization of the moduli space of projective surfaces by cross ratios. He develops (together with S. Mizushima and S. P. Tan) a theory of circle packings in projective geometry which can be traced back to works by Andreev and by Thurston.

Workshop: Teichmüller Space (Classical and Quantum)

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Abstracts

Simplicial representations of surface mapping class groups

John D. McCarthy
(joint work with Athanase Papadopoulos)

Let $S = S_{g,b}$ be a connected compact orientable surface of genus g with b boundary components. Let ∂S denote the boundary of S. The *mapping class group of* S, $\Gamma(= \Gamma_{g,b} = \Gamma(S))$, is the group of isotopy classes of orientation-preserving self-homeomorphisms of S. The *extended mapping class group of* S, Γ^*, is the group of isotopy classes of self-homeomorphisms of S. Note that Γ is a normal subgroup of index 2 in Γ^*.

The study of these groups, Γ and Γ^*. has used their action on various abstract simplicial complexes, each of which encodes combinatorial information about the relationship which certain subspaces of S bear to one another. For instance, the curve complex, $C(S)$, which was introduced by W. Harvey [1], captures the combinatorial complexity of the set of isotopy classes of essential unoriented simple closed curves on S.

In recent joint work with Athanase Papadopoulos, we have begun the study of a new complex on which Γ^* acts [5]. This complex is naturally associated to the Thurston theory of surface diffeomorphisms for compact connected orientable surfaces with boundary. The various pieces of the Thurston decomposition of a surface diffeomorphism, thick domains and annular or thin domains, fit into this flag complex, which we call the complex of domains.

More precisely, a *domain on* S is a nonempty connected compact embedded surface in S which is not equal to S and each of whose boundary components is either contained in ∂S or is essential on S. The vertex set $D_0(S)$ of $D(S)$ is the set of isotopy classes of domains on S. An n-simplex of $D(S)$ is a set of $n+1$ distinct vertices of $D(S)$ which can be represented by disjoint domains of S.

The main result discussed in this talk is our computation of the group of automorphisms of $D(S)$. Unlike the celebrated complex of curves introduced by Harvey [1], for which, for all but a finite number of exceptional surfaces, by the works of Ivanov [2], Korkmaz [3], and Luo [4], all automorphisms are geometric (i.e. induced by homeomorphisms), the complex of domains has nongeometric automorphisms, provided S has at least two boundary components. These nongeometric automorphisms of $D(S)$ are associated to natural *biperipheral edges* of $D(S)$.

More precisely, a *biperipheral edge of* $D(S)$ is an edge of $D(S)$ whose vertices are represented by a *biperipheral pair of pants* X *on* S, (i.e. a domain X on S which is homeomorphic to a sphere with three holes having exactly two of its boundary components in ∂S) and a regular neighborhood Y of the remaining boundary component of X on S, the unique essential boundary component of X on S. The corresponding nongeometric automorphism of $D(S)$, which we call a *simple exchange of* $D(S)$ exchanges the two vertices of $D(S)$ corresponding to X and Y and fixes every other vertex of $D(S)$.

We prove for most surfaces that every automorphism of $D(S)$ preserves the set $\mathcal{E}$ of all biperipheral edges of $D(S)$ and, hence, induces an automorphism of the subcomplex of $D(S)$ which is obtained from $D(S)$ by removing each vertex of $D(S)$ corresponding to a biperipheral pair of pants on S, which subcomplex $D^2(S)$ we call the *truncated complex of domains on* S. In this way, we obtain a natural homomorphism $\rho : Aut(D(S)) \to Aut(D^2(S))$.

Studying this homomorphism, $\rho : Aut(D(S)) \to Aut(D^2(S))$, we prove that it is surjective and that its kernel, which we call *the group of exchange automorphisms* $EAut(D(S))$ *of* $D(S)$, consists of involutions $\varphi_{\mathcal{F}} : D(S) \to D(S)$, defined for each subcollection $\mathcal{F}$ of $\mathcal{E}$, which interchange the two vertices of each edge of $D(S)$ in $\mathcal{F}$, and fix every vertex of $D(S)$ which is not a vertex of an edge of $D(S)$ in $\mathcal{F}$. In this way, we see that $EAut(D(S))$ is naturally isomorphic to the Boolean algebra $\mathcal{B}(\mathcal{E})$ of all subsets $\mathcal{F}$ of $\mathcal{E}$ and, thereby, exhibit $Aut(D(S))$ as an extension of $Aut(D^2(S))$ by the Boolean algebra $\mathcal{B}(\mathcal{E})$.

Studying $Aut(D^2(S))$, we prove that the natural representation $\eta : \Gamma^*(S) \to Aut(D^2(S))$, arising by induction from the natural action of $\Gamma^*(S)$ on $D(S)$, is an isomorphism, completing our computation of $Aut(D(S))$, expressed as follows in our main result.

Theorem 1. *Suppose that S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two. Then we have a natural commutative diagram of exact sequences:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{B}(\mathcal{E}) & \longrightarrow & \mathcal{B}(\mathcal{E}) \rtimes \Gamma^*(S) & \longrightarrow & \Gamma^*(S) & \longrightarrow & 1 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ 1 & \longrightarrow & B_{\mathcal{E}} & \longrightarrow & Aut(D(S)) & \longrightarrow & Aut(D^2(S)) & \longrightarrow & 1 \end{array}$$

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Global properties of Teichmüller geodesics

Ursula Hamenstädt

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures where $3g - 3 + m \geq 2$. The *Teichmüller space* $\mathcal{T}(S)$ of S is a complex manifold which is biholomorphic to a bounded domain in $\mathbb{C}^{3g-3+m}$. The *mapping class group* $\mathcal{M}(S)$ is the group of all biholomorphic automorphisms of $\mathcal{T}(S)$. It acts properly discontinuously on $\mathcal{T}(S)$, with the *moduli space* $\mathrm{Mod}(S) = \mathcal{T}(S)/\mathcal{M}(S)$ as its quotient orbifold. The Teichmüller space admits a natural complete $\mathcal{M}(S)$-invariant Finsler metric, the so-called *Teichmüller metric*. Even though this metric is not nonpositively curved in any reasonable sense, it shares many properties with a Riemannian manifold of non-positive curvature. For example, any two points in $\mathcal{T}(S)$ can be connected by a unique Teichmüller geodesic, and *closed* geodesics in moduli space are in one-to-one correspondence with the conjugacy classes of the so-called *pseudo-Anosov* elements of the mapping class group.

However, unlike in the case of a negatively curved manifold of finite volume, closed geodesics in moduli space may escape into the end of moduli space. We discuss the following result [1].

Theorem: *If $3g - 3 + m \geq 4$ then for every compact subset K of $\mathrm{Mod}(S)$ there is a closed Teichmüller geodesic which does not intersect K.*

The proof uses train tracks and the Perron Frobenius theorem.

The unit cotangent bundle $\mathcal{Q}^1(S)$ of Teichmüller space equipped with the Teichmüller metric can naturally be identified with the bundle of all holomorphic quadratic differentials of area one. The *Teichmüller geodesic flow* commutes with the action of $\mathcal{M}(S)$ and projects to a flow Φ^t on the quotient $\mathcal{Q}^1(S)/\mathcal{M}(S)$ preserving a probability measure in the Lebesgue measure class. We briefly discuss some dynamical properties of the Teichmüller geodesic flow related to invariant measures and return probabilities into compact sets [2].

References

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Surface groups and Hermitian symmetric spaces

Anna Wienhard
(joint work with Marc Burger, Alessandra Iozzi)

Let Σ be a connected compact oriented surface of negative Euler characteristic and G a connected semisimple Lie group with finite center. We single out special "components"[1] of the representation variety

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

which might be considered as generalizations of Teichmüller space.

We restrict to the case when the Lie group G is of Hermitian type, that is when the symmetric space $\mathcal{X}$ associated to G carries a G-invariant complex structure J.

This complex structure gives rise to a continuous function, the *Toledo invariant*

$$T : \mathrm{Hom}(\pi_1(\Sigma), G)/G \to \mathbf{R},$$

which in the case when Σ has empty boundary is a characteristic number.

The motivation for considering the Toledo invariant to single out special "components" of the representation variety comes from the case when $G = \mathrm{PSL}(2, \mathbf{R})$ and Σ has no boundary. In this case the Toledo invariant is the Euler number of a representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{R})$. The Euler number satisfies the *Milnor-Wood inequality*

$$|T(\rho)| \leq |\chi(\Sigma)|.$$

Goldman [3] proved that the Euler number separates connected components of the representation variety and that moreover the *Teichmüller components* consisting of faithful representations with discrete image are characterized by having *maximal Euler number* $|T(\rho)| = |\chi(\Sigma)|$.

Note that the G-invariant complex structure J gives rise to a G-invariant differential two form $\omega \in \Omega^2(\mathcal{X})^G$ defined by $\omega(X, Y) := h(X, JY)$, where h is the unique G-invariant metric of minimal holomorphic sectional curvature -1 on $\mathcal{X}$. For surface with empty boundary one way to define the Toledo invariant of a representation $\rho : \pi_1(\Sigma) \to G$ is to consider a smooth section $f : \Sigma \to E_\rho$ of the flat bundle $E_\rho = \mathcal{X} \times_\rho \widetilde{\Sigma}$ associated to ρ. Then f lifts to a ρ-equivariant smooth map $\tilde{f} : \widetilde{\Sigma} \to \mathcal{X}$ and the Toledo invariant is

$$T(\rho) := \frac{1}{2\pi} \int_\Sigma \tilde{f}^* \omega.$$

Before we turn to the definition of the Toledo invariant for arbitrary compact connected oriented surfaces, let us summarize some properties of the Toledo invariant, defined in the general case by Equation (1) below.

[1]When Σ has nonempty boundary, the representation variety is connected, so that there are no nontrivial connected components. We speak nevertheless of "components" since the subset we will single out is a union of connected components of a natural semialgebraic subset of the representation variety

Proposition 1. *(1) The Toledo invariant is a continuous function*

$$T : \mathrm{Hom}(\pi_1(\Sigma), G)/G \to \mathbf{R}.$$

(2) The Toledo invariant satisfies a generalized Milnor-Wood type inequality

$$|T(\rho)| \leq r_{\mathcal{X}} |\chi(\Sigma)|,$$

where $r_{\mathcal{X}}$ is the rank of $\mathcal{X}$.

(3) The Toledo invariant is explicitly computable when the representation ρ is given on the generators of a standard presentation of $\pi_1(\Sigma)$.

(4) When Σ has empty boundary, then $T(\rho) \in \frac{1}{2}\mathbf{Z}$.

In view of statement (2) of Proposition 1 we say that a representation $\rho : \pi_1(\Sigma) \to G$ is *maximal* if $T(\rho) = r_{\mathcal{X}} |\chi(\Sigma)|$ and we denote by

$$\mathrm{Hom}_{max}(\pi_1(\Sigma), G)/G \subset \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

the set of *maximal representations* which gives our special "components" alluded to above.

We obtain the following result about the geometric properties of maximal representations, which was announced for surfaces with empty boundary in [1] and is proven for arbitrary surfaces in [2].

Theorem 2. *Let G be a connected semisimple Lie group with finite center of Hermitian type and let $\mathcal{X}$ be the associated symmetric space. Let $\rho : \pi_1(\Sigma) \to G$ be a maximal representation. Then*

(1) ρ is faithful with discrete image.

(2) The Zariski closure $\mathbf{H} := \overline{\rho(\pi_1(\Sigma))}^Z$ is reductive

(3) The real Lie group $H = \mathbf{H}(\mathbf{R})^\circ$ is reductive with compact center and its associated symmetric space $\mathcal{Y}$ is Hermitian.

(4) H stabilizes a maximal tube type subdomain $\mathcal{T} \subset \mathcal{X}$.

To define now the Toledo invariant in the case when Σ has nonempty boundary we recast the above construction on the level of cohomology. The Kähler form $\omega \in \Omega^2(\mathcal{X})^G$ gives - via the van Est isomorphism - rise to a continuous cohomology class $\kappa \in \mathrm{H}^2_{\mathrm{c}}(G, \mathbf{R})$, which is represented by the homogeneous cocycle $c_\omega : G \times G \times G \to \mathbf{R}$, $(g_0, g_1, g_2) \mapsto \int_{\Delta(g_0 x, g_1 x, g_2 x)} \omega$, where $x \in \mathcal{X}$ is a basepoint and $\Delta(g_0 x, g_1 x, g_2 x)$ a smooth triangle with vertices in $g_0 x, g_1 x, g_2 x$ and geodesic sides. A homomorphism $\rho : \pi_1(\Sigma) \to G$ gives rise to a natural pullback map $\rho^* : \mathrm{H}^2_{\mathrm{c}}(G, \mathbf{R}) \to \mathrm{H}^2(\pi_1(\Sigma), \mathbf{R})$. Since Σ is an Eilenberg-MacLane space we have that $\mathrm{H}^2(\pi_1(\Sigma), \mathbf{R}) \cong \mathrm{H}^2(\Sigma, \mathbf{R})$, and the Toledo invariant is the evaluation of $\rho^*(\kappa) \in \mathrm{H}^2(\Sigma, \mathbf{R})$ on the fundamental class $[\Sigma] \in H_2(\Sigma, \mathbf{R})$.

When Σ has nonempty boundary $\mathrm{H}^2(\Sigma, \mathbf{R}) = 0$, there is no fundamental class $[\Sigma]$, and the above definition breaks down. But the cocycle c_ω is bounded and thus gives rise to a *bounded* continuous cohomology class $\kappa^b \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbf{R})$. Taking the pullback map $\rho^* : \mathrm{H}^2_{\mathrm{cb}}(G, \mathbf{R}) \to \mathrm{H}^2_{\mathrm{b}}(\pi_1(\Sigma), \mathbf{R})$ we obtain a class

$$\rho^* \kappa^b \in \mathrm{H}^2_{\mathrm{b}}(\pi_1(\Sigma), \mathbf{R}) \cong \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbf{R}).$$

The bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbf{R})$ of a surface with boundary does not vanish, but is indeed infinite dimensional. More importantly, since the fundamental groups of the boundary components of Σ are Abelian, we have an isomorphism

$$\mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbf{R}) \cong \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbf{R}).$$

Thus, we obtain a class $\rho^*\kappa^b \in \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbf{R})$ which - forgetting that $\rho^*\kappa^b$ is bounded - can be evaluated on the relative fundamental class $[\Sigma, \partial\Sigma]$ to give the Toledo invariant

$$T(\rho) := \left\langle \rho^*\kappa^b, [\Sigma, \partial\Sigma] \right\rangle. \qquad (1)$$

Having now the definition of the Toledo invariant for an arbitrary compact connected oriented surface we mention two further properties which hold in addition to the properties stated in Proposition 1.

Proposition 3. *(1) The Toledo invariant is additive. Let $\Sigma = \Sigma_1 \# \Sigma_2$ be an (admissible) decomposition of Σ into subsurfaces, $\rho : \pi_1(\Sigma) \to G$ a representation and $\rho_i : \pi_1(\Sigma_i) \to G$, $i = 1, 2$ the induced representations, then*

$$T(\rho) = T(\rho_1) + T(\rho_2).$$

(2) There exists a conjugacy invariant rotation number function $\mathrm{Rot}_\kappa : G \to \mathbf{R}/\mathbf{Z}$ *such that*

$$T(\rho) \equiv -\sum_{i=1}^{n} \mathrm{Rot}_\kappa(\rho(c_i)) \mod \frac{1}{2}\mathbf{Z},$$

where $c_i \in \pi_1(\Sigma)$ are elements freely homotopic to the boundary circles of Σ.

Note that property (1) of Proposition 3 suggests that the space of maximal representations can be equipped with Fenchel-Nielsen type coordinates, whereas property (2) implies that the Toledo invariant is constant on connected components of the space of representation with fixed holonomy around the boundary loops.

We conclude with a remark concerning the relation of our special components of maximal representations with higher Teichmüller spaces defined by Hitchin and Fock-Goncharov in the context of split real semisimple Lie groups. The only group which is at the same time real split and of Hermitian type is $G = \mathrm{Sp}(2n, \mathbf{R})$. For this group the Hitchin-Fock-Goncharov higher Teichmüller spaces are properly contained in the space of maximal representations.

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A parametrization of the Teichmüller space of surfaces with boundary

Rinat Kashaev

Let Σ be a connected compact oriented surface of finite type with non-empty boundary and negative Euler characteristic $\chi(\Sigma) = -\kappa$. We consider the Teichmüller space $\mathcal{T}(\Sigma)$ of all isotopy classes of hyperbolic metrics on Σ with totally geodesic boundary. The *restricted* Teichmüller space $\mathcal{T}_\lambda(\Sigma)$ is a subspace of $\mathcal{T}(\Sigma)$ with fixed total boundary length λ. In this talk a certain parametrization of the space $\mathcal{R}_\lambda(\Sigma) = \mathcal{T}_\lambda(\Sigma) \times H^1(\Sigma;\mathbb{R})$ in terms of ideal triangulations is constructed which permits to construct a mapping class group invariant symplectic structure. The parametrization is similar to those of Penner [1] and Luo [2].

An ideal triangulation τ of Σ is an isotopy class of cell decompositions of Σ with the following properties. All 0-cells (called vertices) are on the boundary $\partial\Sigma$. The 1-cells are of two types called respectively *long* and *short* edges. The long edges do not intersect the boundary, while each short edge is a segment of the boundary. Each 2-cell is a hexagon bounded in alternating order by three long and three short edges. Below, each ideal triangulation will be identified with a chosen representative cell complex. We denote by $F(\tau)$ the set of 2-cells and by $L(\tau)$, the set of oriented long edges. The projection $\varphi_\tau\colon L(\tau) \to F(\tau)$ associates to each oriented long edge the 2-cell which has this edge as its long side with the induced orientation. The *link* $\mathrm{lk}_\tau b$ of a boundary component $b \in \pi_0(\partial\Sigma)$ in τ is the set of the boundary long edges of the union of all 2-cells intersecting b.

Given an ideal triangulation τ of Σ. We define an open polytope $H_\lambda(\tau)$ as the set of functions $f\colon L(\tau) \to \mathbb{R}$ satisfying the conditions $\sum_{a\in\varphi_\tau^{-1}(h)} f(a) = \frac{\lambda}{2\kappa}$, $\forall h \in F(\tau)$ and $\sum_{a\in \mathrm{lk}_\tau b} f(a) > 0$, $\forall b \in \pi_0(\partial\Sigma)$. Note that the mapping class group acts in $H_\lambda(\tau)$ through the action on the set of ideal triangulations.

For any oriented long edge a we denote by $\hat{a}$ and $\check{a}$ the other two long sides and by a' the short edge opposite to a in the 2-cell $\varphi_\tau(a)$, the cyclic order $(a, \hat{a}, \check{a})$ being induced from the orientation of this 2-cell. In the corresponding to τ dual cell decomposition of Σ we denote by τ^* the set of oriented 1-cells dual to the oriented long edges of τ. The correspondence between their orientations is chosen so that each oriented dual pair $(a, a^*) \in L(\tau)\times\tau^*$ induces the orientation of Σ. For any $m \in \mathcal{T}(\Sigma)$ and any m-geodesic path a in Σ we denote by $|a|_m$ the m-length of a.

Given $(m, \alpha) \in \mathcal{R}_\lambda(\Sigma)$. For τ we choose the cell complex where all 2-cells are right-angled m-geodesic hexagons. We choose also a unique cocycle $\alpha_\tau^m\colon \tau^* \to \mathbb{R}$, representing the class α, such that

$$2\sum_{a\in\varphi_\tau^{-1}(h)} \alpha_\tau^m(a^*) = \frac{\lambda}{2\kappa} - \sum_{a\in\varphi_\tau^{-1}(h)} |a'|_m, \quad \forall h \in F(\tau)$$

We define a mapping $\phi_{\lambda,\tau}\colon \mathcal{R}_\lambda(\Sigma) \to H_\lambda(\tau)$ by the formula

$$(m,\alpha) \mapsto f, \quad f(a) = \alpha_\tau^m(\check{a}^*) + \alpha_\tau^m(\hat{a}^*) + |a'|_m + \log\cosh(|\hat{a}|_m/2) - \log\cosh(|\check{a}|_m/2)$$

Theorem 1. *For any $\lambda > 0$ and any ideal triangulation τ the mapping $\phi_{\lambda,\tau}$ is a homeomorphism commuting with the action of the mapping class group.*

Let $p\colon F(\tau) \to L(\tau)$ be a section of the projection φ_τ. Define a two form in $H_\lambda(\tau)$ by the formula: $\omega_\tau = \sum_{h\in F(\tau)} dp(h) \wedge \widehat{dp(h)}$ where the long edges are identified with coordinate functions in the space $H_\lambda(\tau)$.

Theorem 2. *The form ω_τ does not depend on p and for any mapping class f one has $f(\omega_\tau) = \omega_{f(\tau)}$.*

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Teichmüller spaces and simple Lie groups

Vladimir Fock
(joint work with Aleksandr Goncharov)

Given a triangulation of a Riemann surface with punctures one can parametrize Teichmüller spaces of open surfaces by shear coordinates one per each edge of the triangulation and taking values in positive real numbers. In these coordinates the Weil-Petersson Poisson bracket between any two coordinates x^i and x^j is proportional to their product with integer coefficient ϵ^{ij} determined by the combinatorics of the triangulation. Moreover the coordinate change corresponding to the change of the triangulation is given by rational expressions (called mutations) determined by the same coefficients ϵ^{ij}.

The construction is purely algebraic and admits an obvious generalization by replacement of the set of positive numbers by elements of arbitrary semifield. If we take coordinates in a multiplicative group $\mathbb{F}^\times$ of the field $\mathbb{F}$ of a field it gives a parameterization of representation spaces of the fundamental group of the surface into the group $PSL(2,\mathbb{F})$. If we take the coordinates in a tropical semifield the construction gives a parameterization of the spaces of maesured geodesic laminations.

It turns out that very similar construction gives coordinates for simple Lie groups. The role analogous to the one of the triangulation is played by a decomposition of the longest element of the square of the corresponding Weyl group into a product of the standard generators. For every such triangulation one associates a coordinate system on a Zarisky open subset of the group. There exists a Poisson structure on a simple Lie group given by the standard r-matrix. This

Poisson bracket computed between two coordinates x^i and x^j is also proportional to their product with integral coefficients ϵ^{ij}. Changing the decomposition of the longest Weyl group element amounts to a birational coordinate change given exactly by the mutation formulae. Moreover the group product can be expressed as a composition of mutations and projections along coordinate axis thus showing its compatibility with the Poisson structure.

This construction allows to apply techniques developed for Teichmüller spaces to semisimple Lie groups. For example one can use a quantization procedure using quantum dilogarithms [3]. One can also look for the groups analogues of the mapping class groups for Teichmüller spaces. In some examples these groups are computed and turn out to be related to braid groups. One can also look for an analogue of the basis of functions on Teichmüller spaces given by traces of monodromies around closed loops. Conjecturally this coincides with the Lusztig's dual canonical basis.

The construction admits several generalizations. For example one can associate Poisson manifolds not only to the longest elements of the Weyl group but also to any element of the corresponding Weyl group. Given two elements of the braid group there exists a canonical Poisson maps from manifolds corresponding these elements to the manifold corresponding to the product.

Another generalization describes in analogous terms the coordinates on dual Poisson-Lie groups and more generally on the space of Stokes data for irregular singularity of a holomorphic connection. In the latter case the corresponding mapping class group can be shown to contain the corresponding braid group. And finally, although it was our starting point for all the construction, it gives analogous description for the moduli spaces of G local systems on Riemann surfaces for simple groups G [2].

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Quantum Thurston theory

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(joint work with R. C. Penner)

A *fatgraph* or *ribbon graph* is a graph Γ together with a cyclic ordering on the half-edges incident on each vertex, and we canonically associate to Γ a surface $F(\Gamma)$ with boundary obtained by "fattening each edge of the graph into a band" in the natural way; we shall tacitly require all vertices to have valence three.

Recall [1] that the lambda length of a pair of horocycles for punctured surfaces is $\sqrt{2\ e^{\delta}}$, where δ is the signed hyperbolic distance between the horocycles. Lambda lengths give a global real-analytic parametrization of the decorated Teichmüller space as the trivial bundle $\widetilde{\mathcal{T}}_g^s = \mathcal{T}_g^s \times \mathbb{R}_{>0}^s$ over Teichmüller space, where the fiber over a point is the space of all s-tuples of horocycles in the surface, one horocycle about each puncture (parameterized by hyperbolic length). Another coordinate system is provided by *shear coordinates* Z_α which proved to be useful for quantizing the theory and for providing the explicit parameterization of the Fuchsian group.

Given an open Riemann surface F of finite topological type, a neighborhood of an ideal boundary component is either an annulus or a punctured disk; in the former case, the ideal boundary component will be called a "true" boundary component and in the latter will be called a "puncture." We assume the latter a degeneration of the former.

Theorem 1. *Fix any spine $\Gamma \subseteq F$, where Γ is a cubic fatgraph. Then there is a real-analytic homeomorphsim $\mathbb{R}^{E(\Gamma)} \to \mathcal{T}_H(F)$. The hyperbolic length l_γ of a true boundary component γ is given by $l_\gamma = |\sum Z_i|$, where the sum is over the set of all edges traversed by γ counted with multiplicity. Furthermore, $\sum Z_i = 0$ if and only if the corresponding ideal boundary component is a puncture, so $\mathcal{T}_g^s \subseteq \mathcal{T}_H(F)$ is determined by s independent linear constraints.*

The theorem is due to Thurston with a systematic study by Fock [2, 3]. We describe explicitly the homeomorphism in the theorem letting $\alpha = 1, \ldots, E = E(\Gamma)$ to index the edges of Γ and (Z_α) to denote a point of $\mathbb{R}^E$. We associate the Möbius transformation

$$X_{Z_\alpha} = \begin{pmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{pmatrix}. \tag{1}$$

to the edge α. We also introduce the "right" and "left" turn matrices

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L = R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \tag{2}$$

Consider a closed oriented edge-path P in Γ with no "turning back". Choosing an initial base point on P, we may imagine the corresponding curve serially traversing the oriented edges of Γ with coordinates $Z_1, \ldots, Z_n$ turning left or right from Z_i to Z_{i+1}, for $i = 1, \ldots, n$ (with the indices mod n so that $Z_{n+1} = Z_1$). Assign to P the corresponding composition

$$P_{Z_1 \ldots Z_n} = LX_{Z_n} LX_{Z_{n-1}} RX_{Z_{n-2}} \ldots RX_{Z_2} LX_{Z_1}, \tag{3}$$

where the matrices L or R are inserted depending on which turn—left or right—the path takes at the corresponding stage.

Fixing any base point, the assignment $P \mapsto P_{Z_1,\ldots,P_n} \in PSL_2(\mathbb{R})$ gives rise to a representation $\rho \in \mathcal{T}_H(F)$, and this defines the required map $\mathbb{R}^{E(\Gamma)} \to \mathcal{T}_H(F)$.

Proposition 2. *There is a one-to-one correspondence between the set of conjugacy classes of elements of $\pi_1(F)$ and free homotopy classes of closed oriented geodesics in F. For any spine of F, each free homotopy class is uniquely represented by a cyclically defined closed edge-path P with no turning back, and the length of γ is determined by*

$$G_\gamma \equiv 2\cosh(l_\gamma/2) = tr\, P_{Z_1 \dots Z_n} \geq 2. \tag{4}$$

We call G_γ the geodesic function.

Theorem 3. [2] *In the coordinates (Z_α) on any fixed spine, the Weil–Petersson bracket B_{WP} is given by*

$$B_{WP} = \sum_v \sum_{i=1}^{3} \frac{\partial}{\partial Z_{v_i}} \wedge \frac{\partial}{\partial Z_{v_{i+1}}}, \tag{5}$$

where the sum is taken over all vertices v and v_i, $i = 1, 2, 3 \bmod 3$, are the labels of the cyclically ordered half-edges incident on this vertex.

This Weil–Petersson bracket is mapping class group invariant.

Proposition 4. *The center of the Poisson algebra* (5) *is generated by elements of the form $\sum Z_\alpha$, where the sum is over all edges of Γ in a boundary component of $F(\Gamma)$ and the sum is taken with multiplicity.*

Here we construct a quantization $\mathcal{T}^\hbar(F)$ of the Teichmüller space $\mathcal{T}_H(F)$ that is equivariant with respect to the action of the mapping class group $\mathcal{D} = MC(F)$.

Fix a cubic fatgraph Γ as spine of F, and let $\mathcal{T}^\hbar = \mathcal{T}^\hbar(\Gamma)$ be the algebra generated by $Z_\alpha^\hbar$, one generator for each unoriented edge α of Γ, with relations

$$[Z_\alpha^\hbar, Z_\beta^\hbar] = 2\pi i\hbar\{Z_\alpha, Z_\beta\} \tag{6}$$

(cf. (5)) and the $*$-structure

$$(Z_\alpha^\hbar)^* = Z_\alpha^\hbar, \tag{7}$$

where Z_α and $\{\cdot,\cdot\}$ denotes the respective coordinate functions and the Poisson bracket on the classical Teichmüller space. Because of (5), the righthand side of (6) is a constant taking only five values 0, $\pm 2\pi i\hbar$, and $\pm 4\pi i\hbar$ depending upon the coincidences of endpoints of edges labelled α and β and sums $\sum Z_\alpha^\hbar$ over all edges of a boundary component remain Casimir elements of the quantum algebra.

We now quantize the Thurston theory following [4]. We assign a signed quantity, the *(Thurston's foliation-)shear coordinate* of the edge indexed by α to to be

$$\zeta_\alpha = \frac{1}{2}(\mu(A) - \mu(B) + \mu(C) - \mu(D)),$$

where A, B, C, D label the four nearest edges (some of which may coincide) adjacent to the αth edge (ordered in a natural way) and μ are the corresponding (nonnegative) transverse measures. These new quantities are subject to the only restriction (the *face conditions*) that

$$\sum_{\alpha \in I} \zeta_\alpha = 0 \tag{8}$$

for the sum over edges $\alpha \in I$ surrounding any given boundary component.

Definition 5. The *proper length* p.l.(γ) of a closed curve γ in the classical or quantum case is constructed from the quantum ordered operator P_γ associated to a closed oriented edge-path
as

$$\text{p.l.}(\gamma) = \lim_{n\to\infty} \frac{1}{n} \text{ tr } \log 2T_n(P_\gamma/2), \tag{9}$$

where T_n are Chebyshev's polynomials. The p.l.(γ) is half the hyperbolic length of γ in the Poincaré metric in the classical case.

Definition 6. A *graph length* function with respect to the spine Γ is any linear function

$$\text{g.l.}_\Gamma^{\vec{a}}(\hat{C}_{\vec{n}}) = \text{g.l.}_\Gamma^{\vec{a}}(\hat{C}_{\vec{m}}) = \sum_i a_i m_i. \tag{10}$$

In particular, when all a_i are unity, the graph length is just the combinatorial length of $\hat{C}_{\vec{m}}$.

Definition 7. A sequence $\vec{n}^\beta\{n_i^\beta\}$, for $\beta \geq 1$, of integer-valued n_i, for $i = 1, \dots, LB(\tau)$, on τ is an *approximating sequence* for the projectivized measured foliation $P\vec{\zeta}$
if the face conditions (8) hold on $\vec{n}^\beta$ and if $\lim_{\beta\to\infty} n_i^\beta/n_j^\beta = \zeta_i/\zeta_j$ for all i, j with $\zeta_j \neq 0$.

Theorem 8. *Fix a spine Γ of F_1^1 with corresponding freeway τ. Fix any projectivized vector $P\vec{\zeta}$ of foliation-shear coordinates on τ and any graph length function* g.l.. *For any approximating sequence $\vec{n}^\beta$ to $P\vec{\zeta}$, the limit*

$$\lim_{\beta\to\infty} \frac{\text{p.l.}(\hat{C}_{\vec{n}^\beta})}{\text{g.l.}(\hat{C}_{\vec{n}^\beta})} \tag{11}$$

exists both in the classical case as a real number and in the quantum case as a weak operatorial limit.

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A higher analogue of the period matrix of a compact Riemann surface

Nariya Kawazumi

Let $\mathcal{T}_g$ be the Teichmüller space of genus $g \geq 2$, and $\mathfrak{H}_g$ the Siegel upper half space of genus g. Rauch's variational formula [11] tells us the dual of the differential of the period matrix map $\mathrm{Jac} : \mathcal{T}_g \to \mathfrak{H}_g$ at a marked Riemann surface $[C, \alpha]$ is given by the multiplication map $\mathrm{Sym}^2(H^0(C; K)) \to H^0(C; 2K)$. Here α is a marking, i.e., an isotopy class of an orientation preserving diffeomorphism of a fixed closed C^∞ surface Σ_g of genus g onto the Riemann surface C, and $K = K_C = T^*C$ is the canonical bundle of C. Let H and H^* denote the first real homology and cohomology groups of the surface C, respectively. The map $H^* = H^1(C; \mathbb{R}) \to \Omega^1(C)$ assigning each cohomology class the harmonic 1-form representing it can be regarded as a H-valued 1-form $\omega_{(1)} \in \Omega^1(C) \otimes H$. For a 1-form $\varphi \in \Omega^1(C) \otimes \mathbb{C}$ we denote by φ' and φ'' its $(1,0)$- and $(0,1)$-parts, respectively. For example, ${\omega_{(1)}}'$ is holomorphic, and ${\omega_{(1)}}''$ anti-holomorphic. Then Rauch's formula can be written by

$$(d\,\mathrm{Jac})^* = {\omega_{(1)}}'{\omega_{(1)}}' \in T^*_{[C,\alpha]}\mathcal{T}_g \otimes H^{\otimes 2}.$$

The Morita-Mumford classes [10] [8] play an essential role in the topology of the moduli space of compact Riemann surfaces. The odd ones are represented by the pull-backs of $Sp_{2g}(\mathbb{R})$-invariant differential forms on $\mathfrak{H}_g$, but the even ones are *not* represented by such forms. To obtain "canonical" differential forms representing all the Morita-Mumford classes and their higher relations, we construct a higher analogue of the period matrix, the harmonic Magnus expansion $\theta : \mathcal{T}_{g,1} \to \Theta_{2g}$. Here $\mathcal{T}_{g,1}$ is the Teichmüller space of *triples* (C, P_0, v) of genus g. Here C is a compact Riemann surface of genus g, $P_0 \in C$, and v a non-zero tangent vector of C at P_0. For any triple (C, P_0, v) one can define the fundamental group of the complement $C \setminus \{P_0\}$ with the tangential basepoint v denoted by $\pi_1(C, P_0, v)$, which is a free group of rank $2g$. The space Θ_n is the set of all the Magnus expansions of the free group F_n of rank $n \geq 2$ in a wider sense stated as follows.

We denote by $H := H_1(F_n; \mathbb{R})$ the first real homology group of the group F_n, by $H^* := H^1(F_n; \mathbb{R})$ the first real cohomology group of F_n, and by $[\gamma] \in H$ the homology class induced by $\gamma \in F_n$. The completed tensor algebra generated by H, $\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$, has a decreasing filtration of two-sided ideals $\widehat{T}_p := \prod_{m \geq p} H^{\otimes m}$, $p \geq 1$. The subset $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We call a map $\theta : F_n \to 1 + \widehat{T}_1$ *a Magnus expasion of the free group* F_n in a wider sense [5], if $\theta : F_n \to 1 + \widehat{T}_1$ is a group homomorphism, and if $\theta(\gamma) \equiv 1 + [\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$. We denote by Θ_n the set of all the Magnus expansions, which one can endow with a natural strucure of a (projective limit of) real analytic manifold(s). A certain (project limit of) Lie group(s) $\mathrm{IA}(\widehat{T})$ acts on Θ_n in a free and transitive way. This induces a series of 1-forms $\eta_p \in \Omega^1(\Theta_n) \otimes H^* \otimes H^{\otimes(p+1)}$, $p \geq 1$, the Maurer-Cartan forms of the action of $\mathrm{IA}(\widehat{T})$, which are invariant under a natural action of the automorphism group of the group F_n, $\mathrm{Aut}(F_n)$. The 1-forms η_p's represent the twisted Morita-Mumford classes on the group $\mathrm{Aut}(F_n)$ [4] [5] [6].

Let (C, P_0, v) be a triple of genus g. We denote by $\delta_{P_0} : C^\infty(C) \to \mathbb{R}$, $f \mapsto f(P_0)$, the delta 2-current on C at P_0. Then there exists a $\widehat{T}_1$-valued 1-current $\omega \in \Omega^1(C) \otimes \widehat{T}_1$, satisfying the following 3 conditions

(1) $d\omega = \omega \wedge \omega - I \cdot \delta_{P_0}$, where $I \in H^{\otimes 2}$ is the intersection form.
(2) The first term of ω equals to $\omega_{(1)} \in \Omega^1(C) \otimes H$.
(3) $\int_C (\omega - \omega_{(1)}) \wedge *\varphi = 0$ for any closed 1-form φ and each $p \geq 2$.

Here $*$ is the Hodge $*$-operator on $\Omega^1(C)$, which is conformal invariant of the Riemann surface C. Using Chen's iterated integrals [1], we can define a Magnus expansion

$$\theta = \theta^{(C,P_0,v)} : \pi_1(C, P_0, v) \to 1 + \widehat{T}_1(H_1(C;\mathbb{R})), \quad [\ell] \mapsto 1 + \sum_{m=1}^{\infty} \int_\ell \overbrace{\omega\omega\cdots\omega}^{m}.$$

Let a point $p_0 \in \Sigma_g$ and a non-zero tangent vector $v_0 \in T_{p_0}\Sigma_g \setminus \{0\}$ be fixed. Moreover we fix an isomorphism $\pi_1(\Sigma_g, p_0, v_0) \cong F_{2g}$. A marking α of a triple (C, P_0, v) is an orientation-preserving diffeomorphism of Σ_g onto C satisfying the conditions $\alpha(p_0) = P_0$ and $(d\alpha)_{p_0}(v_0) = v$. For any marked triple $[C, P_0, v, \alpha]$ we define a Magnus expansion of the free group F_{2g} by

$$F_{2g} \cong \pi_1(\Sigma_g, p_0, v_0) \stackrel{\alpha_*}{\to} \pi_1(C, P_0, v) \stackrel{\theta^{(C,P_0,v)}}{\longrightarrow} 1 + \widehat{T}_1(H_1(C;\mathbb{R})) \stackrel{{\alpha_*}^{-1}}{\to} 1 + \widehat{T}_1.$$

Consequently the Magnus expansions $\theta^{(C,P_0,v)}$ for all the triples (C, P_0, v) define a canonical real analytic map $\theta : \mathcal{T}_{g,1} \to \Theta_{2g}$, which we call *the harmonic Magnus expansion on the universal family of Riemann surfaces*. The pullbacks of the Maurer-Cartan forms η_p's give the canonical differential forms representing the Morita-Mumford classes and their higher relations [9] [7].

Theorem 1 ([6]). *For any $[C, P_0, v, \alpha] \in \mathcal{T}_{g,1}$ we have*

$$(\theta^*\eta)_{[C,P_0,v,\alpha]} = 2\Re(N(\omega'\omega') - 2{\omega_{(1)}}'{\omega_{(1)}}') \in T^*_{[C,P_0,v,\alpha]}\mathcal{T}_{g,1} \otimes \widehat{T}_3.$$

Here $N : \widehat{T}_1 \to \widehat{T}_1$ is defined by $N|_{H^{\otimes m}} := \sum_{k=0}^{m-1} \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix}$, and the meromorphic quadratic differential $N(\omega'\omega')$ is regarded as a $(1,0)$-cotangent vector at $[C, P_0, v, \alpha] \in \mathcal{T}_{g,1}$ in a natural way.

The third homogeneous term of $N(\omega'\omega')$ equals to the first variation of the (pointed) harmonic volumes of compact Riemann surfaces introduced by Harris [3], which is closely related to an Arakelov-geometirc approach to the Teichmüller space by Hain and Reed [2]. The second term coincides with $2{\omega_{(1)}}'{\omega_{(1)}}'$, which is exactly the first variation of the period matrices given by Rauch's formula [11].

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Mapping class groups of non-orientable surfaces

Nathalie Wahl

Let $S = S_{n,r}$ be a non-orientable surface of genus n with r boundary components, i.e. S is the connected sum of n copies of $\mathbb{R}\mathrm{P}^2$ with r discs removed. The mapping class group of S is

$$\mathcal{M}_{n,r} := \pi_0 \mathrm{Diff}(S_{n,r} \text{ rel } \partial),$$

the group of path components of the space of diffeomorphisms of S which fix its boundary pointwise.

When $r \geq 1$, there are stabilization maps $\alpha : \mathcal{M}_{n,r} \to \mathcal{M}_{n+1,r}$, obtained by gluing a punctured Moebius band (or a twice punctured $\mathbb{R}\mathrm{P}^2$) to the surface and extending the diffeomorphisms by the identity on the added part, and $\beta : \mathcal{M}_{n,r} \to \mathcal{M}_{n,r+1}$, obtained similarly by gluing a pair of pants. Gluing a disc on the added pair of pants defines a right inverse $\delta : \mathcal{M}_{n,r} \to \mathcal{M}_{n,r-1}$ to β. This means in particular that the map induced by β in homology is always injective. Our main theorem is the following.

Theorem 1 (Stability Theorem)**.** *For any $r \geq 1$,*

(1) *$\alpha_i : H_i(\mathcal{M}_{n,r};\mathbb{Z}) \to H_i(\mathcal{M}_{n+1,r};\mathbb{Z})$ is surjective when $n \geq 4i-1$ and an isomorphism when $n \geq 4i+2$.*

(2) *$\beta_i : H_i(\mathcal{M}_{n,r};\mathbb{Z}) \to H_i(\mathcal{M}_{n,r+1};\mathbb{Z})$ is an isomorphism when $n \geq 4i+2$.*

(3) *$\delta_i : H_i(\mathcal{M}_{n,1};\mathbb{Z}) \to H_i(\mathcal{M}_{n,0};\mathbb{Z})$ is an isomorphism when $n \geq 4i+4$.*

An analogous theorem was proved by Harer [1] (improved by Ivanov [2]) in the case of orientable surfaces.

Let $\mathcal{M}_\infty = \mathrm{colim}_{n\to\infty}(\mathcal{M}_{n,1} \to \mathcal{M}_{n+1,1} \to \dots)$ be the stable non-orientable mapping class group. Using the work of Madsen and Weiss [3], we obtain the following consequence of Theorem 1:

Theorem 2 (Stable Homology)**.** $H_*(\mathcal{M}_\infty;\mathbb{Z}) \cong H_*(\Omega_0^\infty \mathbb{G}_{-2};\mathbb{Z})$.

Here $\Omega_0^\infty \mathbb{G}_{-2}$ denotes the 0th component of the infinite loop space of the Thom spectrum $\mathbb{G}_{-2}$ defined by the orthogonal bundle to the tautological bundle over the grassmannians of 2-planes in $\mathbb{R}^{n+2}$—letting n vary. Let Q denote the functor $\Omega^\infty\Sigma^\infty = \mathrm{colim}_{n\to\infty}\Omega^n\Sigma^n$ and Q_0 its 0th component. Looking away from 2 or rationally, the right hand side in Theorem 2 simplifies further:

Corollary 3. $H_*(\mathcal{M}_\infty;\mathbb{Z}[\frac{1}{2}]) \cong H_*(Q_0(\mathrm{BO}(2)_+);\mathbb{Z}[\frac{1}{2}])$

Corollary 4. $H^*(\mathcal{M}_\infty;\mathbb{Q}) \cong \mathbb{Q}[\zeta_1,\zeta_2,\dots]$ *with* $|\zeta_i| = 4i$.

This latter corrolary is the non-orientable analogue of the Mumford conjecture.

The homological stability theorem (Theorem 1) is proved using complexes of arcs in non-orientable surfaces.

Let S be a surface, orientable or not, and let $\overrightarrow{\Delta}$ be a set of *oriented points* in ∂S, that is each point comes with the choice of an orientation of the component of ∂S it lies in. We say that an arc in $(S,\overrightarrow{\Delta})$ is *1-sided* if its boundary points are in $\overrightarrow{\Delta}$ and its normal bundle identifies the orientations of its endpoints. Note that a 1-sided arcs from a point to itself is a 1-sided curves in the usual meaning of the word. If S is orientable, the choice of an orientation for S decomposes $\overrightarrow{\Delta}$ as $\overrightarrow{\Delta} = \Delta^+ \sqcup \Delta^-$, where Δ^+ is the set of "positive" points and Δ^- the set of negative ones. The 1-sided arcs in this case are exactly the arcs with one boundary point in Δ^+ and the other in Δ^-. This complex, in the oriented case, was studied by Harer in [1]. He shows that it is highly connected. We generalize his result in two ways: first to the complex of arcs between two sets of points Δ_0 and Δ_1 in a non-orientable surface, and then to the complex of 1-sided arcs in $(S,\overrightarrow{\Delta})$ for a set of oriented points $\overrightarrow{\Delta}$. Our proof is different from Harer's and uses techniques from [4]. (We also fill in a gap in Harer's argument.)

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The space of smooth surfaces in a manifold

Ralph L. Cohen
(joint work with Ib Madsen)

The goal of this work is to study the topology of the space of surfaces mapping to a background manifold X, with boundary condition γ, $\mathcal{S}_{g,n}(X;\gamma)$. This space is defined as follows.

Let X be a simply connected space with basepoint $x_0 \in X$. Let $\gamma : \coprod_n S^1 \to X$ be n smooth loops in X. Define the space

$$\begin{aligned}\mathcal{S}_{g,n}(X,\gamma) = \{(S_{g,n},\phi,f) : &\text{ where } S_{g,n} \subset \mathbb{R}^\infty \text{ is a smooth oriented surface} \\ &\text{of genus } g \text{ and } n \text{ boundary components,} \\ &\phi : \coprod_n S^1 \xrightarrow{\cong} \partial S \text{ is a parameterization of the boundary,} \\ &\text{and } f : S_{g,n} \to X \text{ is a smooth map with } \partial f = \gamma : \coprod_n S^1 \to X. \ \}\end{aligned}$$

The parameterization ϕ is an orientation preserving diffeomorphism. ∂f is the composition $\coprod_n S^1 \xrightarrow{\phi} \partial S \xrightarrow{f_{|\partial S}} X$.

We think of these spaces as moduli spaces of Riemann surfaces mapping to X, or for short, the moduli space of surfaces in X. Indeed the embedding of the surface in Euclidean space defines an inner product on the tangent space of the surface, which together with the orientation defines an almost complex structure, and hence a complex structure on the surface. In fact when X is a point, this space is homotopy equivalent to the moduli spae, $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g, and n-parameterized boundary components.

We have three main results about these moduli spaces. The first describes the "stable topology" of $\mathcal{S}_{g,n}(X,\gamma)$, the second is a stability result showing the range of dimensions in which the homology of $\mathcal{S}_{g,n}(X;\gamma)$ is in the stable range, and the third is a stability result about the homology of mapping class groups with certain families of twisted coefficients.

We first observe that the spaces $\mathcal{S}_{g,n}(X;\gamma)$ have homotopy types that do not depend on the boundary map γ. This is because the space $\mathcal{S}_{g,n}(X)$ fibers over the space of n-tuples of loops, $(LX)^n$. Since X is simply connected, these loop spaces are connected, and hence the fibers have homotopy types which are independent of the choice of point $\gamma \in (LX)^n$.

Because of this fact, we are free to work with convenient choices of boundary conditions. We will assume our boundary map $\gamma : \coprod_n S^1 \to X$, viewed as n-loops numbered $\gamma_0, \cdots, \gamma_{n-1}$, has the property that $\gamma_0 : S^1 \to x_0 \in X$ is constant at the basepoint.

To state our result about the stable topology of $\mathcal{S}_{g,n}(X;\gamma)$, consider the fixed surface of genus one, $T \subset \mathbb{R}^3$ having two boundary components. Given $(S,\phi,f) \in \mathcal{S}_{g,n}(X,\gamma)$, we "glue in" the surface T along the boundary component labeled by γ_0, to get a surface $T\#S$ of genus $g+1$. The boundary parameterization ϕ now defines a boundary parameterization of $T\#S$, and the map $f : S \to X$ extends to $T\#S$ by letting it be constant at the basepoint on T. This construction defines a map

$$T_{\#} : \mathcal{S}_{g,n}(X;\gamma) \to \mathcal{S}_{g+1,n}(X;\gamma).$$

We now define $\mathcal{S}_{\infty,n}(X;\gamma)$ to be the limit of the map $T_{\#}$,

$$\mathcal{S}_{\infty,n}(X;\gamma) = lim\, \{\mathcal{S}_{g,n}(X;\gamma) \xrightarrow{T_{\#}} \mathcal{S}_{g+1,n}(X;\gamma) \xrightarrow{T_{\#}} \cdots\}$$

We refer to the topology of $\mathcal{S}_{\infty,n}(X;\gamma)$ as the "stable topology" of the moduli spaces, $\mathcal{S}_{g,n}(X;\gamma)$. It can viewed as the moduli space of maps of infinite genus surfaces to X, which, in an appropriate sense, are constant outside a finite genus subsurface.

Our first theorem describes the stable topology of these moduli spaces.

Theorem 1. *Let X be a simply connected, based space. The stable cohomology $H^*(\mathcal{S}_{\infty,n}(X;\gamma);\mathbb{Z})$ is completely known (see* [1] *for a description). The rational cohomology has a particularly nice description. Let $\mathcal{K}$ be the graded vector space over $\mathbb{Q}$ generated by one basis element, κ_i, of dimension $2i$ for each $i \geq -1$. Consider the tensor product of graded vector spaces, $\mathcal{K} \otimes H^*(X;\mathbb{Q})$. Let $(\mathcal{K} \otimes H^*(X;\mathbb{Q}))_+$ denote that part of this vector space that lives in nonnegative grading. Let $A((\mathcal{K} \otimes H^*(X;\mathbb{Q}))_+)$ be the free graded algebra over $\mathbb{Q}$ generated by this vector space. We then have an isomorphism of algebras,*

$$H^*(\mathcal{S}_{\infty,n}(X;\gamma)\,\mathbb{Q}) \cong A((\mathcal{K} \otimes H^*(X;\mathbb{Q}))_+)/(\kappa_0 - 1).$$

In other words, the only relation in this free algebra is the class κ_0 is set equal to 1.

Notice that $H^*(\mathcal{S}_{\infty,n}(point);\mathbb{Q})$ is the stable rational cohomology of moduli space. This algebra was conjectured by Mumford, and proven by Madsen and Weiss in [5], to be the polynomial algebra on the Miller-Morita-Mumford κ-classes. The classes $\kappa_i \in \mathcal{K} \subset H^*(\mathcal{S}_{\infty,n}(X;\gamma)\,\mathbb{Q})$ for $i \geq 1$ are the image of the Miller-Morita-Mumford classes under the map

$$H^*(\mathcal{S}_{\infty,n}(point);\mathbb{Q}) \to H^*(\mathcal{S}_{\infty,n}(X;\gamma)\,\mathbb{Q}).$$

This theorem can be viewed as a parameterized version of the Madsen-Weiss theorem.

Notice that in the statement of Theorem 1, the right hand side does not depend on n, the number of boundary components. This is strengthened by the following theorem, which identifies the stable range of the homology of the individual surface spaces.

Theorem 2. *For X simply connected as above, the homology groups,*

$$H_q(\mathcal{S}_{g,n}(X;\gamma);\mathbb{Z})$$

are independent of the genus g, the number of boundary components n, and the boundary condition γ, so long as $2q+4 \leq g$.

Our last result, which is actually a key ingredient in proving both Theorem 1 and Theorem 2 is purely a statement about the homology of mapping class groups. Our inspiration for this theorem was the work of Ivanov [4] which gave the first stability results for the homology of mapping class groups with certain kinds of twisted coefficients. The following is a generalization of his results.

Let $\mathcal{C}$ be the category whose objects are oriented surfaces, which, if they have boundaries come equipped with boundary parameterizations. The morphisms are isotopy classes of embeddings $e : S_1 \hookrightarrow S_2$, which on the boundary, maps each component of ∂S_1 either diffeomorphically to a boundary component of ∂S_2 (preserving the parameterizations), or to the *interior* of ∂S_2. This way there exist morphisms that change both the genus and the number of boundary components.

Let $\Gamma_{g,n} = \pi_0(Diff^+(F_{g,n}, \partial F_{g,n}))$ be the mapping class group. Notice these are morphisms in the surface category $\mathcal{C}$.

A *coefficient system* is a functor $\mathcal{V} : \mathcal{C} \to Abelian\, groups$. Notice that for any surface S, $\mathcal{V}(S)$ is a representation of the mapping class group, $\Gamma(S)$.

For a fixed surface of genus $F_{g,n}$, let $V_{g,n} = \mathcal{V}(F_{g,n})$. Following Ivanov [4] we define the notion of the degree of a coefficient system. Degree 0 coefficient systems have the property that all the $V_{g,n}$'s are isomorphic, and have trivial $\Gamma_{g,n}$-actions.

A nice example of a coefficent system of degree one is $V_{g,n} = H_1(F_{g,n};\mathbb{Z})$. We also prove that the coefficient system $V^q_{g,n} = H_q(Map(F_{g,n}, \partial F_{g,n}; X, x_0))$ is a degree q-coefficient system. Our last main result is a generalization of stability theorems of Harer [2] and Ivanov [4], [3].

Theorem 3. *If $\mathcal{V}$ is a coefficient system of degree d, then the homology group*

$$H_q(\Gamma_{g,n}; V_{g,n})$$

is independent of g, and n, so long as $2q+d < g$.

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Counting simple closed geodesics on hyperbolic surface

Maryam Mirzakhani

Let X be a complete hyperbolic Riemann surface of genus g with n punctures. In this talk, we address the problem of estimating $s_X(L)$, the number of primitive *simple* closed geodesics of hyperbolic length less than L on X. To explore this problem, we follow two approaches: the first using symplectic geometry of moduli spaces of curves, and the second using ergodic theory of the Teichmüller horocycle flow.

For $X \in \mathcal{M}_g$, let $c_X(L)$ be the number of primitive closed geodesics on X of length $\leq L$. By work of Delsart, Huber, Selberg and Margulis, we have

$$c_X(L) \sim e^L/L$$

as $L \to \infty$. However, very few closed geodesics are *simple* [1] and it is hard to discern them in $\pi_1(S_g)$.

For simplicity, we assume that X is compact. Given $X \in \mathcal{T}_g$, let $H(X) = \{v \in T^1(X) \mid g^t v$ is a simple complete geodesic$\}$. It is easy to check that $H(X)$ is a geodesic flow invariant closed subset of $T^1(X)$. By a result of Birman and Series [1] $H(X)$ has Hausdorff dimension 1. We are interested in understanding the ergodic theory of the geodesic flow on $H(X)$. Let μ_γ be the probability measure supported on a simple closed geodesic γ on X. Using ideas in [2] we show

Theorem 1. *Let $N(L)$ be the number of simple closed geodesic of length $\leq L$ on X. Then there exists a measure μ on $T^1(X)$ such that as $L \to \infty$,*

$$\frac{\sum_{\ell(\gamma_i)\leq L} \mu_{\gamma_i}}{N(L)} \to \mu.$$

Moreover, the support of measure μ is exactly $H(X)$.

The measure μ is very different from the measure of maximal entropy on X. In fact, all the ergodic components of μ have zero entropy.

Let $\mathcal{ML}_g$ be the space of *measured laminations* on S_g [5]. There is a one-to-one correspondence between the integral measured laminations, $\mathcal{ML}_g(\mathbb{Z})$, and unions of disjoint essential simple closed curves on S_g, up to isotopy. There is a natural volume form on $\mathcal{ML}_g$ preserved by the action of the mapping class group Mod_g. For any $X \in \mathcal{T}_g$ and $\lambda \in \mathcal{ML}_g$, let $\ell_\lambda(X)$ denote the hyperbolic length of λ on X.
Counting problems. To understand the growth of $s_X(L)$, it proves fruitful to fix a simple closed curve $\gamma \in \mathcal{ML}_g(\mathbb{Z})$ and consider more generally the counting function $s_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}_g \cdot \gamma \mid \ell_\alpha(X) \leq L\}$. There are only finitely many isotopy classes of simple closed curves on S_g up to the action of the mapping class group. Therefore, summing $s_X(L,\gamma)$ over representatives of these orbits gives $s_X(L)$, and the asymptotics of the $s_X(L,\gamma)$'s determines the asymptotics of $s_X(L)$. In [2] we show :

Theorem 2. *For any* $\gamma \in \mathcal{ML}_g(\mathbb{Z})$*, we have*

$$\lim_{L \to \infty} \frac{s_X(L, \gamma)}{L^{6g-6}} = n_\gamma(X),$$

where $n_\gamma(X)$ *is a smooth proper function of* $X \in \mathcal{M}_g$*.*

In the case of $\mathcal{M}_{1,1}$, this result was previously obtained by McShane and Rivin [8]. The upper and lower estimates for $S_X(L)$ when $X \in \mathcal{M}_g$ were obtained by M. Rees in [7] and I. Rivin in [6].

Idea of the proof. The crux of matter is to understand the density of $\mathrm{Mod}_g \cdot \gamma$ in $\mathcal{ML}_g(\mathbb{Z})$. This is similar to the problem of the density of relatively prime pairs (p, q) in $\mathbb{Z}^2$. Our approach is to use the moduli space $\mathcal{M}_{g,n}$ to understand the average of these densities. We show that the average defined by

$$S(L, \gamma) = \int_{\mathcal{M}_g} s_X(L, \gamma)\, dX$$

is well-behaved; in fact it is a polynomial in L. Here the integral on $\mathcal{M}_g$ is taken with respect to the Weil-Petersson volume form. This polynomial behaviour allows us to use the ergodicity of the action of the mapping class group on $\mathcal{ML}_g$ [9] to prove that these densities exist. Then Theorem 2 follows by a simple lattice-counting argument.

Frequencies of different types of simple closed curves. We now discuss more precisely how $n_\gamma(X)$, the constant in the growth rate of $s_X(L, \gamma)$, depends on X and on the simple closed curve γ.

Let B_X be the unit ball in the space of measured geodeic laminations with respect to the length function at X:

$$B_X = \{\lambda \mid \ell_\lambda(X) \leq 1\} \subset \mathcal{ML}_g.$$

We show that B_X is convex with respect to the piecewise linear structure of $\mathcal{ML}_g$. Let $B(X) = \mathrm{Vol}(B_X)$ with respect to the Thurston volume form on $\mathcal{ML}_g$. We show that $b_g = \int_{\mathcal{M}_g} B(X)\, dX$ is a finite number in $\pi^{6g-6} \cdot \mathbb{Q}$ which can be calculated in terms of the leading coefficients of the volume polynomials.

We show that the contributions of X and γ to $n_\gamma(X)$ separate as follows:

Theorem 3. *For any* $\gamma \in \mathcal{ML}_g(\mathbb{Z})$*, there exists a rational number* c_γ *such that we have:*

$$n_\gamma(X) = \frac{c_\gamma \cdot B(X)}{b_g}.$$

It follows that the relative frequencies of different types of simple closed curves on X are universal rational numbers.

Corollary 4. *For* $X \in \mathcal{M}_g$ *and* $\gamma_1, \gamma_2 \in \mathcal{ML}_g(\mathbb{Z})$*, we have*

$$\lim_{L \to \infty} \frac{s_X(L, \gamma_1)}{s_X(L, \gamma_2)} = \frac{c_{\gamma_1}}{c_{\gamma_2}} \in \mathbb{Q}.$$

The limit is a positive rational number independent of X*.*

Remark. The exact same result holds when the surface X has variable negative curvature.

Let $V_{g,n}(L)$ denote the Weil-Petersson volume of the moduli space of hyperbolic surfaces of genus g with n geodesic boundary components of length $L_1, \ldots, L_n$ [3]. We can calculate c_γ recursively using our recursive formula for $V_{g,n}(L)$. In fact, we can write the number c_γ in terms of the intersection numbers of tautological line bundles over the moduli space of Riemann surfaces of type $S_g - \gamma$ [4].

Remark. Note that the result is in fact a topological statement about $\mathcal{ML}_g$. Therefore one can replace the hyperbolic length function by any continuous function, $F : \mathcal{ML}_g \times \mathcal{T}_g \to \mathbb{R}$ such that $F(t \cdot \lambda, X) = tF(\lambda, X)$; e.g. $F(\lambda, X) = \sqrt{\mathrm{Ext}_\lambda(X)}$.

The growth of the number of simple closed geodesics, $s_X(L)$, can be also investigated via the dynamics of the Teichmüller horocycle flow on moduli space of holomorphic quadratic differentials. Let $\mathcal{QT}_g$ be the bundle of quadratic differentials over the Teichmüller space. Then the space $\mathcal{ML}_g$ parametrizes the space of horospheres of $\mathcal{QT}_g$.

Train tracks and equidistribution results in moduli space of quadratic differentials. Let τ be a train track on a surface S_g. If we assign positive integer weights to the edges of the graph τ satisfying the switch conditions, then the resulting weighted train track determines an isotopy class of a simple closed curve on S_g [10]. We would like to know the probability of getting a connected curve. We answer this question using equidistribution of the level sets of the extremal lengths of closed curves in the moduli space of quadratic differentials. We show that any train track τ defines an open chart in a stratum of the moduli space of quadratic differentials. The special case of this question for a generic train track can be answered by Corollary 4.

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Geometric identities, cross ratios and the Hitchin component

Greg McShane
(joint work with François Labourie)

In [7] McShane established a remarkable identity for lengths of simple closed geodesics on punctured hyperbolic surfaces and, using the same method, M. Mirzakhani [8] extended this identity to hyperbolic surfaces with geodesic boundary. Although it is possible to state and prove identities for surfaces with multiple boundary components, to simplify the exposition, we only consider Σ a complete hyperbolic surface with a single totally geodesic boundary component $\partial\Sigma$. If C is a closed curve then we denote by $\ell(C)$ the infimum of the set of lengths of curves freely homotopic to C with respect to the hyperbolic metric; this extends to a finite set of curves $\{C_i\}_i$ by $\ell(\{C_i\}_i) = \sum_i \ell(C_i)$. With this notation Mirzakhani's version of McShane's identity is

$$\ell(\partial\Sigma) = \sum_{P\in\mathcal{P}} \log\left(\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial\Sigma)}}{e^{\frac{\ell(\partial P)}{2}} + 1}\right), \tag{1}$$

where $\mathcal{P}$ is the setof embedded pants (with marked boundary) up to homotopy such that first the boundary component of the pair of pants is $\partial\Sigma$.

The purpose of the talk is twofold. Firstly, we show that the identity above has a natural formulation in terms of (generalized) cross ratios. Then, using this formulation, we study identities arising from the cross ratios constructed for representations in $SL(n,\mathbb{R})$ by Labourie [6]. We give a brief overview of the main ideas below, see [1] for details.

Cross ratio and periods: Let Σ be a closed surface. and $\partial_\infty\pi_1(\Sigma)$ be the *boundary at infinity* of the fundamental group $\pi_1(\Sigma)$ of Σ A *cross ratio* on $\partial_\infty\pi_1(\Sigma)$ is a $\pi_1(\Sigma)$-invariant Hölder function on

$$\partial_\infty\pi_1(\Sigma)^{4*} = \{(x,y,z,t) \in \partial_\infty\pi_1(\Sigma)^4 \ x \neq t, \text{ and } y \neq z\},$$

satisfying certain rules, the most significant being the *multiplicative cocycle identities* namely

$$\begin{aligned} b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t),\\ b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t). \end{aligned}$$

To every non trivial element γ of the group $\partial_\infty\pi_1(\Sigma)$ we associate a positive number, $\ell_b(\gamma)$, called the *period* of γ

$$\ell_b(\gamma) = \log b(\gamma^-, \gamma y, \gamma^+, y),$$

where γ^+ and γ^- are respectively the attractive and repulsive fixed points of γ in $\partial_\infty\pi_1(\Sigma)$ and where y is any point of $\partial_\infty\pi_1(\Sigma)$ such that $\gamma(y) \neq y$. The archetype of a cross ratio comes from hyperbolic geometry – a complete hyperbolic metric on Σ gives rise to an identification of $\partial_\infty\pi_1(\Sigma)$ with the real projective line. The classical cross ratio on the projective line then gives rise to a cross ratio on

$\partial_\infty \pi_1(\Sigma)$ and the period of γ is just the hyperbolic length of the closed geodesic freely homotopic to γ.

Pant gap function and the generalised formula: A pair of pants P with marked boundary in Σ corresponds to a triple (α, β, γ) of elements of $\pi_1(\Sigma)$, unique up to conjugation, such that $\alpha\gamma\beta = 1$. We define the *pant gap function* G_b at P to be the positive number

$$G_b(P) = \log(b(\alpha^+, \gamma^-, \alpha^-, \beta^+),$$

where b a cross ratio on $\partial_\infty \pi_1(\Sigma)$.

The general form of the McShane identity is:

Theorem 1. *Let Σ be closed surface. Let b be a cross ratio on $\partial_\infty \pi_1(\Sigma)$. Let α be a non trivial element of $\pi_1(\Sigma)$. Let $\mathcal{P}$ be the set of homotopy classes of pair of pants with marked boundary in Σ whose first boundary component is α, then*

$$\ell_b(\alpha) = \sum_{P \in \mathcal{P}} G_b(P).$$

Cross ratios and hyperbolic geometry: For $SL(2, \mathbb{R})$ (hyperbolic geometry) the pant gap function can be computed in terms of the length of the boundary components of pants using trigonometry [8]. We show in [1] how to determine G_b using just Thurston's *shear coordinates* and elementary manipulations involving the classical cross ratio.

Cross ratios and $SL(n, \mathbb{R})$: In [5], Labourie gives an interpretation of *the Hitchin representations*, a connected component of the space of representations of the $\pi_1(\Sigma)$ in $SL(n, \mathbb{R})$, as the space of cross ratios on $\partial_\infty \pi_1(\Sigma)$ satisfying an extra functional identity the form of which depends on n. As an example consider $SL(2, \mathbb{R})$ where the associated cross ratio, i.e. the classical cross ratio on the projective line, satisfies the following (well known) functional identity

$$b(t, y, z, x) = 1 - b(x, y, z, t). \tag{2}$$

Conversely, if we have a cross ratio b on a set A satisfying (2), it is well known that A can be identified with a subset of the projective line such that the cross ratio b is just the restriction of classical cross ratio. If the cross ratio is invariant by $\pi_1(\Sigma)$ then one obtains a representation of $\pi_1(\Sigma)$ into $PSL(2, \mathbb{R})$ and, with a little more work, a bijection between the Teichmüller space of Σ and the set of cross ratios on $\partial_\infty \pi_1(\Sigma)$ satisfying (2). In [5] this correspondence is extended to $SL(n, \mathbb{R})$ – up to conjugation every Hitchin representation of $\pi_1(\Sigma)$ in $SL(n, \mathbb{R})$ determines and is uniquely determined by a cross ratios on $\partial_\infty \pi_1(\Sigma)$.

Unfortunately, for $n \geq 3$ the pant gap function G_b is no longer determined by just the monodromies of three boundary components of the pants –it also depends on "internal parameters" which come from triple ratios [2].

Hitchin representations for open surfaces. We say an element in $SL(n, \mathbb{R})$ is *purely loxodromic* if it is real, split, with simple eigenvalues. Let Σ be a compact surface possibly with boundary. A representation of $\pi_1(\Sigma)$ in $SL(n, \mathbb{R})$ is *Fuchsian* iff it factorises as a discrete faithful representation without parabolics in $SL(2, \mathbb{R})$ composed with the irreducible representation of $SL(2, \mathbb{R})$ in $SL(n, \mathbb{R})$.

A representation of $\pi_1(\Sigma)$ in $SL(n,\mathbb{R})$ is *Hitchin* iff the boundary components have purely loxodromic images by the representation, and if it can be deformed to a Fuchsian representation so that the images of the boundary components stay purely loxodromic. It is shown in [4], [6] and [5], that Hitchin representations are discrete and faithful, that every non trivial element is purely loxodromic and that the mapping class group acts properly on the moduli space of Hitchin representations. We prove [1] a "doubling" theorem which implies that we can always find a closed surface S containing Σ such that every Hitchin representation of $\pi_1(\Sigma)$ is the restriction of a Hitchin representation of $\pi_1(S)$. The restriction of a Hitchin representation to a subsurface is Hitchin and it follows that they are *positive* in the sense of Fock and Goncharov [2].

Pant gap functions in FGT coordinates: In [2] Fock and Goncharov introduced a far reaching generalisations of Thurston's shear coordinates, which we call FGT coordinates, on the (augmented) moduli space of positive representations. We compute the gap functions for Hitchin representations using FGT coordinates for the moduli of pants; since the augmented moduli space is a "covering" of the space of Hitchin representations, we obtain $(n!)^3$ different answers. It turns out that, for a suitable choice of FGT coordinates, the pant gap function has a nice expression. However, using the explicit description of the holonomies [3] in the case of $n = 3$, we see the pant gap function has in general a very complicated expression for some choices of coordinates.

Applications and conclusion: Using the identities, M. Mirzakhani gives a recursive formula for the volume of moduli space of hyperbolic structures, *i.e* the quotient of Teichmüller space by the mapping class group. The formulae we obtain, combined with the use of FGT coordinates should enable one to compute analogous integrals associated to the quotient of the Hitchin representations by the mapping class group. However the volume is not the quite the right quantity to compute since for $n \geq 3$, one can show it is infinite.

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On Thurston's asymmetric metric on Teichmüller space

Guillaume Théret

An *asymmetric metric* on a space X is a function $L : X \times X \to \mathbb{R}_+$ satisfying the standard axioms of a metric except the symmetry axiom, namely,

- $\forall x, y \in X,\ L(x,y) = 0 \iff x = y,$
- $\forall x, y, z \in X,\ L(x,z) \leq L(x,y) + L(y,z),$
- $\exists\, x, y \in X \mid L(x,y) \neq L(y,x).$

W.P. Thurston introduced an example of such a kind of metric on Teichmüller space and initiated a deep study of it (see [5]). We first recall his definition.
Let Σ denote an oriented surface of genus $\mathfrak{g}$ with $\mathfrak{n}$ punctures, having a negative Euler-Poincaré characteristic. The *Teichmüller space* $\mathcal{T}(\Sigma)$ of Σ is the set of isotopy classes of complete hyperbolic metrics on Σ with finite area.
The Teichmüller space $\mathcal{T}(\Sigma)$ is endowed with its classical topology, given for instance by its embedding in the space $\mathbb{R}_+^{\mathcal{S}}$ provided by the length functional $l : \mathcal{T}(\Sigma) \to \mathbb{R}_+^{\mathcal{S}}$, defined by $g \mapsto l_g(\cdot) : \alpha \mapsto l_g(\alpha),\ \forall \alpha \in \mathcal{S}$; Here, $\mathcal{S}$ denote the set of homotopy classes of essential simple closed curves in Σ and the space $\mathbb{R}_+^{\mathcal{S}}$ is endowed with the weak topology.
Thurston's asymmetric metric L is defined by

$$\forall g, h \in \mathcal{T}(\Sigma),\ L(g,h) = \log \inf_{\phi \sim \mathrm{id}_\Sigma} L(\phi),$$

where $L(\phi)$ denotes the Lipschitz constant of the homeomorphism ϕ homotopic to the identity. Thurston showed that L is an asymmetric metric on $\mathcal{T}(\Sigma)$. Moreover, he considered the following quantity

$$\forall g, h \in \mathcal{T}(\Sigma),\ K(g,h) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_h(\alpha)}{l_g(\alpha)},$$

and proved the following

Theorem 1 ([5])**.** $K = L$.

In this talk, we shall first compare the classical topology of $\mathcal{T}(\Sigma)$ with the two topologies induced by L. It turns out that these topologies coincide. We next focus on some geodesics for L, called "stretch lines", and exhibit some features of their convergence towards Thurston's boundary of Teichmüller space.

1. Topology induced by L on $\mathcal{T}(\Sigma)$

An asymmetric metric L defines two natural topologies, namely, one generated by "left balls" ${}_xB(R) = \{y \,|\, L(x,y) < R\}$ and one generated by "right balls" $B_x(R) = \{y \,|\, L(y,x) < R\}$. Concerning Thurston's asymmetric metric L on $\mathcal{T}(\Sigma)$, these topologies turn out to coincide with the classical one.

Theorem 2 ([2])**.** *The topologies induced by L on $\mathcal{T}(\Sigma)$ coincide with the classical topology. More precisely, if (g_n) is a sequence of $\mathcal{T}(\Sigma)$ and $g \in \mathcal{T}(\Sigma)$ is an arbitrary point, one has*

- $\lim_{n\to\infty} L(g_n, g) = 0 \iff \lim_{n\to\infty} L(g, g_n) = 0 \iff \lim_{n\to\infty} g_n = g$,
- $\lim_{n\to\infty} L(g_n, g) = \infty \iff \lim_{n\to\infty} L(g, g_n) = \infty \iff \lim_{n\to\infty} g_n = \infty$.

2. Stretch Lines and Thurston's boundary of $\mathcal{T}(\Sigma)$

2.1. Before describing stretch lines, we make some definitions. Recall that a *geodesic lamination* μ is a union of disjoint simple geodesics such that this union is a closed subset of Σ. The geodesics of μ are called the *leaves* of μ. A geodesic lamination μ is said to be *complete* if every component of $\Sigma \setminus \mu$ is the interior of an ideal triangle. Note that to define a geodesic lamination, one had to fix an underlying hyperbolic structure on Σ. However, it turns out that this notion can be defined in a metric independent way.
A *transverse measure* (of full support) on a geodesic lamination μ is a positive Radon measure defined on each compact arc a transverse to the leaves of μ with support equal to $a \cap \mu$; Moreover, the masses assigned to two transverse arcs a and b are required to be equal if b is obtained from a by an isotopy leaving the leaves of μ invariant. A geodesic lamination μ equipped with a transverse measure is called a *measured geodesic lamination* and μ will be the *support* of the measured geodesic lamination. Let $\mathcal{ML}_0(\Sigma)$ denote the space of measured geodesic laminations of compact support and let $\mathcal{PL}_0(\Sigma)$ be the associated projective space, which consists in identifying two measured geodesic laminations with the same support and with proportional transverse measures.
The *stump* of a geodesic lamination μ is the support of the maximal (in the sense of inclusion) compact measured sublamination of μ.

2.2. Let us now talk about stretch lines.
Let us fix a complete geodesic lamination μ on Σ. The lamination μ is roughly thought of a kind of decomposition of Σ into ideal triangles, and to each hyperbolic metric there is a way to encode how these various ideal triangles are glued together. This encodement is given by a partial measured foliation called the *horocyclic foliation*, which is transverse to μ. Its transverse measure is defined by the requirement that the mass of an arc contained in μ is the length of that arc with respect to the underlying metric on Σ. When one varies the hyperbolic metric in its isotopy class, one obtains a well-defined element $F_g(\mu)$ of $\mathcal{MF}_0(\Sigma)$, the space of equivalence classes of measured foliations. In other words, one gets a map

$$\varphi_\mu : \mathcal{T}(\Sigma) \to \mathcal{MF}_0(\Sigma) \quad g \mapsto F_g(\mu),$$

about which Thurston proved the following

Theorem 3 ([5])**.** *The map φ_μ defined above is a homeomorphism onto its image.*

The *stretch line* directed by μ and passing through $g \in \mathcal{T}(\Sigma)$ is the oriented curve $t \to g_t^\mu$, $t \in \mathbb{R}$, of $\mathcal{T}(\Sigma)$ defined by $g_t^\mu = \varphi_\mu^{-1}(e^t F_g(\mu))$, where $e^t F_g(\mu)$ means that the transverse measure of $F_g(\mu)$ has been multiplied by the factor e^t. (The orientation of the stretch line is given by the classical orientation of $\mathbb{R}$.) The structure g_t^μ is said to be obtained by "stretching the structure g along μ": Indeed,

by definition of the transverse measure of the horocyclic foliation, after a stretch along μ, the arc-length on μ has been multiplied by a factor e^t, which explains, when $t > 0$, the terminology.
Thurston proved that a stretch line directed by a complete geodesic lamination μ with non-empty stump is a geodesic for L.

2.3. A natural question is to understand the asymptotic behavior of such a stretch line; In particular, one wants to know if a stretch line converges or not to a point of Thurston's boundary of Teichmüller space.
Recall that *Thurston's boundary* of Teichmüller space is the space $\mathcal{PL}_0(\Sigma)$ of projective classes of measured geodesic laminations with compact support. The elements of $\mathcal{PL}_0(\Sigma)$ will be denoted between brackets $[\cdot]$.
There are two natural geodesic laminations associated to the stretch line directed by μ and passing through g: the complete geodesic lamination μ and the (support of) the geodesic lamination $\lambda_g(\mu)$ that corresponds to the equivalence class $F_g(\mu)$ of the horocyclic foliation. (Recall that there is a natural one-to-one correspondance between $\mathcal{MF}_0(\Sigma)$ and $\mathcal{ML}_0(\Sigma)$.) Along the stretch line, μ is dilated whereas $\lambda_g(\mu)$ is contracted (see [3]). Concerning the problem of convergence to Thurston's boundary, one can state the following

Theorem 4. *Let $t \mapsto g_t^\mu$ be a stretch line directed by μ and passing through $g \in \mathcal{T}(\Sigma)$. Then,*

- ([1]) $\lim_{t\to+\infty} g_t^\mu = [\lambda_g(\mu)]$.
- ([4]) *If μ has a non-empty stump γ which is uniquely ergodic, then* $\lim_{t\to-\infty} g_t^\mu = [\gamma]$.

(A geodesic lamination is said to be *uniquely ergodic* if it supports a transverse measure which is unique up to positive scalar multiples.)

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Non-realization of the Mapping class group by homeomorphisms

Vladimir Markovic

Let M be a closed surface of genus g, with $g \geq 2$. The mapping class group $MC(M)$ is the group of homotopy classes of orientation preserving homeomorphisms of M (homeomorphisms are assumed to be orientation preserving). By

$Homeo(M)$ we denote the group of homeomorphisms of M. If $\widetilde{f} \in Homeo(M)$ is a homeomorphism, then $[\widetilde{f}] = f \in MC(M)$ denotes the corresponding homotopy class. The group $MC(M)$ can be seen as the quotient of the group $Homeo(M)$ by the subgroup of homeomorphisms of M that are isotopic to the identity (recall that two homeomorphisms from $Homeo(M)$ are isotopic if an only if they are homotopic to each other). There is the associated projection

$$Pr : Homeo(M) \to MC(M).$$

The main result is the following theorem.

Theorem: Let M be a closed surface of genus $g > 5$. Then, there is no homomorphism $E : MC(M) \to Homeo(M)$, such that the composite homomorphism $Pr \circ E$ is the identity.

Let $G < MC(M)$ be a finite group. The question of whether the corresponding homomorphism $E : G \to Homeo(M)$ exists, was known as the Nielsen realization problem. Kerckhoff [4] solved this famous problem by showing the existence of E in this case. There are other subgroups of $MC(M)$ for which the corresponding homomorphism exists (one example are Abelian subgroups of $MC(M)$). But generally, there is no criteria which would help decide whether a given subgroup of $MC(M)$ can be realized by homeomorphisms. We believe that the methods developed in this paper may help obtain further results in this direction.

The version of our main theorem was proved by Morita [7], [8], where the space of homeomorphisms of M is replaced by diffeomorphisms of M, and where the genus of M is at least 5. The projection $Pr : Homeo(M) \to MC(M)$ induces the map between the corresponding cohomology groups. In the diffeomorphic case, it was shown that not all respective cohomology groups of the groups $Diff(M)$ and $MC(M)$ are the same. This shows that there can not exist a homomorphism like E. However, it follows from the work of Mather and Thurston [9], [6], [7],[8], that there is no cohomological obstructions to the existence of E in the general homeomorphisms case. It has been conjectured [5] (also see [7], [8]), that such E does not exists in the general case.

The mapping class group of the torus can be represented by homeomorphisms (that is, the corresponding homomorphism E exists). In fact, it can be represented as the group of affine transformations $\mathbf{SL}_2(Z)$.

Let $\alpha \subset M$ be a simple closed curve. By $t_\alpha \in MC(M)$ we denote the standard twist about α. By $[\alpha]$ we denote the corresponding homotopy class of curves on M. Clearly, if $\alpha_1 \in [\alpha]$ then $t_{\alpha_1} = t_\alpha$. Now, let α and β be two simple closed curves such that neither of them separates M (this means that $M \setminus \alpha$ and $M \setminus \beta$ are connected sets). Assume that the intersection number between the homotopy classes of α and β is one. Then we have the following standard relation in the mapping class groups (see the Ivanov's book [10] for background on the Mapping class groups and more), called the Artin's relation

$$t_\alpha \circ t_\beta \circ t_\alpha = t_\beta \circ t_\alpha \circ t_\beta. \tag{1.1}$$

Set $f = t_\beta \circ t_\alpha$. Then f conjugates t_β to t_α in $MC(M)$. We have

$$E(t_\alpha) \circ E(t_\beta) \circ E(t_\alpha) = E(t_\beta) \circ E(t_\alpha) \circ E(t_\beta). \tag{1.2}$$

Another useful type of relation in $MC(M)$ is the following. Let $\widetilde{f}, \widetilde{g} : M \to M$ be two homeomorphisms that have disjoint supports (a closed set is said to be support of $\widetilde{f}$, if $\widetilde{f}$ is the identity outside that set). In particular, $\widetilde{f} \circ \widetilde{g} = \widetilde{g} \circ \widetilde{f}$, so $\widetilde{f}$ and $\widetilde{g}$ commute. We have

$$E([\widetilde{f}]) \circ E([\widetilde{g}]) = E([\widetilde{g}]) \circ E([\widetilde{f}]).$$

The following is an old theorem (going back to Hurwitz).

Proposition: Let a homeomorphism $\widetilde{f} : M \to M$ be periodic, that is, $\widetilde{f}^k = id$, for some $k \geq 1$. Then, there exists a complex structure on M, and a homeomorphisms $\widetilde{g} : M \to M$, such that $\widetilde{g}^{-1} \circ \widetilde{f} \circ \widetilde{g}$ is a conformal automorphism with respect to this complex structure. Moreover, if a periodic homeomorphism $\widetilde{f}$ is homotopic to a conformal automorphism $e : M \to M$ (with respect to some complex structure), then we can chose $\widetilde{g}$ to be homotopic to the identity homeomorphism of M, and $\widetilde{g}^{-1} \circ \widetilde{f} \circ \widetilde{g} = e$.

Let $e \in MC(M)$ be a periodic element. Assume that for some complex structure on M there exists a conformal automorphism $\widetilde{e}$ of M such that $[\widetilde{e}] = e$. Then $E(e)$ is conjugated to a conformal automorphism, by a homeomorphism $\widetilde{g} : M \to M$ which is homotopic to the identity. We can conjugate the whole homomorphism E by $\widetilde{g}$, to obtain a new homomorphism $E' = \widetilde{g}^{-1} \circ E \circ \widetilde{g}$, for which we have that $E'(e)$ is a conformal automorphism of M (when M is endowed with the appropriate complex structure).

The above relations are important ingredients of the proof of our main. However, one needs to do a bit of analysis before using the Artin's relations. This analysis is intended for proving that every twist $E(t_\alpha)$ is semi-conjugate to the identity mapping outside some closed annulus that is homotopic to α (in fact, this annulus is compactly contained in a bigger open annulus homotopic to the same curve). One important step in establishing this is based on the analysis of Anosov like elements of $MC(M)$. These are elements of $MC(M)$ that are Anosov (or in general, pseudo-Anosov) on some subsurface of M. The methods we deploy have similarity with the theory of global shadowing by Anosov (and in general pseudo-Anosov) homeomorphisms (see the papers by Franks [2], and Handel [3], where some of the these classical results were proved, and for further references). However, there are important differences. The main difference is that in our analysis we are focused on analyzing homeomorphisms that commute with these Anosov-like maps. In any case, the methods we give can be used to give perhaps more direct proof of the theorem which states that a homeomorphism that is homotopic to an Anosov map of the torus (or the punctured torus), is in fact semi-conjugate to that map. It can be generalized to pseudo-Anosov maps as well (in this case the corresponding statement is somewhat different, because a homeomorphism that is homotopic to a pseudo-Anosov map is not necessarily semi-conjugated to that map).

There are various realizations problems of similar flavor. One of these problems asks whether one can geometrically realize the group quasisymmetric maps of the unit circle by quasiconformal homeomorphisms of the unit disc. Dennis Sullivan has named this question the "Dream problem". Such a realization has been proved impossible in my joint work with David Epstein [1].

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Holomorphic sections for certain holomorphic families of Riemann surfaces of genus two

Yoichi Imayoshi

(joint work with Toshihiro Nogi)

We shall find all the holomorphic sections for certain holomorphic families of closed Riemann surfaces of genus two (cf.[2]).

1. Construction of a certain holomorphic family (M, π, R) of genus two and the main result

The idea to construct these families is originated from Kodaira [3], and the following construction is due to Riera [4].

Take a point τ in the upper half-plane $\mathbf{H}$ in the complex plane $\mathbf{C}$. Let $\Gamma_{1,\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbf{C})$ generated by two translations $z \mapsto z+1$ and $z \mapsto z+\tau$. Denote by $\hat{T}$ a torus defiend by the quotient space $\mathbf{C}/\Gamma_{1,\tau} = \{[z] \mid z \in \mathbf{C}\}$. We set $p_0 = [0] \in \hat{T}$ and $T = \hat{T} \setminus \{p_0\}$.

For a point $p \in T$ we take two replicas of the torus $\hat{T}$ cut along a simple arc from p_0 to p, and call them sheet I and sheet II. The cut on each sheet has two edges, labeled + edge and − edge. To construct a Riemann surface X_p, we attach the + edge on sheet I and the − edge on sheet II, and then attach the + edge on sheet II and the − edge on sheet I. Then we obtain a closed Riemann surface X_p of genus two and the two-sheeted covering $X_p \to \hat{T}$ which is branched over p_0 and p with branch order 2. It should be noted that the above procedure depends, of course, not only on the choice of the point p but also on the choice of the "cut" from p_0 to p, and essentially there are four different cuts.

To specify the "cut" we construct a four-sheeted unbranched covering

$$\rho\colon R \to T \tag{1}$$

of T such that R is a torus with four punctures as follows: Let $\Gamma_{2,2\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbf{C})$ generated by two translations $z \mapsto z+2$ and $z \mapsto z+2\tau$. Denote by $\hat{R}$ a torus defiend by the quotient space $\mathbf{C}/\Gamma_{2,2\tau} = \{[z] \mid z \in \mathbf{C}\}$. Let $\hat{\rho}\colon \hat{R} \to \hat{T}$ be the canonical projection given by $\hat{\rho}([z]) = [z]$. We set $R = \hat{\rho}^{-1}(T)$ and $\rho = \hat{\rho}|R$. The good thing is that a point $t = [z] \in R$ determines a point $p = \rho([z]) \in T$ and a "cut" $\alpha = \hat{\rho}(\beta)$ from p to $p_0 = [0]$, where β is a simple arc on R from $[0]$ to t. Denote by S_t the closed Riemann surface of genus two which is a two-sheeted branched covering surface of $\hat{T}$ constructed by a "cut" $\alpha = \hat{\rho}(\beta)$. Note that the two-sheeted branched covering $\Pi_t\colon S_t \to \hat{T}$ is uniquely determined by the choice of $t \in R$ and does not depend on β.

We set

$$M = \bigsqcup_{t \in R} \{t\} \times S_t,$$
$$\pi\colon M \to R,\ \pi(t,q) = t.$$

Then (M, π, R) is a holomorphic family of closed Riemann surfaces of genus two over a fourth punctured torus R.

Our main result is as follows:

Theorem 1. *The holomorphic family (M, π, R) of closed Riemann surfaces of genus two has exactly two holomorphic sections s_1, s_2, which are given by $s_1(t) = (t, p_0)$ and $s_2(t) = (t, \rho(t))$ for every $t \in R$.*

2. A defining equation for (M, π, R)

Now we will give a defining equation for (M, π, R). For any point $t = [\tilde{t}] \in R$, Abel's theorem shows there exists a unique meromorphic function f_t on $\hat{T}$ which has two simple zeros $[0]$ and $\rho(t)$, a pole $q_t = \rho(t)/2$ of order two, and satisfies $(df_t/dz)([0]) = 1$. The function f_t is represented explicitly by theta functions.

The holomorophic map $f_t\colon \hat{T} \to \hat{\mathbf{C}}$ has four branch points q_t (pole), $a(t), b(t)$, and $c(t)$, where

$$a(t) = f_t([(\tilde{t}+1)/2]),$$
$$b(t) = f_t([(\tilde{t}+\tau)/2]),$$
$$c(t) = f_t([(\tilde{t}+1+\tau)/2]).$$

Let g_t be the meromorphic function on $\hat{T}$ of degree 3 which has simple zeros $[(\tilde{t}+1)/2], [(\tilde{t}+\tau)/2], [(\tilde{t}+1+\tau)/2]$, and a pole $[\tilde{t}]$ of order 3, and satisfies $g_t([0]) = i$. The function g_t is also represented explicitly by theta functions.

Setting $x = f_t, y = g_t$, we have a functional relation

$$y^2 = \frac{1}{a(t)b(t)c(t)} (x - a(t)) (x - b(t)) (x - c(t))$$

on $\hat{T}$.

Now we have the following theorem:

Theorem 2. *In the above situation, let*

$$P_t(x) = \frac{(x^2 - a(t))(x^2 - b(t))(x^2 - c(t))}{a(t)b(t)c(t)},$$
$$M_{HE} = \{(t, x, y) \in R \times \hat{\mathbf{C}} \times \hat{\mathbf{C}} \,|\, y^2 = P_t(x)\},$$
$$\pi_{HE} \colon M_{HE} \to R,\ \pi_{HE}(t, x, y) = t.$$

Then the triplet (M_{HE}, π_{HE}, R) is a holomorphic family of closed Riemann surfaces of genus two, and it is isomorphic to (M, π, R). Moreover, (M_{HE}, π_{HE}, R) has exactly two holomorphic sections $s_{HE,1}, s_{HE,2}$, which are given by $s_{HE,1}(t) = (t, 0, i)$ and $s_{HE,2}(t) = (t, 0, -i)$ for every $t \in R$.

3. A sketch of proof for the main result

In order to prove Theorem 1, we need the following two theorems (cf. Imayoshi [1], Theorem 4 and Theorem 5):

Theorem 3. *The holomorphic family (M, π, R) has a canonical completion $(\hat{M}, \hat{\pi}, \hat{R})$, where $\hat{M}$ is a compact two dimensional normal complex analytic space and $\hat{\pi} \colon \hat{M} \to \hat{R}$ is holomorphic. Moreover every holomorphic section $s \colon R \to M$ has a holomorphic extension $\hat{s} \colon \hat{R} \to \hat{M}$.*

Theorem 4. *The holomorphic map $\Pi \colon M = \bigsqcup_{t \in R} \{t\} \times S_t \to \hat{T}$ defined by $\Pi(t, q) = \Pi_t(q)$ has a holomorphic extension $\hat{\Pi} \colon \hat{M} \to \hat{T}$.*

Now we can prove Theorem 1 as follows: Let $s \colon R \to M$ be an arbitrary holomorphic section of (M, π, R). Theorem 3 and 4 imply that the holomorphic map $\varphi = \Pi \circ s \colon R \to \hat{T}$ has a holomorphic extention $\hat{\varphi} = \hat{\Pi} \circ \hat{s} \colon \hat{R} \to \hat{T}$. Let $\tilde{\varphi} \colon \mathbf{C} \to \mathbf{C}$ is a lift of $\hat{\varphi} \colon \hat{R} \to \hat{T}$. Then $\tilde{\varphi}(z) = Az + B, z \in \mathbf{C}$ for some constants $A, B \in \mathbf{C}$. Since $\varphi = \Pi \circ s$, we can show that we may assume that $A = 0, B = 0$, or $A = 1, B = 0$. In the case $A = 0, B = 0$, we have the section $s_1(t) = (t, p_0)$, and in the case $A = 1, B = 0$, we have the section $s_2(t) = (t, \rho[t])$.

4. Moduli map J of (M, π, R) into M_2

Let M_2 be the moduli space of all biholomorphic equivalence classes $[S]$ of closed Riemann surfaces S of genus two. Then we have the following assertion:

Theorem 5. *For the holomorphic family (M, π, R) of closed Riemann surfaces of genus two, the holomorphic map $J\colon R \to M_2$, $J(t) = [S_t]$, is not injective. In particular, $J(t) = J(-t)$ for all $t \in R$.*

Note that it is possible to obtain a condition for $t, t' \in R$ with $J(t) = J(t')$.

Let $\hat{M}_2$ be the Deligne-Mumford compctification of M_2, that is, $\hat{M}_2$ is the set of all closed Riemann surfaces of genus two with or without nodes, which is a three dimensional compact normal complex analytic space. Then the holomorphic map $J\colon R \to M_2$ has a holomorphic extension $\hat{J}\colon \hat{R} \to \hat{M}_2$ (see Imayoshi [1], Lemma 1). It is also possible to get a condition for $t, t' \in \hat{R} \setminus R$ with $\hat{J}(t) = \hat{J}(t')$.

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Quasi-homomorphisms on mapping class groups

D. Kotschick

Let G be a group. A quasi-homomorphism on G is a map $f\colon G \longrightarrow \mathbb{R}$ for which there is a constant $D(f)$ such that

$$|f(xy) - f(x) - f(y)| \leq D(f)$$

holds for all $x, y \in G$. Obviously homomorphisms and bounded maps are quasi-homomorphisms. Every quasi-homomorphism f can be homogenized by defining

$$\varphi(x) = \lim_{n\to\infty} \frac{f(x^n)}{n} \; .$$

Then φ is again a quasi-homomorphism, it is homogeneous in the sense that $\varphi(x^n) = n\varphi(x)$, and it is constant on conjugacy classes. Non-trivial homogeneous quasi-homomorphisms are never bounded, and they vanish on elements of finite order.

When one has a non-trivial homogeneous quasi-homomorphism φ on G, then one can estimate the S-length of any element $g \in G$ provided that $\varphi(g) \neq 0$ and φ is bounded on the subset $S \subset G$. In the case when $S = C$ is the set of commutators, the S-length is the commutator length. It is a theorem of Bavard [1] that for the stable commutator length the estimates obtained from quasi-homomorphisms are sharp, that is they give a precise calculation of the stable commutator length. This is not so for other instances of S-length.

The existence of homogeneous quasi-homomorphisms is also related to (the failure of) a weak version of bounded generation. These applications of quasi-homomorphisms are detailed in [5].

For $SL(2,\mathbb{Z})$, the mapping class group of the torus, many homogeneous quasi-homomorphisms are known from the work of many authors. Polterovich and Rudnick proved that elements of infinite order in $SL(2,\mathbb{Z})$ that are not conjugate to their inverses can be separated by quasi-homomorphisms.

For the mapping class groups of surfaces of higher genera, the existence of quasi-homomorphisms was first proved by Endo and myself in 2000, and published in [3]. In fact, we proved that the stable commutator length does not vanish identically, confirming a conjecture of Morita. The existence of quasi-homomorphisms follows from this by the result of Bavard mentioned above. The argument from [3] was elaborated on by Braungardt and myself, and also by Korkmaz. The final version of it appeared in [5] in the form of the following:

Theorem 1. *Let Γ_h be the mapping class group of a closed oriented surface Σ_h of genus $h \geq 2$. If $g \in \Gamma_h$ is the product of k right-handed Dehn twists along homotopically essential disjoint curves $a_1, \ldots, a_k \subset \Sigma_h$, then the stable commutator length of g is bounded below by*

$$||g|| \geq \frac{k}{6(3h-1)} .$$

By the discussion above, this has immediate applications to various other length problems in Γ_h, see [5]. The proof of Theorem 1 proceeds by considering the symplectic geometry and Seiberg–Witten theory of certain four-manifolds constructed as Lefschetz fibrations associated with expressions of powers of Dehn twists as products of commutators.

Using geometric group theory instead, specifically the weak properness of the action of mapping class groups on the curve complex proved by Mazur and Minsky, Bestvina and Fujiwara proved the following result in 2001:

Theorem 2 ([2])**.** *Let G be any non-virtually Abelian subgroup of a mapping class group. Then the space of homogeneous quasi-homomorphisms on G is infinite-dimensional.*

This shows in particular that Γ_h is not weakly boundedly generated, see [5]. It remains an interesting problem to extract explicit bounds on the stable commutator length from the argument of [2].

Although there are abundant supplies of quasi-homomorphisms on mapping class groups, there are some things one can not do. For example, the analog of the separation theorem of Polterovich and Rudnick mentioned above fails, because mapping class groups contain elements of infinite order that are not conjugate to their inverses, but that nevertheless have zero stable commutator length. Therefore, all homogeneous quasi-homomorphisms vanish on them. The first examples exhibiting this phenomenon were found by Endo and myself in 2003, and will be published in [4].

Theorem 3 ([4]). *For every closed oriented surface of genus at least* 2 *there exist primitive elements* g *of infinite order in its mapping class group of orientation-preserving diffeomorphisms such that* g^k *is not conjugate to* g^{-k} *for all* $k \neq 0$*, but all powers of* g *are products of some fixed number of torsion elements, and are also products of a fixed number of commutators.*

Another thing one can *not* do, is to find homogeneous quasi-homomorphisms on the stable mapping class groups. Let Γ_h^1 be the group of isotopy classes of diffeomorphisms with compact support in the interior of a compact surface of genus h with one boundary component. Attaching a two-holed torus along the boundary defines the stabilization homomorphism $\Gamma_h^1 \to \Gamma_{h+1}^1$. The stable mapping class group Γ_∞ is defined as the union

$$\Gamma_\infty = \bigcup_h \Gamma_h^1 \ .$$

In contrast with Theorem 1 we have:

Theorem 4. *The stable commutator length for* Γ_∞ *vanishes identically.*

As a non-trivial homogeneous quasi-homomorphism forces all elements on which it does not vanish to have positive stable commutator length, this theorem shows that there are no non-trivial homogeneous quasi-homomorphisms on Γ_∞. Consequently the bounded cohomology of mapping class groups does not stabilize. This contrasts sharply with what happens with the usual cohomology according to Harer, Ivanov and others. Theorem 4 fits in nicely with the form of the estimates for the stable commutator length obtained in Theorem 1.

The proof of Theorem 4 applies to many other groups defined as unions of smaller groups that admit different embeddings into each other satisfying certain technical assumptions. Another instance of a group to which the proof of Theorem 4 applies is the stable automorphism group of a free group.

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Wick rotations in 3D-gravity

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(joint work with R. Benedetti)

Let Σ denote a closed orientable surface of genus $g \geq 2$. Solutions of Einstein equation on $\Sigma\times\mathbb{R}$ are Lorentzian metrics of constant curvature (whose sign depends on the sign of the cosmological constant). Among all the solutions a class of particular interest is formed by globally hyperbolic metrics. Roughly speaking, a Lorentzian metric on $\Sigma\times\mathbb{R}$ is said globally hyperbolic if, up to some diffeomorphism isotopic to the identity, can be written in the form

$$-dt^2 + g_t \tag{1}$$

where t denotes the coordinate on $\mathbb{R}$ and g_t is a Riemannian metric on $\Sigma \times \{t\}$.

For $k \in \{0, 1, -1\}$ let $\mathcal{M}_k(\Sigma)$ denote the set of maximal globally hyperbolic Lorentzian metrics on $\Sigma \times \mathbb{R}$ up to diffeomorphisms isotopic to the identity. In his seminal work [6] G. Mess showed that $\mathcal{M}_k(\Sigma)$ is homeomorphic to $\mathbb{R}^{12g-12}$ independently of k. More precisely Mess constructed an identification

$$m_k : \mathcal{T}(\Sigma) \times \mathcal{ML}(\Sigma) \to \mathcal{M}_k(\Sigma) \tag{2}$$

where $\mathcal{T}(\Sigma)$ is the Teichmüller space of Σ and $\mathcal{ML}(\Sigma)$ is the set of measured geodesic laminations on Σ.

In fact, the case $k = 1$ was carried over by Scannell some years later [7]. In that case the parameterization was given in terms of complex projective structures on Σ. On the other hand, Thurston pointed out a way to associate to a pair $(F, \lambda) \in \mathcal{T}(\Sigma) \times \mathcal{ML}(\Sigma)$ a complex projective surface, $Gr_\lambda(F)$, called the grafting of F along λ. If λ is a weighted simple geodesic, $Gr_\lambda(F)$ is obtained by cutting F along λ and gluing a projective annulus of height equal to the weight of λ. The map

$$Gr : \mathcal{T}(\Sigma) \times \mathcal{ML}(\Sigma) \to \mathcal{P}(\Sigma)$$

turns to be an identification (for a proof see e.g. [5]). As a by-product of this theory, a canonical metric, said Thurston metric, is defined on each projective structure.

Eventually the map m_1 in (2) is intended as the composition of Scannell map and Thurston parameterization.

As a corollary of Mess parameterization, implicit identifications

$$\mathcal{M}_0(\Sigma) \to \mathcal{M}_k(\Sigma) \qquad \mathcal{M}_0(\Sigma) \to \mathcal{P}(\Sigma)$$

arise. In [3] an explicit description of such identifications is given.

The main ingredient to get such a description is the *cosmological time*. Given a time-oriented Lorentzian manifold M, its cosmological time is a $\mathbb{R}_{>0} \cup \{+\infty\}$-valued function that returns at p the sup of the lengths of timelike curves with future end-point at p. In general this function could be very degenerated (for instance if M is geodetically complete it takes only the value $+\infty$). On the other hand if $M \in \mathcal{M}_k(\Sigma)$ with $k \geq 0$, then its cosmological time, say τ, is $C^{1,1}$ and its image is the whole interval $(0, +\infty)$ (this result is somehow implicit in Mess and

Scannell original works and was explicitly pointed out in [4] for the flat case). If $k = -1$ then the cosmological time is only a $C^{0,1}$-function. Its image is an interval $(0, a)$ with $\pi/2 < a \leq \pi$, and τ is $C^{1,1}$ on the subset $\tau^{-1}(0, \pi/2)$ that will be called the *past part* of M.

Riemannian structures on the level sets of the cosmological time can be explictly related to Mess-Scannell parameters of M. More precisely, if (F, λ) are the Mess-Scannell parameters of M, then the metric on the level surface $\tau^{-1}(a)(a < \pi/2$ if $k = -1)$ is obtained by rescaling $Gr_{u(a)\lambda}F$ (regarded as a metric space equipped with the Thurston metric) by some factor $v(a)$, where u and v are explicit functions depending only on k. As a corollary, the path of conformal structures $\{\tau^{-1}(a) \in \mathcal{T}_g\}$ coincides (up to some re-parameterization) with the grafting path joining the point in Thurston boundary corresponding to λ to F if $k \leq 0$ and to $gr_\lambda F$ if $k = 1$.

Eventually, the cosmological time allows a homogenous description of spacetimes in $\mathcal{M}_k(\Sigma)$ in terms of Mess-Scannell parameters. In particular the family of Riemannian surfaces corresponding to level sets of the cosmological time is somehow independent of the curvature up to some scaling factors.

Such a remark could be made more precise by introducing an operation on Lorentzian metrics called *rescaling*. In general a rescaling depends on a timelike vector field X, and two positive functions α, β. The field X provides a decomposition of TM in a vertical part (parallel to X) and a horizontal part (orthogonal to X). The rescaling on M along X with rescaling functions α and β is the Lorentzian metric obtained by rescaling the squared norm of vertical vectors by the factor α and the squared norm of horizontal vectors by a factor β.

Let (F, λ) be a fixed element of $\mathcal{T}_g \times \mathcal{ML}_g$, and let M_k be the element of $\mathcal{M}_k(\Sigma)$ associated to (F, λ) via Mess-Scannell parameterization.

In [3] an explicit rescaling on M_0 is pointed out such that the rescaled spacetime is the past part of M_{-1}. Such rescaling is directed along the gradient of the cosmological time τ, and the rescaling functions are explicit functions of τ (namely, $\beta = (1 + \tau^2)^{-1}$, $\alpha = \beta^2$).

In a similar way it is possible to construct a rescaling relating M_0 to M_1. In such a case the rescaling is defined only on the set $\{\tau < 1\}$, it is directed along the gradient of τ with rescaling functions $\beta = (1 - \tau^2)$ and $\alpha = \beta^2$.

Finally, in order to describe explicitly the identification $\mathcal{M}_0(\Sigma) \to \mathcal{P}(\Sigma)$, we need to introduce an operation, the Wick rotation, that transforms Lorentzian metrics into Riemannian metrics. This operation depends on a timelike vector field X and two positive functions α and β. It works like the rescaling except that the squared norm of vertical vectors is rescaled by the factor $-\alpha$.

In [3] it is shown that the Wick rotation on the set $\{\tau > 1\}$ of M_0, directed along the gradient of the cosmological time with rescaling functions $\beta = (\tau^2 - 1)$ and $\alpha = \beta^2$ is a hyperbolic metric. The developing map $D : \widetilde{\{\tau > 1\}} \to \mathbb{H}^3$ extends to a map $D : \widetilde{\{\tau \geq 1\}} \to \mathbb{H}^3 \cup S^2$ sending $\widetilde{\{\tau = 1\}}$ to S^2. Moreover, the restriction of D on the level surface $\widetilde{\{\tau = 1\}}$ is the developing map for $Gr_\lambda(F)$.

The transformations described above extend even if Σ is not closed.

In fact, in order to get a reasonable treatment when Σ is not closed, the definition of the class of interest of Lorentzian structures has to be made more precise.

In particular we require that a Lorentzian metric could be written (up to diffeomorphisms) in the form $-dt^2 + g_t$ with g_t *complete* Riemannian structure on $\Sigma \times \{t\}$ [1]. Moreover a more careful definition of global hyperbolicity is needed (e.g., see [2]).

In [3] it is shown that Wick rotations and rescalings pointed out in the closed case, work in the general case iff the cosmological time is a regular function. On the other hand, if the fundamental group of Σ is not Abelian, then it is proved that the cosmological time on a globally hyperbolic structure on $\Sigma \times \mathbb{R}$ is regular (in fact the same statement as in the closed case holds).

Eventually, for every surface Σ with non-Abelian fundamental group, Wick rotations introduced above lead to an identification

$$\mathcal{P}(\Sigma) \to \mathcal{M}_k(\Sigma) \tag{3}$$

and thus to a parameterization of $\mathcal{M}_k(\Sigma)$ in terms of projective structures on Σ.

In [5] a generalization of Thurston parameterization for Möbius structure in every dimension is pointed out. As a by-product, projective structures on a surface with non-Abelian fundamental group are encoded by triples (F, F', λ), where F is a complete hyperbolic structure on Σ, F' is a convex subset of F with geodesic boundary and λ is a measured geodesic lamination on F', with the property that the total mass of an arc reaching the boundary is infinite.

By combining this theory with (3), elements of $\mathcal{M}_k(\Sigma)$ turn to be encoded by hyperbolic structures on Σ equipped with these more general measured geodesic laminations. In [3] this encoding has been investigated.

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Classification of tree-like diffeomorphisms up to conjugacy

Yvonne Gerber

We are interested in surface diffeomorphisms, that can be constructed out of a rooted planar tree.

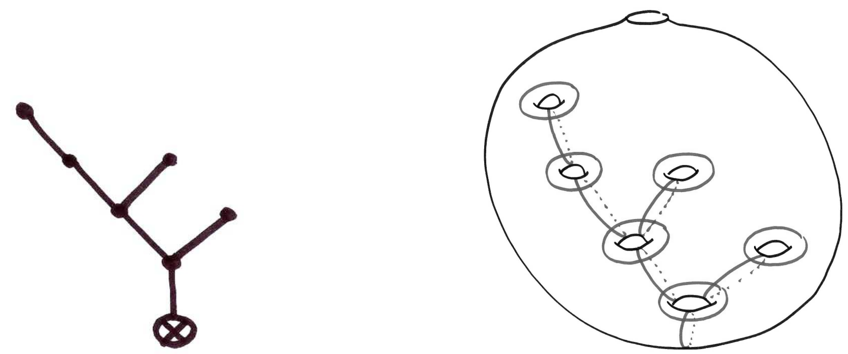

Figure 1. Rooted planar tree [0,1,1,2,2,4] and the associated surface with curves

One valence-one vertex is called the root (marked by $\otimes$), the other valence-one vertices are called crown vertices.

A rooted planar tree gives us a surface with one boundary component and a set of essential simple closed curves, called A-curves and B-curves (see figure 1). The A-curves correspond to the edges of the tree and the B-curves to the vertices except the root vertex. The diffeomorphism T, we want to study is

$$T = T_A \circ T_B,$$

where T_A is a product of positive Dehn twists along all A-curves, and T_B is a product of positive Dehn twists along all B-curves.

Theorem 1. *Tree-like diffeomorphisms, that arise from non-congruent planar trees with more than tree crown vertices, are not conjugate.*

To prove the theorem, we need an important property of T: T is strongly inversive. This means, that there exists an involution C, $C^2 = Id$, such that $CTC = T^{-1}$.

In fact, up to conjugacy of the pair (T, C), there are only two such involutions C and TC. Each of these involutions fixes pointwise an arc on the surface. C and TC can be distinguished using their fixed arc γ and γ' respectively. γ and $T(\gamma)$ has always one intersection point whereas γ' and $T(\gamma')$ always has more than one intersection point.

From the pair (T, C) we reconstruct the rooted planar tree. In fact, to do this, we use only T and the fixed arc γ of C.

This class of diffeomorphisms arises as monodromies of fibered knots. Out of a rooted planar tree, there can be constructed a knot, called slalomknot, and the above diffeomorphisms are their monodromies. Two slalomknots that come from the same abstract tree, but from different planar embeddings are mutant [1]. The tree in figure 1 has two non-congruent planar embeddings. Mutant knots are hard to distinguish. For small examples the quantum invariant can be calculated and separates. Knotscape too, helps us to separate small examples. The two knots for the two embeddings of figure 1 are $15n30444$ and $15n30419$ in the table of knotscape. Sometimes there is also a symmetry argument that can be applied. For the whole class of slalomknots it was not known if the knots are all different. The above theorem tells us, that the monodromies of two slalomknots, that come from non-congruent embeddings of the same abstract tree, are not conjugate, so we get the following theorem.

Theorem 2. *Two slalomknots that come from non-congruent planar trees are different.*

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Witten cycles on the moduli space of Riemann surfaces

Gabriele Mondello

Consider a compact oriented surface S of genus g with a P-marking, that is an injection $P = \{p_1, \ldots, p_n\} \hookrightarrow S$, and assume that $\chi(S \setminus P) = 2 - 2g - n < 0$. Given a complex structure on S and positive weights $a_1, \ldots, a_n$, one can construct a metric fatgraph, that is a metric graph G together with an isotopy class of embeddings $G \hookrightarrow S \setminus P$ which induce a homotopy equivalence. There are (at least) two recipes to do this: the former uses existence and uniqueness results for meromorphic quadratic differentials with closed trajectories (see [18] and [5]), and it explicitly appears first in [4] and [9]; the latter exploits the complete hyperbolic metric with finite volume of $S \setminus P$ and it appears first in [16] (but see also [2] for a different rephrasing). Both constructions commute with the action of the mapping class group $\Gamma(S, P)$ and give a homeomorphism Φ between the moduli space $\mathcal{M}_{g,P} \times \mathbb{R}_+^P$ of weighted P-marked Riemann surfaces of genus g and the space $\mathcal{M}_{g,P}^{comb}$ of metric fatgraphs whose "fattening" is a P-marked oriented surface of genus g. This space of fatgraphs $\mathcal{M}_{g,P}^{comb}$ comes naturally equipped with a cellular structure: each homeomorphism type of fatgraph G corresponds to a cell and the lengths of the edges of G are natural coordinates on the cell.
Thus we have two different presentation of $\mathcal{M}_{g,P}$: the first one from analytic/algebraic geometry; the second one via Φ as a space of graphs. Consequently, we will have two different families of characteristic classes.
From the complex-analytic point of view, we can define the *tautological classes* ([14], [11], [13]) on $\mathcal{M}_{g,P}$ as restriction of the classes ψ_i and κ_r from the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,P}$ to $\mathcal{M}_{g,P}$. The classes ψ_i and κ_r are defined as

follows: $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$ and $\kappa_r = (\pi_q)_*(\psi_q^{r+1}) \in H^{2r}(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$, where $\mathcal{L}_i$ is the holomorphic line bundle on $\overline{\mathcal{M}}_{g,P}$ with stalk $\mathcal{L}_i\Big|_{[S]} \cong \Omega^{1,0}_{S,p_i}$ and $\pi_q : \overline{\mathcal{M}}_{g,P\cup\{q\}} \to \overline{\mathcal{M}}_{g,P}$ is the map that forgets the q-marking, which can be identified to the universal family $\pi : \overline{\mathcal{C}}_{g,P} \to \overline{\mathcal{M}}_{g,P}$.
From the combinatorial point of view, given a sequence $m_* = (m_0, m_1, \dots)$ of non-negative integers, we can define the *Witten cycles* $\mathcal{W}_{m_*}$ ([9], [17]) as (the closure) of the union of all the cells of $\mathcal{M}^{comb}_{g,P}$ corresponding to graphs with m_i vertices of valence $2i+3$. For instance, maximal cells correspond to fatgraphs whose vertices are all trivalent, but the locus (which we denote by $\mathcal{W}_{2r+3}$ for brevity) corresponding to graphs with a vertex of valence $\geq 2r+3$ defines a cycle (with locally finite support) of real codimension $2r$ in $\mathcal{M}^{comb}_{g,P}$ and so a cohomology class in $H^{2r}(\mathcal{M}^{comb}_{g,P};\mathbb{Q}) \cong H^{2r}(\mathcal{M}_{g,P};\mathbb{Q})$, using Poincaré duality. Similarly, Poincaré duals to Witten cycles define classes in $H^{2*}(\mathcal{M}_{g,P};\mathbb{Q})$.

The map Φ can be extended to include the case in which some weights are zero: namely, if $a_i = 0$ then the boundary component corresponding to p_i is collapsed to a vertex of the fatgraph. In this way we can define other *generalized Witten cycles*, in which we ask that the vertex with a certain marking has a certain valence. For instance, we call $\mathcal{W}^q_{2r+3} \subset \mathcal{M}^{comb}_{g,P\cup\{q\}}$ the locus of fatgraphs with a vertex decorated by q of valence $\geq 2r+3$, which defines a class in $H^{2r+2}(\mathcal{C}_{g,P};\mathbb{Q})$. In general, Witten cycles can be extended to a certain combinatorial compactification of the space of fatgraphs, but this compactification is usually badly singular at the boundary, so Poincaré duality does not work well. Here we only discuss what happens in the smooth locus. People might be interested in Witten cycles for a few reasons:

(1) the $\mathcal{W}_{m_*}$'s is the simplest family of cycles on $\mathcal{M}_{g,P}$ arising from A_∞ algebras (see the construction in [10])
(2) integration over $\mathcal{W}_{m_*}$ is governed by matrix models and so related to integrable hierarchies (see [19], [9] and [3])
(3) the cycles $\mathcal{W}_{m_*}$ can naturally arise in enumeration of branched coverings of Riemann surfaces (see for instance [15])
(4) integration over $\mathcal{W}_{m_*}$ governs the asymptotic for $L \to +\infty$ of the number of simple closed geodesics on S with length $\leq L$ that follow a specific pattern (see Mirzakhani's report in the same volume).

Theorem 1 ([6], [12]). *For every $r \geq 1$,*
we have $\mathcal{W}^q_{2r+3} = 2^{r+1}(2r+1)!!\,\psi_q^{r+1}$ in $H^{2r+2}(\mathcal{C}_{g,P};\mathbb{Q})$.
As a consequence, $\mathcal{W}_{2r+3} = 2^{r+1}(2r+1)!!\,\kappa_r$ in $H^{2r}(\mathcal{M}_{g,P};\mathbb{Q})$.

The case of $\mathcal{W}_5$ had already been proven in [17] (using an explicit expression for the Weil-Petersson Kähler form ω_{WP}, which is proportional to κ_1, see [20]) and [1] (using the relation between matrix models and intersection theory on Witten cycles, see (2) above).

For the general case we have the following.

Theorem 2 ([7],[8], [12]). *Generalized Witten cycles and tautological classes generate the same subring of* $H^*(\mathcal{M}_{g,P};\mathbb{Q})$.
There are explicit (though complicated) formulae to express Witten classes as polynomials in the tautological classes and vice versa.

The second assertion of Theorem 1 intuitively follows from the first one remembering that $(\pi_q)_*(\psi_q^{r+1}) = \kappa_r$ (by definition) and noticing that π_q can be identified to the map that "forgets" q, and thus pushes $\mathcal{W}^q_{2r+3}$ down to $\mathcal{W}_{2r+3}$.

To prove the first assertion of Theorem 1, we construct a deformation retraction $\mathcal{H}_q$ of $\mathcal{M}^{comb}_{g,P\cup\{q\}}$ that shrinks the boundary component q to a vertex: we need to show that $(\mathcal{H}_q)_*(\psi_q^{r+1}) = \dfrac{(r+1)!}{(2r+2)!}\mathcal{W}^q_{2r+3}$, that is the integral of ψ_q^{r+1} along the fibers of $\mathcal{H}_q$ is $(r+1)!/(2r+2)!$ over $\mathcal{W}^q_{2r+3}$ and 0 elsewhere. For dimensional reasons, as we discard terms in the boundary, we only have to integrate over maximal cells corresponding to fatgraphs in which the polygon surrounding the point q is made of exactly $2r+3$ edges that do not identify to each other. Using Kontsevich's explicit representative for ψ_q on $\mathcal{M}^{comb}_{g,P\cup\{q\}}$ ([9]), it turns out that the restriction of ψ_q^{r+1} to a fiber of $\mathcal{H}_q$ is exactly $(r+1)!\,\mathrm{Vol}_{\mathrm{Euc}}$. As the fiber of $\mathcal{H}_q$ is isometric to a standard simplex of dimension $2r+2$, whose volume is $1/(2r+2)!$, we obtain our result.

To prove Theorem 2 one needs to shrink more boundary components, namely one for each nontrivalent vertex. A combinatorial analysis is needed to take care of all the possible mutual configurations of the polygons surrounding the boundary components that are shrunk.

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Sections of the elliptic fibration

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(joint work with Burak Ozbagci)

A Lefschetz fibration on a closed oriented smooth four manifold X is a smooth map $f : X \to S^2$ having only finitely many critical points $Q = \{q_1, q_2, \dots, q_n\}$ such that the restriction of f to Q is one-to-one and that for each $i = 1, 2, \dots, n$ there are complex coordinates (z_1, z_2) about q_i compatible with the orientation of X and there is a complex coordinate z about $f(q_i)$ compatible with the orientation of S^2 so that the restriction of f to a neighborhood of q_i is of the form $f(z_1, z_2) = z_1^2 + z_2^2$. It follows that regular fibers are diffeomorphic to a closed orientable surface of genus g. One can assume that they are connected as well. It also follows that the monodromy about each critical value $f(q_i)$ is a right Dehn twist about a simple closed curve, which is called a vanishing cycle.

After fixing a regular value p_0 and choosing simple loops $\alpha_1, \alpha_2, \dots, \alpha_n$ as a generating set for the fundamental group of $S^2 \backslash f(Q)$ each encircling a critical value such that $\alpha_1 \alpha_2 \cdots \alpha_n = 1$ in $\pi_1(S^2 \backslash f(Q))$, one gets the monodromy representation

$$\phi : \pi_1(S^2 \backslash f(Q)) \to \mathrm{Mod}_g,$$

defined by $\phi(\alpha_i) = t_{a_i}$, where t_{a_i} is the right Dehn twist about the vanishing cycle around α_i and Mod_g is the mapping class group of the surface $\Sigma_g = f^{-1}(p_0)$. It follows that $t_{a_1} t_{a_2} \cdots t_{a_n} = 1$ in Mod_g. It turns out that this relation completely determines the four manifold X.

A result of Gompf [2] asserts that if the fiber genus is at least two, then the manifold X is symplectic. Conversely, Donalson [1] proved that every closed symplectic four manifold Y admits a Lefschetz pencil structure. Thus, $Y\#k\,\overline{\mathrm{CP}^2}$ admits a Lefschetz fibration over S^2 for some k.

Let $\mathrm{Mod}_{g,1}$ denote the mapping class group of Σ_g relative to a marked point. There is an epimorphism $\mathrm{Mod}_{g,1} \to \mathrm{Mod}_g$ whose kernel is isomorphic to the fundamental group of Σ_g. If a relation $t_{a_1}t_{a_2}\cdots t_{a_n} = 1$ in Mod_g can be lifted to a relation $t_{b_1}t_{b_2}\cdots t_{b_n} = 1$ in $\mathrm{Mod}_{g,1}$, this means that the corresponding Lefschetz fibration has a section. Moreover, by the short exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g,1} \to 1$$

one can write $t_{b_1}t_{b_2}\cdots t_{b_n} = t_\delta^k$, where Mod_g^1 is the mapping class group of Σ_g minus an open disc and δ is a curve parallel to the boundary component. Then the self intersection number of the corresponding section of the Lefschetz fibration is $-k$.

On the other hand, it is well known that two degree d curves P and Q in CP^2 intersect at d^2 points. Any other point in CP^2 lies on a unique curve $sP+tQ$, for some $[s:t] \in \mathrm{CP}^1$. This gives a map from CP^2 minus d^2 points to $\mathrm{CP}^1 = S^2$, which cannot be extended to CP^2. Blowing up CP^2 at these d^2 points gives a Lefschetz fibration $\mathrm{CP}^2\#d^2\overline{\mathrm{CP}^2} \to S^2$. The genus of regular fibers of this Lefschetz fibration is $g = \frac{d-1)(d-2)}{2}$.

Let us now consider the case $d = 3$. In this case, the manifold $\mathrm{CP}^2\#9\overline{\mathrm{CP}^2}$ is diffeomorphic to the elliptic surface $E(1)$, regular fibers are diffeomorphic to the 2-torus T, and the monodromy of the Lefschetz fibration is $(t_a t_b)^6 = 1$, where a and b are two simple closed curves on T intersecting each other transversely at one point.

Since the Lefschetz fibration $\mathrm{CP}^2\#9\overline{\mathrm{CP}^2} \to S^2$ has nine disjoint sections (the exceptional spheres) of each having self intersection -1, the relation $(t_a t_b)^6 = 1$ can be lifted to a relation in the mapping class group of the torus with nine boundary components. More precisely, there are twelve nonseparating simple closed curves $a_1, a_2, \ldots, a_{12}$ on the torus with nine boundary components such that $t_{a_1}t_{a_2}\cdots t_{a_{12}} = t_{\delta_1}t_{\delta_2}\cdots t_{\delta_9}$, where δ_i is a simple closed curve parallel to the ith boundary component. We determine these twelve curves. It also follows that this relation cannot be lifted to the mapping class group of the torus with ten boundary components for homological reasons. Liftings to the torus with three boundary components were already known. In the case of four boundary components, the relation is new and has particularly simple form.

Starting with the relation $(t_a t_b)^6 = t_{\delta_1}$ in the mapping class group of a torus with one boundary component, the proof of our result is based on a repeated application of the well-known lantern relation.

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The asymptotic dimension of a curve graph is finite

Koji Fujiwara

(joint work with Gregory Bell, Kevin Whyte)

1. Asymptotic dimension

We review the notion of asymptotic dimension from [Gr93]. Let X be a metric space, and $X = \cup_i O_i$ a covering. For $D \geq 0$, we say that the *D-multiplicity* of the covering is at most n if for any $x \in X$, the closed D-ball centered at x intersects at most n elements of $\{O_i\}_i$. The usual notion of multiplicity is exactly the 0-multiplicity.

The *asymptotic dimension* of the metric space X is at most n if for any $D \geq 0$, there exists a covering $X = \cup_i O_i$ such that the diameter of O_i is uniformly bounded (i.e. there exists C such that for all i, $\mathrm{diam} O_i \leq C$), and the D-multiplicity of the covering is at most $n+1$. We say that the asymptotic dimension of X, *asdim* X, is n if the asymptotic dimension of X is at most n, but it is not at most $n-1$. If such n does not exist, then we define the asymptotic dimension of X to be infinite.

It is an easy but important fact that if two metric spaces are quasi-isometric, then they have the same asymptotic dimension, [Gr93].

Let G be a finitely generated group, and S a finite, symmetric (i.e. $S = S^{-1}$) generating set. Let Γ be the Cayley graph of G with respect to S. The asymptotic dimension of G is defined as the asymptotic dimension of Γ. This definition does not depend on the choice of a finite, symmetric generating set S, because Cayley graphs of G are quasi-isometric to each other, and the asymptotic dimension is a quasi-isometry invariant of metric spaces.

Theorem 1 ([FW]). *Let G be a finitely presented group. The asymptotic dimension of G is one if and only if G contains a free group of rank r with $1 \leq r < \infty$ as a subgroup of finite index.*

2. Curve graphs

Let $S = S_{g,p}$ be a compact, orientable surface such that g is the genus and p is the number of the connected components of the boundary of S. The *curve complex* of S was defined by Harvey [Ha]. The 1-skeleton of the curve complex is called the *curve graph* of S, $C(S)$, so that $C(S)$ is a graph whose vertices are isotopy classes of essential, nonperipheral, simple closed curves in S, and two distinct vertices are joined by an edge if the corresponding curves can be realized by disjoint curves.

We remark that the curve complex of S is quasi-isometric to the curve graph of S, so that they have same asymptotic dimension. In certain sporadic cases $C(S)$ as defined above is 0-dimensional, i.e. when $g = 0$, $p \leq 4$ and when $g = 1$, $p \leq 1$. Unless otherwise mentioned, we assume that $3g-4+p > 0$ in this note. Masur and Minsky [MaMi] showed a remarkable result that $C(S)$ is δ-hyperbolic. A geodesic space X is called *δ-hyperbolic* (for some constant $\delta \geq 0$) if for any geodesic triangle in X, each side is contained in the δ-neighborhood of union of the other two sides ([Gr87]).

Theorem 2 ([BeF]). *The asymptotic dimension of $C(S)$ is finite.*

No upper bound of $C(S)$ has been known. If $3g - 4 + p > 0$, $C(S)$ contains an infinite quasi-geodesic (cf.[MaMi]), therefore $1 \leq \text{asdim}C(S)$.

Theorem 3 ([FW]). *Suppose $g \geq 2$. Then the asymptotic dimension of $C(S_{g,1})$ is at least two.*

Among exceptional cases we excluded, if $g = 1$ and $p = 0$ or 1, we modify the definition of $C(S)$ so that we join two vertices if they are represented by simple closed curves which intersect in one point. Then $C(S)$ is a connected graph, which turns out to be the Farey graph. It is not hard to see that the Farey graph is quasi-isometric to a simplicial tree, so that its asymptotic dimension is one (cf.[BeF]).

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Analysis and Geometry of $\mathbb{CP}^1$ Structures on Surfaces

David Dumas

(joint work with Michael Wolf)

Let S be a compact smooth surface of genus $g > 1$. A complex projective structure (or $\mathbb{CP}^1$ structure) on S is a maximal atlas of charts with values in $\mathbb{CP}^1$ and Möbius transition functions. The space $\mathcal{P}(S)$ of marked $\mathbb{CP}^1$ structures on S can be studied through complex analysis or hyperbolic geometry. Our goal is to discuss the two perspectives separately and then compare them.

For background on $\mathbb{CP}^1$ structures, we refer the reader to [GKM] [Gol] [KT] [McM] [SW] [Tan]. The results described in this talk are presented in [D1] [D2] [DW].

From a complex-analytic perspective, the forgetful map $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ gives $\mathcal{P}(S)$ the structure of a bundle over the Teichmüller space $\mathcal{T}(S)$ of complex structures. Using the Schwarzian derivative, the fiber $P(X) = \pi^{-1}(X)$ can be identified with the vector space $Q(X)$ of holomorphic quadratic differentials on X, and the total space $\mathcal{P}(S)$ with the tangent bundle of Teichmüller space

$$\mathcal{P}(S) \simeq T^*\mathcal{T}(S).$$

This is the *analytic parameterization* of $\mathcal{P}(S)$: a $\mathbb{CP}^1$ structure is determined by its underlying complex structure X and its Schwarzian derivative $\phi \in Q(X)$.

There is another way of looking at the space of $\mathbb{CP}^1$ structures using hyperbolic geometry. Thurston showed that there is a natural *projective grafting map*

$$\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$$

which is a homeomorphism [KT]. Here $\mathcal{ML}(S)$ is the space of measured geodesic laminations, a PL-manifold homeomorphic to $\mathbb{R}^{\dim \mathcal{T}(S)}$. The map Gr associates to $(\lambda, Y) \in \mathcal{ML}(S) \times \mathcal{T}(S)$ a projective surface $\mathrm{Gr}_\lambda Y$, the *grafting of Y by λ*, which is obtained from Y by "thickening" the lamination λ to a Euclidean subsurface. For example, when $\lambda = t\alpha$ is a simple closed geodesic α with weight $t \in \mathbb{R}^+$, the surface $\mathrm{Gr}_{t\alpha} Y$ is obtained from Y by removing α and replacing it with a Euclidean cylinder $\alpha \times [0, t]$.

Thus grafting gives a *geometric parameterization* of $\mathcal{P}(S)$: a $\mathbb{CP}^1$ structures is determined by a measured geodesic lamination $\lambda \in \mathcal{ML}(S)$ and a hyperbolic structure $Y \in \mathcal{T}(S)$.

The main motivation for the results in this talk is the following

Question. *How are the complex-analytic and hyperbolic-geometric parameterizations of $\mathcal{P}(S)$ related? That is, how do X and ϕ determined Y and λ?*

One way to approach this question is to take a fiber in the complex-analytic parameterization and look at its grafting coordinates. To that end, define

$$M_X = \mathrm{Gr}^{-1}(P(X)) = \{(\lambda, Y) \in \mathcal{ML}(S) \times \mathcal{T}(S) \mid \pi(\mathrm{Gr}_\lambda Y) = X\}.$$

This is the set of pairs (λ, Y) that determine (via grafting) $\mathbb{CP}^1$ structures with underlying complex structure $X \in \mathcal{T}(S)$.

It turns out that $M_X \subset \mathcal{ML}(S) \times \mathcal{T}(S)$ looks like a graph over each of the factors $\mathcal{ML}(S)$ and $\mathcal{T}(S)$, at least on a large scale. Let $p_{\mathcal{ML}} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{ML}(S)$ and $p_{\mathcal{T}} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$ denote the natural projections. Then we have:

Theorem 1. *For each $X \in \mathcal{T}(S)$, the restrictions $p_{\mathcal{ML}} : M_X \to \mathcal{ML}(S)$ and $p_{\mathcal{T}} : M_X \to \mathcal{T}(S)$ are proper maps of degree* 1.

This result follows from the work of Tanigawa on grafting [Tan] combined with an asymptotic relationship between the two projections. To explain the latter more precisely, let $\overline{\mathcal{ML}(S)}$ denote the projective compactification of $\mathcal{ML}(S)$ and $\overline{\mathcal{T}(S)}$

the Thurston compactification of Teichmüller space. Each of these has boundary $\mathbb{P}\mathcal{ML}(S)$. Given $X \in \mathcal{T}(S)$, there is a natural involution $i_X : \mathbb{P}\mathcal{ML}(S) \to \mathbb{P}\mathcal{ML}(S)$, the *antipodal involution relative to* X, that exchanges laminations corresponding to the vertical and horizontal foliations of holomorphic quadratic differentials on X (see [D1, §4] for details).

Theorem 2 ([D1])**.** *For each* $X \in \mathcal{T}(S)$*, the boundary of* M_X *in* $\overline{\mathcal{ML}(S)} \times \overline{\mathcal{T}(S)}$ *is the the graph* $\Gamma(i_X) \subset \mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$*.*

In other words, a pair $([\lambda], [\mu]) \in \mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$ is a limit point of M_X if and only if there is a holomorphic quadratic differential on X whose vertical and horizontal foliations are equivalent to representatives λ and μ of the projective classes, respectively.

While the previous theorem involves only the underlying complex structure X, there is also a relationship between the Schwarzian derivative of a $\mathbb{CP}^1$ structure on X and its grafting coordinates:

Theorem 3 ([D2])**.** *Let* $\mathrm{Gr}_\lambda Y \in P(X)$ *be a* $\mathbb{CP}^1$ *structure with Schwarzian derivative* $\phi \in Q(X)$*. Let* $\psi \in Q(X)$ *be the unique holomorphic quadratic differential whose horizontal foliation is equivalent to* λ*. Then*

$$\|2\phi - \psi\|_{L^1(X)} \leq C(X).$$

In other words, the measured foliation of X coming from the Schwarzian (suitably normalized) is approximately equal to the one coming from the grafting lamination. Note that the existence of a quadratic differential with any given trajectory structure (i.e. ψ in Theorem 3) is a theorem of Hubbard and Masur [HM].

While the preceding results concerned the large-scale structure of M_X, we can also say something about its local structure. Let us introduce the conformal grafting map $\mathrm{gr} = \pi \circ \mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$. Thus we have $M_X = \mathrm{gr}^{-1}(X)$.

Theorem 4 (Scannell-Wolf [SW])**.** *For each* $\lambda \in \mathcal{ML}(S)$*, the conformal* λ*-grafting map* $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ *is a diffeomorphism.*

Corollary 5. *The projection* $p_{\mathcal{ML}} : M_X \to \mathcal{ML}(S)$ *is a homeomorphism.*

Proof. Its inverse is the map $\lambda \mapsto \mathrm{Gr}_\lambda\left((\mathrm{gr}_\lambda)^{-1}(X)\right)$. □

It would be natural to hope for a similar result about the Y-grafting map $\lambda \mapsto \mathrm{gr}_\lambda Y$. However, the lack of a differentiable structure on $\mathcal{ML}(S)$ complicates matters. In joint work with Wolf, we show:

Theorem 6 (D.-Wolf [DW])**.** *For each* $Y \in \mathcal{T}(S)$*, the conformal* Y*-grafting map* $\mathrm{gr}_{\cdot} Y : \mathcal{ML}(S) \to \mathcal{T}(S)$ *is a tangentiable diffeomorphism (and in particular, it is a homeomorphism).*

Here a *tangentiable* map is one with one-sided derivatives everywhere, and in which the convergence is uniform over the set of tangent rays at a point. Bonahon showed that grafting is a tangentiable map (see [Bon]), making Theorem 6 the natural complement to the Scannell-Wolf result. As before there is a corollary about M_X:

Corollary 7. *The projection $p_{\mathcal{T}} : M_X \to \mathcal{T}(S)$ is a homeomorphism.*

Combining the results above, we have that for each $X \in \mathcal{T}(S)$ the manifold $M_X \subset \mathcal{ML}(S) \times \mathcal{T}(S)$ is properly embedded as a graph over each factor, and its boundary in $\overline{\mathcal{ML}(S)} \times \overline{\mathcal{T}(S)}$ is determined (explicitly) by the complex structure of X. This can be seen as evidence of an overall compatibility between the complex-analytic and hyperbolic-geometric coordinate systems for $\mathcal{P}(S)$.

Finally, we remark that the results described here focus on the case of a fixed underlying complex structure X. It would be interesting to study the relation between the grafting and complex-analytic perspectives when the complex structure is varied, and in particular the limiting behavior as $X \to \infty$.

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Circle Packings on Projective Riemann Surfaces

Sadayoshi Kojima

(joint work with Shigeru Mizushima and Ser Peow Tan)

A projective structure on a surface is, by definition, a geometric structure modeled on the pair of the Riemann sphere $\hat{\mathbb{C}}$ and the projective linear group $\mathrm{PGL}(2, \mathbb{C})$ acting on $\hat{\mathbb{C}}$ by projective transformations. Hence it is in particular a complex structure, but finer than the complex structure up to conformal equivalence. We thus would like to call a surface with a projective structure a *projective Riemann surface* for short.

A circle is a fundamental object in dimension 1, since a projective transforamtion sends a circle on $\hat{\mathbb{C}}$ to a circle on $\hat{\mathbb{C}}$. This is despite the fact that $\mathrm{PGL}(2,\mathbb{C})$ does not preserve any metric on the Riemann sphere. Thus, circles on a projective Riemann surface are not metric circles in the usual sense, but, they are homotopically trivial simple closed curves which develop circles in $\hat{\mathbb{C}}$ via the developing map.

Suppose we are given a closed orientable surface Σ_g of genus $g \geq 2$ without any auxiliary structure, and a graph τ on Σ_g which lifts to an honest triangulation of the universal cover $\widetilde{\Sigma}_g$. We are interested in the moduli space of all pairs (S, P) consisting of a projective Riemann surface S with a reference homeomorphism $h : \Sigma_g \to S$ and a circle packing P on S whose nerve is isotopic to $h(\tau)$.

In [4], we have shown that this moduli space can be identified with, what we call, the cross ratio parameter space $\mathcal{C}_\tau$. $\mathcal{C}_\tau$ contains a unique example of a hyperbolic surface admitting a packing associated with τ, observed in [3, 1, 11], and hence $\mathcal{C}_\tau$ is certainly non-empty. Also it is shown to be a semi-algebraic set, however the geometry and topology of $\mathcal{C}_\tau$ is not quite clear in general. Thus we would like to relate $\mathcal{C}_\tau$ with the other spaces for better understanding.

Let $\mathcal{P}_g$ be the space of all projective Riemann surfaces homeomorphic to Σ_g up to marked projective equivalence. In other words, it is the space of all marked projective structures on Σ_g. To each pair $(S, P) \in \mathcal{C}_\tau$, assign its first component and we obtain the forgetting map,

$$f : \mathcal{C}_\tau \to \mathcal{P}_g.$$

Also assigning the underlying complex structure to each projective Riemann surface, we obtain the uniformization map

$$u : \mathcal{P}_g \to \mathcal{T}_g,$$

of $\mathcal{P}_g$ to the Teichmüller space $\mathcal{T}_g$, the space of all complex structures on Σ_g up to marked conformal equivalence. By taking the Schwarzian derivative of the developing map, a projective structure can be identified with a holomorphic quadratic differential over the underlying Riemann surface, so the uniformization map is a complex affine space bundle of rank $3g - 3$ over $\mathcal{T}_g$.

Motivated by an earlier work of the second author in [8], we would like to conjecture that the composition

$$u \circ f : \mathcal{C}_\tau \to \mathcal{T}_g$$

is a homeomorphism. In the talk, we have reported the following theorem which combines the results in [4, 5, 6]

Theorem : *Let τ be a graph on Σ_g ($g \geq 2$) with one vertex which lifts to an honest triangulation of $\widetilde{\Sigma}_g$ and $\mathcal{C}_\tau$ the cross ratio parameter space associated with τ. Then, $\mathcal{C}_\tau$ is homeomorphic to a real euclidean space of dimension $6g - 6$ and the composition $u \circ f : \mathcal{C}_\tau \to \mathcal{T}_g$ is proper.*

What is missing for the proof of the conjecture even in this special case is the local injectivity of $u \circ f$. This sort of question for the grafting map based on

Tanigawa's properness theorem in [10] was settled by Scannell and Wolf in [9]. See also Faltings [2] and McMullen [7] for earlier proofs of special cases. Dumas has reported some related results obtained with Wolf in the workshop too. However, it is not clear if the proofs in the above cited results can be extended to our setting.

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Teichmüller curves and Veech groups of special translation surfaces

Gabriela Schmithüsen

1. Teichmüller curves and Veech groups

1.1. Teichmüller curves. Teichmüller curves are one-dimensional subvarieties of the moduli space M_g that fit naturally to the complex structure as well as to the Teichmüller metric on Teichmüller space T_g. They are defined as follows.

Definition 1. Let $\iota : \mathbf{H} \hookrightarrow T_g$ be an embedding of the upper half plane $\mathbf{H}$ that is holomorphic as well as isometric with respect to the Poincaré metric on $\mathbf{H}$ and the Teichmüller metric on T_g.

a) Its image $\Delta = \iota(\mathbf{H}) \subseteq T_g$ is called a *Teichmüller disk*.
b) If the image of Δ in M_g under the natural projection $T_g \to M_g$ is an algebraic curve, then this curve is called *Teichmüller curve*.

There are two questions that immediately occur:

(1) How can one decide whether a Teichmüller disk descends to an algebraic curve in the moduli space?

(2) What type of algebraic curves does one get as Teichmüller curves?

A possible way to answer these two questions is provided by a certain subgroup of the mapping class group called the *Veech group* (see below).

1.2. **Teichmüller disks.** Let X be a Riemann surface and let q be a holomorphic quadratic differential on X. The pair (X, q) defines a Teichmüller disk in the following way: The differential q naturally determines a flat structure on X, i.e. an atlas such that all transition maps are of the form $z \mapsto \pm z + c$ (with $c \in \mathbb{C}$ some constant). $\mathrm{SL}_2(\mathbf{R})$-variation of the flat structure defines a map to Teichmüller space in the following way:

$$\mathrm{SL}_2(\mathbf{R}) \to T_g, \quad A \mapsto [(X_A, \mathrm{id})],$$

where X_A is obtained from X by composing each chart of the flat atlas with the affine map $z \mapsto A \cdot z$. This map from $\mathrm{SL}_2(\mathbf{R})$ to T_g factors through $\mathrm{SO}_2(\mathbf{R})$ and thus defines an embedding $\iota : \mathbf{H} \hookrightarrow T_g$ that is in fact holomorphic and isometric.

1.3. **Veech groups.** In order to describe the image of a Teichmüller disk Δ under the natural projection $\mathrm{proj} : T_g \to M_g$, one shall study $\mathrm{Stab}(\Delta)$, the subgroup of the mapping class group consisting of all elements that map Δ to itself. The restriction of the projection proj to Δ factors as follows:

$$\mathrm{proj}|_\Delta : T_g \supseteq \Delta \to \Delta/\mathrm{Stab}(\Delta) \to \mathrm{proj}(\Delta) \subseteq M_g$$

Now, $\mathrm{proj}(\Delta)$ is an algebraic curve C iff this is true for the quotient $\Delta/\mathrm{Stab}(\Delta)$. In this case the map $\Delta/\mathrm{Stab}(\Delta) \to C = \mathrm{proj}(\Delta)$ is a morphism of degree one. Hence, $\Delta/\mathrm{Stab}(\Delta)$ is the normalization of C.

Identifying Δ with $\mathbf{H}$, the stabilizing group $\mathrm{Stab}(\Delta)$ acts as subgroup of $\mathrm{Aut}(\mathbf{H}) = \mathrm{PSL}_2(\mathbf{R})$. For a Teichmüller disk defined by a pair (X, q) as above this subgroup is (almost) equal to the projective Veech group, which is defined as follows.

Definition 2. Let $\mathit{Aff}^+(X, q)$ be the group of diffeomorphisms that are affine with respect to the flat structure defined by q, i.e. locally of the form

$$z \mapsto Az + c \quad \text{with} \quad A \in SL_2(\mathbf{R}), \ \ c \in \mathbb{C}.$$

Note that up to the sign, the matrix A does not depend on the charts.

The *(projective) Veech group* $\Gamma(X, q)$ is the image of $\mathit{Aff}^+(X, q)$ in $PSL_2(\mathbf{R})$.

One has the following fact (see e.g. [6], [1, Thm.1]):

(X, q) defines a Teichmüller curve $C \iff \mathbf{H}/\Gamma(X, q)$ is an algebraic curve

$\iff \Gamma(X, q)$ is a lattice in $\mathrm{PSL}_2(\mathbf{R})$.

In this case, the normalization of C is antiholomorphic to $\mathbf{H}/\Gamma(X, q)$.

2. Particular examples: Origamis

2.1. **Definition.** One way to obtain closed surfaces X together with a flat structure μ is provided by the following construction. Take finitely many copies of the Euclidean unit square in the plane and glue their edges by translations respecting the following rules: Each left edge shall be glued to a right one, each upper edge to a lower one and the resulting surface shall be connected.

The surface X carries a natural flat structure μ_I defined by the Euclidean unit squares. Actually it is even a translation structure; therefore in Definition 2 the matrices are well defined in $\mathrm{SL}_2(\mathbf{R})$ and we may consider the Veech group as subgroup of $\mathrm{SL}_2(\mathbf{R})$. The translation surfaces obtained in this way are called *origamis* (motivated by the idea that they are defined by a few combinatorial data, see [3]) or *square tiled surfaces*.
The squares naturally define a covering p from X to $\mathbb{C}/\Lambda_I$, where Λ_I is the unit lattice $\mathbf{Z} \oplus \mathbf{Z}i$.

It is ramified over at most one point. One may consider the map p as covering between the topological surface underlying X and the torus; one may then, for all lattices Λ_A ($A \in \mathrm{SL}_2(\mathbf{R})$) identify the torus with $\mathbb{C}/\Lambda_A$ and lift this translation structure via p. In this way one obtains precisely the $\mathrm{SL}_2(\mathbf{R})$-variation described above that leads to a Teichmüller disk. It is determined by p regarded as topological covering. This motivates the following definition.

Definition 3. Let X be a closed topological surface and E a torus. An *origami* is a covering $O = (p : X \to E)$ that is ramified over at most one point.

We study the Veech groups $\Gamma(O) = \Gamma(X, \mu_I)$ and the Teichmüller curves defined in this way. The Veech group of $\mathbb{C}/\Lambda_I$ itself is $\mathrm{SL}_2(\mathbf{Z})$, and it is well known that the Veech group of an origami is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$. However, so far there is no general result which ones occur.

2.2. **Some results on Veech groups of origamis.** In our work (see [4], [5]) we develop the following access to origamis: Removing the ramification point ∞ on E and all its preimages on X, one obtains an unramified covering

$$p : X^\star \;\to\; E^\star \quad \text{with} \quad E^\star = E - \{\infty\},\;\; X^\star = X - p^{-1}(\infty).$$

This induces an inclusion $U = \pi_1(X^\star) \hookrightarrow \pi_1(E^\star) = F_2$, where F_2 is the free group on two generators. This description of origamis allows the following characterisation of their Veech groups.

Theorem 4. *Let $O = (p : X \to E)$ be an origami and $U \subseteq F_2$ as above. Define* $\mathrm{Stab}(U) = \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(U) = U\}$ *and let* $\beta : \mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) = \mathrm{SL}_2(\mathbf{Z})$ *be the natural projection. Then* $\Gamma(O) = \beta(\mathrm{Stab}(U)) \subseteq \mathrm{SL}_2(\mathbf{Z})$.

Using this characterisation it was possible to decide for a large class of congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ that they are Veech groups.

Theorem 5. *Let $B := \{b_1, \dots, b_k\}$ be a partition of $(\mathbf{Z}/n\mathbf{Z})^2$ and define* $\mathrm{Stab}(B) = \{A \in \mathrm{SL}_2(\mathbf{Z}) | A \cdot b_i = b_i$ *for all* $i\}$. *Then* $\mathrm{Stab}(B)$ *is a Veech group.*

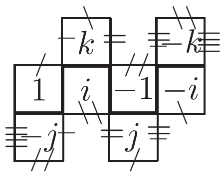
From this one can deduce the following result for $n = p$ prime.

Theorem 6. *Let p be prime. Then each congruence group Γ of level p is a Veech group except (possibly): $p \in \{2, 3, 5, 7, 11\}$ and the index* $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma] = p$.

This statement can be generalized to arbitrary n (see [5]). One might ask, whether all Veech groups are congruence groups. But this is not at all the case, see the following theorem.[1]

Theorem 7. *For each genus $g \geq 2$ there is an origami in M_g whose Veech group is a non congruence group.*

2.3. **The Teichmüller curve of an extraordinary origami (joint work with F. Herrlich).** In [2] we studied the Teichmüller curve of the following origami W of genus 3:



The origami W

W has several nice properties. One of them is that its Veech group is $\mathrm{SL}_2(\mathbf{Z})$. We proved the following result for its Teichmüller curve in M_3.

Theorem 8. *The Teichmüller curve to W is intersected by infinitely many other Teichmüller curves all coming from origamis.*

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Billiards

Martin Möller

(joint work with Irene Bouw)

We are interested in billiard tables, i.e. in planar rational-angled polygons that are *dynamically optimal* in the following sense. For each direction trajectories starting in that direction have one the following properties independently of the starting point: Either the trajectory is closed (or connects two corners) or else the trajectories are uniformly distributed.

Unfolding such a billiard by reflections along its sides yields a Riemann surface X_0 together with a holomorphic one-form ω. Veech has exhibited in [6] the first series of dynamically optimal billiard tables besides the rectangular table and its

[1] An other large class of examples all in genus 2 was given by Hubert and Lelièvre

coverings. Attached to the pair (X_0, ω) there is a Fuchsian subgroup Γ of $\mathrm{PSL}_2(\mathbf{R})$, called the *projective affine group.* Veech shows that the property 'dynamically optimal' is implied by (and in fact not far from equivalent to, see [3] and [5]) the projective affine group Γ being a lattice in $\mathrm{PSL}_2(\mathbf{R})$. The pair (X_0, ω) is then called a *Veech surface.*

Dynamically optimal billiard tables and Veech surfaces are rare. Veech original series was derived from billiards in a $(\pi/n, \pi/2, (n-2)\pi/2n)$-triangle and the projective affine group is the triangle group $\Delta(2, n, \infty)$ for n odd. Ward studied $(\pi/n, \pi/2n, (2n-3)\pi/2n)$-triangles and found the projective affine group $\Delta(3, n, \infty)$. An infinite series of Veech surfaces with genus $g(X_0) = 2$ generated by L-shaped billiard tables was discovered by McMullen (and by Calta independently). He also showed that the projective affine groups of these L-shaped tables are almost never triangle groups. A variant of this construction also yields series of Veech surfaces for $g(X_0) \leq 5$ whose projective affine group is again almost never a triangle group. Up to coverings and the natural action of $\mathrm{SL}_2(\mathbf{R})$ on the set of Veech surfaces these were the only known Veech surfaces besides a small number of sporadic examples. Moreover many other triangles, e.g. all acute triangles but the above, were shown not to yield Veech surfaces.

We show in [1] that the impression 'triangular tables yield triangle groups' that one might get from looking at the first known examples is unjustified. In fact, the following series of tables $T(5, n, \infty)$ is also dynamically optimal. They may be scaled such that $|I_4| = 1$ and they are determined by $\alpha = \beta = \pi/n$, $\gamma = \pi/2n$ and $\mathrm{Re}(I_3) = \cos(\pi/n) + \cos(\pi/5)$. The corresponding projective affine groups are

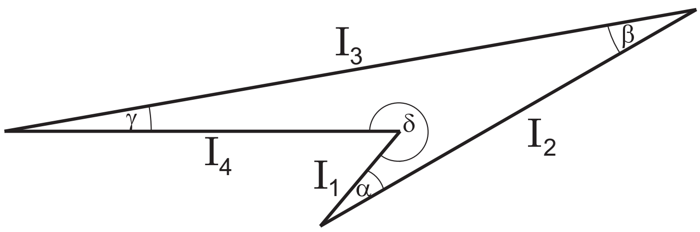

FIGURE 1. Billiard table $T(5, n, \infty)$, for $n = 9$

the triangle groups $\Delta(5, n, \infty)$. More generally, all triangle groups $\Delta(m, n, \infty)$ of hyperbolic signature arise as projective affine groups of Veech surfaces. Remark that projective affine groups are never cocompact ([6]).

Our construction is in fact not geometric but algebraic. The $\mathrm{SL}_2(\mathbf{R})$-orbits of Veech surfaces are curves in the moduli space of curves, called *Teichmüller curves.* These curves have been characterized in [4] by properties of the variation of Hodge structures (VHS) of an associated fibred surface. In [1] we rephrase this criterion as follows:

A stable model of a fibred surface $f : \overline{X} \to \overline{C}$ comes from a Teichmüller curves if and only if the VHS contains a subsystem $\mathbb{L}$ of rank two with Fuchsian monodromy group such that the set of singular fibres coincides with the set of points where the monodromy of $\mathbb{L}$ is infinite. We remark aside that more careful analysis of the

Kodaira-Spencer map of this fibred surface allows us to calculate certain invariants of the geodesic flow on the Teichmüller curves, called Lyapunov exponents.

There is a well-known family of cyclic 4-point coverings of $\mathbb{P}^1$ whose VHS contains a subsystem of rank two with monodromy group equal to $\Delta(m, n, \infty)$. In fact, the corresponding differential equation is a hypergeometric differential equation. Unless $m = n = \infty$ this family of curves does not come from a Teichmüller curve since it has singular fibres at places of $\overline{C}$ where the monodromy of $\mathbb{L}$ is finite. But a subgroup isomorphic to $(\mathbf{Z}/2)^2 \subset \mathrm{Aut}(\mathbb{P}^1 \setminus \{0, 1, t, \infty\})$ lifts to the cyclic covering. The corresponding quotient family still has a suitable local subsystem and fewer singular fibres. It thus satisfies the above criterion for being a Teichmüller curve.

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Mapping Class Groups and Curve Complexes

Elmas Irmak

Let R be a compact, connected, orientable surface of genus g with p boundary components. The mapping class group, Mod_R, of R is the group of isotopy classes of orientation preserving homeomorphisms of R. The extended mapping class group, Mod_R^*, of R is the group of isotopy classes of all (including orientation reversing) homeomorphisms of R. The combinatorial structure of several curve complexes on surfaces are studied to get information about the algebraic structure of the mapping class groups. One of these complexes, introduced by Harvey [H], is defined as follows: Let $\mathcal{A}$ denote the set of isotopy classes of nontrivial simple closed curves on R. The *complex of curves*, $\mathcal{C}(R)$ is an abstract simplicial complex, with vertex set $\mathcal{A}$ such that a set of n vertices $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ forms an $n-1$-simplex if and only if $\alpha_1, \alpha_2, ..., \alpha_n$ have pairwise disjoint representatives.

Definition: A simplicial map $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is called **superinjective** if the following condition holds: if α and β are two vertices in $\mathcal{C}(R)$ such that the geometric intersection number of α and β, $i(\alpha, \beta)$, is not equal to zero, then $i(\lambda(\alpha), \lambda(\beta))$ is not equal to zero.

Superinjective simplicial maps were first defined and used by the author to find a complete description of injective homomorphisms from finite index subgroups of Mod_R^* to Mod_R^* in 2002, [Ir1]. In our talk, we will give a survey of the author's main results in [Ir1], [Ir2], [Ir3], and also her work on the automorphisms of some curve complexes given in [Ir3] and [IrK]. The main results that will be discussed are as follows:

Theorem 1. *Suppose $g \geq 2$. A simplicial map $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is superinjective if and only if λ is induced by a homeomorphism of R.*

Theorem 2. *Let K be a finite index subgroup of Mod_R^* and f be an injective homomorphism $f : K \to Mod_R^*$. If $g \geq 2$ and R is not a closed surface of genus 2, then f has the form $k \to hkh^{-1}$ for some $h \in Mod_R^*$ and f has a unique extension to an automorphism of Mod_R^*. If R is a closed surface of genus 2, then f has the form $k \to hkh^{-1}i^{m(k)}$ for some $h \in Mod_R^*$ where m is a homomorphism $K \to \mathbb{Z}_2$ and i is the hyperelliptic involution on R.*

Let B be the set of isotopy classes of nonseparating simple closed curves on R. The complex of nonseparating curves, $\mathcal{N}(R)$, is the subcomplex of $\mathcal{C}(R)$ with the vertex set B such that a set of n vertices forms an $n-1$ dimensional simplex if and only if they have pairwise disjoint representatives.

Theorem 3. *Suppose that $g \geq 2$ and R has at most $g-1$ boundary components. Then a simplicial map $\lambda : \mathcal{N}(R) \to \mathcal{N}(R)$ is superinjective if and only if λ is induced by a homeomorphism of R.*

Theorem 4. *Suppose that $g \geq 2$. If R is not a closed surface of genus 2, then $Aut(\mathcal{N}(R)) \cong Mod_R^*$. If R is a closed surface of genus 2, then $Aut(\mathcal{N}(R)) \cong Mod_R^*/\mathcal{C}(Mod_R^*)$.*

The author's work on superinjective simplicial maps was motivated by the work of Ivanov [Iv1] and Ivanov-McCarthy [IvMc]. In [Iv1], Ivanov proved that $Aut(\mathcal{C}(R)) \cong Mod_R^*$, and as an application he proved that every isomorphism between finite index subgroups of Mod_R^* is induced by a homeomorphism of R, i.e. it is of the form $k \to hkh^{-1}$ for some $h \in Mod_R^*$ for most surfaces. Ivanov-McCarthy gave a complete description of injective homomorphisms between mapping class groups of surfaces Mod_R and $Mod_{R'}$, when the maxima of ranks of abelian subgroups of Mod_R and $Mod_{R'}$ differ by at most one in [IvMc]. In particular, they showed that an injective homomorphism of Mod_R to itself is of the form $k \to hkh^{-1}$ for some $h \in Mod_R^*$ for most surfaces.

The author's results generalize Ivanov's results since an automorphism of $\mathcal{C}(R)$ is a superinjective map of $\mathcal{C}(R)$, and they also generalize Ivanov-McCarthy's results that we mentioned. We note that an exceptional case appears for injective homomorphisms from finite index subgroups when R is a closed surface of genus two. In this case, our result is similar to McCarthy's explicit description of automorphisms of Mod_R^* for a closed surface of genus two given in [Mc].

Ivanov's above mentioned theorems were extended to most of the surfaces of genus zero and one by Korkmaz in [K], and independently by Luo in [Luo]. Luo gave a proof by using a multiplicative structure on the set of isotopy classes of nonseparating simple closed curves on R. Our results, Theorem 1 and Theorem 2, were extended to surfaces of genus zero by Bell-Margalit in [BMa], and to surfaces of genus 1 by Berhstock-Margalit in [BeMa].

After our work in superinjective simplicial maps of complex of curves, these maps of separating curve complex were studied by Brendle-Margalit to prove injections from finite index subgroup of K to the Torelli group, where K is the subgroup of Mod_R^* generated by Dehn twists about separating curves, are induced by homeomorphisms [BrMa]. Recently, Shackleton proved that local embeddings between two curve complexes whose complexities do not increase from domain to codomain are induced by surface homeomorphisms. From this he deduces a strong local co-Hopfian result for mapping class groups [Sh].

In our talk, we will also give an outline of the proof of our joint work with Korkmaz, about the automorphism group of the Hatcher-Thurston complex $\mathcal{HT}(R)$ given below. This complex was constructed by Hatcher and Thurston in order to find a presentation for the mapping class group [HT]. It was also used by Harer [Ha] in his computation of the second homology group of mapping class group. We note that a similar result was given by Margalit about the complex of pants decompositions $\mathcal{P}(R)$: $Aut(\mathcal{P}(R)) \cong Mod_R^*$ for most closed surfaces [Ma].

Theorem 5. *Suppose $g > 0$. Then $Aut(\mathcal{HT}(R)) \cong Mod_R^*/\mathcal{C}(Mod_R^*)$.*

We use Schaller's result in the proof of this theorem. Schaller considered the graph $\mathcal{G}(R)$: the vertex set of $\mathcal{G}(R)$ is the set of isotopy classes of nonseparating simple closed curves on R. Two vertices are connected by an edge if and only if their geometric intersection number is one. His main result in [Sc] is the following theorem; we state as much as we use (he defines the graph $\mathcal{G}(R)$ for surfaces of genus zero and one as well): If $g \geq 2$ and R is not a closed surface of genus two, then $Aut(\mathcal{G}(R)) \cong Mod_R^*$.

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Piecewise flat Metrics on Surfaces and the Moduli Space

Marc Troyanov

The role of piecewise flat surfaces in Teichmuller theory has been studied by a number of authors the last 20 years. See in particular B. H. Bowditch [1], D.B.A. Epstein and R.C. Penner [2], F. Fillastre [3], W. Thurston [4], and W.A. [7]. In this presentation, we will show how the theory of deformations of geometric structures (development and holonomy) applied to the case of piecewise flat surfaces leads to some interesting geometric structures on the moduli space of a punctured surface. The details and proofs are given in the paper [6].

We define a *punctured surface* $\Sigma_{g,n}$ to be an oriented, closed connected surface Σ of genus g together with a distinguished set of n pairwise distinct points $p_1, p_2, ..., p_n \in \Sigma_{g,n}$, and we denote by $\Sigma'_{g,n} := \Sigma_{g,n} \setminus \{p_1, p_2, ..., p_n\}$ the same surface with the points p'_j removed. The fundamental group $\pi_{g,n} = \pi_1(\Sigma'_{g,n})$ is a free group on $2g + n - 1$ generators.

A flat metrics with conical singularities on $\Sigma_{g,n}$ of is a flat metric m on $\Sigma'_{g,n}$ such that in the neighbourhood of a p_j, we can introduce polar coordinates (r, φ), where $r \geq 0$ is the distance to p and $\varphi \in \mathbb{R}/(\theta_j\mathbb{Z})$ is the angular variable (defined modulo θ_j). The number θ_j is the *total angle* at the singular point p_k and $\beta_j = \theta_j/(2\pi) - 1$

is called the *order* of the singularity. These metrics have been classified by the author in 1986 [5]:

Theorem 1. *Let $\Sigma_{g,n}$ be a punctured surface with punctures $p_1, p_2, \ldots, p_n$. Fix n real numbers $\beta_1, \beta_2, \ldots, \beta_n \in (-1, \infty)$ satisfying the Gauss-Bonnet condition:*

$$\chi(\Sigma) + \sum_i \beta_i = 0\,,$$

For each conformal structure μ on $\Sigma_{g,n}$, there exists a metric m such that

i) m is a flat metric on Σ having a conical singularity of order β_j at p_j ($j = 1, \ldots, n$);

ii) m belongs to the conformal class μ.

This metric is unique up to a dilation (homothety).

Associated to any flat metrics with conical singularities on $\Sigma_{g,n}$, we have a *developing map* and a *holonomy representation.* These invariant are defined as follow: Consider the punctured surface $\Sigma_{g,n}$ with a fixed flat metric m with conical singularity of order β_j at p_j ($j = 1, \ldots, n$). We conformally have $\Sigma'_{g,n} \simeq \mathbb{U}/\Gamma$ where $\mathbb{U}$ is the unit disk and $\Gamma \subset \mathrm{Aut}(\mathbb{U})$ is a Fuchsian group isomorphic to the fundamental group $\pi_{g,n}$. Thus, the unit disk $\mathbb{U}$ inherits a (incomplete) conformal flat metric $\tilde{m}$. If f_0 is a germ of an isometry near a point $\tilde{z}_0$, to the euclidean plane (identified with $\mathbb{C}$), then we obtain a map $f : \mathbb{U} \to \mathbb{C}$ by analytic continuation from f_0. This map is called *the developing map*, it is a local isometry for the metric $\tilde{m}$ on $\mathbb{U}$ and the canonical metric on $\mathbb{C}$. The corresponding *holonomy* is the unique homomorphism $\varphi_m : \Gamma \to \mathrm{SE}(2)$ such that

$$f(\gamma u) = \varphi_m(\gamma) f(u),$$

here, SE(2) is *special Euclidean group*, i.e. the group of orientation preserving isometries of the Euclidean plane.

Thus, to each flat metric m with conical singularities and germ of isometry f_0, we have associated an element

$$\varphi_m \in \mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2)).$$

Changing the developing map (i.e. the germ f_0) does not affect the conjugacy class of φ_m. Hence to each flat metric, the element

$$[\varphi_m] \in \mathcal{R}(\pi_{g,n}, \mathrm{SE}(2)) = \mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2))/\,\mathrm{SE}(2)$$

is well defined.

Remarks. *A)* If h is a diffeomorphism of Σ preserving the punctures and the orientation, and $m' = h^* m$, then $\varphi_{m'}$ is conjugate to φ_m.

B) If $m'' = \lambda m$ is a dilation of m, then $\varphi_{m''} = \lambda \varphi_m$. Thus

$$[\varphi_m] \in \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2)) = \mathcal{R}(\pi_{g,n}, \mathrm{SE}(2))/\mathbb{R}_+$$

is a well defined invariant of the similarity class of the metric m invariant under any isotopy of Σ preserving the punctures.

C) If m has a conical singularity of order β_j at the puncture p_j, then $\varphi_m(c_j)$ is a rotation of angle $\theta_j = 2\pi(\beta_j + 1)$.

Let us denote by $\Xi = \mathcal{SR}_\beta(\pi_{g,n}, \mathrm{SE}(2))$ the set of equivalent classes of representations $\varphi : \pi_{g,n} \to \mathrm{SE}(2)$ such that $\varphi_m(c_j)$ is a rotation of angle $\theta_j = 2\pi(\beta_j + 1)$ for $j = 1, 2, .., n$. We have associated to each flat metric m on Σ with a conical singularity of order β_j at the puncture p_j a well defined element $[\varphi_m] \in \Xi$.

This element is invariant under any dilation of the metric m and any isotopy. Combining this construction with the previous theorem about the existence of flat singular metrics in each conformal class, we obtain a well defined map :

$$\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi = \mathcal{SR}_\beta(\pi_{g,n}, \mathrm{SE}(2)).$$

Theorem 2. *Ξ has a natural structure of real algebraic variety. If $\beta_j \notin \mathbb{Z}$, then we have*

$$\Xi \simeq \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}$$

Theorem 3. *The map* $\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi$ *is a local homeomorphism.*

Any automorphism of $\pi_{g,n}$ acts on $\mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2))$ by twisting the representation. This leads to a natural action of the pure mapping class group $\mathrm{PMod}_{g,n}$ on $\Xi = \mathcal{SR}_\beta(\pi_{g,n}, \mathrm{SE}(2))$, i.e. we have constructed a natural homomorphism

$$\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\Xi) = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}\,\mathbb{C}$$

Theorem 4. *The map* $\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi$ *is Φ-equivariant.*

The previous results taken together give the following

Theorem 5. *Given a punctured surface $\Sigma_{g,n}$ such that $2g+n-2 > 0$ and $\beta_j > -1$ satisfying the Gauss-Bonnet condition and such that no β_i is an integer, there is a well defined group homomorphism*

$$\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}(\mathbb{C}),$$

and a Φ-equivariant local homeomorphism

$$\mathrm{hol} : \mathcal{T}_{g,n} \to \Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}.$$

In other words, the theorem says that

$$\mathcal{M}_{g,n} = \mathcal{T}_{g,n} / \mathrm{PMod}_{g,n}$$

is a good orbifold with a $(\mathcal{G}, \Xi)$-structure.

In the special case of the punctured sphere, a stronger form of this theorem has been obtained by Deligne and Mostow [8] using some techniques from algebraic geometry and by Thurston [4] using an approach closer to ours.

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Problem Session at Oberwolfach June 1, 2006

S. Morita, A. Papadopoulos, R. C. Penner

This note summarizes a problem session held at Mathematisches Forschungsinstitut Oberwolfach on June 1, 2006, as part of the program on Teichmüller theory. This problem session was chaired by Bill Goldman.

1. Splittings of Jacobians; posed by Cliff Earle

Consider one-dimensional loci in the Teichmüller space of the closed surface of genus two. When are they Teichmüller disks? (In this case if they have large stabilizers, then they are Veech surfaces.) When is the Jacobian $J(R)$ of such a Riemann suface R isomorphic to a product $E \times E'$ of two Jacobians of genus one? Here are examples: Take a matrix

$$U = \begin{pmatrix} na & nb \\ nb & d \end{pmatrix},$$

where $a, b, d, n \in \mathbb{Z}_{>0}$ with determinant $n(ad - nb^2) = 1$ and there is no $X \in SL(2, \mathbb{Z})$ with XUX^t diagonal, and define

$$\mathcal{H}^+(U) = \{\tau U : \tau \in \mathbb{C} \text{ with } \mathcal{I}m\ \tau > 0\}.$$

Then:
a) If R is a Riemann surface of genus two whose Jacobian $J(R)$ splits as a product $E \times E'$, then R has a canonical homology basis so that the period matrix lies in some $\mathcal{H}^+(U)$.
b) Conversely, any element $\tau U \in \mathcal{H}^+(U)$ either corresponds to such a Riemann surface or does not arise from a Riemann surface, i.e., if τU is a period matrix for a Riemann surface R, then $J(R)$ splits as $E_\tau \times E_{n\tau}$.

c) The collection of τ so that $\tau U \in \mathcal{H}^+(U)$ is a period matrix forms a dense open set whose complement is infinite and discrete. If $\mathcal{R}e\ \tau = 0$, then τU is actually a period matrix. Is the imaginary axis the image of a Teichmüller geodesic?
d) The stabilizer of $\mathcal{H}^+(U)$ in $Sp(2, \mathbb{Z})$ contains the group

$$\Gamma_0(U) = \left\{ \begin{pmatrix} pI & qU \\ rU^{-1} & sI \end{pmatrix} : \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(n) \right\},$$

and a sufficently large stabilizer gives a surface of finite type.
e) If $a = d = 2$ and $b = 1$, then this indeed gives a Teichmüller disk. In the next cases $a = 2, b = 1, d \geq 2$, are these again Teichmüller disks?

2. Infinite genus surfaces; posed by John Hubbard:

If R is a Riemann surface of finite type and S is a maximal multicurve, then Fenchel-Nielsen coordinates determine a mapping

$$\tau \mapsto (\log \ell_\tau(s), (t_\tau(s)) = (\log \text{ of } \tau - \text{geodesic length of } s, \text{twist along } s)$$

from the Teichmüller space of R to $\mathbb{R}^S \times \mathbb{R}^S$, where $s \in S$. For a Riemann surface of infinite type and some fixed maximal multicurve, there is again a mapping

$$\tau \mapsto (\log\ [\ell_\tau(s)/\ell_{\tau_0}(s)], t_\tau(s)),$$

for any fixed structure τ_0. By equicontinuity of quasiconformal mappings, the first coordinates lie in ℓ^∞, but in which function space should these generalized twist parameters be understood? Notice that this depends upon the length spectrum insofar as "you can twist a lot along a short curve, but only a little along a long one".

Vladimir Marković comments that this is akin to Bill Thurston's question about the image of earthquake rays, where having bounded transverse measure implies that the image is quasi-symmetric.

Leonid Chekhov comments that the Teichmüller space of an infinite genus surface admits a canonical quantization as follows: Take the braid on m strands corresponding to one half twist, and regard it as a fatgraph with $2(m-2)$ trivalent vertices in the natural way (with $m-2$ vertices on each of the top and bottom). There are then observables G_{ij} that correspond to traversing only bands i and j, for distinct $i, j = 1, \ldots, m$, and their Poisson algebra gives examples of so-called $so_q(m)$ algebras. In the continuum limit as the genus $g = [(m-1)/2]$ tends to infinity, there is a natural limiting Poisson algebra plus its quantization, which are presumably related to Hubbard's question.

3. Homology of compactification; posed by Ralph Cohen

The stable rational homology of Riemann's moduli space is generated by the tautological classes by Madsen and Weiss. What are generators for the stable rational homology of the Deligne-Mumford compactification of moduli space?

Gabriele Mondello asks more directly if the stable classes extend to the boundary for integral coefficients. Nariya Kawazumi mentions that the second rational cohomology of the compactified moduli space was determined by Wolpert and depends on the genus. John Hubbard points out that there are many obvious complex varieties in Riemann's moduli space (e.g., those with at least one Weierstrauss point of weight at least two), and he asks for their expressions in terms of tautological or other classes.

4. Canonical forms; posed by Nariya Kawazumi

There are many "canonical" differential forms representing the first Miller-Morita-Mumford class e_1 on Riemann's moduli space $\mathcal{M}_g$ for the closed surface of genus g just as in Buddhism, there are many gods. Since $H^0(\mathcal{M}_g; \mathcal{O}) = \mathbb{C}$ for $g \geq 3$, we may represent the difference of two such canonical forms as $\partial\bar{\partial}f$, for some potential function f, and we should undertake a serious study of thes e canonical functions f. For instance, Faltings' δ function δ_g arose by comparing two metrics on the Hodge bundle, and the Hain-Reed function β_g was defined by comparing a third metric on some multiple of the Hodge bundle.

5. Teichmüller ζ-functions; posed by Ursula Hamenstädt

Let d denote the Teichmüller distance and MC denote the mapping class group of some surface of finite type, and consider the ζ-type function $\sum_{\gamma\in MC} \mathrm{e}^{-sd(x,\gamma x)}$. Find the critical exponent s, i.e., find the smallest s for which this series converges. Does the series converge at the critical exponent? If the series does indeed converge, is the associated measure equivalent to Lebesgue measure? One might also reasonably ask these same questions for subgroups of MC.

6. Baby Teichmüller space; posed by Volodya Fock

Consider the collection of all functions $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{RP}^1$ so that $f(i) \neq f(i+1)$ for any i (taking indices modulo n) and whose image contains at least three points. Conjugacy classes of such functions modulo $SL(2,\mathbb{R})$ comprise the "baby Teichmüller space". What is the cohomology of this space? For instance, how many components does it have?

7. Forgotten punctures; posed by Maryam Mirzakhani

Let $\mathcal{M}_{g,n}$ denote Riemann's moduli space of the surface of genus g with n punctures, so the Weil-Peterssen volume of $\mathcal{M}_{g,n}$ for fixed g grows as $V(g,n) \sim n!c^n$. Let $\pi_n : \mathcal{M}_{g,n} \to \mathcal{M}_{g,0}$ denote the forgetting the punctures morphism, and suppose that $U \subseteq \mathcal{M}_{g,0}$. Thus in the limit, the WP volume $V_n(U)$ is the ratio of the WP volume of $\pi_n^{-1}(U)$ by $V(g,n)$, and we ask for the limiting value of $V_n(U)$ as $n \to \infty$. In words, what is the asymptotic distribution in $\mathcal{M}_{g,0}$ of forgetting the punctures in $\mathcal{M}_{g,n}$? This is closely related to the distribution of punctured arithmetic surfaces.

8. Families of surfaces; posed by Yoichi Imayoshi

Suppose that R is a Riemann surface of finite type and $\pi : M \to R$ is a holomorphic family of surfaces of type (g, n) over R. If two such families have the same monodromy up to conjugacy, then they are isomorphic. Characterize those monodromies $\pi_1(R) \to MC_{g,n}$ arising from such families. Richard Wentworth points out that only finitely many be realized with fixed topology and complex structure.

9. Minimal vs. holomorphic; posed by Richard Wentworth

When does a holomorphic family as in the previous problem give rise to a holomorphic mapping from the universal cover of M to the Teichmüller space of R? This question is equivalent to distinguishing between minimal and holomorphic surfaces in Riemann's moduli space of R.

10. Are a.e. maps pseudo-Anosov; posed by John Hubbard

Choose a set of generators for the mapping class group. As $n \to \infty$, are a.e. words of length n pseudo-Anosov? It seems this problem or one that is closely related has recently been solved by Igor Rivin.

11. Infinite quotients; posed by Koji Fujiwara

Suppose that $g \in MC(R)$ is a pseudo-Anosov mapping class on the Riemann surface R. If you add to a presentation of $MC(R)$ the relation that $g^n = 1$, for some n, then is the corresponding quotient an infinite group for n sufficiently large? Norbert A'Campo points out that if g is contained in the Torelli group (or, more generally, in the kernel of known representation of $MC(R)$ onto an infinite group G), then the quotient is infinite for any n because it factors through the infinite group $Sp(2g, Z)$ (or G above). However, it seems that his argument does not apply to arbitrary pseudo-Anosov mapping.

Does $MC(R)$ have a quotient that is infinite and pure torsion? viz. Baumslag groups. By Gromov's work, the answer is yes for a non-elementary word hyperbolic group for any g with a sufficiently large n.

12. Commutator lengths; posed by Mustafa Korkmaz

Given a group G and some $g \in [G, G]$ in the commutator, let $c(g)$ denote the minimum n so that g can be written as a product of n commutators. Since the stable commutator length of a right Dehn twist t_a along a simple closed curve a is positive for a closed surface R as was discussed by Dieter Kotschick at this conference, it follows that $c(t_a^n) \geq Kn$, for some constant K depending only on the genus. Now consider a collection of curves $a_1, \ldots a_m$, and ask: For R closed, is there a constant K so that $c(t_{a_1} \cdots t_{a_m}) \geq Kn$?

13. Rigid curves; posed by Gabriele Mondello

Consider the Deligne-Mumford compactification $\bar{\mathcal{M}}_{0,n}$ in the planar case of n times punctured stable surfaces of genus zero. Are there any rigid rational curves C that are not contained in the boundary of $\bar{\mathcal{M}}_{0,n}$? There are Veech curves for instance, but we ask for other examples of such classes of curves which are described explicitly.

14. Arc complexes; posed by Bob Penner

Suppose the surface R has genus g with s punctures and $r \geq 1$ boundary components with at least one distinguished point chosen on each boundary component. Let $Arc(R)$ denote the purest possible mapping class group orbits of projectively weighted arc families in R, where the endpoints of each arc lie among the distinguished boundary points. What is the topological type of $Arc(R)$? In particular, the only non-spherical manifolds arise in the following cases: one distinguished point on each boundary component and either a planar surfaces with $r + s = 4$ and $r = 2, 3, 4$ or the surface of genus one with $r = 2, s = 0$; what are these four special manifolds of respective dimensions 5,7,9,7? It is natural to ask corresponding questions about analogous arc complexes for non-orientable surfaces, cf. Nathalie Wahl's lecture at this conference.

15. Even tautological classes; posed by Shigeyuki Morita

It is known that the odd tautological classes vanish (rationally) on the Torelli group because they come from the Siegel modular group. Prove (or disprove) that the even tautological classes are non-zero on the Torelli group in an appropriate stable range.

16. Thurston's asymmetric metric; posed by Athanase Papadopoulos and Guillaume Théret

Let K denote Thurston's asymmetric metric on Teichmüller space as discussed in Guillaume Théret's talk at this conference. Thurston proved that stretch lines are geodesic for K, and we already understand the limiting behaviour of a stretch line in Teichmüller space. Here are three questions concerning this metric: (1) Study of the behaviour of stretch lines in moduli space. (2) Work out an asymptotic formula linking $K(g, h)$ and $K(h, g)$. Along a stretch line, we suspect a formula reminiscent of the collar formula $\sinh(aK(g, h)) \sinh(bK(h, g)) \simeq c$ with constants a, b, c depending only on the topology of the surface and on a "complexity" of the complete lamination directing the stretch line. (3) The mapping class group is a subgroup of the group of K-isometries. Is the group of K-isometries equal to the mapping class group?

Reporters: Shigeyushi Morita, Athanase Papadopoulos and Robert C. Penner

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