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## Transport in Multi-Dimensional Random Schrödinger Operators

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ABSTRACT. Random Schrödinger operators are a topic of common interest in Mathematical Physics that connects to both Functional Analysis and Probability theory. It is believed in Physics that these operators possess a spectral regime with localised states, which do not contribute to electrical transport, and another regime with delocalised or electrically conducting states. While the first regime is understood well in mathematical terms, it is a major challenge for analysts to shed light on the delocalised phase. It is only very recently that some results have been obtained on electrical transport described by random Schrödinger operators. The meeting gathered nearly all main protagonists involved in recent advances in the theory of random Schrödinger operators and provided a forum for intensive discussion.

*Mathematics Subject Classification (2000):* 47B80, 60H25, 82B44.

### Introduction by the Organisers

This (half-size) workshop was attended by 25 participants. About fifty per cent of the attendees could be considered as “non-senior” researchers, among them also two PhD-students. The level of the workshop was high and nearly all main protagonists of the field were present. The organisers feel that the special focus and the familiar atmosphere of a small-size meeting found great appreciation among the participants.

The programme of the workshop consisted of short talks, long talks and some distinct series of lectures. The purpose of having such series – given by MICHAEL AIZENMAN, ABEL KLEIN, LEONID PASTUR and CLAUDE-ALAIN PILLET – was to underline important recent developments and stimulate new ideas.

Random Schrödinger operators play an important rôle in our understanding of electronic properties in disordered materials, such as random alloys, doped semiconductors or amorphous substances. It is well known from Physics that these materials exhibit a metal-insulator transition in three or more space dimensions, separating a conducting phase with a non-zero direct-current conductivity from an insulating phase where the direct-current conductivity vanishes. Furthermore, physicists argue that the conducting (resp. insulating) phase occurs, if the Fermi energy of the system falls into a region corresponding to delocalised (resp. localised) states of the quantum-mechanical energy operator. It was ANDERSON'S achievement in 1958 – rewarded by the 1977 Nobel Prize in Physics – to have given the first heuristic evidence for the existence of both localised and delocalised states for a discrete random Schrödinger operator in three or more space dimensions. For these very reasons, random Schrödinger operators (both discrete and in the continuum) have also attracted a great deal of attention in Mathematics over the past decades with research taking place at the interface of Functional Analysis and Probability theory. Yet, our mathematical understanding of the physical picture is still unsatisfactory.

The workshop covered a broad selection of topics of current interest in the theory of random Schrödinger operators. These include absolutely continuous spectrum on tree graphs, Anderson localisation for two interacting particles in a random environment, an extension of the fractional-moment method in the continuum, persistence of Anderson localisation for models with decaying randomness, spectral properties of random band matrices, localisation and control of Sobolev norms, an extension of Minami's estimate, linear response theory for random Schrödinger operators and Mott's law for the low-temperature behaviour of conductivities. Despite a focus on truly multi-dimensional phenomena, there were also two talks on recent interesting developments related to transport in one-dimensional systems. In addition to the above, the topics of the lecture series, which are summarised very briefly in the next paragraph, received particular attention.

In his lectures, MICHAEL AIZENMAN gave an overview on facts and conjectures which relate spectral types and level statistics for a variety of different models. Particular attention was directed at tree graphs, in the context of which some surprises were presented and elucidated. ABEL KLEIN presented a concentration inequality for functions of random variables. It is the main novel ingredient in a Bourgain-Kenig-like multi-scale analysis to prove localisation at extremal energies for alloy-type random Schrödinger operators with an arbitrary single-site distribution of the random coupling constants. Electrical conductivities and higher-order density correlations in disordered materials stood in the centre of the lectures delivered by LEONID PASTUR. He reviewed a multitude of their characteristic properties, which are expected or known in Physics, thus stimulating many different directions for future mathematical research. Finally, CLAUDE-ALAIN PILLET presented an operator-algebraic formalism for a mathematical description of transport in quasi-free fermionic systems.

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It is the organisers' great pleasure to thank the Oberwolfach institute for providing such a stimulating atmosphere and excellent research infrastructure. Both contributed to the overall success of the meeting.



## Workshop: Transport in Multi-Dimensional Random Schrödinger Operators

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## Abstracts

### On the Level Statistics for Random Operators

MICHAEL AIZENMAN

(joint work with Simone Warzel)

For operators with homogeneous disorder, it is generally expected that there is a relation between the spectral characteristics of a random operator in the infinite setup and the distribution of the energy gaps in its finite volume versions, in corresponding energy ranges. Whereas pure point spectrum of the infinite operator goes along with Poisson level statistics [14, 13], it is expected that purely absolutely continuous spectrum would be associated with level repulsion, spectral rigidity, and gap distribution resembling the corresponding random matrix (RM) ensemble. The RM conjecture echoes the broad numerical evidence and some theoretical arguments indicating that random-matrix gap statistics (GOE/GUE) are of relevance in a wide range of situations [5, 11], including the spacing of zeros of the Riemann zeta function [15]. One may add that also other statistics were noted to appear in situations of interest [4, 12, 6, 7, 9] and the explicit RM form of this conjecture is not universally embraced. At present the only examples of random operators with extensive disorder, of homogeneous strength, which are proven to exhibit ac spectrum are associated with trees [10, 1, 2, 8]. We have therefore undertaken to analyse the gap statistics for that case. The result, which is established under an auxiliary assumption (which for certain cases is proven to be satisfied) is that on finite regular rooted trees the eigenstate point process has Poissonian limit at all energies, even where the infinite regular tree exhibits absolutely continuous (ac) spectrum [3]. Though at first sight this may appear to contradict the conjecture described above, we also find that this is not so – if its statement is carefully interpreted. Upon inspection, one finds that the relevant limit of finite trees is not the infinite homogenous tree graph from which they are ‘carved out’, but rather a single-ended *canopy graph*. For this tree graph, which corresponds to a horoball as a subset of the regular tree, we prove that the random Schrödinger operator has only pure-point spectrum at any strength of the disorder.

#### REFERENCES

- [1] M. Aizenman, R. Sims, S. Warzel, *Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs*, Prob. Theor. Rel. Fields. **136** (2006) 363–394.
- [2] M. Aizenman, B. Sims, S. Warzel, *Fluctuation based proof of the stability of ac spectra of random operators on tree graphs*, In: Quantum Graphs and Their Applications, (Proceedings AMS-IMA-SIAM Joint Res. Conf., Snowbird 2005). Eds.: G. Berkolaiko, R. Carlson, S.A. Fulling, P. Kuchment (eds.). AMS Contemporary Mathematics Series, **415**, (2006) 1–14.
- [3] M. Aizenman, S. Warzel, *The canopy graph and level statistics for random operators on trees*, to appear in: Math. Phys. Anal. Geom.
- [4] S. Albeverio, P. Šeba, *Wave chaos in quantum systems with point interaction*, J. Stat.Phys. **64** (1991) 369–383.
- [5] O. Bohigas, M. J. Giannoni, C. Schmidt, *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, Phys. Rev. Lett. **52** (1984), 1–4.

- [6] G. Berkolaiko, E.B. Bogomolny, J.P. Keating, *Star graphs and Šeba billiards*, J. Phys. A **34** (2001) 335–350.
- [7] J. Breuer, P. J. Forrester, U. Smilansky, *Random discrete Schrödinger operators from random matrix theory*, Preprint math-ph/0507036.
- [8] R. Froese, D. Hasler, W. Spitzer, *Absolutely continuous spectrum for the Anderson model on a tree: geometric proof of Klein’s theorem*, Comm. Math. Phys. **269** (2007) 239–257.
- [9] R. Killip, M. Stoiciu, *Eigenvalue Statistics for CMV Matrices: From Poisson to Clock via Circular Beta Ensembles*, Preprint math-ph/0608002.
- [10] A. Klein, *Extended states in the Anderson model on the Bethe lattice*, Adv. Math. **133** (1998), 163–184.
- [11] T. Kottos, U. Smilansky, *Periodic orbit theory and spectral statistics for quantum graphs*, Ann. Phys. **274** (1999), 76–124.
- [12] Y. Last, B. Simon, *Fine structure of the zeros of orthogonal polynomials, IV. A priori bounds and clock behavior*, Preprint math/0606038.
- [13] N. Minami, *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*, Comm. Math. Phys. **177** (1996), 709–725.
- [14] S. A. Molchanov, *The local structure of the spectrum of the one-dimensional Schrödinger operator*, Comm. Math. Phys. **78** (1981), 429–446.
- [15] H. Montgomery, A. M. Odlyzko, *Gaps between zeros of the zeta function*, Topics in classical number theory, Colloq. Budapest 1981, Vol. II, Colloq. Math. Soc. János Bolyai **34** (1984) 1079–1106.

## Multi-Particle Anderson Localisation

VICTOR CHULAEVSKY

(joint work with Yuri M. Suhov)

We study spectral properties of a system of two quantum particles on an integer lattice  $\mathbb{Z}$  with a bounded short-range two-body interaction, in an external random potential field  $x \mapsto V(x, \omega)$  with independent, identically distributed values. The main result is that if the common probability density  $f$  of random variables  $V(x, \cdot)$  is analytic in a strip around the real line and the amplitude constant  $g$  is large enough (i.e. the system is at high disorder), then, with probability one, the spectrum of the two-particle lattice Schrödinger operator  $H(\omega)$  (bosonic or fermionic) is pure point, and all eigen functions decay exponentially. The proof given in this paper is based on a refinement of a multiscale analysis (MSA) scheme proposed by von Dreifus and Klein [2], adapted to incorporate lattice systems with interaction.

**The model:**

$$H(\omega) = \sum_{j=1}^2 (\Delta_j^{(1)} + g(V(x_j; \omega)) + U(x_1, x_2))$$

**Assumptions:**

- $\{V(x; \omega), x \in \mathbb{Z}^1\}$  is an i.i.d. random field with analytic probability density function (PDF)  $p_V(z)$ :

$$|\hat{p}_V(t)| \leq be^{-a|t|}$$

- $|g| \gg 1$ ; in particular, 1-particle AL holds
- The interaction potential  $U$  is of finite range  $d \geq 0$  and bounded; however, a



hard core component can be allowed (then a minor modification is required)

- quantum statistics: any (Fermi or Bose).

The strategy of the proof of exponential localisation is as follows:

(1) We prove that density of states (DoS) for a given volume  $\Lambda \subset \mathbb{Z}^2$  is analytic (in a strip around real line). This is a direct analog of the Wegner estimate for interacting systems. We also prove that the DoS in a volume  $\Lambda'_L$  of linear size  $2L$  conditioned by the potential in another volume  $\Lambda''_L$  with  $\text{dist}(\Lambda'_L, \Lambda''_L) > 8L$  is analytic, although potential samples in  $\Lambda'_L$  and in  $\Lambda''_L$  are not independent at any distance. Pairs of volumes at distance  $> 8L$  are called  $L$ -distant ( $L$ -D, for short).

(2) Results of (1) allow to prove that

$$\mathbb{P} \{ \text{dist}(\Sigma(H_{\Lambda'_L}), E) < \epsilon \} \leq C L^2 \epsilon.$$

and that

$$\mathbb{P} \{ \text{dist}(\Sigma(H_{\Lambda'_L}), \Sigma(H_{\Lambda''_L}) < \epsilon \} \leq C L^2 \epsilon.$$

(3) Following the strategy of [2], we analyse pairs of volumes  $\Lambda'_k, \Lambda''_k$  of sizes  $L_k = L_0^{\alpha_k}$ ,  $\alpha = 3/2$ , and show by induction in scale  $L_k$  that with sufficiently high probability volumes of any size  $L_k$  feature an exponential decay of Green's functions (such volume are called non-singular, or NS). In addition,

$$\mathbb{P} \{ \text{both } \Lambda'_{L_k} \text{ and } \Lambda''_{L_k} \text{ are singular} \} \leq L_k^{-2p}, \quad p > 1.$$

Compared to [2], the main technical difference of our model is that we have to treat separately pairs  $\Lambda', \Lambda'' \subset \mathbb{Z}^2$  where, respectively, 0, 1, or 2 volumes among  $\Lambda', \Lambda''$  are subject to interaction.

Finally, we show that the above mentioned extensions of Wegner estimate to interacting particle systems, combined with detailed analysis of pairs of singular volumes, imply exponential decay of Green's functions of operator  $H$  with probability one. A fairly general result of [2], [4] shows then that, with probability one, all generalised eigenfunctions of  $H$  decay exponentially at infinity.

## REFERENCES

- [1] R. Carmona, J. Lacroix, Spectral Theory of Random Schrödinger Operators. Birkhäuser, Boston, 1990.
- [2] H. von Dreifus, A. Klein, A new proof of localization in the Anderson tight binding model. Comm. Math. Phys. (1989), **124**, 285-299.
- [3] J. Fröhlich, F. Martinelli, E. Scoppola, T. Spencer, A constructive proof of localization in Anderson tight binding model. Comm. Math. Phys. (1985), **101**, 21-46.
- [4] J. Fröhlich, T. Spencer, Absence of diffusion in the Anderson tight binding model for large disorder or low energy. Comm. Math. Phys. (1983), **88**, 151-184.
- [5] L. A. Pastur, A. L. Figotin, Spectra of Random and Almost Periodic Operators. Springer-Verlag, Berlin, 1992.

## Hyperbolicity, Fractal Dimension, and Quantum Dynamics for the Fibonacci Hamiltonian

DAVID DAMANIK

(joint work with Mark Embree, Anton Gorodetski, Serguei Tcheremchantsev)

We consider the Fibonacci Hamiltonian, which is the discrete one-dimensional Schrödinger operator

$$[Hu](n) = u(n+1) + u(n-1) + \lambda \chi_{[1-\phi^{-1}, 1)}(n\phi^{-1} + \theta \bmod 1)u(n)$$

in  $\ell^2(\mathbb{Z})$ , where  $\lambda > 0$  is the coupling constant,  $\phi = \frac{\sqrt{5}+1}{2}$ ,  $\theta \in [0, 1)$  is the phase, and describe several results obtained in [2] for this operator family.

It follows by minimality of irrational rotations and strong convergence of operators that the spectrum of  $H$  is independent of the phase. That is, for every  $\lambda > 0$ , there is a compact subset  $\Sigma_\lambda$  of  $\mathbb{R}$  such that  $\Sigma_\lambda = \sigma(H)$  for every  $\theta \in [0, 1)$ . Sütő showed in 1989 that the Lebesgue measure of the spectrum is zero [3],

$$\text{Leb}(\Sigma_\lambda) = 0 \quad \text{for every } \lambda > 0.$$

It is therefore natural to study the dimension of this set. Recall that for  $S \subseteq \mathbb{R}$  bounded and infinite, the following two dimensions are of interest. For  $\alpha \in [0, 1]$ , let

$$h^\alpha(S) = \lim_{\delta \rightarrow 0} \inf_{\delta\text{-covers}} \sum_{m \geq 1} |I_m|^\alpha$$

and then define the Hausdorff dimension of  $S$  by

$$\dim_H(S) = \inf\{\alpha : h^\alpha(S) < \infty\} = \sup\{\alpha : h^\alpha(S) = \infty\}.$$

The lower box counting dimension of  $S$  is given by

$$\dim_B^-(S) = \liminf_{\varepsilon \rightarrow 0} \frac{\log N_S(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where  $N_S(\varepsilon) = \#\{j \in \mathbb{Z} : [j\varepsilon, (j+1)\varepsilon) \cap S \neq \emptyset\}$ . The upper box counting dimension,  $\dim_B^+(S)$ , is defined with a  $\limsup$  in place of the  $\liminf$ . When the lower and upper box counting dimensions coincide, we say that the box counting dimension exists and denote it by  $\dim_B(S)$ .

Our first result is the following.

**Theorem 1.** *Suppose that  $\lambda \geq 16$ . Then, the box counting dimension of  $\Sigma_\lambda$  exists and obeys  $\dim_B(\Sigma_\lambda) = \dim_H(\Sigma_\lambda)$ .*

The key to this result is the hyperbolicity of the so-called trace map as established by Casdagli [1] in 1986.

In order to describe the large coupling asymptotics of the dimension of the spectrum, let us introduce the function

$$f(x) = \frac{1}{x} [(2-3x) \log 2 + (1-x) \log(1-x) - (2x-1) \log(2x-1) - (2-3x) \log(2-3x)]$$

on  $(\frac{1}{2}, \frac{2}{3})$ .  $f$  takes its maximum at a unique point  $x^* \in (\frac{1}{2}, \frac{2}{3})$ . Write  $f^* = f(x^*) = \max_{x \in (\frac{1}{2}, \frac{2}{3})} f(x)$ . Numerics show that  $x^* \approx 0.5395$  and  $f^* \approx 0.88137$ . Moreover, let

$$S_u(\lambda) = 2\lambda + 22$$

and

$$S_l(\lambda) = \frac{1}{2} \left( (\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right).$$

**Theorem 2.** (a) *Suppose  $\lambda > 4$ . Then, we have*

$$\dim_B^-(\Sigma_\lambda) \geq \frac{f^*}{\log S_u(\lambda)}.$$

(b) *Suppose  $\lambda \geq 8$ . Then, we have*

$$\dim_H(\Sigma_\lambda) \leq \frac{f^*}{\log S_l(\lambda)}.$$

As an immediate consequence, we obtain the following exact asymptotic result. We write  $\dim$  for either  $\dim_H$  or  $\dim_B$ , which is justified by Theorem 1.

**Corollary 1.** *We have*

$$\lim_{\lambda \rightarrow \infty} \dim(\Sigma_\lambda) \cdot \log \lambda = f^*.$$

Next, we present an application of the dimensional lower bound to the rate of wave packet propagation in this model. The time-averaged moments of the position operator are given by

$$\langle |X|_{\delta_0}^p \rangle(T) = \frac{2}{T} \int_0^\infty e^{-2t/T} \sum_{n \in \mathbb{Z}} |n|^p |\langle e^{-itH} \delta_0, \delta_n \rangle|^2 dt.$$

To describe their power-law behavior, let

$$\beta_{\delta_0}^-(p) = \liminf_{T \rightarrow \infty} \frac{\log \langle |X|_{\delta_0}^p \rangle(T)}{p \log T}$$

and

$$\beta_{\delta_0}^+(p) = \limsup_{T \rightarrow \infty} \frac{\log \langle |X|_{\delta_0}^p \rangle(T)}{p \log T}.$$

Both functions  $\beta_{\delta_0}^\pm(p)$  are nondecreasing in  $p$  and hence the following limits exist,

$$\alpha_u^\pm = \lim_{p \rightarrow \infty} \beta_{\delta_0}^\pm(p).$$

Thus, the exponents  $\alpha_u^\pm$  correspond to the rate of propagation of the fastest (polynomially small) part of the wave packet.

**Theorem 3.** *For every  $\lambda > 0$  and every  $\theta \in [0, 1)$ , we have that*

$$\alpha_u^\pm \geq \dim_B^\pm(\Sigma_\lambda).$$

*Consequently, for  $\lambda > 4$  and every  $\theta$ , we have*

$$\alpha_u^\pm \geq \frac{f^*}{\log S_l(\lambda)}.$$

## REFERENCES

- [1] M. Casdagli, Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation, *Commun. Math. Phys.* **107** (1986), 295–318.
- [2] D. Damanik, M. Embree, A. Gorodetski, and S. Tcheremchantsev, The fractal dimension of the spectrum of the Fibonacci Hamiltonian, Preprint (2007).
- [3] A. Sütő, Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian, *J. Statist. Phys.* **56** (1989), 525–531.

## An Evolution Equation as the WKB-Correction in Long-Time Asymptotics of Schrödinger Dynamics

SERGUEI DENISSOV

The purpose of the talk is to discuss a recent development of the scattering theory for multidimensional Schrödinger operators. There is a hope to build suitable tools to eventually treat the cases when the potential  $V(x)$  is random and decays slowly. For instance, an interesting case is  $V(x) = V_{and}(x)\langle x \rangle^{-\gamma}$ , where  $V_{and}(x)$  is an Anderson potential. The case  $\gamma > 1/2$  was treated in earlier papers by J. Bourgain and the author. Going below the critical value  $1/2$  is interesting since one might expect that the spectrum remains a.c. at least for high dimension (recall that for 1-dim case the spectrum is pure point almost surely for any  $\gamma < 1/2$ ).

Current paper deals with a three-dimensional non-random model in which the potential  $V(x)$  satisfies the following conditions:

$$|V(x)| < C\langle x \rangle^{-\gamma}, |V_r(x)| < C\langle x \rangle^{-\gamma-1}, |V_{rr}(x)| < C\langle x \rangle^{-2\gamma-1}$$

where  $V_r$  denotes the radial derivative. We prove existence of modified wave-operators and present nontrivial WKB-correction to the long-time Schrödinger dynamics. This correction is described by a certain evolution equation which generalizes earlier results (e.g. [1, 2]).

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## REFERENCES

- [1] D. Yafaev, *Wave operators for Schrödinger equation*, Theoret. and Math. Phys. **45**, no. 2, (1981), 992–998.
- [2] D. Yafaev, *Scattering theory: some old and new problems*, Lecture Notes in Mathematics, **1735**, (2000), Springer-Verlag, Berlin.

## Persistence of Anderson Localization in Schrödinger Operators with Decaying Random Potentials

ALEX FIGOTIN

(joint work with François Germinet, Abel Klein, Peter Müller)

We consider Schrödinger operators with a negative and decaying random potential. Our goal is to study the discrete spectrum created by this potential below zero, and to show that a persistence of Anderson localization, and even dynamical localization, occurs. We first prove that if the envelope decays faster than  $|x|^{-2}$  at infinity, then the operator possesses infinitely many eigenvalues below zero. For envelopes decaying as  $|x|^{-\alpha}$  at infinity, we then determine the number of bound states below a given energy  $E < 0$ , asymptotically as  $\alpha \downarrow 0$ . To show that bound states located at the bottom of the spectrum are related to the phenomenon of Anderson localization that occurs for the corresponding homogeneous model, we prove: (1) that these states are exponentially localized with a localization length that is uniform in the decay rate  $\alpha$  and that (2) dynamical localization holds uniformly in  $\alpha$ .

Details can be found in [9].

### REFERENCES

- [1] Aizenman, M., Elgart, A., Naboko, S., Schenker, J. H., Stolz, G.: Moment analysis for localization in random Schrödinger operators. E-print math-ph/0308023, to appear in *Invent. Math.*
- [2] Bourgain, J.: On random Schrödinger operators on  $\mathbb{Z}^d$ , *Discrete Contin. Dyn. Syst.* **8**, 1–15 (2002).
- [3] Bourgain, J.: Random lattice Schrödinger operators with decaying potential: Some higher dimensional phenomena. In Milman, V. D., Schechtman, G. (eds.): *Geometric aspects of functional analysis. Proceedings of the Israel seminar (GAFA) 2001–2002*. Lect. Notes Math. **1807**. Springer, Berlin, 2003, pp.70–98.
- [4] Bourgain, J., Kenig, C.: On localization in the continuous Anderson–Bernoulli model in higher dimension. *Invent. Math.* **161**, 389–426 (2005).
- [5] Boutet de Monvel, A., Stollmann, P., Stolz, G.: Absence of continuous spectral type for certain non-stationary random Schrödinger operators. *Ann. H. Poincaré* **6**, 309–326 (2005).
- [6] Chen, T.: Localization lengths for Schroedinger operators on  $\mathbb{Z}^2$  with decaying random potentials. E-print math-ph/0503064, to appear in *Int. Math. Res. Not.*
- [7] Damanik, D., Hundertmark, D., Simon, B.: Bound states and the Szego condition for Jacobi matrices and Schrödinger operators. *J. Funct. Anal.* **205**, 357–379 (2003).
- [8] Denisov, S.: Absolutely continuous spectrum of multidimensional Schrödinger operator, *Int. Math. Res. Not.* **74**, 3963–3982 (2004).
- [9] Figotin, A., Germinet, F., Klein, A., Müller, P.: Persistence of Anderson localization in Schrödinger operators with decaying random potentials. *Ark. Mat.* **45**, 15–30 (2007).
- [10] Germinet, F., Hislop, P., Klein, A.: Localization for the Schrödinger operator with a Poisson random potential. In preparation.
- [11] Germinet, F., Klein, A.: Bootstrap multiscale analysis and localization in random media. *Commun. Math. Phys.* **222**, 415–448 (2001).
- [12] Germinet, F., Klein, A.: Explicit finite volume criteria for localization in continuous random media and applications. *Geom. Funct. Anal.* **13**, 1201–1238 (2003).

- [13] Germinet, F., Klein, A.: Finite volume criteria and localization at high disorder. *Markov Process. Relat. Fields* **9**, 1201–1238 (2003).
- [14] Germinet, F., Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. E-print math-ph/050317, to appear in *J. Stat. Phys.*
- [15] Germinet, F., Klein, A., Schenker, J. H.: Dynamical delocalization in random Landau Hamiltonians. E-print math-ph/0412070, to appear in *Ann. Math.*
- [16] Kirsch, W., Krishna, M., Obermeit, J.: Anderson model with decaying randomness: Mobility edge. *Math. Z.* **235**, 421–433 (2000).
- [17] Krishna M.: Anderson model with decaying randomness existence of extended states. *Proc. Indian Acad. Sci. Math. Sci.* **100**, 285–294 (1990).
- [18] Rodnianski, I., Schlag, W.: Classical and quantum scattering for a class of long range random potentials. *Int. Math. Res. Not.* **5**, 243–300 (2003).

## A Variant of an Estimate by Minami

GIAN MICHELE GRAF

(joint work with Alessio Vaghi)

We consider the Anderson model in the form

$$(1) \quad H = K + V$$

acting on  $\ell^2(\mathbb{Z}^d)$ , where  $V = \{V_x\}_{x \in \mathbb{Z}^d}$  consists of independent, identically distributed real random variables, whose common density  $\rho$  is bounded. The operator  $K = K^*$  describes short-range hopping of a particle moving on the lattice  $\mathbb{Z}^d$ . It may be the discrete Laplacian or, more generally, have matrix elements which need not be real or, equivalently, symmetric:

$$(2) \quad K(x, y) \neq K(y, x),$$

as it is e.g. the case in presence of a magnetic field.

Let  $G(z) = (H - z)^{-1}$  be the resolvent and  $\text{Im} G(z) = (G(z) - G^*(z))/2i$ . We observe that in view of (2)  $(\text{Im} G(z))(x, y) \neq \text{Im}(G(z; x, y))$ , unless  $x = y$ . The estimate is

$$(3) \quad \mathbb{E} \det((\text{Im} G(z))(x_i, x_j))_{i,j=1}^n \leq \pi^n \|\rho\|_\infty^n,$$

for any points  $x_1, \dots, x_n \in \mathbb{Z}^d$  and for  $\text{Im} z > 0$ . It also holds for the Hamiltonian  $H_\Lambda$  on  $\ell^2(\Lambda)$  obtained by truncating  $H$  to  $\Lambda$  and for  $x_1, \dots, x_n \in \Lambda$ .

The estimate was established in [3] and independently in [1] with a different proof. For  $n = 1$  it is due to [2], where its relation to Wegner's bound was pointed out; for  $n = 2$  and in the case of equality in (2) it was established by Minami for the purpose of proving Poisson distribution of eigenvalues of  $H_\Lambda$  in the localization regime. More precisely, the eigenvalue statistics near an energy  $E \in \mathbb{R}$  is described by the point process

$$\xi(\Lambda; E)(dx) = \sum_{j=1}^{|\Lambda|} \delta_{|\Lambda|(E_j - E)}(dx),$$

where  $E_j$  are the eigenvalues of  $H_\Lambda$ . In the expression  $|\Lambda|(E_j - E)$  they are rescaled by the volume  $|\Lambda|$ , so as to allow for a limiting distribution as  $\Lambda$  grows large. For

$E$  in the localization regime Minami showed that  $\xi(\Lambda; E)$  converges in law weakly to the Poisson point process  $\xi(E)$  of intensity  $n(E)dx$ ,

$$\xi(\Lambda; E)(dx) \xrightarrow[w]{\text{law}} \xi(E)(dx), \quad (\Lambda \uparrow \mathbb{Z}^d),$$

where  $n$  is the density of states.

A consequence of (3) derived in [1] is about the number of eigenvalues contained in an interval  $I$ :

$$\mathbb{P}(\#\{E_j \in I\} \geq n) \leq \frac{\pi^n}{n!} (\|\rho\|_\infty |I| |\Lambda|)^n.$$

The proof of (3) relies on Krein's formula, as did the above mentioned works in the cases  $n = 1, 2$ . Let  $\widehat{G}(z)$  be the resolvent of the Hamiltonian (1) in which  $V_{x_i}$  ( $i = 1, \dots, n$ ) have been set to zero; let  $A$  be the  $n \times n$  matrix defined by  $-(A^{-1})_{ij} = \widehat{G}(z; x_i, x_j)$ . Then the dependence of  $G(z; x_i, x_j)$  on  $v_i \equiv V_{x_i}$  is explicit:

$$(G(z; x_i, x_j))_{i,j=1}^n = (\text{Im}[\text{diag}(v_1, \dots, v_n) - A]^{-1}).$$

Further ingredients of the proof, which proceeds by induction, are the Schur complement formula and the inequality (related to Hadamard's) applying to positive  $n \times n$  matrices  $C$ :

$$\det C \leq c_{nn} \cdot \det \widehat{C},$$

where the r.h.s. refers to the  $(n-1, 1)$ -block decomposition

$$C = \begin{pmatrix} \widehat{C} & c \\ c^T & c_{nn} \end{pmatrix}.$$

The proof given in [1] is by means of a representation of the l.h.s. of (3) in terms of a Gaussian integral.

## REFERENCES

- [1] J.V. Bellissard, P.D. Hislop, G. Stolz, *Correlations estimates in the lattice Anderson model*, mp\_ arc 07-66.
- [2] L. Pastur, A. Figotin, *Spectra of random and almost-periodic operators*, Springer (Grundlehren der mathematischen Wissenschaften **297**, 1992).
- [3] G.M. Graf, A. Vaghi, *A remark on the estimate of a determinant by Minami*, Lett. Math. Phys. **79** (2007), 17–22.
- [4] N. Minami, *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*, Commun. Math. Phys. **177** (1996), 709–725.

## Estimates for Spectral Moments of Random Schrödinger Operators

PETER D. HISLOP

(joint work with Jean Bellissard, Jean-Michel Combes, Frédéric Klopp, Olivier Lenoble, Peter Müller)

The moments of spectral densities and covariant observables are important in the theory of transport for random quantum systems. The first moment of the spectral density describes the density of states measure (DOS) and the second moment of the velocity operator describes the current-current correlation measure. This talk summarizes recent progress on the existence and regularity of these moments. The basic one-particle Hamiltonian describes a charged particle interacting with a disordered environment. The Hamiltonian has the form  $H_\omega = H_0 + V_\omega$  acting on  $L^2(\mathbb{R}^d)$  or  $\ell^2(\mathbb{Z}^d)$ , for the continuum or lattice models, respectively. The background operator  $H_0 = L + V_0$  is a deterministic, periodic Schrödinger operator, where  $L$  is the nonnegative Laplacian for the continuum case, of the discrete Laplacian for the lattice case. In the continuum case, we assume that  $H_0$  satisfies the classical unique continuation principle (UCP). The random potential  $V_\omega$  is Anderson-type having the form

$$(1) \quad V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j).$$

For  $X = \mathbb{R}^d$ , the single-site potential  $u \geq 0$  and  $u \in L^\infty(\mathbb{R}^d)$ , and for  $\mathbb{Z}^d$ , we have  $u(j - k) = \delta_{jk}$ . The random variables  $\{\omega_j \mid j \in \mathbb{Z}^d\}$  form a real-valued, bounded process on  $\mathbb{Z}^d$ . The most common example is the case when the random variables are independent and identically distributed (iid) with a common probability measure  $\mu_0$  with a density  $h_0 \in L^\infty(\mathbb{R})$ .

The spectral density operator  $\rho_\omega(E)$  is defined as the boundary-value of the imaginary part of the resolvent:

$$(2) \quad \rho_\omega(E) = \lim_{\epsilon \rightarrow 0} \Im(H_\omega - E - i\epsilon)^{-1}.$$

On the lattice, the  $N^{\text{th}}$  moment of the spectral density is defined by

$$(3) \quad K_N(E_1, \dots, E_N) \equiv \mathbb{E}\{\langle 0 | \rho_\omega(E_1) A_1 \cdots \rho_\omega(E_N) A_N | 0 \rangle\},$$

with a similar expression for operators on the continuum. These moments define Radon measures on  $\mathbb{R}^N$ , and arise in the transport theory of the quantum system described by the one-particle Hamiltonian  $H_\omega$ . Among the families  $\{A_j\}$  of covariant observables, a special role is played by the velocity operator  $V_j = (-i/2)[H_\omega, X_j]$ .

### First Moment: The Density of States

The DOS measure for lattice operators is defined by (3) with  $N = 1$  and  $A = I$ , so that

$$(4) \quad d\mu(E) = \mathbb{E}\{\langle 0 | \rho_\omega(E) | 0 \rangle\} dE.$$



When it exists, the density of the measure  $\mu$ ,  $n(E)$ , is defined by the expectation of the matrix element on the right in (4). If the random variables are iid with a probability density  $h_0 \in L^\infty(\mathbb{R})$ , Wegner [15] proved  $n(E)$  exist almost surely and is a bounded function. For additional information about the DOS measure  $\mu$  for lattice models, we refer to [2]. Much less is known about this nonnegative measure for continuum models. The DOS measure is related to the integrated density of states (IDS) which is the distribution function of the DOS measure  $\nu(\lambda)$ ,

$$(5) \quad N(E) = \int_{-\infty}^E d\nu(\lambda).$$

The DOS and the IDS can be defined by a Pastur-Shubin formula that expresses these objects as the thermodynamic limit of objects defined for the system in a finite volume. Let  $\Lambda \subset \mathbb{R}^d$  or  $\Lambda \subset \mathbb{Z}^d$  be a cube. We define a finite-volume Hamiltonian  $H_\Lambda \equiv H_\omega|_\Lambda$ , with periodic boundary conditions. We then have

$$(6) \quad N(E) = \lim_{\Lambda \rightarrow X} \frac{1}{|\Lambda|} \#\{\lambda_j(\Lambda) \leq E\},$$

where  $\lambda_j(\Lambda)$  are the eigenvalues of  $H_\Lambda$ , and  $X = \mathbb{R}^d$  or  $X = \mathbb{Z}^d$ . The IDS is known to exist almost surely and is a monotone increasing function. One basic question is: Does the positive measure  $\nu$  have a density? When the operator  $H_\omega$  is ergodic, the IDS exists almost surely. We assume the existence in the general case treated in the first theorem. We define the Levy concentration for the process  $\{\omega_j\}$  as follows. For any  $j \in \mathbb{Z}^d$ , the conditional probability measure  $\mu_j$  is defined, for any measurable  $K \subset \mathbb{R}$ , by

$$(7) \quad \mu_j(K) \equiv \mathbb{P}\{\omega_j \in K \mid (\omega_l)_{l \neq j}\}.$$

We then define

$$(8) \quad s(\epsilon) \equiv \sup_{j \in \mathbb{Z}^d} \mathbb{E}\{\sup_{E \in \mathbb{R}} \mu_j([E, E + \epsilon])\}.$$

Notice that in the iid case, if the probability measure  $\mu_0$  is Hölder continuous with exponent  $0 < \alpha \leq 1$ , then  $s(\epsilon) \sim \epsilon^\alpha$ , as  $\epsilon \rightarrow 0$ .

**Theorem 1.** [5] *For  $X = \mathbb{R}^d$ , and for all intervals  $I \subset \mathbb{R}$ , there is a finite constant  $C_I > 0$  so that for all  $E \in I$  and  $\epsilon > 0$  small, we have*

$$(9) \quad 0 \leq N(E + \epsilon) - N(E) \leq C_I s(\epsilon).$$

As a corollary, we note that if the process is iid with a density  $h_0$ , as above, then the IDS is uniformly, locally Lipschitz continuous. In this case, the DOS exists as a locally bounded function. This theorem follows from a Wegner estimate of the form:

$$(10) \quad \mathbb{E}\{\text{Tr} E_\Lambda([E, E + \epsilon])\} \leq C_E s(\epsilon) |\Lambda|.$$

Wegner estimates such as (10) play an important role in some approaches to Anderson localization. There have been several recent results on the Hölder continuity of the IDS using bounds on the spectral shift function, see, for example, [4, 10, 11]. When the potential  $V_\omega$  is a Gaussian process on  $\mathbb{R}^d$ , the Lipschitz continuity of

the IDS was proved in [12]. See also [14] for addition situations for which the continuity of the IDS has been studied. We remark that similar results are known for Landau Hamiltonians perturbed by Anderson-type random potentials.

The proof of this theorem requires a refined spectral averaging theorem and a quantitative UCP. A basic spectral averaging bound, see [3], states that if  $H_\lambda = H_0 + \lambda B^2$ , where  $B^2 \geq 0$  is a bounded operator, then for any compactly support density  $g_0$ , there exists a finite constant  $C_0 > 0$  so that

$$(11) \quad \sup_{\epsilon > 0} \left\| \int d\lambda g_0(\lambda) B(H_0 + \lambda B^2 - E - i\epsilon)^{-1} B \right\| \leq C_0 \|g_0\|_\infty.$$

The quantitative UCP was proven in [4]. It basically states that if  $W \geq 0$  is a nonnegative periodic function and if  $E_0^\Lambda(\cdot)$  is the spectral family for the periodic Schrödinger operator  $H_0$  restricted to a cube  $\Lambda$  compatible with the periodicity, then for any interval  $I \in \mathbb{R}$ , there is constant  $C_I > 0$ , *independent of  $\Lambda$* , so that

$$(12) \quad E_0^\Lambda(I) W E_0^\Lambda(I) \geq C_0 E_0^\Lambda(I).$$

Lower bounds on the DOS  $n(E)$  are important in Minami's proof [13] that the energy level statistics are Poissonian in the strong localization regime. Although discussed in [15], it is only recently that a proof is given for the lattice model by the author and P. Müller. The result for continuum models is work in progress. On the lattice, the spectrum of the discrete Laplacian  $L$  is  $[-2d, 2d]$ , and we assume that the random variables are iid with a density  $h_0 > c_0 > 0$  on its support  $[W_-, W_+]$ .

**Theorem 2.** [9] *For each  $\delta > 0$ , there is a finite positive constant  $C_\delta > 0$  such that  $n(E) \geq C_\delta > 0$  for Lebesgue almost every  $E \in [-2d, 2d] + [W_- + \delta, W_+ - \delta]$ .*

### Second Moment: Current-Current Correlation Measure

Among the second moments, the second moment of the velocity operator  $\nabla_j H_\omega = (-i/2)[H_\omega, X_j]$  is important in the theory of conductivity. The *current-current correlation measure* can be defined, in analogy with the DOS formula (6), as

$$(13) \quad dm_{ij}(E, E') \equiv \mathbb{E} \{ \langle 0 | \nabla_i H_\omega E_{H_\omega}(dE) \nabla_j H_\omega E_{H_\omega}(dE') | 0 \rangle \},$$

where  $E_{H_\omega}(\cdot)$  is the spectral family for  $H_\omega$  and  $i, j = 1, \dots, d$ . We define the positive current-current correlation measure as  $dm = \sum_{i=1}^d dm_{ii}$ . It is known that this measure exists and can be obtained through a Pastur-Shubin formula expressing it as a thermodynamic limit (see [8] and references therein). One of the open problems is to determine if this measure has a density  $m(E, E')$ . Partial progress was recently made for energies outside of the diagonal  $E = E'$ . It is expected that in the strong localization regime, the density  $m(E, E')$  vanishes as  $E \rightarrow E'$ , and in the transport regime, the density is bounded from below by a positive constant when  $E \rightarrow E'$ . For the lattice model with a disorder parameter  $\lambda$  in from of the potential, we have the following result.

**Theorem 3.** [1] *If random variables in the Anderson-type random potential have a probability density  $h_0$  that admits an analytic continuation to a strip of width  $r > 0$  in the complex plane, then there is a constant  $a_0 > 0$ , depending on  $r$  and*

$d$ , so that for all  $|\lambda| > 0$  large enough, the current-current correlation measure has a density  $m(E, E')$  that is real analytic in  $\{(E, E') \in \mathbb{R}^2 \mid |E - E'| > a_0|\lambda|^{-1}\}$ .

The proof of this theorem uses the random walk expansion of the Green's function as utilized in [7], for example, for proofs of regularity of the DOS under similar conditions.

### Higher-Order Correlations

In [1], higher-order correlation measures for covariant observables were studied. Higher-order moments appear, for example, in the study of the time-evolution of powers of the position operator. The main result for lattice models is the following. Suppose we have iid random variables with a probability density having a continuation to a strip about the real axis. Then, for strong disorder, and any family of  $N$ -covariant operators  $A_j$ , the moment  $K_N(E_1, \dots, E_N)$ , defined in (3), defines a Radon measure that has a real analytic density away from a  $|\lambda|$ -dependent neighborhood of the coincident planes  $E_j = E_k$ ,  $j \neq k$ . The behavior near and at the coincident planes is an open problem.

### REFERENCES

- [1] J. Bellissard, P. D. Hislop, *Smoothness of correlations in the Anderson model at strong disorder*, Ann. H. Poincaré **8**, 1–26 (2007).
- [2] R. Carmona, J. Lacroix, *Spectral theory of random Schrödinger operators*, Boston: Birkhäuser, 1990.
- [3] J. M. Combes, P. D. Hislop, *Localization for some continuous, random Hamiltonians in  $d$ -dimensions*, J. Funct. Anal. **124**, 149 - 180 (1994).
- [4] J. M. Combes, P. D. Hislop, F. Klopp, *Hölder continuity of the integrated density of states for some random Schrödinger operators at all energies*, IMRN **2003**, no. 4, 179–209.
- [5] J. M. Combes, P. D. Hislop, F. Klopp, *An optimal Wegner estimate and its application to the integrated density of states for random Schrödinger operator*, to appear in Duke Mathematics Journal.
- [6] J. M. Combes, P. D. Hislop, S. Nakamura, *The  $L^p$ -theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators*, Commun. Math. Phys. **218**, 113–130 (2001).
- [7] Constantinescu, J. Fröhlich, T. Spencer, *Analyticity of the density of states and replica method for random Schrödinger operators on a lattice*, J. Statist. Phys. **34**, 571–596 (1984).
- [8] P. D. Hislop, O. Lenoble, *Basic properties of the current-current correlation measure for random Schrödinger operators*, J. Math. Phys. **47**, 112106–11228 (2006).
- [9] P. D. Hislop, P. Müller, *Lower bounds on the density of states for random Schrödinger operators*, preprint.
- [10] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, I. Veselić, *Bounds on the spectral shift function and the density of states*, Commun. Math. Phys. **262**, 489–503 (2006).
- [11] D. Hundertmark, B. Simon, *A diamagnetic inequality for semigroup differences*, J. Reine Angew. Math. **571**, 107–130 (2001).
- [12] T. Hupfer, H. Leschke, P. Müller, S. Warzel, *The absolute continuity of the integrated density of states for magnetic Schrödinger operators with certain unbounded random potentials*, Commun. Math. Phys. **221**, 229–254 (2001).
- [13] N. Minami, *Local Fluctuation of the Spectrum of a Multidimensional Anderson Tight Binding Model*, Commun. Math. Phys. **177**, 709–725, (1996).
- [14] I. Veselić, *Wegner estimate and the density of states of some indefinite alloy type Schrödinger operators*, Lett. Math. Phys. **59**, 135–158 (2002).

- [15] F. Wegner, *The density of states for disordered systems*, Zeit. Phys. B **44**, 9–15 (1981).

## Linear Response Theory for General Ergodic Magnetic Schrödinger Operators

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(joint work with Abel Klein)

Let  $H(t) \geq 1$  be a time-dependent self-adjoint operator on a Hilbert space  $\mathcal{H}$  with quadratic form domain  $Q(H(t))$ . If  $Q(H(t))$  is independent of  $t$ , along with other suitable conditions, we construct a unitary propagator that solves weakly the corresponding time-dependent Schrödinger equation. Using this extension of Yosida's Theorem, we justify the linear response theory for an ergodic magnetic Schrödinger operator defined as a quadratic form, and derive a Kubo formula for the electric conductivity.

### REFERENCES

- [1] Bouclet, J.M., Germinet, F., Klein, A., Schenker, J.H. *Linear response theory for magnetic Schrödinger operators in disordered media*, J. Funct. Anal. **226**, 301-372 (2005).
- [2] H. Leinfelder, C.G. Simader, *Schrödinger operators with singular magnetic potentials*, Math. Z. **176**, 1-19 (1981).
- [3] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics I: Functional Analysis*, revised and enlarged edition. Academic Press, 1980.
- [4] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Academic Press, 1975.
- [5] Schulz-Baldes, H., Bellissard, J.: *A Kinetic Theory for Quantum Transport in Aperiodic Media*. J. Statist. Phys. **91**, 991-1026 (1998).
- [6] Yosida, K.: *Functional Analysis, 6th edition*. Springer-Verlag, 1980.

## Concentration Inequalities and the Universal Occurrence of Anderson Localization

ABEL KLEIN

(joint work with François Germinet)

The Anderson Hamiltonian is the random Schrödinger operator

$$(1) \quad H_\omega := -\Delta + V_\omega \quad \text{on} \quad L^2(\mathbb{R}^d),$$

with

$$(2) \quad V_\omega(x) := \sum_{\zeta \in \mathbb{Z}^d} \omega_\zeta u(x - \zeta),$$

where

- The single-site potential  $u$  is a nonnegative bounded measurable function on  $\mathbb{R}^d$  with compact support, uniformly bounded away from zero in a neighborhood of the origin.

- $\omega = \{\omega_\zeta\}_{\zeta \in \mathbb{Z}^d}$  is a family of independent, identically distributed random variables with common probability distribution  $\mu$ , such that
  - $\mu$  is non-degenerate with compact support  $\subset [0, \infty[$ .
  - $0 \in \text{supp } \mu$ .

Without loss of generality we may just assume

$$(3) \quad \{0, 1\} \in \text{supp } \mu \subset [0, 1].$$

The Anderson Hamiltonian  $H_\omega$  is an  $\mathbb{R}^d$ -ergodic family of random self-adjoint operators. It follows from standard results that there exists fixed subsets of  $\mathbb{R}$  so that the spectrum of  $H_\omega$ , as well as the pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one. It follows from our assumptions on  $u$  and  $\mu$  that  $\sigma(H_\omega) = [0, +\infty[$  with probability one.

We prove that the Anderson Hamiltonian with single-site probability distribution  $\mu$  as in (3), but otherwise arbitrary, always exhibits Anderson and dynamical localization at the bottom of the spectrum.

We use  $\chi_x$  to denote the characteristic function of the cube of side 1 centered at 0. We write  $\langle x \rangle = \sqrt{1 + |x|^2}$ .

**Theorem 1.** *Let  $H_\omega$  be the Anderson Hamiltonian on  $L^2(\mathbb{R}^d)$  with single-site probability distribution  $\mu$  as in (3), but otherwise arbitrary. Then there exists  $E_0 > 0$  such that  $v$  exhibits Anderson localization as well as dynamical localization in the energy interval  $[0, E_0]$ . More precisely, we prove*

- (Anderson localization) *There exists  $m > 0$  such that, with probability one the operator  $H_\omega$  has pure point spectrum in  $[0, E_0]$  with eigenvalues of finite multiplicity and exponentially localized eigenfunctions with rate of decay  $m$ , i.e., if  $\phi$  is an eigenfunction of  $H_\omega$  with eigenvalue  $E \in [0, E_0]$  we have*

$$(4) \quad \|\chi_x \phi\| \leq C_{\omega, \phi} e^{-m|x|}, \quad \text{for all } x \in \mathbb{R}^d.$$

- (Dynamical localization) *For all  $s < \frac{3}{8}d$  we have*

$$(5) \quad \mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left\| \langle x \rangle^{\frac{m}{2}} e^{-itH_\omega} \chi_{[0, E_0]}(H_\omega) \chi_0 \right\|_2^{\frac{2s}{m}} \right\} < \infty \quad \text{for all } m \geq 1.$$

The theorem is proved by using the Bourgain-Kenig multiscale analysis [2] with the following concentration estimate for function of independent random variables.

The (Levy) concentration function of a probability measure  $\mu$  on  $\mathbb{R}$  is the function on  $[0, \infty]$  defined by

$$(6) \quad Q_\mu(s) := \max_{x \in \mathbb{R}} \mu \{[x, x + s]\}.$$

Given a random variable  $X$  with probability distribution  $\mu_X$ , we define its concentration function by

$$(7) \quad Q_X(s) := Q_{\mu_X}(s).$$

If  $\mu$  is as in (3), we have

$$(8) \quad 0 < Q_\mu(s) < 1 \quad \text{for all } s \in ]0, 1[, \quad Q_\mu(s) = 1 \quad \text{for all } s \geq 1.$$

Write  $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$  and let  $\{e_j := \{\delta_{ji}\}_{i=1, \dots, n}\}_{j=1, \dots, n}$  denote the standard basis in  $\mathbb{R}^n$ .

**Theorem 2.** *There exists a universal constant  $\Upsilon < \infty$  with the following property: For all random variables  $X$  with probability distribution  $\mu$  as in (3), and all random variables  $Y = F(X_1, X_2, \dots, X_n)$ , where  $n \geq 3$ ,  $\{X_i\}_{i=1, \dots, n}$  are independent copies of  $X$ , and  $F$  is a real-valued Borel function on  $[0, 1]^n$  for which there exist constants  $0 < a \leq b < \infty$  such that*

$$(9) \quad at \leq F(\mathbf{t} + te_j) - F(\mathbf{t}) \leq bt$$

for all  $t \geq 0$ ,  $\mathbf{t}, \mathbf{t} + te_j \in [0, 1]$ ,  $j = 1, 2, \dots, n$ , we have

$$(10) \quad Q_Y(s) \leq \frac{\Upsilon}{(1 - Q_X(\gamma_{n,a,b}s))^2} \frac{(\log n)^2}{\sqrt{n}},$$

where

$$(11) \quad \gamma_{n,a,b} = \frac{3}{a} \left( \frac{2nb}{a} + 2 \right)^{\frac{\log n}{2}}.$$

To prove this theorem, we prove explicit bounds on the maximal probability of antichains in multisets. Given  $k \in \mathbb{N}$ , let  $\mathcal{M}_k := \{0, 1, 2, \dots, k\}$ , a poset with the usual order. Let  $\mathbf{p} = (p_0, p_1, \dots, p_k) \in (0, 1)^{k+1}$  with  $\sum_{j=0}^k p_j = 1$ , and consider the positively weighted poset  $(\mathcal{M}_k, \mathbf{p})$ , a probability space. We set

$$(12) \quad p_+ := \max_{j=0,1,\dots,k} p_j; \quad p_- := \min_{j=0,1,\dots,k} p_j.$$

Given  $n \in \mathbb{N}$ , consider the multiset  $\mathcal{M} = \mathcal{M}_{k,n} := \{0, 1, 2, \dots, k\}^n$ . Elements of  $\mathcal{M}$  will be denoted by  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ . Given  $\mathbf{x}, \mathbf{y} \in \mathcal{M}$ , we let  $\mathbf{x} \leq \mathbf{y}$  if and only if  $x_s \leq y_s$  for all  $s = 1, 2, \dots, n$ . This makes the multiset  $\mathcal{M}$  a poset, the direct product of  $n$  copies of the poset  $\mathcal{M}_k$ . (Note  $\mathcal{M}_k = \mathcal{M}_{k,1}$ .)  $(\mathcal{M}, \mathbb{P})$  is a probability space with the product measure  $\mathbb{P}$ , the weight of  $\mathbf{x}$  is is

$$(13) \quad p_{\mathbf{x}} = \mathbb{P}(\mathbf{x}) = \prod_{s=1}^n p_{x_s}$$

The function  $r(\mathbf{x}) := \sum_{s=1}^n x_s \in [0, kn]$  is a rank function on  $\mathcal{M}$ . We consider the level sets  $L_r := \{\mathbf{x} \in \mathcal{M}; \quad r(\mathbf{x}) = r\}$  for  $r \in [0, kn]$ , and set  $W_r = \mathbb{P}(L_r)$ , the weighted  $r$ -th Whitney number. (See [1, 3].)

A subset  $\mathcal{A} \subset \mathcal{M}$  is called an antichain if it consists of incomparable elements, i.e.,  $\mathbf{x}, \mathbf{y} \in \mathcal{A}$  and  $\mathbf{x} \leq \mathbf{y}$  imply  $\mathbf{x} = \mathbf{y}$ . We define

$$(14) \quad S(\mathcal{M}, \mathbb{P}) := \max \{\mathbb{P}\{\mathcal{A}\}; \quad \mathcal{A} \subset \mathcal{M} \text{ antichain}\}.$$

For fixed  $k$  and  $\mathbf{p}$ , Engel [3, Theorem 7.2.1] gave an asymptotic estimate for  $S(\mathcal{M}_{k,n}, \mathbb{P})$ :

$$(15) \quad \lim_{n \rightarrow \infty} \sigma \sqrt{2\pi n} S(\mathcal{M}_{k,n}, \mathbb{P}) = 1.$$

We derive explicit estimates on  $S(\mathcal{M}_{k,n}, \mathbb{P})$ , the constants depending on  $k$  and  $\mathbf{p}$ . We prove:

**Theorem 3.** *Suppose*

$$(16) \quad \frac{n}{\log n} > 4k \max \left\{ \frac{1}{p_-}, \frac{p_+}{(1-p_+)^2} \right\}.$$

Then

$$(17) \quad S(\mathcal{M}_{k,n}, \mathbb{P}) \leq C \min \left\{ \frac{1}{\sqrt{kn p_-}}, \frac{k}{\sqrt{n(1-p_+)}} \right\} + \frac{C}{np_-} \min \left\{ k \log n, \frac{\log n}{kp_-} \right\} + \frac{2}{\sqrt{kn p_- \log n}}.$$

In particular, we have two useful bounds:

**Corollary 1.** *For all  $n$  such that  $np_- \geq 4k \log n$ ,*

$$(18) \quad S(\mathcal{M}_{k,n}, \mathbb{P}) \leq C \left( \frac{1}{\sqrt{kn p_-}} + \frac{k \log n}{(kp_-)^2 n} \right),$$

and for all  $n$  such that  $n(kp_-)^2 \geq 4k \log n$ ,

$$(19) \quad S(\mathcal{M}_{k,n}, \mathbb{P}) \leq C \frac{k}{\sqrt{n}} \left( \frac{1}{\sqrt{(1-p_+)}} + \frac{\log n}{\sqrt{np_-}} \right) + \frac{2}{\sqrt{kn p_- \log n}}.$$

REFERENCES

[1] Anderson, I.: *Combinatorics of finite sets*. Corrected reprint of the 1989 edition. Dover Publications Inc. Mineola, NY, 2002  
 [2] Bourgain, J., Kenig, C.: On localization in the continuous Anderson-Bernoulli model in higher dimension, *Invent. Math.* **161**, 389-426 (2005)  
 [3] Engel, K.: *Sperner theory*. Encyclopedia of Mathematics and its Applications **65**. Cambridge University Press, Cambridge, 1997

**Renormalization of Certain Exponential Matrix Cocycles**

FRÉDÉRIC KLOPP

(joint work with Alexander Fedotov)

We consider the family of Schrödinger equations with a sparse potential

$$(1) \quad -\psi''(t) + \alpha \sum_{l \geq 0} \delta(t_\phi(l) - t) \psi(t) = E\psi(t), \quad t \geq 0,$$

$$t_\phi(l) = l(l-1)/2 + l\phi_1 + \phi_2 \text{ for } l \in \mathbb{N}$$

and with Dirichlet boundary condition at zero. Here,  $\lambda$  is a positive coupling constant, and  $0 < \phi_1, \phi_2 \leq 1$  are parameters indexing the equations of the family. The analysis of the solutions of (1) can be reduced to the analysis of an ergodic

matrix cocycle that we describe now. Pick  $l \in \mathbb{Z}$ . On the interval  $t_\phi(l - 1) < t < t_\phi(l)$ , any solution of (1) has the form

$$\psi(t) = a_l^+ e^{i\sqrt{E}t} + a_l^- e^{-i\sqrt{E}t}$$

where  $a_l^\pm$  are constant coefficients. The jump conditions at  $t_\phi(l)$  imply the following relations between the coefficients  $(a_{l+1}^\pm)$  and  $(a_l^\pm)$ :

$$\vec{\psi}_{l+1}(x) = M(T_\omega^l(x), A) \vec{\psi}_l(x), \quad \vec{\psi}_l = \begin{pmatrix} a_l^+ \\ a_l^- \end{pmatrix}, \quad l \geq 0,$$

where  $x = (x_1, x_2)$  is a point on  $\mathbb{T}^2$ , the two dimensional torus which is identified to  $[0, 1]^2$ ,

$$x_1 = \left( \frac{\sqrt{E}}{\pi} \phi_1 \right) \bmod 1, \quad x_2 = \left( \frac{\sqrt{E}}{\pi} \phi_2 + \frac{1}{4} \right) \bmod 1,$$

$T_\omega$  is the skew shift on the torus,

$$T_\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + \omega \\ x_2 + x_1 \end{pmatrix}$$

where the frequency  $\omega$  given by

$$\omega = \left( \frac{\sqrt{E}}{\pi} \right) \bmod 1, \quad 0 < \omega \leq 1,$$

and  $M$  is the unimodular matrix

$$(2) \quad M(x, A) = \begin{pmatrix} A & B e(-x_2) \\ B e(x_2) & A^* \end{pmatrix}, \quad e(z) \stackrel{\text{def}}{=} e^{2\pi iz}$$

where the constants  $A$  and  $B$  are defined by

$$A = 1 - i \frac{\alpha}{2\sqrt{E}}, \quad B \geq 0, \quad B^2 = |A|^2 - 1.$$

We see that the analysis of the spectrum (1) is reduced to the analysis of the matrix cocycle

$$(3) \quad M(T_\omega^l(x), A) \dots M(T_\omega(x), A) M(x, A).$$

When  $\omega$  is irrational, the skew shift and, so, the cocycle itself is ergodic. We give a constructive description of the set  $\mathcal{L}$  of all the values of the ergodic parameter  $x$ , for which the Lyapunov exponent

$$\gamma = \lim_{N \rightarrow +\infty} \frac{1}{N} \log \|M(T^l(x), A) \dots M(T(x), A) M(x, A)\|$$

exists. For  $L = 0, 1, 2 \dots$ , define

$$\omega_{L+1} = \frac{1}{\omega_L} \pmod 1, \quad \omega_0 = \omega.$$

and

$$\lambda_{L+1} = \lambda_L^{\frac{1}{\omega_L}}, \quad \lambda_0 = |A|$$



Furthermore, for a given  $x = (x_1, x_2) \in [0, 1]^2$ , let

$$S_{L+1} = \frac{S_L}{\omega_L} \pmod{1}, \quad S_0 = x_1 - \omega - 1/2 - \frac{1}{\pi} \arg A.$$

One has

**Theorem 1.** *Let  $\lambda_L \omega_L \rightarrow \infty$ . For  $x$  as above, the Lyapunov exponent exists if and only if, there exists a sequence  $\{c_L\}_{L=1}^\infty$  of positive numbers that tends to zero such that, for all  $L \in \mathbb{Z}$ , for all  $m = 1, 2 \dots \left[ \frac{1}{\omega_L} \right]$ , one has*

$$|S_L - 1 + m\omega_L| \geq \omega_L e^{-\frac{c_L}{\omega_0 \omega_1 \dots \omega_{L-1}}}, \quad \text{if } L \text{ even}$$

and

$$|S_L - m\omega_L| \geq \omega_L e^{-c_L \log \lambda_L}, \quad \text{if } L \text{ odd}.$$

Moreover, when the Lyapunov exists, it is equal to  $\log \lambda$

The proof of this characterization relies upon a monodromization method developed for skew-shift cocycles [5], which happens to be exact in the special case of the above cocycle. This method should also lead to a description of the self-similar structure of the solutions to (1), in particular, in the case when the Lyapunov exponent does not exist.

We now shortly describe the monodromization (see also [1, 2, 3]). The analysis of the cocycles (3) is equivalent to the analysis of the equation

$$(4) \quad \Psi(T_{\omega_0}x) = M(x)\Psi(x)$$

for a matrix valued function  $\Psi$  defined on  $\mathbb{R} \times \mathbb{T}$ . We now define a *monodromy matrix* for this equation. On the cylinder  $\mathbb{R} \times \mathbb{T}$ , the skew shift  $T_{\omega_0}$  and the translation  $S_1 : (x_1, x_2) \mapsto (x_1 + 1, x_2)$  commute. So, (4) is invariant with respect to the transformation  $\Psi \mapsto \Psi \circ S_1$ . Let  $\Psi$  be a fundamental matrix solution to (4) (i.e.,  $\det \Psi(x) \neq 0$  for all  $x \in \mathbb{R}$ ). Clearly, any other solution to (4), say  $\tilde{\Psi}$ , can be represented in the form

$$\tilde{\Psi}(x) = \Psi(x)P(x),$$

where  $P$  is a matrix-valued function satisfying  $P \circ T_{\omega_0} = P$ .

Now, assume that  $\det \Psi(x) \equiv 1$ . The last two observations imply

$$\Psi(S_1(x)) = \Psi(x)\tilde{M}^t(x), \quad \forall x \in \mathbb{R} \times \mathbb{T},$$

where  $^t$  denotes the transposition, and  $\tilde{M} : \mathbb{R} \times \mathbb{T} \rightarrow SL(2, \mathbb{C})$  satisfies

$$\tilde{M} \circ T_{\omega_0} = \tilde{M}.$$

We call the matrix function  $\tilde{M}$  the monodromy matrix associated to the fundamental solution  $\Psi$ .

The main result of our monodromization method is

**Theorem 2.** *There exists a quadratic change of variable in  $\mathbb{R}^2$ , say  $R_{\omega_0}$ , depending only on the parameter  $\omega_0$ , such that, if one sets  $M_1 = \tilde{M} \circ R_{\omega_0}^{-1}$  and, for  $n_0 \in \mathbb{N}$  and  $x^0 = x \in \mathbb{T}^2$ , one defines*

$$n_1 = [x_1^0 + \omega_0 n_0] \quad \text{and} \quad x^1 = R_{\omega_0}(x^0) \pmod{\mathbb{T}^2},$$

then,

$$(5) \quad M_0(T_{\omega_0}^{n_0-1}(x^0)) \cdots M_0(T_{\omega_0}(x^0)) M_0(x^0) \\ = \Psi((T_{\omega_0}^{n_0} x^0) \pmod{\mathbb{T}^2}) \sigma M_1^{-1}(T_{\omega_1}^{-n_1}(x^1)) \cdots M_1^{-1}(T_{\omega_1}^{-1}(x^1)) \sigma \Psi^{-1}(x^0)$$

and

$$(6) \quad M_0^{-1}(T_{\omega_0}^{-n_0}(x^0)) \cdots M_0^{-1}(T_{\omega_0}^{-1}(x^0)) \\ = \Psi((T_{\omega_0}^{-n_0}(x^0)) \pmod{\mathbb{T}^2}) \sigma M_1(T_{\omega_1}^{n_1-1}(x^1)) \cdots M_1(T_{\omega_1}(x^1)) M_1(x^1) \sigma \Psi^{-1}(x^0),$$

where  $\sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ .

Let us now underline two important facts. First, in this theorem, we do not make use of the special structure of  $M$  given in (2); it holds for any skew-shift cocycle of unimodular matrices. The reduction is quite similar to and was inspired by the one done in [4] for exponential sums.

Second, the main gain provided by Theorem 2 is that, as in general  $\omega_1$  is smaller than 1, the number of terms in the products on the right hand sides of (5) and (6) contain much less terms than those in the left hand sides. One can iterate the procedure and thus further reduce the number of terms.

Clearly, this method can only be effective if one is able to compute  $\Psi$  and  $M_1$ . In the case when  $M$  is given by (2), this can be done exactly and one gets

**Theorem 3.** *Consider the cocycle (3). There exists a fundamental matrix solution to (4) (that is explicitly computable) such that the renormalized monodromy matrix (i.e. the one defined in Theorem 2) is*

$$M_1(x) = M(x, A_1), \quad A_1 = A_{\frac{1}{\omega}}.$$

Again, this exact renormalization is very similar to the renormalization found for quadratic exponential sums in [4].

#### REFERENCES

- [1] V. Buslaev and A. Fedotov. Bloch solutions of difference equations. *St Petersburg Math. Journal*, 7:561–594, 1996.
- [2] V. Buslaev and A. Fedotov. On the difference equations with periodic coefficients. *Adv. Theor. Math. Phys.*, 5(6):1105–1168, 2001.
- [3] A. Fedotov and F. Klopp. Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case. *Comm. Math. Phys.*, 227(1):1–92, 2002.
- [4] A. Fedotov and F. Klopp. Renormalization of exponential sums and matrix cocycles. Proceedings of the Seminaire “Équations aux Dérivées Partielles”, École Polytechnique, 2004–2005.
- [5] A. Fedotov and F. Klopp. In preparation.

## Low-Frequency Conductivities and Correlation Functions in Disordered Systems

LEONID PASTUR

We present a short review of physical and mathematical facts on the Kubo conductivity and binary (density-density and current-current) correlators of the ideal Fermi gas in external random field. An emphasis is made on the low-frequency conductivity and binary correlators for close energies. We discuss, in particular, asymptotically exact results for one-dimensional disordered systems for high and low Fermi energies, stressing their universal form. We also present a method that allows us to find the asymptotic form of various characteristics of disordered systems for any dimension in the strong localization regime, i.e., when either the random potential is big or the energy is close to a spectrum edge. The method is based on the hypothesis that the relevant realizations of the random potential in the strong localization regime have the form of deep random wells that are uniformly and chaotically distributed in space with a sufficiently small density. Assuming this and using the density expansion, we show first that the density of wells coincides in leading order with the density of states. Thus the density of states is in fact the small parameter of the theory in the strong localization regime. Then we derive the Mott formula for the low frequency conductivity and the asymptotic formulas for certain two-point correlators when the difference of the respective energies is small.

## Transport in Quasi-Free Fermionic Systems

CLAUDE-ALAIN PILLET

(joint work with Walter Aschbacher, Vojkan Jakšić, Yan Pautrat)

Equilibrium statistical mechanics is a beautiful piece of knowledge sitting on firmly established conceptual foundations and leaning on a highly developed mathematical framework. In principle, it allows to understand thermodynamic properties of matter and radiation at thermal equilibrium. In contrast, the status of nonequilibrium statistical mechanics is far from being satisfactory both at the conceptual level and regarding its mathematical structure. Even basic issues like linear response theory near equilibrium are still outside the scope of currently available analytic techniques.

Ideas inspired by the mathematical theory of turbulence have recently shed a new light on the mathematical structure of nonequilibrium statistical mechanics (see the nice exposition by D. Ruelle in [1]). In particular the emerging concept of natural nonequilibrium state provides a basis for a mathematical analysis of nonequilibrium phenomena and transport properties. Two mathematical approaches to the construction of nonequilibrium steady states (NESS) of quantum systems have been proposed. The first one by Ruelle [2] is based on the scattering theory of  $C^*$ -dynamical systems. The second one, developed in [4], associates NESS to spectral resonances of a new type of “Liouvilleans” which generate the

dynamics in a GNS representation of the system. These two approaches are particularly well adapted to the study of open quantum systems, i.e., spatially confined systems in contact with extended ideal reservoirs (see [5] for a review). Models of this kind are commonly used in physics to describe transport processes in mesoscopic electronic devices.

In order to make contact between the new NESS approach and more formal techniques routinely used by solid-state physicists we have investigated the simplest possible class of open quantum systems: free Fermi gasses or, in the language of solid state physics, independent electrons models [6]. Our result shows that, for such systems, Ruelle's scattering approach is equivalent to the well known Landauer-Büttiker formalism (see e.g. [3]). More precisely, expectation values of thermal and electric currents in the NESS coincide with steady currents computed from the Landauer-Büttiker formula. I will also discuss linear response theory based on this result.

#### REFERENCES

- [1] Ruelle, D.: *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics*. J. Stat. Phys. **95** (1999), 393.
- [2] Ruelle, D.: *Natural nonequilibrium states in quantum statistical mechanics*. J. Stat. Phys. **98** (2000), 57.
- [3] Datta, S.: *Electronic Transport in Mesoscopic Systems*. Cambridge University Press (1995).
- [4] Jakšić; V., Pillet, C.-A.: *Non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs*. Commun. Math. Phys. **226** (2002), 131.
- [5] Aschbacher, W., Jakšić; V., Pautrat, Y., Pillet, C.-A.: *Topics in non-equilibrium quantum statistical mechanics*. In *Open Quantum Systems III*. S. Attal, A. Joye, C.-A. Pillet editors. Lecture Notes in Mathematics **1882**, Springer, New York (2006).
- [6] Aschbacher, W., Jakšić; V., Pautrat, Y., Pillet, C.-A.: *Transport properties of quasi-free Fermions*. J. Math. Phys. **48** (2007), 032101.

### Spectral Statistics and Localization of Eigenfunctions in Random Band Matrices

JEFFREY SCHENKER

Consider an  $N \times N$  random band matrix  $X_{W;N}$  with distribution

$$e^{-\frac{W}{2} \text{Tr} X_{W;N}^2} dX_{W;N}$$

with  $dX_{W;N}$  the Lebesgue measure on the vector space of  $N \times N$  matrices of band width  $W$ . Thus

$$X_{W;N} = \frac{1}{\sqrt{W}} \underbrace{\begin{pmatrix} d_{1,1} & a_{1,2} \cdots & a_{1,W} & & & & \\ a_{2,1}^* & d_{2,2} & & \ddots & & & \\ \vdots & & & \ddots & \ddots & & \\ a_{W,1} & & & & \ddots & \ddots & \\ & \ddots & & & \ddots & \ddots & \\ & & \ddots & & \ddots & \ddots & \\ & & & \ddots & & \ddots & \ddots \end{pmatrix}}_{N \times N},$$

with  $d_i$  and  $a_{i,j}$  independent families of i.i.d. real and complex Gaussian variables, respectively. (More generally we might take  $d_i$  and  $a_{i,j}$  to be i.i.d. families with a suitably nice distribution.)

As shown in [1] the density of states of  $X_{W;N}$  converges to the semi-circle law as  $W$  and  $N$  tend jointly to infinity, provided  $W/N \rightarrow 0$  or  $1$ . That is, if  $W(N)$  is any diverging sequence with  $W(N)/N \rightarrow 0$  or  $1$  then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} (\text{Tr} f(X_{W(N);N})) = \frac{1}{2\pi} \int_{-2}^2 f(t) \sqrt{4 - t^2} dt.$$

This describes the asymptotic density of states, however, because the convergence is in the weak sense, we have very little information about the density of states at finite  $N$ . For instance, an interesting open question, whose solution would be useful, is to find a good constant  $C$  in the bound

$$\mathbb{E} (\text{Tr} f(X_{W(N);N})) \leq C \int_a^b f(t) dt$$

for functions  $f$  supported in an interval  $(a, b)$ . Exploiting the diagonal terms, as in the Wegner estimate for random Schrödinger operators [7], one obtains this bound with  $C \propto \sqrt{W}$ . However, it seems to the author the bound should hold with  $C$  of order one.

Now, fix an energy  $\lambda \in \mathbb{R}$  and consider the scaled and shifted eigenvalue process

$$(1) \quad \{\tilde{\lambda}_j\}_{j=1}^N = \{N(\lambda_j - \lambda)\}_{j=1}^N,$$

with  $\lambda_j$  the (random) eigenvalues of  $X_{W;N}$ . With  $W$  fixed equal to one, the matrix is diagonal,  $\{\lambda_j\} = \{d_{j,j}\}$ , and  $\{\tilde{\lambda}_j\}$  converges in distribution (as  $N \rightarrow \infty$ ) to a Poisson process. Conversely, with  $W = N$  the matrix  $X_{W;N}$  is sampled from the

Gaussian Unitary Ensemble and the scaled, shifted eigenvalue process converges in distribution to an explicit determinantal point process as shown by Dyson [3]. Based on numerical evidence [2] and a saddle point analysis of a sigma model approximation [4] it is believed these two extremes characterize the behavior in each of the regimes  $W^2/N \sim 0$  and  $W^2/N \sim \infty$ .

Part of this picture can be confirmed by rigorous analysis. What I have obtained is the following theorem

**Theorem 1.** *There exists  $\nu > 0$  such that if  $W = W(N)$  is any sequence with  $W^\nu/N \rightarrow 0$  then the scaled, centered eigenvalue process converges in distribution to a Poisson process.*

Of course, the value of  $\nu$  given by the proof is larger than 2 (larger than 5, in fact), so it remains an open question to establish Poisson statistics in the full regime  $W^2/N \sim 0$ .

The nature of eigenvalue statistics is expected to be related to localization properties of the eigenfunctions – Poisson statistics corresponding to localized eigenfunctions and GUE statistics corresponding to extended eigenfunctions. This is borne out in the extreme cases mentioned above: for  $W = 1$  each eigenfunction is completely localized on a single basis vector while for  $W = N$ , as is well known, the eigen-basis is uniformly distributed ortho-normal frame, so each eigenfunction is typically roughly uniformly distributed over the basis vectors. (To quantify this, one could compute the entropy  $\sum_j |\psi_j|^2 \ln |\psi_j|^2$  of an eigenfunction.)

One must be a little careful with this heuristic, however, since it is certainly possible to concoct random matrices with arbitrary statistics and arbitrary localization properties of the eigenfunctions. Nonetheless, the above result is consistent with the picture as it stems from a localization result for eigenfunctions of the matrices  $X_{N;W}$ , which is most conveniently stated in terms of the resolvent  $(X_{N;W} - \lambda)^{-1}$ , a well defined (though unbounded) random matrix for  $\lambda \in (-2, 2)$ . To state the result, let  $\mathbf{e}_i$  denote the standard basis vectors  $\mathbf{e}_i(j) = \delta_{i,j}$ .

**Theorem 2.** *Given  $s \in (0, 1)$ , there are  $A_s < \infty$  and  $\mu_s > 0$  such that*

$$\mathbb{E} (|\langle \mathbf{e}_i, (X_{N;W} - \lambda)^{-1} \mathbf{e}_j \rangle|^s) \leq A_s W^{s/2} e^{-\mu_s \frac{|i-j|}{W^5}}.$$

One expects based on [2, 4] this result should hold with  $W^2$  in place of  $W^5$  in the denominator in the exponent. That remains an open problem, however, as the proof gives  $W^5$ .

The first theorem follows from the second by an adaptation of an argument of Minami [6]. The proof of the second theorem is based on the Kunz-Souillard proof of localization [5] in 1D systems adapted to a block matrix setting.

#### REFERENCES

- [1] L. V. Bogachev, S. A. Molchanov, L. A. Pastur, *On the density of states of random band matrices.* (Russian) *Mat. Zametki* **50** (1991), 31–42; translation in *Math. Notes* **50** (1992), 1232–1242.
- [2] G. Casati, L. Molinari, F. Izrailev, *Scaling properties of band random matrices,* *Phys. Rev. Lett.* **64** (1990), 1851–1854.

- [3] F. J. Dyson, *Statistical Theory of the Energy Levels of Complex Systems. III*, J. Math. Phys. **3** (1962), 166–175.
- [4] Y. V. Fyodorov and A. D. Mirlin, *Scaling properties of localization in random band matrices: A  $\sigma$ -model approach*, Phys. Rev. Lett. **67** (1991), 2405–2409.
- [5] H. Kunz and B. Souillard, *Sur le spectre des operateurs aux differences finies aleatoires*, Comm. Math. Phys. **78** (1980), 201–246.
- [6] N. Minami, *Local fluctuations of the spectrum of a multidimensional Anderson tight binding model*, Comm. Math. Phys. **177** (1996), 709–725.
- [7] F. Wegner, *The density of states for disordered systems*, Z. Phys. B **44** (1981), 9–15.

## Transfer Matrix Methods for Discrete Random Schrödinger Operators

HERMANN SCHULZ-BALDES

(joint work with Christian Sadel)

We give an overview over recently developed complements to the transfer matrix techniques for one-dimensional and quasi-one-dimensional random Schrödinger operators (random Jacobi matrices). For the strictly one-dimensional case, this concerns in particular the perturbative calculation of the Lyapunov exponents at anomalies and the band edges, which involves the use of a certain Fokker-Planck operator on the space of modified Pruefer phases. For the quasi-one-dimensional situation, particular focus is on a Sturm-Liouville type oscillation theorem for Jacobi matrices with matrix entries, namely self-adjoint block tridiagonal matrices with positive definite blocks on the off-diagonals. This gives a new rotation number calculation for the eigenvalues. The three universality classes of time reversal invariance are dealt with by implementing the corresponding symmetries. Using this theorem one obtains a new formula for the integrated density of states which can be calculated perturbatively in the coupling constant of the randomness with an optimal control on the error terms.

### REFERENCES

- [1] H. Schulz-Baldes, *Perturbation theory for an Anderson model on a strip*, GAFA **14**, (2004), 1089-1117.
- [2] Christian Sadel, H. Schulz-Baldes, *Scaling diagram for the localization length at a band edge*, preprint, 2007.
- [3] H. Schulz-Baldes, *Lyapunov exponents at anomalies of  $SL(2, R)$ -actions*, Operator Theory: Advances and Applications **174**, (2007), 159-172.
- [4] H. Schulz-Baldes, *Rotation numbers for Jacobi matrices with matrix entries*, preprint, 2007.

## Mott Law for a Random Walk in a Random Medium

DOMINIQUE SPEHNER

(joint work with Alessandra Faggionato, Hermann Schulz-Baldes)

We consider a random walk on the support of an ergodic stationary simple point process on  $\mathbb{R}^d$ ,  $d \geq 2$ , which satisfies a mixing condition with respect to translations. Furthermore the point process is furnished with independent random energy marks in the interval  $[-1, 1]$ . The transition rates of the random walk decay exponentially in the jump distances and depend on the energies through a factor of the Boltzmann-type. This is an effective classical model for the phonon-induced hopping of electrons in disordered solids within the regime of Anderson localization. We show that the random walk converges to a Brownian motion after the usual rescaling in time and position. Moreover, the low-temperature behavior of the (logarithm of the) diffusion constant  $D$  is given up to a multiplicative constant by Mott's law for the variable range hopping conductivity at zero frequency of disordered solids. A lower bound on  $D$  has been proven in [1]. It involves estimates for the supercritical regime of an associated site percolation problem. More recently, an upper bound with roughly the same low-temperature behavior has been established in [2].

### REFERENCES

- [1] A. Faggionato, H. Schulz-Baldes, and D. Spohner, *Mott law as lower bound for a random walk in a random environment*, Commun. Math. Phys. **263** (2006), 21-64.
- [2] A. Faggionato and P. Mathieu, *Mott law for Mott variable-range random walk*, arXiv:math-ph/0608033.

## Absolutely Continuous Spectrum on some Tree Graphs

WOLFGANG SPITZER

(joint work with Richard Froese, David Hasler)

We study here the Anderson model on trees,  $\mathcal{T}$ , i.e., connected graphs without loops, and we are primarily interested in tracing its absolutely continuous (ac) spectrum. It was Abel Klein [7] who first proved ac spectrum on the Bethe tree. Recently and by different methods, Aizenman, Sims, and Warzel [1] also proved (the existence of) ac spectrum in a yet slightly more general situation.

In our first paper [4], we constructed deterministic potentials on the Bethe tree that yield ac spectrum using a geometric approach where hyperbolic contractions are the essential tool to control the Green's function. The same idea was also successful [5] for the Anderson model which we sketch here. We want to point out that the proof below is simpler and more direct than our original proof and works much better for large values of the connectivity of the graph (see [6]).

In order to define the Anderson model let us start with the Laplacian  $\Delta$  defined by  $(\Delta f)(x) = \sum_{\langle x, y \rangle} f(y)$ ;  $\langle x, y \rangle$  means that  $x$  and  $y$  are nearest neighbours. The potential  $V$  is random in the sense that  $\{V(x)\}_{x \in \mathcal{T}}$  are iid random variables with



common probability distribution (measure)  $\nu$ , which, for simplicity, has compact support  $I$ . The Anderson model under consideration here is  $H = \Delta + V$ .

Our first example concerns the Bethe tree where the number of forward neighbours is constant, say  $K$ . In this case, the spectrum of the Laplacian is equal to  $[-2\sqrt{K}, 2\sqrt{K}]$  and purely ac. The following is our main result.

**Theorem:** *For any  $0 < E < 2\sqrt{K}$  there exists an interval  $I$  around 0 such that for all probability distributions  $\nu$  with support in  $I$ , the spectrum of  $H$  is purely ac in  $[-E, E]$  with probability one.*

*Sketch of Proof:* For simplicity we set  $K = 2$ . Our main task is to bound a certain moment,  $M_n(\rho)$ , of the Green's probability function  $\rho$  (see below) from which one can conclude by a separate argument that the spectrum is ac. As usual, for  $x \in \mathcal{T}$ , the Green's function  $G(x, \lambda) = (H - \lambda)^{-1}(x, x)$ . Since  $H$  is random, this kernel is a random variable. By  $\rho$  we denote the probability distribution of  $G(x, \lambda)$  induced by  $\nu$ . I.e., for  $A \subseteq \mathbb{H}$ ,  $\rho(A) = \text{prob}(G(x, \lambda) \in A)$ . In the Bethe tree we have translation invariance so that  $\rho$  is independent of  $x \in \mathcal{T}$  but, of course, depends on  $\lambda$  which we suppress.

For  $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$  we have that  $(\Delta - \lambda)^{-1}(x, x) = z_\lambda = -\frac{\lambda}{4} + \frac{i}{4}\sqrt{8 - \lambda^2}$ . Note that for real  $\lambda$ ,  $z_\lambda$  is in the upper half plane  $\mathbb{H}$  iff  $\lambda$  is in the above interval. This is our reference point for  $\rho$ . To measure distances in  $\mathbb{H}$  we use the function  $\text{cd}(z) = \frac{|z - z_\lambda|^2}{\text{Im}(z)}$ ,  $z \in \mathbb{H}$ . Then we define the moments,

$$(1) \quad M_n(\rho) = \int_{\mathbb{H}} \text{cd}^n(z) d\rho(z), \quad n \geq 1.$$

In order that everything is well-defined we should add some positive imaginary part  $i\epsilon$  to  $\lambda$  but our estimates below will be uniform in  $\epsilon$ . We will continue with  $n = 1$  but  $n > 1$  is needed to show that the spectrum is pure ac.

The simplicity of the Anderson model on the tree is based upon the recurrence relation for  $\rho$ ; in fact, this relation only holds for the *truncated* Green's probability distribution but we will not make this distinction here. To this end we introduce the function  $\phi : \mathbb{H}^2 \times \mathbb{H} \times I \rightarrow \mathbb{H}$ ,  $\phi(z_1, z_2, \lambda, q) = -\frac{1}{z_1 + z_2 + \lambda - q}$ . Then,  $\rho = \phi_*(\rho \times \rho \times \nu \times \nu)$ , where  $\phi_*$  is the pull-back operation (see below). This is utilized to rewrite the first moment,

$$\begin{aligned} M_1(\rho) &= \int_{\mathbb{H}} \text{cd}(z) d\rho(z) \\ &= \int_{\mathbb{H}^2 \times I} \text{cd}(\phi(z_1, z_2, \lambda, q)) d\rho(z_1)d\rho(z_2)d\nu(q) \\ &= \int_{\mathbb{H}^2 \times I} \underbrace{\frac{\text{cd}(\phi(z_1, z_2, \lambda, q))}{\frac{1}{2}\text{cd}(z_1) + \frac{1}{2}\text{cd}(z_2)}}_{\mu_2(z_1, z_2, \lambda, q)} \left(\frac{1}{2}\text{cd}(z_1) + \frac{1}{2}\text{cd}(z_2)\right) d\rho(z_1)d\rho(z_2)d\nu(q). \end{aligned}$$

A simple convexity argument shows that  $\mu_2(z_1, z_2, \lambda, 0) \leq 1$  with equality iff  $z_1 = z_2 = z_\lambda$ . So in that sense  $\phi$  is a contraction. Let us *assume* for a moment that  $\mu_2(z_1, z_2, \lambda, q) \leq 1 - \mu_0 < 1$  for  $z_1, z_2$  near the boundary of  $\mathbb{H}^2$  with some constant

$\mu_0$  and  $q$  in some small interval  $I$  around 0. Then for some fixed (hyperbolic) compact ball around  $(z_\lambda, z_\lambda)$  in  $\mathbb{H}^2$ ,

$$\begin{aligned} M_1(\rho) &= \int_{B \times I} \mu_2(z_1, z_2, \lambda, q) \left( \frac{1}{2} \text{cd}(z_1) + \frac{1}{2} \text{cd}(z_2) \right) d\rho(z_1) d\rho(z_2) d\nu(q) \\ &\quad + \int_{(\mathbb{H}^2 \setminus B) \times I} \mu_2(z_1, z_2, \lambda, q) \left( \frac{1}{2} \text{cd}(z_1) + \frac{1}{2} \text{cd}(z_2) \right) d\rho(z_1) d\rho(z_2) d\nu(q) \\ &\leq C + (1 - \mu_0) \int_{(\mathbb{H}^2 \setminus B) \times I} \left( \frac{1}{2} \text{cd}(z_1) + \frac{1}{2} \text{cd}(z_2) \right) d\rho(z_1) d\rho(z_2) d\nu(q) \\ &\leq C + (1 - \mu_0) \int_{\mathbb{H}^2 \times I} \left( \frac{1}{2} \text{cd}(z_1) + \frac{1}{2} \text{cd}(z_2) \right) d\rho(z_1) d\rho(z_2) d\nu(q) \\ &= C + (1 - \mu_0) M_1(\rho), \end{aligned}$$

where  $C$  is some finite constant. Consequently,  $M_1(\rho) < C/\mu_0$ . Our assumption that  $\mu_2(z_1, z_2, \lambda, q) \leq 1 - \mu_0 < 1$  is not true (there are exceptional points even for  $q = 0$ ). But if we apply once more the recurrence relation to the variables  $z_1$  and  $z_2$  (so that we have then three variables  $z_1, z_2, z_3$  to integrate over) we can define an analogous function  $\mu_3$  that is strictly less than 1 near the boundary (and small  $q$ ), and the above argument goes through.  $\square$

*Remarks:*

- (1) By a slight extension we cannot only bound the moment  $M_n(\rho)$  for any  $n \geq 1$  but we can also show that  $\rho(z)$  decays exponentially fast to 0 as  $z$  approaches the boundary of  $\mathbb{H}$ .
- (2) The function  $\mu_2$ , respectively  $\mu_3$ , is a rational function that can be analyzed numerically to plot a phase-diagram for the region where  $\mu_3 < 1$  and thereby locate ac spectrum as a function of the spectral parameter  $\lambda$  and the strength of the disorder.

It is interesting to study trees where the connectivity is not constant, in particular hybrids of the binary Bethe tree with the one-dimensional lattice where we have complete localization. So let us consider a radially symmetric tree and let  $\kappa = (\kappa_n)_n \in \{1, 2\}^{\mathbb{N}}$  be a binary sequence that determines the number of forward neighbours at any vertex in the  $n$ -th sphere. If  $\kappa \neq 1$  then this sequence  $\kappa_n$  determines a unique sequence  $k_n, n \geq 0$  (of binary branching points) so that  $\kappa_{k_n} = 2$  and the sequence  $\ell_n = k_{n+1} - k_n - 1$  for all  $n \geq 0$ . The standard one-dimensional half-line corresponds to setting  $\kappa_n = 1$ , and the usual binary tree is given by  $\kappa_n = 2, k_n = n, \ell_n = 0$ .

Breuer [2] proved that if  $\ell_n = e^{\gamma n}$  (a so-called *sparse tree*) for some constant  $\gamma > 0$ , then the Anderson model has no ac spectrum a.s. for any non-zero disorder like in the one-dimensional case. This is not surprising since already the Laplacian [3] has purely singular spectrum, a fact which is surprising to us. We have analyzed the case when  $\ell_n = 1$  and proved pure ac spectrum for  $\sqrt{2} - 1 \leq |\lambda| \leq \sqrt{2} + 1$  and small disorder. We believe that all constant sequences  $\ell$  should also have pure ac spectrum on the spectral set of the free Laplacian. However, we do not know

whether there exists an increasing sequence  $\ell$  for which there is still ac spectrum in the Anderson model.

The reason why there is no ac spectrum even for the free Laplacian on sparse trees is that by radial symmetry we are effectively in a situation of a sparse potential in dimension one. We can break this symmetry by inserting one extra vertex on each top branch of the binary Bethe tree. In this case, already the function  $\mu_2 \leq 1 - \mu_0 < 1$  (this constitutes the hard part of the whole analysis) and thus one [6] proves ac spectrum for the Anderson model on such a tree as outlined above.

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#### REFERENCES

- [1] M. Aizenman, R. Sims and S. Warzel, *Stability of the Absolutely Continuous Spectrum of Random Schrödinger Operators on Tree Graphs*, Prob. Theor. Rel. Fields **136** (2006), 363–394.
- [2] J. Breuer, *Localization for the Anderson model on trees with finite dimensions*, math.SP/0609474.
- [3] J. Breuer, *Singular continuous spectrum for the Laplacian on certain sparse trees*, math.SP/0604552.
- [4] R. Froese, D. Hasler and W. Spitzer, *Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs*, J. Func. Anal. **230** (2006), 184–221.
- [5] R. Froese, D. Hasler and W. Spitzer, *Absolutely Continuous Spectrum for the Anderson Model on a Tree: A Geometric Proof of Klein’s Theorem*, Commun. Math. Phys. **269** (2007), 239–257.
- [6] F. Halasan, Ph.D. thesis, University of British Columbia, in preparation.
- [7] A. Klein, *Extended States in the Anderson Model on the Bethe Lattice*, Advances in Math. **133** (1998) 163–184.

### Localization near Fluctuation Boundaries via the Fractional Moment Method

PETER STOLLMANN

(joint work with Anne Boutet de Monvel, Serguei Naboko, Günter Stolz)

We report on work that can be accessed as eprint mparc 05-324 and will appear in J. Anal. Math. Building on ideas of the recent adaptation [1] of the Aizenman-Molchanov [2] or Fractional Moment method, we present a version of the FMM that can be applied in situations under fairly general conditions as far as the geometry is concerned.

Therefore, we can treat models without the covering condition, like surface models and models with displacement.

Here are some details: On  $\mathbb{R}^d$  we often consider the supremum norm  $|x| := \max_{i=1,\dots,d} |x_i|$  and write

$$\Lambda_r(x) := \left\{ y \in \mathbb{R}^d : |x - y| < \frac{r}{2} \right\}$$

for the  $d$ -dimensional cube with side length  $r$  centered at  $x$ . For an open set  $G \subset \mathbb{R}^d$  we denote the restriction of the Schrödinger operator  $H$  to  $L^2(G)$  with Dirichlet boundary conditions by  $H^G$ . In our results we assume  $d \leq 3$  and rely upon the following assumptions, which guarantee self-adjointness and lower semi-boundedness of the operators in question:

- (A1) The background potential  $V_0 \in L^2_{\text{loc,unif}}(\mathbb{R}^d)$  is real-valued,  $H_0 := -\Delta + V_0$ .
- (A2) The set  $\mathcal{I} \subset \mathbb{R}^d$ , where the random impurities are located, is uniformly discrete, i.e.,  $\inf\{|\alpha - \beta| : \alpha \neq \beta \in \mathcal{I}\} =: r_{\mathcal{I}} > 0$ .
- (A3) The random couplings  $\eta_\alpha$ ,  $\alpha \in \mathcal{I}$ , are independent random variables supported in  $[0, \eta_{\max}]$  for some  $\eta_{\max} > 0$  and with absolutely continuous distribution of bounded density  $\rho_\alpha$  with a uniform bound  $\sup_\alpha \|\rho_\alpha\|_\infty =: M_\rho < \infty$ .

The single site potentials  $U_\alpha$ ,  $\alpha \in \mathcal{I}$  satisfy

$$c_U \chi_{\Lambda_{r_U}}(\alpha) \leq U_\alpha \leq C_U \chi_{\Lambda_{R_U}}(\alpha)$$

for all  $\alpha$  with  $c_U, C_U, r_U, R_U > 0$  independent of  $\alpha$ .

$$V_\omega(x) = \sum_{\alpha \in \mathcal{I}} \eta_\alpha(\omega) U_\alpha(x)$$

and

$$H := H(\omega) := H_0 + V_\omega \text{ in } L^2(\mathbb{R}^d).$$

The most important condition expresses the fact that the ground state energy comes from those realizations of the potential that vanish on large sets:

- (A4) Denote  $E_0 := \inf \sigma(H_0) \leq \inf \sigma(H(\omega))$  and let

$$H_F := H_0 + \eta_{\max} \sum_{\alpha \in \mathcal{I}} U_\alpha,$$

the subscript  $F$  standing for full coupling.

Assume that  $E_0$  is a *fluctuation boundary* in the sense that

- (i)  $E_F := \inf \sigma(H_F) > E_0$ , and
- (ii) There is  $m \in (0, 2)$  and  $L^*$  such that for  $m_d := 42 \cdot d$ , all  $L \geq L^*$  and  $x \in \mathbb{Z}^d$

$$\mathbb{P}(\sigma(H^{\Lambda_L(x)}(\omega) \cap [E_0, E_0 + L^{-m}] \neq \emptyset) \leq L^{-m_d}.$$

By  $\chi_x$  we denote the characteristic function of the unit cube centered at  $x$ . In the following it is understood that  $\chi_x(H^G - E - i\varepsilon)^{-1}\chi_y = 0$  if  $\Lambda_1(x) \cap G$  or  $\Lambda_1(y) \cap G$  have measure zero.

Our main result is

**Theorem 1.** *Let  $d \leq 3$  and assume (A1)-(A4). Then there exist  $\delta > 0$ ,  $0 < s < 1$ ,  $\mu > 0$  and  $C < \infty$  such that for  $I := [E_0, E_0 + \delta]$ , all open sets  $G \subset \mathbb{R}^d$  and  $x, y \in \mathbb{R}^d$ ,*

$$(1) \quad \sup_{E \in I, \varepsilon > 0} \mathbb{E}(\|\chi_x(H^G - E - i\varepsilon)^{-1}\chi_y\|^s) \leq C e^{-\mu|x-y|}.$$

Exponential decay of fractional moments of the resolvent as described by (1) implies spectral and dynamical localization in the following sense:

**Theorem 2.** *Let  $d \leq 3$ , assume (A1)-(A4) and let  $I$  be given as in Theorem 1. Then:*

- (a) *For all open sets  $G \subset \mathbb{R}^d$  the spectrum of  $H^G$  in  $I$  is almost surely pure point with exponentially decaying eigenfunctions.*  
 (b) *There are  $\mu > 0$  and  $C < \infty$  such that for all  $x, y \in \mathbb{R}^d$  and open  $G \subset \mathbb{R}^d$ ,*
- $$(2) \quad \mathbb{E}(\sup \| \chi_x g(H^G) P_I(H^G) \chi_y \|) \leq C e^{-\mu|x-y|}.$$

where the supremum is taken over all Borel measurable functions  $g$  which satisfy  $|g| \leq 1$  pointwise and  $P_I(H^G)$  is the spectral projection for  $H^G$  onto  $I$ .

Dynamical localization should be considered as the special case  $g(\lambda) = e^{it\lambda}$  in (b), with the supremum taken over  $t \in \mathbb{R}$ .

The proof of Theorem 1 is done by a self-contained presentation of a new version of the continuum fractional moment method. While we use many of the same ideas as [1], due to the lack of a covering condition we can not rely any more on the concept of “averaging over local environments”, heavily exploited in [1]. It is interesting to note that, in some sense, we instead use a global averaging procedure. Technically, this actually leads to some simplifications compared to the method in [1], as repeated commutator arguments can be replaced by simpler iterated resolvent identities. We also mention that exponential decay in (1) will follow from smallness of the fractional moments at a suitable initial length scale (the localization length) via an abstract contraction property.

As technical tools we need Combes-Thomas bounds (in operator norm as well as in Hilbert-Schmidt norm) and a weak- $L^1$ -type bound for the boundary values of resolvents of maximally dissipative operators, which is based on results from [5] and was also central to the argument in [1].

As applications we mention surface models, in a form a little more general than what had been studied in [4] and models including displacements that are more general than those treated in [3, 6].

## REFERENCES

- [1] M. Aizenman, A. Elgart, S. Naboko, J. Schenker and G. Stolz: Moment analysis for localization in random Schrödinger operators, *Invent. Math.* **163** (2006) 343–413  
 [2] M. Aizenman and S. Molchanov: Localization at large disorder and at extreme energies: an elementary derivation, *Commun. Math. Phys.* **157** (1993) 245–278.  
 [3] J.-M. Combes and P. D. Hislop: Localization for some continuous, random Hamiltonians in  $d$ -dimensions, *J. Funct. Anal.* **124** (1994) 149–180.  
 [4] W. Kirsch and S. Warzel: Anderson localization and Lifshits tails for random surface potentials, *J. Funct. Anal.* **230** (2006) 222–250  
 [5] S. N. Naboko: The structure of singularities of operator functions with a positive imaginary part. *Funktsional. Anal. i Prilozhen.* **25**(4) (1991) 1–13.  
 [6] H. Zenk: Anderson localization for a multidimensional model including long range potentials and displacements, *Rev. Math. Phys.* **14**(3) (2002) 273–302.

## Localization and Control of Sobolev Norms

WEI-MIN WANG

The purpose of this talk is to manifest the close relationship between Anderson Localization (A. L.) and control of Sobolev norms for Schrödinger equations.

### *I. Linear time-independent Schrödinger equations.*

We consider the Schrödinger equations:

$$(1) \quad i \frac{\partial}{\partial t} \psi = (-\Delta + V)\psi,$$

where  $V(x, t) = V(x)$ , on  $L^2(\mathbb{T}^d)$ , the periodic Schrödinger or on  $L^2(\mathbb{R}^d)$  when  $V \rightarrow \infty$  as  $x \rightarrow \infty$ , e.g.,  $V = x^2$ , the quantum harmonic oscillator. We also consider the above equation on  $\ell^2(\mathbb{Z}^d)$  when  $V$  is a family of random variables.

The  $L^2$  norms of solutions to (1) are conserved. Therefore the first non-trivial norms to study are the Sobolev norms  $H^s$ ,  $s = 1, 2, \dots$ . In general there are no conservation laws for Sobolev norms. For the linear Schrödinger (both time independent and dependent), there is the a priori bound that the  $H^s$  norm cannot grow faster than  $t^s$  as  $t \rightarrow \infty$ . In order to get better bounds, one usually needs to study the details of the solutions.

Since the RHS of (1) is independent of time, this reduces to the study of dynamical localization properties of eigenfunctions in the Fourier space when we consider (1) on  $L^2$ . When  $V$  is periodic, assume moreover  $V$  is analytic, boundedness of  $H^s$  norms follows from exponential localization properties of the eigenfunctions with respect to the exponentials. When  $V = x^2$ , boundedness of  $H^s$  norms follows from the fact that all the solutions to (1) are periodic in time. When  $V$  is large and random and under appropriate conditions on the probability distribution, we have dynamical localization for (1) in  $\mathbb{Z}^d$ . The first 2 cases are the analogues of A. L. in the Fourier space.

As is well known, (1) is a 0th order approximation to a real quantum system, which consists of many particles. A first order approximation is a nonlinear Schrödinger, e.g.,

$$(2) \quad i \frac{\partial}{\partial t} \psi = (-\Delta + V)\psi + |\psi|^{2p}\psi, \quad p \in \mathbb{N}^+,$$

where the nonlinear term models the particle-particle interactions. The  $L^2$  norms are conserved. But what about the  $H^s$  norms? We note that here contrary to the linear case, there is no a priori bound on  $H^s$  as  $t \rightarrow \infty$ .

Assume the nonlinearity is small and the linear Schrödinger operator has pure point spectrum. Linearizing (2) and using eigenfunction decomposition lead naturally to study time dependent Schrödinger equations. So I first mention results on time dependent Schrödinger equations.

### *II. Linear time-dependent Schrödinger equations.*

Aside from motivations coming from nonlinear equations, time dependent Schrödinger equations occur naturally in physics, where the time-dependence models radiation. The control of  $H^s$  norm is essentially about the stability of bound states under radiation.

In [1], we prove that the one dimensional quantum harmonic oscillator is stable under *nonresonant* time quasi-periodic perturbations of the form:  $e^{-x^2} \sum_{k=1}^{\nu} \cos(\omega_k t + \theta_k)$ , where  $\omega = \{\omega_k\}$  belongs to a subset of Diophantine frequencies of positive measure. In [2], we prove that the bound states of periodic Schrödinger operator are stable under *resonant* perturbations. The motivations for both problems come from nonlinear Schrödinger.

### III. Nonlinear Schrödinger equations.

For the tempered nonlinear random Schrödinger equation:

$$(3) \quad i\dot{q}_j = v_j q_j + \epsilon(q_{j-1} + q_{j+1}) + \lambda_j q_j |q_j|^2$$

where  $|\lambda_j| < \epsilon(|j| + 1)^{-\tau}$ ,  $\tau > 0$ , we prove in [3] that the  $H^1$  norm cannot grow faster than  $t^\kappa$ ,  $\kappa > 0$ , as  $t \rightarrow \infty$ . Finally we remark that for the standard nonlinear random Schrödinger, i.e.,  $\lambda_j = \delta$ ,  $\delta \ll 1$ ,  $\forall j$ , time quasi-periodic solutions were constructed in [4].

### REFERENCES

- [1] W.-M. Wang, *Stability of quantum harmonic oscillator under time quasi-periodic perturbation (Lanzhou lecture notes)*, Séminaires et Congres (to appear).
- [2] W.-M. Wang, *Boundedness of Sobolev norms for the periodic Schrödinger equation under resonant perturbations*, (in preparation).
- [3] J. Bourgain, W.-M. Wang, *Diffusion bound for a nonlinear Schrödinger equation*, Ann. Math. Studies (to appear).
- [4] J. Bourgain, W.-M. Wang, *Quasi-periodic solutions of nonlinear random Schrödinger equations*, J. European Math. Soc. (to appear).

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