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Algebraic K-Theory

Organised by
Daniel R. Grayson (Urbana)
Annette Huber-Klawitter (Leipzig)
Uwe Jannsen (Regensburg)
Marc Levine (Boston)

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Abstract. This is the report on the Oberwolfach workshop *Algebraic K-Theory*, held in July 2006. The talks covered mainly topics from Algebraic Geometry and Number Theory in connection with K-Theory. Special emphasis was placed on motivic cohomology and motivic homotopy of general schemes.

Mathematics Subject Classification (2000): 19Dxx, 19Exx, 19Fxx.

Introduction by the Organisers

Vaguely speaking, K-theory is a way of examining features of systems of polynomial equations by considering the possible ways to associate vector spaces to each solution. While the objects of study come from algebraic geometry or number theory the methods come from algebraic topology. In the last decade motivic cohomology has matured to provide a useful tool for computing algebraic K-theory.

The conference covered a wide range of interconnected topics from classical Milnor K-theory, cyclic homology, algebraic K-theory (in the narrow sense), Chow groups, regulators, homotopy theory of schemes with a certain stress in motivic cohomology and triangulated motives. Algebraic K-theory is the theme holding these questions together. This was reflected both in the selection of the participants as well as in the choice of talks. The 48 participants came mostly from Europe and North America, but also from Japan and China. There was a good mix of participants, from leading experts of the field to younger researchers and even a couple of graduate students. There were 17 mostly 1-hour talks. Priority was given to young researchers, to allow them to present their results. The schedule left enough time for interesting discussions between participants.

Workshop: Algebraic K-Theory

Table of Contents

Abstracts

Duality via Cycle complexes

Thomas Geisser

If $f : X \to S$ is separated and of finite type, then the generic duality statement is the adjointness

$$R\,\mathrm{Hom}_X(\mathcal{F}, Rf^!\mathcal{G}) \cong R\,\mathrm{Hom}_S(Rf_!\mathcal{F}, \mathcal{G})$$

for torsion sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on S. In order to obtain useful statements, one has to identify the complex $Rf^!\mathcal{G}$. We show that in many situations, Bloch's complex [1] of zero-cycles can by used to explicitly calculate $Rf^!\mathcal{G}$. Let S be the spectrum of a field or a Dedekind ring, and for a scheme X essentially of finite type over S¡ let $\mathbb{D}_X$ be the complex of etale sheaves which in degree $-i$ consists of the free abelian group generated by cycles of relative dimension i over S on $X \times_S \Delta^i$ which meet all faces properly, and alternating sum of intersection with faces as differentials. Then for $f : X \to k$ separated and of finite type over a perfect field, and for every torsion sheaf $\mathcal{G}$ on X, there is a quasi-isomorphism

$$R\,\mathrm{Hom}_X(\mathcal{G}, \mathbb{D}_X) \cong R\,\mathrm{Hom}_k(Rf_!\mathcal{G}, \mathbb{Z}).$$

In particular, if $\mathcal{G}$ is constructible, and k is algebraically closed, we obtain a perfect pairing

$$\mathrm{Ext}^{1-i}_X(\mathcal{G}, \mathbb{D}_X) \times H^i_c(X_{\mathrm{et}}, \mathcal{G}) \to \mathbb{Q}/\mathbb{Z},$$

and if k is finite we obtain a perfect pairing

$$\mathrm{Ext}^{2-i}_X(\mathcal{G}, \mathbb{D}_X) \times H^i_c(X_{\mathrm{et}}, \mathcal{G}) \to \mathbb{Q}/\mathbb{Z};$$

This generalizes results of Deninger [3, 4] for curves, Spieß [11] for surfaces, and Milne-Moser [8, 10] for the p-part in characteristic p. We obtain a similar duality theorem for schemes over a local field of characteristic 0.

Assuming the Beilinson-Lichtenbaum conjecture, we also get a duality theorem over the spectrum B of the ring of integers of a number field. For $f : X \to B$ proper and a torsion sheaf $\mathcal{G}$ on X, we define cohomology with compact support $H^i_c(X_{\mathrm{et}}, \mathcal{G})$ as the cohomology of the complex $R\Gamma_c(B, Rf_!\mathcal{G})$, where $R\Gamma_c(B, -)$ is cohomology with compact support as defined in [7]. It differs from $R\Gamma(B_{\mathrm{et}}, \mathcal{F})$ only at the prime 2 and only for those B having a real embedding. Then we have a quasi-isomorphism

$$R\,\mathrm{Hom}_X(\mathcal{G}, \mathbb{D}_X) \cong R\,\mathrm{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_{\mathrm{et}}, \mathcal{G}), \mathbb{Z})[-1],$$

which induce perfect pairings

$$\mathrm{Ext}^{2-i}_X(\mathcal{G}, \mathbb{D}_X) \times H^i_c(X_{\mathrm{et}}, \mathcal{G}) \to \mathbb{Q}/\mathbb{Z}$$

of finite groups for constructible $\mathcal{G}$. This generalizes results of Artin-Verdier [7] for $\dim X = 1$, and Spieß for $\dim X = 2$.

As an application, we generalize and re-prove Rojtman's theorem [9, 12]. Let X be a normal scheme, proper over an algebraically closed field k. Then there is an isomorphism

$$_{\mathrm{tor}}(\mathrm{Pic}^0_{red}(X))^t(k) \cong {}_{\mathrm{tor}}CH_0(X)$$

between the torsion points of the dual abelian variety of $\mathrm{Pic}^0_{red}(X)$, and the Chow group of zero-cycles on X.

For the abelianized algebraic fundamental group we obtain, for a proper scheme X over an algebraically closed field, a short exact sequence

$$0 \to CH_0(X,1)^\wedge \to \pi_1^{ab}(X) \to TCH_0(X) \to 0,$$

and the first group is finite if X is normal. Here $G^\wedge$ is the profinie completion, and TG the Tate module of an abelian group G. If k is finite, and if $\pi_1^{ab}(X)^{geom}$ is the kernel of $\pi_1^{ab}(X) \to \mathrm{Gal}(k)$, then for X normal and proper, we have a short exact sequence

$$0 \to CH_0(\bar{X},1)^\wedge_{\hat{G}} \to \pi_1^{ab}(X)^{geom} \to \mathrm{Pic}^0_{red}(X)(k) \to 0.$$

The key ingredient in the proof of the main theorem is purity for $\mathbb{D}_X$ over an algebraically closed field, i.e. for $i : Z \to X$ a closed embedding over an algebraically closed field, $Ri^!\mathbb{D}_X \cong \mathbb{D}_Z$. In order to prove this, we first show that $\mathbb{D}_X$ has etale hypercohomological descent (i.e. its cohomology and etale hypercohomology agree), and then use purity for the cohomology of $\mathbb{D}_X$ proved by Bloch and Levine [2, 6]. Using purity, an induction and devissage argument is used to reduce to showing the case of a constant sheaf on a smooth and proper scheme, which is known by SGA 4 XVIII and Milne's duality [8].

The contents of this talk will appear in the article [5].

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ℓ-adic realization of triangulated motives over a noetherian separated scheme and a motivic equivalence

Florian Ivorra

In [9, chapter 5], V. Voevodsky set out a construction of mixed motives over the spectrum of a perfect field. Using the general theory of relative cycles of [9, chapter 2], it is possible to extend this construction to a noetherian separated scheme S. The triangulated category $DM^{\text{eff}}_{gm}(S)$ of effective geometrical mixed motives is then the pseudo-abelian hull of the quotient

$$\mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S)/E_{gm}$$

by the thick subcategory E_{gm} generated by homotopy invariance and Nisnevich localization. As in the perfect field case, SmCor_S denotes the additive tensor category of smooth schemes of finite type over S with finite correspondences as morphisms. According to the notation of [9, chapter 2], finite correspondences from X to Y are given by the abelian group $c_S(X,Y) = c_{\mathrm{equi}}(X \times_S Y/X, 0)$. The category $DM_{gm}(S)$ is then obtained from $DM^{\text{eff}}_{gm}(S)$ by inverting the Tate motive.

1. Localization of finite correspondences

Fix a S-scheme X. Denote by $X^{\mathfrak{h}}_{X,x}$ the spectrum of the henselian local ring of X at x and let

$$X^{\mathfrak{h}} := \coprod_{x\in X} X^{\mathfrak{h}}_{X,x} \qquad \check{C}_X(X^{\mathfrak{h}})_n := \underbrace{X^{\mathfrak{h}} \times_X \cdots \times_X X^{\mathfrak{h}}}_{n+1 \text{ copies of } X^{\mathfrak{h}}}$$

We have the augmented Čech complex of Nisnevich sheaves with transfers over S

$$\check{C}_{X^{\mathfrak{h}}/X} : \cdots \to \mathbb{Z}_{\mathrm{tr}}[\check{C}_X(X^{\mathfrak{h}})_n] \xrightarrow{d_n} \mathbb{Z}_{\mathrm{tr}}[\check{C}_X(X^{\mathfrak{h}})_{n-1}] \to \cdots \to \mathbb{Z}_{\mathrm{tr}}[X].$$

The following result is inspired by [9, chapter 5, prop 3.1.3]:

Proposition. *Let $\mathcal{O}$ be an henselian local S-scheme. The complex of abelian groups $\check{C}_{X^{\mathfrak{h}}/X}(\mathcal{O})$ is canonically homotopic to zero* [1].

Using the previous proposition one then proves given a finite correspondence $\alpha \in c_S(X,Y)$ the existence of a Nisnevich local decomposition [2]

$$\alpha \circ [\mathfrak{l}^{\mathfrak{h}}_{X,x}] = \sum_{y\in Y} [\mathfrak{l}^{\mathfrak{h}}_{Y,y}] \circ \alpha_{x,y}$$

where $\mathfrak{l}^{\mathfrak{h}}_{X,x} : X^{\mathfrak{h}}_x \to X$ is the natural map. These decompositions have good properties with respect to composition, tensor product and provide canonical transfers on the Godement resolution of a Nisnevich sheaf with transfers.

[1] The homotopy is functorial for finite correspondences between henselian local S-schemes.

[2] The result is also true for the étale topology if one uses strictly henselian local S-schemes.

2. ℓ-adic realization over a noetherian separated scheme

Fix a prime ℓ invertible on S. One has a symmetric quasi-monoidal ℓ-adic realization functor for smooth scheme of finite type over S

$$\begin{array}{rcl} R_\ell : \mathrm{Sm}_S^{\mathrm{op}} & \to & \mathrm{D}^+(S, \mathbb{Z}_\ell) \\ X & \mapsto & R\pi_{X*}\pi_X^*\mathbb{Z}_S/\ell^* \end{array}$$

with values in T. Ekedahl's category [1] .

Theorem ([5, thm 4.3])**.** *This functor has a canonical extension to a triangulated quasi-tensor functor*

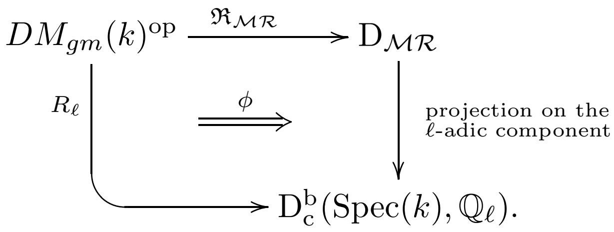
$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^+(S, \mathbb{Z}_\ell).$$

A. Huber has constructed [3, 4] for a field k embeddable into $\mathbb{C}$ a mixed realization functor

$$\mathfrak{R}_{\mathcal{MR}} : DM_{gm}(k)^{\mathrm{op}} \to \mathrm{D}_{\mathcal{MR}}$$

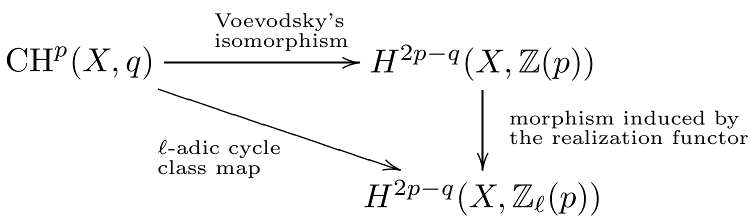
where $\mathrm{D}_{\mathcal{MR}}$ is the triangulated category defined in [2]. The following result [6, thm 1.1] gives the link with A. Huber's approach:

Proposition. *The ℓ-adic component of A. Huber's mixed realization functor* [3, 4] *is isomorphic to the realization functor obtained in* [6, thm 4.3]*: there exists a canonical isomorphism of functor ϕ*



This proposition relies on a « dévissage » which reduces the proof to a comparison between transfers: one has to check that A. Huber's transfers given by Galois theory agree with local transfers. Using a naive description of Voevodsky's isomorphism [10], we prove:

Proposition. *Let X be a smooth quasi-projective scheme over a perfect field k and p, q be integers. The following triangle*



is commutative.

3. A motivic equivalence

Let k be a perfect field. M. Levine's category [8] of triangulated motives $\mathcal{DM}(k)$ is built from generators and relations according to the following two main guiding principles.

- The category $\mathcal{DM}(k)$ should admit a realization functor for a Bloch-Ogus twisted duality cohomology theory.
- The motivic cohomology defined by $\mathcal{DM}(k)$ has to be isomorphic to higher Chow groups.

In [8] M. Levine has proved that $\mathcal{DM}(k)$ and $DM_{gm}(k)$ are equivalent if one has resolution of singularities over k [3]. Using local transfers one may extend M. levine's method of proof to positive characteristic [4]:

Theorem. [7] *Let k be a perfect field. Assume either that*

- $\mathrm{char}(k) = 0$ *and* $A = \mathbb{Z}$, *or*
- $\mathrm{char}(k) > 0$ *and* $A = \mathbb{Q}$.

There exists a triangulated tensor equivalence

$$\mathcal{DM}(k, A) \xrightarrow{\Upsilon} DM_{gm}(k, A).$$

In addition the equivalence Υ induces a triangulated tensor equivalence

$$\mathcal{DM}(k)^{\mathrm{pr}} \xrightarrow{\Upsilon} DM_{gm}(k)^{\mathrm{pr}}$$

between the pseudo-abelian hulls of the integral triangulated tensor subcategories generated by the motives of smooth projective schemes.

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[3] The proof uses a result of Voevodsky that relies on cdh topology

[4] One has to use de Jong's theorem

On Nori's fundamental group scheme

Hélène Esnault
(joint work with Phùng Hô Hai and Xiaotao Sun)

The talk is a report on a recent joint work with Phùng Hô Hai and Xiaotao Sun [4]. The aim is to give two structure theorems on Nori's fundamental group scheme of a proper connected variety defined over a perfect field and endowed with a rational point.

For a proper connected reduced scheme X defined over a perfect field k endowed with a rational point $x \in X(k)$, Nori defined in [9] and [10] a fundamental group scheme $\pi^N(X,x)$ over k. It is Tannaka dual to the k-linear abelian rigid tensor category $\mathcal{C}^N(X)$ of *Nori finite* bundles, that is bundles which are trivializable over a principal bundle $\pi : Y \to X$ under a finite group scheme. The rational point x endows $\mathcal{C}^N(X)$ with a fiber functor $V \mapsto V|_x$ with values in the category of finite dimensional vector spaces over k. This makes $\mathcal{C}^N(X)$ a Tannaka category, thus by Tannaka duality ([1, Theorem 2.11]), the fiber functor establishes an equivalence between $\mathcal{C}^N(X)$ and the representation category $\mathrm{Rep}(\pi^N(X,x))$ of an affine group scheme $\pi^N(X,x)$, that is a prosystem of affine algebraic k-group schemes, which turn out to be finite group schemes. The purpose of the lecture is to study the structure of this Tannaka group scheme.

To this aim, we define two full tensor subcategories $\mathcal{C}^{ét}(X)$ and $\mathcal{C}^F(X)$. The objects of the first one are *étale finite* bundles, that is bundles for which the finite group scheme is étale, while the objects of the second one are *F-finite* bundles, that is bundles for which the group scheme is local. As sub-Tannaka categories they are representation categories of Tannaka group schemes $\pi^{ét}(X,x)$ and $\pi^F(X,x)$. Our first main theorem (see [4, Theorem 4.1]) asserts that the natural homorphism of k-group schemes

$$\pi^N(X,x) \to \pi^{ét}(X,x) \times \pi^F(X,x) \tag{1}$$

is faithfully flat, so in particular surjective. To have a feeling for the meaning of the statement, it is useful to compare $\pi^{ét}(X,x)$ with the more familiar fundamental group $\pi_1(X,\bar{x})$ defined by Grothendieck in ([5, Exposé 5]), where $\bar{x}$ is a geometric point above x. Grothendieck's fundamental group is an abstract group, which is a pro-system of finite abstract groups. One has

$$\pi^{ét}(X,x)(\bar{k}) \cong \pi_1(X \times_k \bar{k}, \bar{x}), \tag{2}$$

thus the étale piece of Nori's group scheme takes into account only the geometric fundamental group and ignores somehow arithmetics. On the other hand, $\pi^F(X,x)$ reflects the purely inseparable covers of X. That k is perfect guarantees that inseparable covers come only from geometry, and not from the ground field.

However (1) is not injective. Raynaud's work [11] on coverings of curves producing a new ordinary part in the Jacobian yields an example.

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The central theorem reported on in the lecture is the determination by its objects and morphisms of a k-linear abelian rigid tensor category $\mathcal{E}$, which is equivalent to the representation category of Ker((1)). This is the most delicate part of the construction. If S is a finite subcategory of $\mathcal{C}^N(X)$ with étale finite Tannaka group scheme $\pi(X, S, x)$, then the total space X_S of the $\pi(X, S, x)$-principal bundle $\pi_S : X_S \to X$ which trivializes all the objects of S has the same property as X. It is proper, reduced and connected. However, if S is finite but $\pi(X, S, x)$ is not étale, then Nori shows that X_S is still proper connected, but it is not the case that X_S is still reduced. We give a concrete example which is due to P. Deligne. However, in order to describe $\mathcal{E}$, we need in some sense an extension of Nori's theory to those non-reduced covers. We define on each such X_S a full subcategory $\mathcal{F}(X_S)$ of the category of coherent sheaves, the objects of which have the property that their push down on X lies in $\mathcal{C}^N(X)$. We show that indeed those coherent sheaves have to be vector bundles, so in a sense, even if the scheme X_S might be bad, objects which push down to Nori's bundles on X are still good. In particular, $\mathcal{C}^N(X_S) = \mathcal{F}(X_S)$ if $\pi(X, S, x)$ is finite étale, so the definition generalizes slightly Nori's theory. For given finite subcategories S and T of $\mathcal{C}^N(X)$, with $\pi(X, S, x)$ étale and $\pi(X, T, x)$ local, one defines a full subcategory $\mathcal{E}(X_{S\cup T}) \subset \mathcal{F}(X_{S\cup T})$ on the total space $X_{S\cup T}$ of the principal bundle $\pi_{S\cup T} : X_{S\cup T} \to X$ consisting of those bundles V, the push down of which on X_S is F-finite. Now the objects of $\mathcal{E}$ are pairs $(X_{S\cup T}, V)$ for V an object in $\mathcal{E}(X_{S\cup T})$. Morphisms are subtle as they do take into account the whole inductive system of such $T' \subset \mathcal{C}^F(X)$.

We now describe our method of proof. We proceed in three steps. To see that $\pi^N(X, x) \to \pi^{ét}(X, s)$ is surjective is very easy, thus we consider the kernel $L(X, x)$ and determine its representation category. The computation is based on two results. The first one of geometric nature asserts that sections of an F-finite bundle can be computed on any principal bundle $X_S \to X$ with finite étale group scheme. The second one is the key to the categorial work and comes from [3, Theorem 5.8]. It gives a criterion for a k-linear abelian rigid category $\mathcal{Q}$, endowed with a tensor functor $q : \mathcal{T} \to \mathcal{Q}$, to be the quotient category of the full embedding of k-linear rigid tensor categories $\mathcal{S} \xrightarrow{\iota} \mathcal{T}$, in the sense that if the three categories are endowed with compatible fiber functors to the category of k-vector spaces, then the Tannaka group schemes $G(?)$ are inserted in the exact sequence

$$1 \to G(\mathcal{Q}) \xrightarrow{q^*} G(\mathcal{T}) \xrightarrow{\iota^*} G(\mathcal{S}) \to 1.$$

The objects of $\mathrm{Rep}(L(X, x))$ are pairs (X_S, V) where $X_S \to X$ is a principal bundle under an étale finite group scheme and V is a F-finite bundle on X_S. Morphisms are defined naturally via Proposition. The second step consists in showing surjectivity $L(X, x) \to \pi^F(X, x)$. To this aim, one needs a strengthening of the geometric proposition which asserts that not only sections can be computed on finite étale principal bundles, but also all subbundles of an F-finite bundle. Finally the last step consists in showing that the category $\mathcal{E}$ constructed is indeed the right quotient category, for which we use the already mentioned criterion [3, Theorem 5.8].

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K-theory of singularities and a conjecture of Vorst

Christian Haesemeyer
(joint work with G. Cortiñas and C. Weibel)

It is a well-known fact that algebraic K-theory is homotopy invariant as a functor on regular schemes; if X is a regular scheme then the natural map $K_n(X) \to K_n(X \times \mathbb{A}^1)$ is an isomorphism for all $n \in \mathbb{Z}$. This is false in general for nonregular schemes and rings.

To express this failure, Bass introduced the terminology that, for any contravariant functor $\mathcal{P}$ defined on schemes, a scheme X is called *$\mathcal{P}$-regular* if the pullback maps $\mathcal{P}(X) \to \mathcal{P}(X \times \mathbb{A}^r)$ are isomorphisms for all $r \geq 0$. If $X = \mathrm{Spec}(R)$, we also say that R is $\mathcal{P}$-regular. Thus regular schemes are K_n-regular for every n. In contrast, it was observed as long ago as [1] that a nonreduced affine scheme can never be K_1-regular. In particular, if A is an Artinian ring (that is, a 0-dimensional Noetherian ring) then A is regular (that is, reduced) if and only if A is K_1-regular. In [6], Vorst conjectured that for an affine scheme X, of finite type over a field F and of dimension d, regularity and K_{d+1}-regularity are equivalent; Vorst proved this conjecture for $d = 1$ (by proving that K_2-regularity implies normality).

In this talk, we sketch a proof of Vorst's conjecture in all dimensions provided the characteristic of the ground field F is zero. Details can be found in our paper [3]. In fact we prove a stronger statement. We say that X is *regular in codimension* $< n$ if $\mathrm{Sing}(X)$ has codimension $\geq n$ in X.

Let $\mathcal{F}_K$ denote the presheaf of spectra such that $\mathcal{F}_K(X)$ is the homotopy fiber of the natural map $K(X) \to KH(X)$, where $K(X)$ is the algebraic K-theory spectrum of X and $KH(X)$ is the homotopy K-theory of X defined in [7]. We write $\mathcal{F}_K(R)$ for $\mathcal{F}_K(\mathrm{Spec}(R))$.

Theorem 1. *Let R be a commutative ring which is essentially of finite type over a field F of characteristic 0. Then:*
(a) If $\mathcal{F}_K(R)$ is n-connected, then R is regular in codimension $< n$.
(b) If R is K_n-regular, then R is regular in codimension $< n$.
(c) (Vorst's conjecture) If R is $K_{1+\dim(R)}$-regular, then R is regular.

It was observed in [7] that if X is K_n-regular then $K_i(X) \to KH_i(X)$ is an isomorphism for $i \leq n$, and a surjection for $i = n+1$, so that $\mathcal{F}_K(X)$ is n-connected. Thus (a) implies (b) in this theorem, and (c) is a special case of (b).

The bounds in (a) and (b) are the best possible, because it follows from Vorst's results ([6, Thm. A], [5, Thm. 3.6]) that for an affine singular seminormal curve X, $\mathcal{F}_K(X)$ is 1-connected, but not 2-connected. The converse of (c) is trivial, but those of (a) and (b) are false. Indeed, affine normal surfaces are regular in codimension 1 but may not be K_{-1}-regular, much less K_2-regular; see [8, 5.8.1].

Finally the analogue of (c) –and thus also of (a) and (b)– for K-theory of general nonaffine schemes is false. Indeed there are examples of nonreduced (and in particular nonregular) projective curves which are K_n-regular for all n.

The proof of Theorem 1 employs results from our paper with M. Schlichting [2] that allow us to describe $\mathcal{F}_K$ in terms of cyclic homology. In fact, $\mathcal{F}_K$ can be identified with the homotopy fiber $\mathcal{F}_{HN}$ of the natural transformation $HN \to \mathbb{H}(-, HN)$, where HN is negative cyclic homology taken over $\mathbb{Q}$ and for any presheaf of spectra E on Sch/F, $\mathbb{H}(-, E)$ denotes a hypercohomology spectrum of E in the *cdh*-topology. Using standard facts about cyclic and Hochschild homology, a transitivity spectral sequence due to Kassel and Sletsjœ(see [4]) and its *cdh*-fibrant version, one arrives at the following result.

Theorem 2. *Let X be an essentially finite type scheme over F. If $\mathcal{F}_K(X)$ is $(n+1)$-connected, then $\mathcal{F}_{HH(/F)}(X)$ is n-connected. Moreover, if $R = \mathcal{O}_{X,x}$ is a local ring of X, which is the localization of an algebra of finite type over some extension E of F (of transcendence degree the codimension of x) at a maximal ideal, then $\mathcal{F}_{HH(/E)}(R)$ is n-connected.*

Here $HH(/k)$ denotes Hochschild homology over k for a field k, and $\mathcal{F}_{HH(/k)}$ is defined as above. Finally, one proves:

Theorem 3. *Suppose R is a local ring of dimension n that can be obtained as localization of a finite type algebra over a field E of characteristic 0 at a maximal ideal. Further assume that $\mathcal{F}_{HH(/E)}$ is n-connected. Then $\Omega^{n+1}_{R/E} = 0$ and R is regular.*

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Operations on algebraic K-theory and regulators *via* the homotopy theory of schemes

Joël Riou

This talk presents the results in my thesis [6]. The main result relies on the following theorem of $\mathbf{A}^1$-homotopy theory (see [4] and [9]):

Theorem 1 (Morel-Voevodsky). *Let S be a regular scheme. For any natural number n and $X \in \mathbf{Sm}/S$, there exists a canonical isomorphism:*

$$\mathrm{Hom}_{H_\bullet(S)}(S^n \wedge X_+, \mathbf{Z} \times \mathbf{Gr}) \simeq K_n(X) \ ,$$

where $\mathbf{Sm}/S$ denotes the category of S-schemes that are smooth, separated and of finite type, $H_\bullet(S)$ the pointed homotopy category of S and $\mathbf{Gr}$ the infinite Grassmannian.

Thanks to this theorem, we may consider thinking of operations on algebraic K-theory as endomorphisms of $\mathbf{Z} \times \mathbf{Gr}$ in $H(S)$. We let $K_0(-)$ be the presheaf of sets on $\mathbf{Sm}/S$ given by Grothendieck groups of vector bundles on schemes in $\mathbf{Sm}/S$. The main result is the following simple theorem:

Theorem 2. *Let S be a regular scheme. There are natural isomorphisms:*

$$\mathrm{End}_{H(S)}(\mathbf{Z} \times \mathbf{Gr}) \xrightarrow{\sim} \mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Sets}}(K_0(-)) \simeq (K_0(S)[[c_1, c_2, \dots]])^{\mathbf{Z}} \ .$$

The proof involves theorem 1, the Milnor exact sequence, computations in [SGA 6, VI 4.10] and Jouanolou's trick.

Corollary 1. *Over regular schemes, any natural transformation (of presheaves of pointed sets) $\tau \colon K_0(-) \to K_0(-)$ naturally extends to maps on higher algebraic K-theory groups $K_n(-) \to K_n(-)$ for any natural number n.*

There is a version of theorem 2 with several variables so that any algebraic structure on $K_0(-)$ comes in a unique way from such a structure on $\mathbf{Z} \times \mathbf{Gr}$ in the category $H(S)$. Using [SGA 6, VI 3.2], we get:

Corollary 2. *The object $\mathbf{Z} \times \mathbf{Gr}$ of $H(S)$ is endowed with a structure of a special λ-Ring.*

We thus get pairings $K_i(X) \times K_j(X) \to K_{i+j}(X)$ for any regular scheme X and $i, j \geq 0$.

Proposition 1. *For a regular scheme X, these pairings coincide with those defined by*

- *Quillen* [5] *if $i = 0$ or $j = 0$;*
- *Loday* [3] *if $i > 0$ and $j > 0$ (and X affine);*
- *Waldhausen* [10] *for all i and j.*

In particular, the pairings defined by Loday and Waldhausen coincide for affine regular schemes (which was already known for all affine schemes, see [11]).

One can also construct a natural map

$$(\mathbf{R}_{\mathbf{Z}}\,\mathrm{GL})^{\mathbf{Z}} \to \mathrm{End}_{H(S)}(\mathbf{Z} \times \mathbf{Gr})$$

that enables one to prove that the operations on higher algebraic K-theory constructed here coincide with the ones defined by Soulé in [8].

We can also consider additive operations. Using the splitting principle, we get:

Proposition 2. *Let S be a regular scheme. There is a canonical isomorphism*

$$\mathrm{End}_{\mathbf{Sm}/S^{\mathrm{opp}}\mathbf{Ab}}(K_0(-)) \simeq K_0(S)[[U]]$$

which maps the Adams operation Ψ^n to $(1+U)^n$.

Definition 1. *Let A be an abelian group. We let A^{Ω} be the following projective system of abelian groups indexed by $\mathbf{N}$:*

$$\cdots \to A[[U]] \overset{\Omega_{\mathbf{P}^1}}{\to} A[[U]] \overset{\Omega_{\mathbf{P}^1}}{\to} A[[U]]$$

where $\Omega_{\mathbf{P}^1}$ is the operator $f \longmapsto (1+U)\frac{df}{dU}$.

In [9], Voevodsky defined an object $\mathbf{BGL}$ representing algebraic K-theory in the stable homotopy category $SH(S)$ of a regular scheme S. Due to the "stably phantom" maps phenomenon, it is not clear that this object is well-defined up to *unique* isomorphisms.

Theorem 3. *Let S be a regular scheme and $n \in \mathbf{Z}$. There exists a short exact sequence:*

$$0 \to \lim{}^1 K_{n+1}(S)^{\Omega} \to \mathrm{Hom}_{SH(S)}(\mathbf{BGL}, \mathbf{BGL}[-n]) \to \lim K_n(S)^{\Omega} \to 0\ .$$

The group $\lim^1 K_{n+1}(S)$ is identified to the set of stably phantom maps $\mathbf{BGL} \to \mathbf{BGL}[-n]$ in $SH(S)$.

One can prove that if A is finite or divisible, then $\lim^1 A^{\Omega} = 0$. In particular, if $S = \mathrm{Spec}\,\mathbf{Z}$, there are no stably phantom endomorphisms of $\mathbf{BGL}$ in $SH(\mathrm{Spec}\,\mathbf{Z})$, so that $\mathbf{BGL} \in SH(\mathrm{Spec}\,\mathbf{Z})$ is well-defined up to *unique* isomorphisms and we can construct a canonical $\mathbf{BGL}$ for any regular scheme S by base change.

With $\mathbf{Q}$-coefficients one can explicitly compute $\mathrm{End}_{SH(\mathrm{Spec}\,\mathbf{Z})}(\mathbf{BGL}_{\mathbf{Q}})$: it is canonically isomorphic to the ring $\mathbf{Q}^{\mathbf{Z}}$ of functions $\mathbf{Z} \to \mathbf{Q}$. We can use this to get the following theorem:

Theorem 4. *Let S be a regular scheme. There is a canonical decomposition in $SH(S)$:*

$$\mathbf{BGL}_{\mathbf{Q}} \simeq \bigoplus_{i\in\mathbf{Z}} \mathbf{H}_B^{(i)} .$$

One can use the same arguments to study maps from $\mathbf{Z} \times \mathbf{Gr}$ to a general object in $H(S)$. In particular, if k is a perfect field, Voevodsky defined motivic Eilenberg-MacLane spaces $K(\mathbf{Z}(n), 2n)$. The set of maps $\mathbf{Z} \times \mathbf{Gr} \to K(\mathbf{Z}(n), 2n)$ in $H(k)$ is isomorphic to the set of natural transformations $K_0(-) \to CH^n(-)$ in the category $\mathbf{Sm}/k^{\mathrm{opp}}\mathbf{Sets}$. There is also a stable version of this which enables one to define a Chern caracter $\mathbf{BGL} \to \mathbf{H}_{\mathbf{Q}}$ in $SH(k)$ where $\mathbf{H}_{\mathbf{Q}}$ is the motivic Eilenberg-MacLane spectrum with $\mathbf{Q}$-coefficients.

We finally make the computation $\mathrm{Hom}_{SH(S)}(\mathbf{BGL}, \mathbf{H}_{\mathbf{Z}}[1]) \simeq \hat{\mathbf{Z}}/\mathbf{Z}$: all these morphisms are stably phantoms, which proves the existence of non-zero stably phantom maps in the stable homotopy theory of schemes.

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Algebraic cycles on products of elliptic curves over p-adic fields

Andreas Rosenschon
(joint work with V. Srinivas)

Let X be a smooth projective variety over a field k, and let $\mathrm{CH}^p(X)$ be the Chow group of algebraic cycles of codimension p modulo rational equivalence. Equivalently, we may identify $\mathrm{CH}^p(X)$ with the motivic cohomology group $\mathrm{H}^p_M(X, \mathbb{Z}(p))$ defined by Voevodsky.

If k has characteristic 0, let $\bar{X} = X \times_k \bar{k}$ for a separable closure $\bar{k}$ of k. Taking the kernel of the cycle maps into ℓ-adic cohomology, we obtain

$$\mathrm{CH}^p_{\mathrm{hom}}(X) = \ker\{\mathrm{CH}^p(X) \to \prod_p \mathrm{H}^{2p}_{et}(\bar{X}, \mathbb{Z}_\ell(p))\}.$$

If $\mathrm{Griff}^p(X)$ is the Griffiths group of homologically trivial cycles modulo cycles which are algebraically equivalent to zero, we have a surjective map

$$\mathrm{CH}^p_{\mathrm{hom}}(X) \to \mathrm{Griff}^p(X),$$

whose kernel is generated by the images of correspondences coming from Picard varieties of smooth projective curves over k. In particular, if k is separably closed, this kernel is divisible. For examples of smooth projective complex varieties such that the Griffiths group in codimension 2 is not finitely generated, see [4], [13], [3] and [17], for instance.

We study the structure of these groups over p-adic fields. To place our results, we recall the following examples, which are due to Schoen [11, 6, 7]:

I. *Let E be an elliptic curve defined over a subfield of $\mathbb{C}$ with j-invariant $j(E) \notin \bar{\mathbb{Q}}$. If $k = \overline{\mathbb{Q}(j(E))}$ and $\ell = 5, 7, 11, 13, 17$, $\# \mathrm{CH}^2_{\mathrm{hom}}(E^3_k)/\ell = \infty$.*

II. *Let $E \subset \mathbb{P}^2_{\mathbb{Q}}$ be the Fermat curve defined by $x_0^3 + x_1^3 + x_2^3 = 0$. Let $k = \overline{\mathbb{Q}}$. If $\ell \equiv 1 \mod 3$, then $\# \mathrm{CH}^2_{\mathrm{hom}}(E^3_k)/\ell = \infty$.*

III. *Let E/k be as in I or II. If F is an elliptic curve with $j(F) \notin k$, let $K = \overline{k(j(F))}$. Then the ℓ-torsion subgroup $\mathrm{CH}^3(E^3 \times_k F_K)[\ell]$ is infinite.*

The examples in I and II exhibit that $\mathrm{CH}^2_{\mathrm{hom}}(X)$, and thus $\mathrm{Griff}^2(X)$, is large and far from being divisible; in particular II shows that this already occurs over $\overline{\mathbb{Q}}$ (in general the Chow groups of a smooth projective variety over a number field are expected to be finitely generated abelian groups). The examples in III show the ℓ-torsion subgroup of the Chow group in codimension 3 is not finite, contrary to the codimension 2 case [1].

Our main theorem shows that all of the above phenomena also occur over p-adic fields, i.e. finite extensions of $\mathbb{Q}_p$.

Theorem 1. *Let p be an odd prime. There exists a p-adic field K_p and an elliptic curve E/K_p with the property that the natural map*

$$\mathrm{CH}^2_{\mathrm{hom}}(E^3_{K_p})/\ell \to CH^2_{\mathrm{hom}}(E^3_{\bar{K}_p})/\ell$$

has infinite image, for each $\ell \in \{5, 7, 11, 13, 17\}$ (where possibly $p = \ell$).

Corollary 2. *If E/K_p is as above, $\# \operatorname{Griff}^2(E^3_{K_p}) = \infty$.*

Corollary 3. *For E/K_p as above, there exist elliptic curves F over K_p such that the ℓ-torsion subgroup $\mathrm{CH}^3((E^3 \times F)_{K_p})[\ell]$ is infinite.*

To our knowledge these are the first examples of smooth projective varieties over a p-adic field with these properties. In fact, we do not know of any conjectures about the general structure of Chow groups for varieties over p-adic fields; our results show that these groups are large, and one cannot hope to have finiteness results as expected in the number field case.

We point out the contrast with the case of zero cycles. If k is a p-adic field, and X is a product of smooth projective curves with split semi-ordinary reduction, Raskind and Spiess [12] have shown that the Chow group $\mathrm{CH}_0(X)/\ell$ is finite for all primes ℓ. A recent theorem of S. Saito and K. Sato [15] shows that this holds for an arbitrary smooth projective variety and almost every prime.

To give an overview of the proof of our main theorem, we recall the two essential steps in Schoen's proof of I:

- Schoen considers a particular elliptic curve E/K, where $K = \mathbb{Q}(t)$, and applies a criterion for non-divisibility of cycles due to Bloch-Esnault [8] to a modified Ceresa cycle Ξ to obtain a non-trivial element in $CH^2_{\mathrm{hom}}(E^3_{\bar{K}})/\ell$. To apply this criterion, one has to verify that the image of a certain cycle under the ℓ-adic Abel-Jacobi map modulo ℓ is non-trivial. This is difficult, and the restriction on ℓ results from the fact that only for these primes has this computation been carried out.

- There is a fine moduli scheme representing the functor 'elliptic curves with a point of order 4', which is such that the generic fiber of the universal elliptic curve is isomorphic to an elliptic curve $\mathcal{E}/K$, that is a quadratic twist of E. For suitable primes q_i one obtains endomorphisms of $\mathcal{E}_{\bar{K}}$ such that for $\ell \neq q_i$ the induced map Γ_{q_i} on $\mathrm{CH}^2_{\mathrm{hom}}(\mathcal{E}^3_{\bar{K}})/\ell$ is an automorphism. Schoen identifies Ξ with a non-trivial element of $\mathrm{CH}^2_{\mathrm{hom}}(\mathcal{E}_{\bar{K}})/\ell$, and adapts an argument due to Nori [13] to show there is an infinite set of primes q_i such that the set of cycles $\{\Gamma_{q_i}(\Xi)\} \subset \mathrm{CH}^2_{\mathrm{hom}}(\mathcal{E}_{\bar{K}})/\ell$ is linearly independent.

Given an odd prime p, we use the cuspidal geometry of modular curves to show that there is a p-adic field K_p such that the variety E^3, the modified Ceresa cycle Ξ, and an infinite set of modular correspondences are defined over this field. It follows that infinitely many of the cycles considered by Schoen are the base change of cycles defined over K_p.

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Weak Bloch-Beilinson conjecture for zero-cycles over p-adic fields

Kanetomo Sato
(joint work with Shuji Saito)

Let V be a smooth projective variety over a field k. Let $\mathrm{CH}_0(V)$ be the Chow group of the zero-cycles on V modulo rational equivalence with $A_0(V) \subset \mathrm{CH}_0(V)$, the subgroup of cycle classes of degree 0. There is a natural map called Albanese mapping

$$\phi_V \ : \ A_0(V) \to \underline{\mathrm{Alb}}_V(k)$$

where $\underline{\mathrm{Alb}}_V$ is the Albanese variety of V. By Abel's theorem ϕ_V is injective if $\dim(V) = 1$. It is Mumford [Mu] who first discovered that the situation is rather chaotic in the case of higher dimension. One of the most fascinating and challenging conjectures in arithmetic geometry is a conjecture of Bloch and Beilinson ([Be]) that $\mathrm{Ker}(\phi_V)$ is torsion in case k is a number field. Here is a folklore conjecture, which can be viewed as an analogue of the Bloch and Beilinson conjecture, over a p-adic field

Conjecture 1. *Let V be a smooth projective variety over a p-adic field k. Then $\mathrm{Ker}(\phi_V)$ is the direct sum of a finite group and a divisible group.*

There are some known cases in which this conjecture is true ([BKL], [Sal], [CTR], [RS]). By a result of Mattuck [Ma], $\underline{\mathrm{Alb}}_V(k)$ is isomorphic to the direct sum of a finite group and ${\mathbb{Z}_p}^{\oplus N}$ for an integer $N \geq 1$. So the above conjecture implies the following:

Conjecture 2. *Let V be as in Conjecture 1. Then $A_0(V)$ is the direct sum of a finite group and a group which is divisible by any positive integer prime to p.*

The main result of the talk is the following affirmative result on this conjecture.

Theorem 3. *Conjecture 1 is true, if V has a model which is QSP over the integer ring $\mathcal{O}_k$ of k.*

Here we say that a scheme X is QSP over $\mathcal{O}_k$, if X is regular projective flat over $\mathrm{Spec}(\mathcal{O}_k)$ and if $X_{s,red}$, the reduced part of its special fiber, is a simple normal crossing divisor on X. Some finiteness results in this direction were known in case $\dim(V) = 2$ ([SaSu]) and some special varieties of higher dimension ([PS], [KoSz], [CT2]). We note a direct consequence of Theorem 3.

Corollary 4. *Let the assumption be as in Theorem 3, and assume further that V is rationally connected in the sense of Kollár, Miyaoka and Mori. Then $A_0(V)$ is the direct sum of a finite group and a p-primary torsion group.*

In [CT2], Colliot-Thélène proved a stronger result than this corollary assuming that V is a compactification of a connected linear group over k but not assuming the existence of a model $\mathcal{V}$. The essential result in our case is the following:

Theorem 5. *Let X be a QSP scheme over $\mathcal{O}_k$, and let d be the relative dimension $\dim(X/\mathcal{O}_k)$. Then the cycle map obtained by the absolute purity of Thomason-Gabber*

$$\rho_X \ : \ \mathrm{CH}^d(X)/n \to \mathrm{H}^{2d}_{\text{ét}}(X, \mu_n^{\otimes d})$$

is bijective for any positive integer n which is prime to p.

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Triangulated mixed motives and the 6 functors formalism

Frédéric Déglise

The aim of the talk was to present the construction of a candidate for the triangulated category of motives over a scheme S together with functoriality and monoidal structural analogous to what we have for the category of l-adic complexes.

This construction was obtained in a joint work with Denis-Charles Cisinski and is still under writing at the time of this report. Thus I will sum up here the rough steps of the construction we have in mind.

Schemes are always supposed to be noetherian, separated and quasi-projective over $\mathbb{Z}$. Smooth means formally smooth of finite type - instead of locally of finite type. We denote by $\mathrm{Sh}(S)$ the category of sheaves of abelian groups on the category of smooth S-schemes for the Nisnevich topology.

The construction of the category $DM(S)$ for a scheme S decomposes into four steps :

(1) First, based on a reformulation of the theory of [1, chap. 2], we define a good notion of correspondances between smooth S-schemes to obtain a category $\mathcal{S}m_S^{cor}$ of smooth S-schemes with morphisms these good S-correspondances. We define a symmetric monoidal structure on $\mathcal{S}m_S^{cor}$. For a morphism of schemes $f : T \to S$, we also define a base change functor $f^* : \mathcal{S}m_S^{cor} \to \mathcal{S}m_T^{cor}$ together with a "forget the base" functor $f_\sharp : \mathcal{S}m_T^{cor} \to \mathcal{S}m_S^{cor}$ when f is smooth. Taking the graph of a morphism yields a functor $\gamma : \mathcal{S}m_S \to \mathcal{S}m_S^{cor}$. (see [3] for more details in the case when S is regular).

(2) We then consider sheaves with tranfers, that is additivite contravariant functors $F : \mathcal{S}m_S^{cor} \to \mathcal{A}b$ such that $F \circ \gamma$ is a Nisnevich sheaf. For any smooth S-scheme X, the functor $c_S(., X)$ is q sheaf with tranfers which we denote by $\mathbb{Z}_S^{tr}(X)$. The theory of sheaves with transfers then goes on almost as in the regular case treated in [3] : The category of sheaves with transfers $\mathrm{Sh}^{tr}(S)$ is an abelian category with generators the sheaves $\mathbb{Z}_S^{tr}(X)$. It has a symmetric monoidal closed structure. For a morphism

$f : T \to S$ of schemes, we have a base change functor $f^* : \mathcal{S}m_S^{cor} \to \mathcal{S}m_T^{cor}$. It has a right adjoint f_* and a right adjoint $f_\sharp$ when f is smooth.

(3) We next consider the category $\mathrm{C}(\mathrm{Sh}^{tr}(S))$ of complexes of sheaves with tranfers and its derived category $\mathrm{D}(\mathrm{Sh}^{tr}(S))$. Let $\mathcal{T}$ be the smallest localizing subcategory of $\mathrm{D}(\mathrm{Sh}^{tr}(S))$ closed under arbitrary direct sums and containing the complexes $\mathbb{Z}_S^{tr}(\mathbb{A}_X^1) \to \mathbb{Z}_S^{tr}(X)$ for any smooth S-schemes. We define the category $DM^{eff}(S)$ of effective triangulated motivic complexes over S as the Verdier quotient $\mathrm{D}(\mathrm{Sh}^{tr}(S))/\mathcal{T}$. An important technical point is that there is a canonical Quillen model category structure on $\mathrm{C}(\mathrm{Sh}_S^{tr})$ such that the canonical extensions of the functors f^*, $f_\sharp$ (resp. f_*) to complexes admits left (resp. right) derived functors. Moreover, this model structure is a monoidal model structure, which means there is a left derived tensor product and a right derived internal Hom on $DM^{eff}(S)$.

(4) The canonical immersion $S \to \mathbb{P}_S^1$ induces a split monomorphism $\mathbb{Z}_S^{tr} \to \mathbb{Z}_S^{tr}(\mathbb{P}_S^1)$. We put $\mathbb{Z}_S^{tr}(1) = \mathbb{Z}_S^{tr}(\mathbb{P}_S^1)/\mathbb{Z}_S^{tr}(\{\infty\})[-2]$ and call it the Tate object. We want to invert the Tate object for the monoidal structure. For that purpose, we use the theory of spectra from algebraic topology. A Tate spectrum is a sequence $(E_n, \sigma_n)_{n\in\mathbb{N}}$ such that E_n is a complex of sheaves with tranfers and $\sigma_n : E_n(1) = E_n \otimes \mathbb{Z}_S^{tr}(1) \to E_{n+1}$ a morphism of complexes. A Morphism of spectra is a sequence of morphisms of complexes compatibles with the structural map in the obvious way. For a smooth S-scheme X, a couple of integer (n, m) and a Tate spectrum as above, we put

$$H^{n,m}(X; E) = \varinjlim_{r>>0} \mathsf{Hom}_{DM^{eff}(S)}(\mathbb{Z}_S^{tr}(X)(r), E_{m+r}[n]).$$

Say (quickly) a map of spectra $E \to F$ is a *stable equivalence* if the induced morphism $H^{n,m}(X; E) \to H^{n,m}(X; F)$ is an isomorphism for any X, n, m as above. Then we can finally define abstractly the category $DM(S)$ as the localisation of the category of Tate spectra with respect to stable equivalence. Again, we construct a suitable Quillen model category structure on spectra so that $DM(S)$ is the associated homotopy category and the obvious extension of the functors f^*, $f_\sharp$, f_* to spectra can be derived. The same apply to the monidal structure - though there is need for an extra construction to obtain the symmetry structure. This final step can be carried out from what preeceed with the help of [4].

The central result in this work is the following theorem :

Theorem (Cisinski, Déglise)**.** *Let $i : Z \to S$ be a closed immersion with complementary open immersiom $j : S - Z \to S$.*

Then, for any Tate spectrum E, the canonical adjunction morphisms

$$\mathrm{L}j_\sharp j^* E \to E \to i_* \mathrm{L}i^* E$$

induce a canonical distinguished triangle in $DM(S)$.

With all these definitions in hands and the preceding theorem, one checks that DM satisfies all the properties of a homotopy functor required in [2]. In particular, we obtain the other adjoint pair of functors $(f_!, f^!)$ and deduce all the properties of the *six functors formalism.*

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On $RO(S^1)$-Graded TR

Teena Gerhardt

In general, algebraic K-theory is very difficult to compute. However, for every ring A, we have a cyclotomic trace map

$$trc : K_q(A) \to TC_q(A)$$

to topological cyclic homology. Results of McCarthy, for example, tell us that we can often understand algebraic K-theory by understanding TC and the cyclotomic trace map [8].

As an approach to understanding TC, Hesselholt and Madsen defined TR (see [5], [6] for details). For a ring A and a fixed prime p, we can define

$$TR^n_q(A;p) = \pi_q(T(A)^{C_{p^{n-1}}})$$

where $T(A)$ denotes the S^1-equivariant topological Hochschild spectrum of A and $T(A)^{C_{p^{n-1}}}$ denotes the $C_{p^{n-1}}$ fixed point spectrum of this spectrum. These TR groups come equipped with several operators. Inclusion of fixed points induces the Frobenius map

$$F : TR^n_q(A;p) \to TR^{n-1}_q(A;p).$$

This map has an associated transfer, the Verschiebung

$$V : TR^{n-1}_q(A;p) \to TR^n_q(A;p).$$

There is also a derivation $d : TR^n_q(A;p) \to TR^n_{q+1}(A;p)$ induced from the circle action on $T(A)$. Lastly, we have a restriction map

$$R : TR^n_q(A;p) \to TR^{n-1}_q(A;p).$$

These maps satisfy the relations $FV = p, VF = V(1)$, and $FdV = d$.

We can define $TC^n(A;p)$ as the homotopy equalizer of the maps

$$R, F : TR^n(A;p) \to TR^{n-1}(A;p).$$

Then $TC(A;p)$ is the homotopy limit of the spectra $TC^n(A;p)$. Thus understanding the TR groups of a ring helps us to understand its topological cyclic homology, and hence its algebraic K-theory.

One observation that we could make about TR-theory is that the operators on TR and relations between them give TR a rather rigid algebraic structure. So, one could ask if it is possible to define an abstract algebraic structure embodying the structure of TR-theory. Indeed it is, and this is the structure of a Witt complex over A [4].

Via standard category theoretic arguments, one can show that the category of Witt complexes over A has an initial object, which we write as $W.\Omega^*_A$. This is the de Rham-Witt complex of A. Hesselholt and Madsen have given a construction of the de Rham-Witt complex for $\mathbb{Z}_{(p)}$-algebras [4], [6] which extends the Bloch-Deligne-Illusie construction for $\mathbb{F}_p$- algebras [1], [7]. Since $W.\Omega^*_A$ is initial, we have a map

$$W.\Omega^*_A \to TR^{\cdot}_*(A;p).$$

This can help us understand TR in terms of the de Rham-Witt complex of A.

Instead of defining TR as an integer graded theory, we could instead define a theory graded by the real representation ring of the circle, $RO(S^1)$. Let $\alpha \in RO(S^1)$. We can write $\alpha = [\beta] - [\gamma]$, a formal difference of isomorphism classes of orthogonal S^1 representations. Then the $RO(S^1)$-graded TR groups are defined as

$$TR^n_\alpha(A;p) = [S^\beta \wedge \mathbb{T}/C_{p^{n-1}}{}_+, T(A) \wedge S^\gamma]_{S^1}.$$

These $RO(S^1)$-graded TR groups arise in computations. For instance, Hesselholt and Madsen [2] used the $RO(S^1)$-graded TR groups of an $\mathbb{F}_p$ algebra A in their computation of the algebraic K-theory of a truncated polynomial algebra $A[x]/(x^e)$.

We would like to ask if we can define an algebraic object embodying the structure of the $RO(S^1)$-graded TR the way we were able to for the integer graded case. In other words, can we define what it means to be an $RO(S^1)$-graded Witt complex? Further, we would like to identify the initial object in this category, which we would call the $RO(S^1)$-graded de Rham-Witt complex.

We first consider the operators and relations that we have in this $RO(S^1)$-graded version of TR. Again, inclusion of fixed points will induce the Frobenius map

$$F : TR^n_\alpha(A;p) \to TR^{n-1}_\alpha(A;p).$$

As in the integer graded case, this map has an associated transfer, the Verschiebung

$$V : TR^{n-1}_\alpha(A;p) \to TR^n_\alpha(A;p).$$

There is also a derivation

$$d : TR^n_{q+\alpha}(A;p) \to TR^n_{q+1+\alpha}(A;p)$$

Note that we have changed our notation slightly and written our representation as $q+\alpha$ where $q \in \mathbb{Z}$ and $\alpha \in RO(S^1)$ has no trivial summands. These maps again satisfy the relations $FV = p, VF = V(1)$, and $FdV = d$. However, our restriction map is different from the integer graded case. Let $\rho_p : S^1 \to S^1/C_p$ be the isomorphism given by the pth root. Then we define a prime operation as follows: for $\alpha \in RO(S^1), \alpha' = \rho_p^*(\alpha^{C_p})$. Then our restriction map is a map

$$R : TR^n_{q+\alpha}(A;p) \to TR^{n-1}_{q+\alpha'}(A;p).$$

We would like to have a fully computed example of $TR^n_\alpha(A;p)$ for some A. The first computation to be done is that of $TR^n_\alpha(\mathbb{F}_p;p)$. We state the result for $TR^n_{q+\alpha}(\mathbb{F}_p;p)$ below, where q is an even integer and α is of the form $\alpha = \lambda$ or $\alpha = -\lambda$ for $\lambda \in RO(S^1)$ an actual representation. We use the notation $\alpha^{(k)}$ to denote the prime operation applied k times to α. The notation $|\alpha|$ denotes the dimension of the representation α.

Proposition 0.1. *Let $\alpha = \lambda$ or $\alpha = -\lambda$, $q \in \mathbb{Z}$, even. Then $TR^n_{q+\alpha} \cong \mathbb{Z}/p^{l_0}\mathbb{Z}$ where l_0 is defined as follows. Let*

$$l_{n-1} = \begin{cases} 0 & \text{if } -|\alpha^{(n-1)}| > q \\ 1 & \text{else} \end{cases}$$

Letting k range from $n-1$ to 0,

$$l_k = \begin{cases} l_{k+1} & \text{if } -|\alpha^{(k)}| > q \\ \min\{l_{k+1} + 1 + \min(\frac{q+|\alpha^{(k)}|}{2}, n-1-k), n-k\} & \text{else} \end{cases}$$

This extends the result of Hesselholt and Madsen [5] for representations of the form $\alpha = q - \lambda$.

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$\mathcal{D}$-Crystals

Clark Barwick

Definition 1 (Grothendieck, [4, §16.8])**.** Suppose k a field of characteristic 0, X/k a smooth scheme.[1] Then the diagonal embedding $X \twoheadrightarrow X \times X$ is given by an ideal $I \lhd O_{X \times X}$, and the *sheaf of differential operators of order* $n \in \mathbf{N}$ is typically defined as the O_X-dual of the quotient $O_{X \times X}/I^{n+1}$:

$$D_{X/k,n} := \underline{Mor}_{O_X}(O_{X \times X}/I^{n+1}, O_X).$$

The resulting filtered sheaf is a sheaf of noncommutative rings, which will simply be denoted $D_{X/k}$, and the category of right $D_{X/k}$-modules that are quasicoherent as O_X-modules will be denoted $\mathrm{Mod}^r(D_{X/k})$.[2]

Theorem 2 (Kashiwara, [6, Theorem 2.3.1])**.** *Suppose $Z \twoheadrightarrow X$ a closed immersion of smooth schemes; then the category of right $D_{Z/k}$-modules is naturally equivalent to the category of right $D_{X/k}$-modules set-theoretically supported on Z.*

Theorem 3 (Hodges, [5])**.** *Suppose k algebraically closed of characteristic 0, and suppose X smooth over k. Then the functor $- \otimes_{O_X} D_{X/k}$ induces an equivalence of K-theory spectra*

$$K(X) \twoheadrightarrow K(D_{X/k}).$$

About the Proof. For affines, this follows from the K'-equivalence of a filtered ring and its 0-th filtered piece [7, Theorem 7]. The general case follows from using Kashiwara's Theorem to devise a localization sequence for $K(D_{-/k})$, which can be compared to the localization sequence for K. □

Example 4 (Bernstein-Gelfand-Gelfand, [2])**.** If X is singular, then $D_{X/k}$ is an unpleasant ring, and neither Kashiwara's nor Hodges' Theorem holds for right $D_{X/k}$-modules. To illustrate, suppose that C is the affine cone over the Fermat curve $x^3 + y^3 + z^3 = 0$ (over $\mathbf{C}$, let us say); then X is normal, and has an isolated Gorenstein singularity at the origin.

Nevertheless, the ring $D(C)$ of differential operators is neither left nor right nœtherian: if e denotes the Euler operator $x\partial_x + y\partial_y + z\partial_z$, and if $D^{(j)}(C)$ (respectively, $D^{(j)}_n(C)$) is the R-module of homogenous differential operators of degree j (resp., and of order n), then the two-sided ideals

$$J_k := \sum_{j>1} D^{(j)} + \sum_{n \geq 0} e^n D^{(1)}_k$$

form an ascending chain that does not stabilize.

[1]For simplicity I will use the term "scheme" for a separated nœtherian scheme of finite type.

[2]I will stick to right D-modules here.

5. The standard method for rectifying this is *defining deviancy down* by forcing Kashiwara's Theorem; namely, for a singular scheme Z, one embeds Z (at least locally) into a smooth scheme X and *defines* the category of right $D_{Z/k}$-modules to be the full subcategory of right $D_{X/k}$-modules set-theoretically supported along Z. One must then show that the resulting category is invariant up to a canonical equivalence of categories.

Definition 6 (Grothendieck, [3, 4.1])**.** The *infinitesimal site* (X_{inf}/k) of X/k is the category of diagrams $X \leftarrow S \rightarrow T$ in which the morphism $S \rightarrow T$ is a closed nilimmersion of k-schemes, and the morphism $S \rightarrow X$ is étale.[3] There is a natural forgetful functor $(S,T) \mapsto T$ to the category of k-schemes; pull back the étale topology along this functor.

7. There is a stack in categories on the infinitesimal site of X:

$$\begin{aligned} \mathrm{Mod}^!_{X/k,\mathrm{qc}} : (X_{\mathrm{inf}}/k)^{\mathrm{op}} &\longrightarrow \mathrm{Cat} \\ (S,T) &\longmapsto \mathrm{Mod}_{\mathrm{qc}}(O_T) \\ (f,g) &\longmapsto H^0 g^!. \end{aligned}$$

Definition 8 (Beilinson-Drinfeld, [1, Definition 7.10.3])**.** A *$\mathcal{D}$-crystal* on X is a cartesian section of the stack $\mathrm{Mod}^!_{X/k,\mathrm{qc}}$. More precisely, a $\mathcal{D}$-crystal M assigns to every object (S,T) a quasicoherent O_T-module $M_{(S,T)}$ and to every morphism $(f,g) : (S,T) \rightarrow (S',T')$ an isomorphism

$$M_{(S,T)} \rightarrow H^0 g^! M_{(S',T')} .$$

The category of such will be denoted $\mathrm{Cris}^!(X/k)$.

Example 9. Suppose X a smooth k-scheme. Then for any object $(S,T) \in (X_{\mathrm{inf}}/k)$, let $p_T : T \rightarrow Speck$ denote the structure morphism of T, and set

$$t\omega_{X/k}(T) := H^n p_T^! \mathcal{O}_{Speck}.$$

It follows from the smoothness property of X that there exists a morphism $q : T \rightarrow X$ of k-schemes, so that $H^n p_T^! \mathcal{O}_{Speck} \cong H^0 q^! \omega_{X/k}$, where $\omega_{X/k}$ is the dualizing sheaf of top-degree differential forms.[4] Thus $t\omega_{X/k}$ is a $\mathcal{D}$-crystal.

Proposition 10 (Beilinson-Drinfeld, [1, Proposition 7.10.12])**.** *If X is a smooth k-scheme, then the category* $\mathrm{Cris}^!(X/k)$ *is equivalent to the category* $\mathrm{Mod}^r(D_{X/k})$.

About the Proof. The question is local, so assume X affine. If pr_1, pr_2 are the projections from the formal completion of the diagonal, $\mathrm{Cris}^!(X/k)$ is equivalent to the category of quasicoherent O_X-modules M equipped with isomorphisms $pr_1^! M \cong pr_2^! M$ satisfying the obvious cocycle condition. There is a natural isomorphism

$$M \otimes_{O_X} D_X pr_{2,\star} pr_1^! M,$$

[3] I can replace "étale" more generally with "quasi-finite" or less generally with "Zariski open immersion;" the resulting theory of $\mathcal{D}$-crystals is the same in each instance.

[4] Observe however that $\omega_{X/k}(T)$ is only a truncation of the dualizing complex $\omega_{T/k}$.

and adjunction then converts the isomorphism $pr_1^! M \cong pr_2^! M$ into the structure of a right D_X-module; the cocycle condition guarantees associativity. □

Theorem 11 (Beilinson-Drinfeld, [1, Lemma 7.10.11])**.** *Kashiwara's Theorem holds for $\mathcal{D}$-crystals; i.e., for any closed immersion $Z \rightarrowtail X$ of schemes (not necessarily smooth), the category of $\mathcal{D}$-crystals on Z is naturally equivalent to the category of $\mathcal{D}$-crystals on X set-theoretically supported on Z.*

12. The appropriate functorialities of $\mathcal{D}$-crystals do not exist in general. It is more natural not to truncate $g^!$, and to consider instead the following $(\infty,1)$-stack:

$$\begin{aligned} \mathrm{HMod}^!_{X/k,\mathrm{qc}}: \ (X_{\mathrm{inf}}/k)^{\mathrm{op}} &\longrightarrow (\infty,1)\mathrm{Cat} \\ (S,T) &\longmapsto \mathrm{Cplx}(\mathrm{Mod}_{\mathrm{qc}}(O_T)) \\ (f,g) &\longmapsto g^!. \end{aligned}$$

Definition 13. A *homotopy $\mathcal{D}$-crystal* on X is a homotopy cartesian section of the stack $\mathrm{HMod}^!_{X/k,\mathrm{qc}}$. The category of such will be denoted $\mathrm{HCris}^!(X/k)$.

Example 14. The assignment $(S,T) \mapsto \omega_{T/k}$ is a homotopy $\mathcal{D}$-crystal on X.

Proposition 15. *If X is a smooth k-scheme, then the category* $\mathrm{HCris}^!(X/k)$ *is equivalent to the category* $\mathrm{Cplx}(\mathrm{Mod}^r(D_{X/k}))$.

Theorem 16. *Kashiwara's Theorem holds for homotopy $\mathcal{D}$-crystals; i.e., if $Z \rightarrowtail X$ is any closed immersion of schemes (not necessarily smooth), there is a natural equivalence between the $(\infty,1)$-category of homotopy $\mathcal{D}$-crystals on Z and the full subcategory of the $(\infty,1)$-category of homotopy $\mathcal{D}$-crystals on X set-theoretically supported on X.*

Conjecture 17. *For any scheme X, the K-theory of the $(\infty,1)$-category of $\mathcal{D}$-crystals on X is naturally equivalent to $K'(X)$.*

Strategy. Again the analogue of Kashiwara's theorem permits a quick reduction to the affine case. In this case it seems possible to work directly with the definition of K-theory of $(\infty,1)$-categories, but since the definition is necessarily complicated, I have not yet managed to check all the details unless X is Cohen-Macaulay. □

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Reflection theorems and the p-Sylow subgroup of $K_{2n}O_F$ for a number field F

Hourong Qin

Let F be a number field, O_F the ring of integers in F. For any prime p and $i \geq 1$, it is interesting to know p-rank$K_{2i}(O_F)$, more generally, the structure of $K_{2i}(O_F)\{p\}$, the p-Sylow subgroup of $K_{2i}O_F$. However, even for $i = 1$, we have no answer in general. On the other hand, some relations between the p-Sylow subgroup of K_2O_F for a number field F and some p-Sylow subgroup of the ideal class group of $F(\zeta_{p^n})$ have been established, see Tate[8], Keune[4], Browkin[1] and among others.

In the study of the ideal class group of a number field, some so called reflection theorems are useful. A classical result, in this direction, is due to Scholz. A generalization of this result exists, see for example Theorem 10.11 in Washington[9]. However, Scholz Theorem can not be viewed as a particular case of this generalization.

We set up a reflection theorem, which contains Scholz Theorem as a special case.

Theorem 1. (A) Let $d > 1$ be a square-free integer, and let $p \neq d$ be a regular odd prime. For any positive integer n, let r be the p-rank of ideal class group of $\mathbb{Q}(\sqrt{d},\ \zeta_{p^n} + \zeta_{p^n}^{-1})$ and s be the p-rank of ideal class group of $\mathbb{Q}(\sqrt{d}(\zeta_{p^n} - \zeta_{p^n}^{-1}))$. Then

$$r \leq s \leq r + \frac{1}{2}p^{n-1}(p-1).$$

(B) Let $d < -1$ be a square-free integer, and let $p \neq -d$ be a regular odd prime. For any positive integer n, let s be the p-rank of ideal class group of $\mathbb{Q}(\sqrt{d},\ \zeta_{p^n} + \zeta_{p^n}^{-1})$ and r be the p-rank of ideal class group of $\mathbb{Q}(\sqrt{d}(\zeta_{p^n} - \zeta_{p^n}^{-1}))$. Then

$$r \leq s \leq r + \frac{1}{2}p^{n-1}(p-1).$$

Note that for $p \equiv 3 \pmod 4$, replacing d by $-pd$, we see that (A) and (B) in the theorem are in fact the same, and if $p \equiv 3 \pmod 4$, then $\mathbb{Q}(\sqrt{d}(\zeta_{p^n} - \zeta_{p^n}^{-1})) = \mathbb{Q}(\sqrt{-pd},\ \zeta_{p^n} + \zeta_{p^n}^{-1})$. So, if we let $p = 3,\ n = 1$, then we get

Corollary (Scholz Theorem). Let $d > 1$ be a square-free integer, and let r, s be the 3-rank of ideal class groups of $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{-3d})$, respectively. Then

$$r \leq s \leq r + 1.$$

The following statements were first conjectured by H.Gangl.

Conjecture. Let d be a negative square-free integer and $F=\mathbb{Q}(\sqrt{d})$. Then
(i) if $d\not\equiv 6 \pmod 9$, then $3|\sharp(K_2O_F)$ implies $3|\sharp\mathrm{Cl}(O_F)$;
(ii) if $d\equiv 6 \pmod 9$, then $9|\sharp(K_2O_F)$ implies $3|\sharp\mathrm{Cl}(O_F)$.

(i) was proved by Browkin[1].

As an application of Theorem 1, we have the following theorem, which implies both (i) and (ii).

Theorem 2. **(A)** Let d be a negative square-free integer and $F=\mathbb{Q}(\sqrt{d})$. Then
(i) if $p\neq 3$ is a regular prime or $p=3, d\not\equiv 6 \pmod 9$, then $p|\sharp K_2O_F$ implies $p|\sharp\mathrm{Cl}(O_{\mathbb{Q}(\sqrt{d},\ \zeta_p+\zeta_p^{-1})})$;
(ii) if $d\equiv 6 \pmod 9$, then $9|\sharp K_2O_F$ implies $3|\sharp\mathrm{Cl}(O_F)$.

(B) Let d be a positive square-free integer, $F=\mathbb{Q}(\sqrt{d})$, and $p\equiv 1 \pmod 4$ a regular prime. Then $p|\sharp K_2O_F$ implies $p|\sharp\mathrm{Cl}(O_{\mathbb{Q}(\sqrt{d(\zeta_p-\zeta_p^{-1})})})$.

We now turn to the higher K-groups of the ring of integers in a number field F. For a finite set of primes of F, we use O_S for S-integers in F. Rognes and Weibel[6] gives the results on $K_n(O_S)\{2\}$. Let $n=2i\geq 2$. Based on [6] and the Voevosky and Rost Theorem, one has

Theorem[10]. If p is an odd prime, then $K_n(O_S)\{p\}\cong H^2(O_S[\frac{1}{p}],\mathbb{Z}_p(i+1))$.

Let F be an abelian extension of $\mathbb{Q}$ with degree d. Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension and $\mu_q\subset F$ with $q=p$ if $p>2$ and 4 if $p=2$. Let q_0 be the largest power of p such that $\mu_{q_0}\subset F$. Let $F_0=F$ and F_n be the unique intermediate field for F_∞/F such that $[F_n:F]=p^n$. Let M/F_∞ be the maximal abelian p-extension unramified outside p. Let $\Gamma=\mathrm{Gal}(F_\infty/F)$ and $\Gamma_n=\Gamma/\Gamma^{p^n}$. From Iwasawa, the Galois group $\mathrm{Gal}(M/F_\infty)$ is a noetherian Γ-module with no non-trivial finite Γ-submodule. Let $\mathrm{Gal}(M/F_\infty)^\bullet$ be the twist Λ-module of the usual Λ-module $\mathrm{Gal}(M/F_\infty)$. Iwasawa showed that $H=\frac{\Lambda^{d/2}}{\mathrm{Gal}(M/F_\infty)^\bullet/t(\mathrm{Gal}(M/F_\infty)^\bullet)}$ is a finite group, here $t(Y)$ denotes the torsion Λ-submodule of any Λ-module Y. We denote by $H(i)$ the Tate twist of the usual Λ-module H. Let $f(T)$ be the characteristic polynomial of the Γ-module $\mathrm{Gal}(M/F_\infty)^\bullet$.

Theorem 3 (Ji and Qin)**.**

$$\sharp K_{2i}(O_{F_n})\{p\}=\sharp H(i)^{\Gamma_n}\cdot\prod_{j=0}^{n}|f(\kappa(\gamma_0)^{-i}\zeta_{p^j}-1)|_{v_j}^{-1},$$

where γ_0 is a fixed generator of Γ and $\kappa:\Gamma\longrightarrow 1+q_0\mathbb{Z}_p$ is the isomorphism such that

$$\gamma(\zeta)=\zeta^{\kappa(\gamma)},\ \text{ for all }\ \gamma\in\Gamma\ \text{ and }\ \zeta\in W=\cup_{n\geq 0}\mu_{p^n},$$

and $|\cdot|_{v_j}$ is the standard valuation on the fields $\mathbb{Q}_p(\zeta_{p^j})$ such that $|\zeta_{p^j}-1|_{v_j}=1/p$ for all $j\geq 1$, and $|\cdot|_{v_0}=|\cdot|_p$ on $\mathbb{Q}_p$ such that $|p|_p=1/p$.

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Rigidity for A^1-representable theories

Jens Hornbostel
(joint work with Serge Yagunov)

This is joint work with Serge Yagunov. The details may be found in [HY].

We establish rigidity results for graded cohomology type functors E on smooth varieties over an infinite base field k. This work generalizes the results of [PY] and [Ya] where the special case of orientable theories E resp. stably $\mathbb{A}^1$-representable theories on smooth varieties over algebraically closed fields have been studied.

Consider some category of schemes $\mathcal{S}$ over a base scheme B together with a cohomology theory $E^*\colon \mathcal{S}^{\text{op}} \to \mathbf{Ab}$. Then we say that E^* satisfies *rigidity* if for every scheme $X \xrightarrow{\chi} B$, any two sections $\sigma_0, \sigma_1\colon B \to X$ of the structure morphism χ induce the same homomorphism $\sigma_0^* = \sigma_1^*\colon E(X) \to E(B)$. In classical topology, the rigidity property is an obvious consequence of homotopy invariance of cohomology theories. However, in algebraic geometry $\mathbb{A}^1$-invariance does not always imply rigidity. It only holds under certain restrictions on $\mathcal{S}$ and the cohomology theory E^*. In particular, rigidity fails for K_1 with integral coefficients. Rigidity results for finite coefficients have been established for algebraic K-theory by Suslin, Gabber and others (see [Su, Ga, GT]).

Over algebraically closed fields, Panin and Yagunov [PY] establish a set of axioms for transfer maps and show they are satisfied for any *orientable theory* over algebraically closed fields (e.g., motivic cohomology, algebraic K-theory or algebraic cobordism). Then they deduce a rigidity theorem for orientable theories with finite coefficients. In [Ya], Yagunov shows that these results carry over to all theories that are representable in Voevodsky's stable $\mathbb{A}^1$-homotopy category. Examples include hermitian K-theory, Balmer Witt groups assuming $\text{char}(k) \neq 2$ [Ho], and stable cohomotopy groups. Stable $\mathbb{A}^1$-representability allows Yagunov to construct algebraic "Becker-Gottlieb transfers" with respect to a class of morphisms $\mathcal{C}_{\text{triv}}$, which is rather small but still large enough to conclude.

Assuming certain additional hypotheses that can be checked in many cases of interest (see Corollary 0.3 below), we generalize these results from algebraically closed fields to arbitrary infinite ones. (The case $E = K$ recovers Gabber's rigidity theorem for algebraic K-theory.)

Theorem 0.1. *Let k be a field, and let R be a henselian local ring over k with an infinite field of fractions $\mathrm{Frac}(R) = F$. Assume that $E = E^{**}$ is a contravariant bigraded functor on the category Sm/k of smooth schemes over k that is representable in the stable $\mathbb{A}^1$-homotopy category and that $\ell E = 0$ for $\ell \in \mathbb{Z}$ invertible in R. Let $f\colon M \to \operatorname{Spec} R$ be a smooth affine morphism of (pure) relative dimension d, and $s_0, s_1\colon \operatorname{Spec} R \to M$ two sections of f such that $s_0(p) = s_1(p)$, where p is the closed point of $\operatorname{Spec} R$. Assume moreover that E is normalized with respect to the field F. Then the two maps $E(M) \xrightarrow{s_i^*} E(\operatorname{Spec} R)$ are equal $(i = 0, 1)$.*

Corollary 0.2. *Let E and k be as in Theorem 0.1, V be a smooth variety over k, $P \in V(k)$ be a k-rational point of V, and $R = \mathcal{O}^h_{V,P}$. Then*

$$E(\operatorname{Spec} R) \xrightarrow{\cong} E(\operatorname{Spec} k)$$

is an isomorphism.

As the classical proofs for K-theory and the proofs of Panin and Yagunov, the proof of the Theorem relies on the existence of *transfer maps* fulfilling certain properties and on *homotopy invariance* (i.e. $E(X) \cong E(X \times \mathbb{A}^1)$ if X smooth) whereas for the Corollary one needs moreover that E commutes with colimits. Nevertheless, some parts of the proof are more considerably more complicated, for example when dealing with divisors having points corresponding to non-separable field extensions. Corollary 0.2 implies the following.

Corollary 0.3. *Let $X \in Sm/k$, V be a smooth variety over k, $P \in V(k)$, and $F = \mathrm{Frac}(\mathcal{O}^h_{V,P})$. Let also (as in Theorem 0.1) E be a representable cohomology theory such that $\ell E = 0$ for some $\ell \in \mathbb{Z}$ invertible in F. If the map $E(\mathbb{P}^2_{X_L}) \to E(\mathbb{P}^1_{X_L})$ is an epimorphism for any finite separable field extension L/F (e.g. $E = MGL$, H_{mot}, or K), then the map*

$$E(X \times_{\operatorname{Spec} k} \operatorname{Spec} \mathcal{O}^h_{V,P}) \to E(X)$$

is an isomorphism. If E is represented by a commutative motivic ring spectrum, then it is sufficient to check the epimorphism condition for $X = \operatorname{Spec} k$.

There is work in progress combining the above techniques with my joint work with B. Calmes [CH] on transfers for Witt groups. This will hopefully imply the above rigidity theorems unconditionally for Witt groups in all degrees.

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On the p-adic Beilinson conjecture for number fields

Rob de Jeu

(joint work with Amnon Besser, Paul Buckingham and Xavier-François Roblot)

1. The p-adic Beilinson conjecture

For a totally real number field k we consider a p-adic analogue of Borel's theorem [6]. Let $d = [k : \mathbb{Q}]$, $\mathcal{O}_k$ the ring of algebraic integers of k, $\sigma_1, \ldots, \sigma_d$ the embeddings of k into $\mathbb{R}$, $\{a_1, \ldots, a_d\}$ a $\mathbb{Z}$-basis of $\mathcal{O}_k$, so that $D_k^{1/2,\infty} = \det(\sigma_i(a_j))$ is a square root in $\mathbb{R}$ of the discriminant of k. If $n \geq 2$ is even then $K_{2n-1}(k)$ is torsion, but if $n \geq 2$ is odd then $K_{2n-1}(k)$ is a finitely generated group of rank d. For such n, let $\{\alpha_1, \ldots, \alpha_d\}$ be a $\mathbb{Z}$-basis of $K_{2n-1}(k)/\text{torsion} \cong K_{2n-1}(\mathcal{O}_k)/\text{torsion}$.

For any $n \geq 2$ Borel defined a regulator map $\text{reg}_\infty : K_{2n-1}(\mathbb{C}) \to \mathbb{R}$, and for every embedding $\sigma : k \to \mathbb{C}$ we get a composition $\text{reg}_\infty^\sigma : K_{2n-1}(k) \to K_{2n-1}(\mathbb{C}) \to \mathbb{R}$. With $R_{n,\infty}(k) = \det(\text{reg}_\infty^{\sigma_i}(\alpha_j))$ for $n \geq 2$ odd, a special case of Borel's theorem is:

Theorem 1. *For $n \geq 2$ odd, $\zeta_k(n) D_k^{1/2,\infty} = q(n,k) R_{n,\infty}(k)$ with $q(n,k)$ in $\mathbb{Q}^*$.*

Now let p be a prime, $F \subset \overline{\mathbb{Q}}_p$ the topological closure of the Galois closure of k embedded in $\overline{\mathbb{Q}}_p$ in any way, and $\mathcal{O}_F$ the valuation ring of F. We define $D_k^{1/2,p} = \det(\sigma_i^p(a_j))$, a square root in F of the discriminant of k. By [2], for any finite extension F' of $\mathbb{Q}_p$ there is a syntomic regulator

$$\text{reg}_p : K_{2n-1}(\mathcal{O}_{F'}) \to H^1_{\text{syn}}(\text{Spec}(\mathcal{O}_{F'})/\mathcal{O}_{F'}, n) \cong F' ,$$

hence for every embedding $\tau : k \to F$ we get a composition

$$\text{reg}_p^\tau : K_{2n-1}(k) \cong K_{2n-1}(\mathcal{O}_k) \to K_{2n-1}(\mathcal{O}_F) \to F .$$

If $\tau_1, \ldots, \tau_d$ are the embeddings of k into F then we let $R_{n,p}(k) = \det(\text{reg}_p^{\tau_i}(\alpha_j))$ in F be the p-adic regulator.

The interpolation formula for p-adic L-functions [1, 8, 9, 11] suggests to replace $\zeta_k(n)$ in Theorem 1 with $L_p(n, \omega_p^{1-n}, k)/\text{Eul}_p(n,k)$, where ω_p is the Teichmüller character for p and $\mathrm{Eul}_p(s,k)$ is the reciprocal of the Euler factor for p in $\zeta_k(s)$. We therefore make the following conjecture (cf. [14]), which is independent of any choices.

Conjecture 2. *For k a totally real number field, $n \geq 2$ odd, and any prime p, the following holds:*

(1) *in F we have*

$$L_p(n, \omega_p^{1-n}, k) D_k^{1/2,p} = q_p(n,k) \mathrm{Eul}_p(n,k) R_{n,p}(k)$$

for some $q_p(n,k)$ in $\mathbb{Q}^*$;

(2) *in fact, $q_p(n,k) = q(n,k)$;*

(3) *$L_p(n, \omega_p^{1-n}, k)$ and $R_{n,p}(k)$ are non-zero.*

2. A motivic version of the conjecture

If E is a number field, $k/\mathbb{Q}$ any (finite) Galois extension with Galois group G then we let $M^E = E \otimes_{\mathbb{Q}} k$ and $K_{2n-1}(k)_E = E \otimes_{\mathbb{Z}} K_{2n-1}(k)$, which we view as left $E[G]$-modules. Then Conjecture 2 can be extended to a conjecture for certain idempotents in $E[G]$ and certain $n \geq 2$.

For an idempotent π in $E[G]$ let $M_\pi^E = \pi M^E$ and $K_{2n-1}(M_\pi^E) = \pi K_{2n-1}(k)_E$. Using fixed embeddings $\phi_\infty : k \to \mathbb{C}$ and $\phi_p : k \to \overline{\mathbb{Q}}_p$ we obtain E-bilinear pairings

$$(\cdot,\cdot)_\infty : E[G]\pi \times M_\pi^E \to E \otimes_{\mathbb{Q}} \mathbb{C}$$

and

$$(\cdot,\cdot)_p : E[G]\pi \times M_\pi^E \to E \otimes_{\mathbb{Q}} F\,,$$

where $F \subset \overline{\mathbb{Q}}_p$ is the topological closure of $\phi_p(k)$. Combining ϕ_∞ and ϕ_p with the maps reg_∞ and reg_p we also get E-bilinear pairings when $n \geq 2$,

$$[\cdot,\cdot,]_\infty : E[G]\pi \times K_{2n-1}(M_\pi^E) \to E \otimes_{\mathbb{Q}} \mathbb{R}$$

and

$$[\cdot,\cdot,]_p : E[G]\pi \times K_{2n-1}(M_\pi^E) \to E \otimes_{\mathbb{Q}} F\,.$$

Then $\dim_E(M_\pi^E) = \dim_E(E[G]\pi)$ and $\dim_E(\pi K_{2n-1}(k)_E) \leq \dim_E(E[G]\pi)$, but for $n \geq 2$ equality in the latter holds in precisely two cases. They are, if k' is the fixed field of the kernel of the representation ρ of G on $E[G]\pi$:

(i) k' is totally real and n is odd;

(ii) k' is a CM field and n is even, and the (unique) complex conjugation in $\mathrm{Gal}(k'/\mathbb{Q})$ acts as multiplication by -1 on $E[G]\pi$.

In those cases we fix ordered E-bases of $E[G]\pi$, M_π^E and $K_{2n-1}(M_\pi^E)$, and for $* = p$ or ∞ we let $D(M_\pi^E)^{1/2,*}$ be the determinant of $(\cdot,\cdot)_*$, $R_{n,*}(M_\pi^E)$ that of $[\cdot,\cdot]_*$, all computed with respect to the chosen bases.

In either case one can define, using the embeddings of E into $\mathbb{C}$ (resp. $\overline{\mathbb{Q}}_p$), an $E\otimes_{\mathbb{Q}}\mathbb{C}$-valued L-function $L(s, \rho\otimes \mathrm{id}, \mathbb{Q})$ (resp. an $E\otimes_{\mathbb{Q}}\mathbb{Q}_p$-valued p-adic L-function $L_p(s, \rho\otimes\omega_p^{1-n}, \mathbb{Q})$) corresponding to ρ, as well as $\mathrm{Eul}_p(s, \rho\otimes \mathrm{id}, \mathbb{Q})$, the reciprocal of an Euler factor for p in $L(s, \rho\otimes \mathrm{id}, \mathbb{Q})$. $L_p(s, \rho\otimes\omega_p^{1-n}, \mathbb{Q})$ is not identically zero.

We can then formulate the following refinement and generalization of Conjecture 2 (cf. [14]), which is again independent of any choices made.

Conjecture 3. *For $n \geq 2$ and p prime, in (i) and (ii) above the following holds:*

(1) *in $E \otimes_{\mathbb{Q}} \mathbb{C}$ we have*

$$L(n, \rho \otimes \mathrm{id}, \mathbb{Q}) D(M_\pi^E)^{1/2,\infty} = e(n, M_\pi^E) R_{n,\infty}(M_\pi^E)$$

for some $e(n, M_\pi^E)$ in $(E \otimes_{\mathbb{Q}} \mathbb{Q})^*$;

(2) *in $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$ we have*

$$L_p(n, \rho \otimes \omega_p^{1-n}, \mathbb{Q}) D(M_\pi^E)^{1/2,p} = e_p(n, M_\pi^E) \mathrm{Eul}_p(n, \rho \otimes \mathrm{id}, \mathbb{Q}) R_{n,p}(M_\pi^E)$$

for some $e_p(n, M_\pi^E)$ in $(E \otimes_{\mathbb{Q}} \mathbb{Q})^*$;

(3) *in fact, $e_p(n, M_\pi^E) = e(n, M_\pi^E)$;*

(4) *$L_p(n, \rho \otimes \omega_p^{1-n}, \mathbb{Q})$ and $R_{n,p}(M_\pi^E)$ are non-zero.*

Remark 4. *Part (1) is a special case of a conjecture by Gros* (*see* [13, p. 210]).

For parts (1)-(3) we can prove the following result, building on [10], [12] and [4].

Proposition 5. *Let $N \geq 2$ and $k = \mathbb{Q}(\mu_N)$. Assume that E contains a root of unity of order equal to the exponent of G and let π be the idempotent corresponding to an irreducible character χ of G. Then parts (1), (2) and (3) of Conjecture 3 hold for $M^E = \pi E[G]$ and $n \geq 2$ whenever it applies, i.e., when χ maps complex conjugation to $(-1)^{n-1}$.*

For $p = 2$ or 3 it follows from irrationality results for some values of certain p-adic L-functions [7, 5] that the last part of the conjecture also holds in a few cases of Proposition 5, but in general one can only verify this numerically.

We also verified Conjecture 3 numerically for certain $k/\mathbb{Q}$, either totally real or CM, with $G \cong S_3$, D_8 or $S_3 \times \mathbb{Z}/3\mathbb{Z}$, and π corresponding to an irreducible non-abelian representation ρ of G. Full details will appear in [3].

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The Gersten conjecture for Milnor K-theory

Moritz Kerz

Define the Milnor K-ring of a semi-local commutative ring A as the graded ring

$$K_*^M(A) = T_*(A^\times)/(\{a \otimes (1-a) | a, 1-a \in A^\times\})$$

where T_* denotes the tensor algebra of an abelian group. Let $\mathbb{Z}(n)$ be Beilinson's motivic complex in the Zariski topology. One of Beilinson's conjectures on motivic cohomology states:

Theorem 1. *For an essentially smooth semi-local ring A over an infinite field and $n > 0$ there exists an isomorphism*

$$\eta : K_n^M(A) \xrightarrow{\sim} H^n(Spec(A), \mathbb{Z}(n)) .$$

If A is a field this theorem was proved by Nesterenko/Suslin [5] and Totaro [8]. The surjectivity of the canonical map $\eta : K_n^M(A) \to H^n(Spec(A), \mathbb{Z}(n))$ under the conditions of Theorem 1 was shown by Gabber [2] and Elbaz-Vincent/Müller-Stach [1]. Furthermore it was known that η is injective modulo torsion.

By what has been proved in the literature it remains to show the exactness of the Gersten complex for $X = Spec(A)$:

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^M(F) \longrightarrow \oplus_{x\in X^{(0)}} K_n^M(\mathbf{k}(x)) \longrightarrow \oplus_{x\in X^{(1)}} K_{n-1}^M(\mathbf{k}(x)) \longrightarrow \cdots$$

The exactness of this complex – except at the first place – was established in [2],[1],[7],[4].

Our main result is:

Theorem 2. *Let A be a regular connected semi-local ring containing an infinite field. Let $F = Q(A)$ be the fraction field of A. Then the natural homomorphism*

$$K_n^M(A) \longrightarrow K_n^M(F)$$

is (universally) injective.

The proof of Theorem 2 is contained in [3]. It is somewhat analogous to Ojanguren's result about the Witt ring [6].

We use two new results from Milnor K-theory of local rings which are of interest in themselves:

The étale excision exact sequence for motivic cohomology motivates the following result which was suggested to hold by Gabber [2]. Let $A \subset A'$ be a local extension of factorial semi-local rings with infinite residue fields and let $f \in A$ be such that $A/(f) = A'/(f)$.

Lemma 3. *The diagram*

$$\begin{array}{ccc} K_n^M(A) & \longrightarrow & K_n^M(A_f) \\ \downarrow & & \downarrow \\ K_n^M(A') & \longrightarrow & K_n^M(A'_f) \end{array}$$

is co-Cartesian.

The second new result is a generalization of the well known short exact sequence

$$0 \longrightarrow K_n^M(F) \longrightarrow K_n^M(F(t)) \longrightarrow \oplus_\pi K_{n-1}^M(F[t]/(\pi)) \longrightarrow 0$$

to local rings.

Let A be a semi-local ring with infinite residue fields. We define in some appropriate way Milnor K-groups (denoted $K_*^t(A,p)$) of the ring $A[t]_{S_p}$ where S_p is the multiplicative set consisting of all monic polynomials coprime to some fixed monic polynomial $p \in A[t]$.

Then we have:

Lemma 4. *There exists a split short exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A,p) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

where the direct sum is over all irreducible monic polynomials $\pi \in A[t]$ which are coprime to p.

Here irreducible monic means that the polynomials cannot be factored into non-trivial monic polynomials in $A[t]$.

Remark 5. *Using Lemma 4 one can construct norms for finite étale extensions of semi-local rings with infinite residue fields. If the rings contain an infinite field these norms are in fact independent of the generators chosen (compare [3]).*

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Transfers for Witt groups and Grothendieck duality

Baptiste Calmès
(joint work with Jens Hornbostel)

This talk is mainly about the definition of transfers (or push-forwards) for Witt groups. They should satisfy the usual requirements: compatibility to composition, to base change, and a projection formula should hold, as it is the case for other theories of cohomological flavor (K-theory, Chow groups ...). We define such transfers [1] and prove those properties in a general abstract setting that could be summarized as follows:

Let us consider a collection of symmetric monoidal triangulated categories, each equipped with an internal Hom adjoint to the tensor product, and some exact monoidal functors f^* between them with reasonable composition properties (2-functors). If these f^* have right adjoints f_* (as usual functors), which themselves have right adjoints $f^!$ and some canonical morphisms (such as the projection formula morphism $f_*A \otimes B \to f_*(A \otimes f^*(B))$) are isomorphisms, then we can define natural transfers between Witt groups associated to some dualities $Hom(-, f^!L)$ and $Hom(-, L)$ where L is a dualizing object. They satisfy a projection formula, are compatible with base change and composition.

We apply these results in algebraic geometry, to Witt groups of the bounded derived category of coherent sheaves over a regular scheme, using results from the theory of duality (Grothendieck, Deligne, Hartshorne, Verdier, see [2] and [4]), which produces a right adjoint $f^!$ to the derived functor Rf_* for a proper morphism f. Thus, when $f : X \to Y$ is a proper map between regular Noetherian schemes of finite Krull dimension and L is a line bundle over Y, we obtain a transfer map

$$W^*(X, f^!L) \to W^*(Y, L)$$

satisfying the above mentioned requirements.

When X and Y are smooth over a regular Noetherian base scheme R of finite Krull dimension, $f^!L$ has a concrete description and this transfer can be reinterpreted as

$$W^{i+\dim X}(X, \omega_X \otimes f^*L) \to W^{i+\dim Y}(Y, \omega_Y \otimes L)$$

where ω is the canonical sheaf. This shows that the line bundles are essential in this framework, and that we cannot restrict to the case where all line bundles are the structural sheaf.

We then obtain two applications. Firstly, this can be used to define a category of Witt-correspondences in the spirit of [3] but in which the objects are pairs (X, L), where X is a smooth projective scheme over R and L is a (maybe shifted) line bundle over X. As in the cases of Chow groups or K-theory, this category is the natural one to express interesting decompositions.

Secondly, this provides a dévissage theorem. Let $f : Z \to X$ be a closed immersion between regular Noetherian schemes of finite Krull dimension, and L a line bundle on X. The transfer mentioned above factorizes through the Witt group of X with support in Z, and it is an isomorphism onto this group:

$$W^*(Z, f^!L) \simeq W^*_Z(X, L)$$

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Modules over Motivic Cohomology

Paul Arne Østvær

(joint work with Oliver Röndigs)

In joint work with Oliver Röndigs we introduce a category of motives for noetherian and separated base schemes of finite Krull dimension. Its construction is based on highly structured models for the motivic stable homotopy category [1], [2]. For fields of characteristic zero we obtain an equivalence with Voevodsky's big category of motives.

Denote by Sm the smooth Nisnevich site of a noetherian and separated scheme S of finite Krull dimension. The category $\mathbf{M}^{\mathrm{tr}}$ of motivic spaces with transfers consists of contravariant additive functors from Cor – the Suslin-Voevodsky category of finite correspondences of S – to simplicial abelian groups. There exists an evident forgetful functor from $\mathbf{M}^{\mathrm{tr}}$ to motivic spaces $\mathbf{M}$ induced by the graph $\mathrm{Sm} \to \mathrm{Cor}$. Its left adjoint functor $\mathbb{Z}^{\mathrm{tr}}$ adding transfers associates to any scheme $U \in \mathrm{Sm}$ the representable motivic space with transfers U^{tr}. Letting $\otimes^{\mathrm{tr}}$ denote the tensor product in $\mathbf{M}^{\mathrm{tr}}$ we note that the transfer functor is determined by $\mathbb{Z}^{\mathrm{tr}}(U \times \Delta^n)_+ = U^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\Delta^n]$. In [5], we construct the (projective) motivic model structure on $\mathbf{M}^{\mathrm{tr}}$ such that the forgetful functor to motivic spaces detects and preserves fibrations and weak equivalences. This model structure represents the starting point of homotopy theory of motivic spaces with transfers, analogously to the (injective) motivic model structure on $\mathbf{M}$ introduced by Morel-Voevodsky [3]. Implicitly in the above we have employed the Nisnevich topology; in particular, whether a map between motivic spaces with transfers is a motivic weak equivalences can be tested on Hensel local schemes.

Let S^1_s denote the simplicial circle. The category $\mathbf{MSS}$ of motivic symmetric spectra is defined by taking the suspension functor with respect to the cofibrant motivic space $T = S^1_s \wedge \mathbb{G}$, where $\mathbb{G}$ is a cofibrant replacement of $(\mathbb{A}^1 \smallsetminus \{0\}, 1)$. Similarly, by applying $\mathbb{Z}^{\mathrm{tr}}$ to T we obtain the category $\mathbf{MSS}^{\mathrm{tr}}$ of motivic symmetric spectra with transfers. Let $\mathbb{G}_m^{\mathrm{tr}}$ denote $\mathbb{Z}^{\mathrm{tr}}(\mathbb{A}^1 \smallsetminus \{0\}, 1)$, and $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}_m^{\mathrm{tr}}[1]}$ denote the symmetric spectra of connective chain complexes of presheaves with transfers with respect to the normalized chain complex of T, i.e. the shifted copy of $\mathbb{G}_m^{\mathrm{tr}}$ in degree one. The symmetric spectrum category $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ is defined by suspending with respect to $\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1, 1)$.

By [5], there exists a zig-zag of symmetric monoidal Quillen equivalences between $\mathbf{MSS}^{\mathrm{tr}}$ and $\mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1}$, and likewise for $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ and $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}_m^{\mathrm{tr}}[1]}$. Now since $\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1, 1)$ is cofibrant and discrete, the Dold-Kan equivalence yields a lax symmetric monoidal Quillen equivalence between $\mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1}$ and $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ [5]. The stable homotopy theoretic forerunner of this result was proved in [7].

Let $\mathrm{M}\mathbb{Z}$ denote the motivic Eilenberg-MacLane symmetric spectrum. On account of the monoid axiom – see [6] – for motivic symmetric spectra [2], the module category of $\mathrm{M}\mathbb{Z}$ acquires a model structure: A map between $\mathrm{M}\mathbb{Z}$-modules is a weak equivalence if the underlying map of motivic symmetric spectra is a stable weak equivalence. In [5] we compare $\mathrm{M}\mathbb{Z} - \mathbf{mod}$ with $\mathbf{MSS}^{\mathrm{tr}}$.

Theorem. *The model categories* $\mathrm{M}\mathbb{Z} - \mathbf{mod}$ *and* $\mathbf{MSS}^{\mathrm{tr}}$ *are Quillen equivalent when the base scheme is a field of characteristic zero.*

Combining this with the Quillen equivalence between $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}_m^{\mathrm{tr}}[1]}$ and $\mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}[1]}$ – which holds for arbitrary base schemes as above – we get an equivalence between the homotopy category $\mathrm{Ho}(\mathrm{M}\mathbb{Z} - \mathbf{mod})$ and Voevodsky's big category of motives consisting of $\mathbb{G}_m^{\mathrm{tr}}[1]$-spectra of non-connected chain complexes of sheaves with transfers having homotopy invariant cohomology sheaves [9].

The proof of the theorem makes use of assembly maps of motivic functors [1] and dualizability of the generators of the motivic stable homotopy category. We refer to the note [4] for an outline of the proof and to [5] for complete proofs.

In conclusion, having established the connection between modules over $\mathrm{M}\mathbb{Z}$ and Voevodsky's theory of motives, we mention some additional properties of $\mathrm{M}\mathbb{Z} - \mathbf{mod}$. First, localization for an open embedding and its closed complement holds in view of localization for motivic symmetric spectra. Second, there exists a six functor formalism on the level of $\mathrm{M}\mathbb{Z}$-modules. These properties are closely related to the joint work of Cisinski and Deglise communicated at the meeting. Moreover, by making use of localization, it appears that the equivalence in the above theorem extends to arbitrary base schemes over fields of characteristic zero.

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Algebraic Cycles and Additive Chow Groups

Spencer Bloch

"Mighty oaks from little acorns grow."

Introduction

Additive algebraic K-theory means roughly "replace the algebraic group GL by the lie algebra $\mathfrak{g}l$ where ever you see it", [L]. Given the central role of algebraic cycles in motivic cohomology, one may ask for an algebraic cycle interpretation of additive K-theory. More than a simple restatement of the theory, such a geometric reformulation suggests new problems. What are motivic sheaves over $k[t]/(t^2)$? What is the tangent space to the space of motives?

One should, I believe, have the following picture in mind. An algebraic circle is represented by the pair $\mathbb{A}^1, \{0, t\}$ for any $t \neq 0$. It is natural, geometrically, to think of the limiting situation $t \to 0$ as represented by

$$\mathbb{A}^1, 2(0) \tag{1}$$

As a simple example, for k a field

$$H^1_M(k, \mathbb{Z}(1)) = Pic(\mathbb{A}^1_k, \{0, t\}) = \mathbb{G}_m(k) = k^\times, \tag{2}$$

It is natural to write for the corresponding additive group

$$TH^1_M(k, \mathbb{Z}(1)) := Pic(\mathbb{A}^1_k, 2(0)) = \mathbb{G}_a(k) = k. \tag{3}$$

A word of warning. The notation TH_M suggests tangent space, but this is perhaps not the precise analogy. A better picture would be a sort of non-semi-stable degeneration, with a group like $\mathbb{G}_m$ which is constant in the parameter t degenerating to an additive group for $t = 0$.

Hyperbolic versus Euclidean

Another analogy is the picture of the hyperbolic metric on a disk of expanding radius. As the radius tends to ∞, the metric tends to the euclidean metric. The link with motivic cohomology is summarized by the following 4-term exact sequence

$$\begin{array}{ccccccccc} 0 \to & H^1_M(\mathbb{C},\mathbb{Z}(2)) & \to & B_2(\mathbb{C}) & \xrightarrow{\delta} & \mathbb{C}^\times \otimes \mathbb{C}^\times & \to K_2(\mathbb{C}) & \to 0 \\ & \downarrow{\scriptstyle reg} & & \downarrow{\scriptstyle vol} & & & & \\ & \mathbb{R} & \xrightarrow{=} & \mathbb{R} & & & & \end{array} \tag{4}$$

Here $B_2(\mathbb{C})$ has generators $[x]_2,\ x \in \mathbb{C} - \{0,1\}$, and $vol[x]_2$ is the hyperbolic volume of the tetrahedron in hyperbolic 3-space with vertices at infinity at the points $0, 1, \infty, x \in \mathbb{P}^1(\mathbb{C})$. One defines $\delta[x]_2 = x \otimes (1-x)$, and the symbol $[x]_2 \in B_2$ satisfies the classical 5-term relation.

The additive analogue of 4 was worked out in [BE] using a K-theoretic definition of the additive motivic group $TH^1_M(k,\mathbb{Z}(2))$ involving the relative K-group $K_2(\mathcal{O}_{\mathbb{A}^1,0},(t^2))$:

$$\begin{array}{ccccccccc} 0 \to & TH^1_M(k,\mathbb{Z}(2)) & \to & TB_2(k) & \xrightarrow{T\delta} & k \otimes k^\times & \xrightarrow{a\otimes b \mapsto a\,db/b} & \Omega^1_k & \to 0 \\ & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & & & & \\ & k & \xrightarrow{=} & k & & & & & \end{array} \tag{5}$$

By definition, $TB_2(k)$ is a $k^\times$-module (not a k-vector space!) with generators $\langle x \rangle_2$ which satisfy the 4-term information-theory equation first identified in this context by Kontsevich

$$\langle x\rangle_2 - \langle y\rangle_2 + x \star \langle y/x\rangle_2 + (1-x) \star \langle (1-y)/(1-x)\rangle_2 = 0 \tag{6}$$

The regulator map ρ in (5) is defined by $\rho\langle x\rangle_2 = x(1-x)$. One might hope that $\rho\langle x\rangle_2$ represents some Euclidean polyhedron with Euclidean volume $x(1-x)$. The action of $k^\times$ on the target of ρ is by the cube of the standard character, suggesting an interpretation in terms of volume in $\mathbb{R}^3$.

Cycles

We consider algebraic cycles on $\mathbb{A}^1 \times (\mathbb{P}^1)^n$, where $\mathbb{A}^1$ has parameter t and the $\mathbb{P}^1$ have parameters $t_i,\ 1 \le i \le n$. We have divisors $\sigma_i : t_i = 1$ and $\delta_i = (t_i)_0 - (t_i)_\infty$. Fix $m \ge 2$. A closed subvariety $Z \subset \mathbb{A}^1 \times (\mathbb{P}^1)^n$ will be said to be congruent to 1 mod t^m if scheme-theoretically

$$Z \cdot \{t^m = 0\} \subset \bigcup_{i=1}^{n} Z \cdot \sigma_i \tag{7}$$

We consider algebraic cycles $z = \sum n_\nu Z_\nu$ such that all faces $Z_\nu \cdot \delta_I$ (I multi-index) have codim. $\geq \#(I)$ in Z_ν and are congruent to 1 mod m. Write

$$T_m CH^p(k,q) = T_m H_M^{2p-q}(k, \mathbb{Z}(p)) \tag{8}$$

for the resulting higher Chow groups, generated by cycles of codim. p on $\mathbb{A}^1 \times (\mathbb{P}^1)^{q-1}$. The case of 0-cycles has been worked out by K. Rülling.

Theorem 1 (K. Rülling, [R]). *$T_m H_M^n(k, \mathbb{Z}(n)) \cong W_{m-1}\Omega^{n-1}$, the de Rham-Witt groups built from the "big" Witt ring on k.*

Remark 2. Rülling uses a slightly different version of the congruence condition (7). I have checked casually that his results hold under (7), but this should perhaps be verified more carefully.

A cycle-theoretic version of (5) involves cycles of dimension 1. This has been considered by J. Park [P]. His main result is the construction of a non-trivial regulator map

$$\rho_{m,n} : T_m CH^{n-1}(k,n) \to \Omega_k^{n-3}. \tag{9}$$

(Of particular interest is the case $n = 3$, $\rho_{m,3} : T_m CH^2(k,3) \to k$.)

An abstract regulator construction

In trying to generalize Park's regulator construction to cycles of dim. $r > 1$, one is led to consider meromorphic differential forms

$$\frac{t_{i_0} - 1}{t^{m+1}} dt_{i_1}/t_{i_1} \wedge \ldots \wedge dt_{i_r}/t_{i_r} \tag{10}$$

on $\mathbb{A}_t^1 \times (\mathbb{P}_{t_i}^1)^n$. The most subtle aspect of his work, the careful control of signs necessary to verify that the regulator he constructs is trivial on boundaries of two-dimensional cycles on $\mathbb{A}^1 \times (\mathbb{P}^1)^{n+1}$, will not be attempted here; but let me sketch the construction of a generalized regulator on cycles.

Let X be a smooth, projective variety, and let $D_0, \ldots, D_r$ be effective Cartier divisors on X. Let $Z \subset X$ be a closed subvariety of dimension r. We assume all intersections $Z \cdot D_I$ are either empty or have the correct dimension. In particular, $Z \cap \bigcap_0^r D_i = \emptyset$.

For $z \in Z$ a closed point, we have surjections

$$H_z^r(Z, \Omega_{Z/k}^r) \twoheadrightarrow H^r(Z, \Omega_{Z/k}^r) \overset{\deg}{\twoheadrightarrow} k \tag{11}$$

(The group on the left is local cohomology. We do not assume Z smooth.)

Let ω be a meromorphic Kähler r-form on X which is regular on $X - \bigcup_0^r D_i$. (For example, if $X = \mathbb{A}_t^1 \times (\mathbb{P}_{t_i}^1)^n$ one might take $\omega = \frac{t_{i_0}-1}{t^{m+1}} dt_{i_1}/t_{i_1} \wedge \ldots \wedge dt_{i_r}/t_{i_r}$ with $D_0 : t = 0$ and $D_j = (t_{i_j})_0 + (t_{i_j})_\infty$, $1 \leq j \leq r$.

Chose $i \neq j \in [0, r]$ and define $X(ij) = (D_i + D_j) \cap \bigcap_{h \neq i,j} D_h$ (resp. $Z(ij) = X(ij) \cap Z$). The r open sets $X - (D_i + D_j),\ X - D_h,\ h \neq i, j$ cover $X - X(ij)$, and we may view ω as a Cech $r-1$ cocycle representing a class $\omega(ij) \in H^{r-1}(X - X(ij), \Omega^r_{X/k})$ (resp. by restriction $\omega_Z(ij) \in H^{r-1}(Z - Z(ij), \Omega^r_{Z/k})$.)

$Z(ij)$ is a finite set of points which we can write as a disjoint union $Z(ij) = Z_j(i) \amalg Z_i(j)$, where $Z_j(i) \subset Z - Z \cap D_j$. Write $\deg_j(i) \in k$ for the image of $\omega_Z(ij)$ under the composition

$$H^{r-1}(Z - Z(ij), \omega^r) \xrightarrow{\partial} H^r_{Z(ij)}(Z, \omega^r) \xrightarrow{proj} H^r_{Z_j(i)}(Z, \omega^r) \xrightarrow{(11)} k \tag{12}$$

Clearly, $\deg_j(i) = -\deg_i(j)$.

Now take $\nu \in [0, r]$ with ν, i, j all distinct. Note the sets $Z_j(\nu)$ and $Z_j(i)$ coincide. (They are the intersection of Z with all the $D_h,\ h \neq j$.) Furthermore, the open coverings

$$\begin{gathered} X - (D_i + D_j),\ X - D_h,\ h \neq i, j \\ X - (D_\nu + D_j),\ X - D_h,\ h \neq \nu, j \end{gathered} \tag{13}$$

agree upto reordering as open coverings of $X - X(ij) - D_j = X - X(\nu j) - D_j$. It follows that

$$\deg_i(j) = -\deg_j(i) \stackrel{(*)}{=} \pm \deg_j(\nu) = \mp \deg_\nu(j). \tag{14}$$

But now, for a fourth index μ, the identity (*) above gives

$$\deg_i(j) = \pm \deg_\nu(j) = \pm \deg_\nu(\mu) \tag{15}$$

We conclude that, upto a sign which depends on i, j and the ordering of the divisors D_h, the quantity

$$\deg_j(i) \in k \tag{16}$$

is independent of the choice of i, j.

Example 3. When Z is a curve ($r = 1$) the poles of $\omega|_Z$ are supported on $Z \cap (D_0 + D_1)$. Our construction then amounts to taking residues along those poles lying on $Z \cap D_0$.

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Weil-étale cohomology of mixed motives

Stephen Lichtenbaum

Let M be a motive over Q, with L-function $L_M(s)$. We would like to describe the special value $L^*_M(0)$ up to a rational number in terms of Weil-étale Euler characteristics and relate this to the conjectures of Deligne-Beilinson-Scholl (DBS) which give an alternative description of $L^*_M(0)$. (In later work, we hope to extend this to a comparison of the Weil-étale conjecture with the more refined conjectures of Bloch-Kato-Fontaine -Perrin-Riou).

The Weil-étale story is highly conjectural, since we cannot define the cohomology groups except for $M = Z$, but we can make compelling guesses.

In the DBS world we associate with M finite-dimensional Q-vector spaces $H^0_f(M), H^1_f(M), H^1_c(M), H^2_c(M), H_B(M)^+$ and t_M and a map $\alpha_M : H_B(M)^+_R \to (t_M)_R$.

Note that $H^i_c(M) = Hom_Q(H^{2-i}_f(M^*(1), Q)$, where M^* is the dual motive to M.

In the DBS world the folowing exact sequence of finite-dimesional real vectoer spaces is conjectured to exist:

$$0 \to H^0_f(M)_R \to Ker\alpha \to H^1_c(M)_R \to H^1_f(M)_R \to Coker\alpha \to H^2_c(M)_R \to 0\ .$$

The "determinant" of this sequence with respect to the rational structures on its component real vector spaces should be equal to $L^*_M(0)$, up to a rational number. (note that $Ker\alpha$ and $Coker\alpha$ do not have natural rational structures, but their difference does, thanks to the exact sequence:

$$0 \to Ker\alpha \to H_B(M)^+_R \to (t_M)_R \to Coker\alpha \to 0\ .$$

We conjecture that there exist finite-dimensional vector spaces

$$H^1_{Wc}(M), H^2_{Wc}(M), H^1_{Wc}(M_a), H^2_{Wc}(M_a)$$

and isomorphisms $\beta : H^1_{Wc}(M)_R \to H^2_{Wc}(M)_R$ and $\gamma : H^1_{Wc}(M_a) \to H^2_W c(M_a)$ given by cup-product with a fixed element θ in $H^1_W(R)$ such that $L^*_M(0)$ is equal to $det(\beta)/det(\gamma)$ up to a rational number. Here H_{Wc} denotes Weil-étale cohomology with compact support. These groups are vey closely related to cohomology groups of the "critical" motive E functorially attached to M by Scholl.

This conjecture is analogous to conjectures on zeta-functions of varieties over finite fields made by the author and Thomas Geisser.

These vector spaces should be related by the exact sequences (defined over Q):

$$0 \to H^0_f(M) \to H_B(M)^+ \to H^1_{Wc}(M) \to H^1_f(M) \to 0$$

$$0 \to H^1_c(M) \to H^2_{Wc}(M) \to H^2_{Wc}(M_a) \to H^2_c(M) \to 0$$

and an isomorphism $H^1_{Wc}(M_a) \to t_M$.

These relations and the cup-product isomorphisms imply the DBS eact sequence and the equivalence of the two formulas for $L_M^*(0)$.

Explicit descriptions of these groups and maps can be given when M ia a 1-motive and when M is a cohomology motive coming from a projective non-singular variety.

Reporter: Moritz Kerz

Participants

Prof. Dr. Joseph Ayoub
61 rue Gabriel Peri
F-94200 Ivry sur Seine

Dr. Clark Barwick
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Shahram Biglari
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld

Prof. Dr. Spencer J. Bloch
Department of Mathematics
The University of Chicago
1118 E 58th St. University Ave
Chicago IL 60637
USA

Dr. Baptiste Calmes
Department of Mathematical and
Statistical Sciences
University of Alberta
CAB 632
Edmonton AB T6G 2G1
Canada

Dr. Frederic Deglise
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
99 Av. J.-B. Clement
F-93430 Villetaneuse

Prof. Dr. Helene Esnault
FB 6 - Mathematik
Universität Duisburg-Essen
45117 Essen

Prof. Dr. Eric M. Friedlander
Dept. of Mathematics
Lunt Hall
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
USA

Prof. Dr. Wojciech Gajda
Faculty of Mathematics and Computer
Science
A. Mickiewicz University
ul. Umultowska 87
61-614 Poznan
POLAND

Dr. Herbert Gangl
Dept. of Mathematical Sciences
The University of Durham
Science Laboratories
South Road
GB-Durham, DH1 3LE

Prof. Dr. Thomas Geisser
Department of Mathematics
University of Southern California
DRB 1042 W 36th Place
Los Angeles, Ca 90089-1113
USA

Teena Gerhardt
Department of Mathematics
Massachusetts Institute of
Technology
Cambridge, MA 02139-4307
USA

Dr. Stefan Gille
Mathematisches Institut
Universität München
Theresienstr. 39
80333 München

Prof. Dr. Henri Gillet
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
USA

Prof. Dr. Daniel R. Grayson
Dept. of Mathematics, University of
Illinois at Urbana-Champaign
273 Altgeld Hall MC-382
1409 West Green Street
Urbana, IL 61801-2975
USA

Dr. Joseph Gubeladze
Dept. of Mathematics
San Francisco State University
1600 Holloway Avenue
San Francisco, CA 94132
USA

Dr. Vladimir Guletskii
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
USA

Prof. Dr. Christian Haesemeyer
Dept. of Mathematics, University of
Illinois at Urbana-Champaign
273 Altgeld Hall MC-382
1409 West Green Street
Urbana, IL 61801-2975
USA

Dr. Jens Hornbostel
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Annette Huber-Klawitter
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig

Dr. Florian Ivorra
Inst. de Mathematiques de Jussieu
Universite Paris VII
175 rue du Chevaleret
F-75013 Paris

Prof. Dr. Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Prof. Dr. John Frederick Jardine
Department of Mathematics
The University of Western Ontario
London ONT N6A 5B7
CANADA

Dr. Rob de Jeu
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Dr. Seva Joukhovitski
UCLA
Department of Mathematics
Los Angeles, CA 90095-1555
USA

Prof. Dr. Bruno Kahn
Inst. de Mathematiques de Jussieu
Universite Paris VII
175 rue du Chevaleret
F-75013 Paris

Moritz C. Kerz
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Dr. Bernhard Köck
School of Mathematics
University of Southampton
Highfield
GB-Southampton, SO17 1BJ

Prof. Dr. Aderemi O. Kuku
International Centre for
Theoretical Physics, International
Atomic Energy Agency, UNESCO
P. O. B. 586 Miramare
I-34100 Trieste

Prof. Dr. Klaus Künnemann
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Marc Levine
Dept. of Mathematics
Northeastern University
567 Lake Hall
Boston, MA 02115-5000
USA

Prof. Dr. Stephen Lichtenbaum
Department of Mathematics
Brown University
Box 1917
Providence, RI 02912
USA

Grace Lyo
UC Berkeley
Department of Mathematics
970 Evans Hall # 3840
Berkeley CA 94720-3840
USA

Prof. Dr. Carlo Mazza
Dipartimento di Matematica
Universita di Genova
Via Dodecaneso 35
I-16146 Genova

Prof. Dr. Paul-Arne Ostvar
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
N-0316 Oslo

Prof. Dr. Claudio Pedrini
Dipartimento di Matematica
Universita di Genova
Via Dodecaneso 35
I-16146 Genova

Oliver Petras
Fachbereich Mathematik
Johannes Gutenberg Universität
Mainz
Staudingerweg 9
55099 Mainz

Prof. Dr. Hourong Qin
Department of Mathematics
Nanjing University
Nanjing 210093
CHINA

Joel Riou
Inst. de Mathematiques de Jussieu
Universite Paris VII
175 rue du Chevaleret
F-75013 Paris

Dr. Oliver Röndigs
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld

Prof. Dr. Andreas Rosenschon
Department of Mathematics
The State University of New York
University at Buffalo
Buffalo NY 14260-2900
USA

Prof. Dr. Shuji Saito
Dept. of Mathematical Sciences
University of Tokyo
Komaba, Meguro
Tokyo 153-8914
JAPAN

Dr. Kanetomo Sato
Graduate School of Mathematics
Nagoya University
Chikusa-Ku
Furo-cho
Nagoya 464-8602
JAPAN

Dr. Marco Schlichting
Dept. of Mathematics
Louisiana State University
Baton Rouge, LA 70803-4918
USA

Prof. Dr. Alexander Schmidt
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Dr. Christian Serpe
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Charles A. Weibel
Dept. of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08903
USA

Prof. Dr. Kirill Zainoulline
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld