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Graph Theory

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Abstract. This week broadly targeted both finite and infinite graph theory, as well as matroids, including their interaction with other areas of pure mathematics. The talks were complemented by informal workshops focussing on specific problems or particularly active areas.

Mathematics Subject Classification (2000): 05Cxx.

Introduction by the Organisers

This conference was one of a series of Oberwolfach conferences on the same topic, held every two years. There were 55 participants, including about twenty graduate students and postdocs.

Since graph theory is a broad and many-faceted field, we need to focus the workshop on a specific domain within the field (as we did in previous years). A dominant area within graph theory today is extremal graph theory: the study of the asymptotic and probabilistic behaviour of various graph parameters. While this is an exciting area, it is of a different character from much of the remainder of graph theory, and its inclusion would lead to an undesired division of the participants. Since extremal graph theory is adequately covered by the Oberwolfach combinatorics conference, we decided to minimize the extremal content and to focus the conference on other fundamental areas in graph theory, their interaction, and their interaction with mathematics outside combinatorics. In particular, we focus on structural aspects of graphs like decomposability, embeddability, duality, and noncontainment of substructures and its relations to basic questions like colourability and connectivity, and on the applicability of methods from algebra, geometry, and topology to these areas.

The conference was organized along lines similar to the earlier Oberwolfach Graph Theory conferences of 2003 and 2005. There is a reduced number of formal talks, to give space for informal workshops on various topics in the area. As before, on the first day, we asked everyone to make a five-minute presentation of their current interests. This was designed also to promote contact between participants early in the meeting, and turns out to work very well.

As for the talks intended for all the participants, there were six 50-minute and twenty-one 25-minute talks. We selected these from the abstracts submitted before the meeting, and we chose them to be of scientific relevance and general interest, as best we could. Also, we deliberately chose younger speakers, and tried for a wide range of topics. Among the highlights of the week were the presentation of a proof of Berge's strong path partition conjecture for k=2, a new method to apply chip-firing games in graphs to derive a Riemann-Roch theorem in tropical geometry, and constructions of limits of graphs forming a topological space that encompass Szemeredi's regularity lemma.

The workshops are intended to be informal — no formal speakers or time slots, with a 'convenor' to manage the workshop — and are focused on areas with specific recent results, conjectures and problems. It is our experience that such workshops can be the best part of a conference. We wanted these to be really informal, so that anyone in the group who wanted to contribute could spontaneously get up and say his or her piece. Before the meeting, we selected a few topics that seemed appropriate for workshops, some of which were suggested by participants. This time there were workshops on graph width, matroids, graph limits, flows and cycles in infinite graphs, and paths and minors. Sometimes, the workshops were scheduled in parallel, and some were extended in evenings later in the week.

We were very satisfied with the way the conference worked out. Although the participants had varied interests, they were not so far apart that they polarized into separate camps. Most of the talks were of interest to almost all of of the participants, and the workshop format satisfied the desire of some for more focus.

Again, we regard this as a very successful conference. If we organize another Graph Theory meeting at Oberwolfach, we would run it on the same lines. We are very thankful to the Oberwolfach management and staff for the opportunity to organize this meeting and for their smooth running and support of the meeting.

Reinhard Diestel,
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Paul Seymour.

Workshop: Graph Theory

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Abstracts

Two structures on the same set

Ron Aharoni

The talk concerned a way of casting combinatorial duality results, in which there are two structures given on the same vertex set - two graphs, or two matroids, or a graph and a matroid, or most generally - two, or even more, simplicial complexes (a *simplicial complex* is a closed down hypergraph). The classical result of this type is Edmonds' two matroids intersection theorem [2]:

Theorem 1 (Edmonds)**.** *Given two matroids, $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same ground set V, we have:*

$$\max\{|I| : \ I \in \mathcal{M}_1 \cap \mathcal{M}_2\} = \min\{|X| + |Y| : \ sp_{\mathcal{M}_1}(X) \cup sp_{\mathcal{M}_1}(Y) = V\}$$

The first part of the talk presented a version of this theorem which is valid for general complexes, obtained in joint work with Eli Berger, Ron Holzman and Ori Kfir (in writing). To present it we shall need a few definitions.

For a simplicial complex $\mathcal{C}$ write $\mu(\mathcal{C}) = \max\{|\sigma| : \ \sigma \in \mathcal{C}\}$. Define a "span" operation, acting on subsets X of $V(\mathcal{C})$, as

$$sp_{\mathcal{C}}(X) = X \cup \{v : \ \sigma + v \notin \mathcal{C} \quad \text{for some } \sigma \in \mathcal{C}\}.$$

Note that for matroids this is the usual "span" operation. For complexes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ on the same vertex set V write

$$\gamma_{\cup}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m) = \min\{\sum |X_i| : \ \bigcup sp_{\mathcal{C}_i}(X_i) = V\}.$$

Given a complex $\mathcal{C}$, let $\Omega(\mathcal{C})$ be the convex hull of the incidence vectors of simplices in $\mathcal{C}$. Also write:

$$\mu^*_{\cap}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m) := \max\{\vec{x} \cdot \vec{1} \mid \vec{x} \in \cap\Omega(\mathcal{C}_i)\}.$$

For a graph G let $\mathcal{I}(G))$ be the complex of independent sets of G and $\mathcal{N}(G)$ the complex of non-punctured neighborhoods (namely the set of subsets of the closed neighborhoods of vertices in G). As is customary, we denote $\mu(\mathcal{I}(G))$ by $\alpha(G)$.

For graphs $G_1, G_2, \ldots, G_m$ on the same set define:

$$\alpha^*_{\cap}(G_1, G_2, \ldots, G_m) = \mu^*_{\cap}(\mathcal{I}(G_1), \mathcal{I}(G_2), \ldots, \mathcal{I}(G_m))$$

$$\gamma_{\cup}(G_1, G_2, \ldots, G_m) = \gamma_{\cup}(\mathcal{I}(G_1), \mathcal{I}(G_2), \ldots, \mathcal{I}(G_m))$$

Theorem 2 (A+Berger+Holzman+Kfir)**.** *For any pair of graphs (G_1, G_2) on the same vertex set*

$$\alpha^*_{\cap}(G_1, G_2) \geq \gamma_{\cup}(G_1, G_2)$$

For a complex $\mathcal{C}$ we write $\chi(\mathcal{C})$ for the minimal number of simplices needed to cover $V(\mathcal{C})$. Note that $\chi(\mathcal{I}(G))$ is the chromatic number of G, more commonly denoted by $\chi(G)$. On the other hand, $\chi(\mathcal{N}(G))$ is the domination number of G, usually denoted by $\gamma(G)$. Note also that

$$\gamma_{\cup}(G_1, G_2, \ldots, G_m) = \chi(\mathcal{N}(G_1), \mathcal{N}(G_2), \ldots, \mathcal{N}(G_m)).$$

Theorem 3. *For any system of graphs $(G_1, G_2, \ldots, G_m)$ on the same vertex set*

$$\alpha^*_{\cap}(G_1, G_2, \ldots, G_m) \geq \frac{2}{m}\gamma_{\cup}(G_1, G_2, \ldots, G_m).$$

The theorem is sharp asymptotically, but for $m > 2$ strict inequality holds.

(The case of partition graphs, namely graphs that are disjoint unions of cliques, is a theorem of Lovász: in an r-partite hypergraph $\tau \leq \frac{r}{2}\tau^*$). Theorem 2 has a generalization to complexes:

Theorem 4.

$$\mu^*_{\cap}(\mathcal{C}_1, \mathcal{C}_2) \geq \gamma_{\cup}(\mathcal{C}_1, \mathcal{C}_2).$$

This means that Edmonds' theorem holds for general complexes, if we replace the "max" side by its fractional version. Theorem 3 has a similar generalization.

Next we note the following hierarchy of inequalities.

- For one graph $\alpha \geq \gamma$.
- For two graphs $\alpha^*_{\cap} \geq \gamma_{\cup}$.
- For a general number of graphs $\alpha^*_{\cap} \geq \gamma^*_{\cup}$

The underlying reason behind these inequalities is a duality between the independence and neighborhood complexes. A first glimpse of this duality can be seen in the duality between the two basic inequalities on the chromatic numbers of the two complexes:

$$\chi(\mathcal{I}(G)) \leq \mu(\mathcal{N}(G))$$

(which is usually stated as "$\chi(G) \leq \Delta(G) + 1$") and:

$$\chi(\mathcal{N}(G)) \leq \mu(\mathcal{I}(G))$$

(usually formulated as "$\gamma \leq \alpha$").

The fact which entails all these is:

Theorem 5. $\Omega(\mathcal{N}(G)) \supseteq \overline{\Omega(\mathcal{I}(G))}$

Here $\bar{P}$ is the *anti-blocker* of P,

$$\bar{P} := \{\vec{x} : \ \forall \vec{y} \in P \quad \vec{y} \cdot \vec{x} \leq 1\}.$$

The rest of the talk concentrated on topological extensions of Edmonds' theorem. Of these, we mention just the main one, from [1]:

Theorem 6 (A+Berger)**.** *For a matroid $\mathcal{M}$ and a complex $\mathcal{C}$ on the same vertex set,* $\mu(\mathcal{M} \cap \mathcal{C}) \geq \min_{X \subseteq V}(\rho(\mathcal{M}|X) + \eta(\mathcal{C}|(V \setminus X))$

Here $\eta(\mathcal{C})$ is the topological connectivity of $\mathcal{C}$, plus 2. In other words, it is the minimal dimension of a "hole" in the geometric realization of $\mathcal{C}$ (and if there is no hole at all, η is defined as ∞).

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Proof of Berge's Strong Path Partition Conjecture for $k = 2$

Eli Berger

(joint work with Irith Ben-Arroyo Hartman)

Let $G = (V, E)$ be a directed graph containing no loops or multiple edges, defined by a set V of vertices and a set $E \subseteq V \times V$ of directed edges. A *path* P in G is a sequence of distinct vertices $(v_1, v_2, \ldots, v_l)$ such that $(v_i, v_{i+1}) \in E$, for $i = 1, 2, \ldots, l-1$. The set of vertices $\{v_1, v_2, \ldots, v_l\}$ of a path $P = (v_1, v_2, \ldots, v_l)$ is denoted by $V(P)$, and the set of edges by $E(P)$. The *cardinality* of P, denoted by $|P|$, is $|V(P)|$. A path of cardinality one is called a *trivial* path.

A family $\mathcal{P}$ of paths is called a *path partition* of G if its members are vertex disjoint and $\cup\{V(P); P \in \mathcal{P}\} = V$. A directed graph, or for brevity, *digraph*, may have many path partitions. The *trivial* path partition, where every path is a trivial path, is an example of a path partition. For each nonnegative integer k, the k-norm $|\mathcal{P}|_k$ of a path partition $\mathcal{P} = \{P_1, \ldots, P_m\}$ is defined by

$$|\mathcal{P}|_k := \sum_{i=1}^{m} \min\{|P_i|, k\}. \tag{1}$$

A partition which minimizes $|\mathcal{P}|_k$ is called *k-optimal.* Note that a 1-optimal path partition is a partition that contains a minimum number of paths and $|\mathcal{P}|_1 = |\mathcal{P}|$. Denote by $\mathcal{P}^{\geq k}$ the set of paths in $\mathcal{P}$ of cardinality at least k, (which we also call *long* paths), and by $\mathcal{P}^{<k}$ the set of paths in $\mathcal{P}$ of cardinality less than k, (called *short* paths). Then equation (1) can be alternatively written as

$$|\mathcal{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\} = k|\mathcal{P}^{\geq k}| + |V[\mathcal{P}^{<k}]|$$

A *k-colouring* is a family $\mathcal{C}^k = \{C_1, C_2, \ldots, C_k\}$ of k disjoint independent sets called *colour classes.* (Some of the colour classes may be empty). The *cardinality* of a k-colouring is the sum of the sizes of the colour classes, i.e., $|\mathcal{C}^k| = \sum_{i=1}^{k} |C_i|$ and $\mathcal{C}^k$ is said to be *optimal* if $|\mathcal{C}^k|$ is as large as possible. A path partition $\mathcal{P}$ and a k-colouring $\mathcal{C}^k$ are *orthogonal* if every path P_i in $\mathcal{P}$ meets $\min\{|P_i|, k\}$ different colour classes of $\mathcal{C}^k$. Note that this is the maximum number of different colour classes that a path can intersect a k-colouring.

Conjecture 1 (Berge's strong path partition conjecture [3]). *Let G be a digraph and let k be a positive integer. Then for every k-optimal path partition $\mathcal{P}$ there exists a k-colouring orthogonal to it.*

The conjecture holds for $k = 1$ for all digraphs by the Gallai-Milgram Theorem [6]. Berge's strong path partition conjecture has also been proved for acyclic digraphs (see [4], [11], [2], [1], and [9]). It is not difficult to see, as was shown in [3], that the conjecture is true when $k \geq \lambda$, where λ is the cardinality of the longest path in G, and when the k-optimal path partition contains only short paths, (i.e., paths of cardinality less than k). In [1] it was proved that the conjecture holds also in the case that the given k-optimal path partition contains only long paths, i.e., $\mathcal{P} = \mathcal{P}^{\geq k}$. For a survey of Berge's conjecture and related problems see [8]. See also [5], [12], and [13] for related results.

Denote by $\alpha_k(G)$ the cardinality of an optimal k-colouring in G, and by $\pi_k(G)$ the k-norm of a k-optimal path partition in G. Conjecture 1 implies the following:

Conjecture 2 (Weak path partition conjecture -Linial [10]). *For any digraph G and positive integer k, $\alpha_k(G) \geq \pi_k(G)$.*

If G is transitive and acyclic (i.e., the graph of a partially ordered set), then Green-Kleitman's theorem [7] states that $\alpha_k(G) = \pi_k(G)$, implying Conjecture 2.

In this talk we prove Conjecture 1 for $k = 2$ for all graphs. This is done by first itroducing an algorithmic proof for the case $k = 1$ and than adjasting it to the case $k = 2$.

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Cycles in dense digraphs

Maria Chudnovsky
(joint work with Blair D. Sullivan, Paul Seymour)

We begin with some terminology. All digraphs in this paper are finite and have no parallel edges; and for a digraph G, $V(G)$ and $E(G)$ denote its vertex- and edge-sets. The members of $E(G)$ are ordered pairs of vertices, and we abbreviate (u, v) by uv. For integer $k \geq 0$, let us say a digraph G is *k-free* if there is no directed cycle of G with length at most k. A digraph is *acyclic* if it has no directed cycle.

We are concerned here with 3-free digraphs. It is easy to see that every 3-free tournament is acyclic, and one might hope that every 3-free digraph that is "almost" a tournament is "almost" acyclic. That is the topic of this paper.

More exactly, for a digraph G, let $\gamma(G)$ be the number of unordered pairs $\{u, v\}$ of distinct vertices u, v that are nonadjacent in G (that is, both $uv, vu \notin E(G)$). Thus, every 2-free digraph G can be obtained from a tournament by deleting $\gamma(G)$ edges. Let $\beta(G)$ denote the minimum cardinality of a set $X \subseteq E(G)$ such that $G \setminus X$ is acyclic. We already observed that every 3-free digraph with $\gamma(G) = 0$ satisfies $\beta(G) = 0$, and our first result is an extension of this.

1. *If G is a 3-free digraph then $\beta(G) \leq \gamma(G)$.*

Proof. We proceed by induction on $|V(G)|$, and we may assume that $V(G) \neq \emptyset$. Let us say a *2-path* is a triple (x, y, z) such that $x, y, z \in V(G)$ are distinct, and $xy, yz \in E(G)$, and x, z are nonadjacent. For each vertex v, let $f(v)$ denote the number of 2-paths (x, y, z) with $x = v$, and let $g(v)$ be the number of 2-paths (x, y, z) with $y = v$. Since $V(G) \neq \emptyset$ and $\sum_{v \in V(G)} f(v) = \sum_{v \in V(G)} g(v)$, there exists $v \in V(G)$ such that $f(v) \leq g(v)$. Choose some such vertex v, and let A, B, C be the set of all vertices $u \neq v$ such that $vu \in E(G)$, $uv \in E(G)$, and $uv, vu \notin E(G)$ respectively. Thus the four sets $A, B, C, \{v\}$ are pairwise disjoint and have union $V(G)$. Let G_1, G_2 be the subdigraphs of G induced on A and on $B \cup C$ respectively. Since $g(v)$ is the number of pairs (a, b) with $a \in A$ and $b \in B$ such that a, b are nonadjacent, it follows that $\gamma(G) \geq \gamma(G_1) + \gamma(G_2) + g(v)$. From the inductive hypothesis, $\beta(G_1) \leq \gamma(G_1)$ and $\beta(G_2) \leq \gamma(G_2)$; for $i = 1, 2$, choose $X_i \subseteq E(G_i)$ with $|X_i| \leq \beta(G_i)$ such that $G_i \setminus X_i$ is acyclic. Let X_3 be the set of all edges $ac \in E(G)$ with $a \in A$ and $c \in C$; thus $|X_3| = f(v)$. Since there is no edge $xy \in E(G)$ with $x \in A$ and $y \in B$ (because G is 3-free), it follows that every edge xy with $x \in A$ and $y \in \{v\} \cup B \cup C$ belongs to X_3, and so $G \setminus X$ is acyclic, where $X = X_1 \cup X_2 \cup X_3$. Hence

$$\begin{aligned}\beta(G) \leq |X| = |X_1| + |X_2| + |X_3| \\ = \beta(G_1) + \beta(G_2) + f(v) \leq \gamma(G_1) + \gamma(G_2) + g(v) \leq \gamma(G).\end{aligned}$$

This proves the theorem. □

Unfortunately, Theorem 1 does not seem to be sharp, and we believe that the following holds.

2. Conjecture. *If G is a 3-free digraph then $\beta(G) \leq \frac{1}{2}\gamma(G)$.*

If true, this is best possible for infinitely many values of $\gamma(G)$. For instance, let G be the digraph with vertex set $\{v_1 \dots v_{4n}\}$, and with edge set as follows (reading subscripts modulo $4n$):

- $v_i v_j \in E(G)$ for all i, j, k with $1 \leq k \leq 4$ and $(k-1)n < i < j \leq kn$
- $v_i v_j \in E(G)$ for all i, j, k with $1 \leq k \leq 4$ and $(k-1)n < i \leq kn < j \leq (k+1)n$.

It is easy to see that this digraph G is 3-free, and satisfies $\beta(G) = n^2$ (certainly $\beta(G) \geq n^2$ since G has n^2 directed cycles that are pairwise edge-disjoint), and $\gamma(G) = 2n^2$.

The reason for our interest in Conjecture 2 was originally its application to the Caccetta-Häggkvist conjecture [1]. A special case of that conjecture asserts the following:

3. Conjecture. *If G is a 3-free digraph with n vertices, then some vertex has outdegree less than $n/3$.*

This is a challenging open question and has received a great deal of attention. Any counterexample to Conjecture 3 satisfies $\gamma(G) \leq \frac{1}{2}|E(G)|$, so our Conjecture 2 would tell us that $\beta(G) \leq \frac{1}{4}|E(G)|$, and this would perhaps be useful information towards solving Conjecture 3.

We have not been able to prove Conjecture 2 in general, but we obtained two partial results. We showed that Conjecture 2 holds for every 3-free digraph G such that either

- $V(G)$ is the union of two cliques, or
- the vertices of G can be arranged in a circle such that if distinct u, v, w are in clockwise order and $uw \in E(G)$, then $uv, vw \in E(G)$.

Incidentally, Kostochka and Stiebitz [2] proved that in any minimal counterexample to Conjecture 2, every vertex is nonadjacent to at least three other vertices, and the conjecture is true for all digraphs with at most 8 vertices.

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Nowhere-zero 3-flows

Genghua Fan
(joint work with Chuixiang Zhou)

A nowhere-zero k-flow in a graph G with an orientation is an integer-valued function f on $E(G)$ such that $0 < |f(e)| < k$ for each $e \in E(G)$, and for each $v \in V(G)$,

$$\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e),$$

where $E^+(v)$ is the set of non-loop edges with tail v, and $E^-(v)$ is the set of non-loop edges with head v. The following is the well-known 3-flow conjecture of Tutte.

Conjecture. *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

Let H_1 and H_2 be two subgraphs of a graph G. We say that G is the 2-sum of H_1 and H_2, denoted by $H_1 \oplus_2 H_2$, if $E(H_1)\cup E(H_2) = E(G)$, $|V(H_1)\cap V(H_2)| = 2$, and $|E(H_1) \cap E(H_2)| = 1$. A triangle-path in a graph G is a sequence of distinct triangles $T_1T_2\cdots T_m$ in G such that for $1 \le i \le m-1$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i)\cap E(T_j) = \emptyset$ if $j > i+1$. A connected graph G is triangularly connected if for any two edges e and e', which are not parallel, there is a triangle-path $T_1T_2\cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$.

Theorem 1. *Let G be a triangularly connected graph with $|V(G)| \ge 3$. Then G has no nowhere-zero 3-flow if and only if there is an odd wheel W and a subgraph G_1 such that $G = W \oplus_2 G_1$, where G_1 is a triangularly connected graph without nowhere-zero 3-flow.*

Repeatedly applying the theorem, we obtain a complete characterization of triangularly connected graphs which have no nowhere-zero 3-flow. As a consequence, G has a nowhere-zero 3-flow if it contains at most three 3-cuts, extending an earlier result of DeVos et al. [1]. Also, by the characterization, we obtain extensions to earlier results on locally connected graphs (Lai [3]), chordal graphs (Lai [2]) and squares of graphs (DeVos et al. [1]).

Let G be a simple graph on n vertices, $n \ge 3$. It is well known that if G satisfies the Ore-condition: $d(x) + d(y) \ge n$ for every pair of non-adjacent vertices x and y in G, then G has a hamiltonian circuit, which implies that G has a nowhere-zero 4-flow. But, it is not necessary for G to have a nowhere-zero 3-flow. We have that

Theorem 2. *Let G be a simple graph on n vertices, $n \ge 3$. If $d(x) + d(y) \ge n$ for for every pair of non-adjacent vertices x and y in G, then G has no nowhere-zero 3-flow if and only if G is one of the six completely described graphs on at most 6 vertices.*

For $n \geq 6$, let $K^+_{3,n-3}$ denotes the simple graph on n vertices obtained from the complete bipartite graph $K_{3,n-3}$ by adding an edge between two vertices of degree $n-3$. It is easy to show that in the graph $K^+_{3,n-3}$, $d(x)+d(y) \geq n$ for each edge xy, but $K^+_{3,n-3}$ has no nowhere-zero 3-flow. We prove that

Theorem 3. *Let G be a 2-edge-connected simple graph on n vertices. If $d(x) + d(y) \geq n$ for each edge $xy \in E(G)$, then G has no nowhere-zero 3-flow if and only if G is either $K^+_{3,n-3}$ or one of the five completely described graphs on at most 6 vertices.*

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Graph Separator Theorems

Jacob Fox

(joint work with János Pach)

We describe some old and new generalizations of the Lipton-Tarjan separator theorem for planar graphs. We conclude with three conjectures in combinatorial geometry that we recently settled using the new separator theorems.

1. The Lipton-Tarjan separator theorem and its extensions

A *weight* for a graph $G = (V, E)$ is a function $w : V \to \mathbb{R}_{>0}$. For $S \subset V$, we let $w(S) := \sum_{v \in S} w(v)$. A *separator* for a finite graph $G = (V, E)$ with respect to a weight w is a subset $V_0 \subset V$ for which there is a partition $V = V_0 \cup V_1 \cup V_2$ with $w(V_1), w(V_2) \leq \frac{2}{3}w(V)$ and no vertex in V_1 is adjacent to a vertex in V_2.

The classical Lipton-Tarjan separator theorem [7] states that for every planar graph G with n vertices and weight function w for G, there is a separator for G with respect to w of size $O(\sqrt{n})$.

Gilbert, Hutchinson, and Tarjan [5] generalized the Lipton-Tarjan separator theorem to graphs embeddable in an orientable surface. They proved that for every graph G with n vertices embeddable in an orientable surface of genus g and for every weight w for G, there is a separator for G with respect to w of size $O(\sqrt{gn})$. This result is tight up to the implied constant.

The well-known Kuratowski-Wagner theorem states that a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor. Alon, Seymour, and Thomas [1] proved that for every graph G with no K_h-minor and weight w for G, there is a separator for G with respect to w of size $O(h^{3/2}n^{1/2})$. It is an interesting open problem to improve this bound to $O(hn^{1/2})$. If true, this would imply the result of Gilbert, Hutchinson, and Tarjan, since K_h has genus $\Omega(h^2)$.

By an important theorem of Koebe [6], every planar graph can be represented as the intersection (incidence) graph of nonoverlapping closed disks in the plane. Miller, Teng, Thurston, and Vavasis [8] proved that for every $d \geq 2$, graph G that is an intersection graph of a collection of n balls in $\mathbb{R}^d$ such that no point belongs to more than k of them, and weight w for G, there is a separator for G with respect to w of size $O(dk^{1/d}n^{1-1/d})$.

A *Jordan region* is a subset of the plane that is homeomorphic to a closed disk. We say that a Jordan region R *contains* another Jordan region S if S lies in the interior of R. A *crossing* between R and S is either a crossing between their boundaries or a containment between them. The following result is a generalization of the separator theorems of Lipton and Tarjan and of Miller, Teng, Thurston, and Vavasis [8] in two dimensions.

Theorem 1. *If G is an intersection graph of a collection of Jordan regions with a total of m crossings and w is a weight function for G, then G has a separator of size $O(\sqrt{m})$.*

To see that Theorem 1 implies the planar version of the theorem of Miller, Teng *et al.* mentioned above, and hence the original Lipton-Tarjan theorem, it is enough to notice that, given a system of n disks in the plane such that no k of them have a point in common, the number of crossing pairs is $O(kn)$.

Similar results hold for intersecting graphs in other orientable surfaces. Unlike Theorem 1, the following result does not depend directly on the number of crossings, but only on the number of edges and on the clique number of the intersection graph.

Theorem 2. *If G is a K_k-free intersection graph of a collection of convex sets in the plane with m edges, and w is a weight function for G, then G has a separator of size $O(\sqrt{km})$.*

For proofs of Theorems 1 and 2, see [2]. Theorem 2 does not hold in higher dimensions, as Tietze [12] showed that every finite graph can be obtained as the intersection graph of convex polytopes in $\mathbb{R}^3$. One can even assume that these polytopes have no interior points in common!

2. Three applications

A. The following result, conjectured by Pach and Sharir [9], can be deduced both from Theorem 1 and Theorem 2 (see [2]).

Theorem 3. *For each bipartite graph H, there is a constant $c(H)$ such that every intersection graph of n convex sets in the plane that does not contain H as a subgraph has at most $c(H)n$ edges.*

We cannot decide whether the theorem remains true for H-free intersection graphs of not necessarily convex Jordan regions.

B. The proof of the following result, conjectured by Pach and G. Tóth [11], was found by the authors and Cs. D. Tóth [4], and is also based on Theorem 1.

Theorem 4. *For any positive integer k, there is $c_k > 0$ such that if G is an intersection graph of n curves in the plane with no pair of curves intersecting in more than k points, then G or its complement $\bar{G}$ contains a bi-clique of size at least $c_k n$.*

It is interesting to note that this statement does not remain true if we drop the assumption on the number of times two curves may cross.

C. The next result, proved in [3], strengthens a conjecture by Pach and G. Tóth [10]. A *topological graph* is a graph drawn in the plane with vertices as points and edges as curves connecting the corresponding pairs of vertices and not passing through any other vertex.

Theorem 5. *For any $\epsilon > 0$ and for any positive integer k, there are $\delta > 0$ and n_0 such that every topological graph with $n \geq n_0$ vertices, at least $n^{1+\epsilon}$ edges, and no pair of edges intersecting in more than k points contains n^{δ} pairwise intersecting edges.*

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Hadwiger's conjecture for quasi-line graphs

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(joint work with Maria Chudnovsky)

All graphs we consider are finite and for a graph G, $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. For $X \subseteq V(G)$, let $G|X$ denote the subgraph of G induced on X. We say that $X \subseteq V(G)$ is a *claw* if $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. A graph is then *claw-free* if no subset of $V(G)$ is a claw.

Hadwiger's conjecture states that for every loopless graph G and every integer $t \geq 0$, either G is t-colorable, or G has a clique minor of size $t+1$. The case $t = 4$ is equivalent to the four color theorem and the case $t = 5$ was proved by Robertson, Seymour, and Thomas with the use of the four color theorem. For $t > 5$, the conjecture remains open.

Hadwiger's conjecture has also been proved for some special classes of graphs. In a recent work, Reed and Seymour [2] proved Hadwiger's conjecture for line graphs. We prove Hadwiger's conjecture for a class of graphs that is a proper superset of line graphs and a proper subset of claw-free graphs, the set referred to as *quasi-line graphs.* A graph G is a quasi-line graph if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques. Note that this is a partition of the vertex set of the neighborhood of v. Our main result is the following:

Theorem 1. *Let G be a quasi-line graph with chromatic number χ. Then G has a clique minor of size χ.*

Our proof of Theorem 1 uses a structure theorem for quasi-line graphs that appears in [1]. We introduce some definitions and then state the theorem.

Let Σ be a circle and let $F_1, \ldots, F_k$ be subsets of Σ, each homeomorphic to the closed interval $[0, 1]$. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only if $v_1, v_2 \in F_i$ for some i. Such a graph is called a *circular interval graph.* Let $\mathbb{F} = \{F_1, \ldots, F_k\}$. Then we call the pair $(\Sigma, \mathbb{F})$ a *representation* of G. A subset $S \subset V$ is a *block* if $S = F_i \cap V$ for some $F_i \in \mathbb{F}$. We then call S *the block of* F_i. A *linear interval graph* is constructed in the same way as a circular interval graph except we take Σ to be a line instead of a circle. It is easy to see that all linear interval graphs are also circular interval graphs.

The structure theorem that we use states that there are two types of quasi-line graphs. The first subclass is a generalization of the class of circular interval graphs and we proceed to describe it below. Once again, we start with a few definitions.

Let X, Y be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* if no vertex of X is adjacent to a member of Y. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then v is *A-complete* if it is adjacent to every vertex in A, and *A-anticomplete* if it has no neighbor in A. A pair A, B of disjoint subsets of $V(G)$ is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete.

Let G be a circular interval graph with $V(G) = \{v_1, \ldots, v_n\}$ in order clockwise. An edge joining v_j to v_k with $j < k$ is called a *maximal edge* if $\{v_j, v_{j+1}, \ldots, v_k\}$ is a block. In this case the following operation produces another quasi -line graph: replace v_j and v_k by two cliques A and B, respectively, such that every member of A has the same neighbors as v_j and every member of B has the same neighbors as v_k in $V(G) \setminus \{v_j, v_k\}$, and the edges between A and B are arbitrary. The pair (A, B) is then a *homogeneous pair of cliques.* Let H be a graph obtained from a circular interval graph by choosing a matching of maximal edges and replacing each of them by a homogeneous pair of cliques as described above. Then H is called a *fuzzy circular interval graph.*

Let (A, B) be a homogeneous pair of cliques in a circular interval graph. We say that (A, B) is *non-trivial* if there exists an induced 4-cycle in G with exactly two vertices in A and exactly two vertices in B. It is easy to see that if a fuzzy circular interval graph is not a circular interval graph, then it has a non-trivial homogeneous pair.

We proceed with the construction of graphs that belong to the second subclass of quasi-line graphs. A vertex $v \in V(G)$ is *simplicial* if the set of neighbors of v is a clique. A claw-free graph S together with two distinguished simplicial vertices a, b is called a *strip* (S, a, b), with *ends* a and b. If S is a linear interval graph with $V(S) = \{v_1, \ldots, v_n\}$ in order and with $n > 1$, then v_1, v_n are simplicial, and so (S, v_1, v_n) is a strip, called a *linear interval strip.* Since linear interval graphs are also circular interval graphs, we can define *fuzzy linear interval strips* by introducing homogeneous pairs of cliques in the same manner as before.

Let (S, a, b) and (S', a', b') be two strips. Then they can be composed as follows. Let A, B be the set of neighbors of a, b in S respectively, and define A', B' analogously. Consider the disjoint union of $S \setminus \{a, b\}$ and $S' \setminus \{a', b'\}$, and make A complete to A' and B complete to B'.

This method of composing two strips described above can be used as follows. Let S_0 be a graph which is the disjoint union of complete graphs with $|V(S)| = 2n$. We arrange the vertices into pairs $(a_1, b_1), \ldots, (a_n, b_n)$. For $i = 1, \ldots, n$, let (S'_i, a'_i, b'_i) be a strip and let S_i be the graph obtained by composing (S_{i-1}, a_i, b_i) and (S'_i, a'_i, b'_i). The resulting graph S_n is then called a *composition* of the strips (S'_i, a'_i, b'_i).

We are finally ready to state the structure theorem for quasi-line graphs [1] that we use to prove our main result.

Theorem 2. *Let G be a connected, quasi-line graph. Then G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

The proof of Theorem 1 is by induction on the number of non-trivial homogeneous pairs. We first show that the theorem holds if G is a quasi-line graph with no non-trivial homogeneous pairs, that is, G is a circular interval graph or a combination of linear interval strips. We next show that if G has a non-trivial homogeneous pair then there exists a graph H with the following properties:

(1) H is a quasi-line graph with one fewer non-trivial homogeneous pair than G.
(2) $\chi(H) = \chi(G)$.
(3) H is a minor of G.

Then inductively, since H has one fewer homogeneous pair than G, H has a clique minor of size $\chi(H) = \chi(G)$. Since H is a minor of G, every clique minor of H is also a clique minor of G. Hence, G has a clique minor of size $\chi(G)$, which completes the proof of Theorem 1.

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Fleischner's theorem for infinite graphs

Angelos Georgakopoulos

The traditional approach to hamiltonicity in infinite graphs is to take a 'Hamilton cycle' to be a double ray containing all vertices. However, it is easy to see that no graph with more than two ends can contain such a double ray. Recently, a way was found to overcome this difficulty, and many more, opening the way for studying extremal-type problems for infinite graphs without restricting the scope to a special class of graphs: Diestel [3, 4] suggested the use of topological paths and circles as analogues of finite paths and cycles, respectively, in order to translate finite extremal graph theory to locally finite graphs. The topology he uses is the Freudenthal compactification $|G|$ of G obtained by adding its ends (see [2]). In this sense, an infinite cycle, called a *circle*, is a homeomorphic image of S^1 in $|G|$, and a *hamilton circle* is a circle containing all vertices (and hence, since circles are closed subsets, all ends).

This approach of using topological circles as an analog for finite cycles has already enjoyed considerable success in the study of the cycle space of a locally finite graph, see [3]. Our main result, proving a conjecture of Diestel [2], suggests that it is very promising for the study of extremal problems as well:

Theorem 1 ([6])**.** *If G is a locally finite 2-connected graph, then $|G^2|$ has a Hamilton circle.*

(G^n is the graph on $V(G)$ where xy is an edge if $d(x, y) \leq n$ in G.)

For finite graphs this is a well known theorem of Fleischner [5], and in fact one of the ideas used for the proof of Theorem 1 led to a short proof of Fleischner's theorem, see [7]. For 1-ended graphs it is a theorem of Thomassen [8].

When one tries to prove Theorem 1, the standard compactness tools fail; applying them one can easily obtain a continuous map from S^1 to $|G|$ visiting every vertex exactly once, but such a map is in general not a circle because it does not have to be injective at the ends.

In order to overcome this difficulty and construct a map that visits not only every vertex but also every end exactly once, the proof of Theorem 1 reduces the problem to that of proving the existence of a suitable Euler tour in an auxiliary graph, where a *(topological) Euler tour* of a locally finite graph is defined as a continuous image of S^1 in $|G|$ traversing every edge exactly once.

The rough structure of the proof of Theorem 1 is as follows. Firstly, an eulerian auxiliary graph G' is obtained from G by deleting some of its edges and by doubling some others. Then, an Euler tour σ of G' is chosen, and it is transformed into a Hamilton circle of G^2 by performing "leaps" over vertices, that is, by replacing pairs of subsequent edges uv, vw in σ by the edge uw of G^2. It is possible to perform enough leaps of this kind to ensure that every vertex will eventually have degree precisely 2, but only if G' is carefully constructed to make this possible. This task, already difficult in the finite case, is made more complicated by the fact that G' must have the same end-structure as G in order for the rest of the proof to work. In any case, the resulting infinite walk, i.e. the alleged Hamilton circle, will be injective at the ends if and only if σ is. Now σ can indeed be chosen to be injective at the ends by the following result:

Theorem 2 ([6]). *If a locally finite multigraph has a topological Euler tour, then it also has one that is injective at ends.*

Having seen Theorem 2 and the way it is applied to prove Theorem 1, a general approach for generalising sufficient conditions for hamiltonicity to locally finite graphs suggests itself: one could try to reduce the problem of finding a Hamilton cycle to that of finding an Euler tour in some auxiliary graph, extend this reduction to the infinite case, and then use Theorem 2. The following two conjectures are good candidates for trying this approach, since the corresponding finite proofs do use Euler tours:

Conjecture 1. *Every locally finite* 7*-connected line graph has a Hamilton circle.*

For finite graphs this is a theorem of Zhan [10].

Conjecture 2. *If G is a* 4*-edge-connected locally finite graph then $|L(G)|$ contains a Hamilton circle.*

For finite graphs there is an easy proof of this fact also using Euler tours ([1]). The last two results are special cases of a conjecture of Thomassen [9], stating that every finite 4-connected line graph is hamiltonian. Of course, one can also ask if Thomassen's conjecture is true for infinite graphs.

Up to now we restricted our attention to locally finite graphs, but our hamiltonicity problems can also be stated for non-locally-finite graphs, at least for countable ones:

Conjecture 3. *If G is a countable 2-connected graph then $|G^2|$ contains a Hamilton circle.*

A further result from [6] that could be extended to non-locally-finite graphs is the following:

Theorem 3. *If G is a connected locally finite graph, then $|G^3|$ contains a Hamilton circle.*

Conjecture 4. *If G is a countable connected graph then $|G^3|$ contains a Hamilton circle.*

See [6] for a discussion of Conjecture 4.

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The rotational dimension of a graph

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(joint work with Markus Wappler, Christoph Helmberg)

Fiedler [1] introduced an interesting graph invariant - the absolute algebraic connectivity being the maximized (over all non-negative edge weightings that sum up to the number of edges) second smallest eigenvalue of the laplacian of a graph G. By semidefinite duality, the underlying semidefinite optimization problem leeds to the problem of embedding G into n-space such that the variance of the vertex-positions is maximized, while the distances of incident vertices are bounded by one

$$(1)\qquad \begin{array}{rl} \max & \sum_{i\in N} \|v_i\|^2 \\ \text{s.t.} & \sum_{i\in N} v_i = 0 \\ & \|v_i - v_j\| \le 1 \qquad \text{for } ij \in E \\ & v_i \in \Re^n \text{ for } i \in N. \end{array}$$

(see [2]).

For connected graphs G let $\dim_a(G)$ be the minimum dimension of an optimal embedding (for other graphs let $\dim_a(G) = \max\{\dim_a(C) | C$ is a component of $G\})$.

We are interested in upper bounds for $dim_a(G)$.

First we realize, that $tw(G)+1 \geq dim_a(G)$, where $tw(G)$ denotes the treewidth of G. Especially we show, that this bound is tight in the sense, that for each positive integer t there is a graph with $tw(G) = t$ and $dim_a(G) = t+1$.

Unfortunately it turns out, that this bound is not very sharp for some wellknown graph classes (i.e. if G is a planar grid, we get $dim_a(G) = 1$).

Thus better upper bounds for $dim_a(G)$ are wanted.

To make this problem more tractable, we generalize it in the following way: We attach a non-negative value to each edge (the edge length) and each vertex (the vertex weight) and maximize the weighted variance of the vertex positions with respect to the constrained, that the distance between positions of incident vertices is bounded by the edge length:

$$
\begin{array}{rll}
\max & \sum_{i\in N} s_i \|v_i\|^2 & \\
\text{s.t.} & \sum_{i\in N} s_i v_i = 0 & \\
& \|v_i - v_j\| \leq l_{ij} & \text{for } ij \in E \\
& v_i \in \Re^n \text{ for } i \in N. &
\end{array} \tag{2}
$$

Again we look for the minimum dimension of an optimal embedding (in case G is connected). The largest such minimum dimension maximized over all possible edge lengths and vertex weights we call rotational dimension of G (short $dim_{rot}(G)$). By construction, $dim_{rot}(G)$ is an upper bound for $dim_a(G)$.

An optimal solution of (2) can be interpreted as follows: Imagine a graph, the edges being massless cords of fixed nonnegative lengths connecting vertices of possibly different nonnegative masses, rotating uniformly (each vertex having the same angular speed) around its barycenter. The centrifugal force tends to maximize the weighted variance of the positions of its vertices.

This interpretation motivates the chosen name of the invariant as the following theorem

Theorem 1 (Separator-Shadow). *Let $v_i \in \Re^n$ for $i \in N$ be an optimal solution of (2) for a connected graph $G = (N, E)$ with vertex weights s_i and edge lengths l_{ij}. Furthermore, let $K_1 \dot\cup S \dot\cup K_2$ be a partition of N with no node in K_1 adjacent to a node in K_2. Then, for at least one $j \in \{1, 2\}$, for every $i \in K_j$ the straight line segment $[0, v_i]$ intersects the convex hull of the points in S, i.e., $\forall i \in K_j$: $[0, v_i] \cap conv\{v_s : s \in S\} \neq \emptyset$.*

Using Theorem 1 we give an idea of the proof of

Theorem 2. *$dim_{rot}(G) \leq tw(G) + 1$.*

Furthermore we note, that the rotational dimension is a minor monotone graph property. For small values we got the following result:

Theorem 3. *Let G be a graph. Then*

- $\dim_{rot}(G) = 0$ *iff G is edgeless,*
- $\dim_{rot}(G) \leq 1$ *iff G has maximum degree 2, and*
- $\dim_{rot}(G) \leq 2$ *iff G is outerplanar.*

We conjecture $dim_{rot}(G) \leq \mu(G)$, where $\mu(G)$ denotes the Colin de Verdiére number of the graph (see [3]). Because we could prove $\dim_{rot}(K_{3,3}) = 3 < \mu(K_{3,3}) = 4$ it is clear, that the rotational dimension and the Colin de Verdiére number are different graph properties.

Computational experiments give strong evidence to the following conjectures:

- $\dim_{rot}(M_8) = 3$ if M_8 is the moebius ladder on 8 vertices (also known as wagner graph).
- $\dim_{rot}(P) \leq 3$ if P is a planar graph.

Furthermore, we conjecture the following:

- $\dim_{rot}(G_1 \cup G_2) \leq \max\{\dim_{rot}(G_1), \dim_{rot}(G_2), |G_1 \cap G_2| + 1\}$ if $G_1 \cap G_2$ is complete.
- $\dim_{rot}(G) \leq 3$ iff G has no K_5-minor.

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On tree width, bramble size, and expansion

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(joint work with Dániel Marx)

Tree width is a fundamental graph invariant with many applications in graph structure theory and graph algorithms. Tree width has a dual characterisation in terms of brambles [2, 3]. A *bramble* in a graph G is a family of connected subgraphs of G such that any two of these subgraphs have a nonempty intersection or are joined by an edge. The *order* of a bramble is the least number of vertices required to cover all subgraphs in the bramble. Seymour and Thomas [3] proved that a graph has tree width k if and only if the maximum order of a bramble of G is $k + 1$.

Such a dual characterisation of a graph invariant can be very useful in algorithmic or complexity theoretic applications. However, the bramble characterisation of tree width has a serious drawback, because brambles may be exponentially large and therefore it is not even possible to "guess" a bramble of large order in polynomial time and use it as a witness for large tree width in a nondeterministic algorithm. Motivated by such considerations, we address the question of how large brambles actually need to be. It will be important in the following to distinguish between the *size* of a bramble, that is, the number of subgraphs it consists of, and its order. We establish an exponential lower bound on the size of brambles of maximum order. Actually, we prove a stronger result that applies also to brambles of order smaller than the maximum: There is a family $(G_k)_{k\geq 1}$ of graphs such that for every $\epsilon > 0$ and every k, the tree width of G_k is k, and every bramble of G_k of order at least $\Omega(k^{1/2+\epsilon})$ has size exponential in n. Conversely, we prove that every graph of tree width k has a bramble of order $\Omega(k^{1/2}/\log^2 k)$ and size polynomial in n and k.

To establish the lower bound, we need sparse graphs with tree width linear in the number of vertices. We observe that graphs with positive vertex expansion have this property, hence bounded-degree expander graphs can be used for the lower bound. Furthermore, we prove the following converse statement: if all graphs in a class $\mathcal{C}$ has tree width linear in the number of vertices, then they contain subgraphs of linear size (again in the number of vertices) with vertex expansion bounded from below by a constant.

For the upper bound, we give a novel characterization of large tree width, which might be of independent interest. Let $G^{(q)}$ denote the graph obtained by replacing every vertex by a clique of size q and replacing every edge with a complete bipartite graph. Let L_t be the line graph of a complete graph on k vertices. We show that if the tree width of G is large, then $G^{(q)}$ has an $L_k^{(r)}$ minor, for appropriate values of q, r, and t. Thus, rather than characterizing the tree width of G with its minors, we characterize it with the minors of $G^{(q)}$. This way, we can obtain a characterization that is tight up to an $O(\log k)$ factor, where k is the tree width of G. The proof is based on the connection between tree width and the existence of certain separators. The main technical tool that we use is an intergrality gap result of Feige et al. [1] for the balanced separator problem.

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Independent dominating sets and Hamilton cycles

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(joint work with Ben Seamone, Jacques Verstraete)

A famous conjecture of Sheehan [6] from 1975 states that every four-regular hamiltonian graph has at least two hamilton cycles. In the direction of this conjecture, Thomassen proved that all hamiltonian d-regular graphs contain at least two hamilton cycles, provided $d > 71$. A graph is uniquely hamiltonian if it contains exactly one hamilton cycle. Here we extend this result as follows:

Theorem 1. *There are no d-regular uniquely hamiltonian graphs when $d > 22$.*

The proof of Theorem 1 uses Thomassen's [10] sufficient condition for a second hamilton cycle: a hamiltonian graph has at least two hamilton cycles if it contains an independent set of vertices in a hamilton cycle C which is a dominating set in $G - E(C)$. In this paper we call such a set of vertices a *C-independent dominating set.* Since by Thomason [8] every d-regular graph with d odd is not uniquely hamiltonian, to prove Theorem 1, it suffices to show that in a d-regular hamiltonian graph with a hamilton cycle C and $d > 23$, there is a C-independent dominating set. On the other hand one may ask if a smaller value of d is possible. We will give examples of four-regular graphs G with hamilton cycle C that do not contain any C-independent dominating set. However the following question is open: does there exists a five-regular graph with a hamilton cycle C that does not contain a C-independent dominating set?

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Intrinsically knotted graphs

Hein van der Holst

Consider an embedding of a graph G in 3-space. Each circuit of G corresponds to an embedding of the 1-sphere in 3-space, and so is a knot. We call the embedding knotless if each circuit of G is embedded knotlessly. If there is no knotless embedding, we say that the graph is intrinsically knotted.

It is not difficult to see that if G has a knotless embedding, then each minor of G has a knotless embedding. By a theorem of Robertson and Seymour [2], we know that the class of graphs that have a knotless embedding can be characterized by a finite collection of forbidden minors. Conway and Gordon [1] showed that K_7 is intrinsically knotted. They showed that $\sum_C \alpha(C) = 1$ for each embedding of K_7 in 3-space, where the sum is over all Hamilton circuits and where α is the Arf invariant. Can we find more forbidden minors? Instead of restricting ourselves to just the Arf invariant, we use Vassiliev's invariants.

A singular knot of order m is an immersed circle in 3-space with m transversal self intersections. If V is an invariant on singular knots with at most $m-1$ self intersection, then V can be extended to singular knots with m self intersection by defining it as the difference of V on the two singular knots obtained from making one self intersection an overcrossing and an undercrossing. Recursively, we can then extend each knot invariant to singular knots. A Vassiliev invariant of order m is a knot invariant that vanishes on singular knots with at least $m+1$ self intersections (see [3] for more on Vassiliev's invariants).

Let $G = (V, E)$ be a graph and let $\mathcal{C}$ be the collection of all circuits in G. Orient each of the circuits in $\mathcal{C}$ and let V be a Vassiliev invariant. For each embedding $\mathcal{G}$ of G, we define a vector $\overline{V} \in \mathbb{Z}^{\mathcal{C}}$ by $\overline{V}_C = V(\mathcal{G}(C))$. Here $\mathcal{G}(C)$ denotes the embedding of C induced by $\mathcal{G}$. Then $\overline{V}$ is a topological invariant of $\mathcal{G}$. A singular graph of order m is an immersion of a graph in 3-space with m transversal self intersections. Call two singular graphs of order m equivalent if can obtain one from the other by pulling edges through other edges. Extend $\overline{V}$ to singular graphs. If V is a Vassiliev invariant of order m, then $\overline{V}$ vanishes on each singular graph with at least $m+1$ self intersections. If $\mathcal{G}$ is a knotless embedding, then $\overline{V}(\mathcal{G}) = 0$.

To test whether a graph G has no embedding $\mathcal{G}$ in 3-space such that $\overline{V}(\mathcal{G}) = 0$, we can use the following matrix M. For each integer $m > 1$ and each equivalence class of singular graphs of order m, take one representative $\mathcal{H}$ and place $\overline{V}(\mathcal{H})$ as a row in M. Take an arbitrary embedding $\mathcal{G}$ of the graph G. If $\overline{V}(\mathcal{G})$ is not an integer combination of the rows of M, then G has no embedding $\mathcal{G}$ in 3-space such that $\overline{V}(\mathcal{G}) = 0$.

The converse need not be true. If $\overline{V}(\mathcal{G})$ is an integer combination of the rows of M, then G need not have an embedding $\mathcal{G}$ in 3-space such that $\overline{V}(\mathcal{G}) = 0$. To find a system of equations such that also the converse holds, is work progress.

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Pin-collinear body-and-pin frameworks and the Molecular Conjecture

Tibor Jordán
(joint work with Bill Jackson)

Informally, a body-and-hinge framework in $\mathbb{R}^d$ consists of large rigid bodies articulated along affine subspaces of dimension $d-2$ which act as hinges i.e. bodies joined by pin-joints in 2-space, line-hinges in 3-space, plane-hinges in 4-space, etc. This notion may be formalized by using the facts that the infinitesimal motions of a rigid body in d-space can be coordinatized using screw centers (real vectors of length $\binom{d+1}{2}$ which represent $(d-1)$-tensors in projective d-space), and that rotations correspond to particular kinds of screw centers called $(d-1)$-extensors, see [1]. A *d-dimensional body-and-hinge framework* (G,q) is a multigraph $G=(V,E)$ together with a map q which associates a $(d-2)$-dimensional affine subspace $q(e)$ of $\mathbb{R}^d$ with each edge $e \in E$. An *infinitesimal motion* of (G,q) is a map S from V to $\binom{d+1}{2}$-space such that, for every edge $e=uv$, $S(u)-S(v)$ is a scalar multiple of $P(e,q)$, where $P(e,q)$ is a $(d-1)$-extensor which corresponds to a rotation about $q(e)$. An infinitesimal motion S is *trivial* if $S(u)=S(v)$ for all $u,v \in V$ and (G,q) is said to be *infinitesimally rigid* if all its infinitesimal motions are trivial.

These definitions can be motivated by considering each vertex $v \in V$ as being represented by a large rigid body B_v in d-space and each edge $e=uv \in E$ as being represented by the 'hinge' $q(e)$ attached to B_u and B_v. Each body B_v can move continuously subject to the constraints that, for each edge $e=uv \in E$, the relative motion of B_u with respect to B_v is a rotation about the hinge $q(e)$. At any given instant, the motion of B_v is represented by the screw center $S(v)$. The constraint concerning the relative motion of B_u with respect to B_v is represented by the condition that $S(u)-S(v)$ is a scalar multiple of $P(e,q)$. We refer the reader to [7, 9] for a more detailed account of body-and-hinge frameworks in $\mathbb{R}^d$. We will only be concerned with the case $d=2$.

Multigraphs which can be realized as infinitesimally rigid body-and-hinge frameworks are characterized by the following theorem, proved independently by Tay [5] and Whiteley [7]. Given a multigraph G and a positive integer k, we use kG to denote the multigraph obtained by replacing each edge of G by k parallel edges.

Theorem 1. [5, 7] *A multigraph G can be realized as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2}-1)G$ has $\binom{d+1}{2}$ edge-disjoint spanning trees.*

Tay and Whiteley jointly conjecture that the same condition characterizes when a multigraph can be realized as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^d$ with the additional property that all the hinges incident to each body are contained in a common hyperplane.

Conjecture 2. [6] *Let G be a multigraph. Then G can be realized as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^d$ if and only if G can be realized as an infinitesimally rigid body-and-hinge framework (G, q) in $\mathbb{R}^d$ with the property that, for each $v \in V$, all of the subspaces $q(e)$, e incident to v, are contained in a common hyperplane.*

Conjecture 2 is known as the Molecular Conjecture because of its implications for the rigidity of molecules when $d = 3$.[1] It has been verified by Whiteley [8] when $d = 2$ for the special case when $2G$ is the union of three edge-disjoint spanning trees.

Our main result is a complete solution of the conjecture when $d = 2$.

Theorem 3. [4] *Let $G = (V, E)$ be a multigraph. Then the following statements are equivalent.*
(a) G has a realization as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^2$.
(b) G has a realization as an infinitesimally rigid body-and-hinge framework (G, q) in $\mathbb{R}^2$ with each of the sets of points $\{q(e) : e \in E_G(v)\}$, $v \in V$, collinear.
(c) $2G$ contains three edge-disjoint spanning trees.

Our proof relies on a new formula for the maximum rank of a pin-collinear body-and-pin realization of a multigraph as a 2-dimensional bar-and-joint framework.

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On Steiner rooted-orientations of graphs and hypergraphs

Tamás Király
(joint work with Lap Chi Lau)

Introduction. Let $H = (V, \mathcal{E})$ be an undirected hypergraph. An *orientation* of H is obtained by assigning a direction to each hyperedge in H. In our setting, a *hyperarc* (a directed hyperedge) is a hyperedge with a designated *tail vertex* and other vertices as *head vertices*. Given a set $T \subseteq V$ of *terminal vertices* (the vertices in $V - T$ are called the *Steiner vertices*) and a *root vertex* $r \in T$, we say a directed hypergraph is *Steiner rooted k-hyperarc-connected* if there are k hyperarc-disjoint paths from the root vertex r to each terminal vertex in T. Here, a *path* in a directed hypergraph is an alternating sequence of distinct vertices and hyperarcs $\{v_0, a_0, v_1, a_1, \ldots, a_{k-1}, v_k\}$ so that v_i is the tail of a_i and v_{i+1} is a head of a_i for all $0 \leq i < k$. The Steiner Rooted-Orientation problem is to find an orientation of H so that the resulting directed hypergraph is Steiner rooted k-hyperarc-connected, and our objective is to maximize k.

When the Steiner Rooted-Orientation problem specializes to graphs, it is a common generalization of some classical problems in graph theory. When there are only two terminals ($T = \{r, v\}$), it is the edge-disjoint paths problem solved by Menger. When all vertices in the graph are terminals ($T = V$), it can be shown to be equivalent to the edge-disjoint spanning trees problem solved by Tutte [12] and Nash-Williams [11]. An alternative common generalization of the above problems is the Steiner Tree Packing problem studied in [7, 4, 8]. Notice that if a graph G has k edge-disjoint *Steiner trees* (i.e. trees that connect the terminal vertices T), then G has a Steiner rooted k arc-connected orientation. The converse, however, is not true. As we shall see, significantly sharper approximate min-max relations and also approximation ratio can be achieved for the Steiner Rooted-Orientation problem, especially when we consider hyperarc-connectivity and element-connectivity. This has implications in the network multicasting problem.

Given a hypergraph H, we say T *is k-hyperedge-connected in H* if there are k hyperedge-disjoint paths between every pair of vertices in T. It is not difficult to see that for a hypergraph H to have a Steiner rooted k-hyperarc-connected orientation, T must be at least k-hyperedge-connected in H. The main focus of this paper is to determine the smallest constant c so that the following holds: If T is ck-hyperedge-connected in H, then H has a Steiner rooted k-hyperarc-connected orientation.

Previous Work. Graph orientations is a well-studied subject in the literature, and there are many ways to look at such questions (see [1]). Here we focus on graph orientations achieving high connectivity.

In the following $\lambda(x, y)$ denotes the maximum number of edge-disjoint paths from x to y, which is called the *local-edge-connectivity* from x to y. Nash-Williams [10] proved the following deep generalization of Robbins' theorem which achieves optimal local-arc-connectivity for all pairs of vertices: "Every undirected graph G has an orientation D so that $\lambda_D(x, y) \geq \lfloor \lambda_G(x, y)/2 \rfloor$ for all $x, y \in V$".

For the Steiner Rooted-Orientation problem, the only known result follows from Nash-Williams' orientation theorem: if T is $2k$-edge-connected in an undirected graph G, then G has a Steiner rooted k-arc-connected orientation. For hypergraphs, there is no known orientation result concerning Steiner rooted-hyperarc-connectivity.

For orientation results concerning vertex-connectivity, very little is known even for global rooted-vertex-connectivity (when there are no Steiner vertices). Frank [3] made a conjecture on a necessary and sufficient condition for the existence of a strongly k-vertex-connected orientation, which in particular would imply that a $2k$-vertex-connected graph has a strongly k-vertex-connected orientation (and hence a rooted k-vertex-connected orientation). The only positive result along this line is a sufficient condition due to Jordán [6] for the case $k = 2$: Every 18-vertex-connected graph has a strongly 2-vertex-connected orientation.

Results. Our main result is the following approximate min-max theorem on hypergraphs, which is tight in terms of the connectivity bound.

Theorem 1. *Suppose H is an undirected hypergraph, T is a subset of terminal vertices with a specified root vertex $r \in T$. Then H has a Steiner rooted k-hyperarc-connected orientation if T is $2k$-hyperedge-connected in H.*

The proof is constructive, and also implies a polynomial time constant factor approximation algorithm for the problem. When the above theorem specializes to graphs, this gives a new and simpler algorithm (without using Nash-Williams' orientation theorem) to find a Steiner rooted k-arc-connected orientation in a graph when T is $2k$-edge-connected in G. On the other hand, we prove that finding an orientation which maximizes the Steiner rooted-arc-connectivity in a graph is NP-complete.

Following the notation on approximation algorithms on graph connectivity problems, by an *element* we mean either an edge or a Steiner vertex. For graph connectivity problems, element-connectivity is regarded as of intermediate difficulty between vertex-connectivity and edge-connectivity (see [5, 2]). A directed graph is *Steiner rooted k-element-connected* if there are k element-disjoint directed paths from r to each terminal vertex in T. We prove the following approximate min-max theorem on element-connectivity, which is tight in terms of the connectivity bound. We also prove the NP-completeness of this problem.

Theorem 2. *Suppose G is an undirected graph, T is a subset of terminal vertices with a specified root vertex $r \in T$. Then G has a Steiner rooted k-element-connected orientation if T is $2k$-element-connected in G.*

Concluding Remarks. The questions of generalizing Nash-Williams' theorem to hypergraphs and obtaining graph orientations achieving high vertex-connectivity remain wide open. We believe that substantially new ideas are required to solve these problems. The following problem seems to be a concrete intermediate problem which captures the main difficulty: If T is $2k$-element-connected in an undirected graph G, is it true that G has a Steiner strongly k-element-connected orientation? We believe that settling it would be a major step towards the above questions.

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Colorings of quadrangulations of the torus and the Klein bottle

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(joint work with Robin Thomas)

Our main motivation comes from the following classical theorem of Grötzsch [3].

Theorem 1. *Every triangle-free planar graph is* 3*-colorable.*

Until the work of Thomassen [5, 6], this result has been regarded as a very difficult theorem—Thomassen [5, 6] found two reasonably simple proofs, and extended Theorem 1 to other surfaces. In the talk, we considered triangle-free quadrangulations of the torus or the Klein bottle, and, more generally, embeddings of graphs in the torus or the Klein bottle with all faces of even size.

Let us now introduce some basic notation. *Graphs* may have parallel edges, but no loops. By a *surface* we mean a compact 2-dimensional manifold with no boundary. A *drawing* of a graph G in a surface Σ refers to an embedding of G in Σ with no crossings, and a *subdrawing* is a restriction of the embedding to a subgraph of G. We apply standard graph-theoretic terminology to drawings and speak about cycles in drawings, colorings of drawings, etc. Two drawings G_1 and G_2 are *isomorphic* if there exists an isomorphism between the graphs of G_1 and G_2 that preserves face boundaries. A drawing G in a surface Σ is a *quadrangulation* if every face is bounded by a walk of length four, and we say that a drawing G is *even-faced* if every face is bounded by a walk of even length.

Thomassen [7] conjectured that every triangle-free quadrangulation of the torus is 3-colorable. We prove that this conjecture holds, with the following exception: the quadrangulation $Q_{13,5,1}$ depicted in Figure 1 is a counterexample, as pointed out by Archdeacon, Hutchinson, Nakamoto, Negami and Ota [1]. However, our first main result states that $Q_{13,5,1}$ is essentially the only counterexample, even for the more general class of even-faced drawings.

Theorem 2. *A triangle-free even-faced drawing in the torus is* 3*-colorable if and only if it has no subdrawing isomorphic to* $Q_{13,5,1}$.

The *edge-width* of a drawing is the length of the shortest non-contractible cycle, or infinity if the drawing has no non-contractible cycle. The *representativity* of a drawing G in a surface Σ is the maximum integer k such that every non-contractible simple closed curve in Σ meets G at least k times. Since $Q_{13,5,1}$ has a non-contractible cycle of length five, we obtain the following:

Corollary 3. *Every even-faced drawing in the torus of edge-width at least six is* 3*-colorable.*

This corollary settles a conjecture of Archdeacon, Hutchinson, Nakamoto, Negami and Ota [1], who proved the same result for drawings of representativity at least 9. An earlier result of Hutchinson [4] proves this with 6 replaced by 25.

Let us turn to nonorientable surfaces now. From the vertex-coloring point of view triangle-free drawings in the projective plane are completely understood. First of all, Euler's formula implies that they have a vertex of degree at most three,

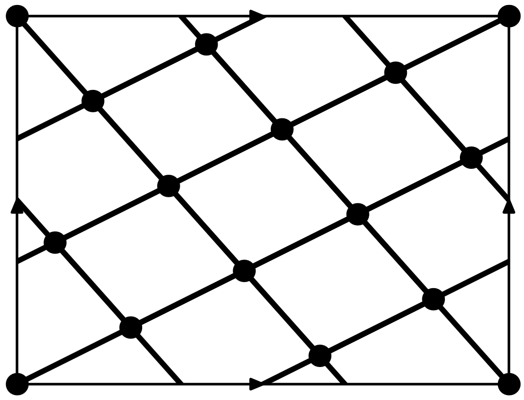

FIGURE 1. The drawing $Q_{13,5,1}$ in the torus.

and hence they are always 4-colorable. Youngs [8] discovered the remarkable fact that no quadrangulation of the projective plane has chromatic number exactly three, and Gimbel and Thomassen [2] extended that result to a characterization of 3-colorable triangle-free drawings in the projective plane:

Theorem 4. *A drawing in the projective plane with no contractible cycles of length three is* 3*-colorable if and only if it has no subdrawing isomorphic to a non-bipartite quadrangulation of the projective plane.*

Thus there are infinitely many triangle-free quadrangulations of the Klein bottle that are not 3-colorable: take two quadrangulations of the projective plane such that at least one of them is not bipartite, in each of them select a facial cycle, and identify those cycles. The resulting quadrangulation of the Klein bottle is not 3-colorable, because it has a subdrawing isomorphic to a nonbipartite quadrangulation of the projective plane.

There is another reason why a quadrangulation of the Klein bottle may fail to be 3-colorable. A closed walk or cycle in a drawing G in the Klein bottle is called *meridian* if it is homotopic to a 2-sided simple closed curve that does not separate the surface. It has been proved in [1] that if a quadrangulation G of the Klein bottle contains an odd meridian walk, then G is not 3-colorable.

We establish that the above two obstructions are the only ones to 3-colorability of quadrangulations of the Klein bottle. By an *equator* in a drawing G in the Klein bottle we mean a non-contractible cycle in G that separates the surface.

Theorem 5. *A non-bipartite quadrangulation of the Klein bottle is* 3*-colorable if and only if*

(1) *it has no equator of length at most four, and*
(2) *it has no odd meridian walk.*

And we have the following corollary for even-faced drawings:

Corollary 6. *Let G be an even-faced drawing in the Klein bottle with no equator of length at most four and no odd meridian walk. Then G is 3-colorable.*

This settles another conjecture of Archdeacon, Hutchinson, Nakamoto, Negami and Ota [1], who proved the same result for drawings of representativity at least 7.

Corollary 7. *Let G be an even-faced drawing in the Klein bottle with edge-width at least five and no odd meridian walk. Then G is 3-colorable.*

By Theorem 5 the bound of five is best possible.

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Property testing, extremal graph theory, and graph limits

László Lovász

The notion of convergent sequences of dense graphs and limit objects for them was introduced in a number of papers by C. Borgs, J. Chayes, L. Lovász, V.T. Sós, B. Szegedy and K. Vesztergombi. The main goal of this talk was to show that applying these methods we obtain simple proofs of various results in property testing and extremal graph theory.

Let $\hom(G, H)$ denote the number of homomorphisms (adjacency-preserving mappings) from a graph G into a graph H. We also define the *homomorphism densities*

$$t(F, G) = \frac{\hom(F, G)}{|V(G)|^{|V(F)|}}.$$

($t(F, G)$ is the probability that a random map of $V(F)$ into $V(G)$ is a homomorphism).

A graph sequence (G_n) of simple graphs with $|V(G_n)| \to \infty$ is *convergent*, if the sequence $t(F, G_n)$ has a limit for every simple graph F. Let $\mathcal{W}_0$ denote the set of symmetric measurable functions $W : [0,1]^2 \to [0,1]$. To every convergent graph sequence we can assign a limit object in the form of a function from $\mathcal{W}_0$, which describes the limits of subgraph densities:

$$t(F, G_n) \to t(F, W),$$

where

$$t(F, W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j)\, dx.$$

Various other characterizations of convergent sequences and the limit object can be found in [7, 4, 5]. Here we sketch two applications of this construction to derive combinatorial results through means of functional analysis.

Example 1. Alon and Stav [1] proved that for every hereditary property $\mathcal{P}$, a random graph with appropriate density is (asymptotically) farthest from the property in edit distance. Define the closure $\overline{\mathcal{P}}$ of the property is the set of limits of of convergent sequences of graphs with this property. Then the result of Alon and Stav says that the maximum L_1 distance of a function $W \in \mathcal{W}$ from $\overline{\mathcal{P}}$ is attained by a constant.

In this context, there is a very simple proof. It is quite easy to check that the set $\mathcal{W}_0 \setminus \overline{\mathcal{P}}$ is convex, and hence the maximum of the distance from its complement is attained on a convex subset. Furthermore, this subset is invariant under measure preserving transformations of $[0,1]$, and hence it is easy to argue that it must contain a constant function.

Example 2. A sequence $(G_n : n = 1, 2, \dots)$ of graphs is called *quasirandom with density* p (where $0 < p < 1$), if for every simple finite graph F, we have $t(F, G_n) \to p^{|E(F)|}$. Using the notion of limits, we can state this as G_n tends to the identically p function. One of the most surprising facts proved in [6] is that it is enough to require the homomorphism density condition for just two graphs: if $t(K_2, G_n) \to p$ and $t(C_4, G_n) \to p^4$, then (G_n) is quasirandom. Since the definition fits so well in the graph limit framework, it is not surprising that it also provides a very simple proof. One such proof was described in [3], but an even simpler one can be given. The following sketch should illustrate the technique.

If the sequence (G_n) is not quasirandom, then we can select a convergent subsequence whose limit is a function $U \not\equiv \text{const}$. For this function we have $t(K_2, U) = p$ and $t(C_4, U) = p^4$. Two simple applications of the Cauchy-Schwarz inequality tells us that

$$t(C_4, U) \geq t(P_3, U)^2 \geq t(K_2, U)^4$$

(where P_3 is the path with length 3). Hence $t(P_3, U) = p^2$. Now U can be viewed as a kernel operator on $L_2([0,1]^2)$, acting by $f \mapsto \int_0^1 U(., x) f(x)\, dx$. The function $U_2(x, y) = \int_0^1 U(x, y) U(z, y)\, dz$ corresponds to its square, and satisfies

$$\int_{[0,1]^2} (U_2(x, y) - p^2)^2 = t(C_4, U) - 2p^2 t(P_3, U) + p^4 = 0,$$

and so $U_2 \equiv p^2$. From this it follows by simple operator algebra that $U \equiv p$, a contradiction.

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Openly disjoint circuits through a vertex in digraphs

Wolfgang Mader

First, we introduce some terminology and notation. All *digraphs* are assumed to be finite without loops and without multiple edges of the same direction. All *paths* and *circuits* are continously directed. Instead of *strongly connected* we say *connected*. Let $\kappa(D)$ be the connectivity number of D. The vertex number of a digraph D is denoted by $|D|$, and $x \in D$ means that x is a vertex of D. Let g(D) be the girth of D, i.e. the length of a shortest circuit. A digraph is *r-outregular*, if every vertex has outdegree r, and it is *r-regular*, if it is r-outregular and r-inregular. Furthermore, $\delta^+(D)$ denotes the minimum outdegree of D, and we define $\delta(D) := min\{\delta^+(D), \delta^-(D)\}$. For an $x \in D$, let $\kappa_D(x,x)$ denote the maximum number of openly disjoint circuits through x, i.e. circuits in D having pairwise only x in common.
The starting point for our studies were the following two conjectures.

Conjecture 1 (M.Behzad, G. Chartrand, and C. Wall [1]). *Every r-regular digraph D has* $|D| \geq r(g(D)-1)+1$.

Conjecture 2 (L. Caccetta and R. Häggkvist [2]) *Every r-outregular digraph D has* $|D| \geq r(g(D)-1)+1$.

If an r-regular or r-outregular digraph D always has a vertex x with $\kappa_D(x,x) = r$, this would easily imply Conjecture 1 or Conjecture 2, respectively. For $r \leq 2$, this is true.

Theorem 3 (C. Thomassen [8]). *Every 2-outregular digraph D has a vertex x with* $\kappa_D(x,x) = 2$.

But for $r \geq 3$, r-outregularity is not enough for the existence of such a vertex.

Theorem 4 (C. Thomassen [9]). *For every integer* $r \geq 3$ *, there is a digraph D with* $\delta(D) \geq r$ *such that for every* $x \in D$, $\kappa_D(x,x) \leq 2$ *holds.*
(Notice that every D with $\delta^+(D) \geq r$ has an r-outregular subdigraph.)

The digraphs constructed by C. Thomassen for the proof of Theorem 4 are neither r-regular nor connected.
Let us first consider digraphs of high connectivity. Of course, a digraph D with $\kappa(D) \geq r$ has $\kappa_D(x,x) \geq r$ for every $x \in D$ by Menger's Graph Theorem. But we can weaken the connectivity condition a bit.

Theorem 5 (W. Mader [5]). *Let D be a digraph with* $\delta(D) \geq r \geq 3$ *and* $\kappa(D) \geq r-2$. *Then D contains a vertex x with* $\kappa_D(x,x) \geq r$. *For each integer* $r \geq 3$ *, this does not remain true, if we weaken* $\delta(D) \geq r$ *to* $\delta^+(D) \geq r$ *or* $\kappa(D) \geq r-2$ *to* $\kappa(D) \geq r-3$.

It is easily seen that Theorem 5 implies the following result proved in [7]. Herein, $D_i(x)$ is the set of all $y \in D$ which have distance exactly i from the vertex x in D.

Theorem 6 (Jian Shen and D.A. Gregory [7]). *If* $\kappa(D) \geq \delta(D) - 1$ *for a digraph D, then there is an* $x \in D$ *with* $|D_i(x)| \geq \delta(D)$ *for all i = 1, ... , g(D) - 1.*
It might be possible that Conjecture 1 is provable by the existence of an x with $\kappa(x,x) = r$.

Question 7 (P.D.Seymour [6]). *Does every r-regular digraph D contain a vertex x with* $\kappa_D(x,x) = r$?

This is true for $r \leq 2$ by Theorem 3. Since every component of the underlying graph of an r-regular digraph D induces an r-regular, connected subdigraph, Theorem 5 answers Question 7 positively for $r = 3$. For $r \geq 4$, the question remains open.
For the class of all vertex-transitive digraphs, Conjecture 1 is proved.

Theorem 8 (Y.O. Hamidoune [3]). *For every vertex-transitive digraph D,* $|D| \geq \delta(D)(g(D)-1)+1$ *holds.*

It seems that Hamidoune's proof cannot be developed further to give the following stronger result, which we proved using other methods.

Theorem 9 (W. Mader [5]). *For every vertex x in a vertex-transitive digraph D, $\kappa_D(x,x) = \delta(D)$ holds.*

Theorem 9 implies a recent result of Y.O.Hamidoune (cf. [4]) that for every x in a vertex -transitive digraph D, $|D_i(x)| \geq \delta(D)$ holds for $i = 1, ..., g(D) - 1$.

Added in proof: In the meantime, I have found some examples which show that Question 7, in general, has a negative answer.

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Small separations in symmetric graphs

Bojan Mohar
(joint work with Matt DeVos)

We prove a rough structure theorem for small separations in symmetric graphs. Let $G = (V, E)$ be a vertex transitive graph, let $A \subseteq V$ be finite vertex set with $|A| \leq \frac{|V|}{2}$ and set $k = |\{v \in V \setminus A : u \sim v \text{ for some } u \in A\}|$. We show that whenever the diameter of G is at least $31(k+1)^2$, then either $|A| \leq 2k^3$, or G has a (bounded) ring-like structure and A is efficiently contained in an interval of this structure. This theorem may be viewed as a rough analogue of earlier results of Mader, Lovász, Tindell, Watkins, and others. This subject is also closely connected to the study of product sets and expansion in groups, and our theorem has some applications in this context as well. For graphs, we apply this to get a new proof of a theorem of Babai on the structure of vertex transitive graphs with no K_n minor. Improving on the original work of Babai, whose proof did not give any explicit bounds, our argument yields explicit structural bounds.

From chip-firing games to Riemann-Roch theorem in tropical geometry

Serguei Norine
(joint work with Matthew Baker)

A finite graph can be viewed, in many respects, as a discrete analogue of a Riemann surface. We pursue this analogy in the context of linear equivalence of divisors.

Let G be a finite, loopless, connected graph, possibly with multiple edges. Choose an ordering $v_1, \ldots, v_n$ of the vertices of G. The *Laplacian matrix* Q associated to G is the $n \times n$ matrix $Q = D - A$, where D is the diagonal matrix whose $(i,i)^{\text{th}}$ entry is the degree of vertex v_i, and A is the adjacency matrix of the graph.

Let $\text{Div}(G)$ be the free abelian group on the set of vertices of G. We think of elements of $\text{Div}(G)$ as formal integer linear combinations of elements of $V(G)$. By analogy with the Riemann surface case, elements of $\text{Div}(G)$ are called *divisors* on G. For convenience, we define $D(v)$ to be the coefficient of $v \in V(G)$ in D. We define the *degree* $\deg(D)$ of the divisor D to be equal to $\sum_{v \in V(G)} D(v)$. We say that a divisor E is *effective* if $E(v) \geq 0$ for every $v \in V(G)$.

We let $\mathcal{M}(G) = \text{Hom}(V(G), \mathbb{Z})$ be the abelian group consisting of all integer-valued functions on the vertices of G. One can think of $\mathcal{M}(G)$ as analogous to the space of meromorphic functions on a Riemann surface. Using our ordering of the vertices, we obtain isomorphisms between $\text{Div}(G), \mathcal{M}(G)$, and the space of $n \times 1$ column vectors having integer coordinates. We define the subgroup $\text{Prin}(G)$ of $\text{Div}(G)$ consisting of *principal divisors* to be the image of $\mathcal{M}(G)$ under multiplication by the Laplacian matrix Q.

Define an equivalence relation $\sim$ on the group $\text{Div}(G)$ by declaring that $D \sim D'$ if and only if $D - D' \in \text{Prin}(G)$. Borrowing again from the theory of Riemann surfaces, we call this relation *linear equivalence*.

This equivalence relation can be conveniently restated in terms of a game on G, similar to the chip-firing game studied by Björner, Lovász, and Shor in [2]. The initial configuration of the game assigns to each vertex v of G an integer number of dollars. A vertex which has a negative number of dollars assigned to it is said to be in *debt*. A *move* consists of a vertex v either borrowing one dollar from each of its neighbors or giving one dollar to each of its neighbors. The object of the game is to reach, through a sequence of moves, a configuration in which no vertex is in debt. We will call such a configuration a *winning position*, and a sequence of moves which achieves such a configuration a *winning strategy*. Configurations in the game can be identified with divisors in $\text{Div}(G)$, and winning positions can be identified with effective divisors. It follows from the definitions that two divisors D and D' on G are linearly equivalent if and only if there is a sequence of moves in the game taking D to D'.

Let $g = |E(G)| - |V(G)| + 1$. We have the following description of the configurations for which there exists a winning strategy.

Theorem 1. *Let $N = \deg(D)$ be the total number of dollars present at any stage of the game.*

1. *If $N \geq g$, then there is always a winning strategy.*
2. *If $N \leq g-1$, then there is always an initial configuration for which no winning strategy exists.*

For $D \in \mathrm{Div}(G)$, we define the *linear system associated to* D to be the set $|D|$ of all effective divisors linearly equivalent to D:

$$|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0,\ E \sim D\} \ .$$

We define the *dimension* $r(D)$ of the linear system $|D|$ by setting $r(D)$ equal to -1 if $|D| = \emptyset$, and then declaring that for each integer $s \geq 0$, $r(D) \geq s$ if and only if $|D-E| \neq \emptyset$ for all effective divisors E of degree s. It is clear that $r(D)$ depends only on the linear equivalence class of D. There is a winning strategy in our game whose initial configuration corresponds to D if and only if $r(D) \geq 0$. By Theorem 1 we have $r(D) \geq 0$ for every $D \in \mathrm{Div}(G)$ with $\deg(D) \geq g$.

In fact, a stronger result, namely a graph-theoretic analogue of the classical Riemann-Roch theorem, holds. Let the *canonical divisor* on G be the divisor K such that $K(v) = \deg(v) - 2$ for every $v \in V(G)$.

Theorem 2 (Riemann-Roch for Graphs)**.** *Let G be a graph, and let D be a divisor on G. Then*

$$r(D) - r(K-D) = \deg(D) + 1 - g \ .$$

The proof of Theorem 2 is purely combinatorial and is based on the analysis of the chip-firing game described above. In [3] the authors use Theorem 2 to derive the Riemann-Roch theorem for tropical curves. Further analogies between Riemann surfaces and graphs that arise in this context are explored in [1].

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A generalization of planarity and linkless embeddability of graphs

R.A. Pendavingh
(joint work with H. van der Holst)

We propose a topological graph invariant $\sigma(G)$ and show that it has the following properties:

- σ is *minor-monotone*, that is if G is a minor of H, then $\sigma(G) \leq \sigma(H)$;
- $\sigma(K_n) = n - 1$;
- G is planar if and only if $\sigma(G) \leq 3$; and
- G is linklessly embeddable if and only if $\sigma(H) \leq 4$.

The immediate motivation for the study of this invariant was to find an general upper bound for the graph invariant $\mu(G)$ that was introduced by Yves Colin de Verdière [1, 2]. Given an undirected graph $G = (V, E)$, $\mu(G)$ is defined as the maximum corank of a symmetric $V \times V$ matrix M with the following properties:

- if $u \neq v$, then $M_{uv} < 0$ if $uv \in E$ and $M_{uv} = 0$ if $uv \notin E$;
- M has exactly one negative eigenvalue; and
- if X is a symmetric $V \times V$ matrix such that
  - $X_{uv} = 0$ whenever $u = v$ or $uv \in E$, and
  - $MX = 0$,

  then $X = 0$

Colin de Verdiére showed that μ is minor-monotone and that $\mu(K_n) = n - 1$, and he characterized the graphs with $\mu(G) \leq k$ for $k = 1, 2, 3$. In view of the nature of the definition of μ it remains a striking fact that $\mu(G) \leq 3$ if and only if G is planar.

A high point in the subsequent study of μ was the characterization of the graphs with $\mu \leq 4$ as the linklessly embeddable graphs. Robertson, Seymour and Thomas [7] had characterized such linklessly embedable graphs in 1995 as the graphs without a minor in the *Petersen family.* Each of the members of this family has $\mu = 5$, and by the minor-monotonicity of μ it follows from their theorem that if $\mu(G) \leq 4$, then G is linkless. The converse implication was then shown by Lovász, and A. Schrijver in 1998 [5].

Inspired by the proof of the latter theorem, we have shown:

Theorem 1. *If G is an undirected graph, then $\sigma(G) \geq \lambda(G)$.*

Here λ is a graph invariant related to μ, introduced by van der Holst, Laurent, and Schrijver [3], with the property that $\lambda(G) + 2 \geq \mu(G)$ [6]. So our theorem indirectly gives an upper bound on μ.

The definition of $\sigma(G)$ is as follows. A *closure* of a graph $G = (V, E)$ is a CW-complex $\mathcal{C}$ so that

- the 1-skeleton of $\mathcal{C}$ is equal to G, and
- the k-skeleton of $\mathcal{C}[U]$ is $(k-1)$-connected for each $U \subseteq V$ such that $G[U]$ is connected, and each integer k with $2 \leq k \leq |V|$.

We say that a continuous mapping $\phi : \mathcal{C} \to \mathbb{R}^n$ is *even* if

- $\phi(c_1) \cap \phi(c_2) = \emptyset$ for every nonadjacent pair of cells c_1, c_2 of $\mathcal{C}$ with $\dim c_1 + \dim c_2 = n - 1$, and
- the intersection number of $\phi(c_1)$ and $\phi(c_2)$ is even for every pair of non-adjacent cells c_1, c_2 of $\mathcal{C}$ with $\dim c_1 + \dim c_2 = n$.

Then, $\sigma(G)$ is the smallest integer n so that a closure of G has an even mapping into $\mathbb{R}^n$.

Looking at this definition it would appear that determining $\sigma(G)$ for a single graph G is already a daunting task, but we have shown that to determine whether there exists an *even* mapping of *some* closure of G into $\mathbb{R}^n$, it suffices to inspect *any* generic continuous mapping of *any* closure of G into $\mathbb{R}^n$. In a way, the existence of such mappings depends only on the homology of the deleted product of some closure of G. Even mappings are well-behaved objects, and hence our claim that they are useful in the study of μ. We could show, for example, that if G embeds on the torus, then $\sigma(G) \leq 6$.

We have $\sigma(G) \leq t$ iff $\mu(G) \leq t$ for $t \leq 4$, but there is a graph T with $\mu(T) \leq 18 < 20 \leq \lambda(T)$ [6] and thus for that graph $\mu(T) < \lambda(T) \leq \sigma(T)$. We nevertheless hazard the following conjecture. For any graph G, the *panelling* $\tilde{G}$ of G is the 2-complex that arises by attaching a disk to each circuit of G (by glueing the boundary of the disk to the circuit).

Conjecture 1. *Let G be an undirected graph. Then the following are equivalent:*

(1) $\mu(G) \leq 5$*;*
(2) $\sigma(G) \leq 5$*; and*
(3) $\tilde{G}$ *can be embedded in* $\mathbb{R}^4$*.*

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Infinite locally random graphs

Alexander D. Scott
(joint work with Pierre Charbit)

The *Rado graph* $\mathcal{R}$ is the unique graph with countably infinite vertex set such that, for any disjoint pair X, Y of finite subsets of vertices, there is a vertex z that is joined to every vertex in X and no vertex in Y. If $0 < p < 1$, and G is a random graph in $\mathcal{G}(\mathbb{N}, p)$, then with probability 1 we have $G \cong \mathcal{R}$. For this reason, the Rado graph is also known as *the infinite random graph* (see [5] for a survey).

The Rado graph can be obtained deterministically by beginning with any finite (or countably infinite) graph G and iterating the following construction:

[E1] For every finite subset X of $V(G)$ add a vertex y with neighbourhood $N(y) = X$.

Here $N(x) = \{y \in V(G) : xy \in E(G)\}$ is the *neighbourhood* of x; we also write $N[x] = N(x) \cup \{x\}$ for the *closed neighbourhood* of x.

Motivated by copying models of the web graph, Bonato and Janssen [3] (see also [1] and [4]) introduced the following interesting construction. For a finite graph G, the *pure extension* $PE(G)$ of G is obtained from G by the following construction:

[E2] For every $x \in V(G)$ and every finite $X \subseteq N[x]$ add a vertex y with neighbourhood $N(y) = X$.

Iterating this construction, we obtain a limit graph, denoted by $\uparrow G$.

Bonato and Janssen ([3], Theorem 3.3) claimed that $\uparrow G \cong \uparrow H$ for every pair G, H of finite graphs. The (claimed) unique limit graph has become known [1] as the *infinite locally random graph* (the name follows from the fact that, as is easily shown, the neighbourhood of any vertex in $\uparrow G$ induces a copy of the Rado graph).

We show that Bonato and Janssen's claim is incorrect. There are in fact infinitely many limit graphs G: for instance, $\uparrow C_5, \uparrow C_6, \uparrow C_7, \ldots$ are all distinct. Furthermore, we give a simple criterion that determines when $\uparrow G \cong \uparrow H$, as follows.

We shall say that a vertex x of a graph G is *inessential* if there exists $y \in V(G)$, $y \neq x$ such that $N(x) \subseteq N[y]$. A graph is *essential* if it contains no inessential vertices. Given a graph G, a sequence of vertices $x_1, \ldots, x_k$ is a *maximal sequence of removals* if x_i is inessential in $G \setminus \{x_1, \ldots, x_{i-1}\}$ for each i, and $G \setminus \{x_1, \ldots, x_k\}$ is an essential graph. It turns out that the resulting graph is well-defined (up to isomorphism), and we shall write it as $\downarrow G$. For instance, $\downarrow C_4 \cong \downarrow K_n \cong K_1$, but $\downarrow C_5, \downarrow C_6, \ldots$ are all distinct and nontrivial.

We can now state our classification result.

Theorem 1. *Let G and H be finite graphs. Then* $\uparrow G \cong \uparrow H \Longleftrightarrow \downarrow G \cong \downarrow H$

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Limits of Discrete Structures

Balázs Szegedy

Let $\mathcal{O}$ be a set of objects and let let $\mathcal{S}$ be a set of sampling procedures from the elements of $\mathcal{O}$. Each element $s \in \mathcal{S}$ associates a (Borel) probability measure $s[A]$ on a compact space $C(s)$ with every $A \in \mathcal{O}$. A sequence $A_1, A_2, \dots \in \mathcal{O}$ is called convergent if for every $s \in \mathcal{S}$ the distributions $s[A_1], s[A_2], \dots$ converge. This means that for every continuous function $f : C(s) \to \mathbb{R}$ the sequence

$$\int_{C(s)} f \; d(s[A_i])$$

is convergent. We denote by $\bar{\mathcal{O}}$ the completion of $\mathcal{O}$ with respect to this convergence notion. By definition, the sampling procedures extend from $\mathcal{O}$ to $\bar{\mathcal{O}}$.

Example 1. (Subsets of integers) This example is the first step in ergodic theory. Let $\mathcal{O} := \{X \mid n \in \mathbb{N},\ X \subseteq \mathbb{Z}/n\mathbb{Z}\}$. For every natural number k we define the sampling procedure in which we pick a random element x from $\mathbb{Z}/n\mathbb{Z}$ and then we take the intersection $I = X \cap \{x, x+1, \dots, x+k-1\}$. We represent I by a subset of $\{1, 2, \dots, k\}$ and so the sampling procedure is a probability distribution on $\{0,1\}^k$. The elements of the completion $\bar{\mathcal{O}}$ can be represented as Borel probability measures on the compact space $\{0,1\}^{\mathbb{Z}}$ which are invariant under the coordinate-wise shift. This language allows one to study certain properties of integer subsets through measure preserving systems.

Example 2. (Bounded degree graphs) Let d be a fixed natural number and let $\mathcal{O}$ be the set of graphs with maximum degree d. For every natural number k we define a sampling procedure in which we pick a random vertex v and then look at the neighborhood of v of radius k. The corresponding convergence notion was introduced and studied by Benjamini and Schramm [6].

Example 3. (Dense Graphs) Let $\mathcal{O}$ be the set of finite graphs. For every natural number k we have a sampling procedure in which we pick k (independent) random vertices $v_1, v_2, \dots, v_k$ from the vertex set of a graph G and then we look at the graph induced by these vertices. This procedure gives a probability distribution of graphs on $\{1, 2, 3, \dots, k\}$. The analysis on $\bar{\mathcal{O}}$ was studied by Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi [1],[2],[3],[4]. It turns out [3] that the elements of $\bar{\mathcal{O}}$ can be represented by equivalence classes of measurable functions $w : [0,1]^2 \to [0,1]$ which are symmetric in the two coordinates.

Example 4. (Dense Hypergraphs) Let $\mathcal{O}$ be the set of d-uniform hypergraphs. For every natural number k we define the sampling procedure in which we pick k random points from the vertex set of a hypergraph H and look at the induced sub-hypergraph. The analysis on $\bar{\mathcal{O}}$ was studied by Elek and Szegedy [7] using the measure and integral theory on the ultra products of finite sets (Another approach is by Tao [8]). It turns out that elements of $\bar{\mathcal{O}}$ can be represented by $2^k - 2$ variable measurable functions. The method yields analytic proofs for the so-called hypergraph removal lemma and hypergrah regularity lemma.

Example 5. (Weighted graphs) Let us fix a real number $d > 0$ and let $\mathcal{O}$ denote the set of those weighted complete graphs in which the edge weights are in the interval $[-d, d]$. The sampling procedures are the same as in Example 3. The corresponding convergence notion was studied by Lovász and Szegedy [5]. In this setting, elements of $\bar{\mathcal{O}}$ can be represented by three variable measurable functions.

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Coloring triangle-free graphs on surfaces

Robin Thomas
(joint work with Zdeněk Dvořák and Daniel Král')

In this talk we are concerned with coloring graphs that embed in a fixed surface. This restriction often makes coloring problems tractable, and the purpose of this talk is to describe a new result along these lines.

The classical theorem of Grötzsch [2] states that every triangle-free planar graph is 3-colorable. Thus deciding whether a triangle-free planar graph is 3-colorable is trivial. However, Grötzsch's Theorem does not generalize to any other surface Σ, and, in fact, the 3-colorability of triangle-free graphs embedded in Σ is an interesting problem. When Σ is the projective plane it has been solved by Gimbel and Thomassen [1], but it was open for all other surfaces. In this talk we announce solution for all surfaces, as follows.

Theorem 1. *For every surface Σ there exists a polynomial-time algorithm that given an input triangle-free graph G embedded in Σ correctly determines whether G is* 3*-colorable.*

In fact, our result is more general in two respects. The input graph is allowed to have triangles, as long as they are not trivial. (We say that a cycle C is *trivial* if it bounds a disk, and non-trivial otherwise.) Second, a bounded number of vertices may be precolored.

We prove Theorem 1 by means of the following structural result. We need a definition first. If G is a graph embedded in a surface Σ and f is a face of a subgraph of G, then by $G[f]$ we denote the subgraph of G consisting of all vertices and edges of G embedded in the closure of f. Furthermore, we regard $G[f]$ as embedded in the surface $\Sigma[f]$ obtained from f by capping off each component of the boundary of f by a disk.

Theorem 2. *For every surface Σ there exists an integer N such that every triangle-free graph G embedded in Σ is either 3-colorable, or has a subgraph H on at most N vertices such that for every face f of H*
(i) *f is a disk, or*
(ii) *f is a cylinder, or*
(iii) *the graph $G[f]$ embedded in the surface $\Sigma[f]$ is "locally planar".*

We omit the definition of local planarity. Instead, let us remark that we have proven a coloring extension theorem that under the assumption of local planarity gives an easily checkable necessary and sufficient condition for a coloring of the boundary of f to extend to $G[f]$. This condition is in terms of "winding number" of the precoloring, and takes a different form depending on whether $\Sigma[f]$ is orientable or not.

The proof of Theorem 2 can be converted to a polynomial-time algorithm to find the graph H. Starting with the null graph, if the current graph does not satisfy any of the conclusions of the theorem, then we find a way to enlarge H while simplifying the faces of H, and repeat. The simplification guarantees that there will be only a constant number of iterations.

Once the graph H is found we use Theorem 2 to test whether some 3-coloring of H extends to $G[f]$ for all faces f of H. We have developed separate algorithms to do that when f is a disk or a cylinder. If condition (iii) holds, then the extension question can be easily decided using the coloring extension theorem mentioned above.

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Submodular partition functions

Stéphan Thomassé
(joint work with Omid Amini, Frédéric Mazoit, Nicolas Nisse)

In their seminal paper Graph Minors X [3], Robertson and Seymour introduced the notion of branch-width of a graph and its dual notion of tangle. Their method is based on the definition of bias and tree-labellings. Later on, Seymour and Thomas [4] found a dual notion to tree-width, the bramble number (named after Reed [2]). The proof of the bramble-number/tree-width duality makes use of Menger's theorem to reconnect partial tree-decompositions, see for instance the textbook of Diestel [1]. Our aim in this paper is to show how the classical dual notions of width-parameters can be deduced from the original method of Graph Minors X.

Let E be a set. A *partitioning tree* on E is a tree T in which the leaves are identified with the elements of E in a one-to-one way. Every edge e of T thus corresponds to a bipartition T_e of E: the leaves of the two subtrees obtained by deleting e. In a similar way, every internal node v of T gives a partition T_v: the leaves of the subtrees obtained by deleting v.

An obvious way of forming a partitioning tree is simply to fix a central node which is linked to all elements of E - a partitioning star. But what if we are not permitted to do so? Precisely, assume that a restricted set of partitions of E, called *admissible partitions*, is given. Is it possible to form an admissible partitioning tree (i.e. all partitions T_e and T_v are admissible)?

An obstruction to the existence of such a tree is the dual notion of bramble. An *admissible bramble* is a set of pairwise intersecting subsets of E which contains an element of every admissible partition of E. It is routine to form an admissible bramble: just pick an element e of E, and collect, for every admissible partition, the element of the partition which contains e. Such a bramble is called *principal.* The crucial fact is that if there is a non-principal admissible bramble $\mathcal{B}$, there is no admissible partitioning tree. To see this, assume for contradiction that T is an admissible partitioning tree. For every internal node u of T, there is an element X of T_u which belong to $\mathcal{B}$. Let v be the neighbour of u which belong to the component of $T \setminus u$ having set of labels X. Now orient the edge uv of T from u to v. Note that every internal node becomes the origin of an oriented edge. Observe also that an edge of T incident to a leaf never gets an orientation since $\mathcal{B}$ is non-principal. The contradiction follows from the fact that one edge of T carries two orientations, which is impossible since the elements of $\mathcal{B}$ are pairwise intersecting.

Unfortunately, if no principal admissible bramble exist, there is not necessarily an admissible partitioning tree. However, for some particular families of admissible partitions (e.g. generated by a submodular partition function) we have the following:

- Either there exists an admissible partitioning tree.
- Or there exists a non-principal admissible bramble.

Let us now properly define submodular partition functions. Let E be a non-empty set. A *partition* of E is a set $\mathcal{X} = \{X_1, \dots, X_n\}$ of subsets of E satisfying $X_1 \cup \cdots \cup X_n = E$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$. The order in which appears the X_i's is irrelevant. We authorize degenerate partitions (i.e. the sets X_i can be empty). Let F be a subset of E disjoint from X_i. The partition

$$\mathcal{X}_{X_i \to F} := \{X_1 \cap F, \dots, X_{i-1} \cap F, E \setminus F, X_{i+1} \cap F, \dots, X_n \cap F\}$$

is the partition obtained from $\mathcal{X}$ by *pushing X_i to F*.

Let Φ be a function defined from the set of partitions of E into the reals. Let $\mathcal{X}$ be a partition of E. We call $\Phi(\mathcal{X})$ the Φ-*width*, or simply *width*, of $\mathcal{X}$. Let k be an integer. A *k-partition* is a partition of width at most k. When F is a subset of E, the *width* of F is denoted by $\Phi(F) := \Phi(\{F, F^c\})$. A *k-subset* is a subset of width at most k. The function Φ is a *partition function* if $\Phi(\mathcal{X}) \geq \Phi(X_i)$ for all partition $\mathcal{X} = \{X_1, \ldots, X_n\}$ and for all $i = 1, \ldots, n$. Observe that when Φ is a partition function, k-partitions consist of k-subsets (but all partitions into k-subsets are not necessarily k-partitions). A partition function Φ is *submodular* if for every pair of partitions $\mathcal{X} = \{X_1, \ldots, X_n\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_l\}$ with $X_i \cap Y_j = \emptyset$, we have:

$$\Phi(\mathcal{X}) + \Phi(\mathcal{Y}) \geq \Phi(\mathcal{X}_{X_i \to Y_j}) + \Phi(\mathcal{Y}_{Y_j \to X_i})$$

Let Φ be a partition submodular function and k be an integer such that every singleton has width at least k. A partition $\mathcal{X}$ being *admissible* if $\Phi(\mathcal{X}) \leq k$, we have the following:

Theorem 1. *There exists either an admissible partitioning tree or a non-principal admissible bramble.*

The key-example of a submodular partition function is the function *border*, denoted by δ, defined on the set of partitions of the edge set E of a graph $G = (V, E)$ by letting:

$$\Delta(\mathcal{X}) = \{x \in V \ : \ \exists xy \in X_i \text{ and } \exists xz \in X_j,\ i \neq j\} \text{ and } \delta(\mathcal{X}) = |\Delta(\mathcal{X})|.$$

The tree-width of a graph $G = (V, E)$ corresponds to the partition function δ. To avoid technicalities, we assume that G has minimum degree two. Since δ is partition submodular, there is, for every $k \geq 2$, either an exact partitioning tree or a non-principal bramble. Here is how we translate each of these cases into the classical notions of tree-decomposition and bramble:

If E has an exact partitioning tree T. The restriction of T to its internal nodes, each of these nodes u corresponding to the bag $\Delta(T_u)$, is a tree-decomposition of G. Indeed, for every edge xy of G, there is a leaf v of T labelled by xy. Denoting by u the unique neighbor of v in T, it follows that x and y both belong to $\Delta(T_u)$, since the minimum degree in G is two. Furthermore, if a vertex of G both belongs to $\Delta(T_u)$ and $\Delta(T_v)$, it also belongs to $\Delta(T_w)$ for every node w in the (u, v)-path of T. Since every bag has size at most k, the tree-width of G is at most $k - 1$.

If E has a non-principal bramble $\mathcal{B}$, we form a bramble $\mathcal{B}'$ (in the usual sense) as follows: Let S be a subset of V with $|S| \leq k$. We associate to S the partition $\{E_1, \ldots, E_n\}$ of E where the sets E_i are the (nonempty) sets of edges minimal with respect to inclusion for the property $\delta(E_i) \subseteq S$. Observe that this is indeed a partition since $\delta(E_i \cap E_j) \subseteq \delta(E_i) \cup \delta(E_j) \subseteq S$. Since $\mathcal{B}$ is a non-principal bramble, one of the E_i, with at least two edges, is in $\mathcal{B}$. This means that $X_i = V(E_i) \setminus S$ is a nonempty set of vertices. In other words, E_i is the set of edges incident to at least one vertex of X_i (such a set is denoted by $E(X_i)$). We now collect, for all subsets S with $|S| \leq k$, these sets X_i to form our $\mathcal{B}'$. Observe first that, by minimality of E_i, every element X_i of $\mathcal{B}'$ induces a connected subgraph of G. We have now to prove that for every pair X_i, X_j of elements of $\mathcal{B}'$, $X_i \cup X_j$ also induces a connected subgraph of G. Indeed, let $E_i = E(X_i)$ and $E_j = E(X_j)$. Since the elements of $\mathcal{B}$ are pairwise intersecting, there is an edge xy of G in $E_i \cap E_j$. Without loss of generality, we can assume that $x \in X_i$. If we also have $x \in X_j$, X_i and X_j have a nonempty intersection, and thus their union is connected. If $x \notin X_j$, we necessarily have $y \in X_j$, hence there is an edge of G connecting X_i and X_j. Thus $\mathcal{B}'$ is a bramble, and the minimum size covering set of $\mathcal{B}'$ has at least $k+1$ elements. In this case the bramble-number of G is at least $k+1$.

The previous proof do not use Menger's theorem. Moreover, this technique provides duals to the notions of branch-width, path-width, rank-width, and also to matroid tree-width.

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Decomposing Berge graphs and detecting balanced skew partitions

Nicolas Trotignon

A hole in a graph is an induced cycle on at least four vertices. A graph is Berge if it has no odd hole and if its complement has no odd hole [2]. In 2002, Chudnovsky, Robertson, Seymour and Thomas [5] proved a decomposition theorem for Berge graphs saying that every Berge graph either is in a well understood basic class or has some kind of decomposition. Then, Chudnovsky proved stronger theorems [3, 4]. One of them restricts the allowed decompositions to 2-joins and balanced skew partitions.

We prove that the problem of deciding whether a graph has a balanced skew partition is NP-hard. We give an $O(n^9)$-time algorithm for the same problem restricted to Berge graphs. Our algorithm is not constructive: it only certifies whether a graph has a balanced skew partition or not. It relies on a new decomposition theorem for Berge graphs, that is more precise than the previously known theorems. Our theorem also implies that every Berge graph can be decomposed in a first step by using only balanced skew partitions, and in a second step by using only 2-joins. Our proof of this new theorem uses at an essential step one of the theorems of Chudnovsky.

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Even-hole-free graphs that do not contain diamonds: structure theorem and β-perfection

Kristina Vušković
(joint work with Ton Kloks, Haiko Müller)

We consider finite and simple graphs. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is $\mathcal{F}$*-free* if it is F-free, for every $F \in \mathcal{F}$.

Many interesting classes of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. Most famous such example is the class of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ denotes the chromatic number of H and $\omega(H)$ denotes the size of a largest clique in H. The famous Strong Perfect Graph Theorem (conjectured by Berge [2] and proved by Chudnovsky, Robertson, Seymour and Thomas [3]) states that a graph is perfect if and only if it does not contain an odd hole nor an odd antihole (where a *hole* is a chordless cycle of length at least four, it is *odd* or *even* if it contains odd or even number of nodes, and an *antihole* is a complement of a hole).

In the last 15 years a number of other classes of graphs defined by excluding a family of induced subgraphs have been studied, perhaps originaly motivated by the study of perfect graphs. The kinds of questions this line of research was focused on were whether excluding induced subgraphs affects the global structure of the particular class in a way that can be exploited for putting bounds on parameters such as χ and ω, constructing optimization algorithms (problems such as finding the size of a largest clique or a minimum coloring) and recognition algorithms. A number of these questions were answered by obtaining a structural characterization of a class through their decomposition (as was the case with the proof of the Strong Perfect Graph Theorem). Each of these (often very complicated) decomposition theorems, resolved some question, but interestingly very few of them answered the question of how one might construct all graphs in a particular class starting from basic blocks by some well defined operations. In [9] we show how such an explicit construction can be obtained for the class of (even-hole,diamond)-free graphs (i.e. graphs defined by excluding even holes and diamonds as induced subgraphs). For this class we also show how they can be properly colored using a greedy algorithm on a particular, easily constructable, ordering of vertices.

The structure of even-hole-free graphs was first studied by Conforti, Cornuéjols, Kapoor and Vušković in [4], where a decomposition theorem is obtained for this class, that was then used in [5] for constructing a polynomial time recognition algorithm. One can find a maximum weight clique of an even-hole-free graph in polynomial time, since as observed by Farber [8] 4-hole-free graphs have $\mathcal{O}(n^2)$ maximal cliques and hence one can lisy them all in polynomial time. In [6] da Silva and Vuškovic show that every even-hole-free graph contains a vertex whose neighborhood is triangulated (i.e. does not contain a hole), and in fact they prove this result for a larger class of graphs that contains even-hole-free graphs. This characterization leads to a faster algorithm for computing a maximum weight clique in an even-hole-free graph. More recently, Addario-Berry, Chudnovsky, Havet, Reed and Seymour [1], settle a conjecture of Reed, by proving that every even-hole-free graph contains a *bisimplicial vertex* (a vertex whose set of neighbors induces a graph that is the union of two cliques). This imediately implies that if G is a non-null even-hole-free graph, then $\chi(G) \leq 2\omega(G) - 1$.

The study of even-hole-free graphs is motivated by their connection to β-perfect graphs introduced by Markossian, Gasparian and Reed [10]. For a graph G, let δ_G be the minimum degree of a vertex in G. Consider the following total order on $V(G)$: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order gives the upper bound: $\chi(G) \leq \beta(G)$, where $\beta(G) = \max\{\delta_{G'} + 1 : G'$ is an induced subgraph of $G\}$. A graph is *β-perfect* if for each induced subgraph H of G, $\chi(H) = \beta(H)$.

It is easy to see that β-perfect graphs belong to the class of even-hole-free graphs. A *diamond* is a cycle of length 4 that has exactly one chord. A *cap* is a cycle of length greater than four that has exactly one chord, and this chord forms a triangle with two edges of the cycle.

Markossian, Gasparian and Reed [10] show that (even-hole, diamond, cap)-free graphs are β-perfect. They show that a minimal β-imperfect graph that is not an even hole contains no simplicial extreme (where a vertex is *simplicial* if its neighborhood set induces a clique, and it is a *simplicial extreme* if it is either simplicial or of degree 2). Then they prove that (even-hole,diamond,cap)-free graphs must always have a simplicial extreme.

This result was then generalized by de Figuiredo and Vušković [7], who show that (even-hole,diamond,cap on 6 vertices)-free graphs contain a simplicial extreme, and hence are β-perfect.

In [9] we obtain a decomposition theorem for (even-hole,diamond)-free graphs that uses 2-joins and a special type of a star cutset. This decomposition theorem is then used to show that (even-hole,diamond)-free graphs contain simplicial extremes, implying that they are β-perfect (which was conjectured in [7]). This is now the largest class of graphs known to be β-perfect. We note that there are (even-hole,cap)-free graphs that are not β-perfect. Total characterization of β-perfect graphs remains open, as well as their recognition.

The fact that (even-hole,diamond)-free graphs have simplicial extremes implies that for such a graph G, $\chi(G) \leq \omega(G)+1$ and also leads to an explicit construction of all graphs in this class.

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The binary matroids with no $M(K_{3,3})$-minor

Geoff Whittle

(joint work with Dillon Mayhew, Gordon Royle)

Seymour's decomposition of regular matroids [2] is a celebrated result in matroid theory. He proves that a sufficiently connected regular matroid is either graphic, cographic or the matroid R_{10}. Evidence is emerging that a qualitative version of Seymour's theorem holds for all proper minor-closed classes of binary matroids [1]. This in turn motivates the task of finding exact structural characterizations of natural minor-closed classes other than regular matroids. By extension from graphs, it seems particularly natural to consider the classes of binary matroids obtained by excluding cylce matroids of Kuratowski Graphs.

In the talk I discussed a recent characterization of the binary matroids with no $M(K_{3,3})$-minor. It turns out that a sufficiently connected member of this class must be close to being cographic. In particular, an internally 4-connected member of the class is either cographic, one of a set of 18 sporadic matroids, or a matroid obtained by taking a single-element extension of the dual of the matroid of a Möbius Graph. Arbitrary members of the class can be obtained by taking 3-sums, 2-sums and direct sums of such matroids with the proviso that 3-sums are taken across triads only.

The theorem has algorithmic consequences. For an internally 4-connected matroid given by a rank oracle, it can be decided in polynomial time whether it belongs to the class of binary matroids with no $M(K_{3,3})$-minor. The condition that the matroid is internally 4-connected is necessary as, if it is removed, it is provably exponential to determine membership of the class.

The proof of the theorem relies heavily on computers. Most of the sporadic matroids in the class were revealed by a computer search undertaken by Gordon Royle. Moreover, the extensive case checking in the proof relied heavily on the program MACEK, developed by Petr Hliněný.

Results of this nature rely heavily on having appropriate inductive tools. For this theorem we developed a splitter-type theorem for finding internally 4-connected minors of vertically 4-connected matroids.

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Progress on Removable Path Conjectures

Paul Wollan
(joint work with Ken-ichi Kawarabayashi, Orlando Lee, Bruce Reed)

Lovász has made the following conjecture.

Conjecture 1. *There exists a function $f = f(k)$ such that the following holds. For every $f(k)$-connected graph G and two distinct vertices s and t in G, there exists a path P with endpoints s and t such that $G - V(P)$ is k-connected.*

Conjecture 1 is known to be true for several values of k. In the case $k = 1$, a path P connecting two vertices s and t such that $G - V(P)$ is connected is called a *non-separating path*. It follows from a theorem of Tutte any 3-connected graph contains a non-separating path connecting any two vertices, and consequently, $f(1) = 3$. When $k = 2$, it was independently shown by Chen, Gould, and Yu [1] and Kriesell [3] that $f(2) = 5$.

We will consider here two distinct weakenings of Lovasz' conjecture. The first asks the following:

Question 1. *Does there exist a function $f_1(k)$ such that for every $f_1(k)$-connected graph G and every pair s and t of vertices in G, there exists k internally disjoint non-separating paths $P_1, \ldots, P_k$ such that the endpoints of P_i are s and t?*

The existence of such a function f_1 was first shown by Chen, Gould, and Yu in [1] where they show $f_1(k) = 22k + 2$ would suffice. We outline a new proof based on bridge analysis. Given a system of internally disjoint $P_1, \ldots, P_t$ in a graph G, let P be the subgraph given by the union of all the paths P_i. Then a *bridge* of the path system is either an edge e with both endpoints contained in $V(P)$ or a connected component H of $G - V(P)$ along with all the edges with one endpoint in $V(H)$ and one endpoint in $V(P)$. A bridge consisting of a single edge is a *trivial* bridge. The *vertices of attachment* of a trivial bridge are simply the endpoints of the edge. For a connected component H of $G - V(P)$, the vertices of attachment of the bridge B containing H are simply the vertices of B contained in P.

Theorem 1. *There exists a constant c satisfying the following statement. Let G be a ck-connected graph $P_1, \ldots, P_t$ be collection of internally disjoint paths contained in G where the endpoints of P_i are s_i and t_i. Then there exist internally paths $P'_1, \ldots, P'_t$ such that the endpoints of P'_i are s_i and t_i such that for every non-trivial bridge B, there do not exist $k-1$ paths containing all vertices of attachment of B. Furthermore, no path P'_i contains both ends of any trivial bridge.*

Theorem 1 generalizes a result of Tutte showing the same result for $k = 2$. From Theorem 1, we immediately get the following theorem.

Theorem 2. *There exists a constant c' such that in every $c'k$-connected graph G and for every pair of vertices s and t of G there exists internally disjoint paths $P_1, \ldots, P_k$ where the endpoints of P_i are s and t and moreover, for any $I \subseteq \{1, \ldots, k\}$, the graph $G - \left(\bigcup_{i \in I} V(P_i)\right)$ is connected.*

Theorem 2 is a slight strengthening of the result of Chen, Gould, and Yu finding many internally disjoint non-separating paths connecting a given pair of vertices, although with a larger constant in the theorem.

An alternate weakening of Lovász' conjecture is to ask the following. What if instead of deleting the vertices on the path connecting s and t, one asks to delete the edges of the path and maintain vertex connectivity. We answer this question in the affirmative with the following theorem.

Theorem 3. *There exists a function $f_2(k)$ such that for every $f_2(k)$-connected graph G and every pair of vertices u and v of G, there exists a path P such that $G - E(P)$ is k-connected.*

This answers a conjecture of Kriesell [4]. When the methods used to prove Theorem 3 are applied to the original conjecture of Lovász, we are led to the following conjecture.

Conjecture 2. *There exists a function $f_3(k)$ such that for every $f_3(k)$-connected graph G and every three vertices s, t, and x in G, there exists an s-t path P and a k- connected subgraph H with $x \in V(H)$ such that $V(H) \cap V(P) = \emptyset$.*

The methodology used in the proof of Theorem 3 shows that an affirmative answer to Conjecture 2 would imply that Lovász' conjecture is also true.

Much of the material presented here appears in [2]

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