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Mini-Workshop: Positional Games

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Abstract. Positional games is one of rapidly developing subjects of modern combinatorics, researching two player perfect information games of combinatorial nature, ranging from recreational games like Tic-Tac-Toe to purely abstract games played on graphs and hypergraphs. Though defined usually in game theoretic terms, the subject has a distinct combinatorial flavor and boasts strong mutual connections with discrete probability, Ramsey theory and randomized algorithms. This mini-workshop was dedicated to summarizing the recent progress in the subject, to indicating possible directions of future developments, and to fostering collaboration between researchers working in various, sometimes apparently distinct directions.

Mathematics Subject Classification (2000): 05C65, 05D40, 91A24, 91A43, 91A46.

Introduction by the Organisers

The mini-workshop was organized by Michael Krivelevich (Tel Aviv) and Tibor Szabó (ETH Zürich).

The theory of Positional Games is a branch of Combinatorics, whose main aim is to develop systematically an extensive mathematical basis for a variety of two player perfect information games, ranging from such commonly popular games as Tic-Tac-Toe and Hex to purely abstract games played on graphs and hypergraphs. Though a close relative of the classical Game Theory of von Neumann and of Nim-like games popularized by Conway and others, Positional Games are quite different and are of much more combinatorial nature. The first papers on the subject appeared in the 60's and the 70's. József Beck turned it into a well established mathematical discipline through a series of papers spanning the last quarter century. Positional games are strongly related to several other branches of Combinatorics like Ramsey Theory, Extremal Graph and Set Theory, the Probabilistic Method. The subject has proven to be quite instrumental in deriving important results in Theoretical Computer Science, in particular in derandomization and algorithmization of important probabilistic tools.

Recently the field of Positional Games has been experiencing an explosive growth with quite a few new and important results in different directions (new versions of game definitions; analysis of biased games; exact results in Ramsey-type games; games of geometric nature; fast winning strategies etc.) appearing. The purpose of this mini-workshop was two-fold: it was aimed to provide an opportunity for leading researchers in the field to present and discuss their recent results on a systematic basis; it was also meant to attract new researchers, including students, to this exciting and rapidly developing field.

17 scientists from different countries participated in the meeting. The organizers and participants would like to thank the Mathematisches Forschungsinstitut Oberwolfach for providing an inspiring setting for this event.

Mini-Workshop: Positional Games

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Abstracts

The game domination number of graphs

József Balogh

The game domination number of a (simple, undirected) graph is defined by the following game. Two players, A and D, orient the edges of the graph alternately until all edges are oriented. Player D starts the game, and his goal is to decrease the domination number of the resulting digraph, while A is trying to increase it. The *game domination number* of the graph G, denoted by $\gamma_g(G)$, is the domination number of the directed graph resulting from this game. This is well-defined if we suppose that both players follow their optimal strategies.

The *k-domination number*, $\gamma_k(G)$, is the minimal cardinality of a k-dominating set. It is easy to see that $\gamma_1(G) \leq \gamma_g(G) \leq \gamma_2(G)$. Both $\gamma_2(G)$ and $\gamma_2(G)$ are monotone decreasing, in the sense that addition of edges do not increase them. An easy application of the pairing strategy implies that adding even number edges does not increase $\gamma_g(G)$. However, adding one edge might have a different effect, as the following examples shows:

Example Let G be obtained from the complete bipartite graph $K_{t,4}$ $(t \geq 6)$ as follows. Let $K_{t,4} = K(M, N)$ with $M = \{e, f, g, \ldots, z\}$, $N = \{a, b, c, d\}$), and $G = K_{t,4} + ab + cd - dz$. Then $\gamma_g(G) = 2$, while $\gamma_g(G + dz) = 3$.

Furthermore, $(k-1)(G + dz) + G$ has game domination number $2k$ and adding only one edge to the graph, $k(G + dz)$ has game domination number $3k$.

We proved additionally the following (sharp) inequalities:

Theorem 1. *For any tree T on n vertices* $\gamma_g(T) \geq \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2. *If a graph G has minimal degree at least 2, then* $\gamma_g(G) \leq \lfloor \frac{n}{2} \rfloor$.

The upper bound for the game domination number can be further strengthened if the minimal degree is larger using a probabilistic argument.

Theorem 3. *For every graph $G = (V, E)$ with n vertices and minimum degree $\delta \geq 2$ and for every real number p between 0 and 1, $\gamma_g(G) \leq np + 2n(1-p)^\delta + 1 + n\delta p(1-p)^\delta$. Therefore, $\gamma_g(G) \leq (1 + o(1))\frac{n \ln(\delta+1)}{\delta+1}$, where the $o(1)$-term tends to zero as δ tends to infinity, and the above the estimate is tight, up to the $o(1)$ error term.*

Tic-Tac-Toe like games and the Surplus

József Beck

Traditional Game Theory (J. von Neumann, J. Nash, etc.) is about games of incomplete information, like Poker where a player doesn't know the opponent's cards, and the main problem is how to compensate for the lack of information by Random Play. A key result is Neumann's minimax theorem about mixed strategies (involving Random Play).

Traditional game theory doesn't really say anything interesting about real games like Chess, Go, Checkers, Tic-Tac-Toe and its grown-up versions, Hex, and so on. The reason why "real" game theory doesn't exist yet is the immense space of possibilities, or combinatorial chaos. Brute force case study is totally impractical even for the simplest games. Is there an escape from the combinatorial chaos?

Perhaps the most natural model to start with is Generalized Tic-Tac-Toe. Nobody knows what generalized Chess or Go are supposed to mean, but it is obvious how to define generalized Tic-Tac-Toe (TTT). In ordinary TTT the "board" consists of $3 \times 3 = 9$ cells and there are 8 winning triples. One can play Generalized TTT on an arbitrary finite hypergraph, where the underlying set is called the "board" and the hyperedges are the "winning sets". The players take turns and occupy new elements of the board; the winner is the player who occupies a whole winning set first, otherwise the play ends in a draw.

Generalized TTT is hopelessly complicated, mainly for two reasons: (1) Exponentially Long Play (for a typical hypergraph), and because (2) Winning is non-monotonic (Extra Set Paradox)!

This forces us to study Weak Win instead: Weak Win simply means to occupy a whole winning set, but not necessarily first. I wrote a whole book about Weak Win. It is 700 pages long and the title is "Tic-Tac-Toe Theory". It will appear in Cambridge University Press in 2007. We have a good understanding of Weak Win, which is summarized in the so-called Meta-Conjecture. I can prove the Meta-Conjecture for large classes of hypergraphs; this is the subject of the book. The main open problem is how to extend the Meta-Conjecture for biased games (we have an analog conjecture, but cannot fully prove it).

The biased case seems far too hard, so what to do next? Weak Win requires a very dense hypergraph: the Set/Point ratio has to be exponentially large. What happens if the hypergraph the players are playing on is sparse? Can one still hope for any positive result? Of course the "complete occupation" has to be replaced by some kind of weaker "majority" concept. This is exactly what motivates the introduction of the Row-Column Game and the concept of Surplus. There is a perfectly natural way to generalize the "geometric" Row-Column Game for arbitrary graphs. This leads to the Degree Game on Graphs. I have a 100 pages long manuscript (unpublished yet) about this. The main result is that "every dense graph has a large Surplus". I can generalize this even for the biased case.

There is a simple conjecture which predicts the exact order of the Surplus in terms of the local density of the graph ("skew core density"). I have partial results, but the general case remains wide open. This is a wonderful research project, which seems more doable than the biased case in Tic-Tac-Toe Theory.

Winning fast in sparse graph construction games

Ohad N. Feldheim
(joint work with Michael Krivelevich)

1. Introduction

A Graph Construction Game is a Maker-Breaker game. In the graph construction game (K_N, G) Maker and Breaker take turns in choosing previously unoccupied edges of the complete graph K_N - (also referred to as the Board of the game). Maker's aim is to claim a copy of a given graph G while Breaker's aim is to prevent Maker from doing so. It is not difficult to show that for every graph G there exists large enough N such that (K_N, G) is won by Maker. One may also notice that if Breaker can prevent Maker from winning the Graph construction game (K_N, G) before the m-th turn, then for every $N' < N$ Breaker can delay Maker's victory in $(K_{N'}, G)$ at least for m turns. In this talk, we study how long can breaker delay graph construction games on an infinite board. Our goal is identify what kind of sparseness is useful for Maker in order to win fast, and to find an explicit Maker's strategy for constructing such sparse G-s quickly. We will also address the question of finding lower bounds for the number of turns required for constructing a specific G on an infinite board. Our interest in these results stems in part from their relation to the famous Burr-Erdős conjecture [3] as will be further explained.

2. Definitions and results

One can expect different graphs with the same number of vertices to differ significantly in the difficulty of their construction, and it is not surprising that denser graphs are harder (i.e. take more turns) for Maker to build. For example, due to Pekec and to Beck [5], [2] we know that the minimal N such that the Clique Game (K_N, K_n) is won by Maker is $N = P(n)2^{n/2}$ where P is a polynomial[1], and that for N large enough Maker can win (K_N, K_n) in less than 2^{n+2} turns, but cannot win in less than $2^{n/2}$ turns. However, constructing a star of order n can easily be done on K_{2n+1} as the board and in n rounds. These examples show us that the order of G is far from being enough to determine both boards size and game length required for Maker to win.

Another class of graphs that has been discussed in the literature in this respect is that of bounded degree graphs. Beck proved in [1] that if $N \geq P(d)3^d \cdot n$, then in a Maker-Breaker game played on the edges of K_N Maker can create a universal graph for the class of graphs on n vertices with maximum degree d, i.e., a graph that contains all such graphs. This immediately implies that a boardsize linear in n (with a coefficient of order 3^d) is sufficient to construct a graph on n vertices of

[1] Actually Beck describes the exact boardsize in [2].

maximum degree d. Yet, Beck's approach can only give an upper bound on the game length quadratic in n. In this talk we will show an upper bound on game length for such graphs which is linear in n.

In order to understand our construction one must be familiar with the definition of degenerate graphs:

Definition 2.1. (d-degenerate graph) *A graph G is called d-*degenerate *if every subgraph of G contains some vertex with degree d or less. We call the minimal d such that G is d-degenerate the* degeneracy *of G.*

The most important property of a d-degenerate graph is that it has a *d-degenerate ordering.*

Definition 2.2. (d-degenerate ordering) *Let $G = (V, E)$ be a graph. An ordering $\sigma = v_1 ... v_n$ of V is called a d-*degenerate ordering *if every vertex v_i has at most d neighbors among $\{v_{i+1}, ..., v_n\}$.*

Note that if a graph has maximum degree at most d, then it is d-degenerate, and every ordering of its vertices is a d-degenerate ordering.

in this talk we show that d-degenerate graphs can be constructed by Maker in linear time:

Theorem 2.3. (Quick victory theorem) *Let G be a d-degenerate graph G on n vertices. For every natural $N > d^{11}2^{2d+9} \cdot n$, Maker can win the game (K_N, G) in at most $d^{11}2^{2d+7} \cdot n$ rounds.*

We learn from this result that the order of the constructed graph has a relatively small impact on the length of the construction game. We will see that the reason for this fact is that graphs which are sparse (in the sense of degeneracy) can be built fast using a local strategy.

Obviously the bound stated in Theorem 2.3 applies to every graph G of maximum degree d on n vertices.

We will also show a simple lower bound for the length of a sparse graph construction game:

Theorem 2.4. (Long game theorem) *Let G be a graph of order n with m edges. Let $k = |aut(G)|$ be the number of automorphisms of G. For every $N < (2^{m-1}k)^{1/n}$ the game (K_N, G) is won by Breaker. Also, the game (K_M, G) cannot be won by Maker in less than $\frac{1}{2}(2^{m-1}k)^{1/n}$ rounds, for every M.*

Theorem 2.4, which is proved through direct application of a theorem by P. Erdős and J. Selfridge[4] gives us a method to bound from below the board-size and the game length required for Maker to win. Applying the bound to complete bipartite graphs, we obtain a family of d-degenerate graphs that cannot be constructed by Maker much faster than our strategy suggests:

Corollary 2.5. (The bounds are tight) *For every pair $1 \leq d < n$ there exists a d-degenerate graph G on n vertices such that for $N < 2^{d-\frac{d^2}{n}-\frac{1}{n}}(d!(n-d)!)^{\frac{1}{n}}$, the game (K_N, G) is won by Breaker. Also, for every M the game (K_M, G) cannot be won by Maker in less than $2^{d-1-\frac{d^2}{n}-\frac{1}{n}}(d!(n-d)!)^{\frac{1}{n}}$ rounds.*

Observe that for a fixed d and a large n, the last quantity behaves as $c2^dn$ for an absolute constant $c > 0$, showing that the upper bound obtained in Theorem 2.3 is not far from being tight.

Our interest in these results stems in part from their relation to the famous Burr-Erdős conjecture [3], and the research on *Ramsey Numbers* of different graph families, in particular of d-degenerate graphs and graphs with bounded degree. The Ramsey Number of a graph G, denoted by $r(G)$ is the smallest integer n such that the edges of K_n can not be divided into two disjoint sets, neither of which contains a copy of G. We know that if $N \geq r(G)$ then a draw in (K_N, G) is impossible. We can then deduce using the Strategy Stealing argument that Maker has a winning strategy for (K_N, G) (though we do not necessarily know how to describe it explicitly). The Burr-Erdős conjecture suggests that for d-degenerate graphs, $r(G) < Cn$ where n is the number of vertices of G, and C is a constant depending only on the degeneracy d of G. So far this conjecture has been settled for several cases including graphs with bounded degree. Our results thus provide a certain support for this conjecture. The value of our construction in the case of graphs with bounded degrees, or in other cases where the Burr-Erdős conjecture is settled, is partly given by the fact that our proof uses an explicit construction, unlike its Ramsey theory-based counterpart (which relies on the so called strategy stealing argument, inherently non-constructive in nature); it also deals with the dependency between C and the degeneracy d of G.

3. Concluding remarks and open problems

It appears that in graph construction games when the graph's structure is relatively regular (for example - if it has many automorphisms), a local construction, immediately designating every touched vertex to potentially function as a specific vertex in Maker's claimed copy G may asymptotically be the fastest possible - or at least asymptotically the fastest. This speculation is backed by Corollary 1. However, when the graph is very irregular it might be possible for Maker to use a global strategy, and to decide which vertex in K_N shall function as which vertex in G very late in the game, making Breaker's life a bit more difficult. Whether this is true remains an open problem.

Problem 3.1. *For a fixed $d > 0$, are there infinitely many d-regular graphs G for which Maker can win the game (K_N, G) in at most $f(d)|V(G)|$ rounds for large enough board-size N, where $f(d) \leq (1 + o(1))^d$?*

Induced graphs versus copies. In the proof of Theorem 2.3 we show that Maker's claimed graph not only contains a copy of G in K_N but also contains an *induced copy* of it. This follows from the fact that we never claim an edge between two candidates for vertices which have no edge between them in G. The reason we chose not to define the graph construction game for induced copies of G is that such a game cannot be represented as a weak game on a hypergraph.

Biased games. It may be of interest to extend our method to biased graph construction games. In these games Maker claims one edge per turn as before, but

Breaker answers by taking $b \geq 1$ edges. Our proof cannot be easily extended to handle this change.

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Fast and slow winning strategies in Positional Games

Dan Hefetz

(joint work with Michael Krivelevich, Miloš Stojaković, Tibor Szabó)

1. Introduction

In this talk, we study games which are played on the edges of the complete graph on n vertices. For quite a few Maker-Breaker and Avoider-Enforcer games it is rather easy to determine the winner. For example, in the connectivity game played on the edges of the complete graph K_n on n vertices, Maker can easily construct a spanning tree by the end of the game, in fact he just needs $n-1$ moves. The Avoider-Enforcer planarity game, played on the edges of K_n for n sufficiently large, is an even more convincing example – Avoider creates a non-planar graph and thus loses the game in the end, irregardless of his strategy, the prosaic reason being that every graph on n vertices with more than $3n-6$ edges is non-planar. Thus, for games of this type, a more interesting question to ask is not who wins but rather how long it should take the winner to reach a winning position. This is the type of question we address in this talk.

2. Definitions and results

For a $(1, 1, \mathcal{H})$ Maker-Breaker game, let $\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win, then set $\tau_M(\mathcal{H}) = \infty$).

Similarly, for a $(1, 1, \mathcal{H})$ Avoider-Enforcer game, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t rounds (if the game is an Avoider's win, then set $\tau_E(\mathcal{H}) = \infty$).

2.1. **Fast strategies for Maker and slow strategies for Breaker.** We study τ_M for specific games with board $E(K_n)$.

Let $\mathcal{M}_n$ be the hypergraph whose hyperedges are all perfect matchings of K_n (or matchings that cover every vertex but one, if n is odd). Let $\mathcal{D}_n$ be the hypergraph whose hyperedges are all spanning subgraphs of K_n of positive minimum degree. We find the *exact* number of moves that Maker needs, in order to win the $(1, 1, \mathcal{M}_n)$ game and the $(1, 1, \mathcal{D}_n)$ game. Obviously, Maker needs to make at least $\lfloor \frac{n}{2} \rfloor$ moves, as this is the size of a member of $\mathcal{M}_n$. We show that if n is odd, then he does not need more moves, whereas if n is even, then he needs just one more move. A similar result, showing the tightness of the obvious lower bound for the minimum degree game $\mathcal{D}_n$, easily follows.

Theorem 2.1. (i)

$$\tau_M(\mathcal{M}_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even} \end{cases}$$

(ii)

$$\tau_M(\mathcal{D}_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

As mentioned earlier, Chvátal and Erdős [6] proved that Maker can win the $(1, 1)$ Hamilton cycle game on K_n within $2n$ rounds. Here we show that for sufficiently large n, Maker can win the $(1, 1)$ Hamilton cycle game within $n + 2$ rounds. This bound is now only 1 away from the obvious lower bound.

Theorem 2.2. *For sufficiently large n,*

$$n + 1 \leq \tau_M(\mathcal{H}_n) \leq n + 2.$$

A corollary of the proof of the previous theorem is that Maker can win the "Hamilton path" game within $n - 1$ moves, which is clearly best possible.

Theorem 2.3. *For sufficiently large n,*

$$\tau_M(\mathcal{H}P_n) = n - 1,$$

where $\mathcal{H}P_n$ is the hypergraph whose hyperedges are all Hamilton paths of K_n.

Let $\mathcal{V}_n^k$ be the hypergraph whose hyperedges are all spanning k-vertex-connected subgraphs of K_n. The classical theorem of Lehman [12] asserts that Maker can build a 1-connected spanning graph in $n - 1$ moves. From Theorem 2.2 it follows that Maker can build a 2-vertex-connected spanning graph for the price of spending just 3 more (that is, in $n + 2$) moves.

In the following, we obtain a generalization of the latter fact for every $k \geq 3$. As every k-connected graph has minimum degree at least k, Maker needs at least $kn/2$ moves just to build a member of $\mathcal{V}_n^k$ (even if Breaker doesn't play at all). The next theorem shows that this trivial lower bound is asymptotically tight, that is, there is a strategy for Maker to build a k-vertex-connected graph in $kn/2 + o_k(n)$ moves.

Theorem 2.4. *For every fixed $k \geq 3$ and sufficiently large n,*

$$kn/2 \leq \tau_M(\mathcal{V}_n^k) \leq kn/2 + (k+4)(\sqrt{n} + 2n^{2/3}\log n).$$

An easy consequence of Theorems 2.1, 2.2 and 2.4, is that for every fixed $k \geq 1$ Maker can build a graph with minimum degree at least k within $(1+o(1))kn/2$ moves. This is clearly asymptotically optimal.

2.2. **Slow strategies for Avoider and fast strategies for Enforcer.** We study τ_E for specific games with board $E(K_n)$.

In the Avoider-Enforcer non-planarity game, Avoider loses the game as soon as his graph becomes non-planar. Clearly, Enforcer can win this game within $3n-5$ moves no matter how he plays; that is, $\tau_E(\mathcal{N}P_n) \leq 3n-5$, where $\mathcal{N}P_n$ is the hypergraph whose hyperedges are all non-planar subgraphs of K_n. On the other hand, Avoider can keep from losing for $\frac{3}{2}n - 3$ moves by simply fixing any triangulation and claiming its edges arbitrarily for as long as possible.

The following theorem asserts that the trivial upper bound is essentially tight, that is, Avoider can refrain from building a non-planar graph for at least $(3-o(1))n$ moves. More precisely,

Theorem 2.5.

$$\tau_E(\mathcal{N}P_n) > 3n - 28\sqrt{n}.$$

In the Avoider-Enforcer non-k-coloring game $\mathcal{N}C_n^k$, Avoider loses the game as soon as his graph becomes non-k-colorable. Avoider can play for at least $(1 - o(1))\frac{(k-1)n^2}{4k}$ moves without losing by simply fixing a copy of the k-partite Turán-graph and claiming half of its edges. On the other hand, it is not hard to see that the game is an Enforcer's win if it is played until the end (see [10]), so Avoider will lose after at most $\frac{1}{2}\binom{n}{2} \approx \frac{n^2}{4}$ moves. In our next theorem we essentially close the gap between the two bounds for the case $k=2$ (the "non-bipartite game"). We also improve the trivial lower bound and establish the order of magnitude of the second order term of $\tau_E(\mathcal{N}C_n^2)$.

Theorem 2.6.

$$\frac{n^2}{8} + \frac{n-2}{12} \leq \tau_E(\mathcal{N}C_n^2) \leq \frac{n^2}{8} + \frac{n}{2} + 1.$$

Next, we look at two Avoider-Enforcer games that turn out to be of similar behavior. In the game $\mathcal{D}_n$ Enforcer wins as soon as the minimum degree in Avoider's graph becomes positive, and in the game $\mathcal{T}_n$ Enforcer wins as soon as Avoider's graph becomes connected and spanning. Enforcer wins both games (see [9]), entailing $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n}{2}$. On the other hand, Avoider can choose an arbitrary vertex v, and, for as long as possible, claim only edges which are not incident with v, implying $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) > \frac{1}{2}\binom{n-1}{2}$. This determines the first order term for both parameters. In the following theorem we determine the second order term and the order of magnitude of the third.

Theorem 2.7.

$$\frac{1}{2}\binom{n-1}{2} + \left(\frac{1}{4} - o(1)\right)\log n < \tau_E(\mathcal{D}_n) \le \tau_E(\mathcal{T}_n) \le \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1.$$

3. Concluding remarks and open problems

- It was proved in Theorem 2.4 that Maker can win the $(1,1)$ k-vertex-connectivity game on K_n within $kn/2+o(n)$ moves. It would be interesting to decide whether the $o(n)$ term can be replaced with some function of k, if not for this game, then for the k-edge-connectivity game or the minimum-degree-k game.
- It would be interesting to find the exact value of $\tau_M(\mathcal{H}_n)$.
- It was proved in Theorem 2.6 that $\tau_E(\mathcal{N}C_n^2) \le \frac{n^2}{8} + \Theta(n)$. For $k \ge 3$, we know just the trivial bounds $\frac{(k-1)n^2}{4k} \le \tau_E(\mathcal{N}C_n^k) \le \frac{1}{2}\binom{n}{2}$. It would be interesting to close, or at least reduce, the gap between these bounds. It seems reasonable that, as in the case $k=2$, the truth is closer to the trivial lower bound, and maybe $\tau_E(\mathcal{N}C_n^k) \le (1+o(1))\frac{(k-1)n^2}{4k}$ for every $k \ge 3$.
- It was proved in Theorem 2.7 that $\tau_E(T_n)$ and $\tau_E(D_n)$ are "almost the same". This is reminiscent of the well-known property of random graphs, that the hitting time of being connected and the hitting time of having minimum positive degree are a.s. the same, and it motivates us to raise the following conjecture.

  Conjecture 3.1. $\tau_E(D_n) = \tau_E(T_n)$.
- It would be interesting to obtain good estimates on $\tau_E(M_n)$ and $\tau_E(H_n)$.

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On-line Ramsey theory

H. A. Kierstead
(joint work with Goran Konjevod)

For a positive integer n and a set S, let $[n]$ denote the set $\{1, \ldots, n\}$ and $\binom{S}{n}$ denote $\{X \subseteq S : |X| = n\}$. An *s-uniform hypergraph*, or *s- graph* for short, is a structure $H = (V, E)$, where $E \subseteq \binom{V}{s}$. Elements of E are called *edges* or *s-edges*. Let K_s^t denote the complete s-graph on t vertices defined by $K_s^t = \left([t], \binom{[t]}{s}\right)$. For s-graphs G and H we write $G \to_c H$, if every c-coloring of the s-edges of G results in a monochromatic copy of the *target* H.

For positive integers s, c, t the *Ramsey number* $\mathrm{Ram}_c^s(t)$ is the least integer n such that $K_s^n \to_c K_s^t$. Estimating Ramsey numbers, even for graphs, is a notoriously difficult problem. This has led researchers to consider other versions of the problem. Suppose that f is an increasing graph parameter. For positive integers s, c, t, define the *f-Ramsey number,* f-$\mathrm{Ram}_c^s(t)$, to be the least integer n for which there exists an s-graph G such that $f(G) = n$ and $G \to_c K_s^t$. Trivially, we have

$$f(K_s^t) \leq f\text{-}\mathrm{Ram}_c^s(t) \leq f(K_s^n), \text{ where } n = \mathrm{Ram}_c^s(t).$$

Erdős, Faudree, Rousseau and Schelp [1] studied the *size Ramsey number*, obtained when $f(G) = \mathrm{size}(G)$, the number of edges of G. Erdős et al. showed, with a proof attributed to Chvatál, that the size Ramsey number for graphs is exactly the trivial upper bound $\binom{n}{2}$, $n = \mathrm{Ram}_c^s(t)$. In other words, allowing more vertices will not reduce the number of edges needed to force a monochromatic clique. The *coloring number* of a graph $G = (V, E)$ is the least integer d such that its vertices can be ordered as $v_1 \prec \cdots \prec v_n$ so that

$$|\{v_i : i < j \wedge \{i, j\} \in E\}| < d \text{ for all } j \in [n].$$

Let $\chi(G)$ be the chromatic number of G. Clearly, $\chi(G) \leq \mathrm{col}(G)$, since its vertices can be colored with $\mathrm{col}(G)$ colors using First-Fit on the enumeration $v_1, \ldots, v_n$. Recently, Kurek and Ruciński [3] considered the *chromatic* and *coloring* Ramsey numbers obtained when f is χ and col, respectively. They observed that the trivial upper bound for the chromatic Ramsey number is again tight, and thus the trivial upper bound is also tight for the coloring Ramsey number.

From a Ramsey theoretic perspective, these results are disappointing. Kurek and Ruciński suggested a more promising line of inquiry might be to study on-line versions of f-Ramsey numbers. This seems encouraging because many Ramsey theoretic constructions are inherently on-line.

In the on-line setting we consider a process in which edges (not vertices) are generated one at a time and then immediately and irrevocably colored by an on-line algorithm. This is best understood as a game played between two players, Builder and Painter. For positive integers c, s, t the (c, s, t)-*Ramsey game* (with target K_s^t) is played as follows. Play begins with an empty s-graph $G_0 = (V, E_0)$ on an arbitrarily large, but finite, vertex set V determined by Builder. (So $E_0 = \emptyset$.) The game is played in rounds. At the beginning of the ith round Builder will have constructed an s-graph $G_{i-1} = (V, E_{i-1})$ with $|E_{i-1}| = i - 1$ and Painter will have constructed a coloring $f_{i-1} : E_{i-1} \to [c]$. On the ith round Builder constructs a new edge e_i (distinct from previous edges) and sets $G_i = (V, E_i)$, where $E_i = E_{i-1} \cup \{e_i\}$. Painter responds by coloring e_i to obtain a coloring $f_i : E_i \to [c]$ with $f_{i-1} \subseteq f_i$. Builder wins if Painter eventually creates a monochromatic copy of K_s^t; otherwise Painter wins when she has colored all $\binom{|V|}{s}$ edges.

For positive integers s, c, t and an increasing graph parameter f, define the *on-line f-Ramsey number,* $f\text{-oRam}_c^s(t)$, to be the least integer n such that Builder can win the (c, s, t)-Ramsey game while constructing a graph G with $f(G) = n$. Trivially,

$$f(K_s^t) \leq f\text{- oRam}_c^s(t) \leq f\text{- Ram}_c^s(t) \leq f(K_s^n), \text{ where } n = \text{Ram}_c^s(t).$$

Kurek and Ruciński conjectured:

Conjecture 1. *For all positive integers c and t,* $\lim_{t\to\infty} \frac{\text{size - oRam}_c^2(t)}{\text{size - Ram}_c^2(t)} = 0.$

We consider the *on-line coloring Ramsey number*, $\text{col-oRam}_c^2(t)$. First we prove the following Theorem that shows that the trivial *lower* bound on $\text{col-oRam}_c^2(t)$ is tight even though the trivial *upper* bound is tight for $\chi\text{-Ram}_c^2(t)$.

Theorem 2. *For all positive integers c, t,* $\text{col-oRam}_c^2(t) = \chi(K_2^t) = \text{col}(K_2^t) = t.$

Next we extend the definition of coloring number to hypergraphs in a natural way so that $\chi(G) \leq \text{col}(G)$ for all hypergraphs G. Finally we prove our main result:

Theorem 3. *For all positive integers c, s, t,* $\text{col-oRam}_c^s(t) = \chi(K_s^t) = \text{col}(K_s^t).$

Our techniques were first used in [2], where it is shown that $\chi\text{-oRam}_2^2(t) = t$ for every positive integer t and $\text{col-oRam}_c^2(3) = 3$ for every positive integer c. As in [2] our main tool is the analysis of an auxiliary game called *survival*, which seems to be interesting in its own right. The novelty of the current paper is that our previous analysis of survival for graphs is extended to hypergraphs. This is needed, even in the case of graphs (Theorem 2) to extend the results of [2] to arbitrary c and t.

Let p, s, t be positive integers with $s \leq p$. The (p, s, t)- *survival game* is played by two players, *Presenter* and *Chooser*. Play begins with the s-graph $H_0 = (S_0, E_0)$, where S_0 is an arbitrarily large, but finite, set of vertices determined at the beginning of the game by Presenter and $E_0 = \emptyset$. The game is played in *rounds*. At the beginning of the ith round the players will have constructed an s-graph $H_{i-1} = (S_{i-1}, E_{i-1})$. During the ith round they construct $H_i = (S_i, E_i)$ as follows. Presenter plays by presenting a p-subset $P_i \subseteq S_{i-1}$. Chooser responds by choosing an s-set $X_i \subseteq P_i$. The remaining vertices in $P_i \setminus X_i$ are discarded, leaving $S_i = S_{i-1} \setminus (P_i \setminus X_i)$ and $E_i = (E_{i-1} \cup \{X_i\})$. The vertices in S_i are called *surviving* vertices. Presenter wins if H_i contains a copy of K_s^t for some i. Otherwise Chooser wins when eventually $|S_i| < t$ as then Presenter cannot make a play.

Theorem 4. *For all positive integers p, s, t with $s \leq p$, Presenter has a winning strategy in the (p, s, t)-survival game.*

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Positional games and probabilistic considerations

Michael Krivelevich

Positional games are two player perfect information games. Being such, they appear to leave no room for probabilistic considerations of any sort. Yet, as accumulated research experience has convincingly shown, probabilistic intuition and arguments are in fact omnipresent, when analyzing and even playing positional games, especially those where players' roles are non-symmetric (like Maker-Breaker and Avoider-Enforcer games), and also biased games. This is certainly one of the key qualitative discoveries of the research in the field during the last three decades, supported by an array of recent results.

In this survey-type talk I discuss several important aspects of applying probability to positional games. The topics to be discussed include:

- First moment method: Maker-Breaker (unbiased) games, Erdős-Selfridge criterion for Breaker's win [8] and its probabilistic interpretation;
- Second moment method [4]: Beck's analysis of the maximum degree game [3];
- Maker-Breaker biased games; the Erdős paradigm – connection between a critical bias and a threshold for property's appearance in the random graph $G(n, m)$;
- applying the Erdős paradigm: the connectivity [7], [1], Hamiltonicity [7], [2] [11], non-planarity [9], giant component [6] and creating-a-copy-of-H [7], [5] games;
- Random strategies for positional games: creating a copy of H [5], creating a k-connected spanning subgraph [10].

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Biased positional games and small hypergraphs with large covers

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(joint work with Tibor Szabó)

We prove that in the biased $(1 : b)$ Hamiltonicity and k-connectivity Maker-Breaker games ($k > 0$ is a constant), played on the edges of the complete graph K_n, Maker has a winning strategy for $b \leq (\log 2 - o(1))n/\log n$. Also, in the biased $(1 : b)$ Avoider-Enforcer game played on $E(K_n)$, Enforcer can force Avoider to create a Hamilton cycle when $b \leq (1 - o(1))n/\log n$. These results are proved using a new approach, relying on the existence of hypergraphs with few edges and large covering number. The main combinatorial tool is a recent sufficient condition for Hamiltonicity, derived by Dan Hefetz and the authors [8].

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The diameter game

Ryan Martin
(joint work with József Balogh, András Pluhár)

1. Introduction

In a generalized Maker-Breaker positional game, Maker and Breaker play in turns. Maker makes a moves in each turn and Breaker makes b moves in each turn. We choose Maker to be the first player, although this nearly always makes no difference in the outcome of the game. We call such games $(a : b)$-games. If $a = b$, the game is *fair*, otherwise it is *biased.* If $a = b > 1$, the game is *accelerated.* In [3], Beck asked about the behavior of accelerated versus unaccelerated games, having observed a game in which Maker can win the $(1 : 1)$-game, but the result of the $(2 : 2)$-game is inconclusive. It is well-studied in the literature that a game may have completely different outcomes if it is played as $(1 : 1)$ or $(a : b)$ game, where either a or b is greater than 1, see [1, 5, 6, 7].

Here, we investigate the so-called 2-diameter game and, in particular, we prove that in the $(1 : 1)$-game, Breaker wins; but, in the $(2 : 2)$-game, Maker wins.

1.1. **Probabilistic intuition.** An important guide to understanding such games is the so-called *probabilistic intuition*, see [2]. In the probabilistic intuition, the perfect players can be thought of as being replaced by players with a random strategy. See, for example, the papers [2, 4, 8]. In this extended abstract, we consider the property that the graph has diameter at most d. We denote the corresponding d-diameter game by $\mathcal{D}_d(a : b)$, or more briefly, by $\mathcal{D}_d$ if $a = b = 1$.

It should be noted that while it has been known that acceleration may change the outcome somewhat, for the diameter game, it is shown that the $(1 : 1)$-game and the $(2 : 2)$-game have completely different outcomes. More specifically, this is the first non-trivial case in which the probabilistic intuition fails completely in the $(1 : 1)$-game, and is at least partially restored in the $(2 : 2)$-game. Note further that this is the first non-artificial case in which it is shown that the $(1 : 1)$ and the $(2 : 2)$-game have a different outcomes, when the **minimum** size of a winning set of both players is large (say, at least $n - 1$).

2. Results

A simple pairing strategy gives that Breaker wins the diameter 2 game unless the graph is trivially small:

Proposition 2.1. *If $n \leq 3$, then Maker wins the game $\mathcal{D}_2$. If $n \geq 4$, then Breaker has a winning strategy for the the game $\mathcal{D}_2$.*

A little acceleration of the game changes the outcome completely. In the case where $a = 2$, we expect the probabilistic intuition to work but we can only prove a weaker result:

Theorem 2.2. *Maker wins the game* $\mathcal{D}_2(2 : \frac{1}{8}n^{1/8}/(\log n)^{1/2})$*, and Breaker wins the game* $\mathcal{D}_2(2 : (2+\epsilon)\sqrt{n/\ln n})$ *for any* $\epsilon > 0$*, provided* n *is large enough.*

Note that $G_{n,1/2}$ has diameter 3 almost surely if $p^{-1/3+\epsilon}$, and it does not have diameter 3 almost surely if $p = n^{-1/3-\epsilon}$, for an arbitrary $\epsilon > 0$. The game $\mathcal{D}_3(1 : b)$ defies the probabilistic intuition again.

Theorem 2.3. *Maker wins the game* $\mathcal{D}_3(1 : c_1\sqrt{n/\ln n})$*, and Breaker wins the game* $\mathcal{D}_3(1 : c_2\sqrt{n})$*, for some* $c_1, c_2 > 0$*, provided* n *is big enough.*

We suspect, however, that for the $\mathcal{D}_3(3 : b)$ game the breaking point should be $b_0 \approx n^{2/3} \times \text{polylog}(n)$, satisfying the probabilistic intuition, but we do not have a conjecture for the breaking point for the $\mathcal{D}_3(2, b)$ game.

Theorem 2.3 is implied by the following more general theorem.

Theorem 2.4. *There exists a constant* $c_0 > 0$ *such that if* d *is an integer,* $3 \le d \le c_0 \ln n/\ln\ln n$*, then there is a* $c_1 = c_1(d) > 0$*, such that Maker wins the game* $\mathcal{D}_d(1 : c_1(n/\ln n)^{1-1/\lceil d/2\rceil})$ *if* n *is large enough.*

Furthermore, for every $a > 1$*, there is a constant* $c_2 > 0$*, depending only on* a *such that if* d *is an integer,* $3 \le d \le c_2 \ln n/(\ln\ln n)$*, then there exist* $c_3 = c_3(d) > 0$ *and* $c_4 = c_4(a, d) > 0$ *such that Breaker wins the games* $\mathcal{D}_d(1 : c_3 n^{1-1/(d-1)})$ *and* $\mathcal{D}_d(a : c_4 n^{1-1/d})$*, provided* n *is big enough.*

Note that in Theorem 2.4, Maker achieves diameter $2k$ by achieving diameter $2k-1$ for any integer $k \ge 2$. We conjecture that the correct break point is close (up to polylog factor) to the "Breaker" bounds.

3. Degree game

A useful tool in proving Theorems 2.2 and 2.4 is the so-called Degree Game. In this game, Maker and Breaker play an $(a : b)$ game on the edges of G. Maker wins by getting at least d edges incident to each vertex. For $G = K_n$ and $a = b = 1$ this game was investigated thoroughly in [9] and [2]. It was shown that Maker wins if $d < n/2 - \sqrt{n \log n}$, and Breaker wins if $d > n/2 - \sqrt{n}/12$, satisfying probabilistic intuition. The general case is analogous.

Lemma 3.1. *Let* $a \le n/(4\ln n)$ *and* n *be large enough. Then Maker wins the* $(a : b)$ *degree game on* K_n *if* $d < \frac{a}{a+b}n - \frac{6ab}{(a+b)^{3/2}}\sqrt{n \ln n}$.

3.1. Expansion game.

In the Expansion Game, Maker attempts to ensure that for every pair of disjoint sets R and S, where $|R| = r$ and $|S| = s$, there is an edge between R and S. We may assume that $s \ge r$. This game is used to ensure that vertices with large neighborhoods have very large second neighborhoods.

Lemma 3.2. *Maker wins the* $(a : b)$*-Expansion Game with parameters* r *and* s *if one of the following holds:*

(a) $2b \ln n < r \ln(a+1)$,
(b) $b \ln n < r \ln(a+1) < 2b \ln n$ *and* $s > \frac{rb \ln n}{r \ln(a+1) - b \ln n}$,
(c) $r \ln(a+1) < b \ln n$ *and* $n - s < \frac{nr \ln(a+1)}{b \ln n + r \ln(a+1)}$.

4. Proof idea for Theorem 2.2

Maker plays in two phases. Phase I lasts for $2nr \approx n^{3/2}(\ln n)^{1/2}$ rounds. There are four subgames that Maker plays in successive rounds. The first subgame is a ratio game. This ratio game ensures that if the degree of vertex x in Breaker's graph is large, then the fraction of the degree of x in Breaker's graph over the degree of x in Maker's graph is small. Maker uses the Degree Game strategy in these rounds.

The second game ensures that, at any vertex, the degree in Maker's graph, at the end of Phase I, is at least $r \approx \sqrt{n \ln n}$. The third uses the Expansion Game to ensure that every vertex in Maker's graph, at the end of Phase I, has second-degree at least $n - s \approx n - n^{3/4}/lnn$.

The fourth game connects pairs of vertices, each vertex having high degree in Breaker's graph, with a Maker's path of length 2.

At the beginning of Phase II, therefore, many pairs of vertices in Maker's graph have a path of length at most two between them. Moreover, if a pair of vertices does not have a Maker's path between them, then at least one of the vertices has very few edges incident to it both in Breaker's graph and in Maker's graph. So, Maker will have many paths of length 2 to put between such a pair of vertices. Before Phase II concludes, Maker can easily connect each of those pairs of vertices with paths of length 2.

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Game Chromatic Index of Graphs with Given Restrictions on Degrees

Oleg Pikhurko

(joint work with Małgorzata Bednarska, Andrew Beveridge, Tom Bohman, Alan Frieze)

Let a graph G and a positive integer k be given. Two players, called Alice and Bob, alternatively color a previously uncolored edge of G in one of the colors from $[k] = \{1, \ldots, k\}$ so that no two adjacent edges have the same color. Thus, at any moment of the game, the current partial coloring of $E(G)$ is a proper edge coloring. The game can end in two different ways. Either all edges of G are colored (and then Alice is the winner) or the uncolored edge picked by a player cannot be properly colored (and then Bob wins).

Let us agree that Bob starts the game. (In fact, all theorems stated here will remain valid for the version where we let Alice to start the game.) The *game chromatic index* $\chi'_g(G)$ is the smallest k such that Alice has a winning strategy. This parameter has been previously studied by Lam, Shiu and Xu [9], Cai and Zhu [6], Erdős, Faigle, Hochstättler, and Kern [8], Andres [1], Bartnicki and Grytczuk [2], and others. Unfortunately, the game chromatic index seems hard to analyze.

This is a variation of the *game chromatic number* which is analogously defined for a game where nodes (not edges) are colored. The latter parameter is much better studied; we refer the reader to Bohman, Frieze, and Sudakov [5] for some history and references on the game chromatic number.

The trivial bounds on the game chromatic index are

$$\Delta(G) \le \chi'_g(G) \le 2\Delta(G) - 1, \tag{1}$$

where $\Delta(G)$ denotes the maximal degree of G.

We show that for any $\mu > 0$ there is $\epsilon > 0$ such that any graph G of order n with $\Delta(G) \ge (\frac{1}{2} + \mu)n$ satisfies

$$\chi'_g(G) \le (2 - \epsilon)\Delta(G). \tag{2}$$

Surprisingly, this is done by letting Alice to play randomly as follows. Fix small constants $\sigma \gg c \gg \epsilon$ depending on μ. Let G be a graph as above and let $k = \lfloor (2 - \epsilon)\Delta(G) \rfloor$. Suppose that Bob colored an edge $\{x, y\}$ in the previous move. With probability σ, Alice picks a random $u \in \{x, y\}$ and then a random uncolored edge containing u. With probability $1 - \sigma$, Alice picks a random uncolored edge in the whole graph. Having selected an edge e (either way), Alice uses a color, chosen uniformly at random from the set of colors currently available for e. We show that with probability $1 - o(1)$ as $n \to \infty$, every two vertices of G share at least $\epsilon\Delta(G)$ common colors after the first cn^2 rounds of the game. If this is so, then every edge e gets eventually colored (and Alice wins) because the number of colors forbidden at e cannot be larger than $2(\Delta(G) - 1) - \epsilon d$.

While probabilistic intuition and reasoning often help in the analysis of positional games, see e.g. Beck [3], there are not many examples where non-trivial results are obtained by actually introducing randomness into a player's strategy. Such examples were discovered by Spencer [11], Bednarska and Łuczak [4], Pluhár [10], and some others. Our proof of the upper bound fits into this category.

We make the following conjecture.

Conjecture 1. *There is $\epsilon > 0$ such that for an arbitrary non-empty graph G we have $\chi'_g(G) \leq (2-\epsilon)\Delta(G)$.*

Also, we construct, for every sufficiently large d, an example of a graph G with $\Delta(G) \leq d$ and $\chi'_g(G) \geq 1.003d$. This answers in the negative a question posed by Lam, Shiu and Xu [9, Question 1], who asked whether there is a constant C such that $\chi'_g(G) \leq \Delta(G) + C$ for an arbitrary graph G.

On the other hand, the lower bound in (1) is attainable for some graphs. A trivial example is $G = K_{1,d}$. However, we believe that the large minimal degree $\delta(G)$ will force $\chi'_g(G)$ to be well above $\delta(G)$. Namely, we make the following conjecture.

Conjecture 2. *There are $\epsilon > 0$ and d_0 such that any graph G with $\delta(G) \geq d_0$ satisfies $\chi'_g(G) \geq (1+\epsilon)\delta(G)$.*

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On Chooser-Picker Positional Games

András Pluhár
(joint work with András Csernenszky, C. Ivett Mándity)

The Positional Games may be defined as follows. Given an arbitrary hypergraph $\mathcal{F} = (V, \mathcal{F})$, the first and second players take elements of V in turns. The player, who takes all elements of an edge $A \in \mathcal{F}$ first wins the game. Easy to see that tic-tac-toe, 5-in-a-row, Qubic, y-game etc fit into that framework.

In the Maker-Breaker version of a game Maker wins by taking all elements of an $A \in \mathcal{F}$, while Breaker wins otherwise. A wellknown example is the game of hex.

Note that if Breaker wins (as a second player) then the original game is a draw, while if the first player wins the original game then Maker wins the Maker-Breaker version. See the detailed theory and further examples in [3, 2, 6, 7].

In order to understand the very hard clique games, Beck introduced the Picker-Chooser and the Chooser-Picker version of Maker-Breaker games in [1].

In these versions Picker takes an unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. When $|V|$ is odd, the last element goes to Chooser. Beck demonstrates in several cases that Picker may win easily the Picker-Chooser game if Maker wins the corresponding Maker-Breaker game. He also notices that, considering the same hypergraph, Picker has more control than Maker.

Something similar must hold for Chooser-Picker games, since Picker has seem to have an easier job in the Chooser-Picker version than Breaker has in the corresponding Maker-Breaker game. The other reason of such belief is that one can always consider $(V, \mathcal{F}^*)$, the transversal hypergraph of $(V, \mathcal{F})$. That is $\mathcal{F}^*$ contains those minimal sets $B \subset V$ such that for all $A \in \mathcal{F}$, $A \cap B \neq \emptyset$. Note that Breaker as a first (second) player wins the Maker-Breaker $(V, \mathcal{F})$ iff Maker as a first (second) player wins the Maker-Breaker $(V, \mathcal{F}^*)$.

We refer to both of these paradigms as *Beck's conjecture:*

Conjecture 1. *Picker wins a Picker-Chooser (Chooser-Picker) game on $(V, \mathcal{F})$ if Maker (Breaker) as a second player wins the corresponding Maker-Breaker game.*

We prove Conjecture 1 for the Picker-Chooser version of Shannon's switching game in the generalized version as Lehman did in [8]. Let $(V, \mathcal{F})$ be a matroid, where $\mathcal{F}$ is the set of bases, and Picker wins by taking an $A \in \mathcal{F}$. Note that the Chooser-Picker version of that game would mention the cutsets, the transversals of the bases.

Theorem 2. *Let $\mathcal{F}$ be collection of the bases of a matroid on V. Picker wins the Picker-Chooser $(V, \mathcal{F})$ game, if and only if there are $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$.*

The Erdős-Selfridge theorem gives a very useful condition for Breaker's win in a Maker-Breaker $(V, \mathcal{F})$ game, see [5].

Theorem 3 (Erdős-Selfridge [5]). *Breaker as the second player has a winning strategy in the Maker-Breaker* $(V, \mathcal{F})$ *game when* $\sum_{A\in\mathcal{F}} 2^{-|A|} < 1/2$.

Using a stronger condition, Beck proves Picker's win in a Chooser-Picker $(V, \mathcal{F})$ game, see [1]. Let $||\mathcal{F}|| = \max_{A\in\mathcal{F}} |A|$ be the rank of the hypergraph $(V, \mathcal{F})$.

Theorem 4. *Picker wins the Chooser-Picker game on the hypergraph* $(V, \mathcal{F})$ *if* $\sum_{A\in\mathcal{F}} 2^{-|A|} < \Big\{8(||\mathcal{F}|| + 1)\Big\}^{-1}$.

We improve on his result by showing:

Theorem 5. *Picker wins the Chooser-Picker game on the hypergraph* $(V, \mathcal{F})$ *if* $\sum_{A\in\mathcal{F}} 2^{-|A|} < \Big\{e\sqrt{\pi(||\mathcal{F}|| + 1)}\Big\}^{-1}$.

Chooser-Picker games may be played on an infinite hypergraph $(V, \mathcal{F})$, too. In that case Chooser selects a finite sub-hypergraph of $(V, \mathcal{F})$ first, then the game proceeds as before. We show that Picker wins the Chooser-Picker version of the game 8-in-a-row by modifying the method of [6]. The key element in such proofs is the monotonicity of these games. It formalizes as follows:

Given the hypergraph $(V, \mathcal{F})$ let $(V \setminus X, \mathcal{F}(X))$ denote the hypergraph where $\mathcal{F}(X) = \{A \in \mathcal{F}, A \cap X = \emptyset\}$.

Lemma 6. *If Picker wins the Chooser-Picker game on* $(V, \mathcal{F})$*, then Picker also wins it on* $(V \setminus X, \mathcal{F}(X))$.

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The 7-in-a-row game

András Pluhár
(joint work with András Csernenszky)

The board of the classical 5-in-a-row game is a graph paper or the 19×19 Go board, and the players goal is to get k squares in a row vertically, horizontally or diagonally first.

Theoretically the board might be the infinite square grid, and the generalization k-in-a-row is considered. In that case the number of required squares is $k \in \mathbb{N}$.

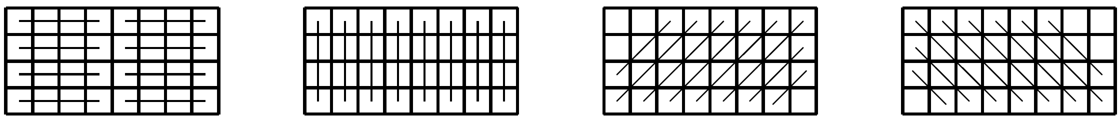
A delicate case study by Allis shows that the first player wins for $k = 5$ on the 19×19 or even on the 15×15 board, see [1].

By the strategy stealing argument the first player wins or achieves a draw for any $k \in \mathbb{N}$. Moreover the first player wins if $k \leq 4$, and the game is a blocking draw if $k \geq 9$, Shannon and Pollak, see e. g. [3, 4], and even for $k = 8$, T. G. L. Zetters, see [6]. While the $k = 5$ is still open on the infinite board, Allis' result implies that Maker wins for $k = 5$ in the Maker-Breaker version. Besides this, the open cases left in k-in-a-row are $k = 6, 7$.

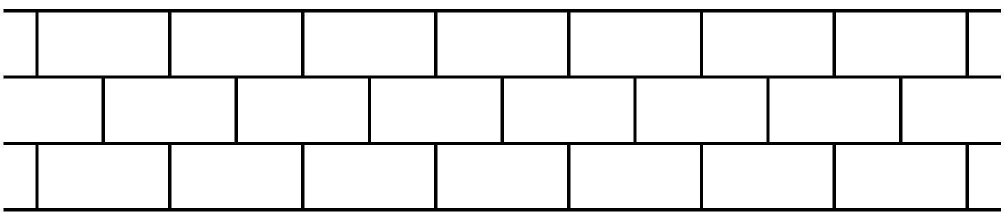
Here we announce that Breaker wins the Maker-Breaker version of the 7-in-a-row game, consequently the original 7-in-a-row game is a draw.

Theorem 1. *The 7-in-a-row game is a blocking draw.*

The proof consists of two parts, and follows the ideas that Shannon and Pollak used for the case $k = 9$, see [4], or T. G. L. Zetters for the case $k = 8$, see [6]. The plane is tiled with 4×8 rectangles; if T is the base rectangle, all the other are in the form $T + iu + jv$, where $u = (8, 0)$, $v = (4, 4)$ vectors and $i, j \in \mathbb{Z}$.



The winning sets of a tile



The tesselation of the plane

It is easy to see that if Breaker wins in every sub-board, then he wins on the whole plane. It is far from obvious though that Breaker can win on T. The proof of this relies on brute force computer search.

According to Beck's conjecture, see [2, 5] one would expect that the Chooser-Picker version of the 7-in-a-row game is a Picker win. See the definition and the monotonicity property in [5]. Indeed, this is the case.

Theorem 2. *Picker wins the Chooser-Picker version of the 7-in-a-row game.*

The proof in this case is a medium size case study, since because of the monotonicity it is enough to prove the claim for T. The following observation is also a great help.

Observation. In a Chooser-Picker game if a winning line contain no elements of Picker, and has only two unclaimed elements, x, y then Picker has an optimal strategy that starts with picking the set $\{x, y\}$.

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Online Ramsey games in random graphs

Reto Spöhel

(joint work with Martin Marciniszyn, Angelika Steger)

1. Online vertex colorings

Consider the following one-player game: The vertices of a random graph $G_{n,p}$ are revealed one by one to a player called Painter, along with all edges induced by the vertices revealed so far. The player has to assign one of r available colors to each vertex immediately, without creating a monochromatic copy of some fixed graph F. For which values of p can the player asymptotically almost surely (a.a.s.) color the entire random graph $G_{n,p}$? We say that $p_0(n)$ is a threshold for this game if there is a strategy such that the player a.a.s. succeeds if $p \ll p_0$, but a.a.s. fails with any strategy if $p \gg p_0$.

In [6] we proved an explicit threshold function $p_0 = p_0(F, r, n)$ for a large family of graphs F including cliques and cycles of arbitrary size, and an arbitrary number r of colors. For any graph F, let

$$m_1(F) := \max_{H \subseteq F} \frac{e_H}{v_H - 1} \quad . \tag{1}$$

Moreover, let

$$\overline{m}_1^1(F) := \max_{H \subseteq F} \frac{e_H}{v_H} \quad ,$$

and for $r \geq 2$,

$$\overline{m}_1^r(F) := \max_{H \subseteq F} \frac{e_H + \overline{m}_1^{r-1}(F)}{v_H} \quad . \tag{2}$$

With these definitions at hand, the result from [6] reads

Theorem 1.1. *Let F be a nonempty graph that has an induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying*

$$m_1(F^\circ) \leq \overline{m}_1^2(F) \quad .$$

Then for all $r \geq 1$, the threshold for the online vertex-coloring game with respect to F and with r available colors is

$$p_0(F, r, n) = n^{-1/\overline{m}_1^r(F)} \quad .$$

The side condition $m_1(F^\circ) \leq \overline{m}_1^2(F)$ is required by our approach to show that $n^{-1/\overline{m}_1^r(F)}$ is an upper bound for the online vertex-coloring game. As a lower bound, we prove this formula in full generality.

There are obvious lower and upper bounds for the threshold of the online vertex-coloring game. Clearly, Painter cannot lose the game if the underlying random graph contains no copy of F. The following well-known theorem of Bollobás, which is a generalization of a result of Erdős and Rényi [2] to arbitrary graphs F, states a threshold for this event.

Theorem 1.2 ([1])**.** *Let F be a nonempty graph, and let $\mathcal{P}$ =‘G contains a copy of F’. Then*

$$\lim_{n \to \infty} \mathbb{P}[G_{n,p} \in \mathcal{P}] = \begin{cases} 1 & \text{if } p \gg n^{-1/m(F)} \\ 0 & \text{if } p \ll n^{-1/m(F)} \end{cases} \quad ,$$

where

$$m(F) := \max_{H \subseteq F} \frac{e_H}{v_H} \quad .$$

Thus, for $p \ll n^{-1/m(F)}$ there is a.a.s. no copy of F in $G_{n,p}$, and finding a proper vertex-coloring is trivial. In fact, the case $r = 1$ of Theorem 1.1 is a mere reformulation of Theorem 1.2.

More interesting is the connection to the obvious upper bound for the duration of the game. Obviously, knowing the entire graph in advance would ease Painter's situation. An upper bound for the online game thus follows from a result of Łuczak, Ruciński, and Voigt about offline Ramsey properties of random graphs.

Theorem 1.3 ([4]). *Let $r \geq 2$ and F be a nonempty graph that in the case $r = 2$ is not a matching. Moreover, let $\mathcal{P}$ =‘every r-vertex-coloring of G contains a monochromatic copy of F’. Then there exist positive constants $c = c(F,r)$ and $C = C(F,r)$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \in \mathcal{P}] = \begin{cases} 1 & \text{if } p > Cn^{-1/m_1(F)} \\ 0 & \text{if } p < cn^{-1/m_1(F)} \end{cases} ,$$

where $m_1(F)$ is defined as in (1).

Hence, there is no hope of finding an r-vertex-coloring avoiding a monochromatic copy of F whenever $p \gg n^{-1/m_1(F)}$. Since the parameter $\overline{m}_1^r(F)$ is strictly increasing in r and satisfies

$$\lim_{r\to\infty} \overline{m}_1^r(F) = m_1(F) ,$$

the online threshold depends on the number of colors r, in contrast to Theorem 1.3, and approaches the offline threshold as the number of colors grows.

2. Online edge colorings

Online Ramsey games in random graphs were first considered for edge-colorings. Investigating algorithmic Ramsey properties of triangles, Friedgut *et al.* introduced and solved the online edge-coloring game with respect to triangles and with two available colors in [3]. In [7, 8], we extended this to a a result analogous to Theorem 1.1, but only covering the case of two colors.

Similarly to (1), let for every nonempty graph F on at least three vertices

$$m_2(F) := \max_{H\subset F} \frac{e_H - 1}{v_H - 2} .$$

This density measure replaces m_1 in the edge-coloring analogon of Theorem 1.3, which is due to Rödl and Ruciński [9, 10].

Similarly to (2), we define

$$\overline{m}_2^1(F) := m(F) ,$$

and for $r \geq 2$,

$$\overline{m}_2^r(F) := \max_{H\subseteq F} \frac{e_H}{v_H - 2 + 1/\overline{m}_2^{r-1}(F)} .$$

Theorem 2.1. *Let F be a graph that is not a forest, and that has a subgraph $F_- \subseteq F$ with $e_F - 1$ edges satisfying*

$$m_2(F_-) \leq \overline{m}_2^2(F) .$$

Then the threshold for the online edge-coloring game with respect to F and with two available colors is

$$p_0(F,n) = n^{-1/\overline{m}_2^2(F)} .$$

As in the vertex case, the side condition $m_2(F_-) \leq \overline{m}_2^2(F)$ stems from our approach for the upper bound. In fact, we can prove a lower bound of $n^{-1/\overline{m}_2^r(F)}$ in full generality for all non-forests F and an arbitrary number of colors r.

3. Concluding remarks and open questions

- The generalization of Theorem 2.1 to an arbitrary number $r \geq 2$ of colors remains as a challenging open problem.
- Very recently, we found a graph F and a strategy which shows that our lower bound of $n^{-1/\overline{m}_1^2(F)}$ for the vertex-coloring game is not tight in general. This means that the side condition $m_1(F^\circ) \leq \overline{m}_1^2(F)$ is more than just an artifact of our proof method, and that the parameter $\overline{m}_1^r(F)$ does not define a general threshold formula, as one might conjecture.
- In the edge-coloring game, the lower bound given by $\overline{m}_2^r(F)$ does not hold for forests. For F a tree, our methods yield a general lower bound which depends on the size of a *minimum vertex cover* of F. For some small examples, we were able to establish a matching upper bound by ad hoc methods, but in general it remains open whether our formula yields the correct threshold.
- The colorings obtained by optimal strategies for the games considered here are typically very unbalanced. What happens if Painter gets r vertices (edges) in every step and has to use each color for exactly one of these vertices (edges)? Marciniszyn, Mitsche, and Stojaković gave some preliminary results for the edge case [5]; we are currently working on the vertex case.

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Unbiased positional games on random graphs

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(joint work with Dan Hefetz, Michael Krivelevich, Tibor Szabó)

Let p and q be two positive integers, X a finite set, and $\mathcal{F} \subseteq 2^X$ a hypergraph. In the positional game $(X, \mathcal{F}, p, q)$, two players take turns claiming previously unclaimed elements of X. In every move, the first player claims p elements, and then the second player responds by claiming q elements. The set X is called the "board", and p and q are the biases of the first and second player, respectively. For the purposes of this paper $\mathcal{F}$ is assumed to be monotone increasing. In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and $\mathcal{F}$ is referred to as the family of winning sets. Maker wins the game if the subset of X he claims by the end of the game (that is, when every element of the board has been claimed by one of the players) is a winning set, that is, an element of $\mathcal{F}$; otherwise Breaker wins. Since $\mathcal{F}$ is monotone increasing, Maker wins if and only if he occupies an inclusion-minimal element of $\mathcal{F}$.

The study of positional games on the set of edges of a (complete) graph was initiated by Lehman [5] who, in particular, proved that Maker can easily win the $(E(K_n), \mathcal{T}_n, 1, 1)$ game, where the family $\mathcal{T}_n$ consists of the edge-sets of all connected and spanning subgraphs of K_n (by "easily" we mean that he can do so within $n-1$ moves, which is clearly optimal). Chvátal and Erdős [4] suggested to "even out the odds" by giving Breaker more power, that is, by increasing his bias. They determined that the connectivity game $(E(K_n), \mathcal{T}_n, 1, b)$ is won by Maker even when the bias b of Breaker is as large as $cn/\log n$, for some small constant $c > 0$, whereas for a constant $C > 0$, Breaker wins the game if his bias is at least $Cn/\log n$. They also showed that the $(E(K_n), \mathcal{H}_n, 1, 1)$ Hamiltonicity game, in which Maker's goal is to build a Hamiltonian cycle (that is, the family $\mathcal{H}_n$ of winning sets consists of the edge-sets of all Hamiltonian subgraphs of K_n), is won by Maker for sufficiently large n. Moreover, they conjectured that in fact Maker can win the $(E(K_n), \mathcal{H}_n, 1, b)$ game for some b that tends to infinity with n. This was proved by Bollobás and Papaioannou [2], who showed that Maker wins Hamiltonicity against a bias of $O(\log n/\log\log n)$. Finally, Beck [1] gave a winning strategy for Maker against a bias of $O(n/\log n)$.

Following [7] we give Breaker more power, not by increasing his bias, but by "thinning out" the board before the game starts. Formally, let $(X, \mathcal{H})$ be a hypergraph and let $0 \leq p \leq 1$ be a real number. We define $(X_p, \mathcal{H}_p)$ to be the hypergraph whose set of vertices X_p is obtained from X by removing every vertex of X with probability $1-p$, independently for each vertex, and whose set of hyperedges is $\mathcal{H}_p = \{A \in \mathcal{H} \ : A \subseteq X_p\}$. Note that $(X_p, \mathcal{H}_p)$ is actually a probability space of hypergraphs. Looking at the $(X_p, \mathcal{H}_p, 1, 1)$ game, we can discuss the probability that Maker (Breaker) wins the game.

The *threshold probability* $p_{\mathcal{F}_n}$ for the family of games $\mathcal{F}_n$, $n \in \mathbb{N}$ is defined to be the probability for which an almost sure Breaker's win turns into an almost sure

Maker's win, that is,

$$Pr[(X_p, (\mathcal{F}_n)_p, 1, 1) \text{ is a Breaker's win}] \to 1 \text{ for } p = o(p_{\mathcal{F}_n}),$$

and

$$Pr[(X_p, (\mathcal{F}_n)_p, 1, 1) \text{ is a Maker's win}] \to 1 \text{ for } p = \omega(p_{\mathcal{F}_n}),$$

when $n \to \infty$. Such a threshold $p_{\mathcal{F}_n}$ exists [3], since being a Maker's win is an increasing property.

In [7] the threshold probability for the connectivity game, the perfect matching game was determined. Moreover, it was proved that the threshold probability for the Hamiltonicity game satisfies $\frac{\log n}{n} \leq p_{\mathcal{H}_n} \leq \frac{\log n}{\sqrt{n}}$, with the conjecture that $p_{\mathcal{H}_n} = \frac{\log n}{n}$. This was verified in [6]. Here we strengthen this result and show that the the property that Maker wins the Hamiltonicity game has a *sharp* threshold.

Theorem 1. *There exists a constant $c' > 0$ such that Maker a.s. wins the $(1,1)$ Hamiltonicity game on $G(n, \frac{\log n + (\log\log n)^{c'}}{n})$.*

This statement is obviously very close to being best possible, as

$$G(n, \frac{\log n + 3\log\log n - \omega(1)}{n}),$$

where the $\omega(1)$ term tends to infinity with n arbitrarily slowly, has at least two vertices of degree at most three (and thus Breaker easily wins).

For a graph H, let $\mathcal{F}_H$ be the set of all copies of H in K_n. In the *k-clique-game* $\mathcal{F}_{K_k}$ Maker's goal is to build a complete subgraph on k vertices. In [7] the exponent of the main factor of the threshold-probability $p_{\mathcal{F}_{K_k}}$ for the k-clique-game with constant k was determined. For $k \geq 4$ it was found that for every $\epsilon > 0$,

$$n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{F}_{K_k}} \leq n^{-\frac{2}{k+1}}.$$

Here the exponent $\frac{2}{k+1}$ is the reciprocal of the so-called 2-density of K_k. The *maximum 2-density* of an arbitrary graph H, defined by

$$m_2(H) = \max_{\substack{H' \subseteq H \\ v(H') \geq 3}} \frac{e(H) - 1}{v(H) - 2},$$

is well-known parameter in random graph theory. For example, $n^{-1/m_2(G)}$ is the threshold probability that every edge of the random graph is contained in a copy of H. Intuitively this means that above this probability the copies of H are "densely and uniformly distributed" in $G(n,p)$. Then it doesn't come as a big surprise that Maker is also able to win the H-game.

Theorem 2. *Let H be a graph containing a cycle. There exists a real number $c_0 > 0$ such that for $p > \frac{1}{c_0} n^{-\frac{1}{m_2(H)}}$ Maker wins the H-game $\mathcal{F}_H$ on $G(n,p)$.*

It turns out that the methods from [7] for dealing with the clique game can be generalized to the H-game. Similarly to the case of cliques K_k for $k \geq 4$, for some graphs H we are able to prove a lower bound for the threshold probability of the

H-game, which essentially matches the upper bound stated in Theorem 2. Let $dgn(G) = \max\{\delta(G') : G' \subseteq G\}$ denote the *degeneracy* of G.

Theorem 3. *Let H be a graph such that $m_2(H) \leq dgn(H) - \frac{1}{2}$. Then for an arbitrarily small $\varepsilon > 0$ and for $p = n^{-\frac{1}{m_2(H)-\varepsilon}}$, Breaker a.s. has a winning strategy for the H-game on $G(n,p)$.*

This theorem includes the case of cliques and even more.

Corollary 4. *The above theorem is valid for every regular graph H with degree at least 4.*
The above theorem is also valid for all graphs H for which $m_2(H) - \lfloor m_2(H) \rfloor \in (0, \frac{1}{2}]$.

Note that Theorem 3 is not applicable for $H = K_3$ as $m_2(K_3) = dgn(K_3) = 2$. In fact, in[7] it was shown that the K_3-game is somewhat of an anomaly in the sense that Theorem 2 *can be* strengthened for it. While $m_2(K_3) = 2$, still it was proved in [7] that the threshold probability for the triangle game is $n^{-\frac{5}{9}}$. Observe that $\frac{5}{9} > \frac{1}{2} = m_2(K_3)^{-1}$.

In [7] it was asked for which graphs H we have $p_{\mathcal{F}_H} = \tilde{\Theta}\left(n^{-1/m_2(H)}\right)$ and for which graphs can Theorem 2 be improved.

For an arbitrary tree $T \neq K_2$ we have $m_2(T) = 1$ and $dgn(T) = 1$, which means that Theorem 3 cannot be applied. We show that Theorem 2 can be improved for trees.

Theorem 5. *Let $T \neq K_2$ be an arbitrary tree. Then there is an $\epsilon(T) > 0$, such that Maker wins the game $\mathcal{F}_T$ on $G(n,p)$ for $p = n^{-1-\epsilon(T)}$.*

We obtain relatively precise estimates for the threshold probability in special tree-games, like the path-game and the star-game. We show that $\epsilon(P_d)$ is exponential, while $\epsilon(S_d)$ is linear in d. It would be interesting to determine $\epsilon(T)$ more precisely for other trees.

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Maker-Breaker games on the complete graph

Tibor Szabó
(joint work with Dan Hefetz, Michael Krivelevich)

Positional games involve two players who alternately occupy the elements of a given set V, the *board* of the game. The focus of their attention is a given family $\mathcal{F} = \{e_1, \dots, e_k\} \subseteq 2^V$ of subsets of V, usually called the family of *winning sets*. The players exchange turns occupying one previously unoccupied element of V. The game ends when there are no unoccupied elements of V.

There are several types of positional games depending on how the identity of the winner is determined. In this talk we restrict our attention to Maker-Breaker games. In a *Maker-Breaker* game the first player, called Maker, wins if he completely occupies one of the winning sets by the end of the game; the second player, called Breaker, wins otherwise, i.e., if he manages to occupy at least one element of (i.e., "to break into") every winning set by the end of the game.

Following Erdős and Chvátal [7] we study games are played on the edge set of the complete graph K_n, and the winning sets are defined by some graph theoretic property like connectivity or Hamiltonicity. In the *connectivity game* Maker wins if he creates a spanning tree by the end of the game. In the *Hamiltonicity game* Maker wins if his graph contains a Hamilton cycle in the end.

It seems that Maker, partly because he has so many (i.e., $\frac{1}{2}\binom{n}{2}$) edges by the end, should be able to win both of these games easily. This is indeed the case; for the connectivity game the winning strategy is a triviality, for the Hamiltonicity game it requires a one-page argument [7]. Motivated by the easy success of Maker, Chvátal and Erdős suggested to make the game more "balanced" by introducing a *bias*: at each turn Breaker is allowed to occupy b edges instead of just one, where $b \geq 1$ is an integer. In these games, as well as in other $(1 : b)$ games, the most natural question is to determine what is the largest bias $b_{\mathcal{F}}$ against which Maker still has a strategy to beat Breaker.

Chvátal and Erdős proved [7] that if $b < \left(\frac{1}{4} - o(1)\right)\frac{n}{\log n}$ then Maker can still occupy a spanning tree and thus win the connectivity game. They also proved [7] that the order of magnitude of the bias is best possible. In fact they showed that if $b > (1 + o(1))\frac{n}{\log n}$ then Breaker can occupy all edges incident to some vertex, so Maker loses the connectivity game, since his graph is disconnected. Later Beck [1] improved the constant factor in the result of Chvátal and Erdős and established that Maker wins the connectivity game even if $b < (\log 2 - o(1))\frac{n}{\log n}$.

For the Hamiltonicity game Chvátal and Erdős conjectured that there is function $b_{\mathcal{H}}(n)$ tending to infinity such that Maker can still build a Hamilton cycle if he plays against a bias $b_{\mathcal{H}}(n)$. Their conjecture was verified by Bollobás and Papaioannou [6] who proved that Maker is able to build a Hamilton cycle even if Breaker's bias is as large as $\frac{c \log n}{\log \log n}$. Beck improved greatly on this [2] and established that the order of magnitude of the critical bias is the same for the

Hamiltonicity game and the connectivity game. He showed that Maker wins the Hamiltonicity game provided Breaker's bias is at most $\left(\frac{\log 2}{27} - o(1)\right)\frac{n}{\log n}$.

In this talk we discuss how to improve the constant factor in Beck's result and achieve the same lower bound as is known for the connectivity game.

Theorem 1 ([9]). *Maker wins the $(1 : b)$ Hamiltonicity game for every $b < (\log 2 - o(1))\frac{n}{\log n}$.*

Our proof technique provides the same lower bound for the critical bias in the k-connectivity game, where the family $\mathcal{C}_k$ of winning sets consists of the edgesets of k-connected graphs. As far as we know the main term of earlier lower bounds on $b_{\mathcal{C}_k}$ depended on k.

Theorem 2 ([9]). *Maker wins the $(1 : b)$ k-connectivity game for every $b < (\log 2 - o(1))\frac{n}{\log n}$.*

The proof of our theorems is based on the combination of our basic "thinning" trick and the following quasirandom Hamiltonicity criterion derived recently in [8].

Lemma 3 ([8]). *Let $12 \le d \le e^{\sqrt[3]{\log n}}$ and let G be a graph on n vertices satisfying properties P1, P2 below:*

- **P1:** *For every $S \subset V$, if $|S| \le k_1(n,d) := \frac{n \log\log n \log d}{d \log n \log\log\log n}$ then $|N(S)| \ge d|S|$;*
- **P2:** *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \ge k_2(n,d) := \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$.*

Then G is Hamiltonian, for sufficiently large n.

Open problems. The new results unify the known lower bounds for a large family of games. Denote by $b_{\mathcal{D}_k}$, $b_{\mathcal{C}_k}$, $b_{\mathcal{H}}$, $b_{\mathcal{T}}$ the critical biases for the minimum degree k, k-connectivity, Hamiltonicity and connectivity games, respectively. We now have that

$$(\log 2 - o(1))\frac{n}{\log n} \le b_{athcalH} \le b_{\mathcal{T}} \le b_{\mathcal{D}_1} \le (1 + o(1))\frac{n}{\log n}$$

and

$$(\log 2 - o(1))\frac{n}{\log n} \le b_{\mathcal{C}_k} \le b_{\mathcal{D}_k} \le b_{\mathcal{D}_1} \le (1 + o(1))\frac{n}{\log n}.$$

Hence the foremost obstacle standing in the way of the asymptotic determination of the critical bias for the connectivity or Hamiltonicity game is the inability of our current techniques to deal with the mindegree-1 game. In other words, what is the smallest bias of Breaker which allows him to isolate a vertex in Maker's graph?

Problem 4. *Determine $b_{\mathcal{D}_1}$ asymptotically.*

Ever since the paper of Chvátal and Erdős, random graph intuition plays an important role in the theory of positional games. In the model of random graph process there is a very strong dependence between, say, the properties of connectivity and mindegree-1. In an informal language one could say that *the main reason* *a random graph is not connected is that there exists an isolated vertex*. This motivates our question whether a similar phenomenon holds in the theory of biased positional games. We think the answer is yes, i.e., the only reason Maker cannot win the connectivity game is that Breaker is able to isolate a vertex in Maker's graph.

Conjecture 5. *For large n*

$$b_{\mathcal{T}} = b_{\mathcal{D}_1}.$$

We are curious whether one can show anything of this sort without obtaining the asymptotic value of these critical biases. Other natural questions motivated by known facts on random graph processes are

- Is it true that $b_{\mathcal{H}} = b_{\mathcal{D}_2}$?
- Is it true that $b_{\mathcal{C}_k} = b_{\mathcal{D}_k}$?

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Avoider-Enforcer games on the complete graph

Tibor Szabó

(joint work with Dan Hefetz, Michael Krivelevich)

In this talk we discuss the misère version of Maker-Breaker games. In an *Avoider-Enforcer game* positional game $\mathcal{F} \subseteq 2^X$ the first player, called Avoider, wins if he does *not* occupy completely any of the members of $\mathcal{F}$; otherwise the second player, called Enforcer, wins. Although Avoider-Enforcer games appear naturally in several situation related to Maker-Breaker games, our understanding of them is much less satisfactory. Avoider-Enforcer versions of graph games often behave completely differently from their Maker-Breaker counterparts.

The very first surprise comes when one realizes that the Avoider-Enforcer threshold bias $f_{\mathcal{F}}$ of a game $\mathcal{F}$, defined analogously to its Maker-Breaker counterpart $b_{\mathcal{F}}$, does not necessarily exist! Formally let $f_{\mathcal{F}}$ be the integer, such that (i) the $(1 : f)$-game is Enforcer's win for every $f \leq f_{\mathcal{F}}$ and (ii) the $(1 : f)$-game is

Avoider's win for every $f > f_{\mathcal{F}}$. In [2] a hypergraph is constructed, such that in the $(1 : f)$ Avoider-Enforcer game the identity of the winner alternates depending on the *parity* of the bias f of Enforcer, and thus the critical bias does not exist in a strong sense. For this reason we define the *lower threshold bias* $f_{\mathcal{F}}^-$ as the largest integer such that the $(1 : f)$-game is Enforcer's win for every $f \leq f_{\mathcal{F}}^-$ and the *upper threshold bias* $f_{\mathcal{F}}^+$ as the smallest integer such that the $(1 : f)$-game is Avoider's win for every $f > f_{\mathcal{F}}^+$.

In terms of lower bounds, it was shown in [2, 4, 6] that for a number of natural graph games we have $f_{\mathcal{F}} = O(f_{\mathcal{F}}^-)$. It would be very interesting to decide whether this is true in general. In terms of upper bounds we are doing much worse. Often the only available upper bound on $f_{\mathcal{F}}^+$ is the trivial one and in no cases are they expected to provide the truth.

The second surprise of Avoider-Enforcer games is that the random graph intuition fails badly for such a natural game like connectivity. More precisely, we showed [2] that in the $(1 : b)$ Avoider-Enforcer connectivity game the critical bias exists and is *linear* in n: Avoider wins if and only if the bias of Enforcer is at least $(1 + o(1))n/2$. This is in striking contrast with the order $\frac{n}{\log n}$ of the critical bias in the Maker-Breaker connectivity game.

For the game of hamiltonicity, Beck raised the question [1] whether Enforcer, playing with a bias of order $n/\log n$, can force Avoider to build a Hamilton cycle. This was analyzed in [2], and an almost affirmative answer, short of only a $\log\log\log n/\log\log\log\log n$-factor, was proved. Recently we improved this further to the conjectured bias.

Theorem 1 ([6])**.** *Enforcer wins the* $(1 : b)$ *Hamiltonicity game for every*

$$b < (1 - o(1))\,\frac{n}{\log n}.$$

Note that the order of magnitude is the same as for the Maker-Breaker games — with a better constant. A major difference we have here compared to the Maker-Breaker counterpart is the complete lack of results from the other direction. While we know the order of magnitude of the critical bias in the Maker-Breaker Hamiltonicity game, for Avoider's win the best available strategy is the trivial one: Avoider (clearly) wins if he has less then n edges by the end of the game (which happens for bias $b = (1 + o(1))n/2)$.

Similarly to Maker-Breaker games, our technique carries the bound through to k-connectivity games.

Theorem 2. *Enforcer wins the* $(1 : b)$ *k-connectivity game for every*

$$b < (1 - o(1))\,\frac{n}{\log n}.$$

The quasirandom Hamiltonicity criterion of [3] makes the proof of these theorems very similar to their Maker-Breaker counterpart: Enforcer will make sure that Avoider's graph satisfies properties **P1** and **P2** for Hamiltonicity. The only major difference in guaranteeing this is that instead of the generalized Erdős-Selfridge

criterion "$\sum_{A\in\mathcal{F}} 2^{-|A|/p} < \frac{1}{2}$" of Beck for Breaker's win in the $(p:1)$-game, Enforcer uses criterion "$\sum_{A\in\mathcal{F}} \left(1+\frac{1}{p}\right)^{-|A|} < \left(1+\frac{1}{p}\right)^{-p}$." for Avoider's win in the $(p:1)$-game (obtained recently in [2] (see also [1])).
Open Problems. Prove that nice games are asymptotically monotone in the bias.

Conjecture 3. *Prove that for the perfect matching, the hamiltonicity, the non-planarity, the non-k-colorability, and the K_k-minor games we have $f_{\mathcal{F}}^- = \Theta(f_{\mathcal{F}}^+)$.*

The following, admittedly modest conjecture could be a first step in this direction and highlights our lack of understanding of Avoider-Enforcer games.

Conjecture 4. *Prove that Avoider has a winning strategy in the* $(1 : \frac{n}{10})$ *hamiltonicity game.*

A remedy? A possible approach for making up for the lack of existence of a threshold bias in Avoider-Enforcer games is to relax the rules on how many elements in one round a player must take. Intuitively, taking *more* elements than one's bias is "bad" for a player in an Avoider-Enforcer game. The *monotone Avoider-Enforcer* $(a : b)$*-game* is defined by requiring Avoider to take *at least* a elements in each round and requiring Enforcer to take *at least* b elements in each round. Here, the analogously defined *monotone critical bias* $f_{\mathcal{F}}^{mon}$ trivially exists for practically all games $\mathcal{F}$.

Remark. In a Maker-Breaker game it is plausible to assume that taking *less* elements than one's bias is "bad" for a player. This motivates the analogous definition of *monotone Maker-Breaker* $(a : b)$-game, where Maker must take *at most* a (and at least one) elements and Breaker must take *at most* b (and at least one) elements in each round. Short meditation convinces us that the analogously defined monotone bias $b_{\mathcal{F}}^{mon}$ always exists and is equal to the strict bias $b_{\mathcal{F}}$.

Hence the monotone Avoider-Enforcer game seems a comfortable remedy for the non-existence of the critical bias for the original definition; provided the plausible statement $f_{\mathcal{F}}^- \leq f_{\mathcal{F}}^{mon} \leq f_{\mathcal{F}}^+$ is valid. The third surprise of Avoider-Enforcer games is that this is not true in general. Even more discouragingly, our example is not artificial at all. In [5] we show that the connectivity game has *monotone threshold bias* in the order $n/\log n$ compared with the strict bias of linear order.

We determine the monotone threshold bias asymptotically for many games, and find that the random graph intuition is valid even in the constant factor!

Theorem 5 ([5])**.** *For $\mathcal{F} = \mathcal{H}, \mathcal{C}_k, \mathcal{D}_k$, where k is a constant,*

$$f_{\mathcal{F}}^{mon} = (1+o(1))\frac{n}{\log n}.$$

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Problem session

József Beck: It is easy to see that Picker wins the Picker-Chooser game, played on a hypergraph $\mathcal{H}$ that consists of 2^n pairwise disjoint sets of size n each. What happens when $|\mathcal{H}| < 2^n$? What if Picker just wants some surplus?

Dan Hefetz: For the Maker-Breaker $(1, k)$ connectivity game on some graph G, let $c(G, k) = e(G)/(k+1)$ (more precisely $c(G, k)$ in this case is the number of edges that Maker will have in the end of the game - when every element of the board is claimed by some player) if Maker wins the game and $c(G, k) = \infty$ otherwise. Let $C_n(k) = \min c(G, k)$, where the minimum is extended over all graphs on n vertices. Is it true that $C_n(k) \leq C_n(k + 1)$? What about similarly defined functions for other Maker-Breaker games?

Angelika Steger: Consider the following one player game. Edges of K_n arrive in a random order (starting with the empty graph). The player immediately colors every edge with some color $c \in \{1, \dots, r\}$. His goal is to avoid a monochromatic copy of some fixed predetermined graph F for as long as possible. We are looking for a threshold for the duration of play, that is an $N_0 = N_0(F, r, n)$ such that if the number of moves is $N \ll N_0$, then the player has a strategy for avoiding a monochromatic copy of F, whereas if $N \gg N_0$, then no such strategy exists. Does the threshold N_0 exist (this seems plausible, but is only known for $r = 1$)? When F is a tree, one can use a simple greedy strategy which gives $N_0 \geq n^{1-\frac{1}{e_r(F)}}$, where $e_r(F) = r(e_F - 1) + 1$ if F is a star and $e_r(F) = \frac{k^r-1}{k-1}(e_F - 1) + 1$, where k is the size of a minimum vertex cover of F, otherwise. Is this bound tight? That is, is it a threshold? This is true for some special cases.

See the abstract of the talk by Reto Spöhel for relevant information and open problems.

Martin Marciniszyn: Consider the following variation of the previous game. There are two rounds. In the first round, the player colors the edges of $G(n, p)$ in an off-line fashion. In the second round he colors on-line N additional random edges. It is known that if $p = \Omega(n^{-1/2})$ and $N = \omega(1)$, then the player loses. Is it true that, for every $0 \leq \alpha \leq 1/6$, if $p = \Omega(n^{-1/2-\alpha})$ and $N = \omega(n^{8\alpha})$, then the player loses? Note that if $p = cn^{-1/2-\alpha}$ for some constant $c > 0$, and $N = o(n^{8\alpha})$, then the player has a winning strategy.

Hal Kirstead:

(1) Let G be a graph. Alice and Bob take turns coloring a vertex of G from a pool of k colors, such that the obtained partial coloring is proper. Alice wins if by the end of the game, every vertex of G is colored; otherwise Bob wins. The minimal k for which Alice has a winning strategy is called the game chromatic number of G, and is denoted by $\chi_g(G)$. When G is planar, it is known that $11 \leq \chi_g(G) \leq 17$. What is the exact value?

(2) Consider the following on-line game. Builder chooses some vertex set and then, in every turn, he picks some edge such that the graph consisting of his chosen edges is planar. Painter immediately colors this edge either red or blue. Painter wins the game iff he is able to avoid a monochromatic copy of some fixed predetermined graph H. Is it true that Builder wins the game iff H is outer-planar? It is known that Builder wins the game for $H = K_4 \setminus e$. The case $H = K_4$ is open.

József Beck: Maker and Breaker take turns claiming edges of K_∞. Maker wins as soon as he claims all edges of some copy of K_n. Let $RM(n)$ denote the minimal number of moves Maker needs in order to win. It is known that $2^{n/2} \leq RM(n) \leq c2^n$ for an appropriate constant c. Can we close or reduce the gap between the lower and upper bounds?

Ohad Feldheim: (extension to the previous problem) Let G be a d-regular graph on n vertices. It is known that $RM(G) \leq 4^d n$. Moreover, there are graphs G for which $RM(G) \geq 2^{d/2}n$. Is it true that there exists a constant $c > 1$ such that $RM(G) \geq c^d n$ for every d-regular n-vertex graph G?

Miloš Stojaković: Pairs of edges of the complete graph on n vertices arrive in a random order. The player is required to immediately color one edge red and the other blue. His goal is to avoid a monochromatic fixed graph H for as long as possible. The threshold is known for C_ℓ, S_ℓ, P_2, P_3, P_4; however, generally it is not even known whether a threshold exists.

József Beck: Consider the following Maker-Breaker game, played on an $n \times n^2$ grid. Maker's goal is to get as much surplus in a row or column as possible. Trivially, Breaker keep Maker's surplus at most n by only blocking the rows. Can we do better? Is the correct answer $\Theta(\sqrt{n \log n})$?

András Pluhár: Two players (called Maker and Breaker) play a game $(Z, \mathcal{H})$ in two stages. In the first stage the players claim elements of Z alternately. This stage lasts exactly n rounds, where n is some predetermined positive integer. In the second stage (whose length is not bounded), the players "move" their marks alternately; that is, if Maker's marks are $X = \{x_1, \ldots, x_n\}$ and Breaker's marks are $Y = \{y_1, \ldots, y_n\}$, then, in every move, Maker can lift one of his marks $x_i \in X$ and replace it with some $z \in (Z \setminus (X \cup Y)) \cup \{x_i\}$; Breaker plays analogously. Maker wins if he claims all elements of some $A \in \mathcal{H}$, otherwise Breaker wins. It is easy to see that if Breaker has a pairing strategy for the **Maker-Breaker** game $(Z, \mathcal{H})$, then he has a winning strategy for the **recycled** game as well. What if we just know that Breaker has a winning strategy? What if his win is guaranteed by the Erdős-Selfridge Theorem? The answer is known in some special cases.

József Beck: In the *n-in-a-line Kaplansky game* two players take turns picking unclaimed elements of $\mathbb{Z}^2$. The first player to claim n points on some line which is opponent free is the winner. Is 4-in-a-line Kaplansky game a draw?

Reporter: Dan Hefetz

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