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Geometrie

Organised by
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Abstract. The program covered a wide range of new developments in geometry. To name some of them, we mention the topics "Metric space geometry in the style of Alexandrov/Gromov", "Polyhedra with prescribed metric", "Willmore surfaces", "Constant mean curvature surfaces in three-dimensional Lie groups". The official program consisted of 21 lectures and included four lectures by V. Schroeder (Zürich) and S. Buyalo (Sankt-Petersburg) on "Asymptotic geometry of Gromov hyperbolic spaces".

Mathematics Subject Classification (2000): 53-xx.

Introduction by the Organisers

The workshop was organized by V. Bangert (Freiburg), Yu. D. Burago (St. Petersburg) and U. Pinkall (Berlin). Out of the 47 participants 22 came from Germany, 8 from the United States, 7 from Switzerland, 4 from Russia, 4 from England and 2 from France.

The official program consisted of 21 lectures and therefore left plenty of space for fruitful informal collaboration. As a tradition in this meeting there always is a series of talks that constitute a small course on a chosen topic of current interest. This time this course consisted of three lectures by V. Schroeder (Zürich) and S. Buyalo (Sankt-Petersburg) on "Asymptotic geometry of Gromov hyperbolic spaces". In addition there were several informal talks organized by the participants, among them an evening devoted to topics related to visualization.

The program covered a wide range of new developments in geometry. These came from four major topics:

- Eight talks concerned recent progress in the geometry of submanifolds in special geometries. In particular many exiting new results were reported on surfaces in various three- or four-dimensional spaces, including three-dimensional Lie-groups.
- Five talks were devoted to various topics in Riemannian geometry.
- Five talks concerned the extension of ideas from differential geometry to more general spaces like discrete groups, polyhedra or manifolds with curvature bounds in the style of Alexandrov. These five talks also included the mentioned small course.
- Finally there were three talks that do not fit the above categories. They were devoted to integral geometry, billiards and the volume of hyperbolic manifolds.

The wide range of topics provided a particularly pleasing environment for the young participants (among them six Phd-students).

Workshop: Geometrie

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Abstracts

Shape and symmetry — noncompact symmetric spaces

Jürgen Berndt
(joint work with Hiroshi Tamaru)

Elie Cartan classified in the 1930s the isoparametric hypersurfaces in the real hyperbolic space $\mathbb{R}H^n$. This result easily leads to the classification of homogeneous hypersurfaces in $\mathbb{R}H^n$. Homogeneous hypersurfaces arise as principal orbits of cohomogeneity one actions. An action is of cohomogeneity one if the orbit space is one-dimensional. Our aim is the classification of cohomogeneity one actions on Riemannian symmetric spaces of noncompact type.

Let M be a connected irreducible Riemannian symmetric space of noncompact type and H a connected closed subgroup of the isometry group of M acting on M with cohomogeneity one. The orbits of this action form either a Riemannian foliation on M, or there exists exactly one singular orbit F. We denote by $\mathcal{M}$ the set of all cohomogeneity one actions on M up to orbit equivalence. Then $\mathcal{M}$ is a disjoint union of the subsets $\mathcal{M}_F$ and $\mathcal{M}_S$ corresponding to the case of foliation and singular orbit.

Our first result says that $\mathcal{M}_F$ is isomorphic to $(\mathbb{R}P^{r-1} \cup \{1, \dots, r\})/\operatorname{Aut}(DD)$. Here, r is the rank of M and Aut(DD) is the symmetry group of the Dynkin diagram associated to M. Aut(DD) acts naturally on the vertices of the Dynkin diagram, which are simple roots and form a set $\{1, \dots, r\}$ of r elements, and on the projective space of the real vector space spanned by the simple roots.

We decompose $\mathcal{M}_S$ into two disjoint subsets $\mathcal{M}_S^0$ and $\mathcal{M}_S^+$ corresponding to the case when the singular orbit F is totally geodesic or not. We prove that $\mathcal{M}_S^0$ corresponds to the congruence classes of reflective submanifolds F of M for which the perpendicular reflective submanifold $F^\perp$ has rank one, together with five congruence classes of non-reflective totally geodesic submanifolds which are all related to the exceptional Lie group G_2. Finally, we determine $\mathcal{M}_S^+$ explicitly for complex hyperbolic spaces and the Cayley hyperbolic plane. For quaternionic hyperbolic spaces we reduce the classification problem to a problem from quaternionic linear algebra. For higher rank symmetric spaces we show that two cases can occur. Firstly, the action is a natural extension from a cohomogeneity one action on a totally geodesic submanifold determined by a subset of a set of simple roots associated to M. Secondly, the action is constructed in a similar way to the rank one case from a gradation of the Lie algebra of the isometry group of M which is determined by a simple root vector.

The principal kinematic formula in hermitian vector spaces

Andreas Bernig
(joint work with Joseph H. G. Fu)

The classical principal kinematic formula in $\mathbb{R}^n$, proved by Blaschke, Chern, Hadwiger and Santaló, is the following identity for compact convex sets $K, L \subset \mathbb{R}^n$:

$$\int_{\bar{O}(n)} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{i+j=n} c_{ij} \mu_i(K) \mu_j(L). \tag{1}$$

Here $\bar{O}(n)$ is the isometry group of $\mathbb{R}^n$, endowed with a Haar measure, χ is the Euler characteristic (i.e. 1 if $K \cap \bar{g}L \neq \emptyset$ and 0 otherwise), the c_{ij} are explicitly known constants and the μ_i are geometric invariants of convex bodies (the intrinsic volumes).

In 1990 J. Fu asked what happens when one replaces $\mathbb{R}^n$ by $\mathbb{C}^n$ and $\bar{O}(n)$ by $\bar{U}(n)$. We completely solved this question by showing that for $K, L \subset \mathbb{C}^n$

$$\int_{\bar{U}(n)} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{K+l=2n,p,q} c_{k,l,p,q} \mu_{k,p}(K) \mu_{l,q}(L), \tag{2}$$

with explicitly known constants $c_{k,l,p,q}$ and geometric invariants $\mu_{k,p}$ depending on the hermitian structure.

The proof of (1) can be sketched as follows: fixing L, the left hand side of (1) is a *valuation*, i.e. a map which is additive and continuous in K. Moreover, it is $\bar{O}(n)$-invariant. By a famous theorem of Hadwiger [7], the space of invariant valuations is generated by the μ_i. From there it is easy to derive formula (1) with unknown constants, and these constants can be fixed by plugging in balls of various radii for K and L (*template method*).

In the hermitian case, the analogous first step was worked out by Alesker [1], who computed the dimension of the space $\mathrm{Val}^{U(n)}$ of $\bar{U}(n)$-invariant valuations and gave a basis. This yields (2) with unknown constants. However, the template method is too weak to determine the constants, except in small dimensions (Heunggi Park computed them for $n = 2, 3$ [8]).

Our method is more algebraic and uses the deep result (proved by Alesker [2]) that the space of valuations is a graded algebra satisfying Poincaré duality and versions of the hard Lefschetz theorem. The kinematic formula is in some sense the *inverse* of the product structure [6].

In order to compute the product structure of $\mathrm{Val}^{U(n)}$, we introduce an $\mathfrak{sl}_2$-representation on this space. The Lefschetz decomposition then yields a diagonalization for the Alesker-Poincaré pairing. Computing the eigenvalues is rather difficult and is based on the hard Lefschetz theorems and some surprising combinatorial formulas.

As first applications, we can generalize the Kang-Tasaki-Poincaré formulas for complex projective spaces of all dimensions of all dimensions and prove a conjecture concerning the positivity of the Alesker-Poincaré pairing. We also derive a whole array of kinematic formulas (where χ is replaced by another valuation $\mu_{m,r}$ on the left hand side of (2)).

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On constrained Willmore tori in the 4–sphere

Christoph Bohle

This report gives a short account of results on constrained Willmore tori in S^4 that are obtained by an integrable systems approach similar to the one used by Hitchin in his study [3] of harmonic tori. As a consequence of our main theorem, all constrained Willmore tori in S^4 can be constructed rather explicitly by methods of complex algebraic geometry. For a detailed presentation of our results, see [2].

Definition. A conformal immersion $f\colon M \to S^4 = \mathbb{R}^4 \cup \{\infty\}$ of a compact Riemann surface M into the conformal 4–sphere is called *constrained Willmore* if it is a critical point of the Willmore functional $\mathcal{W} = \int |\mathbb{I}^\circ|^2 dA$ under infinitesimal conformal variations (with $\mathbb{I}^\circ$ denoting the tracefree second fundamental form).

Constrained Willmore surfaces are the solutions to a constrained variational problem. The notion of constrained Willmore surfaces generalizes that of Willmore surfaces, the critical points of $\mathcal{W}$ under all variations. The fact that both the functional and the constraint are conformally invariant suggests an investigation within the framework of Möbius geometry—like the quaternionic projective model for the conformal geometry of S^4 used in [2]—instead of its metric subgeometries. The metric subgeometries occur in our study mainly as a source of several interesting classes of examples of constrained Willmore surfaces, e.g. CMC surfaces in 3–dimensional space–forms, minimal surfaces in 4–dimensional space–forms and Hamiltonian stationary Lagrangian surfaces in $\mathbb{R}^4$.

A prototype for our main theorem is the following result on harmonic maps: a harmonic map $f\colon T^2 \to S^2$ is either of finite type or conformal, i.e., holomorphic or anti–holomorphic. More precisely, it is of finite type if $\deg(f) = 0$ (Pinkall, Sterling) and it is conformal if $\deg(f) \neq 0$ (Eells, Wood). In the finite type case there is a compact Riemann surface Σ attached to f, the so called *spectral curve*, and f is obtained by "algebraic geometric" or "finite gap" integration, which means that it is the composition of a linear map into the (generalized) Jacobian of Σ and a holomorphic map on (a subset of) the Jacobian.

Main Theorem. *A constrained Willmore immersion $f\colon T^2 \to S^4$ is either of finite type or "holomorphic", i.e., super–conformal or Euclidean minimal.*

In the "holomorphic" case the immersion f (or its differential) is given in terms of meromorphic functions on the torus itself: every *super–conformal* immersion f is the twistor projection $\mathbb{CP}^3 \to \mathbb{HP}^1$ of a holomorphic curve in $\mathbb{CP}^3$ and *Euclidean minimal* means that there is a point $\infty \in S^4$ such that $f\colon T^2\backslash\{p_1, ..., p_n\} \to \mathbb{R}^4 = S^4\backslash\{\infty\}$ is a Euclidean minimal surface with planar ends p_1,...,p_n (and its differential is the real part of a meromorphic 1–form). As in the above prototype result, the algebraic geometry needed to construct an immersion f that is of finite type is more involved since f is given in terms of holomorphic functions on an auxiliary higher dimensional abelian variety, the Jacobian of the spectral curve. The main theorem generalizes the following previous results:

- CMC tori are of finite type (Pinkall, Sterling 1989, [5])
- Constrained Willmore in S^3 are of finite type (Schmidt 2002, [6])
- Willmore tori in S^4 with topologically non–trivial normal bundle are of "holomorphic" type (Leschke, Pedit, Pinkall 2003, [4])

The proof of the main theorem is based on a quaternionic version of Hitchin's method for studying harmonic tori in S^3 which provides a uniform and geometric approach to proving and generalizing these previous results. The proof consists of the following steps:

0.) Formulation of the Euler–Lagrange equation of constrained Willmore surface as a zero–curvature equation with spectral parameter. This arises in the form of an associated family ∇^μ of flat connections on a trivial complex rank 4 bundle which depends on a spectral parameter $\mu \in \mathbb{C}_*$.
1.) Investigation of the holonomy representations $H^\mu\colon \Gamma \to \mathrm{SL}_4(\mathbb{C})$ of ∇^μ.
2.) Non–trivial holonomy implies the existence of a polynomial Killing field (which implies that f is of finite type).
3.) Trivial holonomy implies that f is of "holomorphic" type.

The main difficulty in implementing the strategy of [3] for constrained Willmore tori is Step 1 of the proof. This is due to the fact that one has to deal with a variety of degenerate cases of collapsing eigenvalues that might occur for the family of $\mathrm{SL}_4(\mathbb{C})$–holonomies of ∇^μ (in contrast to the case of harmonic maps into S^3, whose associated family of flat connections has $\mathrm{SL}_2(\mathbb{C})$–holonomy). The possible holonomy representations are given by the following lemma which is proven using methods of quaternionic holomorphic geometry, in particular the geometric approach [1] to the spectral curve based on the notion of Darboux transforms.

Lemma. *The associated family ∇^μ of a constrained Willmore torus $f\colon T^2 \to S^4$ has a holonomy representation $H^\mu(\gamma)$ belonging to one of the following cases:*

I. *generically $H^\mu(\gamma)$ has 4 different eigenvalues that are non–constant as functions of μ*

II. *generically $H^\mu(\gamma)$ has $\lambda = 1$ as an eigenvalue of multiplicity 2 and 2 simple eigenvalues that are non–constant as functions of μ*

IIIa. *all holonomies $H^\mu(\gamma)$ are trivial, or*

IIIb. *generic holonomies $H^\mu(\gamma)$ have two 2×2 Jordan blocks with eigenvalue 1.*

If f has topologically non–trivial normal bundle, its holonomy belongs to Case III.

In Cases I and II one can define a so called eigenline curve, i.e., the Riemann surface $\Sigma \xrightarrow{\mu} \mathbb{C}_*$ that is the 4– or 2–fold branched covering of $\mathbb{C}_*$ parametrizing the non–trivial eigenlines of the holomorphic family of matrices $\mu \mapsto H^\mu(\gamma)$ for $\gamma \in \Gamma\backslash\{0\}$ (this is independent of γ because Γ is abelian). To show that Σ can be compactified we prove the existence of a polynomial Killing field, i.e., a family of sections ξ of the endomorphism bundle that is polynomial in μ and satisfies $\nabla^\mu \xi(\mu, .) = 0$.

In Cases IIIa and IIIb we prove the existence of a polynomial family of ∇^μ–parallel sections of the rank 4–bundle over the torus (in Case IIIa) or the existence of a polynomial Killing field ξ that is nil–potent (in Case IIIb). Investigating the asymptotics for $\mu = 0$ or $\mu = \infty$ of such polynomial families of sections reveals that the immersion has to be super–conformal or Euclidean minimal. Thus:

Theorem A. *For a constrained Willmore immersion $f\colon T^2 \to S^4$ one of the following holds:*

I. *f is of finite type and μ extends to a covering $\Sigma \xrightarrow[4:1]{\mu} \mathbb{CP}^1$,*

II. *f is of finite type and μ extends to a covering $\Sigma \xrightarrow[2:1]{\mu} \mathbb{CP}^1$,*

IIIa. *all holonomies are trivial and f is super–conformal or an algebraic Euclidean minimal surface (i.e., the dual minimal surface has no periods),*

IIIb. *all holonomies are of Jordan type and f is a non–algebraic Euclidean minimal surface with planar ends.*

If the normal bundle is topologically trivial and the immersion f is not Euclidean minimal it belongs to Cases I or II. If the normal bundle of an immersion f is topologically non–trivial, it belongs to the "holomorphic" Cases IIIa or IIIb.

In particular, a Willmore torus $f\colon T^2 \to S^4$ that is not Euclidean minimal either has trivial normal bundle and is of finite type (belonging to Case I) or it has non–trivial normal bundle and is super–conformal (Case IIIa). Examples of constrained Willmore tori belonging to Case II are provided by:

Theorem B. *If a conformal immersion $f\colon T^2 \to S^4$ admits a point $\infty \in S^4$ at infinity such that the (Euclidean) Gauss map $T^2 \to Gr^+(2,4) = S^2 \times S^2$ of the corresponding immersion into $\mathbb{R}^4 = S^4\backslash\{\infty\}$ has a harmonic factor, then it* *is constrained Willmore. Moreover, it belongs to Case II above if the harmonic factor is non–conformal and to Case III if it is conformal.*

Examples of constrained Willmore immersions $f: T^2 \to S^4$ belonging to Case II are CMC tori in $\mathbb{R}^3$ and S^3, Hamiltonian stationary Lagrangian tori in $\mathbb{R}^4$ and Lagrangian tori with conformal Maslov form in $\mathbb{R}^4$.

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Bending the helicoid

Matthias Weber
(joint work with William Meeks III)

In [1], Colding and Minicozzi consider the question of the compactness of a sequence $\{M_n\}_{n\in\mathbb{N}}$ of embedded minimal surfaces in a Riemannian three-manifold N which are *locally simply connected* in the following sense: for each small open geodesic ball in N and for each n sufficiently large, M_n intersects the ball in disk components, with each disk component having its boundary in the boundary of the ball. They prove that every such sequence of minimal surfaces has a subsequence which converges to a possibly singular limit minimal lamination $\mathcal{L}$ of N. In certain cases, the minimal lamination $\mathcal{L}$ is nonsingular and is a minimal foliation of N. In this case, they prove that the singular set of C^1-convergence consists of a properly embedded locally finite collection $S(\mathcal{L})$ of Lipschitz curves that intersect the leaves of $\mathcal{L}$ transversely; we call such a limit foliation $\mathcal{L}$ a *Colding-Minicozzi limit minimal lamination.*

An application by Meeks [3, 2] of this local picture for a minimal disk centered at a point of large almost-maximal curvature demonstrates that the singular curves $S(\mathcal{L})$ of a Colding-Minicozzi lamination $\mathcal{L}$ have class $C^{1,1}$ and are orthogonal to the leaves of $\mathcal{L}$.

In all previously considered examples (see e.g. the recent examples in [4]) of sequences of locally simply connected minimal surfaces which converge to a minimal foliation $\mathcal{L}$ with nonempty singular set of C^1-convergence $S(\mathcal{L})$, the set $S(\mathcal{L})$ consisted of geodesics.

In the following, we provide a complete characterization as to what curves can occur as singular curves:

Theorem. *Every properly embedded $C^{1,1}$-curve α in an open set O in $\mathbb{R}^3$ has a neighborhood foliated by a particular Colding-Minicozzi limit minimal lamination $\mathcal{L}$ with singular set of C^1-convergence being α. The minimal leaves of this lamination $\mathcal{L}$ are a $C^{1,1}$-family of pairwise disjoint flat disks of varying radii. The disks are centered along and orthogonal to α. More generally, if N is a closed regular neighborhood of α formed by disjoint flat disks orthogonal to α and N' is a similarly defined foliation in the interior of N, then N' is contained in a Colding-Minicozzi minimal lamination which lies in N.*

The main step in the proof of our Theorem is to first prove the theorem when α is analytic with a compact exhaustion $\alpha(1) \subset \alpha(2) \subset \ldots \subset \alpha(n) \subset \ldots$, where $\alpha(i)$ is a compact connected arc in α. We do this by giving an essentially explicit construction of a sequence of embedded compact *bent helicoids* $H_{\alpha,n}$ which contain $\alpha(n) \subset \alpha$ as an "axis" and whose Gauss maps rotate faster and faster along $\alpha(n)$ as $n \to \infty$. In this case, the $H_{\alpha,n}$ converge to a family of pairwise disjoint flat disks of varying radii orthogonal to α. The construction of the $H_{\alpha,n}$ is based on the classical Björling formula. Our main difficulty in proving our Theorem in the analytic case is to demonstrate the embeddedness of the $H_{\alpha,n}$ in a *fixed* neighborhood of $\alpha(n)$. The general case of the theorem follows from the analytic case by approximating α by a sequence of embedded analytic curves with uniformly locally bounded curvature, which is always possible for $C^{1,1}$-curves.

In the special case that α is the unit circle in the (x_1, x_2)-plane, then, for all $n \in \mathbb{N}$, we can choose $\alpha(n) = \alpha$ and each compact annular bent helicoid $\overline{H}_n = H_{\alpha,n}$ contains α and is the image of a compact portion of a globally defined explicit periodic complete minimal immersion $f_n \colon \mathbb{C} \to \mathbb{R}^3$. In this case, we let H_n denote the image complete minimal annulus $f_n(\mathbb{C})$ and define compact embedded annuli $\overline{H}_n \subset H_n$ which converge to the limit minimal foliation $\mathcal{L}$ of $\mathbb{R}^3 - x_3$-axis by vertical half planes and with $S(\mathcal{L}) = \alpha$. We also describe the analytic Weierstrass data for their image finite total curvature annuli H_n in terms of simple rational functions on the punctured complex plane $\mathbb{C} - \{0\}$.

The complete minimal annulus H_n has finite total curvature $-4\pi(n+1)$ with the dihedral group $D(2n)$ of symmetries and contains n lines in the (x_1, x_2)-plane passing through the origin. The large symmetry group and the explicit representation of H_n allows us to define the compact *embedded* annuli $\overline{H}_n \subset H_n$ which converge to the minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$. By way of approximation, this special case of a circle plays a key role in the proof of our Theorem in the more general case where α is an arbitrary properly embedded analytic curve in an open set O. This is because at every point of the analytic curve the related bent helicoids that we construct are closely approximated by the related bent helicoids of the second order approximately osculating circle at the point.

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Alexandrov's theorem, weighted Delaunay triangulations and mixed volumes

Ivan Izmestiev
(joint work with Alexander I. Bobenko)

I present the results of our work posted on the web as `http://www.arxiv.org/abs/math.DG/0609447`.

Theorem 1 (A.D.Alexandrov, 1942)**.** *Let M be a 2-sphere equipped with a convex polyhedral metric. Then there is a convex polytope P in $\mathbb{R}^3$ with boundary isometric to M. Besides, P is unique up to a rigid motion.*

The uniqueness part of this theorem is essentially due to Cauchy. The proof of the existence part given by Alexandrov is much more involved. In [2] he cites Minkowski's theorem on the existence and uniqueness of a polytope with given face normals and face areas and gives two proofs of it. One is Minkowski's original proof based on the concavity properties of the volume, the other is by Alexandrov's method used in the proof of Theorem 1. Alexandrov poses the problem of finding a proof of Theorem 1 similar to the variational proof of Minkowski's theorem.

We present a proof that fulfills these requirements probably as much as possible. The proof is based on the properties of the *total scalar curvature functional* H for a certain kind of polyhedral metrics on the ball. We use the fact that the Hessian of this functional is non-degenerate. This is proved using the rather unexpected equality (1). This equality establishes a link between our proof and the variational proof of Minkowski's theorem. The functional H was studied by Volkov, a student of Alexandrov, see the Supplement to [2]. Volkov gave also another proof of Alexandrov's theorem in his PhD dissertation in 1955. Volkov's proof does not make use of the total scalar curvature and is conceptually different from ours.

An outline of our approach follows.

Let T be a geodesic triangulation of M with the vertices at the singularities. Construct a pyramid over every triangle of T and glue them together to form a polyhedral ball P with the boundary M. If the dihedral angles at the boundary edges of P are less or equal π, then P is called a *generalized convex polytope*. A generalized convex polytope is defined by its *radii* r_i which are lengths of the interior edges, and it has the *curvatures* κ_i around the interior edges. Our aim is to make the curvatures vanish.

To achieve this, we proceed as follows. First, we prove the local rigidity of generalized convex polytopes (under some assumptions). This allows us to realize any small deformation of the curvatures through an appropriate deformation of radii. Then, we construct a global deformation that ends with a zero-curved generalized, that is a usual, convex polytope. To construct the deformation, we study the space of generalized convex polytopes and show that no degeneration occurs when we go our way.

Here is an outline of the first step (local rigidity). Define the total scalar curvature of P as

$$H(P) = \sum_i r_i \kappa_i + \sum_e \ell_e(\pi - \theta_e),$$

where ℓ_e is the length of, θ_e is the dihedral angle at, a boundary edge e. The Schläfli formula implies that the Jacobian of the map $r \mapsto \kappa$ equals the Hessian of the function H:

$$\frac{\partial \kappa_i}{\partial r_j} = \frac{\partial^2 H}{\partial r_i \partial r_j}.$$

Thus the local rigidity is equivalent to the non-degeneracy of the Hessian of H.

In order to prove the non-degeneracy of H, we introduce the *dual polyhedron* P^* by generalizing the concept of the polar duality between polytopes and polyhedra. We establish the following equality.

$$\frac{\partial^2 H}{\partial r_i \partial r_j}(P) = \frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}(P^*), \tag{1}$$

where the h_i's are the altitudes of the generalized polyhedron P^*. In the classical situation, the signature of the latter Hessian is known, and this information is expressed in the *Alexandrov-Fenchel inequalities.* By generalizing the classical proof, we establish the following

Theorem 2. *The Hessian* $\left(\frac{\partial^2 H}{\partial r_i \partial r_j}\right)$ *is non-degenerate if* $0 < \kappa_i < \delta_i$. *Here* δ_i *is the curvature of the* i*-th singularity in the metric of* M.

For the second step of the proof, we prove the following lemma.

Lemma. *If* (T, r) *is a convex generalized polytope, then* T *is the weighted Delaunay triangulation of* S *with weights* r_i^2 *at singularities. The converse is true provided that pyramids over triangles of* T *with side lengths* r_i *exist.*

We describe the space of weights of weighted Delaunay triangulations. The deformation used to obtain the convex polytope P with boundary M starts with a generalized polytope (T, r), where T is the Delaunay triangulation of M, and all of the radii r_i are equal to a sufficiently large R. Then we deform the radii so that the curvatures decrease proportionally: $\kappa_i(t) = t \cdot \kappa_i(1)$. The proof that no degenerations occur involves several technical lemmas.

Our proof gives rise to an algorithm for constructing the polytope with the given metric on the boundary. The algorithm was implemented by Stefan Sechelmann. The program is available at `http://www.math.tu-berlin.de/geometrie/ps/software.html`

Figure 1 shows two examples of its output.

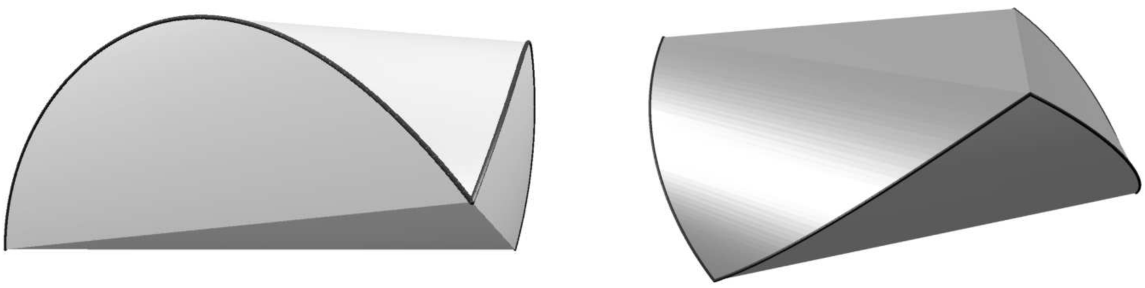

Figure 1. Convex surfaces glued from two Euclidean pieces identified along the boundary. *Left:* Disc and equilateral triangle. *Right:* Two Reuleaux triangles (triangles of constant width); the vertices of one are identified with the midpoints of the sides of the other.

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Quasi-minimizing varieties in spaces of nonpositive curvature

Urs Lang

(joint work with Bruce Kleiner)

The goal of this work is to establish "higher rank analogs" of some well-known phenomena from hyperbolic geometry like the stability of quasigeodesics, the visibility property, and the boundary homeomorphism induced by a quasi-isometry. Throughout this note, $X = (X, d)$ denotes a proper geodesic metric space satisfying Busemann's nonpositive curvature condition globally: for every pair of (constant speed) geodesics $\sigma, \tau \colon [0,1] \to X$, the function $t \mapsto d(\sigma(t), \tau(t))$ is convex. Each pair of points in X is then connected by a unique geodesic segment, and X is contractible. Moreover, we impose the following large-scale rank condition on X: every asymptotic cone X_ω of X (ω a non-principal ultrafilter on $\mathbb{N}$) has "rectifiable" dimension n in the sense that n is the maximal number for which X_ω receives a Lipschitz map from some subset of $\mathbb{R}^n$ whose image has nonzero n-dimensional Hausdorff measure. For instance, these conditions are satisfied if X is a cocompact Hadamard manifold (or CAT(0)-space) containing an n-flat (isometrically embedded copy of $\mathbb{R}^n$) but no $(n+1)$-flat, or, more specifically, if X is a nonpositively curved symmetric space of rank n. In fact, our results are new even in this last case.

We investigate n-dimensional surfaces in X with polynomial volume growth of order n. As we want to prove, among other things, the existence of complete minimizing surfaces with prescribed asymptotic behavior, the space of surfaces in question should possess suitable compactness properties. An adequate chain complex is provided by the integral currents of geometric measure theory. In the general context of metric spaces, such a theory was established by Ambrosio–Kirchheim in [1]. Thus $\mathbf{I}_n(X)$ will denote the abelian group of n-dimensional integral currents in that sense. An element $S \in \mathbf{I}_n(X)$ should roughly be viewed as an n-dimensional Lipschitz surface with an integer-valued multiplicity function. By definition, S has finite total mass $\mathbf{M}(S)$ (area counting multiplicity), and its boundary ∂S belongs to $\mathbf{I}_{n-1}(X)$. It is possible to adapt the Ambrosio–Kirchheim theory so as to accommodate locally integral currents in X, cf. [6]; associated with $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is a Radon measure (locally finite mass) $\|S\|$, and $\partial S \in \mathbf{I}_{n-1,\mathrm{loc}}(X)$. We will denote by $\mathbf{Z}_n(X)$ and $\mathbf{Z}_{n,\mathrm{loc}}(X)$ the respective spaces of (locally) integral cycles in X.

Now let $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$. S is called *quasi-minimizing with constant* $Q \geq 1$ or just Q*-minimizing* if

$$\mathbf{M}(S') \leq Q\mathbf{M}(T)$$

whenever $S', T \in \mathbf{I}_n(X)$, $\|S\| = \|S'\| + \|S - S'\|$ (i.e. S' is a “piece” of S), and $\partial T = \partial S'$. A 1-minimizing current is *minimizing.* For $a \geq 0$, $c > 0$ and $p \in X$, we say that S is (c, a)*-controlled at* p if

$$\|S\|(\bar{\mathrm{B}}(p, r)) \leq cr^n$$

for all $r \geq a$; note that the exponent n equals the dimension of S. If this holds for every $p \in X$, then we say that S is (c, a)*-controlled on* X.

Theorem 1 (persistence of (quasi-)minimizers)**.** *Given* X *and* Q, a, c, L*, there exist* $\bar{a}, \bar{c}, b$ *such that the following holds. Suppose* $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ *is* Q*-minimizing and* (c, a)*-controlled at some point* p *in* X*. Let* d' *be any metric on* X *that is* L*-bi-Lipschitz equivalent to* d*. Then there is a current* $S' \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ *such that* S' *is minimizing with respect to* d'*, both* S *and* S' *are* $(\bar{c}, \bar{a})$*-controlled on* X*, and there is a uniform bound* $\mathrm{Hd}(\mathrm{spt}(S), \mathrm{spt}(S')) < b$ *on the Hausdorff distance of their supports. Moreover, there exists a current* $V \in \mathbf{I}_{n+1}(X)$ *such that* $\partial V = S - S'$ *and* $\|V\|(\bar{\mathrm{B}}(x, r)) \leq \bar{c}r^n$ *for all* $x \in X$ *and* $r \geq \bar{a}$*.*

Note that S' is quasi-minimizing with constant $Q = L^{2n}$ with respect to the original metric d. This type of result has a long history, starting with the work of Morse on the hyperbolic plane and leading to rather general results on (quasi)-minimizing surfaces of arbitrary dimension in spaces of negative curvature. We refer to [5] for one of the most recent contributions and an account of these developments. There is a parallel circle of results on periodic metrics, initiated by the work of Hedlund on the two-dimensional torus and including the investigation of minimal hypersurface laminations and properties of the stable norm of compact riemannian manifolds. See [2], [3], and the references there. To the best of our knowledge, the above theorem is now the first general “non-periodic” result in this direction for spaces of nonpositive curvature.

Denote by $\mathrm{R}X$ the set of all geodesic rays $\sigma \colon \mathbb{R}_+ \to X$ of constant (possibly zero) speed. Two elements $\sigma, \tau \in \mathrm{R}X$ are *asymptotic* if $\sup_{t \geq 0} d(\sigma(t), \tau(t)) < \infty$; this defines an equivalence relation $\sim$ on $\mathrm{R}X$. The *Tits cone* $\mathrm{C_T}X$ of X is the set

R$X/\!\sim$ equipped with the metric defined by

$$d([\sigma],[\tau]) := \lim_{t\to\infty} d(\sigma(t),\tau(t))/t;$$

recall that $t \mapsto d(\sigma(t),\tau(t))$ is convex. For every $r > 0$, the map $h_r\colon \mathrm{C_T}X \to \mathrm{C_T}X$, $h_r([\sigma(\cdot)]) := [\sigma(r\,\cdot)]$, dilates the metric by the factor r. For every basepoint $p \in X$, there is a natural 1-Lipschitz map $\exp_p\colon \mathrm{C_T}X \to X$ satisfying $\exp_p([\sigma]) = \sigma(1)$ for every $\sigma \in \mathrm{R}X$ with $\sigma(0) = p$. Moreover, $\mathrm{C_T}X$ admits a natural isometric embedding into any asymptotic cone X_ω of X with fixed basepoint. In particular, the rank assumption on X implies that $\mathbf{I}_{m,\mathrm{loc}}(\mathrm{C_T}X) = \{0\}$ for all $m > n$. This implies further that every $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)$ is conical with respect to the vertex o of $\mathrm{C_T}X$, i.e. invariant under h_r for every $r > 0$, $h_{r\#}\Sigma = \Sigma$, and hence Σ is $(c,0)$-controlled at o for some c. For $p \in X$ and $0 < t \le 1$, define $h_{p,t}\colon X \to X$ such that $h_{p,t}(x) = \sigma(t)$ for the geodesic $\sigma\colon [0,1] \to X$ from p to x.

Theorem 2 (unique tangent cone at infinity)**.** *Suppose $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is Q-minimizing and (c,a)-controlled at $p \in X$, for some Q,a,c,p. As $t \to 0$, $h_{p,t\#}S$ converges weakly to a current $S_{\downarrow p} \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ which is conical with respect to p, i.e. invariant under $h_{p,t}$ for every $t \in (0,1]$. Moreover, there is a unique element $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)$ such that $\exp_{p\#}\Sigma = S_{\downarrow p}$, and Σ is $(c,0)$-controlled at o. In fact, for every $q \in X$, the weak limit $S_{\downarrow q} := \lim_{t\to 0} h_{q,t\#}S$ exists, and $\exp_{q\#}\Sigma = S_{\downarrow q}$.*

We denote this unique tangent cone $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)$ of S by $\mathrm{C_T}S$ and call it the *Tits cone of* S. We also show that $\mathrm{spt}(S)$ and $\mathrm{spt}(S_{\downarrow p})$ lie within "sublinear" distance of each other, in terms of the distance from p.

Theorem 3 (asymptotic Plateau problem)**.** *Given X and $c > 0$, there exist $\bar a, \bar c$ such that the following holds. Whenever $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)$ is $(c,0)$-controlled at o, then there exists a minimizing current $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ such that S is $(\bar c,\bar a)$-controlled on X and $\mathrm{C_T}S = \Sigma$.*

Combining the last two theorems, we obtain precise information on the geometry at infinity of a quasi-isometric embedding f of X into another space $\bar X$ of the same type (i.e., there exist constants L,a such that $L^{-1}d(x,y) - a \le d(f(x),f(y)) \le Ld(x,y) + a$ for all $x,y \in X$).

Theorem 4 (quasi-isometric embedding)**.** *Let $\bar X$ be another space like X. Then every quasi-isometric embedding $f\colon X \to \bar X$ naturally induces a monomorphism $\mathrm{C_T}f\colon \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X) \to \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}\bar X)$ of abelian groups. Moreover, if $\mathcal{L}$ denotes the lattice of subsets of $\mathrm{C_T}X$ generated by $\{\mathrm{spt}(\Sigma)\colon \Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)\}$, and $\bar{\mathcal{L}}$ denotes the respective lattice of subsets of $\mathrm{C_T}\bar X$, then there is a unique injective lattice homomorphism $F\colon \mathcal{L} \to \bar{\mathcal{L}}$ with the property that $F(\mathrm{spt}(\Sigma)) = \mathrm{spt}((\mathrm{C_T}f)(\Sigma))$ for every $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)$. Finally, for every $A \in \mathcal{L}$, A is L-bi-Lipschitz homeomorphic to $F(A)$, where L is the multiplicative quasi-isometry constant of f.*

The map $\mathrm{C_T}f$ is determined in the following way. Given $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}X)$, let $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ be a solution of the asymptotic Plateau problem for Σ according to Theorem 3. Then it is possible to replace f by a Lipschitz map g that coincides with f on $\mathrm{spt}(S)$ up to a uniformly bounded error. Now the push-forward $\bar S :=$

$g_\# S \in \mathbf{Z}_{n,\mathrm{loc}}(\bar{X})$ is quasi-minimizing in $\bar{X}$ in a coarse sense, for which Theorem 2 is still valid. Thus $\bar{S}$ possesses a well-defined Tits cone $\bar{\Sigma} = \mathrm{C_T}\bar{S} \in \mathbf{Z}_{n,\mathrm{loc}}(\mathrm{C_T}\bar{X})$, and $\mathrm{C_T}f$ is characterized by the property that $(\mathrm{C_T}f)(\Sigma) = \bar{\Sigma}$. When specialized to a quasi-isometry between two nonpositively curved symmetric spaces, Theorem 4 yields an isomorphism of the associated Tits buildings. As an application, one obtains a relatively quick proof of the rigidity theorem of Kleiner–Leeb [4] for symmetric spaces of noncompact type without rank one de Rham factors. Some of our results also extend to non-proper spaces.

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Boundary rigidity and filling volume minimality

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(joint work with Dmitri Burago)

Let M be a compact Riemannian manifold with boundary ∂M. The *boundary distance function* of M is the restriction of the geodesic distance d_M of M to $\partial M \times \partial M$. The term "boundary rigidity" means that the metric is uniquely determined by its boundary distance function. More precisely, we say that M is *boundary rigid* if every compact Riemannian manifold M' with the same boundary and the same boundary distance function is isometric to M via a boundary preserving isometry.

R. Michel [7] conjectured that every simple Riemannian manifold is boundary rigid where "simple" means that the boundary ∂M is strictly convex, every two points $x, y \in M$ are connected by a unique geodesic, and geodesics have no conjugate points. In is easy to see that all simple manifolds are topological discs. The conjecture has been proved in a number of partial cases including all two-dimensional simple manifolds [8] and regions in certain symmetric spaces [7, 5, 1].

One of our main results asserts that if M is C^2-close to a region in Euclidean space, then M is boundary rigid. To the best of our knowledge, this is the first known example of boundary rigid metrics in higher dimensions which are not locally-symmetric. Our result also requires only C^2-smoothness.

We treat boundary rigidity as the equality case of the minimal filling problem. We say that M is a *minimal filling* if, for every orientable M' with $\partial M' = \partial M$, the inequality

$$(1) \qquad d_{M'}(x,y) \geq d_M(x,y) \quad \text{for all } x,y \in \partial M$$

implies that

$$(2) \qquad vol(M') \geq vol(M).$$

In other words, a minimal filling is a Riemannian manifold which realizes the filling volume of its own boundary.

We conjecture that every simple manifold M is a minimal filling, furthermore, equality in (2) implies that M' satisfying (1) is isometric to M via a boundary-preserving isometry. This would imply Michel's rigidity conjecture since the volume of a simple manifold is determined by its boundary distance function (by means of an integral formula due to Santaló).

So far we were able to carry out this plan for metrics close to a Euclidean one. Our main result is the following:

Theorem [4]**.** *Let $D \subset \mathbb{R}^n$ be a compact region with a smooth boundary. There exists a C^2-neighborhood U of the Euclidean metric on D such that for every $g \in U$, the Riemannian manifold $M = (D, g)$ is a minimal filling and boundary rigid.*

Actually U can be defined explicitly in terms of curvature. If the boundary is strictly convex, it suffices to assume that the boundary of M is strictly convex and $\max |K| \cdot \operatorname{diam}(M) \leq c(n)$ where K is the sectional curvature of M.

In the theorem itself, we do not assume convexity of the boundary. The non-convex case is reduced to the convex one by a cut-and-paste argument (that is, by attaching an appropriate "collar" to both manifolds).

Plan of the proof. Our approach to boundary rigidity grew from [2, 6, 3] where we study minimality in normed spaces and ellipticity of surface area functionals. Even though the proof is not directly based on Finsler geometry, it is strongly motivated by Finsler considerations.

Suppose M and M' are as above, namely M is close to a convex Euclidean region, and M' is such that $\partial M' = \partial M$ and $d_{M'} \geq d_M$ on ∂M. We denote their common boundary by S.

Step 1. Construct maps $\Phi : M \to X$ and $\Phi' : M' \to X$ to an appropriate Banach space X so that Φ and Φ' agree on S, Φ is smooth and distance preserving, and Φ' is nonexpanding. This is possible due to simplicity of M and the inequality between the boundary distances in M and M'.

In fact, we set $X = L^\infty(S)$, write down Φ in terms of distance functions in M and construct Φ' as a Lipschitz extension from the boundary. Using the assumption that M is close to a flat region, we make Φ C^1-close to a standard isometric linear map from $\mathbb{R}^n$ to $X \simeq L^\infty(S^{n-1})$. This is needed for Step 3.

Assuming that a notion of n-dimensional area in X is defined, the problem now reduces to the following: prove that the surface $\Phi(M)$ has area no greater than that of $\Phi'(M')$, and in case of equality the two surfaces coincide. In other words, we need to prove that $\Phi(M)$ is a unique global area minimizer among the surfaces with the same boundary.

Step 2. Define a suitable notion of n-dimensional area in X and prove that $\Phi(M)$ is a minimal surface (with respect to a suitable class of variations).

Unfortunately the standard definitions of area (e.g., the Hausdorff measure or the Holmes–Thompson area) do not serve our purposes. So we introduce a special (not translation-invariant) area function in X. The area comes from a "Riemannian structure" (that is, a point-wise scalar product) in X defined as follows. For an $x \in M$, define a measure μ_x on S as a push-forward under the geodesic flow of a suitably normalized Haar measure on the unit sphere in T_xM. Then a scalar product at $\Phi(x) \in X$ is defined as the L^2-product with respect to μ_x. This structure is extended from $\Phi(M)$ to the whole X so that it is invariant under translations along certain codimension-n subspace transversal to $\Phi(M)$.

The area defined this way is not natural from geometric viewpoint but it has the key features needed in Step 1: all Lipschitz-1 maps are area-nonexpanding and Φ is area-preserving. A straightforward but cumbersome computation shows that $\Phi(M)$ is indeed a minimal surface with respect to this area.

Step 3. Show that $\Phi(M)$ is a global area-minimizer. The argument here models a proof of the following fact: if $f : D^n \to \mathbb{R}^m$ is a function such that $|df| \le c(n)$ and the graph of f is a minimal surface in $\mathbb{R}^{n+m}$, then this graph is a global area-minimizer. The proof is based on the convexity of area with respect to linear variations in "almost orthogonal" directions.

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Nilpotency and almost nonnegative curvature

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(joint work with Vitali Kapovitch and Anton Petrunin)

Almost nonnegatively curved manifolds were introduced by Gromov in the late seventies [Gro80], with the most significant contributions to their study made by Yamaguchi in [Yam91] and Fukaya and Yamaguchi in [FY92]. We establish several new properties of these manifolds which yield, in particular, new topological obstructions to almost nonnegative curvature.

Recall that a closed smooth manifold is said to be almost nonnegatively curved if it can Gromov-Hausdorff converge to a single point under a lower curvature bound. By rescaling, this definition is equivalent to the following one:

Definition. *A closed smooth manifold M is called almost nonnegatively curved if it admits a sequence of Riemannian metrics $\{g_n\}_{n\in\mathbb{N}}$ whose sectional curvatures and diameters satisfy $sec(M, g_n) \geq -1/n$ and $diam(M, g_n) \leq 1/n$.*

Almost nonnegatively curved manifolds generalize almost flat as well as nonnegatively curved manifolds. One main source of examples comes from a theorem of Fukaya and Yamaguchi. It states that if $F \to E \to B$ is a fiber bundle over an almost nonnegatively curved manifold B whose fiber F is compact and admits a nonnegatively curved metric which is invariant under the structure group, then the total space E is almost nonnegatively curved [FY92]. Further examples are given by closed manifolds which admit a cohomogeneity one action of a compact Lie group (compare [ST04]).

To put our work into perspective, let us briefly recall some previously known results: Let $M = M^m$ be an almost nonnegatively curved m-manifold.

* Gromov proved in [Gro78] that the minimal number of generators of the fundamental group $\pi_1(M)$ of M can be estimated by a constant $C_1(m)$ depending only on m, and in [Gro81] that the sum of Betti numbers of M with respect to any field of coefficients does not exceed some uniform constant $C_2 = C_2(m)$.
* Yamaguchi showed that, up to a finite cover, M fibers over a flat $b_1(M;\mathbb{R})$-dimensionsal torus and that M^m is diffeomorphic to a torus if $b_1(M;\mathbb{R}) = m$ [Yam91].
* Fukaya and Yamaguchi proved that $\pi_1(M)$ is almost nilpotent, i.e., contains a nilpotent subgroup of finite index, and also that $\pi_1(M)$ is $C_3(m)$-solvable, i.e, contains a solvable subgroup of index at most $C_3(m)$ [FY92].
* If a closed manifold has negative Yamabe constant, then it cannot volume collapse with scalar curvature bounded from below (see [Sch89, LeB01]). In particular, no such manifold can be almost nonnegatively curved.
* The $\hat{A}$-genus of a closed spin manifold X of almost nonnegative Ricci curvature satisfies the inequality $\hat{A}(X) \leq 2^{\dim(X)/2}$ ([Gal83], [Gro96, page 41]).

In [KPT06] we study almost nonnegatively curved manifolds by combining collapsing techniques with a non-smooth analogue of the gradient flow of concave functions which we call the "gradient push". This notion, which plays a key role in the proofs of our results, is based on the construction of gradient curves of λ-concave functions used in [PP96] and bears many similarities to the Sharafutdinov retraction [Sha78].

The first main result of [KPT06] concerns the hitherto unexplored relation between curvature bounds and the actions of the fundamental group on the higher homotopy groups. Recall that an action by automorphisms of a group G on an abelian group V is called nilpotent if V admits a finite sequence of G-invariant subgroups

$$V = V_0 \supset V_1 \supset \ldots \supset V_k = 0$$

such that the induced action of G on V_i/V_{i+1} is trivial for any i. A connected CW-complex X is called *nilpotent* if $\pi_1(X)$ is a nilpotent group that operates nilpotently on $\pi_k(X)$ for every $k \geq 2$.

Theorem A (Nilpotency Theorem)**.** *Let M be a closed almost nonnegatively curved manifold. Then a finite cover of M is a nilpotent space.*

Example. Let $h : S^3 \times S^3 \to S^3 \times S^3$ be defined by $h : (x, y) \mapsto (xy, yxy)$. This map is a diffeomorphism with the inverse given by $h^{-1} : (u, v) \mapsto (u^2v^{-1}, vu^{-1})$. The induced map h_* on $\pi_3(S^3 \times S^3)$ is given by the matrix $A_h = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Notice that the eigenvalues of A_h are different from 1 in absolute value. Let M be the mapping cylinder of h. Clearly, M has the structure of a fiber bundle $S^3 \times S^3 \to M \to S^1$, and the action of $\pi_1(M) \cong \mathbb{Z}$ on $\pi_3(M) \cong \mathbb{Z}^2$ is generated by A_h. In particular, M is not a nilpotent space and hence, by Theorem A, it does not admit almost nonnegative curvature. This fact doesn't follow from any previously known results.

Our next main result provides an affirmative answer to a conjecture of Fukaya and Yamaguchi [FY92, Conjecture 0.15].

Theorem B (C-Nilpotency Theorem for π_1)**.** *Let M be an almost nonnegatively curved m-manifold. Then $\pi_1(M)$ is $C(m)$-nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup of index at most $C(m)$.*

Example. For any $C > 0$ there exist prime numbers $p > q > C$ and a finite group G_{pq} of order pq which is solvable but not nilpotent. In particular, G_{pq} does not contain any nilpotent subgroup of index less than or equal to C. Whereas none of the results mentioned so far excludes G_{pq} from being the fundamental group of some almost nonnegatively curved m-manifold, Theorem B shows that for $C > C(m)$ none of the groups G_{pq} can be realized as the fundamental group of such a manifold.

Notice that Theorem B is new even for manifolds of nonnegative curvature. Moreover, Vitali Kapovitch and Burkhard Wilking proved recently that Theorem B does in fact also hold for manifolds of almost nonnegative Ricci curvature.

In [FY92] Fukaya and Yamaguchi also conjectured that a finite cover of an almost nonnegatively Ricci curved manifold M fibers over a nilmanifold with a fiber which has nonnegative Ricci curvature and whose fundamental group is finite. This conjecture was later refuted by Anderson [And92]. It is, on the other hand, very natural to consider this conjecture in the context of almost nonnegative *sectional* curvature.

Theorem C (Fibration Theorem). *Let M be an almost nonnegatively curved manifold. Then a finite cover $\tilde{M}$ of M is the total space of a fiber bundle*

$$F \to \tilde{M} \to N$$

over a nilmanifold N with a simply-connected fiber F. Moreover, the fiber F is almost nonnegatively curved in the sense of the following definition.

Definition. A closed smooth manifold M is called *almost nonnegatively curved in the generalized sense* if for some nonnegative integer k there exists a sequence of complete Riemannian metrics g_n on $M \times \mathbb{R}^k$ and points $p_n \in M \times \mathbb{R}^k$ such that
(1) the sectional curvatures of the metric balls of radius n around p_n satisfy

$$\sec(B_n(p_n)) \geq -1/n;$$

(2) for $n \to \infty$ the pointed Riemannian manifolds $((M \times \mathbb{R}^k, g_n), p_n)$ converge in the pointed Gromov-Hausdorff distance to $(\mathbb{R}^k, 0)$;
(3) the regular fibres over 0 are diffeomorphic to M for all large n.

Due to Yamaguchi's fibration theorem [Yam91], manifolds which are almost nonnegatively curved in the generalized sense play the same central role in collapsing under a lower curvature bound as almost flat manifolds do in the Cheeger-Fukaya-Gromov theory of collapsing with bounded curvature (see [CFG92]).

Clearly, if $k = 0$, the above definition reduces to the standard one. It is, however, an open question whether all manifolds which are almost nonnegatively curved in the generalized sense are indeed almost nonnegatively curved in the strict sense.

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Asymptotic geometry of hyperbolic spaces I & III

Viktor Schroeder

This abstract outlines the first and third lectures of a three part minicourse given jointly with Sergei Buyalo. In this minicourse we discussed recent new aspects of the asymptotic geometry of Gromov hyperbolic spaces. In particular we focussed on the following embedding theorem [BDS].

Theorem (S. Buyalo, A. Dranishnikov, V. Schroeder). *Every Gromov hyperbolic group* Γ *admits a quasi-isometric embedding into the product of* $n+1$ *copies of the binary metric tree where* $n = \dim \partial_\infty \Gamma$ *is the topological dimension of the boundary at infinity.*

The new and surprising feature of our result is that the trees involved are of uniformly bounded valence. It is much easier to construct embeddings into products of infinite valence trees. One can for example constructs quasi-isometric embeddings of the hyperbolic plane H^2 into the product of two trees where the embeddings are equivariant with respect to actions of some surface group. However, these trees have infinite valence at every vertex.

It has to be mentioned, that the embeddings we construct are by no means equivariant with respect to Γ. In fact some hyperbolic groups do not admit nontrivial actions on trees [dlHV].

The result should be compared with the Bonk-Schramm embedding theorem [BoS], which implies in particular:

Theorem (Bonk-Schramm). Γ *be a Gromov hyperbolic group. Then there is a number* $N \in \mathbb{N}$ *such that* Γ *admits a roughly similar embedding into the standard hyperbolic space* H^N.

The advantage of the Bonk-Schramm embedding is that the target space is (in contrast to a product of trees) itself a hyperbolic space, and the property of the embedding map (rough-similarity) is quite strong. The dimension N of the target space depends however on Γ.

The advantage of our embedding is that the dimension of the target space is optimal and depends only on the topological dimension of Γ. This becomes clear in the following examples.

Consider hyperbolic buildings $X(p,q)$, $p \geq 5$, $q \geq 2$, whose apartments are hyperbolic planes with curvature -1, whose chambers are regular hyperbolic p-gons with angle $\pi/2$, and whose link at each vertex is the complete bipartite graph with $q+q$ vertices as studied by Bourdon [Bou]. Indeed there are infinitely many quasi-isometry classes of these buildings (distinguished by the conformal dimension of their boundary). However all of them admit cocompact group actions and hence are quasi-isometric to some hyperbolic group $\Gamma(p,q)$. The topological dimension of their boundary is 1 (actually $\partial_\infty X(p,q)$ is the Menger curve). Thus by our result, they all allow quasi-isometric embeddings into the product of two binary trees.

On the other hand, if X and Y are hyperbolic and $f : X \to Y$ is a quasiisometric embedding, then the conformal dimensions satisfy $\dim_C(\partial_\infty X) \leq \dim_C(\partial_\infty Y)$ (see [Bou, 1.7]). Thus the existence of a quasi-isometric embedding of $X(p,q)$ into a hyperbolic space H^N implies $N-1 = \dim_C(\partial_\infty \mathrm{H}^N) \geq \dim_C(\partial_\infty X(p,q))$. The cited paper contains the estimate

$$\dim_C(\partial_\infty X(p,q)) \geq \frac{\log(q-1)+\log p}{2\log p},$$

hence we see that the dimension of the target space H^N has to be arbitrarily large as $q \to \infty$.

The proof of our embedding result consists of three main steps:

Step 1: Construction of a quasi-isometric embedding $\Gamma \to \prod_c T_c$ of Γ into a finite product of trees T_c, where each tree T_c has in general infinite valence.

Step 2: Construction of a map $T_c \to T$ of the infinite valence tree T_c into the binary tree T. Hence we also have a product map $\prod_c T_c \to \prod_c T$.

Step 3: Proof that the composition $\Gamma \to \prod_c T_c \to \prod T$ is quasi-isometric.

The most inventive part of our proof is Step 2, the construction of the map $T_c \to T$. The construction is based on what we call *Alice's diary* (compare this also to the prepublication [DS2]). Certainly any map from an infinite valence tree to a finite valence tree looses information. In particular the map is in no way quasi-isometric. The delicate point is that Alice's diary still contains enough information in order to prove in the last step that the composition $\Gamma \to \prod_c T$ is quasi-isometric.

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Manifolds with large systolic volume

Stéphane Sabourau

Consider a nonsimply connected closed n-manifold M endowed with a Riemannian metric g. The systole of (M, g), denoted by $\mathrm{sys}(M, g)$, is defined as the length of the shortest noncontractible loop in M. Define the systolic volume of M as

$$\sigma(M) = \inf_g \frac{\mathrm{vol}(M, g)}{\mathrm{sys}(M, g)^n}$$

where g runs over the space of all metrics on M.

The topological conditions that ensure the positivity of the systolic volume are well understood. In [4], M. Gromov proved that the systolic volume of essential manifolds (in particular, of aspherical manifolds and of connected sums of aspherical manifolds) is bounded away from zero. More precisely, there exists a positive constant C_n such that every essential n-manifold M satisfies

$$\sigma(M) \geq C_n. \tag{1}$$

Conversely, I. Babenko [1] showed that a closed oriented manifold with positive systolic volume is essential.

The systolic inequality (1) can be improved by taking into account the topology of the manifold. One of the goals of systolic geometry is to better understand how the systolic volume depends on this topology. There is some evidence suggesting that the systolic volume of manifolds with "complicated" topology or fundamental group is large. For instance, M. Gromov [4, 6.4.D'], [5, 3.C.3] showed that the systolic volume of a closed manifold with large simplicial volume is large. Specifically, there exists a positive constant C_n depending only on n such that every closed n-manifold M satisfies

$$\sigma(M) \geq C_n \, \frac{||M||}{\log^n(1 + ||M||)}\,, \tag{2}$$

In this note based on the articles [6] and [7], we first present two other systolic inequalities for aspherical manifolds using, on the one hand, the minimal entropy and, on the other hand, the algebraic entropy of the fundamental group. We refer to [6] for precise definitions.

Recall that the minimal entropy of M is defined as

$$\mathrm{MinEnt}(M) = \inf_g \mathrm{Ent}(M, g) \, \mathrm{vol}(M, g)^{\frac{1}{n}}$$

where g runs over the space of all metrics on M.

Our first result shows how the systolic volume and the minimal entropy are related, *cf.* [6].

Theorem A. *Every closed orientable aspherical manifold M of dimension $n \neq 3$ satisfies*

$$\sigma(M) \geq C_n \frac{\text{MinEnt}(M)^n}{\log^n(1 + \text{MinEnt}(M))},$$

where C_n is a positive constant depending only on n.

M. Gromov showed in [3] that every n-manifold M with simplicial volume $||M||$ satisfies

$$\text{MinEnt}(M)^n \geq C_n ||M||,$$

where C_n a positive constant depending only on n. Thus, Theorem A, which is a consequence of a slightly more general result, *cf.* [6], provides a partial generalization of inequality (2).

Our second result shows how the systolic volume of some manifolds is related to the algebraic entropy of their fundamental groups, *cf.* [6].

Theorem B. *Every closed orientable aspherical n-manifold M satisfies*

$$\sigma(M) \geq C_n \frac{\text{Ent}_{alg}(\pi_1(M))}{\log(1 + \text{Ent}_{alg}(\pi_1(M)))},$$

where C_n is a positive constant depending only on n.

Examples show that the topological conditions on M cannot be completely dropped, even though they can be slightly relaxed, *cf.* [6].

In the following, we study the systolic volume of sequences of connected sums of aspherical manifolds and sequences of hyperbolic manifolds, where none of the previous systolic inequalities yield effective estimates in general.

Before stating our results, recall that the systolic area of a closed surface Σ_k of genus k goes to infinity as k goes to infinity by inequality (2). The surface Σ_k can be described as a connected sum of k tori or as a surface admitting a hyperbolic metric (of area $4\pi(k-1)$). In higher dimension, no manifold satisfies these two features, but we can separately study the systolic volume of the manifolds satisfying either one. In each case, we obtain manifolds with large systolic volume, as explained below.

Our first result deals with connected sums, *cf.* [7].

Theorem C. *Let M be a closed orientable aspherical n-manifold. Then, the systolic volume of the connected sums $\#_k M = M \# \ldots \# M$ of k copies of M is unbounded. More precisely,*

$$\sigma(\#_k M) \geq C_n \, \frac{k}{\exp(C'_n \sqrt{\log k})} \, ,$$

where C_n and C'_n are two positive constants depending only on n.

In particular, the systolic volume of the connected sum of a large number of n-dimensional tori is large, even though its simplicial volume vanishes when $n \geq 3$. We can also replace $\#_k M$ by the connected sum $M_1 \# \cdots \# M_k$ of k closed oriented aspherical n-manifolds in the theorem.

Theorem C provides a partial answer to a question raised in [5, p. 330] asking for the asymptotic behaviour of $\sigma(\#_k M)$. Note that a sublinear *upper* bound on $\sigma(\#_k M)$ has been established in [2]. A more precise asymptotic estimate still needs to be found.

Our next and last result deals with hyperbolic manifolds, *cf.* [7].

Theorem D. *Let $\{M_i\}$ be a sequence of infinitely many, non-homeomorphic, closed hyperbolic n-manifolds. Then, the systolic volume of the M_i's is unbounded, that is*

$$\lim_{i \to \infty} \sigma(M_i) = \infty.$$

This result was already known for $n = 2$ and $n \geq 4$ as an application of H. C. Wang's finiteness theorem and inequality (2). In the three-dimensional case, there exist infinitely many closed hyperbolic manifolds whose simplicial volume, minimal entropy and algebraic entropy of their fundamental groups are bounded. Therefore, Theorem D applies in cases not covered by the systolic inequalities previously stated.

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Gauss Equation for subspaces in metric spaces of curvature bounded above

Stephanie B. Alexander
(joint work with Richard L. Bishop)

Alexandrov spaces are metric spaces with curvature bounds in the sense of local triangle comparisons with constant curvature spaces (see [6]). Analogues of the Gauss Equation, governing the passage of curvature bounds to subspaces from ambient spaces, tend to be challenging in Alexandrov spaces. For instance, a major unsolved problem in the theory of spaces of curvature bounded below (CBB) is whether the boundary of a convex set inherits the curvature bound. Somewhat more is known for curvature bounded above (CBA). A classical theorem of Alexandrov states that a curvature bound above is inherited by ruled surfaces [5]. It is an open problem whether saddle surfaces inherit such a bound, but "metric minimizing" surfaces do so by a theorem of Petrunin [13].

Say N *is a subspace of extrinsic curvature* $\leq A$ in M if there is a length-preserving map $F : N \to M$ between intrinsic metric spaces, where N is complete, and intrinsic distances $d_N = s$ and extrinsic distances $d_M = r$ satisfy

$$s - r \leq \frac{A^2}{24} r^3 + o(r^3) \tag{1}$$

for all pairs of points in N having s sufficiently small. For Riemannian submanifolds, this is equivalent to a bound, $|II| \leq A$, on the second fundamental form. For Riemannian subsets, the corresponding condition is *positive reach* [10]. In CAT(K) spaces (the global version of CBA by K), curves of extrinsic curvature $\leq A$ satisfy global arc/chord comparisons with curves of constant curvature A in the simply connected, 2-dimensional space form S_K of curvature K [1]. Recently Lytchak proved that if M has CBA by K, then a subspace N of bounded extrinsic curvature has *some* intrinsic curvature bound above [8].

We extend the Gauss Equation to CBA spaces by proving the following sharp bound on the intrinsic curvature of subspaces of bounded extrinsic curvature.

Theorem 1 (Gauss Equation)**.** *Suppose N is a subspace of extrinsic curvature $\leq A$ in an Alexandrov space of CBA by K. Then N is an Alexandrov space of CBA by $K + A^2$.*

This bound is realized by constantly curved hypersurfaces of Euclidean, spherical and hyperbolic spaces. At first one might think that Riemannian submanifolds of higher codimension offer a counterexample to this theorem, and that the correct bound should be $K + 2A^2$. On closer inspection, however, one sees that for any plane section, normals to the submanifold may be chosen so that at most two of the corresponding subdeterminants of II are nonzero and one of them is nonpositive. Therefore for Riemannian submanifolds, while the sharp *lower* bound is $K - 2A^2$ when ambient curvature is $\geq K$, the sharp *upper* bound is $K + A^2$ when ambient curvature is $\leq K$.

There are important classes of subspaces for which we can compute sharp extrinsic curvature bounds, hence sharp intrinsic curvature bounds by Theorem 1. Fibers of warped products are such a class. Warped products of Alexandrov spaces extend standard cone and suspension constructions from 1-dimensional to arbitrary base, and gluing constructions from 0-dimensional to arbitrary fiber [3], and we expect them to be a major source of constructions and counter-examples in the Alexandrov setting. Theorem 1 allows us to calculate the intrinsic curvature bound of the fiber of a $\mathrm{CAT}(K)$ warped product.

Another significant application of Theorem 1 is to injectivity radii. Our next theorem gives a sharp estimate on the injectivity radius of a subspace of bounded extrinsic curvature, in terms of the circumference $c(A,K)$ of a circle of curvature A in the model space S_K. Set $\pi/k = \infty$ if $k \leq 0$.

Theorem 2. *Suppose N is a subspace of extrinsic curvature $\leq A$ in a CAT(K) space. Then*

$$\mathrm{inj}_N \geq \min\{\frac{\pi}{\sqrt{K+A^2}}, \frac{1}{2}c(A,K)\}. \tag{2}$$

Even in the case of Riemannian manifolds, this estimate on the injectivity radius of a submanifold is new as far as we know. Much weaker dimension-dependent estimates have been used in [7] and [16]. The existence of some dimension-independent bound in the general case is proved in [10].

The following corollary holds, in particular, for Riemannian submanifolds with $|II| \leq A$ in a Hadamard manifold.

Corollary. *Let N be a subspace of extrinsic curvature $\leq A$ in a CAT(0) space M. Then N has injectivity radius at least π/A, and any closed ball of radius $\pi/2A$ in N is CAT(A^2). If M is CAT($-A^2$), then N is CAT(0) and embedded.*

Finally we show how the geometry of Alexandrov spaces provides an abundance of *almost convex* subspaces whose extrinsic curvature bounds can be computed. In what follows, suppose M is complete with either CBA or CBB.

A locally Lipschitz continuous function $f : M \to \mathbf{R}$ is λ-*concave* (or *semiconcave*) if $(f \circ \gamma)'' \leq \lambda$ in the barrier sense for every unit-speed geodesic γ, i.e., $f(\gamma(x)) - \lambda x^2/2$ is concave. We say f is *almost concave* if $\lambda > 0$, and *very concave* if $\lambda < 0$. (Then $-f$ is $-\lambda$-convex, so studying almost concave functions is equivalent to studying almost convex ones.) Gradient curves of λ-concave functions were used to study CBB spaces by Perelman-Petrunin in [12]. The Soul Theorem and Sharafudtinov retraction [15] provided inspiration. See Petrunin's survey [14] for a beautiful exposition of CBB spaces using semiconcave functions. Lytchak has shown that gradient curves also work in CBA and more general metric spaces [9].

We give some examples to illustrate how semiconcave and semiconvex functions arise naturally in spaces with curvature bounds. Here *CS* denotes a convex set, and "$f'' \geq f$" means this inequality holds on every geodesic.

(1) $f = d_p^2/2$ is 1-convex on CAT(0), 1-concave on CBB by 0.
(2) On CAT(-1): $f = \cosh d_p$ and $f = \sinh d_{\mathrm{CS}}$ satisfy $f'' \geq f$.
(3) On CAT(1): $f = \sin(\min\{\text{distance to a } \pi\text{-CS}, \frac{\pi}{2}\})$ satisfies $f'' \geq -f$.

(4) On CBB(-1): $f = \cosh d_p$ and $f = \sinh d_{\text{bdry}}$ satisfy $f'' \leq f$.
(5) On CBB(1): $f = \sin d_{\text{bdry}}$ satisfies $f'' \leq -f$.

Perelman proved in [11] that d_{bdry} is concave in a complete space of CBB by 0; the examples above involving d_{bdry} have the same proof (also see [14]). See [2] for a discussion of all the constructions above.

We prove the following bound on the extrinsic curvature of almost convex subsets, using gradient curve methods. The existence of some bound in this setting was proved by Lytchak [9]. At a point where the gradient length is continuous, our bound becomes $|\lambda|/|\nabla f|_p$, which is the sharp bound on extrinsic curvature in the smooth case.

Theorem 3. *On any CBA or CBB space, a superlevel (sublevel) set of a λ-concave (λ-convex) function f at p has extrinsic curvature $\leq |\lambda|\sqrt{\frac{2}{|\nabla f|_p^2} - \frac{1}{G_p^2}}$, where $G_p = \limsup_{q \to p} |\nabla f|_q$.*

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Constructing discrete K-surfaces

Ivan Sterling
(joint work with Tim Hoffmann and Ulrich Pinkall)

Old and new approaches to the theoretical, computational and physical construction of discrete pseudospherical surfaces in $\mathbb{R}^3$ ("Discrete K-Surfaces") were presented.

1. Twisted metal

G. T. Bennett introduced a "new mechanism" in 1903 [1]. The relation between this mechanism and discrete K-surfaces (including a mechanical description of the Bäcklund transformation) is in [2]. The theory of discrete K-Surfaces was developed by Sauer [6], Wunderlich [8] and Bobenko-Pinkall [3]. In 1951 Wunderlich built a model which still exists today (Figure 1). It is also possible to build deformable prototypes by attaching ring terminals to the ends of butt splices for the edges; small bolts and wing nuts for the vertices. Besides assisting in research, these models are also useful in teaching. For example, the 1-dimensional curves on Bennett's isogram (the asymptotic coordinate lines) model discrete curves of constant torsion.

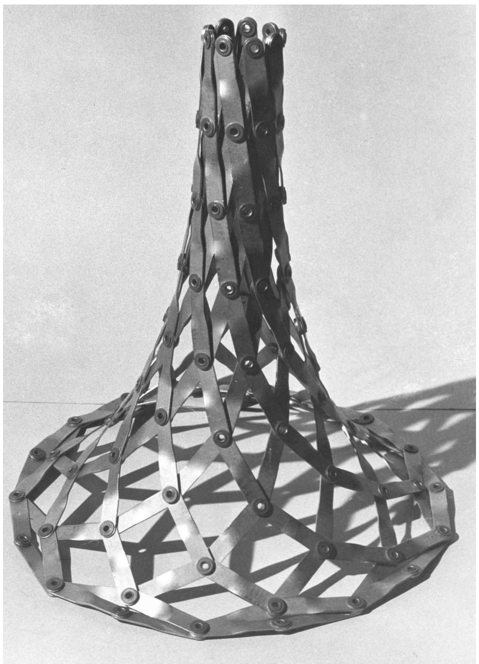

Figure 1. Wunderlich's Model (1951)

2. Scissors-paper-tape and crocheting

On the other hand, in the early 1970's Thurston [7] outlined a completely different way of constructing certain discrete K-surfaces. These surfaces became well-known because of Henderson's paper models and Taimina's crocheted models [4]. A rather generic example is in Figure 2.

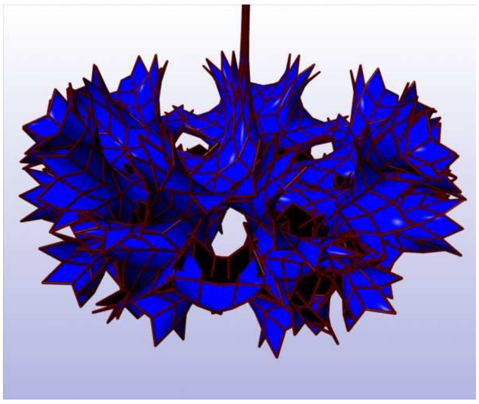

FIGURE 2. Henderson-Taimina-Thurston Discrete K-Surface

3. New examples

There are very few examples of discrete K-surfaces with higher genus or several ends. Of particular interest are those which approximate K-surfaces. Figure 3 shows an example by Tim Hoffmann based on work in his dissertation [5]. By

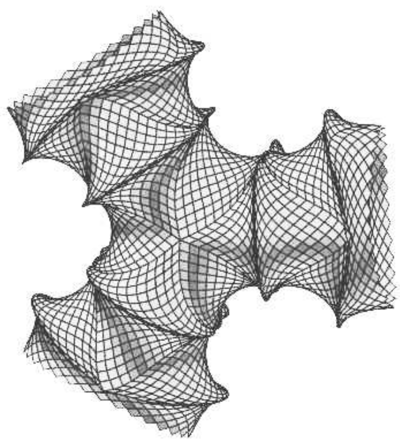

FIGURE 3. Hoffman Discrete K-Surface

adjusting the parameters in the Thurston construction it is possible to find other examples (Figure 4).

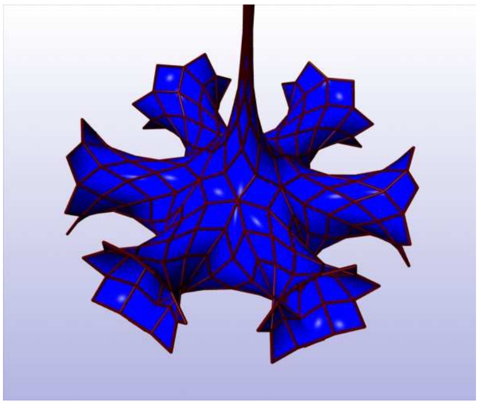

FIGURE 4. Hoffman-Sterling Discrete K-Surface

4. Computer Graphics

The computer graphics used here were generated using jReality. jReality is a Java 3D viewer for mathematics. A demonstration of some of its features were presented by Ulrich Pinkall. More details and examples can be found at www.jreality.de.

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Asymptotic geometry of hyperbolic spaces II

Sergei Buyalo

This is the second lecture of a three part minicourse given jointly with Viktor Schroeder. In this lecture, we discuss two of the three major ingredients of the following binary embedding theorem [BDS] which is the main topic of this minicourse.

Theorem (S. Buyalo, A. Dranishnikov, V. Schroeder). *Every Gromov hyperbolic group* Γ *admits a quasi-isometric embedding into the product of* $(n+1)$ *copies of the binary metric tree where* $n = \dim \partial_\infty \Gamma$ *is the topological dimension of the boundary at infinity.*

The notion of the linearly controlled dimension or the ℓ-dimension of a metric space Z, $\ell\text{-}\dim Z$, is introduced in [Bu1] (under the name capacity dimension), and turns out to be useful in many questions, [Bu2]. Its definition is close to that of the classical covering dimension with only additional point that we require that the Lebesgue number of coverings involved in the definition is linearly controlled by the mesh of the coverings. The first step in the proof of the binary embedding theorem is the following

Theorem 1. *Let* X *be a visual hyperbolic space with finite* ℓ*-dimension, of the boundary at infinity,* $\ell\text{-}\dim(\partial_\infty X) = n < \infty$*. Then there exists a quasi-isometric* $f : X \to T_1 \times \cdots \times T_{n+1}$*, where* $T_1, \ldots, T_{n+1}$ *are simplicial metric trees.*

It is a remarkable feature of the proof that both the source space X and the target trees $T_1, \ldots, T_{n+1}$ are actually recovered from of the boundary at infinity $Z = \partial_\infty X$ via constructions related to various coverings of Z. The initial space X is replaced in the proof by the hyperbolic approximation of Z, and the target trees appear as combinatorial objects associated with an appropriately constructed infinite sequence of colored coverings of Z arising from the notion of the ℓ-dimension.

In this theorem, the target trees $T_1, \ldots, T_{n+1}$ typically have infinite valence of vertices, see [BS]. Furthermore, The ℓ-dimension is larger than or equal to the topological dimension, $\dim Z \le \ell\text{-}\dim Z$ for every metric space Z, and there are simple examples of compact metric spaces with ℓ-dimension arbitrarily larger than the topological dimension.

Thus it is important to know for which spaces equality holds. Let $\lambda \ge 1$ and $R > 0$ be given. A map $f : Z \to Z'$ between metric spaces is λ*-quasi-homothetic with coefficient* R if for all z, $z' \in Z$, we have

$$R|zz'|/\lambda \le |f(z)f(z')| \le \lambda R|zz'|.$$

This property can be regarded as a perturbation of the property to be homothetic, and the coefficient λ describes the perturbation.

A metric space Z is *locally similar* to a metric space Y, if there is $\lambda \ge 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \le \frac{1}{R}$ there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R. If a metric space Z is locally similar to itself then we say that Z is *locally self-similar*. Major examples of compact, locally self-similar metric spaces are the boundaries at infinity of Gromov hyperbolic groups (regarded with visual metrics).

The second ingredient of the proof of the binary embedding theorem is the following

Theorem 2 (S. Buyalo, N. Lebedeva). *The* ℓ*-dimension of every compact, locally self-similar metric space* Z *is finite and coincides with the topological dimension,* $\ell\text{-}\dim Z = \dim Z$.

We note that Theorems 1 and 2 yield the following result conjectured by M. Gromov, [Gr, Section 1.E_1'], see [BL].

Theorem 3. *The asymptotic dimension of any hyperbolic group G equals topological dimension of its boundary at infinity plus* 1, $\operatorname{asdim} G = \dim \partial_\infty G + 1$.

According to Theorems 1 and 2, we have constructed a quasi-isometric embedding of every hyperbolic group G into the product $T_1 \times \ldots, \times T_{n+1}$ of metric simplicial trees, where $n = \dim \partial_\infty G$. However, the nature of the construction is that these trees have infinite valence of the vertices. The most interesting and deep part of the binary embedding theorem deals with passing from the embedding into the product of infinite valence trees to an embedding where the target trees have uniformly bounded valence of the vertices. This is explained in the third lecture of the minicourse.

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Bryant surfaces with smooth ends

G. Paul Peters
(joint work with Christoph Bohle)

Bryant surfaces—the surfaces of constant mean curvature 1 in hyperbolic space—have been studied intensively since Robert Bryant's influential paper [4], cf. [13] and the surveys [8, 10]. The following observation led us to introduce the notion of Bryant surfaces with smooth ends: the simplest nontrivial Dirac sphere [7] which is a surface of revolution related to a 1–soliton solution of the mKdV equation [12, 6, 1], is an immersed sphere that, besides the two points on the axis of rotation, is a Bryant surface in the Poincaré ball model of hyperbolic space (see the left surface in Figure 1).

Definition. *A Bryant surface E in the Poincaré ball model $\mathbf{B}^3 \subset \mathbb{R}^3$ of hyperbolic space is a* smooth Bryant end *if there is a point $p_\infty \in \partial \mathbf{B}^3$ on the asymptotic boundary such that $E \cup \{p_\infty\}$ is a conformally immersed open disc in $\mathbb{R}^3$.*

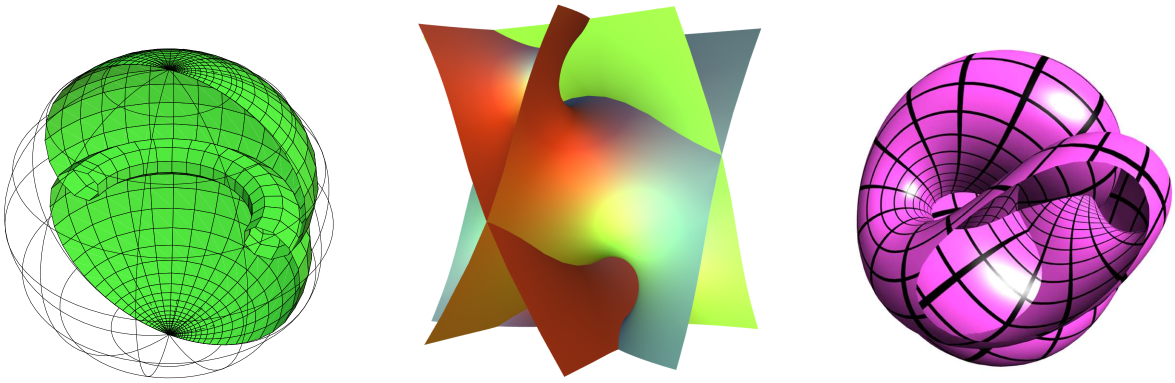

FIGURE 1

From the Möbius geometric point of view, smooth Bryant ends correspond to planar minimal ends in $\mathbb{R}^3$: both can be smoothly extended through the ideal boundary, i.e., the 2–sphere at infinity in the case of hyperbolic space and the point at infinity in the case of $\mathbb{R}^3 = S^3 \setminus \{\infty\}$. We obtain the following analog to the theorem that a planar minimal end may be parametrized by the real part of a holomorphic $\mathbb{C}^3$–valued map with a pole at the end.

Theorem 1. *A surface E in the Poincaré ball is a smooth Bryant end if and only if there exists a holomorphic null immersion $F\colon \Delta \setminus \{0\} \to \mathrm{SL}(2,\mathbb{C})$ with a pole at 0 such that $F'F^{-1}$ has a pole of order 2 and E is parametrized by Bryant's representation formula*

$$\text{(BRF)} \qquad f = \frac{1}{x_0+1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \begin{pmatrix} x_0 + x_3 & x_1 + x_2 i \\ x_1 - x_2 i & x_0 - x_3 \end{pmatrix} = F\bar{F}^t.$$

A deeper analogy becomes apparent when one considers compact surfaces in S^3 obtained by extending Bryant surfaces with smooth ends or minimal surfaces with planar ends through the respective ideal boundary. As Robert Bryant proved in [3, 5], the inversion of a complete finite total curvature minimal surface with planar ends extends to a compact Willmore surface (critical point of the Willmore energy $W = \int H^2 dA$) whose Willmore energy is 4π times the number of ends. Moreover, all Willmore spheres in S^3 are extended minimal surfaces with planar ends and the possible Willmore energies of Willmore spheres are $W \in 4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$. For Bryant surfaces, we prove:

Theorem 2. *Compact Bryant surfaces with smooth ends have Willmore energy $W = 4\pi n$, where $n \in \mathbb{N}^*$ is the total pole order of the Bryant representation, which is greater or equal to the number of Bryant ends. The possible Willmore energies of Bryant spheres with smooth ends are $4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$.*

Note that the Willmore energy W of the compact surface of genus g and the total curvature of the minimal or Bryant surface are related by $W + \int K dA = 4\pi(1-g)$.

In order to prove Theorem 2 we use the fact that Bryant spheres with smooth ends are related to rational null immersions into the non–degenerate quadric $\mathbf{Q}^3 \subset$

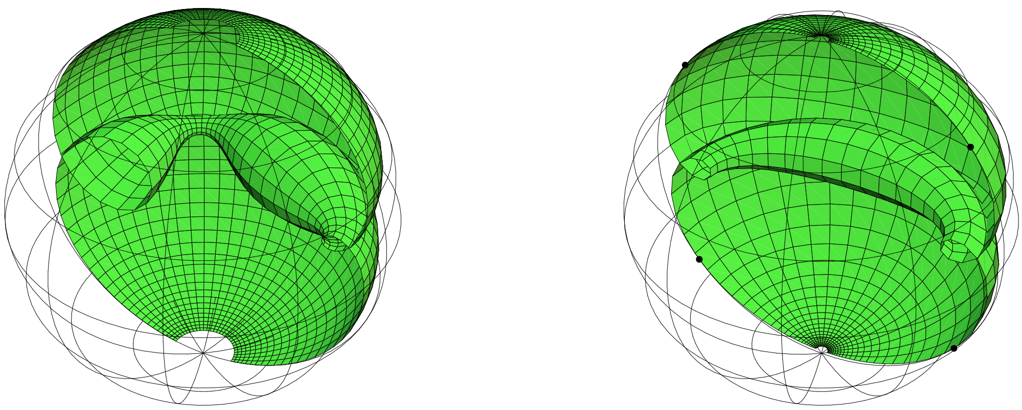

Figure 2

$\mathbb{CP}^4$ via Bryant's representation formula (BRF) and

$$\begin{aligned}\mathrm{SL}(2,\mathbb{C}) &= \left\{ F = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \right\} \\ &\cong \mathbf{Q}^3 \setminus \{e = 0\} = \{ \Phi = [a,b,c,d,e] \in \mathbb{CP}^4 \mid ad - bc - e^2 = 0,\ e \neq 0 \}.\end{aligned}$$

With this identification Theorem 1 becomes:

Theorem 1'. *If E is a smooth Bryant end, then there exists a holomorphic null immersion $\Phi\colon \Delta \to \mathbf{Q}^3$ that parametrizes E via (BRF). Conversely, if $\Phi\colon \Delta \to \mathbf{Q}^3$ is a holomorphic null immersion that intersects $\{e = 0\}$*

(1) *transversely at 0, then F and F^{-1} (restricted to $\Delta \setminus \{0\}$) parametrize smooth horospherical Bryant ends, or*
(2) *non–transversely at 0, then either F or F^{-1} (restricted to $\Delta \setminus \{0\}$) parametrizes a smooth catenoidal Bryant end.*

We then show that (similar to minimal surfaces with planar ends) the degree d of the null immersion Φ into $\mathbf{Q}^3$ and the Willmore energy of the corresponding Bryant surface with smooth ends are related by $W = 4\pi d$. This proves the first statement of Theorem 2. The second statement follows from the fact that rational null immersions into $\mathbf{Q}^3$ exist for every degree, except $2, 3, 5$, and 7, cf. [5] .

Figure 1 shows three surfaces obtained from one degree 4 rational null immersion into $\mathbf{Q}^3$: a Bryant sphere with smooth ends and Willmore energy $W = 16\pi$ (which is a catenoid cousin), a minimal sphere with 4 planar ends, and its inversion, which is a Willmore sphere with $W = 16\pi$.

Figure 2 shows two Bryant spheres with smooth ends that are obtained applying an orthogonal transformation to the rational null immersion used for the surfaces in Figure 1. For the first surface the orthogonal transformation fixes the hyperplane $\{e = 0\}$ at infinity. Such transformations were also studied by Wayne Rossman, Masaaki Umehara, and Kotaro Yamada [9, 11]. The transformation for the second surface does not fix the hyperplane at infinity, such that both catenoidal ends open up, and one gets a surface with 4 horospherical ends (marked points).

Both Bryant spheres with smooth ends and Willmore spheres in S^3 are examples of a more general class of surfaces in S^3 which we call soliton spheres, cf. [1].

Generalizing Robert Bryant's result about Willmore spheres and the second statement of Theorem 2, we prove that the quantization $W \in 4\pi\mathbb{N}^* \setminus \{2, 3, 5, 7\}$ holds for all soliton spheres in S^3.

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Critical metrics of the Schouten functional

Udo Simon

(joint work with Z. Hu, S. Nishikawa,)

Using the Schouten tensor, we introduce a new functional on the space of Riemannian metrics on a compact manifold and study its critical points, in particular in the subclass of conformally flat metrics, for all dimensions $n \geq 3$. We present some of our major results with emphasis on $n = 4$. A full version of the paper with additional results, proofs and references has been submitted for publication.

Notation. Let M^n, $n \geq 3$, be an n-dimensional compact, connected, smooth manifold, $\mathcal{M}(M)$ the space of Riemannian metrics and $\mathcal{D}(M)$ the group of diffeomorphisms. For a fixed metric g let Riem_g denote its Riemannian curvature tensor, Ric_g its Ricci tensors, R_g its (non-normed) scalar curvature, W the Weyl conformal curvature tensor and $E_g := \mathrm{Ric}_g - (R_g/n)g$ its trace-free Ricci tensor.

Definitions. (i) $A_g := \mathrm{Ric}_g - \frac{R_g}{2(n-1)}\, g$ is called the *Schouten tensor*, it controls the non-conformally invariant part of Riem_g:

$$\mathrm{Riem}_g = W + \frac{1}{n-2} A \wedge g.$$

(ii) $\sigma_k(A_g)$ denotes the k-th elementary symmetric function of eigenvalues of A_g (with respect to g) and

$$\mathcal{F}_k[g] := \left(\int_{M^n} \mathrm{dvol}_g\right)^{(2k-n)/n} \int_{M^n} \sigma_k(A_g)\, \mathrm{dvol}_g$$

is called the *k-th Yamabe functional.*
(iii)

$$\mathcal{S}[g] := \left(\int_{M^n} \mathrm{dvol}_g\right)^{(4-n)/n} \int_{M^n} |A_g|^2 \ \mathrm{dvol}_g.$$

is called the *Schouten functional.*

Remark. $\mathcal{S}[g] = 0$ if and only if (M^n, g) is Ricci-flat.

Euler-Lagrange equations for $\mathcal{S}[g]$. $g \in \mathcal{M}(M)$ critical point $\Longleftrightarrow$ (1) and (2) are satisfied:

$$(1) \quad \Delta_g E_{ij} - \frac{(n-2)(2n-3)}{2(n-1)^2} R_{,ij} + \frac{(n-2)(2n-3)}{2n(n-1)^2} \Delta_g R g_{ij} + 2E^{kl} W_{kilj} - \frac{4}{n-2} E_i^k E_{kj} + \frac{n^2-8n+8}{2n(n-1)^2} R E_{ij} + \frac{4}{n(n-2)} |E_g|^2 g_{ij} = 0$$

and

$$(2) \quad \frac{(n-2)^2}{n-1} \Delta_g R - (n-4)|E_g|^2 - \frac{(n-2)^2(n-4)}{4n(n-1)^2} R_g^2 + (n-4)\,\mathrm{Vol}(M^n, g)^{-4/n} \mathcal{S}[g] = 0.$$

Corollary. (i) Any Einstein metric is critical for $\mathcal{S}$.
(ii) $n = 4$: If $g \in \mathcal{M}(M)$ is critical for $\mathcal{S}$ then $R_g = \mathit{const.}$

Conformal flatness in dimension $n = 4$

Theorem. Let M^4 be compact. A locally conformally flat metric $g \in \mathcal{M}(M)$ is critical for $\mathcal{S}$ if and only if (i) or (ii) is satisfied:
(i) The scalar curvature of (M^4, g) is zero.
(ii) (M^4, g) is a space form with $R_g = \mathit{const.} \neq 0$.

Proposition. (i)

$$\mathcal{F}_2[g] = -\frac{1}{4}\int_{M^4} |W|^2 \, d\mathrm{vol}_g + 8\pi^2\chi(M^4),$$

Thus $\mathcal{F}_2[g]$ is a conformal invariant.
(ii) g conformally flat $\Longrightarrow$ g critical for $\mathcal{F}_2[g]$ (there is no variational characterization of 4-dim. space forms in terms of $\mathcal{F}_2[g]$); in case of conformal flatness $\mathcal{F}_2[g]$ depends only on the topology.

Conformal flatness in dimension $n \neq 4$

Theorem. Let (M^n, g), $n \neq 4$, be compact, locally conformally flat with $R_g = 0$. Then g is critical for $\mathcal{S}$ if and only if (i) or (ii) is satisfied.

(i) If n odd, then (M^n, g) flat space form.
(ii) If $n = 2m$ even, then (M^n, g)
 – either is a flat space form
 – or its universal cover $(\tilde{M}^n, \tilde{g})$ is $(\tilde{M}^n, \tilde{g}) = \mathbb{S}^m(c) \times \mathbb{H}^m(c)$.

Theorem. Let (M^n, g), $n \neq 4$, be compact, locally conformally flat with $R_g \neq 0$ constant. Then g is critical for $\mathcal{S}$ if and only is (M^n, g) is a space form.

Critical metrics of $\mathcal{S}$ on $\mathbb{S}^3$

Using a 1-parameter family of Berger metrics we constructed an example of a critical metric of the Schouten functional $\mathcal{S}$ on $\mathbb{S}^3$ that is neither Einstein nor locally conformally flat.

$\mathcal{S}$-optimal metrics on four-manifolds

Definitions. (i) Let M^n be a compact n-manifold, $n \geq 3$. g is called an $\mathcal{S}$-*optimal metric* (LeBrun) if it is an absolute minimizer of the functional $\mathcal{S}$: $\mathcal{S}[\tilde{g}] \geq \mathcal{S}[g]$ for every metric $\tilde{g} \in \mathcal{M}(M)$.

(ii) $\quad \mathcal{I}_{\mathcal{S}}(M^n) := \inf_{g \in \mathcal{M}(M)} \mathcal{S}[g] \qquad$ (invariant under action of $\mathcal{D}(M)$).

(iii) (M^4, g) oriented. We call:
g anti-self-dual $\Longleftrightarrow$ $W_+ = 0$ (self-dual Weyl tensor);
g self-dual $\Longleftrightarrow$ $W_- = 0$ (anti-self-dual Weyl tensor).

Theorem. (1) For every $k \geq 6$, the Schouten functional, defined on the simply connected 4-manifold $k\overline{\mathbb{CP}}^2$, admits a scalar flat and anti-self-dual critical metric which is neither locally conformally flat nor Einstein. Moreover,

$$\mathcal{I}_{\mathcal{S}}(k\overline{\mathbb{CP}}^2) = 8(k-4)\pi^2.$$

(2) For every $k \geq 14$, the Schouten functional defined on the simply connected 4-manifold $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$ admits a scalar flat and anti-self-dual critical metric which

is neither locally conformally flat nor Einstein. Moreover,

$$\mathcal{I}_{\mathcal{S}}(\mathbb{CP}^2\sharp k\overline{\mathbb{CP}}^2) = 8(k-9)\pi^2.$$

Theorem. Let M be a compact oriented 4-manifold. If $g \in \mathcal{M}(M)$ is a Kähler metric for some complex structure of M^4, then we have

$$\mathcal{S}[g] \geq \begin{cases} 8\pi^2[7\tau - 2\chi] \text{ for } & \tau \geq 0, \\ -8\pi^2[3\tau + 2\chi], & \tau < 0, \end{cases}$$

where χ is the Euler-characteristic and τ the signature; equality holds if and only if g is a self-dual or anti-self-dual Kähler metric.

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Parabolic submanifold geometry

Francis Burstall
(joint work with David Calderbank)

Parabolic geometry is Cartan geometry modelled on a real flag manifold, that is, a homogeneous space G/P with G a real semisimple Lie group and P a parabolic subgroup. It is our contention that a uniform approach to the geometry of submanifolds of such a G/P can be obtained as a *relative Cartan geometry* modelled on a homogeneous inclusion $H/Q \to G/P$ of flag manifolds. Our theory applies to (among others): arbitrary submanifolds of the conformal n-sphere; generic hypersurfaces in projective space; generic submanifolds in projective space of dimension k and codimension $k(k+1)/2$; generic CR submanifolds of S^{2n+1}.

Parabolic geometry. Let Σ^k be a k-manifold and G/P a k-dimensional real flag manifold. Suppose we are given:

- a principal G-bundle $E \to \Sigma^k$ with G-connection $\theta \in \Omega^1_E \otimes \mathfrak{g}$;
- a reduction $F \subset E$ of E to structure group P.

In this situation, $\theta_{|F} \mod \mathfrak{p}$ descends to a bundle-valued 1-form, the *solder form* $\beta^\theta \in \Omega^1_{\Sigma^k} \otimes [\mathfrak{g}/\mathfrak{p}]$ where $\mathfrak{p}$ is the Lie algebra of P and, for any P-module W, $[W] = F \times_P W$.

If β^θ is an isomorphism, we say that θ is a *Cartan connection* and call the whole package (E, F, θ) a *Cartan geometry.* The intuition is that the reduction F associates a copy of G/P to each $x \in \Sigma^k$ and then the solder form identifies $T_x\Sigma^k \cong$

$\mathfrak{g}/\mathfrak{p} \cong T_{eP}G/P$. Thus $T\Sigma^k$ inherits the first order structure of G/P: typically this is a filtration (induced by the central descending series of the nilradical of $\mathfrak{p}$) along with a G_0-structure and family of G_0-connections on the associated graded tangent bundle, for G_0 the quotient of P by its nilradical.

Many well-known geometries arise this way such as conformal geometry (G/P a real quadric); projective geometry (G/P a projective space) and CR geometry (G/P a Hermitian quadric). However, the problem arises that many Cartan connections induce the same structure on Σ^k. To get a unique choice requires a digression into Lie algebra homology.

The nilradical of $\mathfrak{p}$ is its Killing polar $\mathfrak{p}^\perp$ which is, via the Killing form, isomorphic to $(\mathfrak{g}/\mathfrak{p})^*$. Given a P-module W, there is a P-invariant chain complex

$$\xrightarrow{\partial} \textstyle\bigwedge^n \mathfrak{p}^\perp \otimes W \xrightarrow{\partial} \bigwedge^{n-1} \mathfrak{p}^\perp \otimes W \xrightarrow{\partial} \cdots \to 0$$

whose homology $H_n(\mathfrak{p}^\perp, W)$ is computed by Kostant's version of the Borel–Bott–Weil theorem [6]. In the presence of a Cartan connection θ, we use the solder form to get a chain complex of bundle morphisms

$$\xrightarrow{\partial} \textstyle\bigwedge^n T^*\Sigma^k \otimes [W] \xrightarrow{\partial} \bigwedge^{n-1} T^*\Sigma^k \otimes [W] \xrightarrow{\partial} \cdots \to 0$$

and homology bundles $H_n(\Sigma^k, [W])$.

We apply these ideas to the curvature $R^\theta \in \Omega^2_{\Sigma^k} \otimes [\mathfrak{g}]$ of our Cartan connection and say that the connection is *normal* if $\partial R^\theta = 0$.

Theorem ([4, 7])**.** *Under favourable circumstances, there is a unique normal Cartan connection, up to isomorphism, inducing the given structure on Σ^k.*

Moreover, thanks to the Bianchi identity, the homology class $[\![R^\theta]\!]$ controls the entire curvature:

Theorem ([4])**.** $[\![R^\theta]\!] = 0$ *if and only if* $R^\theta = 0$ *if and only if* Σ^k *is locally isomorphic to* G/P.

For example, the theorem applies to conformal manifolds Σ^k when $k \geq 3$ (in this case, the result goes back to Cartan) and here $[\![R^\theta]\!]$ is the Weyl curvature when $k \geq 4$ and the Cotton–York tensor when $k = 3$. By contrast, when $k = 2$, the uniqueness assertion fails: the normal Cartan connections parametrise Möbius structures in the sense of Calderbank [2].

Parabolic submanifold geometry. Now fix an n-dimensional real flag manifold G/P and contemplate immersions $\Sigma^k \to G/P$ or, equivalently, reductions of the trivial bundle $E = \Sigma^k \times G$ to structure group P. Our strategy is to model this situation on a homogeneous inclusion $H/Q \to G/P$ with H a semisimple subgroup of G such that $Q := H \cap P$ is a parabolic subgroup of H. Thus, for such an H, we have a fixed reduction F_P of E to P and we contemplate reductions F_H to H so that $F_H \cap F_P$ is a reduction of F_H to Q.

The left Maurer–Cartan form of G gives a flat G-connection $\theta \in \Omega^1_E \otimes \mathfrak{g}$ which decomposes

$$\theta = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}$$

according to the H-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then $\theta_{\mathfrak{h}}$ is an H-connection on F_H while $\theta_{\mathfrak{m}}$ descends to an $[\mathfrak{m}]$-valued 1-form on Σ^k.

Any Q-module has an invariant filtration indexed by *weight* (the eigenvalues of a certain *grading element* in the centre of the Levi factor $\mathfrak{q}_0$) for which $\mathfrak{q}^\perp$ acts by lowering operators.

Definition. A reduction F_H is *good* if $\theta_{\mathfrak{h}}$ is a Cartan connection and $\theta_{\mathfrak{m}}$ has negative weight.

A reduction $\Sigma^k \to G/H$ should be viewed as a map into the space of H/Q's in G/P. Thus, for H/Q a conformal k-sphere in S^n, a reduction is a map into the space of k-spheres in S^n (thus a *congruence* of k-spheres). It is a good reduction when the spheres have first order contact at the corresponding points of Σ^k (thus Σ^k envelops the congruence). For H/Q an $(n-1)$-quadric in real projective n-space, a reduction is a congruence of quadrics and it is good if the quadrics have *second* order contact at corresponding points of Σ^k.

The existence of a good reduction can have implications for the geometry of the submanifold and the question of when good reductions exist is still under consideration: in all the sample geometries we have investigated, ad hoc arguments can be used to establish existence but we presently lack a general understanding.

However, once good reductions are available, one can again normalise via Lie algebra homology. We say that a good reduction is *normal* if $\partial\theta_{\mathfrak{m}} = 0$ and prove:

Theorem. *In favourable circumstances, there is a unique normal reduction F_{Σ^k} and a canonical choice $\hat{\theta}_{\mathfrak{h}}$ of normal Cartan connection thereon.*

In fact, the geometry is completely controlled by $\hat{\theta}_{\mathfrak{h}}$ and the homology class $[\![\theta_{\mathfrak{m}}]\!]$. In examples, these primitive data can be found in the classical literature: for submanifolds of the conformal n-sphere, the normal reduction is the *central sphere congruence* of Blaschke [1] while $[\![\theta_{\mathfrak{m}}]\!]$ is simply the trace-free part of the second fundamental form. Again, for hypersurfaces of projective space, the normal reduction is the congruence of Lie quadrics (see [1]) and $[\![\theta_{\mathfrak{m}}]\!]$ is the Darboux cubic form.

We can recover the immersion from the primitive data $\hat{\theta}_{\mathfrak{h}}, [\![\theta_{\mathfrak{m}}]\!]$. Indeed, given a normal Cartan geometry $(F_{\Sigma^k}, F_{\Sigma^k,Q}, \hat{\theta}_{\mathfrak{h}})$ on Σ^k along with a negative weight section $[\![\theta_{\mathfrak{m}}]\!]$ of the homology bundle $H_1(\Sigma^k, [\mathfrak{m}])$, one can recover $\theta_{\mathfrak{m}} \in \Omega^1_{\Sigma^k} \otimes [\mathfrak{m}]$ via the Čap–Slovák–Souček theory of differential lifts of sections of homology bundles [5]. Further, one can then recover $\theta_{\mathfrak{h}}$ and so construct a connection θ on $F_{\Sigma^k} \times_H G$. The whole issue now is the flatness of θ: if θ is flat, we have a local isomorphism of $(F_{\Sigma^k} \times_H G, \theta)$ with $\Sigma^k \times G$ equipped with the trivial connection. The reduction of F_{Σ^k} to Q will give us a map $\Sigma^k \to G/Q$ and thus the immersion $\Sigma^k \to G/P$. So when is θ flat? This amounts to differential equations on $[\![\theta_{\mathfrak{m}}]\!]$ of a universal character: for each $k \geq 0$ Calderbank–Diemer [3] define k-linear operators $\mu_k : \otimes^k C^\infty(\Sigma^k, H_1(\Sigma^k, [\mathfrak{g}])) \to C^\infty(\Sigma^k, H_2(\Sigma^k, [\mathfrak{g}]))$ with $\mu_0 = [\![R^{\hat{\theta}_{\mathfrak{h}}}]\!]$ and μ_1 the curved Bernstein–Gelfand-Gelfand operator of Čap–Slovák–Souček [5].

Theorem. *θ is flat if and only if*

$$\sum_{k\geq 0} \mu_k([\![\theta_{\mathfrak{m}}]\!], \ldots, [\![\theta_{\mathfrak{m}}]\!]) = 0. \tag{1}$$

Thus (1) is the Gauss–Codazzi–Ricci equation of our submanifold geometry. Since $\theta_{\mathfrak{m}}$ has negative weight and the operators μ_k preserve weight, the sum in (1) is finite. In fact, in both conformal submanifold geometry and projective hypersurface geometry, all summands vanish for $k \geq 3$.

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Configuration spaces of plane polygons, sub-Riemannian geometry and periodic orbits of inner and outer billiards

Serge Tabachnikov

This is a report on recent work by a number of authors concerning periodic trajectories of inner and outer billiards and using ideas from sub-Riemannian geometry and the theory of exterior differential systems [1, 2, 4, 6, 11].

The main motivation is an old conjecture that the set of periodic billiard trajectories has zero measure (with respect to the canonical invariant measure of the billiard transformation). This conjecture is open for about 25 years; it is relevant to the spectral geometry (the Weyl asymptotics of the eigenvalues of the Laplace operator). For 2-periodic billiard trajectories this result is easy, and for 3-periodic ones it was proved in different ways in [8, 9, 13, 12] (the first proof was computer-assisted).

On the other hand, there is an abundance of convex billiard tables possessing an invariant curve consisting of 2-periodic trajectories: any smooth curve of constant width bounds such a table. In [5], a billiard table is constructed possessing an invariant curve consisting of 3-periodic trajectories. A natural question is whether such examples are rare or common and how to construct them systematically. Each of the questions can be asked for outer billiards as well (see [3, 10] for a survey of outer – also known as dual – billiards); no results for outer billiards were previously known.

An n-periodic billiard trajectory is a plane n-gon, and the law of billiard reflection determines the directions of the billiard curve at the vertices of the polygon. This defines an n-dimensional distribution D on the space of non-degenerate plane n-gons. The distribution D is tangent to the level hypersurfaces of the perimeter length function; fixing the perimeter, we obtain an n-dimensional distribution on $2n-1$-dimensional manifold M of polygons with unit perimeter. A similar construction applies to outer billiards with the perimeter function replaced by the area one.

The first result is that D is totally non-integrable: the tangent space to M is spanned by D and the first commutators of the vector fields tangent to D. This result holds in the inner and the outer settings.

A billiard table possessing an invariant curve consisting of n-periodic trajectories can be interpreted as a curve $g(t) = (P_1(t), \ldots, P_n(t)) \subset M$ with $t \in [0, 1]$, tangent to D (such curves are called horizontal) and satisfying the monodromy condition: $(P_1(1), \ldots, P_n(1)) = (P_2(0), \ldots, P_n(0), P_1(0))$. Starting with a circular billiard, one can perturb the respective horizontal curves to obtain new billiard tables possessing an invariant curve consisting of n-periodic trajectories. One proves a theorem that, for each n and every rotation number k coprime with n, this construction yields a Hilbert manifold worth of billiard tables with an invariant curve consisting of n-periodic trajectories having rotation number k. This applies to both inner and outer billiards, see [1, 2, 4]. In particular, one has explicit formulas for outer billiards with an invariant curve consisting of 3-periodic trajectories [4]. A subtle point is to prove that the horizontal curves enjoy sufficient flexibility; as explained in [7], this is not true in general.

A problem arises whether invariant curves consisting of periodic trajectories can coexist for different periods (or for the same period but different rotation numbers). Of course, a circle or an ellipse provides an example of such coexistence; are there other examples? A negative answer would imply the famous Birkhoff conjecture: the only integrable plane billiard is the one inside an ellipse.

If the set of n-periodic trajectories has a non-empty interior then the distribution D admits a horizontal 2-dimensional disk. For $n = 3$, one can prove that such disks do not exist, in both inner and outer settings, and this implies that the set of 3-periodic inner or outer billiard trajectories is a null set. For inner billiards, this provides a new proof of a known result, see [1, 2, 6], and for outer ones, this is a new result proved in [4]. In a recent preprint [11], Tumanov and Zharnitsky obtain the same result for $n = 4$ in the outer set-up, unless the billiard table has four corners that form a parallelogram.

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Mean curvature of surfaces in $\mathbf{TS^2}$

Wilhelm Klingenberg
(joint work with Brendan Guilfoyle)

We consider here the neutral Kähler metric on $\mathbb{L} = \mathrm{TS}^2$, considered as the space of oriented affine lines in $\mathbb{R}^3$ [1] [2]. In terms of local holomorphic coordinates (ξ, η) on TS^2, this metric has expression:

$$ds^2 = \frac{2i}{(1+\xi\bar{\xi})^2}\left(d\eta\, d\bar{\xi} - d\bar{\eta}\, d\xi + \frac{2(\xi\bar{\eta} - \bar{\xi}\eta)}{1+\xi\bar{\xi}}\, d\xi\, d\bar{\xi}\right), \tag{1}$$

We will look at surfaces in $\mathbb{L}$ which arise as the graph of a local section of the bundle $\mathbb{L} \to \mathrm{S}^2$, that is, a map $\xi \to (\xi, \eta = F(\xi, \bar{\xi}))$.

Proposition 1. *The metric induced on the graph of a section by the Kähler metric is given in coordinates $(\xi, \bar{\xi})$ by*

$$\mathbb{G} = \frac{2}{(1+\xi\bar{\xi})^2}\begin{bmatrix} i\sigma & -\lambda \\ -\lambda & -i\bar{\sigma}\end{bmatrix}$$

with inverse

$$\mathbb{G}^{-1} = \frac{(1+\xi\bar{\xi})^2}{2\Delta}\begin{bmatrix} i\bar{\sigma} & -\lambda \\ -\lambda & -i\sigma\end{bmatrix},$$

where $\sigma = -\partial\bar{F}$, $\lambda = \mathbb{I}m\ (1+\xi\bar{\xi})^2\partial\left(F(1+\xi\bar{\xi})^{-2}\right)$ and $\Delta = \lambda^2 - \sigma\bar{\sigma}$. Here, and throughout, ∂ represents differentiation with respect to ξ.

In particular, the determinant of the induced metric is $|\mathbb{G}| = 4\Delta(1+\xi\bar{\xi})^{-4}$. Thus, the metric is lorentz if $\lambda^2 < \sigma\bar{\sigma}$, riemannian if $\lambda^2 > \sigma\bar{\sigma}$ and degenerate if $\lambda^2 = \sigma\bar{\sigma}$.

The area form of the induced metric is $\sqrt{|\mathbb{G}|}\, d\xi\wedge\bar{\xi}$, and the following proposition deals with stationary values of the area functional:

Proposition 2. *A surface* $\Sigma \hookrightarrow \mathbb{L}$ *which is given by the graph of a function* $\xi \to (\xi, \eta = F(\xi, \bar{\xi}))$ *is area-stationary iff*

$$i\partial\left(\frac{\lambda}{\sqrt{|\Delta|}}\right) - (1+\xi\bar{\xi})^2\bar{\partial}\left(\frac{\sigma}{(1+\xi\bar{\xi})^2\sqrt{|\Delta|}}\right) = 0.$$

Area-stationary surfaces have the following property:

Proposition 3. *On an area-stationary graph*

$$\sqrt{|\Delta|}\mathbb{G}^{jk}\nabla_j\nabla_k\sqrt{|\Delta|} = 2\lambda$$

where ∇ *is the Levi-Civita connection associated with the induced metric* $\mathbb{G}$*.*

Proof. This hinges on the (derived Codazzi-Minardi) identity:

$$-(1+\xi\bar{\xi})^2\bar{\partial}\left(\frac{\sigma}{(1+\xi\bar{\xi})^2}\right) = \partial\bar{\rho} + \frac{2\bar{F}}{(1+\xi\bar{\xi})^2}$$

where $\rho = (1+\xi\bar{\xi})^2\partial\left(F(1+\xi\bar{\xi})^{-2}\right)$. □

Let $f : \Sigma \to \mathbb{L}$ be an immersed surface. Then the mean curvature flow is given by the equation

$$\dot{f}^{\perp} = \left(\mathbb{G}^{jk}\nabla^{\mathbb{L}}_{\frac{\partial f}{\partial x^k}}\frac{\partial f}{\partial x^k}\right)^{\perp}$$

where $\nabla^{\mathbb{L}}$ is the Levi-Civita connection associated with the ambient metric on $\mathbb{L}$ and $^{\perp}$ is projection perpendicular to the tangent space of Σ.

Proposition 4. *For a graph in* $\mathbb{L}$*, the mean curvature flow is*

$$\dot{F} = \mathbb{G}^{jk}\partial_j\partial_k F + \frac{i\bar{\sigma}}{\Delta}\left((\sigma\xi - \bar{\rho}\bar{\xi})(1+\xi\bar{\xi}) + \bar{F} - \bar{\xi}^2 F\right).$$

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The fcc lattice and the cusped hyperbolic 4-orbifold of minimal volume

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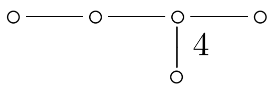
(joint work with Thierry Hild)

Let Q be a cusped hyperbolic n-orbifold of finite volume, that is, Q is the quotient of hyperbolic space H^n by a cofinite discrete group $\Gamma < Iso(H^n)$ containing parabolic elements. A result of Kazdan-Margulis-Heintze implies that the volume spectrum of all cusped hyperbolic n-orbifolds contains a minimum value $v_n > 0$. We discuss the problem to determine v_n and to describe all those n-orbifolds whose volumes realise v_n.

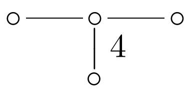
We present a fairly general method to deal with this problem and show that the quotient space Q_* of H^4 by the hyperbolic Coxeter group $\Gamma_*^4 = [4, 3^{2,1}]$ with diagram

Σ_* :



is the unique non-compact hyperbolic 4-orbifold of minimal volume. The orbifold Q_* is isometric to a hyperbolic Coxeter 4-simplex of volume v_4 equal to $\pi^2/1,440$ with precisely one vertex at infinity. Its vertex neighborhood is a cone over the euclidean tetrahedron Δ_{fcc} which is a fundamental domain for the action of the symmetry group of the famous fcc lattice given by the parabolic Coxeter group $\Gamma_{fcc} < \Gamma_*^4$ with diagram

Σ_{fcc} :



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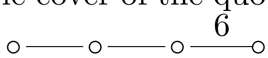
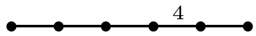
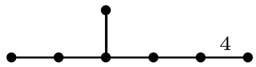
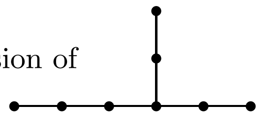
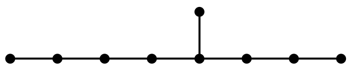
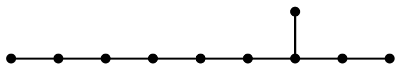
By a well known result of C. F. Gauss, the fcc packing is the unique lattice packing of E^3 with maximal density $\pi/\sqrt{18}$. Indeed, our methods are based on results about crystallographic groups and lattice packings in E^3 as well as horoball geometry in hyperbolic space. In particular, a conjugacy class of a subgroup of parabolic type in Γ gives rise to a canonical cusp in $Q = H^n/\Gamma$. It turns out that for a given sufficiently small upper volume the canonical cusp is maximal within the nested set of cusped neighborhoods.

Our theorem generalises the area minimality property of the triangle group $(2, 3, \infty)$. By [5], it is known that the quotient of H^2 by the group $\Gamma_*^2 = (2, 3, \infty)$ with fundamental triangle of non-zero angles $\pi/2, \pi/3$ is the unique 2-orbifold of area v_2 which equals to $\pi/6$.

Furthermore, R. Meyerhoff [4] showed that among the non-compact oriented hyperbolic 3-orbifolds the oriented double cover of the quotient of H^3 by the Coxeter group $\Gamma_*^3 = [3, 3, 6]$ with diagram  is of minimal volume. The methods which we develop allow us to conclude that the space $H^3/[3, 3, 6]$ is the hyperbolic 3-orbifold of minimal volume and as such is unique. Therefore, v_3 is equal to $\frac{1}{8}\,Л(\frac{\pi}{3}) \simeq 0.04229$ where Л denotes the Lobachevsky function.

Recently, T. Hild [2] extended the methods in order to resolve analoguous questions for cusped hyperbolic orbifolds of higher dimensions. The following table contains parts of his results. For $5 \leq n \leq 9$, he shows that the minimal volume n-orbifold is a quotient of H^n by a simplex Coxeter group Γ_*^n up to one exception: The group Γ_*^7 has a fundamental domain which arises by halfening the Coxeter simplex with flip-invariant diagram as shown in the table by means of the obvious hyperplane. The relevant values v_n were determined in [3] and are explicit rational multiples of Riemann's zeta function, a particular L-function, and some power of π in the even dimensional cases, respectively.

Recently, T. Hild [2] extended the methods in order to resolve analoguous questions for cusped hyperbolic orbifolds of higher dimensions. The following table

n	Γ_*^n	v_n
5		$\frac{7\ \zeta(3)}{46{,}080} \approx 1.83 \cdot 10^{-4}$
6		$\frac{\pi^3}{777{,}600} \approx 3.98 \cdot 10^{-5}$
7	$\mathbb{Z}_2$-extension of 	$\frac{\sqrt{3}\ L(4,3)}{1{,}720{,}320} \approx 9.46 \cdot 10^{-7}$
8		$\frac{\pi^4}{4{,}572{,}288{,}000} \approx 2.13 \cdot 10^{-8}$
9		$\frac{\zeta(5)}{22{,}295{,}347{,}200} \approx 4.65 \cdot 10^{-11}$

contains parts of his results. For $5 \le n \le 9$, he shows that the minimal volume n-orbifold is a quotient of H^n by a simplex Coxeter group Γ_*^n up to one exception: The group Γ_*^7 has a fundamental domain which arises by halfening the Coxeter simplex with flip-invariant diagram as shown in the table by means of the obvious hyperplane. The relevant values v_n were determined in [3] and are explicit rational multiples of Riemann's zeta function, a particular L-function, and some power of π in the even dimensional cases, respectively.

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Surfaces in three-dimensional Lie groups

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(joint work with Dmitry A. Berdinsky)

We give an exposition of the Weierstrass (or spinor) representation for surfaces in three-dimensional Lie groups. For the commutative Lie group $\mathbb{R}^3$ this is the known representation for surfaces in the three-dimensional Euclidean space studied since the middle of the 1990's and for the unit sphere in four-space $G = SU(2)$ it was earlier developed by the speaker (see [4]). Here we are mostly concerned with the Lie groups Nil (a nilpotent group), Sol (a solvable group), and $\widetilde{SL}_2$ endowed with Thurston's geometries.

The derivational equations for a surface $f : M \to G$ with a given conformal parameter z are written in terms of $\Psi = f^{-1}f \in \mathbf{g}$ where $\mathbf{g} = T_1G$ is the Lie algebra of G. These equations take the form

$$\partial\Psi^* - \bar{\partial}\Psi + \nabla_\Psi\Psi^* - \nabla_{\Psi^*}\Psi = 0, \quad \partial\Psi^* + \bar{\partial}\Psi + \nabla_\Psi\Psi^* + \nabla_{\Psi^*}\Psi = e^{2\alpha}Hf^{-1}(N),$$

where N is the unit normal vector, and expanding $\Psi = \sum_{k=1}^{3} Z_k e_k$ in the linear base e_1, e_2, e_3 for $\mathbf{g}$ we rewrite these equations in the form of the Dirac equation $\mathcal{D}\psi = 0$ where

$$\mathcal{D} = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right]$$

and $Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), Z_3 = \psi_1\bar{\psi}_2$. The induced metric takes the form $e^{2\alpha}\,dz\,d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)^2\,dz\,d\bar{z}$. For surfaces in $\mathbb{R}^3$ we have $U = V = \frac{1}{2}He^\alpha$ and for surfaces in $SU(2)$ we have $U = \bar{V} = \frac{1}{2}(H + i)e^\alpha$ where H is the mean curvature.

The Willmore functional for a surface in a general three-space equals $\int(H^2 + \widehat{K})d\mu$ where $\widehat{K}$ is the sectional curvature of the ambient space along the tangent plane and $d\mu$ is the induced measure. In terms of the Weierstrass representation for surfaces in $\mathbb{R}^3$ and $SU(2)$ the Willmore functional equals $4\int UV\frac{idz\wedge d\bar{z}}{2}$. Its relation to the spectral theory of the Dirac operator $\mathcal{D}$ pointed out by the speaker in 1995 was studied in the last years (see the survey [4]).

These methods allow us to obtain the analogue of the Weierstrass representation of minimal surfaces:

Theorem 1 ([2]). *For the Weierstrass representation of surfaces in Lie groups the potentials U and V take the form*

$$U = V = \frac{He^\alpha}{2} + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2) \quad \textit{for } G = \mathrm{Nil},$$

and

$$U = \frac{He^\alpha}{2} + i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right),$$

$$V = \frac{He^\alpha}{2} + i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right) \qquad \textit{for } G = \widetilde{SL}_2.$$

The potentials for the case of $G = \mathrm{Sol}$ are defined outside the zero measure set and this representation is more complicated [2].

Corollary 1 ([2])**.** *For minimal surfaces in G the spinor ψ meets the equations*

$$\bar\partial\psi_1 = \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_2,\ \partial\psi_2 = -\frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_1 \quad \textit{for } G = \mathrm{Nil},$$

$$\bar\partial\psi_1 = i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right)\psi_2,\ \partial\psi_2 = -i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right)\psi_1 \quad \textit{for } G = \widetilde{SL}_2,$$

$$\bar\partial\psi_1 = \frac{1}{2}\bar\psi_1^2\bar\psi_2,\ \partial\psi_2 = -\frac{1}{2}\bar\psi_1\bar\psi_2^2 \quad \textit{for } G = \mathrm{Sol}.$$

Let us define the (spinor) energy functional for compact oriented surfaces without boundary as

$$E(M) = \int_M UV\frac{idz\wedge d\bar z}{2}.$$

For surfaces in $\mathbb{R}^3$ and $SU(2)$ it is a quarter of the Willmore functional however in other cases it differs.

Corollary 2 ([2])**.** *The spinor energy $E(M)$ of a surface M equals*

$$\frac{1}{4}\int_M\left(H^2 + \frac{\widehat K}{4} - \frac{1}{16}\right)d\mu \quad \textit{for } G = \mathrm{Nil},$$

$$\frac{1}{4}\int_M\left(H^2 + \frac{5}{16}\widehat K - \frac{1}{4}\right)d\mu \quad \textit{for } G = \widetilde{SL}_2.$$

The Hopf quadratic differential equals $A\,dz^2 = (\nabla_{f_z}f_z, N)dz^2$ and, by the Hopf theorem, for surfaces in $\mathbb{R}^3$ it is holomorphic if and only if the surface has constant mean curvature.

Methods of the general Weierstrass representation allows us to derive the following

Theorem 2 ([2])**.** *1) A surface in* Nil *has constant mean curvature if and only if the quadratic differential $\widetilde A\,dz^2 = \left(A + \frac{{Z_3}^2}{2H+i}\right)dz^2$ is holomorphic.*

2) If surface in $\widetilde{SL}_2$ has constant mean curvature then the quadratic differential $\widetilde A\,dz^2 = \left(A + \frac{5}{2(H-i)}Z_3^2\right)dz^2$ is holomorphic.

In both cases the e_3 axis is chosen to be the axis of a rotational symmetry.

Thus for the special cases of surfaces in Nil and $\widetilde{SL}_2$ we obtain another derivation to the result by Abresch–Rosenberg that certain generalizations of the Hopf differential for constant mean curvature (CMC) surfaces in three-spaces with four-dimensional isometry group are holomorphic [1]. Their differential is equal to $(H + i\tau)\widetilde A\,dz^2$ where τ is the bundle curvature of the one-dimensional bundle over two-manifold with constant curvature such that this fibration is locally isometric to the three-dimensional ambient space.

Recently Fernandez and Mira showed that for all other ambient spaces with four-dimensional isometry group which are different from the space forms and

Nil there are non-CMC surfaces for which the Abresch–Rosenberg differential is holomorphic.

Let us return to the spinor energy functional $E(M)$. It appears that for surfaces in Nil it resembles the Willmore functional in many geometrical aspects. In particular, we have

Theorem 3 ([3]). *For all CMC spheres in* Nil *the spinor energy is equal to* π.

Theorem 4 ([3]). *Given a closed surface* M *in* Nil *obtained by revolving a curve* γ *in the half-lane* $\{\rho \geq 0, z \in \mathbb{R}\} = \mathrm{Nil}/SO(2)$ *around the* z*-axis, the spinor energy of* M *equals*

$$E(M) = \frac{\pi}{8}\int_\gamma \left(\dot{\sigma} - \frac{\sin\sigma}{\rho}\right)^2 \sqrt{4\rho^2+\rho^4}ds + \frac{\pi\chi(M)}{2} \tag{1}$$

where σ *is the angle between* γ *and* $\frac{\partial}{\partial\rho}$ *(in the metric* $d\rho^2 + \frac{4\rho^2}{4\rho^2+\rho^4}dz^2$ *for which the projection* $\mathrm{Nil} \to \mathrm{Nil}/SO(2)$ *is a submersion) and* $\chi(M)$ *is the Euler characteristic of* M. *Moreover if* $\dot{\sigma} = \frac{\sin\sigma}{u}$ *everywhere on the surface then it is a CMC sphere.*

Corollary 3. *For spheres of revolution* $E(M) \geq \pi$ *and the equality is attained exactly at CMC spheres. For tori of revolution* $E(M) > 0$.

We remark that for surfaces in $\mathbb{R}^3$ the CMC spheres are the round spheres and they give solutions to the isoperimetric problem for all volumes, the Willmore functional attains on these spheres minimal possible value which is 4π (i.e., $E = \pi$) and these are all closed umbilic surfaces in $\mathbb{R}^3$.

It is conjectured that the CMC spheres are isoperimetric surfaces in Nil and actually that holds for small volumes. The spinor energy functional is constant on the CMC spheres, attains on them its minimal possible value for surfaces of revolution (and we think for all surfaces) and these spheres are not umbilic but another similar quantity which is $(\dot{\sigma} - \frac{\sin\sigma}{\rho})$ vanishes exactly on them.

We think that the spinor energy functional is the right generalization of the Willmore functional for surfaces in Lie groups since it respects solutions to the isoperimetric problem (at least for small volumes). We have a guess that in general the isoperimetric surfaces in homogeneous spaces are detected as kind of instantons for a certain functional similar to the Willmore functional and this functional is equal on these surfaces to some topological term. (We remark that for the CMC spheres in $S^2 \times \mathbb{R}$ we have $\int_M (H^2 + \widehat{K} + 1)d\mu = 16\pi$.)

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