

Report No. 46/2006

**Arbeitsgemeinschaft mit aktuellem Thema:
Twisted K-Theory**

Organised by
Ulrich Bunke (Göttingen)
Dan Freed (Austin)
Thomas Schick (Göttingen)

Oktober 8th – Oktober 14th, 2006

ABSTRACT. The “Arbeitsgemeinschaft mit aktuellem Thema ‘Twisted K-Theory’ ” gave an introduction to several aspects of twisted K-theory. It started with a couple of different definitions of twisted K-theory, suitable in situations of varying complexity from spaces to topological stacks. Then the AG presented tools of calculations like Chern characters with values in the corresponding twisted ordinary cohomology theories, and example calculations e.g. for Lie groups, or via T-duality. The program culminated in the theorem of Freed-Hopkins-Teleman calculating equivariant twisted K-theory of simple Lie groups in term of the Verlinde algebra of loop group representations.

Mathematics Subject Classification (2000): 18xx, 19xx, 22xx, 55xx, 57xx, 79xx.

Introduction by the Organisers

The Arbeitsgemeinschaft mit aktuellem Thema “Twisted K-theory”, organized by Ulrich Bunke (Georg-August-Universität Göttingen), Dan Freed (University of Texas at Austin), and Thomas Schick (Georg-August-Universität Göttingen), was held from October 8 through October 14, 2006.

Ordinary (co)homology theories always come along with twisted versions of themselves; the most basic example is cohomology twisted by the orientation bundle, which shows up when one discusses Poincaré duality (and more general push-forward maps) for non-orientable manifolds.

Twisted K -theory is important from this point of view. Even if a manifold is orientable in the ordinary sense, its K -theory does not satisfy Poincaré duality. However, this is the case if one considers twisted K -theory (one has to twist with the bundle of complex Clifford algebras of the cotangent bundle). In fact, some

constructions of twisted cohomology theories are quite classical and can be done in the context of parametrized stable homotopy theory (which on the other hand is itself constantly developing further).

The modern interest in twisted K -theory, however, stems from mathematical physics, in particular from string theory. In this theory D -branes are objects whose charges are measured, in the presence of a B-field, by twisted K -theory. The topological backgrounds of B-fields β on a space X are classified by three dimensional integral cohomology classes. Representatives of the B-field (which were called twists during the Arbeitsgemeinschaft) are precisely the data needed to define twisted K -theory $K^\beta(X)$. In one version of the theory one associates to a twist a non-commutative C^* -algebra whose K -theory is by definition the twisted K -theory.

Twisted K -theory in mathematics has evolved to an interdisciplinary area which combines elements of topology, non-commutative geometry, functional analysis, representation theory, mathematical physics and other. In the Arbeitsgemeinschaft we presented various aspects of the foundations of twisted K -theory and the key calculations. We discussed the construction of twisted equivariant K -theory in different contexts (e.g. homotopy theory, non-commutative geometry, groupoids or stacks) and the verification of the basic functorial properties.

In order to get used to the definitions we made example calculations of twisted K -theory using methods from algebraic topology (Mayer-Vietoris sequences and some spectral sequences). The Umkehr- (or integration or Gysin map) for twisted K -theory is of particular importance and was illustrated through an interpretation of the classical Borel-Weil-Bott theorem. The culmination of the Arbeitsgemeinschaft was the calculation of the equivariant twisted K -theory of compact Lie groups due to Freed-Hopkins-Teleman and the interpretation of this result in the context of representation theory of loop groups.

In the string theory context, some aspects of mirror symmetry are reflected in twisted K -theory; under certain situations a non-commutative space-time (i.e. a space with B-field) will have a dual with isomorphic K -theory, possibly in shifted degrees. The mathematical formulation of this is T -duality, which was worked out at the AG, and was studied also as a computational tool.

In connection with T -duality, but also in the equivariant situation not only ordinary spaces but more singular objects naturally show up. This results in the need to work out equivariant twisted K -theory, twisted K -theory for orbifolds, and for even more singular spaces. A convenient framework to develop this in the necessary generality is the language of stacks, which was introduced and used during the AG.

A Chern character was constructed which relates twisted K -theory to twisted (de Rham) cohomology. During the AG, it was worked out that sheaf theory is important here and developments in twisted K -theory are in fact topological versions of similar results and constructions in algebraic geometry.

An interesting feature of the definition of twisted K -theory in terms of cycles and relations is that those cycles appear naturally in geometric and analytic constructions. As mentioned above, starting from representations of compact Lie groups, an explicit construction of cycles given by families of Dirac type operators was discussed in connection with the Borel-Weil-Bott theorem. In a more elaborate way, this also works for (projective) representations of loop groups, as was first discovered in the physics related literature. Using this idea the calculation of the equivariant twisted K -theory of a compact Lie group G (acting on itself by conjugation) by Freed-Hopkins-Teleman (FHT) can be explicitly interpreted in terms of cycles. In this way the twisted K -theory is identified with the K -group of projective positive energy representations of the loop group $\mathcal{L}G$. The twist corresponds to the “level” of the representation. It was one of the major goals of the AG to prove this FHT theorem.

It turns out that the twisted K -theory on the one side, and the K -group of projective positive energy representations on the other side, both have a subtle products (in K -theory the Pontryagin product induced by the multiplication map, and the Fusion product (Verlinde algebra) on the other side), and the FHT-isomorphism respects these multiplications. The construction of the K -theoretic product was addressed, but because of lack of time we were not able to prove multiplicativity of the FHT-isomorphism. In the Arbeitsgemeinschaft we actually discussed a related product for the twisted K -theory of orbifolds with is closely related to the quantum product in orbifold cohomology.

Altogether, there were 17 talks by the participants, two sessions where questions left open during the talks were discussed, and ample free interaction between the participants.

The conference was attended by 45 participants coming mainly from all over Europe, Northern America and Australia. It is a pleasure to thank the institute for providing a pleasant and stimulating atmosphere.

Workshop: Arbeitsgemeinschaft mit aktuellem Thema: Twisted K-Theory

Table of Contents

Jesper Grodal	
<i>Talk 1: Equivariant K-theory</i>	2763
José Manuel Gómez	
<i>Talk 2: Twisted K-theory - Basic Definitions</i>	2764
Johannes Ebert	
<i>Talk 3: Equivariant twisted K-theory</i>	2765
Michael Joachim	
<i>Talk 4: Cubic Dirac operators, Thom isomorphisms, and the orbit correspondence</i>	2768
Michael Anthony Hill	
<i>Talk 5: The twisted K-homology of simple Lie groups</i>	2769
Niko Naumann	
<i>Talk 6: Introduction to topological and differentiable stacks</i>	2771
Jeffrey Giansiracusa	
<i>Talk 7: The Twisted K-theory of topological stacks</i>	2772
Sebastian Goette, Guido Kings	
<i>Talk 8: The Twisted Chern Character via Noncommutative Geometry</i> ..	2777
Elmar Schrohe	
<i>Talk 9: The Chern Character for Twisted K-theory of Orbifolds</i>	2777
Moritz Wiethaup	
<i>Talk 10: Twisted K-theory via parametrized homotopy theory and twisted integration</i>	2779
Arthur Bartels	
<i>Talk 11: Products in twisted K-theory</i>	2782
Nathalie Wahl	
<i>Talk 12: Twisted K-theory and T-duality</i>	2784
Kate Gruher	
<i>Talk 13: Equivariant twists of compact Lie groups</i>	2786
Gregory D. Landweber	
<i>Talk 14: Construction of twisted K-theory classes from positive energy representations</i>	2788

Markus Spitzweck

Talk 15: The FHT-Theorem I 2791

Jochen Heinloth

Talk 16: The FHT-theorem II: Computation of $K_G^{\tau}(G)$ (after Freed, Hopkins and Teleman) 2794

André Henriques

Talk 17: The FHT-Theorem III 2795

Alexander Alldridge, Jeffrey Giansiracusa

Appendix: Contravariant functoriality for twisted K-theory of stacks ... 2800

Abstracts

Talk 1: Equivariant K -theory

JESPER GRODAL

I introduced equivariant K -theory as an example of an equivariant cohomology theory. I started by giving the axioms of a G -equivariant cohomology theory for G a compact group, as e.g., presented in [1]. I gave several examples of equivariant cohomology theories. These included how to obtain a Borel equivariant cohomology theory from a non-equivariant cohomology theory, Bredon equivariant cohomology, as well as of course equivariant K -theory. I described basic properties of equivariant K -theory, such as the equivariant K -theory of homogeneous spaces, module structure over the representation ring, etc. I then went on to state the Atiyah-Segal completion theorem [2], illustrated with the example of Σ_3 , showing the difference between equivariant K -theory and its completion.

The next part of my talk was on the Thom homomorphism which I described in some detail following [3]. I stated the Thom isomorphism theorem, and used this to write up the six term exact sequence in equivariant K -theory, establishing that equivariant K -theory is an equivariant cohomology theory on the category of finite G -CW-complexes.

I then went on to describe the dual C^* -algebra picture, following the book of Blackadar [4]. I recalled the notion of a G - C^* -algebra, and defined the equivariant K -theory of a G - C^* -algebra. I briefly mentioned the equivariant Swan theorem, linking topological equivariant K -theory and equivariant K -theory for C^* -algebras.

I explained how the equivariant K -theory of a G - C^* -algebra can be identified with the non-equivariant K -theory of the corresponding crossed-product C^* -algebra, illustrating the usefulness of the methods of non-commutative topology.

Finally I mentioned that equivariant K -theory can naturally be extended to an equivariant cohomology theory on all G -CW-complexes by exhibiting a suitable G -classifying space, but time prevented me from providing any details of these constructions.

REFERENCES

- [1] J. P. May et al. *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, No. 91.
- [2] M. Atiyah and G. Segal *Equivariant K -theory and completion*. J. Differential Geometry, 3:1–18, 1969.
- [3] G. Segal *Equivariant K -theory* Inst. Hautes Études Sci. Publ. Math., (34):129-151, 1968.
- [4] B. Blackadar *K -theory for operator algebras*. volume 5 of MSRI Publications, Cambridge University Press, Second Edition, 1998.

Talk 2: Twisted K-theory - Basic Definitions

JOSÉ MANUEL GÓMEZ

In general cohomology theories come along with twisted versions of them. If h^* is a cohomology theory then by Brown's representation theorem, h^* is represented by an Ω -spectrum $\{E_n\}$ i.e, a sequence of spaces together with homotopy equivalences $E_n \rightarrow \Omega E_{n+1}$. A multiplicative cohomology theory such as K-theory is represented by a ring spectrum. If E is a ring spectrum then $Z = E_0$ is an E_∞ -ring space and $\pi_0(Z)$ is a ring. If we write $Z = \coprod_{\alpha \in \pi_0(E_0)} Z_\alpha$ then we can consider $Z_\otimes = \coprod_{\alpha \in \pi_0(E_0)^\times} Z_\alpha$. Z_\otimes is an infinity loop space and the corresponding spectrum is the spectrum of units associated to h^* . In general BZ_\otimes classifies the twistings for h^* .

For the case of K-theory we have the representing spectrum $\{E_n\}$ is $E_n = BU \times \mathbb{Z}$ if n is even and $E_n = U$ if n is odd. In this case we have that $Z_\otimes \simeq \{\pm 1\} \times BU_\otimes$ and thus for a spaces X twisting in K-theory are classified by homotopy classes of maps $X \rightarrow B(\{\pm 1\} \times BU_\otimes)$. We have a factorization as infinite loop spaces $BU_\otimes = K(\mathbb{Z}, 2) \times BSU_\otimes$. It follows that twistings in K-theory are classified by maps to $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_\otimes$. Thus for a compact space X we have twistings corresponding to elements in $H^1(X, \mathbb{Z}/2)$, $H^3(X, \mathbb{Z})$ and $[X, B(BSU_\otimes)]$.

In this talk, I presented a model for twisted K-theory for the twistings corresponding to $H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z})$ following [3]. This model is based on bundles of projective spaces $P \rightarrow X$ whose fibers are of the form $\mathbb{P}(H)$ and have structure group $PU(H)$ with the compact open topology (Here H is a complex infinite dimensional separable Hilbert space) and that come along with an unitary involution in each fiber P_x . An involution of projective spaces P_x expresses it as the joint of two disjoint projective spaces \mathbb{P}_x^+ , \mathbb{P}_x^- which we assume to fit together as to form tame embeddings (i.e. one that locally looks like the inclusion $X \times \mathbb{P}(H_o) \hookrightarrow X \times \mathbb{P}(H)$ for a closed Hilbert subspace H_o of H). To such P we can associate a double cover and hence a cohomology class ξ_P . Also via the Dixmer-Douady class we can associate to P a third cohomology class η_P . We show that P is uniquely determined, up isomorphism, by ξ_P, η_P . In a similar way one can classify the automorphisms of such a bundle $P \rightarrow X$. In fact one can associate to each connected component of the automorphism group of $P \rightarrow X$ a class in $H^2(X, \mathbb{Z})$ that uniquely determines it.

If for a bundle $P \rightarrow X$ the structure group can be reduced to $PU(H)$ with norm topology, then as this group acts continuously by conjugation on $Fred(H)$, then to the bundle $P \rightarrow X$ we can associated a bundle $Fred(P) \rightarrow X$ whose fiber at x is $Fred(H_x)$ with $P_x = \mathbb{P}(H_x)$. (The space $Fred(H_x)$ is canonically determined even though H_x is not). In this situation we can define $K_P^0(X) = \text{Set of homotopy classes of sections of } Fred(P) \rightarrow X$. In the general case we show how to find a space $Fred'(H)$ homotopy equivalent to $Fred(H)$ on which $PU(H)$ with compact open topology acts continuously. Thus as in the previous situation we can associate to $P \rightarrow X$ a bundle $Fred'P \rightarrow X$ and define $K_P^0(X) = \text{Set of homotopy classes of sections of } Fred'P \rightarrow X$.

In the talk it was discussed how to define the higher groups, the existence of Bott periodicity theorem and Mayer-Vietoris long exact sequence in the twisted case. Also we discussed the Atiyah-Hirzebruch spectral sequence and the computation of the d_3 differential in this case. To finish we show how the higher differentials after tensoring with \mathbb{Q} of this spectral sequence in general are not trivial and are given by the Massey higher products.

REFERENCES

- [1] M. F. Atiyah. K-theory. Benjamin, New York, 1967.
- [2] M. F. Atiyah and M.I. Hopkins. A variant of K-theory, K_Σ . In *Topology, Geometry, and Quantum Field Theory*. Ed. U. Tillmann, Cambridge Univ. Press, 2004.
- [3] M. F. Atiyah and G. Segal. Twisted K-theory. *Ukr. Mat. Visn.* 1 (2004), no. 3, 287–330; translation in *Ukr. Math. Bull.* 1 (2004), no. 3, 291–334 55N15 (19Kxx 46L80 55N91)
- [4] M. F. Atiyah and G. Segal. Twisted K-theory and Cohomology.
- [5] U. Bunke and T. Schick. On the Topology of T-duality. *Rev. Math. Phys.* 17(1):77-112,2005
- [6] J. Rosenberg. Continuous-trace algebras from the bundle theoretic point of view. *J. Austral Math. Soc. Ser. A.* 47(3): 368-381, 1981.

Talk 3: Equivariant twisted K-theory

JOHANNES EBERT

The main source for this talk was [1]. The goals are the definition of equivariant twisted K -theory and the classification of equivariant twists. Throughout this talk, G will be a compact Lie group and X a left G -space.

Definition 1. A G -equivariant projective bundle on X is a triple $(P; \pi; \rho)$, where $\pi : P \rightarrow X$ is a projective bundle with structural group $\mathbb{P}U(\mathcal{H})$, endowed with the compact-open-topology; $\rho : G \times P \rightarrow P$ an action by projective isometries, such that π is G -equivariant. There are additional technical conditions.

A G -equivariant projective bundle P is called *stable* or an *equivariant twist* if $P \otimes L^2(G) \cong P$ as a projective G -bundle.

Let $P \rightarrow X$ be an equivariant twist on X . Let \mathcal{H} be a separable stable G -Hilbert space (i.e. $\mathcal{H} \otimes L^2(G) \cong \mathcal{H}$). Let $Q \rightarrow X$ be the associated G -equivariant $\mathbb{P}U(\mathcal{H})$ -principal bundle. Let $\text{Fred}(P) := Q \times_{\mathbb{P}U(\mathcal{H})} \text{Fred}(\mathcal{H})$, which is a G -equivariant bundle on X (one has to choose an appropriate model for the space $\text{Fred}(\mathcal{H})$).

Definition 2. Let X be a G -space and let $P \rightarrow X$ be a G -equivariant twist. Then the *equivariant twisted K-theory* is $K_{P;G}^0(X) := \pi_0(\text{Sect}(X; \text{Fred}(P))^G)$.

We can form the fiberwise loop space $\Omega^n \text{Fred}(P) := Q \times_{\mathbb{P}U(\mathcal{H})} \Omega^n \text{Fred}^0(\mathcal{H})$. For $n \geq 0$, define $K_{P;G}^{-n}(X) := \pi_0(\text{Sect}(X; \Omega^n \text{Fred}(P))^G)$.

There is an alternative definition of the K -theory in terms of algebraic K -theory. Let X be compact. Let \mathcal{K} be the C^* -algebra (without unit) of compact operators on \mathcal{H} and consider the bundle of C^* -algebras $Q \times_{\mathbb{P}U(\mathcal{H})}$. The algebra $\Gamma(\mathcal{K}_P)$ of sections $X \rightarrow \times_{\mathbb{P}U(\mathcal{H})}$ is a C^* -algebra with a G -action.

Proposition 3. *For all $n \geq 0$, there is a natural isomorphism $K_{G;P}^{-n}(X) \cong K_n^G(\Gamma(\mathcal{K}_P))$.*

For the notion of G -equivariant K -theory of C^* -algebras, see [2], chapter 11. Because Bott periodicity holds for the equivariant K -theory of C^* -algebras, we conclude that equivariant twisted K -theory has Bott periodicity. This can be used to define equivariant twisted K -theory functors $K_{G;P}^n$ for all values of n .

If X is a G -space, the *Borel construction* is the fiber bundle $E(G; X) := EG \times_G X \rightarrow BG$. The Borel-equivariant cohomology groups of X with coefficients in R are defined by $H_G^n(X; R) := H^n(E(G; X); R) = [X; C(EG; K(R; n))]^G$. Let $\text{Pic}_G(X)$ denote the group of G -isomorphism classes of complex G -line bundles on X and $\text{Proj}_G(X)$ the group of isomorphism classes of equivariant twists on X . The group $\text{Pic}_G(X)$ is of interest in this context because it is the group of G -homotopy classes of automorphisms of an equivariant twist and thus it acts naturally on the K -groups. An element in $\text{Pic}_G(X)$ has a G -equivariant Chern class in $H_G^2(X; \mathbb{Z})$; similarly for $\text{Proj}_G(X)$.

Theorem 4. *The homomorphisms defined by the invariants*

- (1) $\text{Pic}_G(X) \rightarrow H_G^2(X; \mathbb{Z})$,
- (2) $\text{Proj}_G(X) \rightarrow H_G^3(X; \mathbb{Z})$.

are isomorphisms.

The second statement of the theorem is already interesting if X is a point. The group $\text{Proj}_G(*)$ is isomorphic to the group $\text{Ext}(G; \mathbb{T})$ of central extensions $\mathbb{T} \rightarrow \tilde{G} \rightarrow G$. The theorem says that this group is isomorphic to $H^3(BG; \mathbb{Z})$, which is a finite group by a famous theorem of Cartan (see [3], p.114).

The proof of Theorem 4 employs the notion of *hypercohomology* of simplicial spaces. Let X_\bullet be a simplicial topological space and let A be an abelian topological group. Let $sh(A) \rightarrow \mathcal{F}^{p,0} \rightarrow \mathcal{F}^{p,1} \rightarrow \mathcal{F}^{p,2} \dots$ be an injective resolution of the sheaf $sh(A)$ on X_p . Let $C^{p,q} := \Gamma(X_p; \mathcal{F}^{p,q})$ (global sections). The groups $C^{p,q}$ form a double complex, the vertical differentials coming from the resolution and the horizontal differentials from the face maps of X_\bullet . The hypercohomology $\mathbb{H}(X_\bullet; sh(A))$ of X_\bullet is by definition the cohomology of the total complex. There is a spectral sequence $E_1^{p,q} = H^q(X_p, sh(A)) \Rightarrow \mathbb{H}^{p+q}(X_\bullet; sh(A))$. If either X or A is discrete, then we obtain the usual cohomology of the geometric realization of X_\bullet with coefficients in A .

If G acts on X , then we can build the *translation category* $G \int X$. The *nerve* $N_\bullet G \int X$ is then a simplicial topological space, whose geometric realization is homotopy-equivalent to $EG \times_G X \rightarrow BG$.

If G is compact and $A = \mathbb{R}$, then the hypercohomology $\mathbb{H}^*(N_\bullet(G \int X); sh(\mathbb{R}))$ is trivial in positive degrees. This is a consequence of the existence of invariant integration on compact groups. Consequently; $H^p(EG \times_G X; \mathbb{Z}) \cong \mathbb{H}^p(N_\bullet(G \int X); sh(\mathbb{Z})) \cong \mathbb{H}^{p-1}(N_\bullet(G \int X); sh(\mathbb{T}))$ for $p \geq 2$.

The first assertion in Theorem 4 and the injectivity part of the second one follow

from a careful study of the spectral sequence. If $X = *$, then the surjectivity of $\text{Proj}_G(*) \rightarrow H_G^3(*; \mathbb{Z})$ also follows from the spectral sequence.

The surjectivity for general X is more difficult. For this, one constructs a universal G -space \mathcal{Q} and a G -projective bundle on \mathcal{Q} , whose invariant defines a G -map $\mathcal{Q} \rightarrow C(EG; K(\mathbb{Z}; 3))$. One shows that (for a suitable \mathcal{Q}), this map is a G -homotopy equivalence, which implies the statement of the theorem. To do this, it suffices by a theorem of James and Segal, to prove that the induced map on H -fixed points $\mathcal{Q}^H \rightarrow C(EG; K(\mathbb{Z}; 3))^H \simeq C(BH; K(\mathbb{Z}; 3))$ is an ordinary homotopy equivalence for any closed subgroup H of G .

The construction of \mathcal{Q} is as follows. Let \mathcal{O} be the orbit category of the group G (the objects are the homogeneous spaces G/H and the morphism are the G -maps) and let $F : \mathcal{O} \rightarrow \mathcal{TOP}$ be a contravariant functor. From this data one constructs a related topological category \mathcal{O}_F . This category has a left- G -action, which defines a left G -action on the classifying space $B(\mathcal{O}_F)$. There is an obvious homotopy equivalence $F(G/H) \rightarrow B(\mathcal{O}_F)^H$. In order to prove the theorem, we introduce the functor $Q(G/H) := \coprod_{\mathcal{H} \in \text{Ext}(H; \mathbb{T})} B((\mathbb{P}U(\mathcal{H}))^H)$. As it stands, Q is not functorial because of the usual coherence problems, but $Q(G/H)$ can be replaced by homotopy-equivalent spaces which are functorial on \mathcal{O} . There is also a functorial projective bundle on $Q(G/H)$. The whole construction gives the desired space \mathcal{Q} and the projective G -bundle on it. Furthermore, the homotopy type of $Q(G/H)$ is quite straightforward to determine (using the statement of the theorem if $X = *$, which was established earlier) and it can be checked that the map $\mathcal{Q}^H \simeq Q(G/H) \rightarrow C(BH; K(\mathbb{Z}; 3))$ is an ordinary homotopy equivalence. This completes the proof of Theorem 4.

An important and interesting example of a G -space is the group G itself, acted on by conjugation. For the classification of equivariant twists, we need to compute the groups $H_G^k(G)$ for $k \leq 3$. This is done by the Leray-Serre spectral sequence of the fibration $G \rightarrow E(G; G_{\text{conj}}) \rightarrow BG$. We restrict to the case that G is semisimple and connected. Then $\pi_1(G)$ is finite and $H^1(G) = H^1(BG) = H^2(BG) = 0$. The Leray-Serre spectral sequence yields a split-exact sequence

$$0 \rightarrow H^3(BG) \rightarrow H_G^3(G_{\text{conj}}) \rightarrow H^3(G).$$

If G is also simply-connected, then the result is even simpler: the map

$$\text{Proj}_G(G) \cong H_G^3(G_{\text{conj}}) \rightarrow H^3(G)$$

is an isomorphism.

There is an interesting relation to the loop group $\mathcal{L}G$ of G . Let $\mathcal{P}G$ be the space of all smooth maps $f : \mathbb{R} \rightarrow G$, such that $t \mapsto f(t + 2\pi)f(t)^{-1}$ is constant. The loop group $\mathcal{L}G$ acts from the right on $\mathcal{P}G$ by pointwise multiplication. The map $\mathcal{P} \rightarrow G; f \mapsto f(2\pi)f(0)^{-1}$ makes \mathcal{P} into a $\mathcal{L}G$ -principal bundle. Moreover, G acts from the left on \mathcal{P} , and the bundle map $\mathcal{P}G \rightarrow G$ is G -equivariant. Thus there is a classifying map $\lambda : E(G; G_{\text{conj}}) \rightarrow B\mathcal{L}G$. A surprising, but elementary result is the following:

Proposition 5. *If G is connected, then λ is a homotopy equivalence.*

Now let $\rho : \mathcal{L}G \rightarrow \mathbb{P}U(H)$, H some Hilbert space, be a projective representation. The invariant of the G -equivariant projective bundle $\mathcal{P} \times_{\mathcal{L}G} \mathbb{P}(\mathcal{H}) \rightarrow G$ is given by the homotopy class of the composition $G \rightarrow B\mathcal{L}G \rightarrow B\mathbb{P}U(\mathcal{H}) \simeq K(\mathbb{Z}; 3)$. The projective representation ρ determines an extension of the loop group, and its underlying line bundle gives a class $u_\rho \in H^2(\mathcal{L}G)$, whose relation to $B\rho \in H^3(B\mathcal{L}G)$ is as follows. From a consideration of the Leray-Serre spectral sequence of $\mathcal{L}G \rightarrow E\mathcal{L}G \rightarrow B\mathcal{L}G$ one concludes that u lies in the domain of the partially defined map $d_3 : H^2(\mathcal{L}G) \rightarrow H^3(B\mathcal{L}G)$ and the image is $B\rho$.

These arguments boil down the determination of the cohomology class of an equivariant twist to an understanding of the inclusion $G \rightarrow B\mathcal{L}G$ in cohomology.

REFERENCES

- [1] M. F. Atiyah, G. Segal: *Twisted K-Theory*; preprint, arXiv:math.KT/0407054.
- [2] B. Blackadar: *K-Theory for Operator Algebras*, Mathematical Science Research Institute Publications, 5, Springer, 1986.
- [3] J. Dupont: *Curvature and characteristic classes*, Lecture Notes in Mathematics 640, 1979.

Talk 4: Cubic Dirac operators, Thom isomorphisms, and the orbit correspondence

MICHAEL JOACHIM

The goal of the talk was to introduce the cubic Dirac operator acting on spinors on a compact Lie group G , and to demonstrate how it gives rise to elements in twisted G -equivariant K -theory of \mathfrak{g}^* , the dual of the Lie algebra of G . To do so we first gave a very quick review of the theory of Clifford algebras and their representations. We then introduced all the relevant notions to actually define the cubic Dirac operator D_0 , gave its definition and emphasized how it differs from the classical Dirac operator. Using character theory we explicitly computed the kernel of the μ -shifted cubic Dirac operator $D_\mu = D_0 + \mu$ ($\mu \in \mathfrak{g}^*$) to see that the family of kernels piece together to give a vector bundle $\text{Ker}D$ over a co-adjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. Finally we interpreted the result in terms of G -equivariant twisted K -theory; in particular we showed that the family D of shifted cubic Dirac operators represents an element $[D]$ in the G -equivariant twisted K -theory of \mathfrak{g}^* and that $[D]$ is the image under the Thom push forward map (induced by $\mathcal{O} \subset \mathfrak{g}^*$) of the isomorphism class of a suitable line bundle over \mathcal{O} which is canonically obtained from the bundle $\text{Ker}D$. The source for this talk was [1].

REFERENCES

- [1] D. Freed, M.J. Hopkins, and C. Teleman *Loop groups and twisted K-theory II*, <http://arxiv.org/math.AT/0511232>

Talk 5: The twisted K -homology of simple Lie groups

MICHAEL ANTHONY HILL

In this talk, we illustrate how to compute the twisted K -homology of the simple, connected, simply connected compact Lie groups, largely following Douglas' computation thereof [3]. Since $H^3(G; \mathbb{Z}) = \mathbb{Z}$ and the cohomology and K -theory of these spaces are nice, the simple, simply connected Lie groups form a useful starting point for computations. Our goal is to prove part of one of the main results of [3]: If G is a simple, simply connected, compact Lie group of rank n and $\tau \in H^3(G)$ is non-zero, then

$$K_*^\tau(G) = E(y_1, \dots, y_{n-1}) \otimes \mathbb{Z}/c(G, \tau),$$

where $E(y_1, \dots, y_{n-1})$ denotes the exterior algebra on classes y_1 through y_{n-1} and $c(G, \tau)$ is an integer depending on G and τ .

We begin rationally, running the twisted Atiyah-Hirzebruch spectral sequence to compute the rational twisted K -theory of a space [1]. We know that $H(G; \mathbb{Q})$ is an exterior algebra on a class in dimension 3 tensored with other exterior factors. If our chosen twisting $\tau \in H^3(G)$ is non-zero, then the analysis of the d_3 differential in [1] shows that the Atiyah-Hirzebruch E_2 term is an acyclic complex, since rationally, $d_3(x) = -x \smile \tau$. This implies that if we choose a non-zero twisting class, then the twisted K -theory of G is all torsion for G of our form.

For technical reasons, it will be easier for us to compute the twisted K -homology of G , rather than the twisted K -cohomology. To define twisted K -homology, we consider bundles over a space X with fiber the K -theory spectrum, satisfying certain technical conditions. The homology is then naturally defined as the homotopy of the total space of the bundle, relative to the base, and this is a homology theory on the category \mathcal{K} of pairs (X, E) , where $E \rightarrow X$ is a bundle of the desired form, and maps $(X, E) \rightarrow (Y, F)$ are bundle maps that are fiberwise equivalences [3]. Since our notion of twisted homology is a homology theory, if S_\bullet is a simplicial object in \mathcal{K} , we have a spectral sequence of the form

$$E_2 = H_*(E_*(S)) \Rightarrow E_*(|S|).$$

Here $E_*(S)_\bullet$ is the simplicial graded abelian group that in position p is just $E_*(S_p)$.

We apply this machinery to a special, computable case. Fix a twisting $\tau \in H^3(G) = H^2(\Omega G)$, where ΩG is the loop space of G . Let $S_\bullet = B(*, \Omega G, *_\tau)_\bullet$ be the simplicial object in \mathcal{K} where

$$S_n = ((\Omega G)^{\times n}, (\Omega G)^{\times n} \times K).$$

The simplicial maps are somewhat trickier. The maps on the underlying base spaces are just the ordinary maps in the two-sided bar complex $B(*, \Omega G, *)_\bullet$. On the bundles, all of the face maps but the last are just the obvious lifts of the maps on spaces to the trivial K -bundle over them. The last face map is twisted by the composite

$$\Omega G \times K \xrightarrow{\tau \times \text{Id}} K(\mathbb{Z}, 2) \times K \rightarrow K,$$

where the last map is the action of $K(\mathbb{Z}, 2)$ on K . This construction is entirely analogous to Brown's simplicial construction of the Serre spectral sequence, building fibrations as twisted products in a way that the twisting is controlled simplicially [2]. The geometric realization of this simplicial object in \mathcal{K} is therefore just the twisted K bundle over $B\Omega G = G$ determined by the twisting class τ .

Since ΩG has only even cells, we know that

$$K_*(\Omega G^{\times n}) = K_*(\Omega G)^{\otimes n}.$$

This lets us identify the spectral sequence defined above as a twisted form of the Rothenberg-Steenrod / homology Eilenberg-Moore spectral sequence:

$$E_2 = \text{Tor}^{K_*(\Omega G)}(K_*, K_*^\tau) \Rightarrow K_*^\tau(G),$$

where K_*^τ is the $K_*(\Omega G)$ -module induced by the last face map. The computation now varies depending on the group.

If G is $SU(n+1)$ or $Sp(n)$, then the homology of ΩG is polynomial on n generators of even degree. This implies that the Atiyah-Hirzebruch spectral sequence computing K -homology also collapses, and we see that $K_*(\Omega G)$ is polynomial on n generators. Since this is acting on K_* , we see that for the module K_*^τ , the generator x_k acts as multiplication by a number c_k . At this point, we deviate slightly from the presentation in [3]. For these classical groups (and indeed for $Spin(n)$ as well), we use the fibration in topological groups

$$\Omega S^{2n+1} \rightarrow SU(n) \rightarrow SU(n+1)$$

and the analogue for $Sp(n)$. If we consider the two sided bar complex in \mathcal{K} given by $B(*, \Omega S^{2n+1}, SU(n)_\tau)$, where ΩS^{2n+1} and $*$ have the trivial bundle associated to them and $SU(n)_\tau$ is $SU(n)$ with the twisted K -bundle over it corresponding to the twisting τ , then the realization is $SU(n+1)_\tau$. This gives us a spectral sequence

$$E_2 = \text{Tor}^{K_*(\Omega S^{2n+1})}(K_*, K_*^\tau(SU(n))) \Rightarrow K_*^\tau(SU(n+1)).$$

As a ring, $K_*(\Omega S^{2n+1})$ is polynomial on one generator that corresponds to the last polynomial generator of $K_*(\Omega SU(n+1))$. In other words, this slight recasting allows us to isolate the contribution of each generator of $K_*(\Omega SU(n+1))$. The Tor groups are easy to compute, since we consider only a polynomial generator, and for degree reasons, we conclude that the spectral sequence collapses with no possible extensions. This shows Douglas' theorem for $SU(n)$ and $Sp(n+1)$.

For the exceptional groups, we return to Douglas' initial formulation. We need to understand the K -homology of ΩG . Here we use the fact that the map $G \rightarrow K(\mathbb{Z}, 3)$ inducing $1 \in H^3(G)$ is an equivalence through a range that increases with the rank of G . We can use this, together with the computation of $K_*(\mathbb{C}P^\infty)$, to compute the K -homology of ΩG as a ring [4]. Feeding this into the twisted Rothenberg-Steenrod spectral sequence allows us to see Douglas' aforementioned theorem for the exceptional groups.

REFERENCES

- [1] M. Atiyah and G. Segal, *Twisted K-theory and cohomology*
- [2] E. H. Brown, Jr., *Twisted tensor products I*, Annals of Mathematics **69** (1959), 223–246.
- [3] C. Douglas, *On the twisted K-homology of simple Lie groups*, Topology **45** (2006), 955–988.
- [4] P. Duckworth, *The K-theory Pontrjagin rings for the loop spaces on the exceptional Lie groups* Quart. J. Math. **35** (1984), 253–262.

Talk 6: Introduction to topological and differentiable stacks

NIKO NAUMANN

This talk provided an introduction to the language of stacks with a special emphasis on differentiable and topological stacks, i.e. stacks over the large site of smooth manifolds resp. topological spaces endowed with the topology generated by open covers which satisfy the analogue of the condition to be algebraic, c.f. [2]. In the first section we introduced stacks as certain presheaves of groupoids and discussed the version(s) of the Yoneda lemma in this context, fiber product of stacks and the example of a quotient stack $[X/G]$. Except for the final point, this is basic material on stacks, amply covered for example in [3].

In the second section we defined representable morphisms and properties of those and introduced topological resp. differentiable stacks as those stacks which admit an epimorphism from a representable stack. We discussed in detail why the stacks $[X/G]$ meet these assumptions and performed some simple computations with them, e.g. if G acts on X and $G \subseteq G'$ is a subgroup then $[X \times_G G'/G'] \simeq [X/G]$ and if $H \subseteq G$ is a normal subgroup such that $X \rightarrow X/H$ is an H -bundle, then $[X/G] \simeq [(X/H)/(G/H)]$. These basic geometric results provide a nice implementation of the well known induction/restriction-structure in equivariant K -theory since one has $K_G(X) \simeq K([X/G])$.

In the final section we introduced gerbes and their relation to coarse moduli spaces. This is again standard, covered for example in [2]. We concluded by explaining J. Giraud's result that second sheaf-cohomology with coefficients in an (abelian) band G classifies gerbes with band G up to G -equivalence [1, Chapitre IV, Théorème 3.4.2].

REFERENCES

- [1] J. Giraud, *Cohomologie non abélienne*, Grundlehren der mathematischen Wissenschaften, 179, Springer, Berlin, 1971.
- [2] G. Laumon, L. Moret-Bailly, *Champs algébrique*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 39. Springer-Verlag, Berlin, 2000.
- [3] A. Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, available at: <http://www.dm.unibo.it/vistoli/papers.html>.

Talk 7: The Twisted K-theory of topological stacks

JEFFREY GIANSIRACUSA

Our aim is to describe the construction from [1] of the twisted K-theory of a topological stack. The roadmap to the construction is as follows. Let \mathfrak{X} be a topological stack. Choosing an atlas $A_0 \rightarrow \mathfrak{X}$ leads to a topological groupoid $A_0 \times_{\mathfrak{X}} A_0 = A_1 \rightrightarrows A_0$. Choosing a Haar system λ on this groupoid allows one to endow $C_c(A_1)$ with a convolution product and its completion with respect to the reduced norm is a C^* -algebra $C_r^*(A, \lambda)$. The K -theory of this C^* -algebra is the (untwisted) K -theory of the stack \mathfrak{X} . To add a twist, one begins with a S^1 -banded gerbe $\mathfrak{G} \rightarrow \mathfrak{X}$. The process of choosing an atlas now yields an S^1 -central extension of groupoids, $S^1 \rightarrow \tilde{A} \rightarrow A$. After choosing a Haar system λ on \tilde{A} , one forms a reduced C^* -algebra $C_r^*(\tilde{A}, \lambda)^{S^1}$. The K -theory of \mathfrak{X} twisted by the gerbe \mathfrak{G} is defined to be the K -theory of this C^* -algebra. There are two important tasks left: (1) this definition required two choices, so we must verify that the resulting K -theory is well-defined up to a *canonical* isomorphism, (2) K -theory is supposed to be a contravariant functor with respect to pulling back gerbes, so we must understand how to see the functoriality in this definition.

First consider the choice of an atlas. By assumption \mathfrak{X} admits an atlas, $A_0 \rightarrow \mathfrak{X}$; this is a representable morphism with local sections for a space A_0 . The pullback $A_1 := A_0 \times_{\mathfrak{X}} A_0$ is then a space, and it comes equipped with two maps to A_0 and a diagonal map in the other direction; one checks that this is indeed topological groupoid A_{\bullet} . Note that one can recover the original stack (up to noncanonical isomorphism) from the groupoid in two ways: either as the quotient stack $[A_0/A_1]$ or as the moduli stack of principal A_{\bullet} -bundles.

If \mathfrak{X} has an S^1 -banded gerbe over it then one can choose a covering $A'_0 = \coprod U_i$ of A_0 which trivializes the gerbe so that the morphism $A'_0 \rightarrow \mathfrak{X}$ lifts to $A'_0 \rightarrow \mathfrak{G}$ (up to a 2-isomorphism). Then

$$S^1 \rightarrow A'_0 \times_{\mathfrak{G}} A'_0 \rightarrow A'_0 \times_{\mathfrak{X}} A'_0$$

is an S^1 -central extension of groupoids over A'_0 .

How much can different atlases for a given stack vary? Suppose that we have a second atlas $B_0 \rightarrow \mathfrak{X}$. Then the diagonal arrow in the pullback square

$$\begin{array}{ccc} A_0 \times_{\mathfrak{X}} B_0 & \xrightarrow{\pi_2} & B_0 \\ \pi_1 \downarrow & \searrow & \downarrow \\ A_0 & \longrightarrow & \mathfrak{X} \end{array}$$

is again an atlas for \mathfrak{X} since it is a composition of representable morphisms with local sections. Furthermore, the projection π_1 is a principal right B_{\bullet} -bundle with respect to π_2 , and π_2 is a principal left A_{\bullet} -bundle with respect to π_1 , and the two actions commute—it is an $(A_{\bullet} - B_{\bullet})$ -bibundle. Such a bibundle between two groupoids is called a *Morita equivalence of groupoids*. We have shown that, as long as one remembers the maps to the stack, any two atlases lead to canonically Morita

equivalent groupoids. One can easily show (see [2]) that a converse also hold: a Morita equivalence of groupoids induces an isomorphism between the stacks they represent.

If there is an S^1 -banded gerbe over \mathfrak{X} and $S^1 \rightarrow \tilde{A}_\bullet \rightarrow A_\bullet$, $S^1 \rightarrow \tilde{B}_\bullet \rightarrow B_\bullet$ are two atlases representing this gerbe then the same reasoning as above leads to a *canonical* $(\tilde{A}_\bullet - \tilde{B}_\bullet)$ -bibundle Z which is S^1 equivariant in the sense that the S^1 inside \tilde{A}_\bullet acts on Z exactly as the S^1 inside \tilde{B}_\bullet . This is a *Morita equivalence of S^1 -central extensions*. Thus any two atlases for a gerbe lead to canonically Morita equivalent central extensions of groupoids. Again one can also show that conversely a Morita equivalence of extensions induces an isomorphism of gerbes.

We now consider the choice of a Haar system on a groupoid. Suppose A_\bullet is a locally compact groupoid. A *Haar system* λ on \tilde{A} is a family of positive measures $\{\lambda^x$ supported on the arrows A_1^x ending at $x|x \in A_0\}$ which is continuous and translation invariant. This concept generalizes that of Haar measures on groups. See [5] for details. For example, a Haar system on a vector bundle $V \rightarrow X$ (considered as a groupoid with respect to addition of vectors) is a continuous family of Lebesgue measures on the fibres. Haar systems can be highly non-unique—e.g. rescaling by any nonvanishing translation invariant function will produce a new Haar system.

For each $x \in A_0$ the convolution algebra $C_c(A_1, \lambda)$ acts by convolution on $L^2(A_1^x, \lambda^x)$, and so there is a $*$ -homomorphism $\pi_x : C_c(A_1, \lambda) \rightarrow B(L^2(A_1^x, \lambda^x))$. The *reduced C^* -algebra* $C_r^*(A_1, \lambda)$ is the completion of $C_c(A_1, \lambda)$ with respect to the norm $\|f\|_r = \sup_x \|\pi_x(f)\|$. For an S^1 -central extension we form the C^* -algebra $C_r^*(\tilde{A}, \lambda)^{S^1} := C_c(\tilde{A}, \lambda)^{S^1} \subset C_r^*(\tilde{A}, \lambda)$.

A *strong Morita equivalence* between C^* -algebras \mathcal{A} and \mathcal{B} is an $(\mathcal{A} - \mathcal{B})$ -Hilbert bimodule M (meaning it has inner products $\langle, \rangle_{\mathcal{A}}$, $\langle, \rangle_{\mathcal{B}}$ taking values in \mathcal{A} , \mathcal{B} resp. which are compatible with the module structures and have dense images). The functor $-\otimes_{\mathcal{B}} M$ (with the correct notion of tensor product) induces an equivalence of categories from right \mathcal{B} -modules to right \mathcal{A} -modules. Hence a strong Morita equivalence of C^* -algebras induces an isomorphism of their K -theory groups.

According to [3] a Morita equivalence of groupoids produces a Morita equivalence of their reduced C^* -algebras. More precisely, if $A_0 \leftarrow Z \rightarrow B_0$ is an $A_\bullet - B_\bullet$ -bibundle then $C_c(Z)$ is a $C_c(B_1, \lambda_B) - C_c(A_1, \lambda_A)$ -bimodule which can be canonically completed into a strong Morita equivalence bimodule. Similarly, if Z is a Morita equivalence of S^1 -central extensions of groupoids then $C_c(Z)^{S^1}$ completes into a $C_r^*(\tilde{B}_1, \lambda)^{S^1} - C_r^*(\tilde{A}_1, \lambda)^{S^1}$ strong Morita equivalence bimodule.

In summary, (1) any two representations of a gerbe by S^1 groupoid extensions are canonically Morita equivalent; (2) given any choice of Haar systems this bibundle produces a Morita equivalence bimodule between the associated C^* -algebras; (3) the Morita equivalence bimodule induces an isomorphism of the K -groups.

We now turn to contravariant functoriality for twisted K-theory of stacks. The Morita equivalence bibundles that we have already seen are actually the isomorphisms in a category. This category has groupoids as objects and (isomorphism classes of) *generalized morphisms* as arrows. A generalized morphism

$A_\bullet \rightarrow B_\bullet$ consists of a space Z with maps $A_0 \xleftarrow{\tau} Z \xrightarrow{\sigma} B_0$ and commuting actions of A_\bullet on the left and B_\bullet on the right (with respect to τ, σ) such that $A_0 \leftarrow Z$ is a principal right B_\bullet -bundle. An easy fact is that any representable morphism of topological stacks can be represented as a generalized morphism between atlas groupoids. A similar statement holds with twists.

Let us suppose that the generalized morphism is *proper* (in the sense of [4]). If one chooses Haar systems on A_\bullet and B_\bullet , write \mathcal{A}, \mathcal{B} for the associated C^* -algebras. Under the properness assumption $C_c(Z)$ can be completed into a \mathcal{B} - \mathcal{A} -bimodule, and it can be endowed with an \mathcal{A} -valued inner product with dense image. There is no \mathcal{B} -valued inner product but one can still show that the action of \mathcal{B} is via \mathcal{A} -compact operators. Such an object represents a (degree 0) class $[Z]$ in Kasparov's KK -theory group $KK(\mathcal{B}, \mathcal{A})$. Recall that there are associative products $KK(U, V) \times KK(V, W) \rightarrow KK(U, W)$, and $KK(\mathbb{C}, S^i U) \cong K_i(U)$. Hence we have a product map

$$K_i(\mathcal{B}) \cong KK(\mathbb{C}, S^i \mathcal{B}) \xrightarrow{[Z]} KK(\mathbb{C}, S^i \mathcal{A}) \cong K_i(\mathcal{A}).$$

Using this construction one sees that our definition of twisted K -theory of stacks is contravariantly functorial with respect to morphisms of stacks which are represented by proper generalized morphisms of atlas groupoids.

There is also a description of twisted K -theory in terms of families of Fredholm operators. Fix a separable infinite dimensional Hilbert space \mathbb{H} . Recall that S^1 -gerbes over a stack \mathfrak{X} with atlas groupoid A_\bullet are classified by $H^2(A_\bullet; S^1)$. Similarly principal $PU(\mathbb{H})$ bundles over \mathfrak{X} are classified by $H^1(A_\bullet; PU(\mathbb{H}))$. The central extension $S^1 \rightarrow U(\mathbb{H}) \rightarrow PU(\mathbb{H})$ provides a connecting homomorphism $H^1(A_\bullet; PU(\mathbb{H})) \rightarrow H^2(A_\bullet; S^1)$. This homomorphism has a canonical section. Given a gerbe with class α , choose an extension $S^1 \rightarrow \tilde{A}_\bullet \rightarrow A_\bullet$ representing α and pick a Haar system on \tilde{A}_\bullet . For each $x \in A_0$ there is a Hilbert space $L^2(\tilde{A}_1^x) \otimes \mathbb{H}$. These glue together to form a Hilbert bundle over the groupoid \tilde{A}_\bullet , which is the same as a projective Hilbert bundle over A_\bullet . The class of this projective bundle maps to the class α , so any gerbe can be represented by a $PU(\mathbb{H})$ -bundle, and there is a canonical choice of representative.

Let P_α denote the project bundle constructed from a gerbe class α . The group $PU(\mathbb{H})$ acts continuously by conjugation on the bounded operators $\mathcal{B}(\mathbb{H})$ (in the $*$ -strong topology) and on the compact operators $\mathcal{K}(\mathbb{H})$ (with the norm topology). Let \mathcal{B}_α and \mathcal{K}_α be the $\mathcal{B}(\mathbb{H})$ and $\mathcal{K}(\mathbb{H})$ bundles associated to P_α .

One of the main theorems of [1] is a description of twisted K -theory as certain types of families of Fredholm operators.

Theorem 1. *The twisted K -theory $K_\alpha^i(A_\bullet)$ is canonically isomorphic to the group of homotopy classes of norm bounded A_\bullet -invariant sections of \mathcal{B}_α (self-adjoint if i is odd) and invertible up to an A_\bullet -invariant section of \mathcal{K}_α which vanishes at infinity in the orbit space A_0/A_1 .*

REFERENCES

- [1] J-L. Tu, Ping Xu, C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Scient. Ec. Norm. Sup. **37** (2004), 841–910.
- [2] J. Heinloth, *Some notes on differentiable stacks*, Mathematisches Institut Seminars, (Y. Tschinkel, ed.) Universität Göttingen (2004-2005), 1–32.
- [3] P. Muhly, J. Renault, D. Williams, *Equivalence and isomorphism for groupoid C^* -algebras*, J. Operator Theory **17** (1987), 3–22.
- [4] J-L. Tu, *Non-Hausdorff groupoids, proper actions and K-theory*, Documenta Math. **9** (2004), 565–597.
- [5] J. Renault, *A groupoid approach to C^* -algebras*, Lecture Notes in Mathematics, 793. Springer, Berlin, 1980.

Talk 8: The Twisted Chern Character via Noncommutative Geometry

SEBASTIAN GOETTE, GUIDO KINGS

The classical Chern character is a rational isomorphism from topological K -theory to cohomology. Using the Serre-Swan theorem and the Connes-Hochschild-Kostant-Rosenberg theorem, the Chern character may be understood as a map from K -theory to periodic cyclic homology of the smooth algebra $C^\infty(M; \mathbb{C})$ if M is a smooth manifold.

$$\begin{array}{ccc}
 K_\bullet(C^\infty(M; \mathbb{C})) & \xrightarrow{\text{Ch}} & HP_\bullet(C^\infty(M; \mathbb{C})) \\
 \text{Serre-Swan} \downarrow & & \downarrow \text{Connes-Hochschild-Kostant-Rosenberg} \\
 K^\bullet(M) & \xrightarrow{\text{ch}} & H_{\text{dR}}^\bullet(M) \quad .
 \end{array}$$

Following Mathai and Stevenson [2], we generalize the upper right hand way through this diagram to construct a Chern character for twisted K -theory.

Regard the central extension

$$S^1 \xrightarrow{\iota} U \xrightarrow{q} PU ,$$

where U is the unitary group of a Hilbert space \mathcal{H} . In this talk, a twist is represented as a PU -principal bundle $p: P \rightarrow M$ with a smooth structure and connection Θ . Since PU acts on $\mathcal{L}^1(\mathcal{H})$, one may form the algebra $\mathcal{A}(P) = \Gamma^\infty(P \times_{PU} \mathcal{L}^1(\mathcal{H}))$. Then $K_\bullet(\mathcal{A}(P))$ is a model for the P -twisted K -theory of M .

Let $\tilde{\mathcal{A}}(P) = \mathcal{A}(P) + \mathbb{C}$ be the unitisation of $\mathcal{A}(P)$, and let u be a formal variable of degree -2 . We consider the Hochschild complex (C_\bullet, b) and the periodic cyclic complex $(CC_\bullet, b_u B)$, where $C_k(\mathcal{A}(P)) = \tilde{\mathcal{A}}(P) \otimes \mathcal{A}(P)^{\otimes k}$ and $CC_\bullet(\mathcal{A}(P)) = C_\bullet(\mathcal{A}(P)) \otimes \mathbb{C}((u))$, and b and B are the Hochschild and the cyclic boundary operator, respectively, see [1]. Let $\tilde{p} \in M_n(\tilde{\mathcal{A}}(P))$ be an idempotent representing an element $[p] \in K_0(\mathcal{A}(P)) = \tilde{K}_0(\tilde{\mathcal{A}}(P))$, and let p be its image in $\mathcal{A}(P)$. Then its Connes-Chern character is given by

$$\text{Ch}(\tilde{p}) = \sum_{k=0}^{\infty} (-u)^k \frac{(2k)!}{k!} \text{tr} \left(\left(\tilde{p} - \frac{1}{2} \right) \otimes \tilde{p}^{\otimes 2k} \right) \in CC_0(\mathcal{A}(P)) .$$

To define twisted de Rham cohomology $H_{\text{dR}}^\bullet(M, c)$ and the CHKR-isomorphism $HP_\bullet(\mathcal{A}(P)) \rightarrow H_{\text{dR}}^\bullet(M, c)$, we use connections on bundle gerbes following [3]. Over $P \times_M P \cong P \times PU$, there exists a canonical S^1 -bundle $L \cong P \times U$, which forms a groupoid over P . Then L admits a connection ϑ that is compatible with the groupoid product, and these connections bijectively correspond to PU -equivariant splittings $\vartheta^0: P \rightarrow \text{Hom}(\mathfrak{u}, i\mathbb{R})$ with $\vartheta^0 \circ \iota_* = \text{id}_{i\mathbb{R}}$, or equivalently to PU -equivariant $\sigma: P \rightarrow \text{Hom}(\mathfrak{p}\mathfrak{u}, \mathfrak{u})$ with $q_* \circ \sigma = \text{id}_{\mathfrak{p}\mathfrak{u}}$. Since the Lie-bracket on \mathfrak{u} descends to $\mathfrak{p}\mathfrak{u}$, there is a two-form

$$f = d\vartheta^0 \circ \Theta - \frac{1}{2} \vartheta^0([\Theta, \Theta]_{\mathfrak{u}}) \in \Omega^2(P; i\mathbb{R}) ,$$

such that $(p_2^* - p_1^*)f \in \Omega^2(P \times_M P)$ is the curvature of the connection ϑ on L . Finally, let Ω denote the curvature of Θ . Then df is basic, so define $c \in \Omega^3(M, i\mathbb{R})$ by

$$df = -d\vartheta^0 \circ \Omega = \nabla\sigma \circ \Omega = -p^*c .$$

Note that the data (P, σ, Θ) are up to isomorphism in bijective correspondence with classes in the third integral Deligne cohomology of M .

In the second part of the talk the proof of the following generalization of the theorem of Connes-Hochschild-Kostant-Rosenberg was sketched. To formulate this we define the twisted de Rham cohomology $H_{\text{dR}}^\bullet(M, c)$ as the cohomology of $\Omega^\bullet(M) \otimes \mathbb{C}((u))$ with differential $ud - u^2c$.

Theorem[Mathai-Stevenson] Let M be a compact manifold. Consider the $\mathbb{C}((u))$ -linear map

$$J : CC_\bullet(\mathcal{A}(P)) \rightarrow \Omega^\bullet(M) \otimes \mathbb{C}((u)),$$

which is given by mapping $(\tilde{a}_0, \dots, a_k)$ to

$$\int_{\Delta^k} \text{tr}(\tilde{a}_0 e^{-s_0\sigma(\Omega)u} \nabla(a_1) \dots \nabla(a_k) e^{-s_k\sigma(\Omega)u} ds_1 \wedge \dots \wedge ds_k$$

if $k > 0$ and to $\text{tr}(a_0 e^{\sigma(\Omega)u})$ if $k = 0$. Then J is a chain map

$$J \circ (b + uB) = (ud - u^2c) \circ J,$$

which induces isomorphisms

$$HH_n(\mathcal{A}(P)) \cong \Omega^n(M)$$

$$HP_n(\mathcal{A}(P)) \cong \begin{cases} H_{\text{dR}}^{\text{even}}(M, c) & \text{if } n \text{ is even} \\ H_{\text{dR}}^{\text{odd}}(M, c) & \text{if } n \text{ is odd.} \end{cases}$$

REFERENCES

- [1] J.-L. Loday, *Cyclic Homology*, Grundlehren 301, Springer, Berlin, 1992.
- [2] V. Mathai, D. Stevenson, *On a generalized Connes-Hochschild-Kostant-Rosenberg theorem*, Adv. Math. **200** (2006), 303–335.
- [3] M. K. Murray, *Bundle gerbes*, J. London Math. Soc. (2), **54** (1996), 403–416.

Talk 9: The Chern Character for Twisted K -theory of Orbifolds

ELMAR SCHROHE

This talk was based on the results of Tu and Xu in [5].

The Setup. Given are an orbifold X and an S^1 -gerbe $\alpha \in H^2(X, \mathcal{S}^1)$. The orbifold is represented by a proper étale Lie groupoid

$$\Gamma \rightrightarrows M.$$

Recall that a groupoid is called a *Lie groupoid* if the base space M and the arrow space Γ are smooth manifolds and both the source map s and target map t are smooth. A Lie groupoid is *étale*, if s and t are local diffeomorphisms. The properness refers to the mapping $(s, t) : \Gamma \rightarrow M \times M$.

The representation of X by a proper étale Lie groupoid is not unique. However, it is determined up to Morita equivalence, i.e., there is a one-to-one correspondence between Morita equivalence classes of proper étale Lie groupoids and orbifolds.

We denote by Γ_n the set of n -tuples of composable arrows in Γ :

$$(1) \quad \Gamma_n = \{(g_n, \dots, g_2, g_1) : g_j \in \Gamma, t(g_j) = s(g_{j+1})\}.$$

Clearly, $\Gamma_0 = M$ and $\Gamma_1 = \Gamma$.

Next we represent the S^1 -gerbe in the framework of groupoids. We recall the notion of S^1 -central extensions.

1. Definition. An S^1 -central extension of the Lie groupoid $\Gamma \rightrightarrows M$ consists of

(1) a Lie groupoid $\tilde{\Gamma} \rightrightarrows M$, together with a morphism of Lie groupoids

$$(\pi, id) : (\tilde{\Gamma} \rightrightarrows M) \rightarrow (\Gamma \rightrightarrows M);$$

(2) a left S^1 -action on $\tilde{\Gamma}$, making $\pi : \tilde{\Gamma} \rightarrow \Gamma$ a left principal S^1 -bundle.

Moreover, these two structures are compatible in the sense that $(s_1 \cdot x)(s_2 \cdot y) = s_1 s_2 \cdot xy$ for all s_1, s_2 in S^1 and all composable $x, y \in \tilde{\Gamma}$.

There is a notion of Morita equivalence of S^1 -central extensions. It implies the Morita equivalence of the underlying groupoids. Moreover, it allows us to identify the gerbe $\alpha \in H^2(X, \mathcal{S}^1)$ with the Morita equivalence class of an S^1 -central extension

$$S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M.$$

K-theory. The natural S^1 -action on \mathbb{C} gives us an associated complex line bundle

$$L = \tilde{\Gamma} \times_{S^1} \mathbb{C}$$

over Γ , which is equipped with an associative bilinear product

$$(2) \quad L_g \otimes L_h \rightarrow L_{gh}, \quad \xi \otimes \eta \mapsto \xi \cdot \eta$$

for composable g, h , and an antilinear involution

$$(3) \quad L_g \rightarrow L_{g^{-1}}, \quad \xi \mapsto \xi^*.$$

We also obtain a scalar product: For $\xi, \eta \in L_g$, let $\langle \xi, \eta \rangle = \xi^* \cdot \eta \in L_{s(g)} \cong \mathbb{C}$.

As a consequence we can define a convolution product on $\mathcal{A} = C_c^\infty(\Gamma, L)$ by

$$(4) \quad (\xi_1 * \xi_2)(g) = \sum_{t(h)=t(g)} \xi_1(h) \cdot \xi_2(h^{-1}g),$$

and an adjoint

$$(5) \quad \xi^*(g) = (\xi(g^{-1}))^*.$$

Defining the action by convolution, this in turn gives us, for each $x \in M$, a $*$ -representation π_x of \mathcal{A} on the Hilbert space H_x which is the completion of \mathcal{A} with respect to the scalar product

$$\langle \xi, \eta \rangle_x = (\xi^* \eta)(x) = \sum_{s(g)=x} \langle \xi(g), \eta(g) \rangle.$$

We denote by $C_r^*(\Gamma, L)$ the associated reduced C^* -algebra, i.e., the completion of \mathcal{A} with respect to the norm $f \mapsto \sup_x \|\pi_x(f)\|$.

2. Definition. *The twisted K -theory groups of X are defined by*

$$K_\alpha^i(X) = K_i(C_r^*(\Gamma, L)).$$

The K -class indeed depends only on X and α as the choice of a Morita equivalent S^1 -central extension results in isomorphic K -groups, see [4, Section 3.2].

Connection, curving, 3-curvature. We now introduce the the differential $\partial : \Omega^k(\Gamma_{n-1}) \rightarrow \Omega^k(\Gamma_n)$ as the alternating sum of the pullback maps under the $n + 1$ canonical maps $\Gamma_n \rightarrow \Gamma_{n-1}$.

3. Definition. *A connection on $\tilde{\Gamma}_\bullet \rightarrow \Gamma_\bullet$ is a connection 1-form $\theta \in \Omega^1(\tilde{\Gamma})$ such that $\partial\theta = 0$. Given θ , a curving is a 2-form B on M such that $d\theta = \partial B$. We then call $\Omega = dB \in \Omega^3(M)$ the 3-curvature.*

Note that then $\partial\Omega = \partial dB = 0$, so that Ω is Γ -invariant.

It can be shown [3] that within the Morita equivalence class of $\Gamma \rightrightarrows M$ there is a groupoid for which a connection and a curving exist.

Twisted cohomology. Associated to Γ is the space $S\Gamma = \{g \in \Gamma : s(g) = t(g)\}$ of closed loops. Γ acts on $S\Gamma$ by conjugation. The associated transformation groupoid $\Lambda\Gamma = S\Gamma \times_M \Gamma \rightrightarrows S\Gamma$ is called the *inertia groupoid*.

The line bundle L induces a line bundle L' over $\Lambda\Gamma \rightrightarrows S\Gamma$. Let θ be a connection with curving B and 3-curvature Ω . It can be shown that θ induces a covariant derivative $\nabla' : \Omega_c^*(S\Gamma, L') \rightarrow \Omega_c^{*+1}(S\Gamma, L')$ which is invariant under the action of Γ . The twisted cohomology $H_c^*(X, \alpha)$ is defined as the homology of the complex

$$(\Omega_c^*(S\Gamma, L')^\Gamma((u)), \nabla' - 2\pi i \Omega u \wedge \cdot).$$

Here $\Omega_c^*(S\Gamma, L')^\Gamma$ are the Γ -invariant forms in $\Omega_c^*(S\Gamma, L')$ on which both ∇' and $\Omega \wedge$ act. Moreover, u is a formal variable of degree -2 and $((u))$ denotes formal Laurent series in u .

$H_c^*(X, \alpha)$ indeed only depends on X and α , not on the choice of Γ and $\tilde{\Gamma}$.

The Chern character for twisted K-theory. \mathcal{A} is a dense subalgebra of $C_r^*(\Gamma, L)$ and closed under holomorphic functional calculus. Hence, $K_i(\mathcal{A}) = K_i(C_r^*(\Gamma, L)) = K_\alpha^i(X)$. The noncommutative Chern character of Connes and Karoubi induces a map

$$\text{Ch} : K_i(\mathcal{A}) \rightarrow HP_i(\mathcal{A}).$$

Given θ and B , Tu and Xu now define a mapping

$$\tau_{\theta, B} : HP_i(\mathcal{A}) \rightarrow H_c^{\text{even/odd}}(X, \alpha).$$

In fact, $\tau_{\theta, B}$ is obtained as a chain map

$$CC_k(\mathcal{A})((u)) \longrightarrow \Omega_c^*(S\Gamma, L')^\Gamma((u)).$$

An important ingredient in its construction is the definition of a trace

$$\text{Tr} : \Omega_c^*(\Gamma, L) \longrightarrow \Omega_c^*(S\Gamma, L')^\Gamma.$$

Extending results by Baum and Connes, Tu and Xu then show an Atiyah-Hirzebruch type theorem:

4. Theorem. The mappings Ch and $\tau_{\theta, B}$ yield isomorphisms

$$K_\alpha^*(X) \otimes \mathbb{C} \xrightarrow{\text{Ch}} HP_*(\mathcal{A}) \xrightarrow{\tau_{\theta, B}} H_c^*(X, \alpha).$$

The idea of the proof is that $K_\alpha^*(X) \otimes \mathbb{C}$, $HP_*(\mathcal{A})$, and $H_c^*(X, \alpha)$ agree locally, since an orbifold is locally a crossed product of a manifold by a finite group, so that the results of Baum and Connes [1] can be applied. Moreover, each of these functors admits Mayer-Vietoris sequences; hence the groups agree globally.

REFERENCES

- [1] P. Baum and A. Connes. *Chern character for discrete groups*. A Fête of Topology, 163-232, Academic Press, Boston, MA 1988.
- [2] I. Moerdijk. *Orbifolds as groupoids: an introduction*. Orbifolds in Mathematics and Physics (Madison WI 2001), 205-222, Contemp. Math. **310**. Amer. Math. Soc., Providence, RI, 2002.
- [3] I. Moerdijk, D. Pronk. *Orbifolds, sheaves and groupoids*. K-theory **12** (1997), 3-21.
- [4] J.-L. Tu, Ping Xu. *Twisted K-theory of differentiable stacks*, Ann. Sci. ENS **37** (2004), 841-910.
- [5] J.-L. Tu, Ping Xu. *Chern character for twisted K-theory of orbifolds*, Preprint math.KT/0505267.

Talk 10: Twisted K-theory via parametrized homotopy theory and twisted integration

MORITZ WIETHAUP

The goal of this talk was to explain how twisted K -theory (and twisted cohomology in general) can be understood from the point of view of (stable) parametrized homotopy theory. The main reference for the talk was [3], which uses the foundational work of May-Sigurdsson [2] and (a suitable modification of) the explicit model of the K -theory spectrum obtained in [1] to construct a homotopical version of twisted K -theory. One of the advantages of this approach is that it allows

for the construction of pairings $K_\sigma^*(X) \otimes K_\tau^*(X) \longrightarrow K_{\sigma+\tau}^*(X)$ between twisted K -theory groups, as well as pushforward maps in twisted K -theory.

We started by outlining the appropriate framework for parametrized homotopy theory, as developed in [2]. Let us recall the basic definitions: A space over B is a space equipped with a map to B . An ex-space over a base space B is a space X together with maps $p : X \longrightarrow B$ and $s : B \longrightarrow X$ such that $ps = \text{id}_B$. We have obvious categories of spaces over B (resp. ex-spaces over B), which we will denote by $\text{Top}_{/B}$ (resp. Top_B).¹ The forgetful functor from ex-spaces to spaces over has a left adjoint

$$(-)_+ : \text{Top}_{/B} \longrightarrow \text{Top}_B$$

given by adding a disjoint copy of B , i.e. $X_+ = X \amalg B$. A map $f : B \longrightarrow C$ induces adjoint pairs of base-change functors

$$f_! : \text{Top}_B \rightleftarrows \text{Top}_C : f^*$$

and

$$f_! : \text{Top}_{/B} \rightleftarrows \text{Top}_{/C} : f^*$$

with $f_!$ left adjoint to f^* . In addition, we have two kinds of smash products of ex-spaces: The internal smash product

$$\wedge_B : \text{Top}_B \times \text{Top}_B \longrightarrow \text{Top}_B$$

and the external smash product

$$\overline{\wedge}_{B,C} : \text{Top}_B \times \text{Top}_C \longrightarrow \text{Top}_{B \times C}.$$

There is also a notion of ex-spectrum over B . The ex-spectra of [2] are orthogonal spectra parametrized by B , but for the purpose of this talk it is sufficient to think of an ex-spectrum as a sequence of ex-spaces E_n over B together with structure maps

$$\Sigma_B E_n = E_n \wedge_B (S^1 \times B) \longrightarrow E_{n+1}.$$

In any case, the ex-spectra over B form a category \mathcal{S}_B and we have stable versions of the base-change and smash product functors above. We also have a parametrized version of the usual adjunction between spaces and spectra (as well as its shifted variants):

$$\Sigma_B^\infty : \text{Top}_B \rightleftarrows \mathcal{S}_B : \Omega_B^\infty$$

Theorem 1. [2] *There are model structures on $\text{Top}_{/B}$, Top_B and \mathcal{S}_B such that all the adjunctions above become Quillen adjunctions and the external smash product $\overline{\wedge}$ becomes a left Quillen bifunctor.*

Note that it is not asserted that the internal smash product be a Quillen bifunctor, and in fact it is not.

Now let us explain the relationship between parametrized homotopy theory and twisted cohomology. If G is a topological group acting on a spectrum F , we can

¹Here we are glossing over some subtleties: it is not at all obvious what we should mean by "space" here, since the construction of mapping ex-spaces allows for serious point-set topological pathologies.

use the Borel construction to form the ex-spectrum $EG \times_G F$ over BG . For a space X equipped with a map $q : X \rightarrow BG$ (which we conceive as a twist on X) we can now define the twisted cohomology groups by

$$F_q^n(X) = \text{Hom}_{\text{Ho}\mathcal{S}_{BG}}(\Sigma_{BG}^\infty(X, q)_+, \Sigma^n(EG \times_G F)),$$

at least for maps q making X into a cofibrant object of $\text{Top}/_{BG}$, which can always be achieved by deforming q . Here $\text{Ho}\mathcal{S}_{BG}$ is the homotopy category of ex-spectra over BG .

If F is a ring spectrum and the action of G is compatible with the multiplicative structure on F , one would expect to obtain products on the twisted cohomology groups. The products one wants are of the kind mentioned in the first paragraph, i.e. the product of two twisted cohomology classes should lie in the twisted cohomology group associated to the sum of the two twists. From now on we restrict to the case of PU , the projective unitary group of Hilbert space, acting on K -theory. To realize these products homotopically, one has to introduce a new smash product in $\text{Ho}\mathcal{S}_{BPU}$. Let $p : BPU \times BPU \rightarrow BPU$ be one of the maps given by the action of the linear isometries operad on BPU . Then it is proved in [3] that the left derived functor of the composition

$$\mathcal{S}_{BPU} \times \mathcal{S}_{BPU} \xrightarrow{\bar{\wedge}} \mathcal{S}_{BPU \times BPU} \xrightarrow{p!} \mathcal{S}_{BPU}$$

is a symmetric monoidal product \wedge_p on the homotopy category $\text{Ho}\mathcal{S}_{BPU}$, and that the Borel construction on a PU -spectrum F yields a monoid in $(\text{Ho}\mathcal{S}_{BPU}, \wedge_p)$, if F is an algebra over the linear isometries operad in a way compatible with the action of PU . This applies in particular to (a slight modification) of the K -theory spectrum constructed in [1].

With this multiplicative version of the twisted K -theory spectrum at hand, one can now go ahead and construct twisted Thom isomorphisms and twisted push-forward maps very much as in the classical (untwisted) theory, and one can show that the twisted pushforward maps enjoy the expected properties (functoriality, projection formula, compatibility with pullbacks). As for the Thom isomorphism, one has the following theorem:

Theorem 2. [3] *Let $\pi : X \rightarrow M$ be a real vector bundle of rank n over the finite CW-complex M , let $q : M \rightarrow BPU$ be a map, and suppose u is a (K, q) -orientation of π . Then, for all $r : M \rightarrow BPU$, the map*

$$K^i(M, r) \rightarrow \tilde{K}^{i+n}(Th_{BPU}(X, r + q)), x \mapsto \pi^* x \cup u$$

is an isomorphism.

Here $r + q$ denotes the composition $p \circ (r, q) : X \rightarrow BPU \times BPU \rightarrow BPU$, and $Th_{BPU}(X, r + q)$ is the parametrized Thom space of the bundle $X \rightarrow M$, which is the ex-space over BPU whose fiber over a point b is the ordinary Thom space of the restriction of X to $(r + q)^{-1}(b)$. A (K, q) -orientation of the bundle is defined to be class in $\tilde{K}^n(Th_{BPU}(X, q))$, which restricts to a generator of the free $K^*(pt)$ -module $\tilde{K}^*(S^n)$ on each fiber of π .

REFERENCES

- [1] U. Bunke, M. Joachim and S. Stolz, *Classifying spaces and spectra representing the K-theory of a graded C*-algebra*, High-dimensional manifold topology, 80–102, World Sci. Publ., River Edge, NJ, 2003.
- [2] J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, arXiv:math.AT/0411656.
- [3] R. Waldmueller, *Products and push-forwards in parametrised cohomology theories*, arXiv:math.AT/0611225

Talk 11: Products in twisted K-theory

ARTHUR BARTELS

In this talk I discussed results from the preprint [1] by Adem, Ruan and Zhang. Let \mathcal{G} be an almost complex orbifold and let $\Lambda\mathcal{G}$ be its inertia groupoid. Then for certain twists τ over $\Lambda\mathcal{G}$ Adem-Ruan-Zhang construct a product

$$(1) \quad K^\tau(\Lambda\mathcal{G}) \otimes K^\tau(\Lambda\mathcal{G}) \rightarrow K^\tau(\Lambda\mathcal{G})$$

on twisted K-theory. The crucial point is that this product preserves the twisting. A good warm-up example is the Pontryagin product for finite groups.

Let G be a finite group. Define maps $e_1, e_2, e_{12}: G \times G \rightarrow G$ by $e_1(g_1, g_2) = g_1$, $e_2(g_1, g_2) = g_2$ and $e_{12}(g_1, g_2) = g_1g_2$. One definition of the Pontryagin product is

$$a \star b := (e_{12})_*(e_1^*(a) \cdot e_2^*(b))$$

where \cdot denotes the usual multiplication in K-theory. The following computation is used to establish associativity of this product.

$$(2) \quad \begin{aligned} (a \star b) \star c &= (e_{12})_*(e_1^*((e_{12})_*(e_1^*(a) \cdot e_2^*(b))) \cdot e_2^*(c)) \\ &= (e_{12})_*((e_1^* \circ (e_{12})_*)(e_1^*(a) \cdot e_2^*(b)) \cdot e_2^*(c)) \\ &= (e_{12})_*(((e_{12,3})_* \circ e_{1,2}^*)(e_1^*(a) \cdot e_2^*(b)) \cdot e_2^*(c)) \\ &= (e_{12})_*(((e_{12,3})_*((e_{1,2}^* \circ e_1^*)(a) \cdot (e_{1,2}^* \circ e_2^*)(b)) \cdot e_2^*(c)) \\ &= (e_{12})_*(((e_{12,3})_*((e_{1,2}^* \circ e_1^*)(a) \cdot (e_{1,2}^* \circ e_2^*)(b) \cdot (e_{12,3}^* \circ e_2^*)(c)) \\ &= (\tilde{e}_{123})_*(\tilde{e}_1^*(a) \cdot \tilde{e}_2^*(b) \cdot \tilde{e}_3^*(c)). \end{aligned}$$

Here $e_{1,2}, e_{12,3}: G \times G \times G \rightarrow G \times G$ are defined by $e_{1,2}(g_1, g_2, g_3) = (g_1, g_2)$ and $e_{12,3}(g_1, g_2, g_3) = (g_1g_2, g_3)$; $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_{123}: G \times G \times G \rightarrow G$ are defined by $\tilde{e}_1(g_1, g_2, g_3) = g_1$, $\tilde{e}_2(g_1, g_2, g_3) = g_2$, $\tilde{e}_3(g_1, g_2, g_3) = g_3$, $\tilde{e}_{123}(g_1, g_2, g_3) = g_1g_2g_3$. The square

$$(3) \quad \begin{array}{ccc} G \times G \times G & \xrightarrow{e_{12,3}} & G \times G \\ e_{1,2} \downarrow & & \downarrow e_1 \\ G \times G & \xrightarrow{e_{12}} & G \end{array}$$

is a pull-back square and the computation (2) uses that thus

$$(4) \quad e_1^* \circ (e_{12})_* = (e_{12,3})_* \circ e_{1,2}^*$$

Also used is the following compatibility of push-forward and multiplication

$$(5) \quad f_*(a) \cdot b = f_*(a \cdot f^*b).$$

The proof of associativity is then concluded by computing $a \star (b \star c)$ by similar manipulations arriving at the same result as in (2).

For $k = 1, 2, 3$ the groupoid \mathcal{G}^k of k -sectors is defined by

$$\begin{aligned} (\mathcal{G}^k)_0 &= \{(a_1, \dots, a_k) \in (\mathcal{G}_1)^{\times k} \mid s(a_1) = t(a_1) = \dots = s(a_k) = t(a_k)\} \\ (\mathcal{G}^k)_1 &= \{(a_1, \dots, a_k, u) \in (\mathcal{G}_1)^{\times k+1} \mid \\ &\quad s(a_1) = t(a_1) = \dots = s(a_k) = t(a_k) = s(u)\} \end{aligned}$$

with $s(a_1, \dots, a_k, u) = (a_1, \dots, a_k)$ and $t(a_1, \dots, a_k, u) = (u^{-1}a_1u, \dots, u^{-1}a_ku)$. Note that $\mathcal{G}^1 = \Lambda\mathcal{G}$. Similar as in the case of a finite group there are maps $e_1, e_2, e_{12}: \mathcal{G}^2 \rightarrow \mathcal{G}^1$ and so on.

In order to define the product (1) cocycle formulas for twists are important. Let $C^k(\mathcal{G}, U(1))$ be the abelian group of continuous $U(1)$ -valued maps on the space of composable k -tuples of morphisms in \mathcal{G} . The usual formula [1, p.7] defines a differential $\delta: C^k(\mathcal{G}, U(1)) \rightarrow C^{k+1}(\mathcal{G}, U(1))$. This complex can be used to describe twists for K -theory: For every $\tau \in C^2(\mathcal{G}, U(1))$ with $\delta(\tau) = 0$ there is twisted K -theory $K^\tau\mathcal{G}$; for every $\sigma \in C^1(\mathcal{G}, U(1))$ there is an isomorphism $I_\sigma: K^\tau\mathcal{G} \rightarrow K^{\tau+\delta(\sigma)}\mathcal{G}$; if $\rho \in C^0(\mathcal{G}; U(1))$ then $I_\sigma = I_{\sigma+\delta(\rho)}$.

The space of k -tuples of composable morphisms in \mathcal{G}^n can be identified with the set of all $(n+k)$ -tuples $(a_1, \dots, a_n, u_1, \dots, u_k)$ of morphisms in \mathcal{G} that satisfy

$$s(a_1) = t(a_1) = \dots = s(a_n) = t(a_n) = s(u_1), t(u_1) = s(u_2), \dots, t(u_{k-1}) = s(u_k).$$

The *inverse transgression* $\theta: C^k(\mathcal{G}, U(1)) \rightarrow C^{k+1}(\mathcal{G}^1, U(1))$ is defined [1, p.14] by

$$\begin{aligned} \theta(\phi)(a, u_1, \dots, u_k) = \\ (-1)^k \phi(a, u_1, \dots, u_k) + \sum_{i=1}^k (-1)^{i+k} \phi(u_1, \dots, u_i, a_i, u_{i+1}, \dots, u_k), \end{aligned}$$

where $a_i = (u_1 \dots u_i)^{-1}a(u_1 \dots u_i)$. It is a chain map, i.e., $\delta \circ \theta = \theta \circ \delta$. There is also a map $\mu: C^{k+2}(\mathcal{G}, U(1)) \rightarrow C^k(\mathcal{G}^2, U(1))$ satisfying

$$(6) \quad \mu \circ \delta + \delta \circ \mu = e_1^* \circ \theta + e_2^* \circ \theta - e_{12}^* \circ \theta,$$

[1, p.15]. This map μ is defined similarly to θ , but now the formula for

$$\mu(\phi)(a, b, u_1, \dots, u_k)$$

sums over all possible ways to distributes conjugates of both a and b between the u_i while always keeping a before b . These formulas have further natural generalizations and there is $\eta: C^{k+3}(\mathcal{G}; U(1)) \rightarrow C^k(\mathcal{G}^3, U(1))$ such that

$$(7) \quad \delta \circ \eta + \eta \circ \delta = e_{1,2}^* \circ \mu + e_{12,3}^* \circ \mu - e_{2,3}^* \circ \mu - e_{1,23}^* \circ \mu.$$

A further important ingredient in the definition of (1) is the construction of a push-forward map in twisted K -theory

$$f_*: K^{f^*\tau}\mathcal{G} \rightarrow K^\tau\mathcal{H}$$

for homomorphisms $f: \mathcal{G} \rightarrow \mathcal{H}$ satisfying suitable conditions, see [1, Theorem 3.10]. This push-forward satisfies (4) for pull-back diagrams and (5).

In the discrete case, i.e., if \mathcal{G} is the transport groupoid of a finite group on a finite set, the product (1) can be defined for twistings of the form $\tau = \theta(\tilde{\tau})$ by

$$a \star b := ((e_{12})_* \circ I_{\mu(\tilde{\tau})})(e_1^*(a) \cdot e_2^*(b)),$$

where \cdot denotes the usual product $K^{e_1^*(\tau)}\mathcal{G}^2 \otimes K^{e_2^*(\tau)}\mathcal{G}^2 \rightarrow K^{e_1^*(\tau)+e_2^*(\tau)}\mathcal{G}^2$ that adds the twists and is induced from the tensor product. From (6) one obtains $\delta(\mu(\tilde{\tau})) + e_{12}^*(\theta(\tilde{\tau})) = e_1^*(\theta(\tilde{\tau})) + e_2^*(\theta(\tilde{\tau}))$. Therefore there is an isomorphism

$$I_{\mu(\tilde{\tau})}: K^{e_1^*(\tau)+e_2^*(\tau)}\mathcal{G}^2 \rightarrow K^{e_{12}^*(\tau)}\mathcal{G}^2$$

matching the twists in the definition of $a \star b$. Computations similar to (2) yield

$$(8) \quad (a \star b) \star c = ((\tilde{e}_{123})_* \circ I_{(e_{12,3}^* \circ \mu + e_{1,2}^* \circ \mu)(\tilde{\tau})})(\tilde{e}_1^* a \cdot \tilde{e}_2^* b \cdot \tilde{e}_3^* c),$$

$$(9) \quad a \star (b \star c) = ((\tilde{e}_{123})_* \circ I_{(e_{1,23}^* \circ \mu + e_{2,3}^* \circ \mu)(\tilde{\tau})})(\tilde{e}_1^* a \cdot \tilde{e}_2^* b \cdot \tilde{e}_3^* c).$$

and (7) shows that (8) and (9) agree.

If \mathcal{G} is an almost complex orbifold, but no longer discrete, then there is an additional complication: the analogon of the diagram (3) is no longer a pull-back diagram, but only a clean intersection. Consequently, (4) no longer holds. Instead

$$(e_1^*((e_{12})_*(a))) = (e_{12,3})_*(e_{1,2}^*(a) \cdot e_K(\nu)),$$

where $e_K(\nu)$ is the K -theory Euler class of excess bundle of the clean intersection. However, using the so-called obstruction bundle $E_{\mathcal{G}^2}$ Adem-Ruan-Zhang define the product (1) in this situation by

$$a \star b := ((e_{12})_* \circ I_{\mu(\tilde{\tau})})(e_1^*(a) \cdot e_2^*(b) \cdot e_K(E_{\mathcal{G}^2}))$$

and prove associativity using the obstruction bundle equation [1, Theorem 7.1] that relates $E_{\mathcal{G}^2}$ to the excess bundle ν .

REFERENCES

[1] Alejandro Adem, Yongbin Ruan and Bin Zhang, *A stringy product on twisted orbifold K-theory*, arXiv:math.AT/0605534.

Talk 12: Twisted K-theory and T-duality

NATHALIE WAHL

T-duality is a notion in (super) conformal field theory. More specifically, it is the observation that type IIA and IIB-string theories become equivalent on T-dual targets. If the targets are torus bundles equipped with B-fields, then the corresponding fields are metrics, connections and the B-field. The Buscher rules describe how these fields transform under transition to the T-dual target [1].

The underlying topology was formalized in [2]. Topological T-duality is a relation between pairs (E, η) where $\pi : E \rightarrow B$ is a principal torus bundle and

$\eta \in H^3(E, \mathbb{Z})$ is a twist. A duality between two such pairs is expected to give an isomorphism between the twisted K-theories of the bundles.

In the case of $U(1)$ -bundles, i.e. when the torus is one-dimensional, each pair (E, η) has a unique dual $(\hat{E}, \hat{\eta})$ and there is a T-duality transformation

$$T : K(E, \eta) \xrightarrow{\cong} K(\hat{E}, \hat{\eta}).$$

Moreover, the Chern classes c, \hat{c} of E, \hat{E} , and twists $\eta, \hat{\eta}$ are interchanged according to the following relation: $\pi_!(\eta) = \hat{c}$ and $\hat{\pi}_!(\hat{\eta}) = c$.

Let P denote the contravariant functor that takes a space B to the set $P(B)$ of isomorphism classes of pairs (E, η) with E a $U(1)$ -bundle over B . There is a representing space R for P , i.e. a space R so that $P(B) \cong [B, R]$ and the T-duality transformation can be realized as an automorphism of R .

The T-duality for higher dimensional torus bundles is more complicated: both the existence and uniqueness of T-duals fail in general [3]. One also needs to be more precise about twists in higher dimensions. By a *twist*, we now mean a locally trivial $K(H)$ -bundle $\mathcal{H} \rightarrow E$, where $K(H)$ is the algebra of compact operators on an infinite dimensional Hilbert space H . Morphisms of twists are homotopy classes of algebra bundle isomorphisms. Isomorphism classes of twists over E are classified by $H^3(E, \mathbb{Z})$ and their groups of automorphisms are isomorphic to $H^2(E, \mathbb{Z})$. A *T-duality triple* is a triple $((E, \mathcal{H}), (\hat{E}, \hat{\mathcal{H}}), u)$ where \mathcal{H} is a twist over E , $\hat{\mathcal{H}}$ a twist over \hat{E} and $u : p^*\mathcal{H} \rightarrow \hat{p}^*\hat{\mathcal{H}}$ is a morphism between the twists pulled back along the projections

$$E \xleftarrow{p} E \times_B \hat{E} \xrightarrow{\hat{p}} \hat{E}.$$

The twists and the isomorphism u must satisfy some conditions [3, Def. 2.8]. We say that two pairs (E, η) and $(\hat{E}, \hat{\eta})$ are dual if there exists a T-duality triple $((E, \mathcal{H}), (\hat{E}, \hat{\mathcal{H}}), u)$ with $[\mathcal{H}] = \eta$ and $[\hat{\mathcal{H}}] = \hat{\eta}$. When such a triple exists, it induces an isomorphism $K(E, \mathcal{H}) \xrightarrow{\cong} K(\hat{E}, \hat{\mathcal{H}})$.

T-duality can also be described in terms of C^* -algebras, where a pair (E, η) is replaced by a continuous trace algebra with spectrum E [4]. In the case of T^2 -bundles, Mathai and Rosenberg show that when a “classical dual” does not exist, the T-dual is a “bundle of non-commutative tori”.

REFERENCES

- [1] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: topology change from H-flux*, Comm. Math. Phys. **249**(2) (2004), 383–415.
- [2] U. Bunke and T. Schick, *On the topology of T-duality*, Reviews in Math. Phys. **17** no. 1 (2005), 77–112.
- [3] U. Bunke, P. Rumpf and T. Schick, *The topology of T-duality for T^n -bundles*.
- [4] V. Mathai and J. Rosenberg, *T-duality for torus bundles with H-fluxes via non-commutative topology*, Comm. Math. Phys. **253**(3) (2005), 705–721.

Talk 13: Equivariant twists of compact Lie groups

KATE GRUHER

In this talk we define the twisted loop groups $L_P G$ of a compact Lie group G and describe how a central extension of $L_P G$ gives rise to a twisting for the K -theory of G , equivariant with respect to the action of G on itself by conjugation. We then study certain admissible central extensions and their positive energy representations. The talk closely follows the discussion in [2]; many of the ideas are also discussed in [5] and [4].

Twisted Loop Groups. Principal G -bundles over S^1 are classified by conjugacy classes in $\pi_0(G)$. Given such a bundle $G \rightarrow P \rightarrow S^1$, let $G[P]$ be the union of the components of G in the conjugacy class classifying P , and let $L_P G$ be the group of smooth gauge transformations of P . If P is the trivial G -bundle then $L_P G$ is the loop group LG , so we consider $L_P G$ to be a “twisted” loop group. $L_P G$ is an infinite dimensional Lie group and we denote its Lie algebra by $L_P \mathfrak{g}$. $L_P G$ acts on the space \mathcal{A}_P of connections on P by pulling back the connection.

Proposition 1. *There is an equivalence of stacks $[\mathcal{A}_P/L_P G] \cong [G[P]/G]$.*

Given a quotient stack $[X/H]$ with H a topological group, a central extension $1 \rightarrow S^1 \rightarrow \tilde{H} \rightarrow H \rightarrow 1$ induces an S^1 -gerbe $[X/\tilde{H}] \rightarrow [X/H]$. Hence by Proposition 1 a central extension of $L_P G$ by S^1 induces an S^1 -gerbe on $[G[P]/G]$, ie a twisting of $K^*([G[P]/G])$.

Proposition 2. *If G is connected and semi-simple, then every S^1 -gerbe on $[G/G]$ arises from a central extension of G .*

Note. *If G is connected then $P = S^1 \times G$, $L_P G = LG$ and $G[P] = G$.*

Admissible Central Extensions. Let Π_{rot} be the group of rigid rotations of S^1 , and let $(L_P G)_{\text{rot}}$ be the group of smooth automorphisms of P that cover elements of Π_{rot} . For any connected finite covering $\hat{\Pi}_{\text{rot}} \rightarrow \Pi_{\text{rot}}$ define $\hat{L}_P G$ to be the pullback:

$$\begin{array}{ccc} \hat{L}_P G & \longrightarrow & \hat{\Pi}_{\text{rot}} \\ \downarrow & & \downarrow \\ (L_P G)_{\text{rot}} & \longrightarrow & \Pi_{\text{rot}}. \end{array}$$

Definition 3. A central extension $1 \rightarrow S^1 \rightarrow (L_P G)^\tau \rightarrow L_P G \rightarrow 1$ is admissible if

- (1) There exists a covering $\hat{\Pi}_{\text{rot}}$ and a central extension $(\hat{L}_P G)^\tau$ of $\hat{L}_P G$ which fits into the commutative diagram

$$\begin{array}{ccccccc}
 & & & 1 & & 1 & \\
 & & & \downarrow & & \downarrow & \\
 1 & \longrightarrow & S^1 & \longrightarrow & (L_P G)^\tau & \longrightarrow & L_P G \longrightarrow 1 \\
 & & \parallel & & \downarrow & & \downarrow \\
 1 & \longrightarrow & S^1 & \longrightarrow & (\hat{L}_P G)^\tau & \longrightarrow & \hat{L}_P G \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \hat{\Pi}_{\text{rot}} & \xlongequal{\quad} & \hat{\Pi}_{\text{rot}} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

- (2) There exists an $(\hat{L}_P G)^\tau$ -invariant symmetric bilinear form $\ll \cdot, \cdot \gg_\tau$ on $(\hat{L}_P \mathfrak{g})^\tau$ such that $\ll K, d \gg_\tau = -1$ for all $d \in (\hat{L}_P \mathfrak{g})^\tau$ which project to $i \in i\mathbb{R}_{\text{rot}}$, where K is the central element $i \in i\mathbb{R}$, the Lie algebra of S^1 .

Note. If G is semi-simple than any central extension of $L_P G$ is admissible.

Admissible central extensions of $L_P G$ form a group under the Yoneda product of central extensions. This product corresponds to the addition of twistings of K -theory.

Positive Energy Representations. Fix an admissible graded central extension $(L_P G)^\tau \rightarrow L_P G$. A connection $A \in \mathcal{A}_P$ gives us an operator d_A in $(\hat{L}_P \mathfrak{g})^\tau$ which is a lift of $i \in i\mathbb{R}_{\text{rot}}$.

Definition 4. Let $\rho : (L_P G)^\tau \rightarrow U(V)$ be a unitary representation on a $\mathbb{Z}/2\mathbb{Z}$ -graded complex Hilbert space. Assume ρ is even and the central S^1 acts by scalar multiplication. V has positive energy if:

- (1) ρ extends to a unitary representation $\hat{\rho} : (\hat{L}_P G)^\tau \rightarrow U(V)$.
- (2) For all $A \in \mathcal{A}_P$ the energy operator E_A is self-adjoint with discrete spectrum bounded below, where iE_A is the skew-adjoint operator $\hat{\rho}(d_A)$.

For a positive energy representation V and a connection $A \in \mathcal{A}_P$, we can decompose

$$V = \overline{\bigoplus_{e \in \mathbb{R}} V_e(A)}$$

where E_A acts on $V_e(A)$ as multiplication by e .

Finally, we state the main properties of positive energy representations.

Proposition 5. *Let $(L_P G)^\tau$ be an admissible central extension such that the form $\ll \cdot, \cdot \gg_\tau$ is positive definite on $L_P(\mathfrak{z})$, where \mathfrak{z} is the center of \mathfrak{g} . Then, up to essential equivalence:*

- (1) *Positive energy representations are completely reducible, i.e., a discrete direct sum of irreducible representations.*
- (2) *For any irreducible positive energy representation V , each $V_e(A)$ is finite dimensional.*
- (3) *If V is finitely reducible and the extension $\hat{\rho}$ is given, then if part (2) of the definition holds for one $A \in \mathcal{A}_P$ then it holds for all $A \in \mathcal{A}_P$.*
- (4) *There is a finite number of isomorphism classes of irreducible positive energy representations of $(L_P G)^\tau$.*

REFERENCES

- [1] U. Bunke and I. Schröder. Twisted K-theory and TQFT. In *Mathematisches Institut Universität Göttingen Seminars WS04-05*, (Y. Tschinkel, ed.), 2005, pages 33-80.
- [2] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory II, preprint: math.AT/0511232.
- [3] J. Heinloth. Notes on differentiable stacks. In *Mathematisches Institut Universität Göttingen Seminars WS04-05*, (Y. Tschinkel, ed.), 2005, pages 1-32.
- [4] A. Pressley and G. Segal. *Loop Groups*. Clarendon Press, Oxford, 1986.
- [5] T. Schick. Loop groups and string topology, <http://www.uni-math.gwdg.de/schick/teach>.

Talk 14: Construction of twisted K -theory classes from positive energy representations

GREGORY D. LANDWEBER

Let G be a compact Lie group and let the twisted loop group $L_P G$ be the space of gauge transformations on a principal G -bundle $P \rightarrow S^1$. Letting $R^\tau(L_P G)$ denote the Grothendieck group of positive energy representations of $L_P G$ of level τ , we construct an additive group homomorphism to twisted equivariant K -theory,

$$(1) \quad R^\tau(L_P G) \rightarrow \tau + \sigma K_G^{\dim G}(G[P]),$$

where G acts on itself by conjugation, and the shift σ is the level of the spin representation of $L_P G$. This map can be viewed as providing either a finite dimensional model for the Verlinde algebra (we do not address the product structures here), or an explicit construction for twisted equivariant K -theory classes.

Letting \mathcal{A}_P denote the affine space of connection on the bundle P , the holonomy map $\mathcal{A}_P \rightarrow G$ induces an equivalence of quotient stacks

$$[\mathcal{A}_P / L_P G] \cong [G[P] / G],$$

where G acts on itself by conjugation. The homomorphism (1) factors through the following fictitious Thom isomorphism:

$$“ \tau K_{L_P G}(\text{pt}) \rightarrow \tau + \sigma K_{L_P G}^{\dim L_P G}(\mathcal{A}_P) ”,$$

and the purpose of this talk is to define this Thom homomorphism rigorously. Given a positive energy representation of $L_P G$ of level τ , we construct a $L_P G^{\tau+\sigma}$ -equivariant family of Dirac operators

$$\mathcal{A}_P \ni A \longmapsto \not{D}_A \in \text{Fred}(V \otimes \mathcal{S}_{L_P G}),$$

which ostensibly determines a $L_P G$ -equivariant twisted K -theory class on \mathcal{A}_P . This construction originally appeared in a slightly simpler version in [5], and our treatment follows [1, 2].

For most of this talk we work in greater generality, constructing a Dirac family not just for $L_P \mathfrak{g}$, but rather for a general (infinite dimensional) real Lie algebra \mathfrak{h} , equipped with an ad-invariant (non-degenerate) inner product and a polarization

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}^+ \oplus \mathfrak{h}_{\mathbb{C}}^0 \oplus \mathfrak{h}_{\mathbb{C}}^-,$$

where \mathfrak{h}^0 is finite dimensional and orthogonal to $\mathfrak{h}_{\mathbb{C}}^{\pm}$, both of which are isotropic. We construct the Clifford algebra $\text{Cl}(\mathfrak{h})$, and following [3], we construct the spin representation whose finite elements are given by

$$\mathcal{S}_{\mathfrak{h}} := \mathbb{S}_{\mathfrak{h}^0} \otimes \Lambda_{\mathbb{C}}^{\bullet}(\mathfrak{h}_{\mathbb{C}}^+).$$

To make sense of products in the Clifford algebra, we introduce a normal ordering homomorphism $\text{Cl}(\mathfrak{h}) \rightarrow \Lambda(\mathfrak{h})^*$, and we actually work with the completion of $\text{Cl}(\mathfrak{h})$ in $\text{End } \mathcal{S}_{\mathfrak{h}}$ and its image in $\Lambda(\mathfrak{h})^*$. This allows us to introduce a Clifford product on a subspace of the exterior algebra, as in [3, 4].

For $\xi \in \mathfrak{h}^*$, the 2-form $d\xi \in \Lambda^2(\mathfrak{h})^*$ is given by

$$d\xi(X, Y) = -\frac{1}{2} \xi([X, Y])$$

for $X, Y \in \mathfrak{h}$, and we compute

$$\{X^*, \omega\} = 2 \iota_X \omega, \quad [dX^*, \omega] = 2 \text{ad}_X^* \omega,$$

for $X \in \mathfrak{h}$ and any $\omega \in \Lambda^{\bullet}(\mathfrak{h})^*$. Setting

$$S(X) := \frac{1}{2} dX^*,$$

it follows that

$$[S(X), S(Y)] = S([X, Y]) + \text{central terms},$$

for $X, Y \in \mathfrak{h}$, and thus S is a projective representation of \mathfrak{h} on $\mathcal{S}_{\mathfrak{h}}$, intertwining with the Clifford action to give a projective representation of $\mathfrak{h} \hat{\oplus} \text{Cl}(\mathfrak{h})$ on $\mathcal{S}_{\mathfrak{h}}$.

Given a skew-hermitian derivation d of \mathfrak{h} whose eigenspaces are finite dimensional, we can use it to define a polarization by decomposing $\mathfrak{h}_{\mathbb{C}}$ into its the positive, zero and negative eigenspaces of $-id$. A projective representation of \mathfrak{h} is called *admissible* with respect to d if its projective cocycle is of the form

$$(2) \quad [S(X), S(Y)] - S([X, Y]) = -i \langle\langle dX, Y \rangle\rangle_{\sigma},$$

in terms of some ad-invariant inner product $\langle\langle \cdot, \cdot \rangle\rangle_{\sigma}$ for which d is skew-hermitian (this inner product need not be the same one we started with).

Given an admissible central extension \mathfrak{h}^τ of \mathfrak{h} , the formal Dirac operator is

$$\not\partial_0 := \hat{R} + 1 \otimes \Omega \in U(\mathfrak{h})^\tau \otimes \Lambda^\bullet(\mathfrak{g})^*,$$

where \hat{R} is the tautological $U(\mathfrak{h})^\tau$ -valued 1-form corresponding to the inclusion $R : \mathfrak{h} \hookrightarrow U(\mathfrak{h}^\tau)$, and $\Omega \in \Lambda^3(\mathfrak{h})^*$ is the fundamental 3-form

$$\Omega(X, Y, Z) = -\frac{1}{12} \langle X, [Y, Z] \rangle$$

for $X, Y, Z \in \mathfrak{g}$. Letting

$$T(X) := R(X) \otimes 1 + 1 \otimes S(X)$$

be the total action, the Dirac operator $\not\partial_0$ is characterized by the commutator

$$\{X^*, \not\partial_0\} = 2 \iota_X \not\partial_0 = 2T(X).$$

In addition, if the spin representation S and the representation R are both admissible, then the total representation T is admissible, and we have the identities

$$(3) \quad [T(X), \not\partial_0] = -i(dX)^*, \quad \not\partial_0^2 = 2iT(d) + \text{central terms},$$

provided that we use the inner product at the level $\sigma + \tau$ of T (as in (2)).

For $\mu \in \mathfrak{h}^*$ we define the formal Dirac family $\not\partial_\mu := \not\partial_0 + i\mu$. Then (3) implies

$$(4) \quad [T(X), \not\partial_\mu] = -i(dX)^* + i \operatorname{ad}_X^* \mu,$$

where the right hand side is the affine coadjoint action of $\mathfrak{h}^{\sigma+\tau}$ on \mathfrak{h}^* (see [6]). Thus, the Dirac family is $\mathfrak{h}^{\sigma+\tau}$ -equivariant. In addition, it follows from (3) that

$$(5) \quad \not\partial_\mu^2 = 2iT(d + \mu) + \text{central terms}.$$

Returning to our initial problem, we are interested in the case where $\mathfrak{h} = L_P \mathfrak{g}$, the space of sections of the bundle $P \times_G \mathfrak{g}$, with a polarization determined by the energy operator d_{A_0} corresponding to some fixed connection $A_0 \in \mathcal{A}_P$ on P . A direct computation shows that the spin representation $\mathcal{S}_{L_P \mathfrak{g}}$ is an admissible positive energy representation of $L_P G$ at level σ given by the inner product

$$\langle\langle Y, Z \rangle\rangle_\sigma = -\frac{1}{2} \int_{S^1} \operatorname{Tr}(\operatorname{ad} Y(\theta), \operatorname{ad} Z(\theta)) \frac{|d\theta|}{2\pi}$$

for $Y, Z \in L_P \mathfrak{g}$.

For a general connection, we map

$$\mathcal{A}_P \ni A = A_0 + a \longmapsto a^* \in L_P \mathfrak{g}^*,$$

giving a $L_P G^{\sigma+\tau}$ -equivariant map $\mathcal{A}_P \rightarrow L_P \mathfrak{g}^*$, since the affine coadjoint action on the right hand side of (4) is dual to an infinitesimal gauge transformation. We can therefore construct a $L_P G^{\sigma+\tau}$ -equivariant formal Dirac family

$$\mathcal{A}_P \ni A \longmapsto \not\partial_A \in U(L_P \mathfrak{g})^\tau \otimes \Lambda^\bullet(L_P \mathfrak{g})^*.$$

If V is a positive energy representation of $L_P \mathfrak{g}^\tau$, then the eigenspaces of the energy operators $T(d_A) = T(d_{A_0} + a)$ are all finite dimensional. If V is irreducible, then it follows from (5) and Schur's lemma that $\ker \not\partial_A^2$ is finite dimensional for all A , and since $\not\partial_A$ is self-adjoint, we have $\ker \not\partial_A = \ker \not\partial_A^2$. So, the Dirac family

acting on $V \otimes \mathcal{S}_{L_P \mathfrak{g}}$ is Fredholm. However, these operators are not bounded, and the family is not continuous, which we fix by forcing our operators to be bounded:

$$\mathcal{A}_P \ni A \longmapsto \mathcal{F}_A := \not\partial_A (1 + \not\partial_A^2)^{-1/2} \in \text{Fred}(V \otimes \mathcal{S}_{L_P \mathfrak{g}}).$$

This new family is $L_P G^{\sigma+\tau}$ -equivariant, Fredholm, and continuous, and so it determines a class in the twisted K -theory of the corresponding quotient stacks,

$$\sigma+\tau K^{\dim G}([\mathcal{A}_P/L_P G]) \cong \sigma+\tau K^{\dim G}([G[P]/G]).$$

(The degree shift comes from an additional $\text{Cl}(1)$ -action when $\dim G$ is odd.)

REFERENCES

- [1] D. S. Freed, M. J. Hopkins, C. Teleman, *Loop groups and twisted K-theory II*. [arXiv:math.AT/0511232](#)
- [2] D. S. Freed, M. J. Hopkins, C. Teleman, *Loop groups and twisted K-theory III*. [arXiv:math.AT/0312155](#)
- [3] B. Kostant and S. Sternberg, *Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras*, *Ann. Physics* **176** (1987), 49–113.
- [4] G. D. Landweber, *Multiplets of representations and Kostant's Dirac operator for equal rank loop groups*, *Duke Math. J.* **110** (2001), 121–160. [math.RT/0005057](#)
- [5] J. Mickelsson, *Twisted K-theory invariants*, *Lett. Math. Phys.* **71** (2005), 109–121. [math.AT/0401130](#)
- [6] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986.

Talk 15: The FHT-Theorem I

MARKUS SPITZWECK

In our talk we described the so-called Key Lemma [1, Lemma 5.8] needed in the proof of the FHT-theorem and an application of the Key Lemma to the stack $[N/N]$. Here N is an extension of a finite group by a torus and the action of N on itself is by conjugation.

To formulate the Key Lemma one makes the following Construction: Let X be a compact space, G a compact group acting on X , M a normal subgroup of G acting trivially on X and τ a twisting of $[X/G]$.

Let ${}^\tau \mathbb{P} \rightarrow X$ be a G -equivariant projective bundle representing τ . One extracts the following data:

- (1) A family $S_X^1 \hookrightarrow {}^\tau M \twoheadrightarrow M_X$ of central extensions of M over X ,
- (2) a G/M -equivariant covering $p: Y \rightarrow X$ whose fibers label the irreducible τ -projective representations of M ,
- (3) a G -equivariant tautological projective bundle $\mathbb{P}R \rightarrow Y$, whose fiber over $y \in Y$ is the projectivised τ -representation labelled by y ,
- (4) the corresponding class $[R] \in {}^{\mathbb{P}R} K_G(Y)$,
- (5) a twisting τ' of $[Y/(G/M)]$, together with an isomorphism of G -equivariant twistings $\tau' \cong p^* \tau - \mathbb{P}R$.

Lemma 1. (Key Lemma) *There is a natural isomorphism*

$$\tau' K_{G/M}(Y) \cong^\tau K_G(X),$$

where *K-theory with compact supports is used.*

Idea of proof. One shows that the map given by the composition

$$\tau' K_{G/M}^*(Y) \rightarrow \tau' K_G^*(Y) \cong p^* \tau\text{-PR} K_G^*(Y) \xrightarrow{\otimes[R]} p^* \mathcal{K}_G^*(Y) \xrightarrow{p!} \mathcal{K}_G^*(X)$$

is an isomorphism. By a slice argument and the Mayer-Vietoris property for closed coverings one reduces to the case $X = \text{pt}$. Then a purely representation theoretic argument gives maps in both directions. \square

Example 2. We apply the above construction to the case $X = \text{pt}$, $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, and $M = \mathbb{Z}/n\mathbb{Z} \subset G$ via the diagonal embedding. The possible twists are classified by $H^3(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. The set of irreducible M -representations (twists are not possible) is the character group of M , so $Y \cong \mathbb{Z}/n\mathbb{Z}$ as a set. The action of G on Y is trivial, the G/M -twist τ' of Y as well, hence independent of the given twist τ we get ${}^\tau K_G^*(\text{pt}) \cong \bigoplus_{\mathbb{Z}/n\mathbb{Z}} K_{\mathbb{Z}/n\mathbb{Z}}^*(\text{pt})$.

The proof of the main theorem uses pullback of twisted K-theory classes along maps

$$[T/T] \rightarrow [T/N] \rightarrow [N/N] \rightarrow [G/G],$$

where N is the normalizer in G of a maximal torus $T \subset G$. Hence we put ourselves in the situation where we have an extension

$$1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1,$$

where T is a torus and G a finite group, and we are given a twist of $[N/N]$ represented by a S^1 -gerbe $\mathcal{G} \rightarrow [N/N]$.

For any $f \in N$ let $N(f)$ be the stabilizer in N of the component fT . We have a decomposition of stacks

$$(3) \quad [N/N] \simeq \coprod_f [fT/N(f)],$$

where f runs through a set of representatives of conjugacy classes of W .

Here we are interested in the component of (3) indexed by $f = 1$ ([1, (6.2), (6.3)]).

The twistings of $[T/T]$ are classified by $H_T^3(T)$, T acting trivially on T , and we suppose that \mathcal{G} gives rise to a twisting τ which is classified by a class in the direct summand $H^1(T) \otimes H_T^2(\text{pt})$. We view this class as a map $\kappa_\tau : \Pi := H_1(T) \rightarrow H_T^2$ or equivalently as a map $\kappa_\tau : \Pi \rightarrow H^1(T)$, since $H^2(T) \cong X(T) \cong H^1(T)$. The construction of the Key Lemma applied to the situation $X = T$, $M = T$, $G = N$ yields a family of central extensions ${}^{\mathcal{G}}T$ of T over T , which is locally the trivial central extension. We let P_T be the corresponding locally free sheaf of \mathbb{Z} -lattices of characters of ${}^{\mathcal{G}}T$ and ${}^{\mathcal{G}}P_T$ the affine sublattice of characters of central weight 1. Set ${}^{\mathcal{G}}P := ({}^{\mathcal{G}}P_T)|_{\{1\}}$, which is called the set of \mathcal{G} -affine weights of T .

Claim: The monodromy of $\mathcal{G}P_T$ is given by κ_τ .

This follows from the description of the gerbe \mathcal{G} by gluing data in terms of line bundles (which define local autoequivalences of the stack $[\text{pt}/S^1]$) of the trivial gerbe on $t \times [\text{pt}/T]$, where t is the universal cover of T .

We will assume in the following that κ_τ is *injective*, meaning that the twist defined by \mathcal{G} is topologically regular ([1, Definition 2.2]).

In that case the bundle $p : Y \rightarrow T$ given by the Key Lemma is a disjoint union of copies of t indexed by $\mathcal{G}P/\kappa_\tau(\Pi)$, and we get isomorphisms

$$\tau K_N^*(T) \cong^{\tau'} K_W^*(Y) \cong^{\tau' - \sigma(t)} K_W^{* - \dim T}(\mathcal{G}P/\kappa_\tau(\Pi)).$$

Here $\sigma(t)$ is a twisting (in general involving gradings) defined by a W -equivariant Thom class of t .

Using the equivalence of stacks $[(\mathcal{G}P/\Pi)/W] \simeq [\mathcal{G}P/(\Pi \rtimes W)]$ one can rewrite the latter group as $\tau' - \sigma(t) K_{\Pi \rtimes W}^*(\mathcal{G}P)$.

We briefly sketch the situation for a general component of the decomposition (3) of $[N/N]$ ([1, (6.4)]). We fix $f \in N$.

The Key Lemma is applied to the situation $X = fT$, $G = N(f)$, $M = \underline{T}$, where \underline{T} is the identity component of the stabilizer T^f of f (or equivalently of fT) in T .

The Key Lemma yields

$$(4) \quad \tau K_{N(f)}^*(fT) \cong^{\tau'} K_{N(f)/\underline{T}}^*(Y)$$

where Y is a $X(\underline{T})$ -torsor whose fiber $\mathcal{G}\underline{P}$ over f is called the set of τ -affine weights of \underline{T} . One has an equivalence of stacks

$$[Y/(N(f)/\underline{T})] \simeq [(\mathcal{G}\underline{P} \times \underline{t})/W_{\text{aff}}].$$

Here \underline{t} is the universal cover of \underline{T} , and W_{aff} sits in an extension

$$1 \rightarrow \underline{\Pi} \rightarrow W_{\text{aff}} \rightarrow \widetilde{W}^f \rightarrow 1,$$

where $\underline{\Pi} = H_1(\underline{T})$ and $\widetilde{W}^f = (N(f)/\underline{T})^f$ itself is an extension

$$1 \rightarrow (T/\underline{T})^f \rightarrow \widetilde{W}^f \rightarrow W^f \rightarrow 1.$$

Under the same regularity condition as above this together with (4) yields an isomorphism ([1, Theorem 6.8])

$$\tau K_{N(f)}^*(fT) \cong \tau' - \sigma(\underline{t}) K_{W_{\text{aff}}}^{* - \dim \underline{t}}(\mathcal{G}\underline{P}).$$

REFERENCES

[1] Freed, D. and Hopkins, M. and Teleman, C., *Twisted K-theory and Loop Group Representations*, 2005, arXiv:math.AT/0312155 v2

Talk 16: The FHT-theorem II: Computation of $K_G^\tau(G)$ (after Freed, Hopkins and Teleman)

JOCHEN HEINLOTH

The object of the talk was to explain the calculation of the twisted equivariant K -theory of a compact Lie group given by Freed, Hopkins and Teleman in [1]. To simplify both notation and proof of the theorem we restricted to the case of a compact, connected and simply connected group G (see [1], Theorem 7.10 for the general statement). We tried to point out where the proof in the general situation requires additional arguments.

To state the theorem we need some notation. Fix a maximal torus $T \subset G$. Let N denote its normalizer, $W = N/T$ the Weyl group of G and Π the cocharacter group of T . Let τ be a G -equivariant admissible twisting of G , i. e. an S^1 -gerbe over $[G/G, \text{conj}]$ such that the corresponding class $[\tau] \in H_G^3(G)$ restricts to an element in $H_T^2(\text{pt}) \otimes H^1(T) \subset H_T^3(T)$, which under the canonical isomorphism $H_T^2(\text{pt}) = H^1(T)$ defines a non degenerate form $\kappa^\tau : H_1(T) \rightarrow H^1(T)$. (This condition is automatic under our simplifying assumptions, but not in the general case.) Given this data, the preceding talk defined a central S^1 -extension T^τ of T which is the automorphism group of an object of the gerbe τ over the point $1 \in [T/T]$. We denote by Λ^τ the set of weights of T^τ which restrict to the identity on the central S^1 . The affine space Λ^τ carries an action of the affine Weyl-group $W \rtimes \Pi$, where Π acts via the form κ^τ . And we denote $\Lambda^{\tau, \text{reg}}$ the elements which are not fixed by any reflection in $W \rtimes \Pi$.

Theorem 1 (Freed, Hopkins, Teleman). *There is a canonical isomorphism*

$$K_G^\tau(G) \cong K_W^{\sigma(\mathfrak{t}) - \dim T}(\Lambda^{\tau, \text{reg}} / \kappa^\tau(\Pi)).$$

In particular this group is a free \mathbb{Z} -module of rank equal to the number of elements in an open alcove in Λ^τ .

Here $\sigma(\mathfrak{t})$ is the twisting given by the representation of W on $\mathfrak{t} = \text{Lie}(T)$.

The idea of the proof is to induce classes from the torus T . Consider the natural map

$$\omega : [T/N] \rightarrow [G/G, \text{conj}].$$

Note that this map is surjective, since every element of G is conjugate to an element in T , and it is even an isomorphism over the open set of regular elements $G^{\text{reg}} \subset G$.

Moreover, the map ω is canonically K -oriented, because the G -equivariant normal bundle of the map $T \times_N G \rightarrow G$ turns out to be trivial. Therefore there is a push-forward map $\omega_* : K_N^\tau(T) \rightarrow K_G^\tau(G)$.

The first step of the proof is to show that the composition $\omega_* \omega^*$ is an isomorphism. This is done using a Mayer-Vietoris argument to reduce to the case for the Lie-algebra $[\mathfrak{g}/G]$, where the twisting is trivial, so that we can apply the construction given in the talk of M. Joachim, to see that all K -theory classes are supported on regular orbits. Since ω is an isomorphism on this finishes the first step.

Next we can use the isomorphism $K_N^\tau(T) \cong K_W^{\sigma(\mathfrak{t})-\dim(T)}(\Lambda^\tau/\kappa^\tau(\Pi))$ constructed in the preceding talk. The same argument as before gives a split inclusion $K_W^{\sigma(\mathfrak{t},\text{reg})-\dim(T)}(\Lambda^\tau/\kappa^\tau(\Pi)) \subset K_G^\tau(G)$. And we are left to see that classes supported on singular weights are mapped to 0 under ω_* . This is an analog of the Borel-Weil-Bott theorem. One way to prove this is first to replace W by the subgroup of even elements to remove the grading from $\sigma(\mathfrak{t})$ (i. e. to get a orientable representation) and then apply the Lefschetz formula as in the proof of the Borel-Weil-Bott theorem.

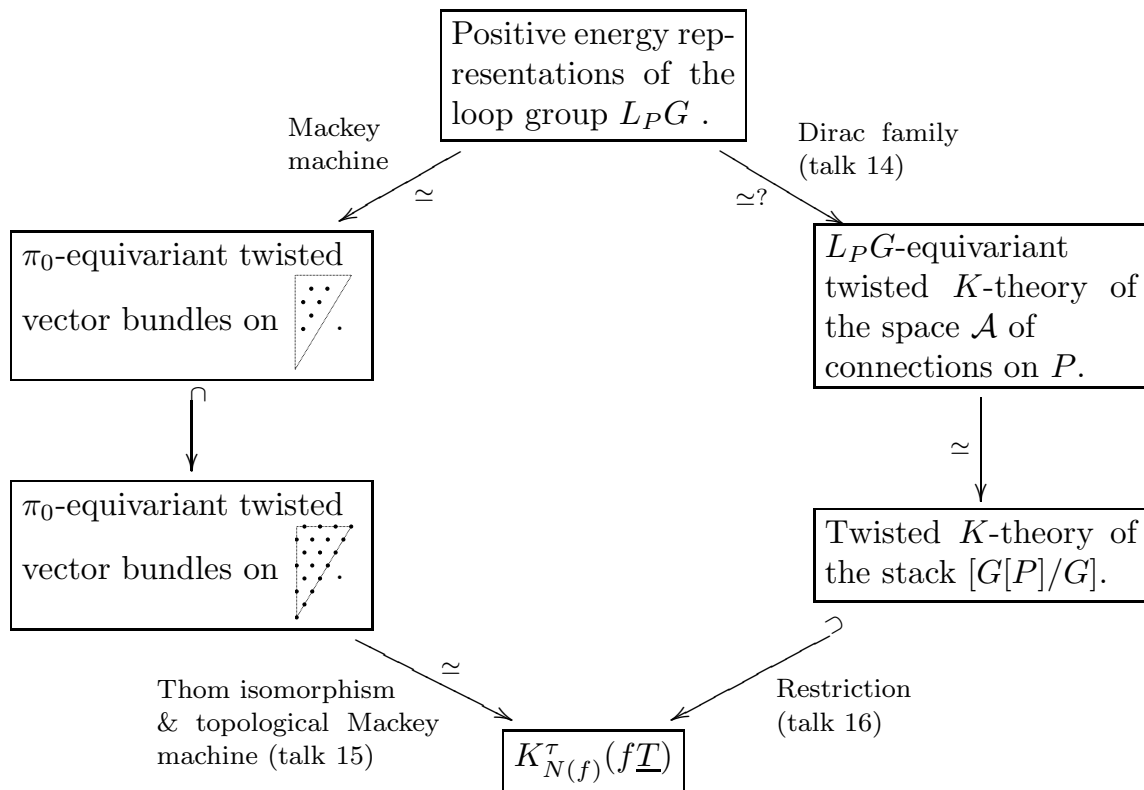
REFERENCES

- [1] D. S. Freed, M. J. Hopkins, C. Teleman, *Loop groups and twisted K-theory III*, preprint 2005, [arXiv:math.AT/0312155](https://arxiv.org/abs/math/0312155), v3
- [2] U. Bunke, I. Schröder, *Twisted K-theory and TQFT*, in *Mathematisches Institut Göttingen, Seminars WS04/05*, ed. Y. Tschinkel, (2005), 33–85.

Talk 17: The FHT-Theorem III

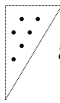
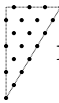
ANDRÉ HENRIQUES

In this last talk, we show the commutativity of the following diagram:



where P is a principal G -bundle over S^1 , $L_P G$ is the group of gauge transformations of P , $G[P] \subset G$ is the set of possible monodromies of connections on P , and π_0 refers to $\pi_0(L_P G)$. The positive energy representations are assumed to have a

fixed projective cocycle k , and all the twistings in the above diagram depend on k .

The symbols  and  refer to the sets of integral points of the open alcove of size $k + h^\vee$, respectively closed alcove of size $k + h^\vee$.

Following [1], we let f be the monodromy of a fixed connection on P , $\underline{T} \subset G$ be a maximal torus of $L_P G$, and $N(f) \subset G$ be the subgroup of elements that conjugate $f\underline{T}$ into itself.

The main theorem of talk 16 identified the image of the restriction map $K^\tau([G[P]/G]) \rightarrow K^\tau_{N(f)}(f\underline{T})$ with the π_0 -equivariant twisted vector bundles supported on the interior of the alcove. The commutativity of the above diagram is then used to show that:

Theorem 1 ([1]). *The Dirac family map $R^k(L_P G) \rightarrow K^\tau_{L_P G}(\mathcal{A})$ is an isomorphism.*

REFERENCES

[1] Dan Freed, Michael Hopkins and Constantin Teleman. Loop groups and twisted K-theory III AT/0312155 v3.

Appendix: Contravariant functoriality for twisted K-theory of stacks

ALEXANDER ALLDRIDGE, JEFFREY GIANSIRACUSA

The purpose of this note is to clarify what is perhaps a small gap in the literature by explaining how the twisted K -theory of topological stacks as defined in [1] is contravariantly functorial. The content here is an exposition of a result from [4], an extension from Morita equivalences to proper generalized morphisms of the main theorem of [3].

Let \mathfrak{X} be a topological stack and $\mathfrak{G} \rightarrow \mathfrak{X}$ an S^1 -banded gerbe, so there is a twisted K -group $K^{\mathfrak{G},*}(\mathfrak{X})$. If

$$\begin{array}{ccc} \mathfrak{H} & \longrightarrow & \mathfrak{G} \\ \downarrow & & \downarrow \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{X} \end{array}$$

is a morphism of S^1 -gerbes then one hopes that there is a well-defined homomorphism

$$f^* : K^{\mathfrak{G},*}(\mathfrak{X}) \rightarrow K^{\mathfrak{H},*}(\mathfrak{Y}).$$

and that these maps should respect composition. We show how to do this when f is *proper* in an appropriate sense. To keep the description simple we present the construction only in the untwisted case; it is entirely straightforward to extend to the twisted setting.

Proper generalized morphisms. By choosing an atlas we can represent a stack by a groupoid. A representable morphism of topological stacks can be given

on the level of atlases by a homomorphism of groupoids if one is allowed to refine the atlas of the domain. The data of a homomorphism out of a refinement of the domain atlas is called a ‘generalized morphism’ of groupoids. See [2] for more details.

For our purposes the bibundle formulation of generalized morphisms is more convenient. Let $A_1 \rightrightarrows A_0$ and $B_1 \rightrightarrows B_0$ be locally compact Hausdorff groupoids.

Definition 1. A generalized morphism $Z : A_\bullet \rightarrow B_\bullet$ is a diagram

$$A_0 \xleftarrow{\alpha} Z \xrightarrow{\beta} B_0$$

where Z has a left A_\bullet -action with respect to α and a commuting right B_\bullet -action with respect to β such that $Z \xrightarrow{\alpha} A_0$ is a principal B_\bullet -bundle.

Definition 2. Such a generalized morphism is said to be proper if the action of A_\bullet on Z is proper and for every compact set $K \subset B_0$ there exists a compact set $L \subset Z$ such that $\beta^{-1}(K)$ is contained in the A_\bullet -orbit of L .

Note that a homomorphism of groupoids $f : A_\bullet \rightarrow B_\bullet$ produces a generalized morphism $Z_f = A_0 \times_{f, B_0, \text{trgt}} B_1$ which is proper if f is a proper homomorphism. Also, Morita equivalences of groupoids are proper.

Locally compact groupoids with *isomorphism classes* of generalized morphisms form a category. The composition of

$$A_0 \leftarrow Z \rightarrow B_0 \text{ and } B_0 \leftarrow W \rightarrow C_0$$

is $A_0 \leftarrow Z \times_{B_\bullet} W \rightarrow C_0$. The composition of two proper generalized morphisms is again proper. For homomorphisms f, g one has $Z_{f \circ g} \cong Z_f \circ Z_g$.

From groupoids to C^* -algebras. Let (A_\bullet, λ) be a locally compact groupoid with Haar system, so $C_c(A_1)$ becomes an algebra by convolution. For each $x \in A_0$ there is a natural $*$ -representation of $C_c(A_1)$ on $L^2(A_1^x, \lambda^x)$. The *reduced norm* of $f \in C_c(A_1)$ is given by taking the supremum of the operator norms of these representations. The completion with respect to the reduced norm is the *reduced C^* -algebra* $C_r^*(A_\bullet, \lambda)$.

Let (B_\bullet, μ) be a second groupoid with Haar systems and consider a proper generalized morphism $A_\bullet \xrightarrow{Z} B_\bullet$. Associated to these two groupoids are reduced C^* -algebras $\mathcal{A} = C_r^*(A_\bullet, \lambda)$ and $\mathcal{B} = C_r^*(B_\bullet, \mu)$. We will now show how to construct a special kind of $(\mathcal{B}, \mathcal{A})$ -bimodule \mathcal{M}_Z from the generalized morphism of groupoids Z .

Start with $C_c(Z)$. This has a left action of $C_c(B_1)$ given by

$$(f \cdot \zeta)(z) = \int_{B_1^{\beta(z)}} f(b)\zeta(zb)d\mu^{\beta(z)}(b),$$

and a right action of $C_c(A_1)$ given by

$$(\zeta \cdot g)(z) = \int_{A_1^{\alpha(z)}} \zeta(a^{-1}z)g(a^{-1})d\lambda^{\alpha(z)}(a).$$

Furthermore, it has a $C_c(A_1)$ -valued inner product given as follows: choose $z \in Z$ with $\beta(z) = \text{src}(a)$ (note that by assumption B_\bullet acts freely and transitively on the fibres of β), and set

$$\langle \zeta, \xi \rangle_{\mathcal{A}}(a) = \int_{B^{\beta(z)}} \overline{\zeta(a^{-1}zb)} \xi(zb) d\mu^{\beta(z)}(b)$$

(Note that we cannot also define a $C_c(B_1)$ -valued inner product unless $Z \rightarrow B_0$ is a principal A_\bullet -bundle.) The bimodule \mathcal{M}_Z is defined as the completion of $C_c(Z)$ with respect to the norm given by the \mathcal{A} -valued norm followed by the norm on \mathcal{A} . Clearly \mathcal{M}_Z is a right \mathcal{A} -Hilbert module with a left action of \mathcal{B} .

Proposition 3. *\mathcal{B} acts by \mathcal{A} -compact operators, i.e. operators contained in*

$$\mathbb{K}_{\mathcal{A}}(\mathcal{M}_Z) = \text{operator norm closure of } \{ |\zeta\rangle\langle \zeta'| |_{\mathcal{A}} \text{ for } \zeta, \zeta' \in \mathcal{M}_Z \} \subset \text{End}(\mathcal{M}_Z).$$

Proof. Let ψ be an element of $C_c(B_1)$ and let $K = \text{supp } \psi$. By the definition of properness for generalized morphisms there exists $L \subset Z$ compact with $\beta^{-1}(K)$ contained in the A_\bullet -orbit of L . One can choose functions $f_1, \dots, f_n \in C_c(Z)$ with

$$\sum_{i=1}^n \int_{A_1^{\alpha(z)}} f_i(a^{-1}z) d\lambda^{\alpha(z)}(a) = 1 \text{ for all } z \in L.$$

Hence for any $z \in Z$ and $\zeta \in C_c(Z)$

$$(4) \quad (\psi \cdot \zeta)(z) = \sum_{i=1}^n \int_{A_1^{\alpha(z)}} f_i(a^{-1}z) \int_{B_1^{\beta(z)}} \psi(b) \zeta(zb) d\mu^{\beta(z)}(b) d\lambda^{\alpha(z)}(a).$$

Since $Z \xrightarrow{\alpha} A_0$ is a B_\bullet principal bundle we can set $F_i(z, zb) = f_i(z)\psi(b)$, which gives a well-defined and compactly supported function on a closed subset of $Z \times Z$. Extend F_i to an element \tilde{F}_i of $C_c(Z \times Z)$. Now the algebraic tensor product $C_c(Z) \otimes C_c(z)$ is dense in $C_c(Z \times Z)$ with respect to the obvious norm, so one can write

$$\tilde{F}_i = \sum_{k=1}^{\infty} g_{ik} \otimes \bar{h}_{ik}.$$

Then

$$(5) \quad f_i(a^{-1}z)\psi(b) = F_i(a^{-1}z, a^{-1}zb) = \sum_{k=1}^{\infty} g_{ik}(a^{-1}z) \overline{h_{ik}(a^{-1}zb)}.$$

Observe that for $\zeta, \zeta', \xi \in \mathcal{M}_Z$ we have

$$(6) \quad |\zeta\rangle\langle \zeta'| |_{\mathcal{A}} \xi(z) = \int_{A_1^{\alpha(z)}} \zeta(a^{-1}z) \int_{B_1^{\beta(z)}} \overline{\zeta'(a^{-1}zb)} \xi(zb) d\mu^{\beta(z)}(b) d\lambda^{\alpha(z)}(a).$$

By inserting (5) into (4), we see from (6) that

$$\psi \cdot \zeta = \sum_{i=1}^n \sum_{k=1}^{\infty} |g_{ik}\rangle\langle h_{ik}| \zeta |_{\mathcal{A}}.$$

(Here we use that convergence of functions \tilde{F}_i in $C_c(Z \times Z)$ implies convergence of the associated finite rank operators in the operator norm.) □

Note that the C^* -algebraic suspension $S\mathcal{M}_Z := \mathcal{M}_Z \otimes C_0(\mathbb{R})$ is naturally a right $S\mathcal{A}$ -Hilbert module with an action of $S\mathcal{B}$ by $S\mathcal{A}$ -compact operators.

Kasparov’s bivariant KK -theory. For C^* -algebras \mathcal{A}, \mathcal{B} , Kasparov has defined a bivariant theory $KK(\mathcal{B}, \mathcal{A})$ such that

- (1) $K_*(\mathcal{A}) \cong KK(\mathbb{C}, S^*\mathcal{A})$.
- (2) There are associative Kasparov products

$$KK(\mathcal{C}, \mathcal{B}) \times KK(\mathcal{B}, \mathcal{A}) \overset{\otimes}{\rightarrow} KK(\mathcal{C}, \mathcal{A}).$$

- (3) A right \mathcal{A} -Hilbert module with a left action of \mathcal{B} by \mathcal{A} -compact operators represents a class in $KK(\mathcal{B}, \mathcal{A})$. (But not all classes are represented by such bimodules.)

Now, suppose $Z : A_\bullet \rightarrow B_\bullet$ is a generalized morphism of groupoids. Pick Haar systems and form the reduced C^* -algebras $\mathcal{A} = C_r^*(A_\bullet, \lambda)$ and $\mathcal{B} = C_r^*(B_\bullet, \mu)$, and form the bimodule \mathcal{M}_Z . We get a morphism $K^*(B_\bullet) \rightarrow K^*(A_\bullet)$ from the diagram

$$K^{-i}(B_\bullet) = K_i(\mathcal{B}) \cong KK(\mathbb{C}, S^i\mathcal{B}) \xrightarrow{(-) \otimes S^i \mathcal{M}_Z} KK(\mathbb{C}, S^i\mathcal{A}) \cong K^{-i}(A_\bullet).$$

These morphisms behave correctly with respect to composition.

Proposition 7. *For generalized morphisms*

$$A_0 \leftarrow Z \rightarrow B_0 \text{ and } B_0 \leftarrow W \rightarrow C_0$$

one has the identity $[\mathcal{M}_{W \circ Z}] = [\mathcal{M}_W] \otimes [\mathcal{M}_Z]$ in $KK(\mathbb{C}, \mathcal{A})$.

Proof. This is because the algebraic tensor product $C_c(W) \otimes_{\mathcal{B}} C_c(Z)$ becomes isomorphic to $C_c(W \times_{B_\bullet} Z)$ after completing both of these into C^* -algebras. \square

Proposition 8. *If Z is the identity generalized morphism then multiplication by \mathcal{M}_Z is the identity on KK -theory.*

Proof. The identity generalized morphisms on A_\bullet is $A_0 \xleftarrow{\text{trgt}} A_1 \xrightarrow{\text{trgt}} A_0$. So \mathcal{M}_{id} is $\mathcal{A} = C_r^*(A_\bullet, \lambda)$, and tensoring with this is clearly the identity. \square

Thus we have a functor from locally compact groupoids with Haar system and proper generalized morphisms to the category of $\mathbb{Z}/2$ -graded abelian groups.

Theorem 9. *The above gives a well-defined functor from (locally compact Hausdorff) topological stacks and representable proper morphisms to $\mathbb{Z}/2$ -graded abelian groups.*

Proof. Given any two atlases for a stack there is a canonical generalized isomorphism between them (i.e. a Morita equivalence). Furthermore, given any choice of Haar systems on these atlas groupoids the associated reduced C^* -algebras \mathcal{A}, \mathcal{B} have a canonical bimodule (constructed from the canonical generalized isomorphism) representing an element in $KK(\mathcal{B}, \mathcal{A})$. Multiplication by this element induces a canonical isomorphism between the K -groups constructed from the two atlases and Haar systems. So the functor is well-defined on objects. It is clearly now also well-defined on arrows. \square

REFERENCES

- [1] J-L. Tu, Ping Xu, C. Laurent-Gengoux, *Twisted K-Theory of Differentiable Stacks*, Ann. Scient. Ec. Norm. Sup. **37** (2004), 841–910.
- [2] J. Heinloth, *Some Notes on Differentiable Stacks*, Mathematisches Institut Seminars, (Y. Tschinkel, ed.) Universität Göttingen (2004-2005), 1–32.
- [3] P. Muhly, J. Renault, D. Williams, *Equivalence and Isomorphism for Groupoid C^* -Algebras*, J. Operator Theory **17** (1987), 3–22.
- [4] J-L. Tu, *Non-Hausdorff Groupoids, Proper Actions and K-Theory*, Documenta Math. **9** (2004), 565–597.

Participants

Johannes Aastrup

Institut für Analysis
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover

Dr. Alexander Alldridge

Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn

Prof. Dr. Bernd Ammann

Universite Henri Poincare
Nancy 1
Institut Elie Cartan
F-54506 Vandoeuvre-les-Nancy

Dr. Arthur Bartels

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Christian Becker

Fachbereich Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Hanno von Bodecker

Mathematik XIII: Topologie
NA 1/26
Fakultät für Mathematik
Universitätsstrasse 150
44780 Bochum

Prof. Dr. Ulrich Bunke

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Vladislav Chernysh

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Christopher S. Dwyer

Dept. of Mathematical Sciences
State University of New York
at Binghamton
Binghamton, NY 13902-6000
USA

Johannes Ebert

Mathematisches Institut der
Universität Bonn
Beringstr. 1
53115 Bonn

Prof. Dr. Gerd Faltings

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Alessandro Fermi

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Martin Fuchssteiner

Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. David Gepner

Dept. of Pure Mathematics
Hicks Building
University of Sheffield
GB-Sheffield S3 7RH

Jeffrey H. Giansiracusa

Merton College
Oxford University
GB-Oxford OX1 4JD

Heiko Gimperlein

Institut für Analysis
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Sebastian Goette

Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Jose Manuel Gomez

Dept. of Mathematics
The University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
USA

Prof. Dr. Jesper Grodal

Department of Mathematics
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen

Ms. Kathleen Gruher

Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Tobias Hartnick

Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Jochen Heinloth

Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen
Universitätsstr. 3
45117 Essen

Dr. Andre Henriques

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Mike Hill

Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
USA

Prof. Dr. Dale Husemöller

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Dr. Michael Joachim

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Troels Roussau Johansen

Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn

Dr. Ivan Kausz

NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Dr. Hyun Jeong Kim

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Ulrich Pennig

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Guido Kings

NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Thomas Schick

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Thilo Kuessner

Universität Siegen
Fachbereich 6 Mathematik
Walter-Flex-Str. 3
57068 Siegen

Ansgar Schneider

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Greg Landweber

Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
USA

Prof. Dr. Elmar Schrohe

Institut für Mathematik
Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Matthias Lesch

Mathematisches Institut
Universität Bonn
Berlingstr. 1
53115 Bonn

Ferdinand Schulz

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Niko Naumann

NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Dr. Zoran Skoda

Rudjer Boskovic Institute
Theoretical Physics Division
PO Box 180
10002 Zagreb
CROATIA

Prof. Dr. Karl-Hermann Neeb

Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Markus Spitzweck

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Andreas Ott

Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Dr. Charlotte Wahl
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Moritz Wiethaup
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Nathalie Wahl
Department of Mathematics
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen

Christoph Wockel
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Bai-Ling Wang
Department of Mathematics
CMA/SMS
Australian National University
Canberra ACT 0200
AUSTRALIA

Dr. Chenchang Zhu
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Herès Cedex