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Applications of Asymptotic Analysis

Organised by
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June 18th – June 24th, 2006

Abstract. This workshop focused on asymptotic analysis and its fundamental role in the derivation and understanding of the nonlinear structure of mathematical models in various fields of applications, its impact on the development of new numerical methods and on other fields of applied mathematics such as shape optimization. This was complemented by a review as well as the presentation of some of the latest developments of singular perturbation methods.

Mathematics Subject Classification (2000): 34Exx, 65-xx,74Pxx.

Introduction by the Organisers

The workshop *Applications of Asymptotic Analysis*, organised by Rupert Klein (Potsdam), Evariste Sanchez-Palencia (Paris), Jan Sokolowski (Nancy) and Barbara Wagner (Berlin) was held June 18th–June 24th, 2006. This meeting was well attended with 46 participants with a broad geographic representation. This workshop was a nice blend of young and senior researchers with various mathematical backgrounds.

The objective of this workshop was to present the new developments of multiple scale asymptotics as they are developed for problems in various fields of application. It brought together experts working in different areas of asymptotic analysis and application fields and initiated exchange of new ideas and discussions of parallel developments.

On the whole the atmosphere of this workshop was very cheerful and characterized by the mutual interest into each others expertise and approach.

The themes of the workshop included:

- Applications in shape optimization, where we discussed new tools like the internal and external topological derivatives and topological variations and their close relationship to basic research in asymptotic analysis of elliptic problems under singular perturbations of boundaries.
- Also, new asymptotic problems that arise in thin shell theory were discussed.
- A major field of application of asymptotic analysis in this workshop turned out to be numerical analysis. Examples within this workshop included new procedures for problems that involve multiple time and spatial scales, where adaptive tools alone are not robust. For boundary layer problems anisotropic finite element methods and the method of asymptotic decomposition of domains point to promising directions to tackle these complex problems. Furthermore, application of asymptotic analysis to finite difference schemes, where the grid spacing is the small parameter, could be shown to be very useful for studying consistency, stability and long-time behavior.
- Naturally, new emerging singular perturbation methods were another focus of discussions. These included methods such as *Gevrey asymptotics* to treat phenomena of *boundary layer resonance* and *logarithmic switchback*. Other new developments that require *asymptotics beyond all orders* analysis were demonstrated for problems that exhibit the Stokes phenomenon, such as for the finger selection problem for Hele-Shaw flow with kinetic undercooling.

Apart from these themes, various other lectures on applications of asymptotic analysis were given, ranging from applications in biology, solid mechanics to fluid mechanics, in particular thin liquid films.

Most of the lectures were given in the morning, with one hour overview lectures, followed by shorter half-hour talks. In the afternoon ample time was left for discussion sessions. Wherever possible the lectures and discussion sessions were grouped according to specific topics, which concerned the interplay of asymptotics and shape optimization (monday), numerical analysis and asymptotics (tuesday), mathematical theory and asymptotic methods (wednesday, thursday) and some new developments in homogenization theory (friday).

Workshop: Applications of Asymptotic Analysis

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Abstracts

Asymptotics of the spectrum for the Steklov problem in a domain with a thin cavity

Serguei A. Nazarov

Let Γ be a simple closed contour on the plane $\mathbb{R}^2 = \{x = (x_1, x_2, x_3) : x_3 = 0\}$ in the space $\mathbb{R}^3$. In its neighborhood $\mathcal{U} \subset \mathbb{R}^3$ we introduce the intrinsic coordinates (n, s, x_3) where s is the arc length and n the orientated distance to Γ. Let also ω be a bounded two-dimensional domain while

$$\Gamma_\varepsilon = \{x \in \mathcal{U} : s \in \Gamma, \eta = (\varepsilon^{-1}n, \varepsilon^{-1}x_3) \in \omega\},$$

where $\varepsilon > 0$ is a small parameter, i.e., Γ_ε implies a thin cavity. Finally, Ω is a domain in $\mathbb{R}^3$ which includes Γ and therefore $\overline{\Gamma_\varepsilon} \subset \Omega$ for a small ε.

The goal of the talk is to describe the asymptotics as $\varepsilon \to +0$ of the eigenpairs $\{\lambda_n(\varepsilon), u_n(\varepsilon, \cdot)\}$ of the Steklov spectral problem

$$\Delta_x u(\varepsilon, x) = 0, \ x \in \Omega(\varepsilon) = \Omega \setminus \Gamma(\varepsilon), \qquad \partial_\nu u(\varepsilon, x) = \lambda(\varepsilon) u(\varepsilon, x), \ x \in \partial\Omega(\varepsilon).$$

Here Δ_x is the Laplace operator and ∂_n the derivative along the outward normal. A distinguishing feature of the problem is that all eigenvalues converge to $+0$ with the same rate $c\varepsilon|\ln\varepsilon|^{-1}$ and, to split them as $\varepsilon \to +0$, it is necessary to construct the two-term asymptotics. Namely, the second asymptotic term of an eigenvalue $\lambda_n(\varepsilon)$ and the main asymptotic term of the corresponding eigenfunction $u_n(\varepsilon, \cdot)$ is influenced by eigenvalues and eigenfunctions of a certain integral operator on the contour Γ of the form

$$J(\gamma; s) = \int\limits_\Gamma \Big(\gamma(\tau) - \gamma(s)\Big) G(\tau, s)\, ds + j(s)\gamma(s)$$

(cf. [1, 2]) where $G(\tau, s)$ is the trace on $\Gamma \times \Gamma$ of the generalized Green function of the Neumann problem in the domain Ω (without a cavity). This is a pseudo-differential operator with the principal symbol $-(2\pi)^{-1}\ln|\xi|$. It is symmetric and if $j(s) < 0$, it is negative definite on the Hörmander function space $H_{ln}(\Gamma)$ generated by the weight function $\mu(\xi) = \big(1+\ln|\xi|+\big|\ln|\xi|\big|\big)^{1/2}$. Thus, the operator J has a countable set of normal real eigenvalues

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \ldots \to -\infty.$$

The corresponding eigenfunctions $\gamma_0, \gamma_1, \cdots \in C^\infty(\Gamma)$ can be orthonormalized in $L_2(\Gamma)$, i.e., $(\gamma_n, \gamma_h)_\Gamma = \delta_{n,h}$ for $n, h = 0, 1, \ldots$. There holds the asymptotic formula $\beta_n = -(4\pi)^{-1}\ln(n/2) + O(1)$ as $n \to \infty$.

Correction asymptotic terms in the first series of eigenvalues depends only on β_n and the length $\mathrm{mes}_1\partial\omega$ of the contour $\partial\omega$. Other series are influenced by the spectrum of the exterior two-dimensional Steklov spectral problem

$$\Delta_\eta w(\eta) = 0, \quad \eta \in \mathbb{R}^2 \setminus \overline{\omega}, \qquad \partial_{\nu(\eta)} w(\eta) = \mu w(\eta), \quad \eta \in \partial\omega,$$

The spectrum of this problem consists of real nonnegative eigenvalues

$$0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots \to +\infty.$$

The corresponding eigenfunctions w_k can be orthonormalized according to $(w_k, w_h)_{\partial\omega} = \delta_{k,h}$, $k, h = 0, 1, \ldots$. There holds the asymptotic decomposition

$$w_k(\eta) = b_k + \widetilde{w}_k(\eta), \quad |\nabla_\eta^j \widetilde{w}_k(\eta)| \le c_{k,j}|\eta|^{-1-j},\ \eta \in \mathbb{R}^2 \setminus \overline{\mathcal{W}},\ j = 0, 1, \ldots,$$

where b_k is a constant and $\mathcal{W}$ a neighborhood of $\overline{\omega}$. In particular, $w_0(\eta) = b_0 = (\mathrm{mes}_1\, \partial\omega)^{-1/2}$.

The asymptotic ansatz for the eigenvalues of the Steklov problem in $\Omega(\varepsilon)$ read

$$\lambda^{kn}(\varepsilon) = \varepsilon \left[\mu_k + M^{kn}(z_n)\right] + O(\varepsilon^{1+\delta}), \quad \delta \in (0, 1),$$

where $k = 1, 2, \ldots$, $n = 1, 2, \ldots$ and M^{kn} is an analytic function in $z_n = \left[(2\pi)^{-1}|\ln\varepsilon| + \beta_n\right]^{-1}$ with $M^{k,n}(0) = 0$. The small asymptotic corrector $M^{kn}(z_n)$ is to be found as a solution of a certain abstract non-linear equation which can be derived in a similar way to [3]. It is possible to construct explicitly several terms of decomposition in $|\ln\varepsilon|^{-1}$ of the first ($k = 0$) series of eigenvalues:

$$\lambda^{0n}(\varepsilon) = \varepsilon\, \mathrm{meas}_1(\partial\omega) \left[\frac{2\pi}{|\ln\varepsilon|} + \frac{4\pi^2}{|\ln\varepsilon|^2}\left(\beta_n - T(\omega)\right)\right] + O\left(\frac{\varepsilon}{|\ln\varepsilon|^3}\right),$$

where $T(\omega)$ is a certain characteristic of the domain ω which vanishes for a circle.

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Numerical methods for shape optimization of variational inequalities

Antoni Żochowski

(joint work with Piotr Fulmański, Antoine Laurain, Jean-Francois Scheid, Jan Sokołowski)

For cases when topology of the domain does not change during shape optimization there exists a well known **speed method**. Otherwise two approaches are used, **material density optimization** and **homogenization method**. Both are related, because admitting fast material density oscillations requires application of the homogenization theory. In our opinion the common drawback of these methods follows from the fact that creation of the void in the domain introduces a non smooth disturbance of the solution. More correct approach should depend on creating a small hole and studying the asymptotic dependence of the solution respect to the diminishing size of the void.

Definition. We consider the an open set $\Omega \subset \mathbb{R}^N$ and the ball $B_\rho(x), x \in \Omega$ such that $\overline{B_\rho(x)} \subset \Omega$. Denote $\Omega_\rho = \Omega \setminus \overline{B_\rho(x)}$ and assume that the state equation (system) satisfies Neumann–like homogeneous condition on $\partial B_\rho(x)$. Let $\mathcal{J}(\Omega)$ be the integral functional depending on the state. Then the **topological derivative** is defined as the following limit if it exists:

$$\mathcal{T}(x) = \lim_{\rho \downarrow 0} \frac{\mathcal{J}(\Omega_\rho) - \mathcal{J}(\Omega)}{|\overline{B_\rho(x)}|}$$

TD provides the information on the infinitesimal variation of the shape functional $\mathcal{J}$ if a small hole is created at $x \in \Omega$. For evaluation of $\mathcal{T}(x), x \in \Omega$ it is sufficient to solve the state equation as well as the appropriate adjoint state equation in the unperturbed domain Ω. This make the approach numerically attractive.

Laplace case - 2D. Let $\Gamma_\rho = \partial B_\rho(x)$ and for simplicity consider the the simple equation in $\mathbb{R}^2$:

$$-\mathrm{div}\nabla u = f \ \text{ in } \ \Omega, \ \ u = g \ \text{ on } \ \Gamma_1, \ \ \partial u_\rho/\partial n = h \ \text{ on } \ \Gamma_2.$$

For the solution u_ρ in Ω_ρ we add condition $\partial u_\rho/\partial n = 0$ on Γ_ρ. The shape functionals we consider are:

$$\mathcal{J}_1(\Omega_\rho) = J_u(\rho) = \int_{\Omega_\rho} F(u_\rho)\, d\Omega, \quad \mathcal{J}_2(\Omega_\rho) = J_g(\rho) = \int_{\Omega_\rho} [\nabla u_\rho \cdot \nabla u_\rho]^p \, d\Omega.$$

Then we get the following result.

Theorem 1. [3] *Assume that $f \in C^1(\Omega)$ and the boundary data (g, h) satisfy some regularity conditions. Then*

$$\mathcal{T}_1(x) = -[\, F(u(x)) + f(x)w(x) + 2\nabla u(x) \cdot \nabla w(x)]$$

$$\mathcal{T}_2(x) = -[\|\nabla u(x)\|^{2p} + f(x)v(x) + 2\nabla u(x) \cdot \nabla v(x)]$$

The functions w, v are the appropriately defined adjoint state variables.

2D elasticity. Let us consider the elasticity equations in the plane domain Ω and the same system in the domain Ω_ρ with the hole $B_\rho(x_0)$. The boundary of the hole is free and the solution is denoted u^ρ. The principal stresses are denoted by $\sigma_I(u), \sigma_{II}(u)$, $\mathrm{tr}\sigma(u) = \sigma_I(u) + \sigma_{II}(u)$ and $a_u = [\sigma_I(u) + \sigma_{II}(u)]_{x=x_0}$, $b_u = [\sigma_I(u) - \sigma_{II}(u)]_{x=x_0}$. The shape functionals are, for isotropic matrix S,

$$J_u(\rho) = \int_{\Omega_\rho} F(u^\rho)\, d\Omega, \quad J_\sigma(\rho) = \int_{\Omega_\rho} \sigma(u^\rho)^T S \sigma(u^\rho)\, d\Omega,$$

Let w and v be adjoint states for these functionals and α, β be angles between principal stress directions for $\sigma(u)$ and $\sigma(w)$ as well as $\sigma(u)$ and $\sigma(v)$ respectively.

Theorem 2. [3] *Assume that $f \in C^1(\Omega; \mathbb{R}^2)$ and satisfy some regularity conditions. Then*

$$\mathcal{T}_u(x_0) = -\,[\, F(u) + f^T w + \frac{1}{E}(\, a_u a_w + 2b_u b_w \cos 2\alpha\,)\,]_{x=x_0},$$

$$\mathcal{T}_\sigma(x_0) = -\,[s_{22}(a_u^2 + 2b_u^2) + f^T v + \frac{1}{E}(\, a_u a_v + 2b_u b_v \cos 2\beta\,)\,]_{x=x_0}.$$

Multiple holes. Consider the Laplace problem in the domain with two balls of radii ρ_1 and ρ_2 removed, denoted $\Omega_{\rho_1\rho_2}$, with domain functionals $I_u(\rho_1,\rho_2)$ and $I_g(\rho_1,\rho_2)$.

Theorem 3. [4] *Assume that $\rho^2=\rho_1^2+\rho_2^2$ is small enough. Then*

$$I_u(\rho_1,\rho_2)=I_u(0,0)+\mathcal{T}I_u(\boldsymbol{x}^1)|B(\boldsymbol{x}^1;\rho_1)|+\mathcal{T}I_u(\boldsymbol{x}^2)|B(\boldsymbol{x}^2;\rho_2)|+o(\rho^2),$$

$$I_g(\rho_1,\rho_2)=I_g(0,0)+\mathcal{T}I_g(\boldsymbol{x}^1)|B(\boldsymbol{x}^1;\rho_1)|+\mathcal{T}I_g(\boldsymbol{x}^2)|B(\boldsymbol{x}^2;\rho_2)|+o(\rho^2).$$

Shape and topology. Let us now change shape and topology simultaneously. We assume that the smooth transformation of the domain, which changes part Γ^V of the boundary (speed method), is given by $T(\tau,\boldsymbol{x})=\boldsymbol{x}+\tau\boldsymbol{\Theta}(\boldsymbol{x})$ As previously, we consider shape functionals of the form

$$I_u(\eta,\tau)=\int_{\Omega_{\rho\tau}}F(u_{\rho\tau})\,dx,\quad I_g(\eta,\tau)=\int_{\Omega_{\rho\tau}}\|\nabla u_{\rho\tau}\|^{2p}\,dx\ ,$$

where $\eta=\pi\rho^2$. The shape derivatives $\mathcal{S}I_u(\Omega;\boldsymbol{\Theta})$ of $I_u(\eta,\tau)$ and $\mathcal{S}I_g(\Omega;\boldsymbol{\Theta})$ for $I_g(\eta,\tau)$, taken at $\eta=0$, $\tau=0$ are well known. The final result gives representation of the variations of shape functionals resulting from both boundary variations and nucleation of small holes.

Theorem 4. [4] *Under some assumptions concerning regularity of data the functionals $I_u(\eta,\tau)$ and $I_g(\eta,\tau)$ have the representation (with $\eta=\pi\rho^2$):*

$$I_u(\eta,\tau)\ =\ I_u(0,0)+\eta\mathcal{T}I_u(\Omega;\boldsymbol{x}^0)+\tau\mathcal{S}I_u(\Omega;\boldsymbol{\Theta})+o(\eta)+o(\tau),$$

$$I_g(\eta,\tau)\ =\ I_g(0,0)+\eta\mathcal{T}I_u(\Omega;\boldsymbol{x}^0)+\tau\mathcal{S}I_g(\Omega;\boldsymbol{\Theta})+o(\eta)+o(\tau).$$

Hence new necessary optimality condition, which are indeed stronger than those given by speed method alone,

$$\mathcal{D}J(\Omega^*;\boldsymbol{\Theta},\boldsymbol{x}^0)(\eta,\tau)=\eta\cdot\mathcal{T}J(\Omega^*;\boldsymbol{x}^0)+\tau\cdot\mathcal{S}J(\Omega;\boldsymbol{\Theta})\geq 0$$

for all $\eta\geq 0$, $\boldsymbol{x}^0\in\Omega^*$ and all admissible $\tau,\boldsymbol{\Theta}$.

Contact problems and Poincare-Steklov operator. The main idea we use to derive the topological derivatives for contact problems is the modification of the energy functional by an appropriate correction term and subsequent minimization of the resulting energy functional over the cone of admissible displacements. We want to take into account the influence of the small hole $B(\rho)$ on the solution of the homogeneous Laplace equation in the annulus $C(\rho,R)$. Assuming the value of u on Γ_R given, it will change the value of $\partial u/\partial n$. In order to evaluate this change we shall use the asymptotic expansions from our earlier works. We introduce the Steklov-Poincaré operator A_ρ for the annulus $C(\rho,R)$ and obtain the expansion

$$A_\rho=A_0+\rho^2B+o(\rho^2)$$

in the operator norm $\mathcal{L}(H^{-1/2}(\Gamma_R),H^{1/2}(\Gamma_R))$. To this end let v_ρ be the solution of

$$\Delta v_\rho=0\ \text{ in }\ C(\rho,R),\quad v_\rho=u\ \text{ on }\ \Gamma_R,\quad \frac{\partial v_\rho}{\partial n}=0\ \text{ on }\ \Gamma_\rho$$

The by definition $A_\rho(u) = \partial v_\rho/\partial n$ on Γ_R. Similarly for the case of elasticity $\mathbf{A}_0(\mathbf{u}) = \sigma(\mathbf{v}).\boldsymbol{n}$ with obvious analogy of BVP's. Then the main result may be summed up as:

Theorem 5. [5] *We have the following expansions with respect to the small parameter* ρ

$$\langle A_\rho(u), u\rangle = \langle A_0(u), u\rangle + +\rho^2\langle B(u), u\rangle + \mathcal{R}(u, u)$$
$$\langle \mathbf{A}_\rho(\mathbf{u}), \mathbf{u}\rangle = \langle \mathbf{A}_0(\mathbf{u}), \mathbf{u}\rangle + \rho^2\langle \mathbf{B}(\mathbf{u}), \mathbf{u}\rangle + \mathcal{R}(\mathbf{u}, \mathbf{u})$$

and $\mathcal{R}(\mathbf{u}, \mathbf{u})$ *as well as* $\mathcal{R}(u, u)$ *are of order* $o(\rho^2)$ *uniformly on bounded subsets of* $\mathbf{H}^{1/2}(\Gamma_R)$.

Explicit and easily computed forms of $B(u)$ and $\mathbf{u}, \mathbf{u})$ are given in [5].

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The Multi-scale Modelling for Thin Structures

Gregory Panasenko

A great number of applied problems contain small parameters. Normally their presence either in the equation or in the domain makes the numerical implementation more complicated, more time and memory consuming. This talk emphasizes the importance of the asymptotic methods studying the behavior of the solution as the small parameter tends to zero. Nevertheless the asymptotic methods are often related to some cumbersome calculations, or they are not too comprehensible for non-specialists. That is why some special numerical methods taking into account the asymptotic behavior of the solution were developed. One of such ideas has been implemented in the numerical schemes uniform with respect to the small parameter [3],[9],[24] or in some projection numerical methods with a special choice of the projection space basis taking into account the regular part of an asymptotic solution [10], [27], (the idea of projection procedure has been widely used in engineering); in the case of multi-scale problems the ideas of the super-elements, of the hierarchic modelling (numerical homogenization) or the two-scale finite element methods is developed (see [26], justified in [6], as well as [2], [8], [23], [28]); another approach is to prescribe some special modified boundary conditions (the so called artificial boundary conditions) in order to increase the accuracy of the

approximate solution [12], [13]. This list of related ideas and methods could be continued.

The method of the partial asymptotic domain decomposition (MAPDD), introduced in [16]-[19], reduces the problem to a **simplified form on some subdomain of regular asymptotic behavior** of the solution (for example, by means of the dimension reduction) **keeping the initial formulation** on a small part of the domain **where the asymptotic behavior is singular** (for example, where the boundary layers are located). Then these two models are **coupled by some special interface conditions** respecting with great accuracy the asymptotic expansion of the solution. These interface conditions are obtained from some projection procedure in the variational formulation where the projection subspace keeps the asymptotic behavior of the solution **out of a boundary layer zone.** It differs this method from some earlier projection methods "imposing" the regular asymptotic behavior in the whole domain. This difference is especially important, for example, in case of thin domains of complex structure (such as the finite rod structures [11], [14], [18]).

One of the important applications of the MAPDD in the homogenization theory is the partial homogenization, which homogenizes the micro-heterogeneous medium at some distance from the boundary keeps the initial formulation near the boundary and prescribes some special interface conditions.

In this talk we remind the method of the partial asymptotic domain decomposition (MAPDD). We start with some model examples; then the general description is given and justified; the main theorem about the estimate of the difference between the exact solution and the solution to the partially decomposed problem is proved. The main theorem is then applied to the modelling of thin structures. The rod structures play an important part in the modelling of the mechanical behavior of the thin structures or thin tube (pipe-wise) structures, in particular of the human or animal blood circulatory system [21]. The dimension reduction in such a modelling is a natural approach, although the full-dimensional models have to be kept in the neighborhoods of the bifurcations or junctions. So the MAPDD gives the asymptotically exact answer what should be the correct interface conditions (some junction conditions appeared in engineering, for example, [25]). Various implementations and applications of the MAPDD will be discussed: micropolar flows in the blood circulatory system [7], flows in thin tube structures with an elastic walls [20], extrusion processes [19], special numerical treatment of boundary layers for heterogeneous rods.

The main goal of the talk is to describe the present state of art and new ideas in the multi-scale modelling in order to establish the links between the specialists in the asymptotic methods and eventually formulate a proposal, for example, in the multi-scale modelling in biology and health (some experience in this field is accumulated as a result of participation of the laboratory LaMUSE in two international cooperation CNRS programms: CNRS-Russian Academy of Sciences PICS no. 3198, CNRS- Academy of Rumania project no. 17033/2004 and in one postdoctoral CNRS programm in the multi-scale modelling for biological structures).

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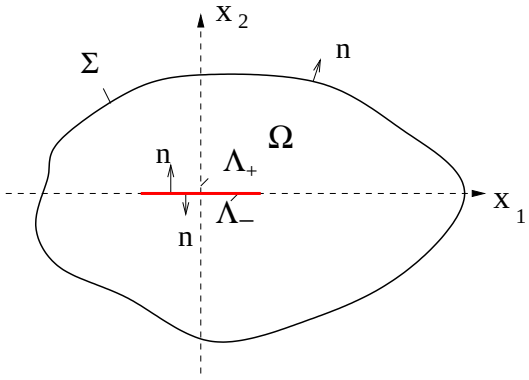
Mathematical models for fracture criteria

Maria Specovius-Neugebauer
(joint work with Sergej A. Nazarov)

The prediction of crack paths obviously is of high practical interest. However, even in the quasistatic case where the process of crack growing is slow in comparison with the wave speed a lot of different criteria are used in praxis. Often they lead to different predictions of the crack path. From the viewpoint of mathematics only the energy criterion of Griffith [3] leads to an axiomatic access: The crack chooses the path in such a way that the total energy is minimal at any moment.



In particular: The crack starts to propagate only if energy is released. Thereby the total energy $T = E - A + S$, where E is the elastic energy, A the work of external forces and S the surface energy which is needed to produce the new crack surface. In this lecture we consider a plane elasticity problem with an initially straight interior crack whose tips are denoted by O^ν, $\nu = 1, 2$, situated in a plane of elastic symmetry:

$$(1) \qquad \begin{aligned} \mathcal{L}u(x) &:= D(-\nabla)^\top A D(\nabla)u(x) &&= f(x), \quad x \in \Omega = \Omega_0 \setminus \Lambda, \\ \mathcal{N}u(x) &:= D(n)^\top A D(\nabla)u(x) &&= g(x), \quad x \in \partial\Omega, \end{aligned}$$

with

$$D(\nabla_x)^\top = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 \end{pmatrix}$$

and A is the symmetric, positive 3×3-Matrix of elastic moduli. Let $\mathbf{R}$ denote the space of rigid motions. It is well known that for $f \in L^2(\Omega)^2$, $g \in H^{1/2}(\partial\Omega)^2$ fulfilling the compatibility condition $\int_\Omega fr\,\mathrm{d}x + \int_{\partial\Omega} gr\,\mathrm{d}o = 0$ for all $r \in \mathbf{R}$ there exists a weak solution u_e uniquely defined up to rigid motions which is the minimizer of the energy functional

$$\mathcal{U}(u; f, g, \Omega) := \frac{1}{2}(D(\nabla)u, AD(\nabla)u)_\Omega - (f, u)_\Omega - (g, u)_{\partial\Omega} = E_e(u) - A(u).$$

We present a mathematical approach which uses a generalized energy functional. It includes a term which can be interpreted as energy concentrated in the tips of the crack. In addition to the usual linear elasticity problem, the Euler equations related to the minimizing problem contain conditions at the tips, which may be nonlinear.

To this end we recall that u_e is not contained in $H^2(\Omega)$ but $u_e \in H^2(\Omega \setminus O_\varepsilon)$ for any small neighborhood O_ε of the crack tips. To describe the behavior of the solution near a tip we need the family of power law solutions solutions to the model problem $\mathcal{L}U = 0$ in $\mathbb{R}^2 \setminus \Lambda_0$, where $\Lambda_0 = \{(x_1, 0) : x_1 < 0\}$, together with the homogeneous boundary conditions $\mathcal{N}U = 0$ on $\Lambda_{0,\pm}$. It is well known that for each $\lambda = k/2$, $k \in \mathbb{Z}$, $k \neq 0$ there correspond two linear independent solutions to the homogeneous model problem of the form $U^{1,2}(r, \phi; \lambda) = r^\lambda \Phi^{1,2}(\phi)$ (r, ϕ polar coordinates in $\mathbb{R}^2$, the index $\nu \in \{1, 2\}$ refers to the tip), while to $\lambda = 0$ there correspond the four solutions e_j, and $T^j(x) = T^{j0} \log r + T^{j1}(\varphi)$, $j = 1, 2$. We choose special bases X^j, Y^j, $j = 1, 2$, corresponding to $\lambda = 1/2$ and $\lambda = -1/2$, respectively, and introduce the enlarged state space $\mathfrak{E}$ consisting of all possible displacement fields $u \in H^2_{loc}(\Omega)^2 \cap L^2(\Omega)^2$ with $\mathcal{N}u = 0$ on $\Lambda_\pm$ and the asymptotic representation

$$u(x) = \tilde{u}(x) + \sum_{\nu,j=1}^{2} \chi(x^\nu)\Big\{c_j^\nu X^j(x^\nu) + b_j^\nu Y^j(x^\nu)\Big\}, \quad \tilde{u} \in H^2(\Omega), \tag{2}$$

the index $\nu = 1, 2$ refers to the tips O^ν, x^ν are local coordinates with center O^ν and χ is a cut-off function with $\chi(x) = 1$ near zero. The space $\mathfrak{E}$ is a Banach space with respect to the norm $\|u; \mathfrak{E}\| = \sum_{\nu,j} |c_j^\nu| + |b_j^\nu| + \|\tilde{u}; H^2(\Omega)\|$. We collect the coefficients c_j^ν, b_j^ν into column vectors b_u, c_u, then with a proper normalization of the bases X^j, Y^j any two fields $u, v \in \mathfrak{E}$ fulfil the generalized Green's formula

$$(\mathcal{L}u, v)_\Omega + (\mathcal{N}u, v)_\Sigma - (u, \mathcal{L}v)_\Omega - (u, \mathcal{N}v)_\Sigma = \langle c_u, b_v\rangle - \langle b_u, c_v\rangle.$$

The subspace in $\mathfrak{E}$ of solutions to the homogeneous problem (1) splits into the rigid motions and an additional space of dimension 4, it is generated by the so called weight functions belonging to $b_u = \mathrm{e}_j$, where e_j is the j-th unit vector in $\mathbb{R}^4$. Thus imposing linear conditions on the coefficient vectors in the form

$$Sb_u + Tc_u = h \in \mathbb{R}^4, \qquad S, T \in \mathbb{R}^{4\times 4}$$

the resulting problem leads to a Fredholm operator with index zero between $\mathfrak{E}$ and $L^2(\Omega) \times H^{1/2}(\Sigma) \times \mathbb{R}^4$. In addition, it is possible to consider nonlinear conditions at the tips. The following theorem is a particular case of a more general result:

Theorem. *Let* $\mathbf{T} : \mathbb{R}^4 \to \mathbb{R}^4$ *a mapping with* $\mathbf{T}(0) = 0$, $\mathbf{T} = \nabla \mathbf{E}$, *and consider the problem* (1) *together with the condition at the tips* $\mathbf{T}(b_u) - c_u = 0$. *Let* Z *denote the so-called polarization matrix composed of the stress intensity factors* [1] *of the weight functions* ζ_j, *i.e. the vectors* $b(\zeta_j)$. *If the mapping* $N \to \mathbf{E}_\Omega(N) := \mathbf{E}(N) - \frac{1}{2}\langle ZN, N\rangle$ *strictly convex and coercive then the problem has a unique solution* $u \in \mathfrak{E}^\perp$, u *is the minimizer of the generalized energy functional*

$$\mathbf{U}(u) = \frac{1}{2}(\mathcal{L}u, u)_\Omega + \frac{1}{2}(\mathcal{N}(u), u)_\Sigma + \mathbf{E}(b_u) - \frac{1}{2}\langle c_u, b_u\rangle - (f, u)_\Omega - (g, u)_\Sigma.$$

Kinking cracks can be modelled within this theory. Further it turns out that the modelling of plastic zones appearing in the Dugdale-criterion, e.g., can be described within this context. The Dugdale criterion itself [4] which is originally a deformation criterion can be interpreted as an energy criterion within this frame.

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Multi-scale stochastic analysis of heterogeneous materials

Adnan Ibrahimegovic

(joint work with Adnan Ibrahimegovic)

The current increase in computational power allows us to completely rethink the modeling of inelastic behavior of engineering materials and the corresponding testing procedure [1-6]. In particular, the traditional phenomenological models are more and more giving way to multi-scale modeling procedures, where one goes down to much smaller scales in order to be able to properly interpret the particular mechanisms of inelastic behavior. One such model, which is built upon the ideas inherited from structural mechanics, is presented in this lecture. The predictive capabilities of this model are illustrated for a couple of challenging problems of dynamic fracture and crack propagation. In the second part of this lecture we address related topics pertinent to problems of material design and testing, as well as the construction of probability bounds for computed results. Moreover, a novel approach to structural optimization that pertains to tailoring the material to best suit the given use is presented, where optimality conditions and equilibrium

equations are placed to the same level. Several important points are addressed in detail regarding the models proposed herein:

i) In this work we consider a strongly coupled multi-scale problem in the context of inelastic structural mechanics. We assume that a finite ratio of scales exists such that we can replace the more standard phenomenological and analytical homogenization approaches by a lower level numerical description of the micro-structural behavior. More specifically, we use continuum damage and plasticity based finite element method (FEM) models to describe the matrix-inclusion type of microstructure (e.g., for a porous or hard inclusion composite). The micro-scale FEM model is then coupled to the macro-scale FEM model through a localized Lagrange multiplier approach. This multi-scale strategy is very well adapted to a parallel computing algorithm using a component template library. The efficiency of the implementation is shown on large scale numerical examples. A more elaborated description of the parallelization procedure is also presented in our paper [1,2]. ii) In our recent work [3] we propose a methodology for dealing with the problem of designing a material microstructure the best suitable for a given goal. The chosen model problem for the design is a two-phase material, with one phase related to plasticity and another to damage. The design problem is set in terms of shape optimization of the interface between two phases. The solution procedure proposed herein is compatible with the multi-scale interpretation of the inelastic mechanisms characterizing the chosen two-phase material and it is thus capable of providing the optimal form of the material microstructure. One can thus achieve the optimal design of the nonlinear behavior of a given two-phase material with respect to the goal specified by a cost function, by computing the optimal form of the shape interface between the phases. The original approach based upon a simultaneous/sequential solution procedure for the coupled mechanics-optimization problem is proposed. Several numerical examples show a very satisfying performance of the proposed methodology. The latter can easily be adapted to other choices of design variables. iii) Our goal in recent paper [4] is to present how to use a part of Stochastic Finite Element Method in mechanical problems. We will focus on the Karhunen-Loève Expansion coupled with Monte Carlo Simulations and its application to describe the size effect encountered in structure involving quasi-brittle materials such as geomaterials. iv) In recent work [5] we discus the finite element model using the embedded discontinuity of strain and displacement field, for dealing with a problem of localized failure in heterogeneous materials by using the structured finite element mesh. On the chosen 1D model problem we develop all the pertinent details of such finite element approximation. We demonstrate the presented model capabilities for representing not only failure states typical of a slender structure, with generalizing linear fracture mechanics crack induced failure in an elastic structure, but also the failure state of a massive structure, with combined diffuse (process zone) and localized cracking. A robust operator split solution procedure is developed for the present model taking into account the subtle difference between the types of discontinuities, where the strain discontinuity iteration is handled within global loop for computing the nodal displacement, while

the displacement discontinuity iteration is carried out within a local, element-wise computation, carried out in parallel with the Gauss-point computations of the plastic strains and hardening variables. The robust performance of the proposed solution procedure is illustrated by a couple of numerical examples. Concluding remarks are stated regarding the class of problems where ED-FEM or X-FEM should be a favorite choice. v) In our recent work [6] we set to develop a model reduction procedure based on statistical modes, which is applicable to nonlinear system under dynamic loads. The system can be characterized by nonlinear inelastic behavior, and contain initially a refined mesh representation which is needed to represent inelastic behavior mechanisms. The reduced model should be capable of achieving the main goal of providing the sufficiently accurate representation of the chosen quantities.

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Variational systems of wave equations

John K. Hunter

(joint work with Giuseppe Alì)

Wave equations. We consider systems of wave equations that are derived from a variational principle in which the Lagrangian is a quadratic function of the derivatives of the wave-field with coefficients depending on the wave-field. The Euler-Lagrange equations consist of a system of second order hyperbolic PDEs whose wave speeds depend on the wave-fields. The PDEs are scale-invariant and nondispersive, but they are not in conservation form, since they include lower-order terms that are quadratic functions of the first-order derivatives of the wave-fields. This class of quasilinear hyperbolic PDEs contrasts with the conservation-form

wave equations that arise in continuum mechanics, such as nonlinear elasticity, where the wave speeds depend on the derivatives of the wave-fields.

The effects of nonlinearity in these variational wave equations are intriguing and differ qualitatively from more familiar phenomena, such as shock waves, in hyperbolic conservation laws. We study the properties of these PDEs by deriving simplified canonical asymptotic equations. This class of equations also provides a useful perspective on the Einstein equations in general relativity.

Director fields. A specific, and somewhat typical, example of such a system is provided by orientation waves in a massive director field. A director field

$$\mathbf{n} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{S}^2$$

is a field of unit vectors. We suppose that the director field has inertia, and that its motion is governed by the variational principle

$$\delta \int_{\mathbb{R}^3} \left\{ \frac{1}{2}\mathbf{n}_t^2 - W\left(\mathbf{n}, \nabla\mathbf{n}\right) \right\} d\mathbf{x}dt = 0, \qquad \mathbf{n}\cdot\mathbf{n} = 1,$$

where $W\left(\mathbf{n}, \nabla\mathbf{n}\right)$ is the Oseen-Frank potential energy density

$$W\left(\mathbf{n}, \nabla\mathbf{n}\right) = \frac{1}{2}\alpha\left(\operatorname{div}\mathbf{n}\right)^2 + \frac{1}{2}\beta\left(\mathbf{n}\cdot\operatorname{curl}\mathbf{n}\right)^2 + \frac{1}{2}\gamma\left|\mathbf{n}\times\operatorname{curl}\mathbf{n}\right|^2.$$

We assume that α, β, γ are distinct positive constants (the coefficients of splay, twist, and bend, respectively).

Up to a null-Lagrangian, $W\left(\mathbf{n}, \nabla\mathbf{n}\right)$ is the most general quadratic function of $\nabla\mathbf{n}$, with coefficients depending $\mathbf{n}$, that is invariant under the transformations $\mathbf{x} \mapsto R\mathbf{x}$, $\mathbf{n} \mapsto R\mathbf{n}$ for all orthogonal maps R. This symmetry involves a simultaneous transformation of the independent and dependent variables. In this respect, these equations are similar to the Einstein equations, whose gauge invariance involves a simultaneous transformation of space-time variables and the metric, and different from gauge field theories on prescribed space-times or wave maps (nonlinear sigma models), which are invariant under independent transformations of the domain and target spaces.

The director-field system has two types of waves, which we call 'splay' and 'twist' waves, respectively. For waves propagating in a direction $\mathbf{k}$ through a constant unperturbed director field $\mathbf{n}_0$, the splay waves carry perturbations of the director field $\mathbf{n}$ in a direction $\mathbf{R}(\mathbf{k}, \mathbf{n}_0)$ that is in the plane of $\{\mathbf{k}, \mathbf{n}_0\}$, while the twist waves carry perturbations in a direction $\mathbf{S}(\mathbf{k}, \mathbf{n}_0)$ normal to $\{\mathbf{k}, \mathbf{n}_0\}$.

We denote the phase velocities of the splay and twist waves by $a(\mathbf{k}, \mathbf{n}_0)$ and $b(\mathbf{k}, \mathbf{n}_0)$ respectively. Then the splay waves are genuinely nonlinear, in the sense that $\nabla_{\mathbf{n}_0} a \cdot \mathbf{R} \neq 0$ (provided that $\mathbf{k}$ is not parallel to $\mathbf{n}_0$, when there is a loss of genuine nonlinearity and strict hyperbolicity), while the twist waves are linearly degenerate, in the sense that $\nabla_{\mathbf{n}_0} b \cdot \mathbf{S} = 0$. The definitions of 'genuinely nonlinearity' and 'linearly degeneracy' for variational wave equations are analogous, but not equivalent, to the corresponding definitions of Lax for first-order systems of hyperbolic conservation laws.

The weakly nonlinear asymptotic expansion for splay waves has the form

$$\mathbf{n}^\varepsilon(\mathbf{x},t) = \mathbf{n}_0 + \varepsilon v\left(\frac{\mathbf{k}\cdot\mathbf{x}-\omega t}{\varepsilon}, t\right)\mathbf{R} + O(\varepsilon^2)$$

as $\varepsilon \to 0^+$, where $\omega = a(\mathbf{k}, \mathbf{n}_0)$. After a normalization of the coefficients, we find that the scalar splay wave amplitude function $v(\theta, t)$ satisfies the Hunter-Saxton (HS) equation [3]

$$\left[v_t + \left(\frac{1}{2}v^2\right)_\theta\right]_\theta - \frac{1}{2}v_\theta^2 = 0.$$

Here, and below, we consider localized waves — for example ones in which v_θ has compact support in θ. (For oscillatory waves, it is necessary to consider additional effects of mean-field interactions.)

The weakly nonlinear asymptotic expansion for the linearly degenerate twist waves has the form

$$\mathbf{n}^\varepsilon(\mathbf{x},t) = \mathbf{n}_0 + \varepsilon^{1/2}u\left(\frac{\mathbf{k}\cdot\mathbf{x}-\omega t}{\varepsilon}, t\right)\mathbf{S} + \varepsilon v\left(\frac{\mathbf{k}\cdot\mathbf{x}-\omega t}{\varepsilon}, t\right)\mathbf{R} + O(\varepsilon^{3/2})$$

as $\varepsilon \to 0^+$, where $\omega = b(\mathbf{k}, \mathbf{n}_0)$. We find that $u(\theta, t)$, $v(\theta, t)$ satisfy a cubically nonlinear asymptotic system,

$$(u_t + vu_\theta)_\theta = 0, \qquad v_{\theta\theta} = u_\theta^2.$$

The twist wave u nonlinearly generates a splay wave v, and the splay wave then affects the propagation speed of the twist wave. Remarkably, the elimination of u from this system leads to a HS-equation for v,

$$\left\{\left[v_t + \left(\frac{1}{2}v^2\right)_\theta\right]_\theta - \frac{1}{2}v_\theta^2\right\}_\theta = 0.$$

General relativity. The vacuum Einstein equations may be obtained from the Einstein-Hilbert variational principle. After an integration by parts, the corresponding Lagrangian is a quadratic function of derivatives $\partial\mathbf{g}$ of the metric $\mathbf{g}$ with coefficients depending on $\mathbf{g}$. The Einstein equations are not hyperbolic owing to their gauge invariance, but they are hyperbolic after fixing a suitable gauge. Thus the Einstein equations belong essentially to the class of variational wave equations considered here.

Gravitational waves are linearly degenerate, in the sense used above, because the wave-speed depends on the metric components in the direction of propagation of the wave, but the waves carry perturbations in the transverse metric components. As a consequence of this linear degeneracy, and the absence of other waves modes, all weakly nonlinear effects in the asymptotic equations for gravitational wave pulses vanish. Thus, the nonlinearity of the Einstein equations is extremely degenerate in comparison with that of the director-field equations.

To capture the direct nonlinear self-interaction of gravitational wave pulses in an asymptotic theory, it is necessary to consider large-amplitude gravitational waves. Such a theory leads to $(1+1)$-dimensional system of nonlinear wave equations that are a generalization of an exact solution of the Einstein equations for

colliding gravitational plane waves [1]. As a further generalization, one may include the effects of diffraction in this expansion [2] leading to a $(1+2)$-dimensional system of nonlinear wave equations. This system should provide insight into the effects of nonlinearity on gravitational waves, such as the formation of space-time singularities.

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Beyond Standard Estimates for Singular Limits

Steve Schochet

In the widely-used theory of singular limits developed by Klainerman and Majda [KM81] at the beginning of the 1980s, uniform estimates for symmetric hyperbolic systems with large constant-coefficient antisymmetric terms are obtained via standard energy estimates. The requirements on the initial data were relaxed in the 1990s (e.g., [Sch94]), but the restrictions that allow the use of standard estimates remained. In this talk I report on a variety of recent results concerning singular limits for which standard estimates do not suffice.

1. Non-Isentropic Flow

Although the equations of isentropic flow have the form

$$A^0(\varepsilon v)v_t + \sum_j A^j(v,\varepsilon)v_{x_j} + \tfrac{1}{\varepsilon}\sum_j C^j v_{x_j} = 0 \tag{1}$$

that [KM81] showed could be treated via standard estimates, the equations of non-isentropic flow have instead the form

$$A^0(v,\varepsilon v)v_t + \sum_j A^j(v,\varepsilon)v_{x_j} + \tfrac{1}{\varepsilon}\sum_j C^j v_{x_j} = 0. \tag{2}$$

The dependence of A^0 on v rather than just on εv means that standard estimates no longer work: Specifically, multiplying (2) by A^{-1}, taking a spatial derivative ∂_{x_k} and multiplying back by A^0 yields an equation that contains the large term $\frac{1}{\varepsilon}A^0\left[\left[(A^0)^{-1}\right]_v v_{x_k}\right]C^j v_{x_j}$ that will not drop out of the energy estimate for $\partial_{x_k}v$.

The following example from [MS01] shows that this breakdown of the method presages the actual nonuniformity of the solution in general: The system

$$S^{-1}(u_t+u_y)+\tfrac{1}{\varepsilon}u_x = 0, \quad S_t+S_y = 0$$

has the form (2) and can be solved explicitly. Assuming for simplicity that the initial data for S depends only on y,

$$S(t,x,y) = S^0(y-t), \qquad u(t,x,y) = u^0\big(x - \tfrac{t}{\varepsilon}S^0(y-t), y-t\big).$$

Hence

$$u_y(t,x,y) = u^0_y\Big(x - \tfrac{t}{\varepsilon}S^0(y-t), y-t\Big) - \tfrac{t}{\varepsilon}S^0_y(y-t)u^0_x\Big(x - \tfrac{t}{\varepsilon}S^0(y-t), y-t\Big),$$

which is not uniformly bounded.

Nevertheless, certain special features of the non-isentropic Euler equations allow uniform estimates to be obtained via a special set of estimates:

Theorem 1. [MS01] *The solution to the non-isentropic Euler equations exists and is uniformly bounded in H^s for a time independent of ε provided that the initial data is uniformly bounded in H^s.*

Some information on asymptotics has also been obtained ([MS01, MS03]). The results have been extended to the full non-barotropic Navier-Stokes equations by Alazard [Ala06].

2. Initial Data in Uniform-Local Sobolev Spaces

A function u belongs to the uniform-local Sobolev space H^s_{ul} if

$$\sup_{x_0} \|\Psi(x-x_0)u(x)\|_{H^s} < \infty, \qquad \text{where } \Psi(y) := e^{-\sqrt{1+y^2}}.$$

The well-posedness of symmetric hyperbolic systems in uniform-local Sobolev spaces was proven in [Kat75].

Are systems of the form (1) uniformly well-posed in H^s_{ul}? Once again, standard estimates do not yield uniform bounds, because the weight function Ψ must be included in the integrals. This difficulty arises even for constant-coefficient equations. In one spatial dimension, Courant-Lax theory yields uniform well-posedness. The following counterexamples show that uniform well-posedness does not hold in higher dimensions in general:

The wave equation $u_{tt} - \frac{1}{\varepsilon^2}\Delta u = 0$ can be written in terms of $v := \varepsilon u_t$ and $w := \nabla u$ as a symmetric-hyperbolic system $v_t - \frac{1}{\varepsilon}\nabla\cdot w = 0$, $w_t - \frac{1}{\varepsilon}\nabla v = 0$. Initial data $v(0,\mathbf{x}) = \cos(|\mathbf{x}|)$, $w(0,\mathbf{x}) = 0$ belong to H^s_{ul} for every s. In spatial dimension $d = 3$, the solution is $u(t,r) = \left[\frac{t}{\varepsilon}\cos\frac{t}{\varepsilon} - \sin\frac{t}{\varepsilon}\right]\frac{\sin r}{r} + \sin\frac{t}{\varepsilon}\cos r$. Hence both u and $w = \frac{\mathbf{x}}{r}u_r$ are of order $O(\frac{t}{\varepsilon})$. The above initial data satisfy the conditions of [Sch94] but not those of [KM81]. However, a similar example in dimension $d = 5$ also satisfies the latter conditions.

3. Non-constant Large Operators

Dutrifoy and Majda ([DM06]) have recently proven uniform well-posedness for a system arising from the shallow water equations in which the large operator includes a variable-coefficient undifferentiated term:

$$v_t + \sum_j A^j(v,\varepsilon)v_{x_j} + \tfrac{1}{\varepsilon}\sum_j C^j v_{x_j} + \tfrac{1}{\varepsilon}D(\vec{x})v = 0.$$

The following simple example illustrates how such a result is possible and emphasizes the essential role of the large constant-coefficient terms.

Let $B(x)$ be a function whose derivatives are all bounded and not identically zero, such as $\sin x$. The solution to the equation

$$v_t - \tfrac{1}{\varepsilon}\left[\lambda v_x + iB'(x)v\right] = 0, \qquad v(0,x) = v_0(x)$$

is

$$v(t,x) = \begin{cases} v_0(x - \lambda t/\varepsilon)e^{i[B(x)-B(x-\lambda t/\varepsilon)]/\lambda}, & \text{if } \lambda \neq 0 \\ v_0(x)e^{iB'(x)t/\varepsilon}, & \text{if } \lambda = 0. \end{cases}$$

When $\lambda = 0$, the k^{th} derivative of the solution grows like $\left(\frac{t}{\varepsilon}\right)^k$. However, for $\lambda \neq 0$, the solution is uniformly bounded in every H^s!

4. A Model Problem for Multiple-Scale Non-isentropic Flow

The non-isentropic Euler equations with multiple spatial scales have been considered in [KBS01]. The following toy equation models some features of that system:

$$a(y,\varepsilon u)\left[u_t + c\left(uu_x + \varepsilon u u_y\right)\right] + \tfrac{1}{\varepsilon}u_x + u_y = 0 \tag{3}$$

When $c = 0$ and $a = a(y)$, the exact solution to the initial-value problem is

$$u(t,x,y) = u_0\left(x - \tfrac{y - A^{-1}(A(y)-t)}{\varepsilon}, A^{-1}(A(y)-t)\right),$$

where $A' = a$, which shows that u_y is not uniformly bounded.

Nevertheless, the solution has the form

$$u(t,x,y,\varepsilon) = v\left(t, x - \tfrac{\mu(t,y)}{\varepsilon}, y, \varepsilon\right), \tag{4}$$

where v does satisfy uniform estimates.

To obtain an analogous result for the full toy equation, substitute (4) into (3) and equate the $O(\frac{1}{\varepsilon})$ terms to zero, which yields

$$-a(y,0)\mu_t + 1 - \mu_y = 0.$$

The profile v then satisfies

$$a(y,\varepsilon v)\left[v_t + vv_x + \mu_y v v_x + \varepsilon v v_y\right] + v_y + \mu_t \tfrac{a(y,0)-a(y,\varepsilon v)}{\varepsilon} v_x = 0. \tag{5}$$

Since (5) contains no large terms, both v and v_t satisfy uniform estimates, which implies the convergence of v to a limit profile satisfying

$$a(y,0)\left[v_t + vv_x + \mu_y v v_x\right] + v_y - \mu_t a_v(y,0) v_x = 0.$$

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Low Mach number flows and the zero electron mass limit

Thomas Alazard

The purpose of this work is to present some recent results in the study of low Mach number flows [1, 2, 3, 6] in connection with the analysis of the zero electron mass limit [4, 5]. The general setting is the analysis of a system depending on a small scaling parameter ε, which is the ratio m_e/m_i of the characteristic mass of an electron an the characteristic mass of an ion. For a plasma made of electrons (charge $q_e = -1$) whose density is denoted by ρ_e and velocity v_e, and ions ($q_i = +1$, density ρ_i, velocity v_i), the governing equations are

$$\begin{cases} \partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha v_\alpha) = 0, & \alpha = e, i, \\ m_\alpha \partial_t(\rho_\alpha v_\alpha) + m_\alpha \operatorname{div}(\rho_\alpha v_\alpha \otimes v_\alpha) + \nabla P_\alpha(\rho_\alpha) = -q_\alpha \rho_\alpha \nabla\phi, & \alpha = e, i, \\ -\Delta\phi = \rho_i - \rho_e - C(x), \end{cases}$$

where C is a given C^∞ function and, for $\alpha = e, i$, P_α is a given non-decreasing C^∞ function.

We are interested in the limit $\varepsilon \to 0$, which is related to the study of the incompressible limit in compressible fluid mechanics. Indeed, at least in the special case where ρ_i and C are constant functions, after the appropriate rescaling (see [5]), we are led to the following quasi-linear system:

$$\begin{cases} g_1(\varepsilon p)(\partial_t p + v \cdot \nabla p) + \varepsilon^{-1} \operatorname{div} v = 0, \\ g_1(\varepsilon p)(\partial_t v + v \cdot \nabla v) + \varepsilon^{-1} \nabla P = \varepsilon^{-1} \nabla\phi, \\ \Delta\phi = \varepsilon^{-1} F(\varepsilon p), \end{cases}$$

where g_1 and g_2 are C^∞ positive functions, and F is a smooth non-decreasing function vanishing at the origin. Note that the system studied in the celebrated work of Klainerman and Majda correspond to $\phi = 0 = F$.

For these problems, a general strategy can be developed. The analysis is in two steps. First, one study the uniform stability of the solutions: one has to prove that the classical solutions exist and are uniformly bounded for a time interval independent of ε. Next, one considers the behavior of the solutions when ε goes to 0: one has to prove that the solutions are the sum of the classical solution of the limit system plus a highly oscillating term that goes to 0 in a weak sense.

The analysis of the zero electron mass limit has been studied by Ali, Chen, Jüngel and Peng [5]. By adapting the method of Klainerman and Majda, they have proved a convergence result when the solutions do not depend on the fast time scale (for well prepared initial data such that the time derivative are uniformly bounded). In [4] we consider the same problem for general initial data such that the solutions do depend on the fast time scale. In addition, we consider initial data which allow for large variations of size $O(1)$ of ρ_i.

The key points of the analysis can be observed on the following simplified system where G is a source term and $g_i = g_i(t, x, \wp) > 0$ are given smooth functions. We prove that the Cauchy problem for the system

$$\left\{\begin{array}{l} g_1(t,x,\varepsilon p)(\partial_t p + v\cdot\nabla p) + \varepsilon^{-1}\mathrm{div}\, v = 0,\\ g_2(t,x,\varepsilon p)(\partial_t v + v\cdot\nabla v) + \varepsilon^{-1}\nabla p = \varepsilon^{-1}\nabla\phi,\\ \Delta\phi = \varepsilon^{-1}F(\varepsilon p) + G(t,x), \end{array}\right. \tag{1}$$

is uniformly well-posed on the Torus or the free space. The source term $G(t, x)$ allows to take into account large density variations. Furthermore, the fact that g_1 and g_2 can depend on the variables (t, x) allows to consider the non-isentropic regime where the density ρ_e depends on the temperature.

Once it is proved that the solutions exist and are uniformly bounded, one can rigorously justify (by using a Theorem of Métivier and Schochet [6, 7]) the limit $\varepsilon \to 0$ for general initial data defined in the whole space.

Also, it is worth noticing that our analysis allows to justify the anelastic approximation.

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Asymptotic analysis of finite difference methods

Michael Junk
(joint work with Martin Rheinländer)

If we consider a finite difference method simply as a set of equations containing a small parameter (the grid spacing), it is evident that the tools of asymptotic analysis can give us useful information about the method. The applicability of this approach for studying consistency, long time behavior and stability is demonstrated. As example, we use a simple lattice Boltzmann scheme for the 1D advection equation with constant advection velocity. Applications of the method to lattice Boltzmann schemes for the Navier-Stokes equation can be found in [1, 2, 3, 5, 6, 7]. It should be stressed that the results are not restricted to lattice Boltzmann methods but can readily be applied to any other finite difference scheme (see [4] for various examples and a short review of asymptotic methods in numerical analysis).

Lattice Boltzmann methods are based on discrete velocity particle models. Concretely, we consider fictitious particles that can move with unit speed in one space dimension either to the left or to the right. Hence the discrete velocities are given by $\mathcal{S} := \{-1, 1\}$. The particle distribution is described by a vector-valued function referred to as the *population function*. The first component represents the density of the particles traveling to the left, while the second component is associated with the other species:

$$\mathsf{F}(t,x) = \big[\mathsf{F}_k(t,x)\big]_{k\in\{1,2\}} = \begin{pmatrix} \mathsf{F}_1(t,x) \\ \mathsf{F}_2(t,x) \end{pmatrix}.$$

A compact notation is obtained with the help of the basis vectors

$$\mathsf{1} := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2 \qquad \mathsf{s} := \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$$

and the componentwise product between vectors, e.g.

$$\mathsf{1f} = \mathsf{f}, \quad \mathsf{s}^2 = \begin{pmatrix} (-1)^2 \\ 1^2 \end{pmatrix} = \mathsf{1}, \quad \mathsf{s}^3 = \mathsf{s}\,.$$

Finally, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product.

The algorithm we are going to investigate has the standard lattice Boltzmann form

$$\mathsf{F}_k(t+h, x+s_k h) = \mathsf{F}_k(t,x) + \big[J\mathsf{F}(t,x)\big]_k, \qquad k = 1,2 \tag{1}$$

where the discretization parameter $h = 1/N$, $N \in \mathbb{N}$ determines the space-time grid. More precisely, t can take the values $t_n = nh$ with $n \in \mathbb{N}_0$ and x ranges in $x_i = ih$ with $i \in \mathbb{Z}$. The collision operator J on the right hand side of (1) models the particle interaction. Here, we choose the simple BGK form

$$J\mathsf{F} = \omega(E - I)\mathsf{F}$$

with relaxation parameter ω and projection matrix

$$E := \tfrac{1}{2}\begin{pmatrix} 1-a & 1-a \\ 1+a & 1+a \end{pmatrix} \tag{2}$$

To keep things simple, we assume 1-periodic initial data which give rise to 1-periodic population functions so that no boundary data have to be prescribed.

In order to understand the behavior of the lattice Boltzmann solution $\mathsf{F}(t,x)$, we try to approximate it in the form of a regular h-expansion

$$\mathsf{F}(t,x) \approx \mathsf{f}^{[\alpha]}(t,x) := \mathsf{f}^{(0)}(t,x) + h\mathsf{f}^{(1)}(t,x) + \ldots + h^\alpha \mathsf{f}^{(\alpha)}(t,x) \tag{3}$$

with $t = t_n = nh$ and $x = x_j = jh$, $n \in \mathbb{N}, j \in \mathbb{Z}$. We refer to $\mathsf{f}^{[\alpha]}$ as *prediction function*. The *asymptotic order functions* $\mathsf{f}^{(\beta)}$ with $0 \le \beta \le \alpha$ are supposed to be h-independent, smooth in t and x and 1-periodic in x. The order functions are determined by inserting (3) into the update rule (1), performing a Taylor expansion and equating orders. Specifically, we find with $D = \partial_t + \mathsf{s}\partial_x$

$$\begin{aligned} (I-E)\mathsf{f}^{(0)} &= 0 \\ (I-E)\mathsf{f}^{(1)} &= -\tfrac{1}{\omega} D\mathsf{f}^{(0)} \\ (I-E)\mathsf{f}^{(2)} &= -\tfrac{1}{\omega} D\mathsf{f}^{(1)} - \tfrac{1}{2\omega} D^2 \mathsf{f}^{(0)} \end{aligned} \tag{4}$$

Since $I-E$ is not invertible (the range of $I-E$ is generated by the vector s which is orthogonal to $\mathsf{1}$), we encounter solvability conditions. In fact, the right hand sides of the equations (4) must be orthogonal to $\mathsf{1}$ which eventually can be cast into conditions on the so called *mass moments* $u^{(\beta)} = \langle \mathsf{f}^{(\beta)}, \mathsf{1} \rangle$ of the order functions. With the abbreviations

$$\mu := (\tfrac{1}{\omega} - \tfrac{1}{2})(1-a^2), \qquad \lambda := 2a(\tfrac{1}{\omega^2} - \tfrac{1}{\omega} + \tfrac{1}{6})(1-a^2).$$

they are

$$\begin{aligned} \partial_t u^{(0)} + a\partial_x u^{(0)} &= 0 \\ \partial_t u^{(1)} + a\partial_x u^{(1)} &= \mu \partial_x^2 u^{(0)} \\ \partial_t u^{(2)} + a\partial_x u^{(2)} &= \mu \partial_x^2 u^{(1)} + \lambda \partial_x^3 u^{(0)} \end{aligned} \tag{5}$$

and the precise form of the leading order coefficients is:

$$\begin{aligned} \mathsf{f}^{(0)} &= \tfrac{1}{2}(\mathsf{1} + a\mathsf{s})u^{(0)} \\ \mathsf{f}^{(1)} &= \tfrac{1}{2}(\mathsf{1} + a\mathsf{s})u^{(1)} - \tfrac{1}{2\omega}(1-a^2)\mathsf{s}\partial_x u^{(0)} \\ \mathsf{f}^{(2)} &= \tfrac{1}{2}(\mathsf{1} + a\mathsf{s})u^{(2)} - \tfrac{1}{2\omega}(1-a^2)\mathsf{s}\partial_x u^{(1)} - \tfrac{1}{2\omega}(\tfrac{1}{\omega} - \tfrac{1}{2})(1-a^2)a\mathsf{s}\partial_x^2 u^{(0)}. \end{aligned} \tag{6}$$

If the algorithm is initialized compatibly to (6), i.e.

$$\mathsf{F}(0,x) = \tfrac{1}{2}(\mathsf{1} + a\mathsf{s})v_0(x) - h\tfrac{1}{2\omega}(1-a^2)\mathsf{s}\partial_x v_0(x) - h^2 \tfrac{1}{2\omega}(\tfrac{1}{\omega} - \tfrac{1}{2})(1-a^2)a\mathsf{s}\partial_x^2 v_0(x)$$

with a 1-periodic function v_0, we deduce initial conditions for the mass moments $u^{(0)}(0,x) = v_0(x)$ and $u^{(1)}(0,x) = u^{(2)}(0,x) = 0$ which completely determine the order functions in view of (5) and (6).

Assuming that $\mathsf{f}^{[2]}$ correctly captures the h-behavior of F up to the expanded order, i.e. $\mathsf{F}(t,x) - \mathsf{f}^{[2]}(t,x) = \mathrm{O}(h^3)$ we find for the mass moment at every grid point

$$U(t,x) = u^{(0)}(t,x) + hu^{(1)}(t,x) + h^2u^{(2)}(t,x) + \mathrm{O}(h^3). \tag{7}$$

In particular, U coincides with the solution $u^{(0)}$ of the advection equation (see (5)) up to an error which is at least proportional to h. In this sense, our lattice Boltzmann algorithm is *consistent* to the advection equation. The order of consistency can also be deduced from (7). If $\omega \neq 2$ and $a^2 \neq 1$ and hence $\mu \neq 0$, the equation for $u^{(1)}$ generally involves a non-zero source term. Thus $u^{(1)}$ will be different from zero and the coincidence of U and $u^{(0)}$ is of first order

$$U(t,x) - u^{(0)}(t,x) = hu^{(1)}(t,x) + \mathrm{O}(h^2).$$

We say that the algorithm is *first order consistent* to the advection equation in that case. If, however, $\omega = 2$ or $a^2 = 1$, the source term in the $u^{(1)}$-equation vanishes and since $u^{(1)}(0,x) = 0$, the solution $u^{(1)}(t,x)$ turns out to be zero everywhere. In this case,

$$U(t,x) - u^{(0)}(t,x) = h^2u^{(2)}(t,x) + \mathrm{O}(h^3)$$

where $u^{(2)}$ is non-zero for non-trivial $u^{(0)}$ and $a^2 \notin \{0,1\}$. Hence, the lattice Boltzmann method is second order accurate in the case $\omega = 2$.

Summarizing these considerations, we can say that a regular expansion of the algorithm essentially amounts to a *consistency* analysis. In contrast to this, information about the *stability* of the method can be obtained by investigating the long-time behavior with the help of a multi-scale expansion.

Starting with an ansatz of the form $\mathsf{F}(t,x) \approx \mathsf{f}^{[\alpha]}(t,ht,x)$, where

$$\mathsf{f}^{[\alpha]}(t_1,t_2,x) := \mathsf{f}^{(0)}(t_1,t_2,x) \ + \ h\mathsf{f}^{(1)}(t_1,t_2,x) \ + \ \ldots \ + \ h^\alpha \mathsf{f}^{(\alpha)}(t_1,t_2,x), \tag{8}$$

we find the following equation for the leading order mass moment

$$\begin{aligned} \partial_{t_1} u^{(0)}(t_1,t_2,x) + a\partial_x u^{(0)}(t_1,t_2,x) &= 0 \\ \partial_{t_2} u^{(0)}(t_1,t_2,x) - \mu\partial_x^2 u^{(0)}(t_1,t_2,x) &= 0 \end{aligned}$$

with the initial value $u^{(0)}(0,0,x) = v_0(x)$. We see that the numerical solution is governed by the advection equation for short times but that the diffusion equation dictates the behavior on the long time scale. In particular, an unwanted behavior of the scheme can be expected for $\mu = (\frac{1}{\omega} - \frac{1}{2})(1-a^2) < 0$ since the analysis leads to the ill-posed backward heat equation in that case. This reflects the findings of a detailed stability analysis which reveals that the scheme runs stable only for $0 \leq \omega \leq 2$ and $a^2 \leq 1$. At the same time, it should be stressed that $\mu > 0$ is possible also in unstable situations (e.g. $a = 2$ and $\omega = 4$) which shows that the long-time asymptotics cannot capture all the instabilities.

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Stabilized Finite Element Methods with Anisotropic Mesh Refinement for the Oseen Problem

Gert Lube

The starting point is the incompressible Navier-Stokes problem

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

for velocity $\mathbf{u}$ and pressure p in a domain $\Omega \subset \mathbf{R}^d$, $d \le 3$. In an outer loop, an A-stable low-order method (possibly with control of the time step Δt) is applied. In an inner loop, we decouple and linearize the resulting system using a Newton-type iteration per time step. This leads to problems of Oseen type:

$$L_{os}(\mathbf{b};\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{4}$$

with an artificial reaction term $c\mathbf{u}$ where $c \sim 1/\Delta t$.

We consider stabilized conforming finite element (FE) schemes with equal-order interpolation of velocity/pressure for problem (3)–(4) with emphasis on anisotropic mesh refinement in boundary layers. The classical streamline upwind and pressure stabilization (SUPG/PSPG) techniques for the incompressible Navier-Stokes problem for equal-order interpolation [4], together with additional stabilization of the divergence constraint (4), are well-understood on isotropic meshes [10]. Much less is known about the analysis in case of equal-order interpolation on anisotropic meshes. The Stokes problem has been considered in [3] for the Q1/Q1-case and in [9] for the P1/P1-case. The extension to the Oseen problem seems to be new. Experiments for the Navier-Stokes problem, e.g. in [6, 5], show the applicability of anisotropic mesh refinement for low-order schemes.

A standard approach to stabilize the Galerkin scheme is a combination of pressure stabilization (PSPG) with streamline-upwind stabilization (SUPG) together

with a stabilization of the divergence constraint, i.e., weighted L_2-residuals

$$\sum_{T\in\mathcal{T}_h} (L_{os}(\mathbf{b}-f;\mathbf{u},p),\ \delta_T((\mathbf{b}\cdot\nabla)\mathbf{v}+\nabla q))_T + \sum_{T\in\mathcal{T}_h} \gamma_T\ (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_T$$

of (3) and (4) are added to the Galerkin discretization.

First we derive an a-priori estimate of Cea-type on arbitrary meshes, e.g. on anisotropic meshes. The result for the stabilized method with $\delta_T = \delta_T^u = \delta_T^p$ is shown w.r.t. the norm

$$[\![V]\!]_{rbs}^2 := \|\nu^{\frac{1}{2}}\nabla\mathbf{v}\|_{L^2(\Omega)}^2 + \|c^{\frac{1}{2}}\mathbf{v}\|_{L^2(\Omega)}^2 + J_{rbs}(V,V), \tag{5}$$

$$J_{rbs}(V,V) := \sum_T \delta_T\|(\mathbf{b}\cdot\nabla)\mathbf{v}+\nabla q\|_{L^2(T)}^2 + \sum_T \gamma_T\|\nabla\cdot\mathbf{v}\|_{L^2(T)}^2 \tag{6}$$

with parameters δ_T, γ_T to be determined. This result is even valid for rather general finite element pairs for velocity and pressure.

The quasi-optimal a-priori result provides no control of the L^2-norm of the pressure. Therefore we analyze the stabilized method w.r.t. the norm

$$|\!|\!|V|\!|\!|_{rbs} := \left([\![V]\!]_{rbs}^2 \ +\ \sigma\|q\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \tag{7}$$

with parameter σ to be determined. We give a discrete inf-sup condition and a quasi-optimal error estimate w.r.t. $|\!|\!|\cdot|\!|\!|_{rbs}$. For simplicity, we restrict ourselves to the case of *equal-oder* interpolation of velocity/pressure.

Of practical interest are *hybrid* meshes with *anisotropic* mesh refinement of tensor product type (in the sense of [1, Chap. 3]) in the boundary layer and a smooth transition to (in general unstructured) shape-regular (isotropic) meshes away from the layer. We restrict ourselves to the case that the boundary layer is located at the hyperplane $x_d = 0$. The advantage of this class of meshes is not only that the coordinate transformation is simplified in regions with anisotropic elements but also that certain edges/faces of the elements are orthogonal/parallel to coordinate axes.

The main result is an extension of the previous Cea-type estimate for the stabilized scheme on hybrid meshes where norm $[\![\cdot]\!]_{rbs}$ is replaced with norm $|\!|\!|V|\!|\!|_{rbs}$. This result is exploited using anisotropic interpolation estimates derived in [1, Chap. 3]. Then the right-hand side of the error estimate is optimized w.r.t. the stabilization parameters δ and γ_T, thus resulting in a proposal for the design of these parameters on isotropic and on anisotropic elements. Full proofs are given in [2].

In the last part of the paper, we provide some numerical experiments for the full Navier-Stokes problem. First, we simulate a "plug flow" in the two-dimensional case in the laminar regime. The results clearly demonstrate the ability of hybrid meshes to produce accurate solutions with much less unknowns than on isotropic meshes. Secondly, we apply the approach to the turbulent flow in a three-dimensional channel [7]. We apply the $k-\epsilon-\overline{v}^2-\overline{f}$-model of Durbin in the "user-friendly" $\varphi-\overline{f}$-version [8] for the RANS version of problem (1)-(2). The

results for relevant quantities in wall units are in reasonable agreement with the DNS data and even better than results presented in [8].

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Point load on a shell

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(joint work with Harri Hakula)

According to the classical two-dimensional models of linear shell theory, the deformation of the midsurface of a thin shell under a given load is obtained by minimizing a quadratic energy functional of the form

$$\mathcal{F}(\mathbf{u}) = \mathcal{A}_m(\mathbf{u},\mathbf{u}) + d^2\mathcal{A}_b(\mathbf{u},\mathbf{u}) - 2Q(\mathbf{u}),$$

where d is the thickness of the shell and the three terms of the energy correspond to the deformation energy due to stretching ($\mathcal{A}_m$), the deformation energy due to bending ($d^2\mathcal{A}_b$) and the external load functional (Q). Further, $\mathbf{u} = (u, v, w)$ is a vector field on the midsurface of the shell that defines the tangential displacements (u, v) and normal deflection (w) of the midsurface in the loaded state.

We consider here the the problem of shell deformation under a normal point load, i.e., the case of a load functional of the form

$$Q(\mathbf{u}) = F\delta_P,$$

where P is a point at the midsurface. This case is of special interest, as the solution is the fundamental solution (Green's function) for normal loads. Localized

loads (idealized as point loads in mathematical models) are rather common also in engineering practice.

Although the point-load problem has been the subject of rather early studies in classical shell theory, cf. [1] and the further references therein, the detailed behavior of the solution, especially in the vicinity of point P, is still an open problem of asymptotic analysis when the thickness d is small. We give here some partial solutions to this problem. In our study, we assume a 'shallow' version of the classical shell model where geometrical simplifications are assumed. Such simplifications are justified when studying local effects in shell deformations, see [2].

Within the simplified shallow shell model, we study first the asymptotic limit solution at zero shell thickness ($d = 0$). In model cases we can work out an explicit limit solution in the sense of distributions. When focusing on the normal deflection (w) of this limit solution we conclude: (1) In all shell geometries w has a component of the form $w = w_0\delta_P$. (2) In the elliptic case the remaining part of w is smooth near P. (3) In hyperbolic and parabolic shell geoemtries, w contains additional line δ-distributions along the characteric lines through P. When aligning the coordinates so that x is the coordinate along the characteristic line (along which the curvature vanishes) and y is the normal coordinate, then the line δ-sistributions take the form $w = A(x)\delta^{(k)}(y)$, where $A(x)$ is relatively smooth (continuous at least) and the possible values of k are $k = 2$ (hyperbolic and parabolic cases) and $k = 4$ (parabolic case).

We assume next that d takes a small positive value and focus again on the transverse deflection w due to the point load at P. Based on the asymptotic solution at $d = 0$ we make two conjectures: (1) When $d > 0$, the asymptotic term $w_0\delta_P$ in w is spread into a 'hot spot' around P of width $\sim \sqrt{Rd}$, where R is the curvature length scale of the shell. We may define R, e.g. as the minimal principal radius of curvature of the midsurface at P. (2) In the hyperbolic and parabolic cases the line δ-distributions are spread to 'ridges' of width $\sim \sqrt[m]{R^{m-2}d}$, where $m = 3$ in the hyperbolic case and $m = 4$ in the parabolic case.

So far we can support the above conjectures only partially based on the earlier results on line layers of shell deformatations, see [2] and also [3, 4]. Numerical solutions in benchmark cases (based on high-order finite elements) also support these conjectures, as we show. What remains to be done is to convert the mentioned qualitative conjectures to more precise theorems. This is an open problem of asymptotic analysis — particularly challenging in hyperbolic and parabolic shell geometries.

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On phase field models: Phase Transition between Ordered and Disordered Lipid Monolayers

Hans Wilhelm Alt
(joint work with Wolfgang Alt)

We present a phase field model for a phase transition between a diffusive high compressible substance with density ρ_1 and a high viscous and limited compressible substance with density ρ_2. The dynamics is driven by a velocity field v. The model is based on balance laws for mass and momentum and a free energy inequality, where the total free energy has the form

$$f_{tot} = \frac{\rho}{2}|v|^2 + f_0(\rho,\varphi) + \frac{\varepsilon^2}{2}|\nabla\varphi|^2 \ . \tag{1}$$

Here the total mass ρ and the phase fraction φ are given by

$$\rho := \rho_1 + \rho_2 \ , \quad \varphi_k := \frac{\rho_k}{\rho} \text{ for } k = 1,2 \ , \quad \varphi := \varphi_2 \ . \tag{2}$$

The model applies to lipid monolayers on the thin water film of lung alveoli (see e.g. [2, 3]). We perform an asymptotic expansion for the interfacial layer.

1. Sharp interface model

We consider the local situation without boundary conditions and assume, that the two substances occupy two time-space regions $\Omega^1, \Omega^2 \subset \mathbb{R}\times\mathbb{R}^n$ separated by an interface $\Gamma \subset \mathbb{R}\times\mathbb{R}^n$. The model is based on strictly convex internal free energies

$$\rho_1 \mapsto f_1(\rho_1) \quad \text{(gas-like)}, \quad \rho_2 \mapsto f_2(\rho_2) \quad \text{(fluid-like)}. \tag{3}$$

Conservation of mass is given by

$$\partial_t\rho_1 + \operatorname{div}(\rho_1 v + J_1) = 0 \quad \text{in } \Omega^1 \text{ (parabolic for } \rho_1), \tag{4}$$

$$\partial_t\rho_2 + \operatorname{div}(\rho_2 v) = 0 \quad \text{in } \Omega^2 \text{ (hyperbolic for } \rho_2), \tag{5}$$

$$\tau_1 := \rho_1(v - v_\Gamma)\bullet\nu + J_1\bullet\nu, \quad \tau_2 := \rho_2(v-v_\Gamma)\bullet\nu \quad \text{on } \Gamma, \tag{6}$$

$$\tau := \tau_2 = -\tau_1 \quad \text{on } \Gamma. \tag{7}$$

Here $J_1 = -\tilde{d}(\rho_1)\nabla\rho_1 = -d(\rho_1)\nabla\mu_1$ is the diffusive flux, where $\mu_k := f_1'(\rho_k)$ for $k = 1,2$ are the chemical potentials. Moreover, v_Γ is the velocity vector of Γ and $\nu := \nu_{\Omega^1} = -\nu_{\Omega^2}$ the unit normal of Γ. Note, that ρ_1 on Ω^1 and ρ_2 on Ω^2 can attain any values.

Conservation of momentum reads (in weak sense)

$$\partial_t(\rho v) + \operatorname{div}(\rho v\otimes v + v\otimes(\mathcal{X}_{\Omega_1}J_1) + \Pi) = \mathbf{f} \quad \text{(parabolic for } v) \tag{8}$$

in $\Omega := \Omega^1 \cup \Gamma \cup \Omega^2$. Here $\rho := \rho_k$ in Ω^k for $k = 1, 2$, $\Pi = p\mathrm{Id} - S$ with pressure

$$p := \begin{cases} \rho_1 f_1'(\rho_1) - f_1(\rho_1) & \text{in } \Omega^1 \ , \\ \rho_2 f_2'(\rho_2) - f_2(\rho_2) & \text{in } \Omega^2 \ , \end{cases} \tag{9}$$

and stress tensor $S = a(\rho)(Dv)^{\mathrm{S}}$.

The main question concerns a missing second interface condition. The aim is, to derive a constitutive equation for τ from a phase field limit.

2. Phase field model

In the phase field model the densities ρ_1 and ρ_2 live in the entire domain Ω, and ρ, φ are defined as in (2). The general balance laws for the mass of each phase and the total momentum are

$$\partial_t \rho_k + \operatorname{div}(\rho_k v + J_k) = \tau_k \ , \quad k = 1, 2, \tag{10}$$

$$\tau_1 + \tau_2 = 0 \ , \tag{11}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + v \otimes J + \Pi) = \mathbf{f} \ . \tag{12}$$

Here J_k are the diffusive mass fluxes, $J := J_1 + J_2$, and Π is the pressure tensor. We consider a free energy as in (1), or more general

$$f_{tot} = f_{kin} + f, \quad f_{kin} := \frac{1}{2}\rho|v|^2, \quad f = \hat{f}(\rho, \varphi, \nabla\varphi) \ .$$

We postulate a

Free energy inequality. With a free energy flux Ψ_{tot}

$$\partial_t f_{tot} + \operatorname{div} \Psi_{tot} \le v \bullet \mathbf{f}$$

for all solutions (ρ_1, ρ_2, v). Thus the free energy production

$$g := \partial_t f_{tot} + \operatorname{div} \Psi_{tot} - v \bullet \mathbf{f}$$

is supposed to be non-positive for all processes under consideration. We split

$$\Psi_{tot} = f_{tot} v + \Pi^{\mathrm{T}} v + \frac{1}{2}|v|^2 J + \Psi$$

in well known kinetic terms and a vector Ψ.

Define the first variation μ of f with respect to φ, chemical potentials μ_1 and μ_2, and a pressure p by

$$\begin{aligned} \mu := \tfrac{\delta f}{\delta \varphi} &= f_{\prime\varphi} - \operatorname{div} f_{\prime\nabla\varphi} \\ \mu_1 &:= f_{\prime\rho} - \tfrac{\varphi_2 \mu}{\rho} \ , \\ \mu_2 &:= f_{\prime\rho} + \tfrac{\varphi_1 \mu}{\rho} \ , \\ p &:= \rho f_{\prime\rho} - f \ . \end{aligned}$$

Using methods from Rational Thermodynamics (see e.g. [1]) one obtains

Theorem. Assume, that the free energy flux is given with

$$\Psi := -\dot{\varphi} f_{\prime\nabla\varphi} + \mu_1 J_1 + \mu_2 J_2$$

and let

$$\Pi = p\mathrm{Id} + \nabla\varphi \otimes f_{\prime\nabla\varphi} - S \ .$$

Then the free energy production becomes, with $\tau := \tau_2$,

$$g = -Dv \bullet S + \frac{\mu}{\rho}\tau + \nabla\mu_1 \bullet J_1 + \nabla\mu_2 \bullet J_2 \ .$$

Therefore the free energy inequality is satisfied, if $Dv \bullet S \geq 0$ and

$$\frac{\mu}{\rho}\tau + \sum_{k=1,2} \nabla\mu_k \bullet J_k \leq 0 \ .$$

It turns out, that the generic model of this type is of fourth order. Only the case $J_1 = \varphi_1 J$, $J_2 = \varphi_2 J$ leads to a second order model. With a free energy

$$f(\rho, \varphi, \nabla\varphi) = f_0(\rho, \varphi) + \frac{\varepsilon^2}{2}|\nabla\varphi|^2$$

it is given by

$$\begin{gathered} J = -d_\varepsilon \nabla f_{0\prime\rho} \ , \quad d_\varepsilon = d_\varepsilon(\rho, \varphi) > 0 \ , \\ S = -aDv^{\mathrm{S}}, \quad a = a(\rho, \varphi) > 0 \ , \\ \tau = \nabla\varphi \bullet J - \rho \cdot c_\varepsilon \mu \ , \quad c_\varepsilon = c_\varepsilon(\rho, \varphi) > 0 \ , \\ \mu = \tfrac{\delta f}{\delta \varphi} = f_{0\prime\varphi} - \varepsilon^2 \Delta\varphi \end{gathered}$$

3. Asymptotic expansion

As an appropriate scaling let

$$c_\varepsilon(\rho, \varphi) = \frac{1}{\varepsilon}, \quad d_\varepsilon(\rho, \varphi) = \frac{d(\rho)}{\frac{1}{\varepsilon}\varphi + (1 - \varphi)} \ .$$

Then an asymptotic expansion of the interfacial layer leads to the following

ODE system. On the real line $\mathbb{R}$ it is $v = \mathrm{const}$ and one has to solve

$$\varphi'' + \lambda\varphi' = f_{0\prime\varphi}(\rho, \varphi) \tag{13}$$

$$d(\rho)(f_{0\prime\rho}(\rho, \varphi))' = -\lambda\varphi(\rho - \rho_2^0) \tag{14}$$

with boundary conditions $\varphi(-\infty) = 0$, $\varphi(+\infty) = 1$, $\rho(-\infty) = \rho_1^0$, $\rho(+\infty) = \rho_2^0$. Here $\lambda := (v_\Gamma - v) \bullet \nu$ is the relative speed of the interface.

The sign of λ can be expressed in terms of the free energy f. Moreover we claim, that for given ρ_1^0 and ρ_2^0 there exists a unique value λ, for which a solution exists. In the sharp interface limit $\varepsilon \to 0$ this leads to a constitutive equation for the production rate τ in (7).

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Asymptotic behavior of the stop hysteresis operator

Jürgen Sprekels

(joint work with Pavel Krejčí)

Let $\Omega \subset \mathbf{R}^n$ be an open bounded Lipschitzian domain. We consider the problem

$$w_t - \varepsilon\,\Delta w + \partial I_K(w) \ \ni \ \gamma(w, u^\varepsilon) \ \text{ in } \ Q_T := \Omega \times (0,T)\,, \tag{1}$$

$$\frac{\partial w}{\partial \nu} \ = \ 0 \ \text{ on } \ \partial\Omega \times (0,T)\,, \tag{2}$$

$$w(x,0) \ = \ \varphi(x) \ \text{ for a.e. } \ x \in \Omega. \tag{3}$$

The system (1)-(3) arises in the phase field modeling of phase transitions with a nonconserved order parameter vector field $w = (w_1, \ldots, w_N) : Q_T \to \mathbf{R}^N$ that attains its values in some closed and convex set $K \subset \mathbf{R}^N$ containing 0. For details of the model, we refer the reader to the paper [1] and the references cited therein. In this connection, $u^\varepsilon : Q_T \to \mathbf{R}^\ell$ is some control variable, which typically represents either the absolute temperature or the strain tensor, and $\gamma : K \times \mathbf{R}^\ell \to \mathbf{R}^N$ is a given globally Lipschitz continuous mapping.

In the physical application, the diffusion parameter ε is often controversial or cannot be identified in a straightforward way; moreover, if u^ε is the absolute temperature or its inverse, then for $\varepsilon > 0$ there are difficulties in proving that u^ε is positive and that the Second Principle of Thermodynamics is satisfied pointwise, while these problems usually do not occur for $\varepsilon = 0$. It is thus natural to study the stability of the model as $\varepsilon \searrow 0$. In the lecture, we have proved the following asymptotic result (see [1, Theorem 2.2]):

Theorem 1. Suppose that $\varphi \in H^1(\Omega)$ satisfies $\varphi(x) \in K$ for almost every $x \in \Omega$, and suppose that $u^\varepsilon \in L^2(Q_T; \mathbf{R}^\ell)$ for all $\varepsilon \geq 0$ and $u^\varepsilon \to u^0$ strongly in $L^2(Q_T; \mathbf{R}^\ell)$. Then problem (1)–(3) has for every $\varepsilon > 0$ a unique solution $w = w^\varepsilon \in L^2(Q_T; \mathbf{R}^N)$ such that $w_t^\varepsilon, \Delta w^\varepsilon \in L^2(Q_T; \mathbf{R}^N)$, and problem (1)+(3) has for $\varepsilon = 0$ a unique solution $w = w^0 \in L^2(Q_T; \mathbf{R}^N)$ such that $w_t^0 \in L^2(Q_T; \mathbf{R}^N)$.

Moreover, it holds

$$\lim_{\varepsilon\to 0+} \varepsilon \int_0^T \int_\Omega \|\nabla w^\varepsilon\|^2 \, dx \, dt = 0 \,, \tag{4}$$

$$\lim_{\varepsilon\to 0+} \sup_{s\in[0,T]} \int_\Omega |w^\varepsilon - w^0|^2(x,s) \, dx = 0 \,, \tag{5}$$

where $\|\cdot\|$ denotes the norm in $\mathbf{R}^{n\,N}$.

The idea of the proof of this result is to apply special properties of the so-called *stop operator* $\mathcal{S}_K$ associated with the convex set K, the main ingredient being the following *partial integration inequality* (see [1, Lemma 4.2]):

Theorem 2. Suppose that $\varphi \in H^1(\Omega)$ satisfies $\varphi(x) \in K$ for almost every $x \in \Omega$, and let $v, w \in L^2(Q_T; \mathbf{R}^N)$ be such that

(i) $v_t, \Delta w \in L^2(Q_T; \mathbf{R}^N)$,

(ii) $w = \mathcal{S}_K[\varphi, v]$,

(iii) $\partial w/\partial \nu(x,t) = 0$ for a.e. $(x,t) \in Q_T$.

Then it holds for every $s \in [0,T]$ that

$$-\int_0^s \int_\Omega v_t(x,t) \cdot \Delta w(x,t) \, dx \, dt \geq \frac{1}{2} \int_\Omega \left[\|\nabla w(x,s)\|^2 - \|\nabla \varphi(x)\|^2 \right] dx \,. \tag{6}$$

This talk reports results that have been obtained jointly with Pavel Krejčí (WIAS Berlin).

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Γ-convergence for evolutionary problems

Alexander Mielke

Many evolutionary problems, such as partial differential equations, display several temporal or spatial scales and it is desirable to find a suitable limit model that describes the macroscopic dynamics correctly. We want to address some general concepts that might be useful for deriving such effective models.

1. Geometric evolution via functionals

Consider a manifold $\mathcal{Q}$ and an energy-storage functional (potential energy) $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R}\cup\{+\infty\}$. For the dynamics we distinguish the dissipative situation and the Hamiltonian one.

In the first case we have Rayleigh's dissipation potential $\mathcal{R} : \mathrm{T}\mathcal{Q} \to \mathbb{R}_\infty$, where $\mathcal{R}(q,\cdot) : \mathrm{T}_q\mathcal{Q} \to \mathbb{R}_\infty$ is assumed to be convex. The evolution law is then given in terms of the balance of the dissipative forces $\partial_{\dot q}\mathcal{R}(q,\dot q)$ and the potential restoring forces $-\mathrm{D}\mathcal{E}(q)$, namely

$$0 \in \partial_{\dot q}\mathcal{R}(q,\dot q) + \mathrm{D}\mathcal{E}(t,q) \subset \mathrm{T}_q^*\mathcal{Q}. \tag{1}$$

(I) The viscous case corresponds to $\mathcal{R}$, which is given in terms of a Riemannian metric $\mathbf{g}$, i.e., $\mathcal{R}(q,v) = \frac12\langle \mathbf{g}(q)v, v\rangle$, and leads to so-called gradient flows

$$\mathbf{g}(q)\dot q = -\mathrm{D}\mathcal{E}(t,q) \quad \Leftrightarrow \quad \dot q = -\nabla_{\mathbf{g}}\mathcal{E}(t,q).$$

(II) Another interesting dissipative situation is the case of rate-independent systems where $\mathcal{R}(q,\cdot)$ is homogeneous of degree 1. Then, $\partial_v\mathcal{R}(q,v) \subset \mathrm{T}_q\mathcal{Q}$ denotes the set-valued subdifferential of the convex function $\mathcal{R}(q,\cdot)$ and (1) is a differential inclusion, which may be reformulated as an evolutionary variational inequality, cf. [4]. In fact, for the rate-independent case there is a weaker *energetic formulation* in terms of a global *stability condition* (S) and the *energy balance* (E). This formulation uses the dissipation distance $\mathcal{D} : \mathcal{Q}\times\mathcal{Q} \to \mathbb{R}_\infty$, that is associated with $\mathcal{R}$ via

$$\mathcal{D}(q_0,q_1) = \inf\{\, \textstyle\int_0^1 \mathcal{R}(\widetilde q(t),\dot{\widetilde q}(t))\,\mathrm{d}t \mid \widetilde q \in \mathrm{W}^{1,1}([0,1];\mathcal{Q}), \widetilde q(0)=q_0, \widetilde q(1)=q_1\,\}.$$

We call a curve $q : [0,T] \to \mathcal{Q}$ an *energetic solution* associated with the functionals $\mathcal{E}$ and $\mathcal{D}$, if for all $t \in [0,T]$ we have

$$\begin{aligned} &\text{(S)} \quad \mathcal{E}(t,q(t)) \le \mathcal{E}(t,\widetilde q) + \mathcal{D}(q(t),\widetilde q) \text{ for all } \widetilde q \in \mathcal{Q},\\ &\text{(E)} \quad \mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q,[0,t]) = \mathcal{E}(0,q(0)) + \textstyle\int_0^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s. \end{aligned} \tag{2}$$

(III) Also classical Hamiltonian systems are driven by two functionals. In addition to the potential energy $\mathcal{E} : \mathcal{Q} \to \mathbb{R}_\infty$ we also have the kinetic energy $\mathcal{K} : \mathrm{T}\mathcal{Q} \to \mathbb{R}_\infty$, which is again given by a Riemannian metric $\mathbf{g}$ in the form $\mathcal{K}(q,\dot q) = \frac12\langle \mathbf{g}(q)\dot q, \dot q\rangle$. The evolution equations in $\mathrm{T}\mathcal{Q}$ (the Lagrangian setting) then read

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{D}\mathcal{K}(q,\dot q)\big) + \mathrm{D}\mathcal{E}(q) = 0 \in \mathrm{T}_q^*\mathcal{Q}. \tag{3}$$

Often the canonical Hamiltonian form is preferred. It is based on the conjugate momentum $p = \mathbf{g}(q)\dot q$ and the Hamiltonian $\mathcal{H}(q,p) = \frac12\langle p, \mathbf{g}(q)^{-1}p\rangle + \mathcal{E}(q)$:

$$\dot q = \mathrm{D}_p\mathcal{H}(q,p) = \mathbf{g}(q)^{-1}p \in \mathrm{T}_q\mathcal{Q}, \quad \dot p = -\mathrm{D}_q\mathcal{H}(q,p) = -\mathrm{D}\mathcal{E}(q) \in \mathrm{T}_q^*\mathcal{Q}. \tag{4}$$

2. Γ-convergence and the limit passage

We now consider sequences of pairs of functionals, namely $(\mathcal{E}_k,\mathcal{R}_k)$ for general dissipative systems, $(\mathcal{E}_k,\mathcal{D}_k)$ for the energetic formulation of the rate-independent

case, and $(\mathcal{E}_k, \mathcal{K}_k)$ for Hamiltonian systems. Additionally we consider an associated sequence of solutions $q_k : [0,T] \to \mathcal{Q}$.

To define a convergence we equip $\mathcal{Q}$ with a Hausdorff topology and write "$\overset{\mathcal{Q}}{\to}$" for the corresponding convergence. The functional $\mathcal{E} : \mathcal{Q} \to \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$ is called Γ-*limit of the sequence* $(\mathcal{E}_k)_{k\in\mathbb{N}}$, if the following two conditions hold:

(i) lower estimate: $q_k \overset{\mathcal{Q}}{\to} q \implies \mathcal{E}(q) \le \liminf_{k\to\infty} \mathcal{E}_k(q_k)$,

(ii) upper estimate: $\forall\, q \in \mathcal{Q}\ \exists\, \widehat{q}_k : \ \widehat{q}_k \overset{\mathcal{Q}}{\to} q$ and $\mathcal{E}(q) = \lim_{k\to\infty} \mathcal{E}_k(\widehat{q}_k)$.

Now assume that both functionals Γ-converge (independently) and that we have solutions q_k with a pointwise limit $q : [0,T] \to \mathcal{Q}$, i.e., $q_k(t) \overset{\mathcal{Q}}{\to} q(t)$. Then , it is a natural question whether q is a solution of the problem defined by the limit functionals. Of course, we cannot expect that the result holds true in sufficient generality. The real task is to identify conditions in the sense of a "joint Γ-convergence" for the two functionals that guarantee the desired result.

(0) In fact, Γ-convergence was introduced for *static problems*. It was developed over the last decade to provide very elegant and strong tools for deriving such macroscopic models, see [2, 1]. In particular, it satisfies the desired convergence property in the following sense: If q_k is a minimizer of $\mathcal{E}_k$ and if $q_k \overset{\mathcal{Q}}{\to} q$, then q is a minimizer of the Γ-limit $\mathcal{E}$.

(I) For *gradient flows*, abstract positive results are contained in [10, 9]. They are based on specific assumptions on the gradients $\nabla\mathcal{E}_k(q_k)$. The following simple example in $\mathbb{R}^2$ shows that the desired result may even fail in finite dimensions. We let $\mathcal{Q} = \mathbb{R}^2$ and

$$\mathcal{E}_k(q) = \tfrac{1}{2}q_1^2 + \tfrac{k^\alpha}{2}(q_2 - q_1/k)^2 \ \text{ and } \ \mathcal{R}_k(\dot q) = \tfrac{1}{2}\dot q_1^2 + \tfrac{k^\beta}{2}\dot q_2^2,$$

where α, β are positive constants. The Γ-limits $\mathcal{E}$ and $\mathcal{R}$ are easily obtained, namely $\mathcal{E}(q) = q_1^2/2$ for $q_2 = 0$ and ∞ otherwise and $\mathcal{R}(\dot q) = \dot q_1^2/2$ for $\dot q_2 = 0$ and ∞ otherwise. The solution with $q(0) = (1,0)^\top$ of the limit problem is obviously $q(t) = (\mathrm{e}^{-t}, 0)^\top$. The solution $q_k : [0,\infty) \to \mathbb{R}^2$ for the functionals $\mathcal{E}_k$ and $\mathcal{R}_k$ with $q_k(0) = (1,0)^\top$ can be written down explicitly in terms of the eigenvalues. The limit $k \to \infty$ shows that the correct limit solution is obtained only if $\min\{\alpha, \beta\} < 2$.

(II) For *rate-independent systems*, Γ-convergence is studied via the energetic formulation (2) in [6, 7]. Since rate-independent systems are very close to static problems (cf. (S), which is a purely static condition), the conditions can be formulated totally in terms of the functionals without using differentials. Again a simple example in $\mathbb{R}^2$ can be constructed to show that the limit passage is not true in general.

The main condition, which provides the positive result, is the existence of *joint recovery sequences*:

$$\begin{aligned}&\forall\, q_k \in \mathcal{S}_k(t) \text{ with } q_k \overset{\mathcal{Q}}{\to} q\ \forall\, \widehat{q}\ \exists\, \widehat{q}_k \text{ with } \widehat{q}_k \overset{\mathcal{Q}}{\to} \widehat{Q}:\\ &\limsup_{k\to\infty} \big(\mathcal{E}_k(t,\widehat{q}_k) + \mathcal{D}_k(q_k,\widehat{q}_k) - \mathcal{E}_k(t,q_k)\big) \le \mathcal{E}(t,\widehat{q}) + \mathcal{D}(q,\widehat{q}) - \mathcal{E}(t,q),\end{aligned}$$

where $\mathcal{S}_k(t) = \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t,q) < \infty,\ \forall\, \widetilde{q} \in \mathcal{Q} : \mathcal{E}_k(t,q) \le \mathcal{E}_k(t,\widetilde{q}) + \mathcal{D}_k(q,\widetilde{q}) \,\}$ denotes the sets of stable states. Based on this condition several applications

are given in [7]. In [8] an application of two-scale homogenization for linearized elastoplasticity is derived.

(III) For *Hamiltonian systems* an abstract theory has not been developed.The oscillatory behavior of the solutions leads to an ongoing exchange between kinetic and potential energy, which is enforced by the underlying symplectic structure. So far, it is unclear how these structures can be used along with Γ-convergence. First preliminary results are given in [5, 3]. There the passage from a discrete lattice system to a macroscopic elastodynamic wave equation is shown by different tools. As a result we obtain that the separate Γ-convergence of $\mathcal{K}_k$ and $\mathcal{E}_k$ in the Lagrangian setting (3) gives the correct limit equation. However, doing the corresponding Γ-limit in the canonical Hamiltonian system (4) leads to a limit equation, which, in general, does not characterize the limits of solutions.

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A hierarchy of theories for thin elastic bodies

Stefan Müller

(joint work with Gero Friesecke, Richard D. James, Maria Giovanna Mora and Maximilian G. Schultz)

In this talk I give an overview on recent progress in understanding the relations between three-dimensional nonlinear elasticity and theories for lower dimensional objects such plates, beams and rods. This has been an outstanding question since

α applied force	β energy	γ in-plane	δ out-of-plane	limit model
$\alpha = 0$	0	0	0	Membrane [6]
$0 < \alpha < 5/3$	α	0	0	Constr. membrane [1, 2]
$\alpha = 2$	α	0	0	Kirchhoff [4, 10, 11]
$2 < \alpha < 3$	$2\alpha - 2$	$2(\alpha - 2)$	$\alpha - 2$	Lin. isometry constr. [5]
$\alpha = 3$	$2\alpha - 2$	$2(\alpha - 2)$	$\alpha - 2$	von Kármán [5]
$\alpha > 3$	$2\alpha - 2$	$\alpha - 1$	$\alpha - 2$	Linearized vK = Germain-Lagrange [5]

TABLE 1. Relation between the scaling exponents α of the applied forces, β of the energy, γ of the in-plane deformation and δ of the out-of-plane deformation. For $\alpha > 2$ we assume that the limit force is normal

the very beginning of the research in elasticity. In fact there is a large variety of lower-dimensional theories. They are usually obtained by making certain (strong) apriori assumptions on the form of the solutions of the full three-dimensional problem and hence their rigorous range of validity is typically unclear. As highlighted already in the work of Fritz John, a key point is the geometric nonlinearity in elasticity, i.e., the invariance of the elastic energy under rotations. In particular thin elastic objects can undergo large rotations even under small loads and this prevents any analysis based on a naive linearization.

The first rigorous results were only obtained by Le Dret and Raoult [6] in the early 90's using a variational approach which guarantees convergence of minimizers to a suitable limit problem. In joint work with G. Friesecke and R.D. James we have derived a hierarchy of limit problems as the thickness h goes to zero. The different limit theories depend on the scaling of the applied force $f^{(h)} \sim h^\alpha f$ which in turn determines the scaling of the energy energy per unit heigth $E^h \sim h^\beta$. For sufficiently small forces (i.e., sufficiently large α) the deformation will be close a rigid motion (which we may assume to be the trivial map after suitable normalization) and one can show that the (height-averaged) in-plane displacements $U^{(h)} = (U_1^{(h)}, U_2^{(h)}) := \frac{1}{h}\int_{-h/2}^{h/2}(v_1^{(h)}(x) - x_1, v_2^{(h)}(x) - x_2)\,dx_3$ and out-of displacement $V^{(h)} := \frac{1}{h}\int_{-h/2}^{h/2} v_3^{(h)}\,dx_3$ converge (at least weakly in the Sobolev space $W^{1,2}$) after suitable rescaling, i.e., $h^{-\gamma}U^{(h)} \to U$, $h^{-\delta}V^{(h)} \to v$, see [5] for details. The results are summarized in Table 1.

The exponent $\beta = 5/3$ is conjectured to be relevant for the crumpling of elastic sheets and its special role was first proposed in the physics literature [7, 12, 2]. Table 1 applies for natural (traction-free) boundary conditions. For clamped boundary conditions one obtains different scaling exponents, corresponding to a much stiffer response (for $0 < \alpha < 3$), see [3].

More recently we started to study the convergence of (possibly non-minimizing) stationary points of the elastic energy functional. To fix the notation let us look

at the variational setting in a bit more detail. Consider a cylindrical domain $\Omega_h = S \times (-h/2, h/2)$, where S is a bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. To a deformation $v : \Omega_h \to \mathbb{R}^3$ we associate the elastic energy (per unit height)

$$E^h = \frac{1}{h} \int_{\Omega_h} W(\nabla v)\, dz. \tag{1}$$

We assume that the stored-energy density function W satisfies the following conditions:

$$W(RF) = W(F) \quad \forall R \in SO(3) \qquad \text{(frame indifference)}, \tag{2}$$

$$W = 0 \quad \text{on } SO(3), \tag{3}$$

$$W(F) \geq c\, \mathrm{dist}^2(F, SO(3)), \quad c > 0, \tag{4}$$

$$W \text{ is } C^2 \text{ in a neighbourhood of } SO(3). \tag{5}$$

Here $SO(3)$ denotes the group of proper rotations. The frame indifference implies that there exists a function $\tilde{W}$ defined on symmetric matrices such that $W(\nabla v) = \tilde{W}((\nabla v)^T \nabla v)$, i.e. the elastic energy depends only on the pull-back metric of v.

To discuss the limiting behaviour as $h \to 0$ it is convenient to rescale to a fixed domain $\Omega = S \times (-1/2, 1/2)$ by the change of variables $x = (z_1, z_2, hz_3)$ and $y(x) = v(z)$. With the notation

$$\nabla_h y = (\partial_1 y, \partial_2 y, \frac{1}{h}\partial_3 y) = (\nabla' y, \frac{1}{h}\partial_3 y) \tag{6}$$

we have

$$E^h(v) = I^h(y) = \int_\Omega W(\nabla_h y)\, dx. \tag{7}$$

We have for $h \to 0$, in the sense of Gamma-convergence,

$$\frac{1}{h^2} I^h \xrightarrow{\Gamma} I_{Ki}. \tag{8}$$

This implies, roughly speaking, that minimizers of I^h (subject to suitable boundary conditions or body forces) converge to minimizers of I_{Ki}, provided I^h evaluated on the minimizers is bounded by Ch^2.

For the limit problem I_{Ki} the natural class of admissible functions is given by $W^{2,2}$ isometric immersions from S to $\mathbb{R}^3$, i.e.,

$$\mathcal{A} := \left\{ y \in W^{2,2}(\Omega, \mathbb{R}^3) : \partial_3 y = 0, (\nabla' y)^T \nabla' y = Id \right\}. \tag{9}$$

The limiting energy functional is

$$I_{Ki}(\bar{y}) = \begin{cases} \frac{1}{24} \int_S Q_2(A)\, dx_1 dx_2, & \text{if } y \in \mathcal{A}, \\ +\infty, & \text{else.} \end{cases} \tag{10}$$

Here A is the second fundamental form and Q_2 is a quadratic form which can be computed from the linearization $\partial^2 W / \partial^2 F(Id)$ of the 3d energy at the identity. If $W = \frac{1}{2}\mathrm{dist}^2(F, SO(3))$ then simply $Q_2(A) = |A|^2$.

We consider convergence of equilibria for the same energy scaling. Instead of studying the reduction from 3d to 2d we focus on the simpler limit from 2d to 1d. Thus we start from a thin strip

$$\Omega_h = (0, L) \times (-h/2, h/2) \tag{11}$$

and after the rescaling $(z_1, z_2) = (x_1, hx_2)$, $\nabla_h = (\partial_1, \frac{1}{h}\partial_2)$ we consider the functional

$$J^h(y) = \int_\Omega W(\nabla_h y) - h^2 g(x_1) \cdot y \, dx. \tag{12}$$

The corresponding Gamma-limit is given by

$$J_2(\bar{y}) = \int_0^L \frac{1}{24} E\kappa^2 - g \cdot \bar{y} \, dx_1, \tag{13}$$

where

$$\bar{y} : (0, L) \to \mathbb{R}^2, \quad \bar{y}' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \kappa = \theta', \tag{14}$$

and where J_2 takes the value $+\infty$ if $\bar{y}$ is not of the above form (here we took the liberty to identify maps on Ω which are independent of y_2 with maps on $(0, L)$). It is convenient to fix one endpoint by requiring $\bar{y}(0) = 0$. Integrating the linear term by parts we easily see that the Euler-Lagrange equation corresponding to the limit functional is given by

$$-\frac{1}{12} E\theta'' + \tilde{g} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = 0, \quad \tilde{g}(x_1) = \int_L^{x_1} g(\xi) \, d\xi. \tag{15}$$

Theorem 6 ([9]). *Assume that (2 – 5) hold, that the energy W is differentiable and the derivative DW is globally Lipschitz. Let $y^{(h)}$ be a sequence of stationary points of J^h (subject to the boundary condition $y^{(h)}(0, x_2) = (0, x_2)$ at $x_1 = 0$ and to natural boundary conditions on the remaining boundaries). Assume that*

$$\int_\Omega W(\nabla_h y^{(h)}) \leq Ch^2. \tag{16}$$

Then

$$y^{(h)} \to \bar{y} \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^2), \tag{17}$$

$$\partial_2 \bar{y} = 0, \quad \partial_1 \bar{y} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{18}$$

and θ solves (15)

Extensions of this result to the 3d to 1d reduction (joint work with M.G. Mora) and to the 3d to 2d reduction in the von Kármán scaling (joint work with R. Pakzad) are in preparation. Mielke [8] used a centre manifold approach to compare solutions in a thin strip to a 1d problem. His approach gives a comparison already for finite h, but it requires that the nonlinear strain $(\nabla_h y)^T \nabla_h y$ is close to the identity in L^∞ (and applied forces g cannot be easily included).

The proof uses in particular the quantitative rigidity estimate in [4] and a compensated compactness argument.

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Boundary Layer Resonance, Canards, and Blowup

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Ackerberg and O'Malley (1970) considered a class of linear singularly perturbed boundary value problems for which the classical asymptotic matching technique fails to provide the asymptotic solution. The Hermite equation $\epsilon y'' - xy' + \beta y = 0$, which is exactly solvable in terms of parabolic cylinder functions, is a typical example for x on the interval $[-1, 1]$. The limiting solution within $(-1, 1)$ can be nontrivial only when $\beta = 0,\ 1,\ 2,\ \ldots$. This phenomena of *boundary layer resonance* provoked much study, especially concerning the necessity and sufficiency of the Matkowsky condition that there be a smooth asymptotic power series solution about the turning point $x = 0$. Resonance relates to the existence of *canards* or *delayed bifurcation* for the corresponding Riccati differential equation (cf. Diener and Diener (1995) and Benoit (1991)). New methods, including *Gevrey asymptotics* and *blowup*, holds considerable promise for understanding this phenomena and others, including *logarithmic switchback*, that remain troublesome for matched expansions (cf. De Maesschalck (2006)).

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Asymptotics beyond all orders for partial differential equations

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Asymptotics beyond all orders (i.e. the subclass of exponential–asymptotic problems that requires the investigation of exponentially–small terms 'hidden' beyond a *divergent* algebraic series) play an important role in a wide range of applications, such as in governing width selection of Saffman–Taylor fingers in a Hele–Shaw cell (it being terms that are exponentially small in the regularising parameter that select from the continuum of fingers that the unregularised problem possesses) and in quantifying front pinning in, say, discrete bistable reaction–diffusion equations, the force needed to overcome the pinning (and thus generate front propagation) being exponentially small in the lattice spacing (so that subtleties arise in taking the continuum limit)[1]. While such behaviour is now relatively well understood in steady–state/travelling–wave contexts (for which references in the papers cited below provide an entry point into the literature), a number of additional complications arise in the consideration of fully–time–dependent problems, for which important open questions remain even in the case of linear PDEs; some of these are spelt out in [BKT], to which we refer for the working out of the technical details of a particular problem – our intention here is instead to highlight in general terms some of the issues that arise. Before doing so, however, we note as an aside some of the striking implications of beyond–all–orders asymptotics for multiple–scales problems, for which the details of a specific case are similarly analysed in [AKT]. Thus while traditional two–scale approaches treat the fast and slow scales (x and $X = \varepsilon x$, say, with $0 < \varepsilon \ll 1$) as distinct independent variables, it turns out in the Stokes–line (SL) calculation of exponentially–small terms that the fact that they are simply scaled versions of the same quantity becomes crucial; one manifestation of this is in the calculation of the phase of (say) a fast–oscillating homoclinic or heteroclinic connection for an autonomous nonlinear ODE; fixing the x origin by specifying a suitable point on the envelope (this slow variation being determined by secularity conditions at algebraic orders) gives no information about the phase of the oscillations because the multiple–scales formulation is separately invariant under translations of x and of X, these being inherited from a single translation

[1] Such studies evidently also have implications for the analysis of numerical methods.

invariant of the original ODE. The phase can, however, be selected (in a fashion akin to that of Saffman–Taylor fingers) from the continuum having arbitrary phase by the calculation of exponentially–small terms, whereby x and X cannot be treated as independent and invariance under a single translation is recovered.

To place the considerations that follow in context, the concept of a Stokes line needs introduction and this is best done through a linear ODE, Airy's equation $d^2y/dx^2 = xy$ providing an instructive such example. One of its solutions, $y = Ai(x)$, has asymptotic behaviour on the real line (as $x \to +\infty$ and $x \to -\infty$)

$$Ai(x) \sim \frac{1}{2(\pi x^{\frac{1}{2}})^{\frac{1}{2}}} e^{-\frac{2}{3}x^{\frac{3}{2}}}, \; Ai(x) \sim \frac{1}{2(\pi(-x)^{\frac{1}{2}})^{\frac{1}{2}}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right).$$

The issue then arises as to how the second exponential in the latter arises as one circles the complex x plane with $|x| \gg 1$ starting from the positive real axis. It emerges that $\exp(2x^{\frac{3}{2}}/3)$ is turned on by $\exp(-2x^{\frac{3}{2}}/3)$ across an SL, i.e. a curve on which (for large $|x|$ in this context) the former is maximally subdominant to the latter, namely $\arg x = \pm 2\pi i/3$. The general recipe for the SL on which an exponential $\exp(-\phi_m/\varepsilon)$ can turn on $\exp(-\phi_l/\varepsilon)$ (where ε is now playing the role of small parameter, rather than $1/|x|$) is

$$(1) \quad m \to l \qquad\qquad \text{Im } \phi_l = \text{Im } \phi_m, \quad \text{Re } \phi_l > \text{Re } \phi_m.$$

There are three key practical issues that arise in undertaking a Stokes–phenomenon calculation, as follows. (I) What exponentials (i.e. what functions $\phi_k(x,t)$ in the case of a 1+1-dimensional PDE) may be present? (II) In which regions of the complex x plane is the curve defined by (1) active, in the sense that $\exp(-\phi_m/\varepsilon)$ turns on a (non–zero) multiple of $\exp(-\phi_l/\varepsilon)$ as it is crossed? (III) If the SL $m \to l$ is active, then what multiple of $\exp(-\phi_l/\varepsilon)$ is switched on?

The answer to (I) typically follows from a JWKB analysis of the PDE that takes careful account of 'diffracted' ray fields for the associated Hamilton–Jacobi equation that arise from singularities in the initial data. For PDEs a considerable number of distinct ϕ_k's can typically arise and this leads to a difficulty with (II) associated with the higher–order Stokes phenomenon, this being shared by higher–order ODEs (at least three distinct ϕ_k being required) and on which we elaborate below. For ODEs, the values of the Stokes multipliers that determine the multiples arising in (III) are simply constants and there are well–established procedures for calculating these; PDEs exhibit the qualitatively new phenomenon that the multipliers are typically functions (of one variable less than the dimension of the PDE, so for 1+1-dimensional PDEs they are functions of one variable). This makes their calculation significantly more difficult for PDEs, typically involving the identification and solution of what amount to canonical diffraction problems. A final remark to distinguish between ODEs and time–dependent PDEs is to observe that in the latter case the asymptotic structure can change rather abruptly as the SLs evolve, in particular when associated turning points cross the real axis.

By default, a SL is expected to be active; an $m \to l$ SL can, however, be inactive for the following reasons. (A) The series associated with $\exp(-\phi_m/\varepsilon)$ is convergent (typically in practice truncating after a finite number of terms) and hence can

never switch anything on via the Stokes phenomenon (which relies intrinsically on the divergence of the relevant series). (B) No ϕ_m contribution is present in the asymptotic series in the region of the complex x plane in which the part of $m \to l$ under consideration lies. (C) Nowhere does the ϕ_m series have in its tail a contribution able to turn on $\exp(-\phi_l/\varepsilon)$; an instructive simple example of how this can arise is when the ϕ_m expansion forms part of the complementary function of a linear ODE and ϕ_l the particular integral – the former then has no knowledge of the latter and cannot switch it on. (D) In part of the complex x plane the divergent ϕ_m series contains in its tail a contribution associated with ϕ_l, but this has been turned off in the region of interest by a different such contribution (associated with ϕ_k, say) via the higher–order Stokes phenomenon.

We note that in (A) and (C) the line $m \to l$ is inactive throughout the complex x plane, while in (B) and (D) it can be active in some parts while being inactive in others.

It remains to give more details of the higher–order Stokes phenomenon. Considering expansions of the form $\exp(-\phi/\varepsilon) \sum_{n=0}^{\infty} \varepsilon^n \Phi_n$, it can be shown that the tail for $\phi = \phi_m$ has

$$\Phi_n^{(m)} \sim \pm \frac{1}{2i} \sum_{k \neq m} \frac{\Gamma(n + \delta_k - \delta_m)}{(\phi_k - \phi_m)^{n+\delta_k-\delta_m}} A_0^{(k)} \qquad \text{as } n \to \infty \tag{2}$$

for some constant δ_k, δ_m and functions $A_0^{(k)}$; (2) can be viewed as a JWKB ansatz involving $\exp\left(-n \log(\phi_k - \phi_m)\right)$ in the limit $n \to \infty$ (rather than $\varepsilon \to 0$). Some of the contributions in (2) can thus switch off others (so the latter are not in fact present) via the higher–order Stokes phenomenon, i.e. by the Stokes phenomenon associated with the limit $n \to \infty$, so (cf. (1)) the higher–order Stokes lines (HOSLs) are given by

$$S(m,k,l) \qquad \begin{array}{l} \mathrm{Im}\left(\log(\phi_l - \phi_m)\right) = \mathrm{Im}\left(\log(\phi_k - \phi_m)\right), \\ \mathrm{Re}\left(\log(\phi_l - \phi_m)\right) > \mathrm{Re}\left(\log(\phi_k - \phi_m)\right), \end{array} \tag{3}$$

this being the HOSL on which the k contribution to the tail of m can turn off the l contribution. Like a SL, a HOSL can naturally be either active or inactive, its inactivation being possible via (in particular) a second–level higher–order Stokes phenomenon[2]; it is an important observation (associated with the resurgence properties of asymptotic expansions) that the second–level HOSLs are again given by (3). Third and higher levels of HOSL activity are naturally also possible, (3) again providing the pertinent recipe.

In summary, then, a procedure for constructing a self–consistent picture of the active SLs, and hence of which contributions ϕ_k are present in each region of the

[2]This results from the $A_0^{(k)}$ in (2) each themselves being the first term in a further expansion that has a tail of the same form as (2) (in particular involving the same functions ϕ_l, in effect because of the (near tautology) that if *all* the ϕ_l's have been correctly identified then any series that arises can contain only those terms that can switch on or off *these* ϕ_l).

complex x plane, can be formulated as follows[3]. (a) Identify all of the possible ϕ_k's and compute the associated SLs and HOSLs from (1), (3); (b) determine the Stokes multipliers (corresponding to the $A_0^{(k)}$ in (2)), for example by solving suitable canonical problems; (c) identify suitable starting points in the complex x plane (for example by obtaining the far field from a small–time analysis or by solving locally to a turning point) and track across the full plane by appropriate bookkeeping of SL activity (typically requiring accounting also for HOSL behaviour at one or more levels). Following various paths to the same destinations can provide important self–consistency checks on such analyses.

Given the constraints of space here we have not even mentioned a number of important aspects of beyond–all–orders problems (for example, the widespread applicability of factorial/power ansätze such as (2), the role of optimal truncation of divergent asymptotic series and the significance of crossing points and of second–generation Stokes switching). We conclude by mentioning that while nonlinear problems are naturally significantly more challenging than related linear ones with regard to beyond–all–orders phenomena, the leading–order problems satisfied by exponentially subdominant terms are of course linear, allowing the superposition of distinct effects (including stochastic ones, say) in investigating the associated exponential sensitivity of such models.

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Nucleation Transients

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(joint work with Yossi Farjoun)

Nucleation refers to the self assemly of identical, initially separated particles (monomers) into clusters. A classical example is precipitation of crystals from a solution. Our analysis starts by modifying the classical nucleation kinetics due to Becker-Döring (1935), which assumes that the concentration of monomers about clusters is uniform. Specifically we allow the local monomer volume fraction $\rho(\mathbf{x}, t)$ to be nonuniform, as naturally arises when the diffusivity of monomers is finite.

[3]It should be remarked that careful account needs to be taken of branch cuts, across which different ϕ_k can swap Riemann sheets, in implementing such a procedure.

Conventional wisdom says that "small" clusters see uniform monomer concentration, while large ones are surrounded by 'diffusion layers' with nonuniform concentration. Our new model identifies a critical particle number h_* that separates 'large' from 'small'

$$h_* = \frac{D^3\rho_s^3}{\omega^3\nu^2}\,. \tag{1}$$

Here, D is the diffusivity of monomers, ρ_s is the monomer concentration in the saturated solution, ω is the time constant for a particle to disassociate from a cluster into the solution, and ν is the monomer volume. Clusters with $h \ll h_*$ particles see asymptotically uniform monomer concentrations, while clusters with $h \gg h_*$ particles are surrounded by strong diffusion layers. Hence, clusters of size $h \ll h_*$ are described by conventional Becker-Döring theory. In particular, the creations of clusters occurs at the rate prdicted by Zeldovich (1942) from classical Becker-Döring:

$$\text{Nucleation rate} = \Omega \exp -\frac{\sigma^3}{2\eta^2}\,. \tag{2}$$

Here, Ω, σ are positive constants made of material parameters, and η is the oversaturation,

$$\eta \equiv \frac{\rho - \rho_s}{\rho_s}\,, \tag{3}$$

which measures the excess of the monomer volume fractions ρ above the saturation value ρ_s. The newly created clusters grow (aquire more particles) at a rate proportional to $h^{2/3}$ (surface area) as long as $h \ll h_*$. When h equals h_* in magnitude, the rate drops below $h^{2/3}$, eventually settling to $h^{1/3}$ when $h \gg h_*$:

$$\dot{h} \sim d\eta h^{1/3} \quad \text{for} \quad h \gg h_*\,. \tag{4}$$

Here, d is a nondimensionalized diffusion, $d \equiv D\rho_s/\nu^{2/3}$.

The physics of cluster creation and growth as described above is summarized in a signaling problem for the density $r(h,t)$ of clusters in the space of cluster size h: $r(h,t)$ satisfies an advection PDE,

$$\partial_t r + \partial_h \left(d\eta h^{1/3} r \right) = 0 \tag{5}$$

for $h \gg h_*$, consistent with the 'diffusion limited' growth rate (4). The creation rate (2) is expressed by an effective boundary condition on r along $h = 0$,

$$d\eta h^{1/3} r \to \Omega \exp\left(-\frac{\sigma^3}{2\eta^2} \right). \tag{6}$$

In these equations, the oversaturation η is a function of time consistent with the conservation of particles. The conservation is expressed by

$$\eta(t) = \eta_* - \Lambda \int_0^\infty h r(h,t)\, dh\,. \tag{7}$$

Here, η_* is the initial concentration before clusters are created, Λ is a positive constant, and the integral represents the number of particles trapped in clusters.

In the limit of small initial oversaturation $\varepsilon \equiv \eta_* \ll 1$, we can derive distinguished limit scalings of h, t, r and the change $\delta\eta = \eta(t) - \varepsilon$ of oversaturation. In particular, the characteristic cluster size and characteristic time are exponentially large in ε,

$$[h] \propto \exp\left(\frac{3}{5}\frac{\sigma^3}{2\varepsilon^2}\right), \qquad [t] \propto \exp\left(\frac{2}{5}\frac{\sigma^3}{2\varepsilon^2}\right). \tag{8}$$

Physically, $g \equiv \sigma^3/2\varepsilon^2$ is the activation energy barrier to nucleation of clusters, so (8) says that is takes time $\exp(2/5g)$ to make clusters of size $\exp(3/5g)$.

Applying the distinguished limit scalings to the signaling problem (5–7) and taking the limit $\varepsilon \to 0$, we find simple reduced equations, which in turn reduce to simple integral equations for the change $\delta\eta$ of oversaturation as afunction of time. From a simple numerical solution for $\delta\eta(t)$, we reconstrauct $r(h,t)$. $r(h,t)$ takes the form of a 'pulse' with a discontinuous jump from zero at the largest cluster size.

The account of the 'nucleation era' described here is not the end of the story. The analysis can be continued in time, all the way to the 'coarsening process', analysed by Lifshitz and Slyosov (LS) (1961). Our analysis selects the discontinuous LS similarity solution in a characteristic time yet longer than $[t]$ in (8). But this too is not the end and we refer you to B. Niethammer's report, which indicates a variety of mechanisms which we believe will select the infinitely smooth solutions on some time scale even longer than the 'coarsening time' of our analysis.

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Propagation in Periodic Dielectric Media

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(joint work with I. D. Abrahams)

We have employed a homogenization procedure to describe the propagation of electromagnetic waves in a dielectric structure which is periodic in the x-y plane and translationally invariant in the direction of propagation, z. The fundamental cell is rectangular and is composed of an arbitrarily shaped pore filled with a dielectric and the host by another. The indices of refraction of the pore and host are N_1^2 and N_2^2, respectively. The pore shape is allowed to depend upon z. The

dimensions of the cell and pore are small compared to a reference wavelength which is commensurate with the variation of the pore shape in the z direction.

Our analysis yields the structure of the electromagnetic fields at the micro level and gives the effective medium equation at the macro level

$$\frac{d^2A}{dz^2} + k^2 \left\{ < N^2 > + \frac{1}{l} \left[N^2\right]_C^2 \oint P \frac{\partial}{\partial n} x \, ds \right\} A = 0 \tag{1}$$

where

$$< N^2 >= \frac{A_1}{l} N_1^2 + \frac{A_2}{l} N_2^2, \tag{2}$$

$\left[N^2\right]_C$ is the jump in the indices of refraction across C, the pore boundary.

The bracketed term in (1) is the effective index of refraction for our periodic structure. It contains a simple arithmetic average of the dielectric constants and a correction term which involves a line integral around the pore. The function P in this integral is harmonic in the cell and along with its first partial derivative are periodic. The function P is smooth across the curve C but its normal derivative there satisfies

$$\left[\frac{\partial}{\partial n} P\right]_C = \frac{\partial}{\partial n} x. \tag{3}$$

The presence of the normal derivative of x in (1) and (3) assumes that the electromagnetic wave is polarized, in the absence of the pore, with the electric in the x direction. If the polarization is such that the electric field was in the y direction, then the normal derivatives of x would be replaced by normal derivatives of y.

At the microscopic level all the electric field components are proportional to A and depend upon the harmonic function P. The same is true for the magnetic fields except they are proportional to $\dfrac{dA}{dz}$.

The implementation of our theory requires finding the harmonic function P. If the pore shape remains fixed in z, then the computational burden in approximating P is minimal. However, if the pore shape changes in z- as would happen in certain microwave assisted chemical vapor infiltration processes- then P needs to be computed at many values of z in order to numerically integrate (1a). This in principle will create a serious computational burden. We are in the process of developing variational approximations to alleviate this cost.

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Regularizations of mean-field theories for Ostwald ripening

Barbara Niethammer
(joint work with J. Velázquez)

Ostwald ripening denotes the late stage coarsening of second phase particles in a first order phase transition. The particles interact by diffusional mass exchange to reduce their total surface area. One expects that the system, after some transient initial stage, evolves in a universal statistically self-similar manner.

The classical theory by Lifshitz, Slyozov and Wagner (LSW) [3] is based on the assumption that in the dilute regime particles interact with each other only via a spatially constant mean-field u_∞, which is determined by the constraint that the volume fraction of particles is conserved. This approach yields a nonlocal evolution law for the particle radius distribution, which has a one-parameter family of self-similar solutions, all with compact support. LSW predict in their classical theory that one particular of those profiles, the one with the largest support, characterizes the large-time behavior of all solutions.

However, the long-time behavior of the LSW-equation is not at all universal but depends very sensitively on the data [1, 7]. Roughly speaking, the dynamics are determined by the details of the initial distribution of largest particles. In addition, all self-similar solutions and the corresponding coarsening rates show significant discrepancies with experimental data.

Several different mechanisms which have been neglected in the LSW theory have been considered to overcome these shortcomings in the LSW model. In [5, 6], it is argued that within Becker-Döring type models for nucleation certain initial data for the coarsening regime are selected which then implies universal self-similar coarsening.

While the selection problem might be solved by taking nucleation into account, this effect cannot overcome the disagreement with experiment. Higher order effects, due to the finiteness of the volume fraction of particles, have been extensively considered in the applied literature (see [8] for an overview). There are at least two possible effects which have to be considered.

First, there are fluctuations in the particle densities due to screening. Screening, analogous to electrostatic screening, implies that the interaction range of a particle is not infinite, as assumed in the mean-field theory, but finite. A simple scaling argument implies that the expected correction in the equation for the particle number density is of order $\phi^{1/2}$ if $\phi \ll 1$ is the volume fraction of the particles. A theory which proposes a closed equation for the one- and two-particle number densities has been developed in [4] and rederived in mathematically more rigorous way in [2]. However, the theory is based on the assumption that correlations are uniformly small in size space, which is not satisfied for the larges particles in

the system. For the largest particles, a boundary layer appears, which has been analyzed in [9]. It is shown that the corresponding correction to the mean-field model is a second-order diffusive type term which is only relevant for the largest particles. It provides a selection criterion for a unique self-similar solution which is a perturbation of the LSW self-similar solution with an exponential tail. The corresponding correction to the mean radius is of order $\phi^{1/4}$.

A second higher order effect which induces corrections to the mean-field model are encounters between particles, which has already been investigated in [3], but a careful analysis of the order of size of the corresponding corrective terms has not been made. On a first glance, the effect of particle collisions is smaller than the corrections due to screening since the fraction of particles involved in collisions is proportional to ϕ. However, it turns out that the relative size of the corrective terms are not the same for all particles, but that they are larger for the largest particles in the system. Since those largest particles determine the coarsening dynamics for large times, this effect is crucial. An ad-hoc model has been suggested in [3], which is the LSW model plus a coagulation type integral term. A rough analysis suggests that also within this model a unique self-similar solution is selected and the correction to the mean radius is of order $\frac{1}{(\ln \phi)^2}$ which is much larger than the corrections due to fluctuations. However, since the right hand side in the encounter model is quadratic in the size distribution, one can expect that the effect is the most relevant only for very large time. The precise time scales over which what effect is most relevant, have still to be identified.

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Long-time asymptotic behavior for coarsening of dewetting fluid films

Thomas P. Witelski
(joint work with Karl Glasner)

The study of instabilities of thin liquid films on solid surfaces is of great importance in understanding coating flows used in many real-world applications. These instabilities lead to rupture, the formation of dry spots, and further morphological changes that promote non-uniformity of coatings; these behaviors in unstable thin films are generally called *dewetting*. To account for these effects, classical lubrication models of fluid flow must incorporate terms describing the influence of material properties of the solid and fluid via a disjoining pressure function, $\Pi(h)$. The resulting nonlinear evolution PDE for the film height $h(x,t)$

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^3 \frac{\partial}{\partial x}\left[\Pi(h) - \frac{\partial^2 h}{\partial x^2}\right]\right)$$

can accurately reproduce the complex physical pattern formation observed in experiments. Following initial transients, the film breaks up into large arrays of near-equilibrium droplets. Energetic considerations favor *coarsening*, that is, the successive re-arrangement and re-grouping of smaller drops into fewer larger drops. The long-time evolution of this PDE problem can be reduced to the solution of a system of coupled ODEs for the masses and positions of the N droplets, $k = 1, 2, \cdots, N$,

$$\frac{dP_k}{dt} = C_P(P_k)(J_{k,k+1} - J_{k-1,k}), \qquad \frac{dX_k}{dt} = -C_X(P_k)(J_{k,k+1} + J_{k-1,k}),$$

where the droplet pressure P_k is inversely related to the droplet mass. The fluxes $J_{m,n}$ are well-defined functions of the pressures and positions, $\{X_m, P_m, X_n, P_n\}$, of neighboring droplets, and is derived from a model for the slowly varying thin film between the droplets. Parameter regimes where droplet coarsening by each of two mechanisms (collision and collapse) are identified, and power laws for the evolution for the number of droplets in the system, $N(t)$, are explained. This is joint work with Karl Glasner, University of Arizona [1, 2]. Further rigorous results on this system have been studied in [3].

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Asymptotics of Dewetting Films in the Presence of Slippage

Andreas Münch
(joint work with P.L. Evans, J.R. King, B. Wagner, T.P. Witelski)

Lubrication models have shown to be extremely useful approximations to the full Navier-Stokes equations for investigating the dynamics of thin liquid films, including the motion and instabilities of their contact lines [1]. For film thicknesses in the range of a few micrometers and larger, the choice of the boundary condition at the solid substrate enters only weakly.

For the dynamics of complex fluids, such as thin polymer films dewetting from hydrophobic substrates, within the the nanofluidic realm, slippage may play the dominant role [2]. In order to describe their behaviour, we have derived and studied a set of lubrication models for the thin film flow of incompressible fluids on solid substrates [3]. The models are obtained as asymptotic limits of the Navier-Stokes equations with the Navier-slip boundary condition

$$u(z=0) = b\partial_z u(z=0), \tag{1}$$

for different orders of magnitude for the slip-length parameter b. The slip-length parameter b can be related to an off-set length such that the fluid velocity at the solid surface is given by the slip-length times the normal derivative of the velocity.

Here u denotes the velocity component parallel to the substrate, z is the coordinate axis pointing normal to the substrate, with the liquid/solid interface located at $z = 0$. For flows of Newtonian fluids on smooth rigid surfaces, the classic no-slip boundary condition, $u(z = 0) = 0$, corresponds to zero slip length. Positive slip lengths have been used to model many different physical systems including effects such as surface roughness and non-Newtonian fluid properties, see e.g. the recent review [4]. We furthermore include an intermolecular potential to represent the chemical and molecular-scale physical properties at the solid.

The family of closed-form lubrication models include the distinguished limits

$$\partial_t h = -\partial_x \left[m(h) \partial_x \left(\partial_{xx} h - \phi'(h) \right) \right], \qquad \text{with} \quad m(h) = \left(h^3 + b h^2 \right), \tag{2}$$

and *strong slip* regimes,

$$\mathrm{Re}^* \left(\partial_t u + u \partial_x u \right) = \frac{4}{h} \partial_x (h \partial_x u) + \partial_x \left(\partial_{xx} h - \phi'(h) \right) - \frac{u}{\beta h}, \tag{3a}$$

$$\partial_t h = - \partial_x \left(h u \right). \tag{3b}$$

We note that in the latter case, u does not depend on z. The parameters in (3) are the (scaled) Reynolds number Re^* and the slip parameter β, a scaled form of b. For other choices of slip scalings, these models reduce to other models that have been used in different parts of fluid dynamics. In particular we note here the no-slip and intermediate slip model, which have the same scalar form as (2) but with mobility $m(h) = h^3$ or $m(h) = h^2$, respectively. In contrast, for very large

values of β, eqns. (3) remains a system of PDEs that describe the motion of a free film (elongational flow).

Matched asymptotics over several time and spatial scales are used to describe the dynamic profiles for dewetting films. The motion of the dewetting front shows transitions from being nearly linear in time for no-slip to $t^{2/3}$ as the slip is increased. For much larger slip lengths the front motion appears to become linear again. Correspondingly, the dewetting profiles undergo a transition from oscillatory to monotone decay into the uniform film layer for large slip. Increasing the slip further, to very large values, is associated with an increasing degree of asymmetry in the structure of the dewetting ridge profile.

We could show that for the *strong slip regime* the morphology of the cross section of the dewetting rim changes from a damped oscillating structure behind the rim and towards the undisturbed film, to a monotone decaying structure, as the slip length is increased. Stability analysis yields the critical slip length for this transition. This property of the strong slip lubrication model captures the recently experimentally observed morphological transition of the ridge when the liquid/solid friction was altered [5].

For even larger slip lengths, the exponential decay rate decreases and the ridge profiles show a pronounced asymmetry [3, 6], a property that has previously been ascribed to the viscoelastic nature of the polymer.

In higher dimensions we could derive, via matched asymptotics, sharp interface models for the *no-slip* $(mh) = h^3)$ and *intermediate slip regimes* $(mh) = h^2)$. Stability analysis yield that the intermediate slip regime shows the characteristic finger instability along the contact-line [8, 7], that has been observed experimentally.

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On the asymptotic behavior of solutions of semilinear parabolic equations of second order

Yu.V. Egorov
(joint work with V.A. Kondratiev)

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitzean boundary $\partial\Omega$, L be an elliptic operator of the form:

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j},$$

where $a_{ij}(x)$ be bounded measurable functions in Ω, $x = (x_1, \dots x_n)$ and $\sum_{i,j=1}^{n} a_{ij}\xi_i\,\xi_j \geq m|\xi|^2$, $|\xi|^2 = \sum_{i=1}^{n} \xi_i^2$, $\quad \xi \in \mathbb{R}^n, \quad x \in \Omega, \quad m = const > 0$.

Consider solutions to the equation :

$$\frac{\partial u}{\partial t} = Lu - f(x,u) \tag{1}$$

in $\Pi_0 = \Omega \times (0,\infty)$, satisfying the nonlinear boundary condition

$$\frac{\partial u}{\partial \nu} + g(x,u) = 0, \quad x \in \Gamma_0 = \partial\Omega \times (0,\infty), \tag{2}$$

where $\frac{\partial u}{\partial \nu} \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} cos(\vec{n}, x_j)$, $\vec{n}$ is the unit outer normal vector to $\partial\Omega$. The functions $f(x,u)$, $g(x,u)$ are supposed to be continuous and differentiable with respect to u.

The asymptotics of the solutions of problem (1)-(2) as $t \to \infty$ were studied by P. Baras, L. Veron [1], Yu.V. Egorov, V.A. Kondratiev, O.A.Oleinik [2], V.A. Kondratiev, L. Veron [3], I.V. Filimonova [4], Yu.V. Egorov, V.A. Kondratiev [5],[6], K.T. Boni [7].

Suppose that

$$|f(x,u)| \leq c \int\limits_{\Omega} |f(x,u)|dx, \quad |g(x,u)| \leq c \int\limits_{\partial\Omega} |g(x,u)|ds \text{ for all } u \in \mathbb{R}^1; \tag{3}$$

$$|f_u(x,u)| \leq c \int\limits_{\Omega} |f_u(x,u)|dx, \quad |g_u(x,u)| \leq c \int\limits_{\partial\Omega} |g_u(x,u)ds \text{ for all } u \in \mathbb{R}^1; \tag{4}$$

$$\frac{f(x,ku)}{k} \geq f(x,u) \text{ for all } k \geq 1, \quad \frac{g(x,ku)}{k} \geq g(x,u) \text{ for any } k \geq 1,\ u > 0; \tag{5}$$

$$\frac{f(x,ku)}{k} \leq f(x,u) \text{ for } k \geq 1, \quad \frac{g(x,ku)}{k} \leq g(x,u) \text{ for } k \geq 1,\ u < 0. \tag{6}$$

Theorem 1. *If conditions* $(3) - (6)$ *are satisfied then any solution of equation* (1), *satisfying* (2) *on* $\Gamma_{a,\infty}$, *tends to zero as* $t \to \infty$ *uniformly in* $\overline{\Omega}$. *Suppose that*

$$\liminf_{v \to +0} \frac{vh'(v)}{h(v)} > 1.$$

Then

$$u(t,x) = \alpha_0(t)(1 + o(1)),$$

where α_0 is the solution to the problem

$$\alpha_0'(t) = -h(\alpha_0(t)), \quad \alpha_0(0) = 1.$$

Other results relate to the case when

$$\liminf_{v \to +0} \frac{vh'(v)}{h(v)} = 1.$$

Some examples are given.

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Asymptotic model of size effects in fracture

Dominique Leguillon

This work relies on an Irwin-like criterion able to predict brittle crack nucleation at corners, v-notches and other situations such as interfaces breaking a free surface (delamination initiation). It is based simultaneously on an energy and a maximum stress criteria. The reason of this dual formulation can be found for instance in Parvizi et al. [7] experiments as analysed in a previous paper of the author [1]. It is shown that, if the singular exponent of the stress elastic field at the concentration point is not 1/2 (a pure crack), the crack nucleation is a brutal process (unstable) at least on a short initiation length ℓ. This distance being used as a small parameter in matched asymptotics, it leads to the following expansion of the energy release rate G and then to the Irwin-like criterion:

$$G = kA\ell^{2\lambda-1} + \ldots \, ; k \geq k_c = \left(\frac{G_c}{A}\right)^{1-\lambda} \sigma_c^{2\lambda-1} \tag{1}$$

where A is a scaling coefficient, k is the generalised intensity factor of the singularity with characteristic exponent λ. Here, G_c and σ_c are two failure parameters:

toughness and strength of the material (the use of two parameters is a usual feature in Cohesive Zone Models). The above criterion matches with the Griffith one for a crack ($\lambda = 1/2$) and with the stress one for a straight edge without stress concentration ($\lambda = 1$).
The matched asymptotic procedures allow analysing neighbouring problems dealing with blunt cracks (the existence of a notch tip radius (figure 1), a soft layer ahead of the crack tip in a bi- or laminated material (figure 2), a small cavity (figure 3)).

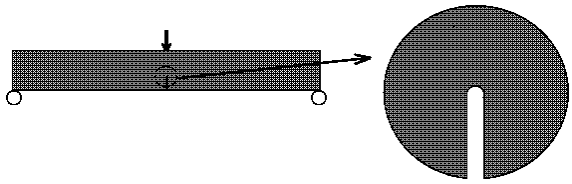

FIGURE 1. A blunted notch.

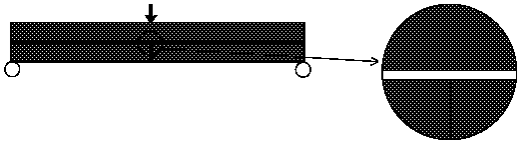

FIGURE 2. A compliant interlayer between stiff substrates.

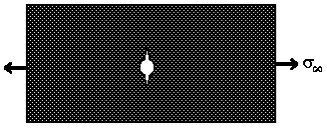

FIGURE 3. A small cavity in a plate in tension.

The main difference is that there are now two competing small parameters. In the examples cited above, in addition to ℓ, it is respectively the notch tip radius, the interlayer thickness or the cavity diameter, they are all noted d. A new expression of the energy release rate comes out:

$$G = kA(\mu)d^{2\lambda-1} + ... \tag{2}$$

where $\mu = \ell/d$ is the ratio of the two small parameters. The scaling coefficient A is a function of μ and the criterion is no longer explicit since the stress field cannot

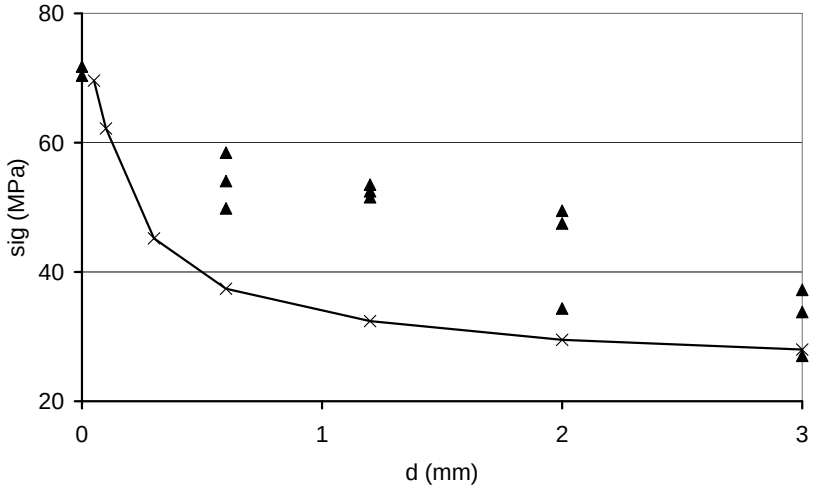

FIGURE 4. Comparison between experiments and prediction for a drilled plate in tension.

be asymptotically expanded directly in power terms. Nevertheless it is possible to give an expression that requires simply the numerical knowledge of the stress field ahead of the primary crack. A FE computation can be carried out once for all on a simplified geometry so-called "inner domain". The role of the characteristic length d of the microstructure at the origin of the size effect is clear in the expression (2). Applications are proposed on the influence of the notch tip radius on the apparent toughness [2, 9]; the step-over mechanism in bedded sediments [8]; the role of an adhesive layer between two steel plates [3]; the crack path in ceramic laminates [4].

To illustrate our purpose let us consider a small circular hole in a plate submitted to a remote tension (figure 3). The two small parameters are the crack increment length and the diameter of the hole. It is known that the stress concentration factor of such a structure is 3 prior to any crack onset. Following a maximum stress criterion, it means that failure would occur for $\sigma_\infty = \sigma_c/3$ whatever the diameter of the cavity (σ_∞ is the remote tension and σ_c the strength of the material). On the contrary experiments show that if the cavity becomes smaller and smaller, the applied load leading to failure increases from $\sigma_c/3$ to σ_c. The above approach, taking into account an initiation length, is able to render (not as well as expected

but this study is in progress) this effect as shown in figure 4 (the solid line is the theoretical prediction and the triangles correspond to experiments on PMMA [5]).

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Homogenization of degenerate elliptic equations

Svetlana E. Pastukhova

1. Consider in $\mathbb{R}^d$ equation

$$-\mathrm{div}\rho\mathrm{A}\nabla\mathrm{u}+\rho\mathrm{u}=\mathrm{f},\quad \mathrm{f}\in\mathrm{C}_0^\infty(\mathbb{R}^\mathrm{d}), \tag{1}$$

where $\rho=\rho(x)\ge 0$, $\quad \rho,\rho^{-1}\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$; A is a measurable symmetric matrice, such that $\lambda\xi^2\le A\xi\xi\le\lambda^{-1}\xi^2\quad\forall\xi\in\mathbb{R}^d,\quad\lambda>0$. Introduce Sobolev weighted space

$$W=W(\mathbb{R}^d,\rho\,dx)=\{u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d):\quad\int\limits_{\mathbb{R}^d}(u^2+|\nabla u|^2)\rho\,dx<\infty\}.$$

The function $u\in W$ is a weak solution, if

$$\int\limits_{\mathbb{R}^d}(A\nabla u\nabla\varphi+u\varphi)\rho\,dx=\int\limits_{\mathbb{R}^d}f\varphi\,dx \tag{2}$$

for each $\varphi\in C_0^\infty(\mathbb{R}^d)$.

If ρ is locally non-degenerate, i.e. $\rho,\rho^{-1}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, then smooth functions are dense in W and the equation (1) is uniquely solvable. Otherwise we define the subspace $H=H(\mathbb{R}^d,\rho\,dx)$ as a closure of $C_0^\infty(\mathbb{R}^d)$ in W. Consider the intermediate space V, such that $H\subseteq V\subseteq W$. By definition, function $u\in V$ is a V-solution (or variational solution) if integral identity (2) holds for test functions $\varphi\in V$. Variational solutions do not exhaust the whole set of weak solutions.

2. Consider approximation ρ^h, such that

$$\rho^h, (\rho^h)^{-1} \in L^\infty_{\rm loc}(\mathbb{R}^d), \quad \rho^h \to \rho, \quad (\rho^h)^{-1} \to \rho^{-1} \text{ in } L^1_{\rm loc}(\mathbb{R}^d). \tag{3}$$

For example, direct and inverse smoothings

$$\rho^h = (\rho)_h \text{ and } \rho^h = ((\rho^{-1})_h)^{-1} \tag{4}$$

are subordinate to condition (3). Here $(f)_h$ denotes the classical smoothing by means of convolution with *delta* -type bounded non-negative finite even kernel, for instanse Steklov smoothing (see below).

Lemma 1. *Let ρ^h be an arbitrary approximation of the type (3), u^h the solution of the problem*

$$u^h \in W(\mathbb{R}^d, \rho^h dx) = H(\mathbb{R}^d, \rho^h dx), \quad -\mathrm{div}\rho^{\mathrm{h}}\mathrm{A}\nabla\mathrm{u}^{\mathrm{h}} + \rho^{\mathrm{h}}\mathrm{u}^{\mathrm{h}} = \mathrm{f}.$$

Then $u^h \rightharpoonup u$, $\nabla u^h \rightharpoonup \nabla u$ in $L^2(\mathbb{R}^d, \rho^h dx)$, where u is a weak solution to the problem (1) called attainable or approximation solution.

It can be easily shown that there exist non-attainable weak solutions which are not variational at the same time. The question arises: maybe, all variational solutions are attainable? The answer in general is not known. Nevertheless, the following fact is true.

Lemma 2. *H- and W-solution of the equation (1) are attained respectively by means of direct and inverse smoothing of the weight ρ, see (4).*

3. Pass to homogenization problems. Suppose, that $\rho(y)$ and $A(y)$ are 1-periodic, $Y = [0,1)^d$ is a periodicity cell; ρ, $\rho^{-1} \in L^1(Y)$. Consider the equation

$$-\mathrm{div}\rho_\varepsilon \mathrm{A}_\varepsilon \nabla \mathrm{u}_\varepsilon + \rho_\varepsilon \mathrm{u}_\varepsilon = \mathrm{f}, \quad \mathrm{f} \in \mathrm{C}_0^\infty(\mathbb{R}^{\mathrm{d}}), \tag{5}$$

where $\rho_\varepsilon = \rho(\varepsilon^{-1}x)$, $A_\varepsilon = A(\varepsilon^{-1}x)$. Our aim is to investigate for $\varepsilon \to 0$ the asymptotic behavior of different type solutions to equation (5) including H_ε- and W_ε-solutions. Here $W_\varepsilon = W(\mathbb{R}^d, \rho_\varepsilon dx)$ and $H_\varepsilon = H(\mathbb{R}^d, \rho_\varepsilon dx)$.

Introduce Sobolev weighted space of periodic functions

$$W_{\rm per} = \{u \in W^{1,1}_{\rm per}(Y): \quad \langle u \rangle = 0, \quad \langle \rho |\nabla u|^2 \rangle < \infty\},$$

and its subspace $H_{\rm per}$ that is the closure of $C^\infty_{\rm per}(Y)$ in $W_{\rm per}$. For generic weight $H_{\rm per} \neq W_{\rm per}$.

Remind the main homogenization attributes.

(i) Problem on the cell of periodicity

$$w \in W_{\rm per}, \quad \mathrm{div}\rho(\mathrm{y})\mathrm{A}(\mathrm{y})(\xi + \nabla \mathrm{w}(\mathrm{y})) = 0, \quad \xi \in \mathbb{R}^{\mathrm{d}}.$$

(ii) Homogenized matrice $\hat{A}$, defined through the solution to cell problem by equality

$$\hat{A}\xi = \langle \rho A(\xi + \nabla w) \rangle.$$

(iii) Homogenized equation

$$u \in H^1(\mathbb{R}^d), \quad -\mathrm{div}\hat{\mathrm{A}}\nabla \mathrm{u} + \mathrm{u} = \mathrm{f}. \tag{6}$$

All these objects are uniquely defined if $H_{\rm per} = W_{\rm per}$ and hence the cell problem is uniquely solvable. Otherwise, there can exist various solutions of cell problem ($W_{\rm per}$-solution, $H_{\rm per}$-solution and other approximation solutions) to which various

homogenized matrices correspond. Thus we have various homogenized equations, depending on the choice of the solution to the cell problem. All homogenized matrices satisfy the relation: $\hat{A}_1 \le \hat{A} \le \hat{A}_2$, where $\hat{A}_1$ and $\hat{A}_2$ correspond to $W_{\rm per}$- and $H_{\rm per}$-solutions respectively.

Theorem 1. *If u_ε is the approximation solution of the equation (5), then $u_\varepsilon \rightharpoonup u$ in $L^2(\mathbb{R}^d, \rho_\varepsilon dx)$, where u is the solution to (6) with appropriate matrice $\hat{A}$. In particular, if u_ε is W_ε-solution (or H_ε-solution) the equation (6) has the matrice $\hat{A}_1$ (or $\hat{A}_2$).*

4. Under additional conditions on weight ρ we prove some estimates.

Theorem 2. *Suppose, that*

$$\rho \in L^r(Y), \quad \rho^{-1} \in L^s(Y), \ \text{where } 2d^{-1} = r^{-1} + s^{-1}, \tag{7}$$

then the following estimate holds

$$\int\limits_{\mathbb{R}^d} |u_\varepsilon - (u)_\varepsilon|^2 \rho_\varepsilon \, dx \le C\varepsilon^2 \int\limits_{\mathbb{R}^d} \rho_\varepsilon^{-1} f^2 \, dx. \tag{8}$$

Here u_ε is the approximation solution of the problem (5), u is the solution of appropriate homogenized problem (6), $(u)_\varepsilon = \int\limits_Y u(\cdot + \varepsilon t)dt$ is Steklov smoothing of u, constant C depends only on dimension d, ellipticity constant λ and norms $\|\rho\|_{L^r}$, $\|\rho^{-1}\|_{L^s}$.

The proof of the theorem 2 is based on the technique from [ZP1],[ZP2], which involves different properties of Steklov smoothing. Some of them are valid if weighted Poincare inequality holds in unit cube in $\mathbb{R}^d$. To this end condition (7) is demanded. More restricted assumption on weight allows to obtain similar estimate with usual zeroth approximation.

Theorem 3. *If $\rho \in L^\infty(Y)$, $\rho^{-1} \in L^s(Y)$, where $2s > d$, it is possible to replace in (8) Steklov smoothing $(u)_\varepsilon$ with the function u itself.*

If instead of zeroth approximation (u or $(u)_\varepsilon$) we take classical "first approximation" or smoothed first approximation, then we obtain W_ε-estimate for its difference with the exact solution. The right-hand side of this estimate will be as in (8).

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Stable finite elements for the Stokes problem on anisotropic meshes

Thomas Apel

Problems with a small parameter lead often to solutions with boundary or interior layers. These are lower-dimensional features where the solution varies heavily only in the direction(s) perpendicular to a lower dimensional manifold. Examples include the Navier-Stokes problem and shell models.

Lower dimensional features in the solution occur not only due to small parameters but also due to edges of three-simensional domains. These so called edge singularities are typical for elliptic problems and lead to unbounded derivatives in directions perpendicularly to the edge in comparison to moderate values of the derivatives in edge direction.

Effective numerical methods take the anisotropic features of the solution into account which leads to the use of *anisotropic meshes*. They are characterized by elements with a large or even asymptotically unbounded aspect ratio. The standard finite element theory excludes such elements since a refined analysis is necessary. Together with several coauthors we developed in the 1990ies the interpolation theory for such elements and applied it to the convergence analysis of scalar elliptic problems with edge singularities and boundary layers, see, e.g., [1, 2, 3, 4] and also the summary in [5].

In solving the Stokes or Navier-Stokes problem, the question arises which pairs of elements are stable uniformly in the aspect ratio. In [6], we succeeded for the lowest order non-conforming element, named after Crouzeix and Raviart, to prove stability without any assumption on the mesh. Moreover, we proved optimal bounds for the discretization on appropriately graded anisotropic meshes for the Stokes problem in a prismatic domain, exploiting only the low regularity of such a solution.

The stability on general meshes is not proved for any other element pair. Contrary, for many pairs we have found pathological families of meshes where the stability constant tends to zero. The challenge is merely to characterize (sufficiently large) classes of meshes such that stability can be proven. One strategy is to consider two-level families of meshes. This is to split the domain into shape regular macro-elements and then to apply local refinement strategies in the macroelements. To our knowledge, this strategy was first devised in [7], where a family of two-dimensional finite elements in the context of the p-version of the finite element method was investigated. For related work we refer to [8].

In [9], we used this strategy to prove stability of some low order discretizations, as known from papers of Fortin and Bernardi/Raugel. This work is restricted to the two-dimensional case; the three-dimensional case is open.

The same strategy works for two families of rectangular elements of arbitrary order that we investigated in [10]. The extension of this work to more irregular meshes and to the three-dimensional case is not straightforward and open.

Finally we refer to [11] for computational studies of further elements with some negative but also some positive results for which a theoretical investigation is still

open. The most prominent example is the family of Taylor-Hood elements which works on some families of meshes but withstand the theoretical investigation up to now.

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Asymptotics of the eigenelements of a "dumb-bell with a thin handle"

Rustem Gadyl'shin

We consider the Neumann boundary-value problem of finding the small-parameter asymptotics of the eigenelements for the Laplace operator in a singularly perturbed domain consisting of two bounded domains joined by a thin "handle". The small parameter is the diameter of the cross-section of the handle. We show that as the small parameter tends to zero the eigenvalues of this perturbed problem converge either to the eigenvalues corresponding to the domains joined or to the eigenvalues of the Dirichlet problem for the Sturm-Liouville operator on the segment to which the thin handel contracts.

The main results of the work are the complete power small-parameter asymptotics of the eigenvalues and the corresponding eigenfunctions and explicit formulae for the first terms of the asymptotics. We consider critical cases generated by the choice of the place where the thin "handle" is joined to the domains, as well as by the multiplicity of the eigenvalues corresponding to the domains joined.

The complete formulations of the statements and its proofs of all results are given in [1] and [2].

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