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Non-Classical Interacting Random Walks

Organised by
Francis Comets (Paris)
Martin Zerner (Tübingen)

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Abstract. The workshop focused on recent developments in the theory of random walks (RWs) in a broader sense. Among the models considered were RWs in random environment, RWs in random potential and random polymers, branching RWs, excited RWs, reinforced RWs, and trapped RWs.

Mathematics Subject Classification (2000): 60xx, 82xx.

Introduction by the Organisers

The workshop was organized by Francis Comets (Paris 7) and Martin Zerner (Tübingen). It was attended by fifty participants of about twenty different nationalities. Among the participants was a relatively large number of young researchers, some of which were supported by the European Union within the Marie Curie Conferences Programme and by the National Science Foundation.

The title of the workshop was *Non-Classical Interacting Random Walks*. The study of random walks (RWs) goes back to the beginnings of probability theory in the seventeenth century. Some of the first probabilists like Bernoulli and Pascal investigated the properties of coin tossing sequences and other simple games of chance, which nowadays are modeled by RWs. Such a RW takes steps at fixed time intervals, and at each step a random direction is chosen, which gives a chaotic trajectory. So in their most stringent definition, RWs come up as sums of independent and identically distributed random variables with real or often integer lattice values. A somewhat wider definition includes time-homogeneous Markov chains whose transition probabilities are in some way adapted to a given geometric-algebraic-combinatorial structure of the underlying state space. In this context, many tools from potential theory, graph theory, harmonic analysis and Fourier analysis are available.

The focus of this workshop was on a quite different area of current research on RWs. This is what the term "non-classical" in the title refers to. Some of the RWs, which are currently heavily investigated, are quite "irregular": Some are not Markovian, but are influenced by their own past. Others are Markovian, but interact with a possibly random environment, which influences the transitions of the walk. Many of the tools used in the more traditional settings of RWs fail in this context.

The main models considered in this workshop were RWs in random environments (RWRE) with talks mainly on Monday, RWs in random potential and random polymers on Tuesday, excited RWs on Wednesday morning, followed by an exciting open problem session in the evening, reinforced RWs on Thursday and various other RWs on Friday. The programme consisted in total of 23 talks of about 50 minutes each and the open problem session. On most days a long lunch break from 12:30 to 16:00 gave plenty of opportunity for interaction between the participants. Following the reinforcement by previous workshops and in spite of the title of the current workshop, on Wednesday afternoon most participants took part in the classical interacting non-random walk to St. Roman.

Workshop: Non-Classical Interacting Random Walks

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Abstracts

Quenched Invariance Principle for ballistic RWRE in $d \geq 4$ dimensions

Ofer Zeitouni
(joint work with Noam Berger)

We prove that every nearest neighbor random walk in i.i.d. environment in dimension greater than or equal to 4 that has an almost sure positive speed in a certain direction, an annealed invariance principle and some mild integrability condition for regeneration times also satisfies a quenched invariance principle. The argument is based on intersection estimates and a theorem of Bolthausen and Sznitman from [2].

Let $d \geq 1$. A Random Walk in Random Environment (RWRE) on $\mathbb{Z}^d$ is defined as follows. Let $\mathcal{M}^d$ denote the space of all probability measures on $\mathcal{E}_d = \{\pm e_i\}_{i=1}^d$ and let $\Omega = \left(\mathcal{M}^d\right)^{\mathbb{Z}^d}$. An *environment* is a point $\omega = \{\omega(x,e)\}_{x\in\mathbb{Z}^d,\, e\in\mathcal{E}_d} \in \Omega$. Let $P = Q^{\mathbb{Z}^d}$ be an i.i.d. probability measure on Ω, with Q a *uniformly elliptic* law such that there exists a $\kappa > 0$ such that for every $e \in \mathcal{E}_d$,

$$Q(\{\omega(0,\cdot) : \omega(0,e) < \kappa\}) = 0.$$

For an environment $\omega \in \Omega$, the *Random Walk* on ω is a time-homogenous Markov chain with transition kernel

$$P_\omega\left(X_{n+1} = x + e \middle| X_n = x\right) = \omega(x,e).$$

The **quenched law** P_ω^x is defined to be the law on $\left(\mathbb{Z}^d\right)^{\mathbb{N}}$ induced by the transition kernel P_ω and $P_\omega^x(X_0 = x) = 1$. Let $\mathcal{P}^x = P \otimes P_\omega^x$ be the joint law of the environment and the walk, and the **annealed** law is defined to be its marginal

$$\mathbb{P}^x(\cdot) = \int_\Omega P_\omega^x(\cdot)\,dP(\omega).$$

We say that the RWRE $\{X(n)\}_{n\geq 0}$ satisfies the law of large numbers with deterministic speed v if $X_n/n \to v$, $\mathbb{P}$-a.s. We say in addition that it satisfies the *annealed* invariance principle with deterministic variance $\sigma^2 > 0$ if the processes

$$B^n(t) = \frac{X([nt]) - [nvt]}{\sqrt{n}}\,, t \geq 0 \tag{1}$$

converge in distribution as $n \to \infty$, under the measure $\mathbb{P}$, to a Brownian motion of variance σ^2. We say the process $\{X(n)\}_{n\geq 0}$ satisfies the *quenched* invariance principle with variance σ^2 if for P-a.e. ω, the above convergence holds under the measure P_ω^0.

Recall the regeneration structure for random walk in i.i.d. environment, developed by Sznitman and Zerner in [3]. We say that t is a regeneration time (in direction e_1) for $\{X(\cdot)\}$ if

$$\langle X(s), e_1\rangle < \langle X(t), e_1\rangle \text{ whenever } s < t$$

and

$$\langle X(s), e_1 \rangle \geq \langle X(t), e_1 \rangle \text{ whenever } s > t\,.$$

When ω is distributed according to an i.i.d. P such that the process

$$\{\langle X(n), e_1 \rangle\}_{n \geq 0}$$

is $\mathbb{P}$-almost surely transient to $+\infty$, it holds by [3] that, $\mathbb{P}$-almost surely, there exist infinitely many regeneration times for $\{X(\cdot)\}$. Let

$$t^1 < t^2 < \ldots\,,$$

be all of the regeneration times for $\{X(\cdot)\}$.

Theorem: *Let $d \geq 4$ and let $Q \in \mathcal{M}^d$ be a uniformly elliptic distribution. Set $P = Q^{\mathbb{Z}^d}$. Assume that the random walk $\{X(n)\}_{n \geq 0}$ satisfies the law of large numbers with a positive speed in the direction e_1, that is*

$$\lim_{n \to \infty} \frac{X(n)}{n} = v\,, \mathbb{P} - a.s \quad \text{with } v \text{ deterministic such that } \langle v, e_1 \rangle > 0\,. \tag{2}$$

Assume further that the process $\{X(n)\}_{n \geq 0}$ satisfies an annealed invariance principle with variance σ^2.

Assume that there exists an $\epsilon > 0$ such that $E(t^1)^\epsilon < \infty$ and, with $r = 1 + \epsilon$,

$$E[(t^2 - t^1)^r] < \infty\,. \tag{3}$$

If $d = 4$, assume further that (3) holds with $r > 8$. Then, the process $\{X(\cdot)\}$ satisfies a quenched invariance principle with variance σ^2.

For $d = 1$, the conclusion of the Theorem does not hold, and a quenched invariance principle, or even a CLT, requires a different centering.

The proof of the Theorem is available in [1].

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Subdiffusivity of the quenched mean and the quenched central limit theorem for random walk in random environment

Firas Rassoul-Agha

(joint work with Timo Seppäläinen)

We first sketch a strategy for proving the quenched invariance principle for random walk in random environment when one can produce a "suitable" invariant measure (as seen from the particle). We then show how this strategy gives a necessary and sufficient condition in the case of a space-time environment. This leads to the necessary regularity condition when more general environments are considered. The second part of the talk discusses the situation when this regularity condition fails to hold, in which case the quenched central theorem degenerates into two central limit theorems, one for the walk with random centering and one for the random centering itself.

Here is a more precise description of the model and results. The walk lives on the integer lattice $\mathbb{Z}^d$. An environment ω is a configuration of probability vectors $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \Omega$ on $\mathbb{Z}^d$. The natural transition group $\{T_z\}$ is defined by $(T_z\omega)_x = \omega_{x+z}$. On Ω we put a probability measure $\mathbb{P}$. Transition probabilities of the walk are given by $\pi_{x,y}(\omega) = \omega_{x,y-x}$. Under environment ω and with initial state $z \in \mathbb{Z}^d$, the random walk $(X_n)_{n\geq 0}$ is the Markov chain on $\mathbb{Z}^d$ whose path measure P_z^ω satisfies

$$P_z^\omega\{X_0 = z\} = 1 \quad \text{and} \quad P_z^\omega\{X_{n+1} = y|X_n = x\} = \pi_{x,y}(\omega).$$

The averaged distribution P_z of the walk is obtained by integrating out the environment: $P_z\{\cdot\} = \int P_z^\omega\{\cdot\}\,\mathbb{P}(d\omega)$. In general P_z is no longer Markovian. However, the crucial observation is that the process $(T_{X_n}\omega)$ is a Markov chain with transition kernel

$$\Pi f(\omega) = \sum_z \pi_{0z}(\omega) f(T_z\omega).$$

When $\mathbb{P}$ is ergodic for this Markov process, one has the following theorem.

Theorem 1. [1] *If $\mathbb{P}$ is a probability measure that is ergodic for the Markov process with kernel Π, $E_0|X_1|^2 < \infty$, and there exists an $\eta > 0$ such that*

$$\mathbb{E}|E_0^\omega[X_n] - E_0[X_n]|^2 = O(n^{1-\eta}),$$

then the processes $B_n(t) = \frac{1}{n}\{X_{[nt]} - ntv\}$ and $\widetilde{B}_n(t) = \frac{1}{n}\{X_{[nt]} - E_0^\omega[X_{[nt]}]\}$ converge weakly to the same Brownian motion (with a deterministic diffusion matrix) under the quenched measure P_0^ω, for $\mathbb{P}$-a.e. environment ω.

When $\mathbb{P}$ is not ergodic for $(T_{X_n}\omega)$, a possible strategy to prove the quenched invariance principle is to find a probability measure $\mathbb{P}_\infty$ that would satisfy the conditions of Theorem 1 and if $\mathbb{P}_\infty$ is "close" to $\mathbb{P}$, then one transfers the a.s. result back to $\mathbb{P}$.

Applying this strategy one can prove the following theorem about space-time random environments.

Theorem 2. [1] *Let e (the "time direction") be an element of the canonical basis of $\mathbb{R}^d$, $d \geq 2$. If $\mathbb{P}$ is a product probability measure on Ω and $\mathbb{P}\{X_1 \cdot e = 1\} = 1$, then the process $B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\}$ converges weakly to a Brownian motion (with deterministic diffusion matrix) under the quenched measure P_0^ω, for $\mathbb{P}$-a.e. environment ω, if and only if $E_0|X_1|^2 < \infty$ and $\mathbb{P}\{\exists z : \pi_{0z} = 1\} < 1$. The same holds for the process $\widetilde{B}_n(t) = n^{-1/2}\{X_{[nt]} - E_0^\omega[X_{[nt]}]\}$.*

The diffusion matrix in Theorem 2 is, of course, the same as that of the averaged invariance principle, which is nothing but Donsker's invariance principle.

When $\mathbb{P}\{X_1 \cdot e_1 = 1\} < 1$, the above regularity condition becomes: the walk must not live in a one-dimensional subspace and $\mathbb{P}\{\exists z : \pi_{0,0} + \pi_{0,z} = 1\} < 1$.

If this regularity condition is violated the quenched central limit theorem degenerates into two. We do not cover the case $d = 1$ and, for convenience, we assume bounded steps. $\mathbb{P}$ is still a product measure.

Theorem 3. [2] *Assume $\mathbb{P}$ is product, there exists a deterministic constant M such that $\mathbb{P}\{\pi_{0z} = 0\} = 1$ if $|z| > M$, and $\mathbb{P}\{\exists z : \pi_{0,0} + \pi_{0,z} = 1\} = 1$. Then*

a) *There exists a deterministic vector v such that $P_0\{n^{-1}X_n \to v\} = 1$. Moreover, the process $B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\}$ converges weakly to a Brownian motion under the averaged measure P_0.*
b) *The process $\widetilde{B}_n(t) = n^{-1/2}\{X_{[nt]} - E_0^\omega[X_{[nt]}]\}$ converges weakly to a Brownian motion (with deterministic diffusion matrix) under the quenched measure P_0^ω, for $\mathbb{P}$-a.e. environment ω.*
c) *The process $n^{-1/2}\{E_0^\omega[X_{[nt]}] - ntv\}$ converges weakly to a Brownian motion (with deterministic diffusion matrix) under the measure $\mathbb{P}$.*
d) *If $\mathbb{P}\{E_0^\omega[X_1] = v\} < 1$, then the process $B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\}$ is not tight under the quenched measure P_0^ω, for $\mathbb{P}$-a.e. environment ω.*

One also has explicit formulæ for the diffusion matrices in Theorem 3; see [2].

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Almost sure functional central limit theorem for ballistic random walk in random environment

Timo Seppäläinen
(joint work with Firas Rassoul-Agha)

We consider a multidimensional random walk in a product random environment with bounded steps, transience in some spatial direction, and high enough moments on the regeneration time. We prove an invariance principle, or functional central limit theorem, under almost every environment for the diffusively scaled centered walk. The main point behind the invariance principle is that the quenched mean of the walk behaves subdiffusively.

Here is a more precise description of the model and the result. The walk lives on the integer lattice $\mathbb{Z}^d$ in dimensions $d \geq 2$. An environment ω is a configuration of probability vectors $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \Omega$ on $\mathbb{Z}^d$. On Ω we put an i.i.d. product measure $\mathbb{P}$. Transition probabilities of the walk are given by $\pi_{x,y}(\omega) = \omega_{x,y-x}$.

Under environment ω and with initial state $z \in \mathbb{Z}^d$, the random walk $(X_n)_{n\geq 0}$ is the Markov chain on $\mathbb{Z}^d$ whose path measure P_z^ω satisfies

$$P_z^\omega\{X_0 = z\} = 1 \quad \text{and} \quad P_z^\omega\{X_{n+1} = y | X_n = x\} = \pi_{x,y}(\omega).$$

The averaged distribution P_z of the walk is obtained by integrating out the environment: $P_z(\cdot) = \int P_z^\omega(\cdot)\,\mathbb{P}(d\omega)$.

We need the following specific assumptions for the theorem. We assume directional transience: namely, there exists a vector $\hat{u} \in \mathbb{Z}^d$ such that

$$P_0\{X_n \cdot \hat{u} \to \infty\} = 1.$$

Define the first regeneration time as the first time τ_1 at which

$$\sup_{n<\tau_1} X_n \cdot \hat{u} < X_{\tau_1} \cdot \hat{u} = \inf_{n\geq\tau_1} X_n \cdot \hat{u}.$$

Our main assumption is a moment bound on τ_1: $E_0(\tau_1^p) < \infty$ for some $p > 176d$. Some minimal regularity on the environments is necessary: the walk must not live in a one-dimensional subspace, and $\mathbb{P}\{\exists z : \pi_{0,0} + \pi_{0,z} = 1\} < 1$. For convenience we assume also bounded steps.

Under these assumption a limiting velocity $v = \lim n^{-1} X_n$ exists and we have the following result on fluctuations.

Theorem. *The centered and diffusively scaled process $B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\}$ converges weakly to a Brownian motion under the quenched measure P_0^ω, for $\mathbb{P}$-a.e. environment ω. The diffusion matrix of the limiting Brownian motion is independent of ω and nondegenerate in some spatial directions.*

Quenched Limits for Transient, Zero Speed One-Dimensional Random Walk in Random Environment

Jonathon Peterson

(joint work with Ofer Zeitouni)

We consider the model of a random walk in random environment (RWRE) in one dimension. An *environment* is an element $\omega = (\omega_x)_{x\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$. Let P be a distribution on the space of environments. In this talk we assume that P is a product measure (i.i.d. environments).

The *quenched* law P_ω^x for a random walk X_n in the environment ω is defined by

$$P_\omega^x(X_0 = x) = 1, \quad \text{and} \quad P_\omega^x\left(X_{n+1} = j | X_n = i\right) = \begin{cases} \omega_i & \text{if } j = i+1, \\ 1-\omega_i & \text{if } j = i-1. \end{cases}$$

Expectations under the law P_ω^x are denoted E_ω^x.

The *annealed* law for the random walk in random environment X_n is defined by averaging the quenched law over all environments:

$$\mathbb{P}^x(\cdot) = \int P_\omega^x(\cdot) P(d\omega).$$

Expectations under the law $\mathbb{P}$ will be written $\mathbb{E}$.
We assume the following conditions on the law of the environment:
Assumption 1. P is a product measure on Ω such that

$$E_P \log \rho < 0 \quad \text{and} \quad E_P \rho^s = 1 \text{ for some } s \in (0,1).$$

Assumption 2. There exists $\rho_{\max} < \infty$ such that $P(\rho < \rho_{\max}) = 1$, and the distribution of $\log \rho$ is non-lattice under P. Due to Assumption 1 and the well known results of Solomon [6], the random walk X_n is transient to the right but with zero speed (i.e. $\lim_{n\to\infty} \frac{X_n}{n} = 0$). Assumption 2 contains technical assumptions needed for our proofs.
Under Assumptions 1 and 2 Kesten, Kozlov, and Spitzer analyzed the annealed law of X_n. They derived the limiting distributions for the walk by first establishing a stable limit law of index s for the hitting times $T_n := \min\{t \geq 0 : X_t = n\}$. In particular, they showed that when $s < 1$ there exists a $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/s}} \leq x\right) = L_{s,b}(x)\,,$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^s} \leq x\right) = 1 - L_{s,b}(x^{-1/s}), \tag{1}$$

where $L_{s,b}$ is the distribution function for a stable random variable with characteristic function

$$\hat{L}_{s,b}(t) = \exp\left\{-b|t|^s\left(1 - i\frac{t}{|t|}\tan(\pi s/2)\right)\right\}.$$

We study the quenched limiting distributions of the random walk and show that the behavior is very different from the annealed behavior. In particular, the stable behavior seen in the annealed results comes from fluctuations in the environment and not the random walk. In particular we prove that $n^{-1/s} E_\omega T_n$ is approximately a stable random variable. We then use this to prove the following two theorems which show that $P-a.s.$ there exist two different random sequences of times (depending on the environment) where the random walk has different limiting behavior. These are the main results of the talk.
Theorem 1. Let Assumptions 1 and 2 hold, and let $s < 1$. Then P-a.s. there exist random subsequences $t_m = t_m(\omega)$ and $u_m = u_m(\omega)$, such that for any $\delta > 0$,

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m} - u_m}{(\log t_m)^2} \in [-\delta, \delta]\right) = 1.$$

Theorem 2. Let Assumptions 1 and 2 hold, and let $s < 1$. Then P-a.s. there exists a random subsequence $n_{k_m} = n_{k_m}(\omega)$ of $n_k = 2^{2^k}$ and a random sequence $t_m = t_m(\omega)$, such that

$$\lim_{m\to\infty} \frac{\log t_m}{\log n_{k_m}} = \frac{1}{s},$$

and

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m}}{n_{k_m}} \le x\right) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x < \infty \end{cases}.$$

Note that Theorems 1 and 2 preclude the possiblity of a quenched analogue of the annealed statement (1).

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On the asymptotic velocity of diffusions in random environment

Laurent Goergen

Within the rich field of random motions in random media, my interest is directed towards diffusions in random environment which are closely related to the discrete random walks in random environment. Although much progress has been made in the last decade, see [12] for a concise overview, the multidimensional setting remains poorly understood. Let alone a few specific situations where for instance the drift is divergence-free or the gradient of a stationary potential, the lack of knowledge about an invariant measure for the so called process of the environment viewed from the particle prevents us from using the highly developed methods from homogenization theory, see for instance [6], [5]. Instead, we build up on more recent techniques that proved successful in the discrete setting of random walks in random environment such as the renewal-type arguments introduced by Sznitman and Zerner, see [16] and also [9] and the so called condition (T) which guarantees the existence of a non-vanishing limit velocity (ballistic behaviour), see [13], [7]. In particular, we obtain in the general framework of multidimensional diffusions in random environment, see below, the existence of a limiting velocity taking at most two values as well as certain zero-one laws, see [2]. Moreover, we prove an effective criterion for ballistic behaviour in the spirit of [14] and use it to construct new examples of ballistic diffusions, see [3]. Before we present our results in greater detail and discuss possible future developments and open questions, let us give the

Definition of the model. The set of random environments is a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ endowed with a jointly measurable group of transformations $(t_x)_{x \in \mathbb{R}^d}$ preserving the probability measure $\mathbb{P}$. The diffusion in the random environment $\omega \in \Omega$ with starting point $x \in \mathbb{R}^d$, whose law is denoted by $P_{x,\omega}$, is then the solution of the following stochastic differential equation:

$$\begin{aligned} dX_t &= \sigma(X_t, \omega) d\beta_t + b(X_t, \omega) dt, \\ X_0 &= x, \ P_{x,\omega}\text{-a.s.}, \end{aligned} \tag{1}$$

where β_t is a d-dimensional Brownian motion. The local characteristics, i.e. the diffusion matrix $\sigma(\cdot, \cdot)$ and the drift $b(\cdot, \cdot)$ are jointly measurable functions on $\mathbb{R}^d \times \Omega$. We assume that they are stationary, bounded and Lipschitz-continuous. Moreover we require σ to be uniformly elliptic, in the sense that for some $\nu > 0$, $\nu |x|^2 \leq x \cdot \sigma(x, \omega) x$, for all $\omega \in \Omega, x \in \mathbb{R}^d$. Finally we impose a finite range dependence condition: there exists some $R > 0$, such that whenever two sets $A, B \subset \mathbb{R}^d$ lie at a mutual distance at least $R > 0$, then the σ-fields generated by the restriction of $\sigma(\cdot, \cdot)$ and $b(\cdot, \cdot)$ to A respectively B are independent under $\mathbb{P}$. Since we are interested in the typical macroscopic behaviour of the diffusion, we introduce the annealed measures $P_x = \mathbb{P} \times P_{x,\omega}$. While they restore some stationarity to the model, they typically destroy the Markov property, which constitutes one of the challenges of the model.

Recent results. In [2], we show the existence of a limiting velocity as well as certain zero-one laws for the annealed process. For any unit vector $\ell \in \mathbb{R}^d$, let A_ℓ denote the event that the diffusion escapes to infinity in direction ℓ. We then obtain for dimension $d \geq 1$, the weak zero-one law $P_0(A_\ell \cup A_{-\ell}) \in \{0, 1\}$ and the existence of a deterministic unit vector ℓ_* and two deterministic numbers $v_+, v_- \geq 0$ such that

$$\lim_{t \to \infty} \frac{X_t}{t} = \left(v_+ 1_{A_{\ell_*}} - v_- 1_{A_{\ell_*}} \right) \ell_*, \ P_0\text{-a.s.} \tag{2}$$

When $d = 2$, we prove in addition the stronger zero-one law $P_0(A_\ell) \in \{0, 1\}$, for all unit vectors $\ell \in \mathbb{R}^2$, by following a similar strategy as in [17]. Together with (2), we therefore obtain a strong law of large numbers when $d = 2$. Although [1] constitutes in the discrete setting a recent step into the direction of a general zero-one law or law of large numbers, their validity for $d \geq 3$ remains a challenging open question.

To prove (2), we show the existence of a limiting velocity for the projections $X_i \cdot e_i, i = 1, \ldots, d$ on basis vectors e_i. In the one case in accordance with the weak zero-one law where $P_0(A_{e_i} \cup A_{-e_i}) = 0$, we follow a similar strategy as in [18] to show that the asymptotic speed of $X_t \cdot e_i$ vanishes. In the other possible case, that is $P_0(A_{e_i} \cup A_{-e_i}) = 1$, we use a certain sequence of regeneration times $\tau_k, k \geq 0$, introduced by Shen [9] in the spirit of [16], which yields a renewal structure for the diffusion under a certain extended annealed measure. We are also able to prove that $X_{\tau_1} \cdot e_i$ has a finite first moment, which is essential for exploiting the renewal structure.

In [3], we show that the so called condition (T') introduced by Sznitman in [13] and by Schmitz in [7] respectively in the discrete and continuous settings is equivalent to an effective condition that can be checked by inspection of the environment in a finite box. Condition (T') is today the most general condition for the existence of a deterministic non-vanishing limit velocity with Gaussian corrections under the annealed measure when $d \geq 2$, see for instance [8]. We now recall the definition of (T'). For any unit vector $\ell \in \mathbb{R}^d$ and any $u \in \mathbb{R}$, let T_u^ℓ and $\tilde{T}_u^\ell$ denote the stopping times where $X_t \cdot \ell$ first goes above respectively below u. We say that condition $(T_\gamma) \mid \ell$ holds if for all unit vectors ℓ' in a neighbourhood of ℓ and for all $b > 0$,

$$\limsup_{L\to\infty} L^{-\gamma} \log P_0 \left[\tilde{T}^{\ell'}_{-bL} < T^{\ell'}_L\right] < 0. \tag{3}$$

Condition (T') relative to ℓ is then the requirement that $(T_\gamma) \mid \ell$ holds for all $\gamma \in (0,1)$. Because of its asymptotic nature (and the difficulty of the exit problem under a non-Markovian measure contained in (3)), basically all examples of diffusions in random environment satisfying (T') originated from a stronger, more handy condition going back to Kalikow, see [4]. On the other hand, the effective condition is essentially the requirement that a certain quantity linked to the exit problem from *some finite* box is smaller than a deterministic polynomial in the size of that box. For an exact definition, we refer to [3], (1.14) therein. With the help of this criterion we are then indeed able to provide new examples of ballistic diffusions that come as special perturbations of Brownian motion when $d \geq 4$.

Moreover, the effective criterion enables us to show that (T') is equivalent to (T_γ) when $\gamma \in (\frac{1}{2}, 1)$. We believe that an improved version of the effective condition should yield the equivalence between all the conditions $(T_\gamma), \gamma \in (0,1]$. Such a version should also allow us to relax the assumptions on the above mentioned perturbation of Brownian motion in view of further examples of ballistic diffusions.

We already pointed out that (T') implies ballistic behaviour when $d \geq 2$. In addition, it is conjectured that the converse statement also holds. If this is the case, the effective criterion would become the multidimensional counterpart of Solomon's characterization of ballistic behaviour in one dimension, see [11]. On the other hand, when $d = 1$, it is known that (T') is equivalent to transience and that it does not imply ballistic behaviour since one can construct examples that are transient with vanishing velocity. In this context, it is an interesting question whether such examples also exist in higher dimensions.

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Central Limit Theorem for Branching Random Walk in Random Environment

Nobuo Yoshida

Let $p(\cdot,\ \cdot)$ be a transition probability for a Markov chain with a countable state space Γ. We write $\mathbb{N}=\{0,1,2,...\}$, $\mathbb{N}^*=\{1,2,...\}$ and $\mathbb{Z}=\{\pm x\ ;\ x\in\mathbb{N}\}$ in the sequel. To each $(t,x)\in\mathbb{N}\times\Gamma$, we associate a distribution

$$q_{t,x}=(q_{t,x}(k))_{k\in\mathbb{N}}\in[0,1]^{\mathbb{N}},\quad \sum_{k\in\mathbb{N}}q_{t,x}(k)=1$$

on $\mathbb{N}$. Then, the branching random walk (BRW) with offspring distribution $q=(q_{t,x})_{(t,x)\in\mathbb{N}\times\Gamma}$ is described as the following dynamics:

- At time $t=0$, there is one particle at the origin $x=0$.
- Suppose that there are $N_{t,x}$ particles at each site $x\in\Gamma$ at time t. At time $t+1$, the ν-th particle at a site x $(\nu=1,..,N_{t,x})$ jumps to a site $y=X^{\nu}_{t,x}$ with probability $p(x,y)$ independently of each other. At arrival, it dies, leaving $K^{\nu}_{t,x}$ new particles there.

We formulate the above description more precisely. The following formulaiton is analogue of [2, section 4.2], where non-random offspring distributions are considered. See also [1, section 5] for the random offspring case.

• *Spatial motion:* A particle at time-space location (t, x) is supposed to jump to some other $(t + 1, y)$ to be replaced by its children there. Therefore, the spactial motion should be described by assignning the destination of the each particle at each time-space location (t, x). So, we are guided to the following definition. We define the measurable space $(\Omega_X, \mathcal{F}_X)$ as the set $\Gamma^{\mathbb{N}\times\Gamma\times\mathbb{N}^*}$ with the product σ-field, and $\Omega_X \ni X \mapsto X^\nu_{t,x}$ for each $(t, x, \nu) \in \Gamma \times \mathbb{N} \times \mathbb{N}^*$ as the projection. We define $P_X \in \mathcal{P}(\Omega_X, \mathcal{F}_X)$ as the product measure such that

$$(1) \qquad P_X(X^\nu_{t,x} = y) = p(x, y) \quad \text{for all } (t, x, \nu) \in \mathbb{N} \times \Gamma \times \mathbb{N}^* \text{ and } y \in \Gamma.$$

Here, we interpret $X^\nu_{t,x}$ as the position at time $t + 1$ of the children born from the ν-th particle at time-space location (t, x).

• *Offspring distribution:* We set $\Omega_q = \mathcal{P}(\mathbb{N})^{\mathbb{N}\times\Gamma}$, where $\mathcal{P}(\mathbb{N})$ denotes the set of probability measures on $\mathbb{N}$:

$$\mathcal{P}(\mathbb{N}) = \{q = (q(k))_{k\in\mathbb{N}} \in [0, 1]^{\mathbb{N}} \ ; \ \sum_{k\in\mathbb{N}} q(k) = 1\}.$$

Thus, each $q \in \Omega_q$ is a function $(t, x) \mapsto q_{t,x} = (q_{t,x}(k))_{k\in\mathbb{N}}$ from $\mathbb{N} \times \Gamma$ to $\mathcal{P}(\mathbb{N})$. $q_{t,x}$ is interpretted as the offspring distribution for each particle which occupies the time-space location (t, x). The set $\mathcal{P}(\mathbb{N})$ is equipped with the natural Borel σ-field induced from that of $[0, 1]^{\mathbb{N}}$. We denote by $\mathcal{F}_q$ the product σ-field on Ω_q.

We define the measurable space $(\Omega_K, \mathcal{F}_K)$ as the set $\mathbb{N}^{\mathbb{N}\times\Gamma\times\mathbb{N}^*}$ with the product σ-field, and $\Omega_K \ni K \mapsto K^\nu_{t,x}$ for each $(t, x, \nu) \in \mathbb{N} \times \Gamma \times \mathbb{N}^*$ as the projection. For each fixed $q \in \Omega_q$, we define $P^q_K \in \mathcal{P}(\Omega_K, \mathcal{F}_K)$ as the product measure such that

$$(2) \qquad P^q_K(K^\nu_{t,x} = k) = q_{t,x}(k) \quad \text{for all } (x, t, \nu) \in \Gamma \times \mathbb{N} \times \mathbb{N}^* \text{ and } k \in \mathbb{N}.$$

Hence, $K^\nu_{t,x}$ is interpretted as the number of the children born from the ν-th particle at time-space location (t, x).

We now define the branching random walk in random environment. We fix a product measure $Q \in \mathcal{P}(\Omega_q, \mathcal{F}_q)$, which describes the i.i.d. offspring distribution assigned to each time-space location. Finally, we define $(\Omega, \mathcal{F})$ by

$$\Omega = \Omega_X \times \Omega_K \times \Omega_q, \quad \mathcal{F} = \mathcal{F}_X \otimes \mathcal{F}_K \otimes \mathcal{F}_q,$$

and $P^q, P \in \mathcal{P}(\Omega, \mathcal{F})$ by

$$P^q = P_X \otimes P^q_K \otimes \delta_q, \quad P = \int Q(dq)P^q.$$

We denote by $N_{t,x}$ the population at time-space location $(t, x) \in \mathbb{N} \times \Gamma$, which is defined inductively by $N_{0,x} = \delta_{0,x}$ for $t = 0$, and

$$(3) \qquad N_{t,x} = \sum_{y\in\Gamma} \sum_{\nu=1}^{N_{t-1,y}} \delta_x(X^\nu_{t-1,y}) K^\nu_{t-1,y}$$

for $t \geq 1$. The total population at time t is then given by

$$N_t = \sum_{x \in \Gamma} N_{t,x} = \sum_{y \in \Gamma} \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^{\nu}. \tag{4}$$

We write

$$m = Q[m_{t,x}] \text{ with } m_{t,x} = \sum_{k \in \mathbb{N}} k q_{t,x}(k). \tag{5}$$

More generally, for $p > 0$,

$$m^{(p)} = Q[m_{t,x}^{(p)}] \text{ with } m_{t,x}^{(p)} = \sum_{k \in \mathbb{N}} k^p q_{t,x}(k) \tag{6}$$

We set

$$\overline{N}_{t,x} = N_{t,x}/m^t \text{ and } \overline{N}_t = N_t/m^t. \tag{7}$$

$\overline{N}_t = N_t/m^t$ is a martingale, and therefore the following limit always exists:

$$\overline{N}_\infty = \lim_t \overline{N}_t, \quad Q\text{-a.s.} \tag{8}$$

Before we state our result, we fix our notation for simple random walk.

• *The random walk:* $(\{S_t\}_{t \in \mathbb{N}}, P_S^x)$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$. More precisely, we let $(\Omega_S, \mathcal{F}_S)$ be the path space $(\mathbb{Z}^d)^{\mathbb{N}}$ with the cylindrical σ-field, and $\Omega_S \ni S \mapsto S_t$, $t \in \mathbb{N}$ be the projection. We define $p : \mathbb{Z}^d \times \mathbb{Z}^d \mapsto \{0, \frac{1}{2d}\}$ by

$$p(x,y) = \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1, \\ 0 & \text{if } |x-y| \neq 1, \end{cases} \tag{9}$$

where $|x| = (|x_1|^2 + .. + |x_d|^2)^{1/2}$ for $x \in \mathbb{Z}^d$. We consider the unique probability measure P_S^x on $(\Omega_S, \mathcal{F}_S)$ such that $S_t - S_{t-1}$, $t = 1, 2, ..$ are independent and

$$P_S^x\{S_0 = x\} = 1, \quad P_S^x\{S_t - S_{t-1} = y\} = p(0,y), \quad \text{for } y \in \mathbb{Z}^d.$$

In the sequel, $P_S^x[X]$ denotes the P_S^x-expectation of a r.v.(random variable) X on $(\Omega_S, \mathcal{F}_S, P_S^x)$,and P_S^0 will be simply written by P_S. We define

$$\pi_d = P_S(S_t = 0 \text{ for some } t \geq 1). \tag{10}$$

As is well-known, $\pi_1 = \pi_2 = 1$, and $\pi_d < 1$ for $d \geq 3$.

To state the central limit theorem for the BRWRE, we assume that $\Gamma = \mathbb{Z}^d$ and that $p(\cdot, \cdot)$ is given by (9).

Theorem Suppose that

$$m > 1, \quad m^{(2)} < \infty, \quad \text{and } d \geq 3. \tag{11}$$

Then, the following are equivalent:

(a): $\dfrac{Q[m_{t,x}^2]}{m^2} < \dfrac{1}{\pi_d}$, where $\pi_d \in (0,1)$ is defined by (10).

(b): $\lim_t \overline{N}_t = \overline{N}_\infty$ in $\mathbb{L}^2(P)$.

(c): $\lim_t \sum_{x\in\mathbb{Z}^d} \overline{N}_{t,x} f\left(t^{-1/2}x\right) = \overline{N}_\infty \int_{\mathbb{R}^d} f(x)\rho_1(x)dx$ in $\mathbb{L}^2(P)$
for all $f \in C_b(\mathbb{R}^d)$, where

$$\rho_t(x) = \left(\frac{d}{2\pi t}\right)^{d/2} e^{-\frac{d|x|^2}{2t}} \quad \text{for } t > 0. \tag{12}$$

Corollary Suppose that

$$m > 1, \;\; m^{(2)} < \infty, \;\; d \geq 3, \;\; \text{and} \;\; \frac{Q[m_{t,x}^2]}{m^2} < \frac{1}{\pi_d}.$$

Then, $P(\overline{N}_\infty > 0) > 0$ and

$$\lim_t P\left(\left| \frac{1}{N_t} \sum_{x\in\mathbb{Z}^d} N_{t,x} f\left(t^{-1/2}x\right) - \int_{\mathbb{R}^d} f(x)\rho_1(x)dx \right| \geq \varepsilon \middle| \overline{N}_\infty > 0 \right) = 0$$

for all $\varepsilon > 0$ and $f \in C_b(\mathbb{R}^d)$.

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Coincidence of Lyapunov exponents for random walks in weak random potentials

Markus Flury

Let $\mathcal{S} = (S(n))_{n\in\mathbb{N}_0}$ be a nearest-neighbor random walk on $\mathbb{Z}^d$, with start at the origin and drift h into the direction of the first axis, evolving under the influence of a random potential given by a family $\mathbb{V} = (V_x)_{x\in\mathbb{Z}^d}$ of nonnegative i.i.d. random variables, independent of the random walk itself and with $\mathbb{E}V_x^d < \infty$.

We distinguish between the so-called *quenched* setting, where the path measures depend on the concrete realization of the potential $\mathbb{V}$, and the *annealed* setting, where the measures depend on averaged values of the potential only. The *quenched path measures* are defined by means of the density functions

$$\frac{dQ^{\text{qu}}_{\mathbb{V},h,\beta,N}}{dP_h} \stackrel{\text{def}}{=} \frac{e^{-\beta\sum_{n=1}^N V_{S(n)}}}{Z_{\mathbb{V},h,\beta}(N)}, \quad N \in \mathbb{N},$$

$$Z_{\mathbb{V},h,\beta}(N) \stackrel{\text{def}}{=} E_h\left[e^{-\beta\sum_{n=1}^N V_{S(n)}}\right], \quad N \in \mathbb{N},$$

at what the *inverted temperature* $\beta \geq 0$ indicates the strength of the potential. The quenched setting defines a discrete-time model for a particle moving in a random media. Here, the path measure is random itself, the randomness coming from the random potential $\mathbb{V}$. Under a concrete realization of the path measure, the walker jumps from site to site, thereby trying to stay in regions where the potential takes on small values. The drift, however, implies a restriction in the search of such an "optimal strategy" by imposing a particular direction to the walk.

While the main interest lies in the quenched setting, the annealed model no longer depends on the realizations of the potential, and is thus easier to handle. The *annealed path measures* are defined by means of the density functions

$$\frac{dQ^{\text{an}}_{h,\beta,N}}{dP_h} \stackrel{\text{def}}{=} \frac{\mathbb{E}\, e^{-\beta\sum_{n=1}^{N} V_{S(n)}}}{\mathbb{E}\, Z_{\mathbb{V},h,\beta}(N)}\,, \quad N \in \mathbb{N}\,.$$

A walker under the annealed measure finds himself in a similar situation as in the quenched setting. To see this, we define

$$\varphi_\beta(t) \stackrel{\text{def}}{=} -\log \mathbb{E}\, \exp(-t\beta V_x)\,, \quad t \in \mathbb{R}^+.$$

A short calculation using the independence of the potential then reveals

$$\frac{dQ^{\text{an}}_{h,\beta,N}}{dP_h} = \frac{e^{-\sum_{x\in\mathbb{Z}^d}\varphi_\beta(\ell_{x,N})}}{\mathbb{E}\, Z_{h,\beta,\mathbb{V}}(N)}\,, \quad N \in \mathbb{N}\,,$$
$$\mathbb{E}\, Z_{h,\beta,\mathbb{V}}(N) = E_h\Big[e^{-\sum_{x\in\mathbb{Z}^d}\varphi_\beta(\ell_{x,N})}\Big]\,, \quad N \in \mathbb{N}\,,$$

at what $\ell_{x,N}$ denotes the number of visits to the site $x \in \mathbb{Z}^d$ by the random walk $\mathcal{S}$ up to time $N \in \mathbb{N}$. The concavity of φ_β implies that the probability is the smaller the more often the random walk intersects its own path. Therefore, on the one hand, it is convenient for the walker to return to places he already visited before, while, on the other hand, he is urged to proceed in the direction of the drift.
In a similar model, namely Brownian motion in Poissonian potential, the contrary influence of drift and potential on the long-time behavior of the walk was first studied by A.S. Sznitman. By means of the powerful method of enlargement of obstacles, he established a precise picture in both quenched and annealed settings (see Chapter 5 of [7]). Among his results there is an accurate description of a phase transition from *localization* for large β to *delocalization* for small β. In the delocalized phase, the random walk is *ballistic*, i.e. the displacement of $S(N)$ from the origin is of order $O(N)$, while in the localized phase, the walk behaves *sub-ballistic*, i.e. the displacement is of order $o(N)$. The analogous results in the discrete setting have been established by M.P.W. Zerner in [8] and M. Flury in [4].

The above results on the transition from sub-ballistic to ballistic behavior are based on large deviation principles for the random walks under the path measures, and on phase transitions for the *quenched*, respectively *annealed free energies*

$$\log Z_{\mathbb{V},h,\beta}(N) \quad \text{and} \quad \log \mathbb{E}\, Z_{\mathbb{V},h,\beta}(N)\,.$$

The free energies are important values for the study of the path measures. For a motivation in the context of random branching processes, and a thorough study of the one-dimensional case, we refer to [5] by A. Greven and F. den Hollander.

We are interested in the long-time behavior of the free energies, measured by the *quenched,* respectively *annealed Lyapunov exponents*

$$m^{\mathrm{qu}}(h,\beta) \stackrel{\mathrm{def}}{=} -\lim_{N\to\infty}\frac{1}{N}\log Z_{\mathbb{V},h,\beta}(N)\,,$$
$$m^{\mathrm{an}}(h,\beta) \stackrel{\mathrm{def}}{=} -\lim_{N\to\infty}\frac{1}{N}\log \mathbb{E}\, Z_{\mathbb{V},h,\beta}(N)\,,$$

for which the existence ($\mathbb{P}$-almost surely and in $\mathcal{L}_1(\mathbb{P})$ for the quenched setting) is derived by subadditive limit arguments. Our main result concerns the ballistic regime in high dimensions:

Theorem. *Suppose $d \geq 4$ and $h > 0$. Then there exists $\beta_0 > 0$, such that*

$$m^{\mathrm{qu}}(h,\beta) = m^{\mathrm{an}}(h,\beta)$$

for all $\beta \leq \beta_0$. Moreover, when V_x is bounded, then there exists $K < \infty$, such that

$$\mathbb{E}\,\big|\log Z_{\mathbb{V},h,\beta}(N) - \log \mathbb{E}\, Z_{\mathbb{V},h,\beta}(N)\big| \leq K\big(1+\beta\sqrt{N}\big)$$

for all $N \in \mathbb{N}$ and $\beta \leq \beta_0$.

The crucial result for the proof the theorem is the following: For $d \geq 4$ and $h > 0$, there exist $\beta_0 > 0$ and $K < \infty$, such that for all $N \in \mathbb{N}$ and $\beta \leq \beta_0$,

$$\mathbb{E}\left(Z_{\mathbb{V},h,\beta}(N)\right)^2 \leq K\left(\mathbb{E}\, Z_{\mathbb{V},h,\beta}(N)\right)^2.$$

In order to achieve a heuristic understanding of this result, we consider two independent copies $\mathcal{S}^1, \mathcal{S}^2$ of the walk $\mathcal{S}$, and we set

$$\Psi_{\beta,N} \stackrel{\mathrm{def}}{=} \sum\nolimits_{x\in\mathbb{Z}^d} \varphi_\beta^{\mathrm{an}}\big(\ell^1_{x,N}\big) - \varphi_\beta^{\mathrm{an}}\big(\ell^2_{x,N}\big) - \varphi_\beta^{\mathrm{an}}\big(\ell^1_{x,N}+\ell^2_{x,N}\big)\,, \quad N \in \mathbb{N}\,, \tag{1}$$

where $\ell^j_{x,N}$ with $j \in \{1,2\}$ denotes the number of visits to the site $x \in \mathbb{Z}^d$ by the random walk $\mathcal{S}^j$ up to time $N \in \mathbb{N}$. A simple calculation then shows that

$$\frac{\mathbb{E}\left(Z_{\mathbb{V},h,\beta}(N)\right)^2}{\left(\mathbb{E}\, Z_{\mathbb{V},h,\beta}(N)\right)^2} = E^{\mathrm{an}}_{h,\beta,N}\big[\exp\left(\Psi_{\beta,N}\right)\big]\,, \quad N \in \mathbb{N}\,, \tag{2}$$

where $E^{\mathrm{an}}_{h,\beta,N}$ stands for the expectation with respect to $Q^{\mathrm{an}}_{h,\beta,N} \otimes Q^{\mathrm{an}}_{h,\beta,N}$. Observe furthermore that the only non-vanishing summands in (1) are the ones associated to those $x \in \mathbb{Z}^d$, that are visited by both random walks up to time N. From the concavity of φ_β, we therefore obtain

$$\Psi_{\beta,N} \leq \varphi_\beta(1) \sum_{x\,\in\,\mathbb{R}^d:\ \ell^1_{x,N}>0,\ \ell^2_{x,N}>0} \big(\ell^1_{x,N}+\ell^2_{x,N}\big)\,, \quad N \in \mathbb{N}\,. \tag{3}$$

This finally gives us the following picture of the situation: In the ballistic regime, $\mathcal{S}^1$ and $\mathcal{S}^2$ under the annealed path measure obey the drift and evolve into the direction of the first axis. Thereby, one expects them to move away from each other in the $(d-1)$-dimensional "vertical" direction, as soon as the dimension of the lattice is large enough. The condition $d \geq 4$ appears to be the right one since the "vertical distance" then is transient. As a consequence, the paths of $\mathcal{S}^1$ and $\mathcal{S}^2$

are supposed to intersect only finitely many times. For β small enough, the right-hand side of (2) then should stay bounded because of (3) and $\lim_{\beta\downarrow 0}\varphi_\beta(1) = 0$.

Coincidence of Lyapunov exponents has been conjectured by A.S. Sznitman in [7]. It emerged from the fact that an analogous result is true for the much simpler case of directed polymers in random potentials. There, $(S(n))_{n\in\mathbb{N}}$ is replaced by $((\xi(n), n))_{n\in\mathbb{N}}$, where $(\xi(n))_{n\in\mathbb{N}}$ is a standard d-dimensional walk. The coincidence of quenched and annealed Lyapunov exponents for $d \geq 3$ and small disorder has first been proved by J. Imbrie and T. Spencer in [6] using cluster expansion techniques, and then by E. Bolthausen in [2] and S. Albeverio and X. Y. Zhou in [1] using martingale techniques. Martingale arguments are also used in the more recent work on directed polymers in [3] by F. Comets, T. Shiga and N. Yoshida.

The situation considered here is much more delicate and it seems not possible to implement martingale techniques. We therefore take recourse to different methods, mainly renewal techniques and arguments from Ornstein-Zernike theory.

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Strong localization for directed polymers in random environment

Vincent Vargas

Directed polymers in random environment is a model of statistical mechanics (introduced by Huse and Henley in [3]) in which stochastic processes interact with a random environment, depending on both time and space: one studies the path of the stochastic process under a random Gibbs measure depending on the temperature (as the temperature increases, the influence of the random environment decreases). More precisely, let $((\omega_n)_{n\in\mathbb{N}}, P)$ denote the simple random walk starting from 0 on $\mathbb{Z}^d$. The random environment on each lattice site is a sequence $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on a probability space $(H, \mathcal{G}, Q)$.

For any $n > 0$, we define the (Q-random) polymer measure μ_n on $(\Omega, \mathcal{F})$ by:

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp(\beta H_n(\omega)) P(d\omega)$$

where $\beta \in \mathbb{R}_+$ is the inverse temperature,

$$H_n(\omega) \stackrel{def.}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

is the hamiltonian and

$$Z_n = P(\exp(\beta H_n(\omega)))$$

is the partition function.

In this talk, we presented strong localization results for the endpoint measure $\mu_{n-1}(\omega_n \in .)$ (cf. [4]). These results are natural extensions of the favorite point localization theorem established in [1] and [2]. Roughly, these results assert that at "low temperature" the polymer measure is asymptotically concentrated at a few points of macroscopic mass (we call these points ϵ-atoms). Unfortunately, it remains an open problem to caracterize geometricaly these points where the polymer measure concentrates. In particular, in dimension $d = 1$, it is conjectured that:

$$\mu_n(|\ \omega_n\ |) \approx n^{2/3}.$$

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Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees

Yueyun Hu

(joint work with Zhan Shi)

1. Branching random walk and martingale convergence

We consider a branching random walk on the real line. Initially, a particle sits at the origin. Its children form the first generation; their displacements from the origin correspond to a point process on the line. These children have children of their own (who form the second generation), and behave – relative to their respective positions – like independent copies of the initial particle. And so on.

We write $|u| = n$ if an individual u is in the n-th generation, and denote its position by $V(u)$. (In particular, for the initial ancestor e, we have $V(e) = 0$.) We assume throughout the paper that there exists a constant $C > 0$ such that $\sup_{|u|=1} |V(u)| \le C$. For technical reasons, we also assume that

$$\mathbf{E}\Big\{\Big(\sum_{|u|=1} 1\Big)^{1+\delta}\Big\} < \infty, \qquad \text{for some } \delta > 0. \tag{1}$$

Let us define $\psi(t) := \log \mathbf{E}\big\{\sum_{|u|=1} e^{-tV(u)}\big\}$ and assume

$$\psi(0) > 0, \qquad \psi(1) = \psi'(1) = 0. \tag{2}$$

In the study of the branching random walk, there is a fundamental martingale, defined as follows: $W_n := \sum_{|u|=n} e^{-V(u)}, n = 0, 1, 2, \cdots$. Since $W_n \ge 0$, it converges almost surely.

When $\psi'(1) < 0$, it is proved by Biggins and Kyprianou (1997) that there exists a sequence of constants (a_n) such that $\frac{W_n}{a_n}$ converges in probability to a non-degenerate limit which is (strictly) positive upon the survival of the system.

The case $\psi'(1) = 0$ is more delicate. In this case, it is known (Lyons (1996)) that $W_n \to 0$ almost surely. The following question is raised in Biggins and Kyprianou (2005): are there deterministic normalizers (a_n) such that $\frac{W_n}{a_n}$ converges?

We aim at answering this question.

Theorem 1. *Assume* (1) *and* (2)*. There exists a deterministic positive sequence* (λ_n) *with* $0 < \liminf_{n\to\infty} \frac{\lambda_n}{n^{1/2}} \le \limsup_{n\to\infty} \frac{\lambda_n}{n^{1/2}} < \infty$*, such that conditionally on the system's survival,* $\lambda_n W_n$ *converges in distribution to* W*, with* $W > 0$ *a.s.*

2. The minimal position in the branching random walk

A natural question in the study of branching random walks is about the position of the leftmost individual in the n-th generation. In the literature, the concentration (in terms of tightness or even weak convergence) of $\inf_{|u|=n} V(u)$ around its median/quantiles had been studied by many authors. See for example Bachmann (2000), Bramson and Zeitouni (2006), as well as Section 5 of the survey paper by Aldous and Bandyopadhyay (2005). We also mention the recent paper of Lifshits (2007+), where an example of branching random walk is constructed such that $\inf_{|u|=n} V(u) - \text{median}(\{\inf_{|u|=n} V(u)\})$ is tight but does not converge weakly.

We are interested in the asymptotic speed of $\inf_{|u|=n} V(u)$. Under assumption (2), it is known from the classical "law of large numbers" (Hammersley (1973), Kingman (1975), Biggins (1976)) that, conditionally on the system's survival, $\frac{1}{n} \inf_{|u|=n} V(u) \to 0$, a.s. Refinements are obtained by McDiarmid (1995) by assuming for e.g.

$$\mathbf{E}\Big\{\Big(\sum_{|u|=1} 1\Big)^{2}\Big\} < \infty, . \tag{3}$$

Theorem 2. *Assume* (2) *and* (3). *Conditionally on the system's survival, we have*

$$\limsup_{n\to\infty} \frac{1}{\log n} \inf_{|u|=n} V(u) = \frac{3}{2}, \qquad \text{a.s.} \tag{4}$$

$$\liminf_{n\to\infty} \frac{1}{\log n} \inf_{|u|=n} V(u) = \frac{1}{2}, \qquad \text{a.s.} \tag{5}$$

$$\lim_{n\to\infty} \frac{1}{\log n} \inf_{|u|=n} V(u) = \frac{3}{2}, \qquad \textit{in probability.} \tag{6}$$

Remark. (i) The most interesting part of Theorem 2 is (4)–(5). It reveals, surprisingly, the presence of fluctuations of $\inf_{|u|=n} V(u)$ on the logarithmic level, which is in contrast with known results of Bramson (1978) and Dekking and Host (1991) stating that for a class of branching random walks, $\frac{1}{\log\log n}\inf_{|u|=n} V(u)$ converges almost surely to a finite and positive constant.

(ii) For a general branching random walk (without assuming (2)), if $t^*\psi'(t^*) = \psi(t^*)$ has a solution $t_* \in (0, \infty)$, then the study will boil down to the case (2) after a linear transformation.

(iii) Under suitable assumptions, Addario-Berry (2006) obtains a very precise asymptotic estimate of $\mathbf{E}[\inf_{|u|=n} V(u)]$, which implies (6).

3. Directed polymers on a disordered tree

Following Derrida and Spohn (1988), we study the associated partition function: $W_{n,\beta} := \sum_{|u|=n} e^{-\beta V(u)}$, $\beta > 0$.

If $0 < \beta < 1$, the study of $W_{n,\beta}$ boils down to the case $\psi'(1) < 0$ which was investigated by Biggins and Kyprianou (1997). In particular, conditionally on the system's survival, $\frac{W_{n,\beta}}{\mathbf{E}\{W_{n,\beta}\}}$ converges almost surely to a (strictly) positive random variable. We study the case $\beta \geq 1$:

Theorem 3. *Assume* (1) *and* (2). *Conditionally on the system's survival, we have*

$$W_n = n^{-1/2+o(1)}, \qquad \text{a.s.}$$

Theorem 4. *Assume* (2) *and* (3), *and let* $\beta > 1$. *Conditionally on the system's survival, we have*

$$\limsup_{n\to\infty} \frac{\log W_{n,\beta}}{\log n} = -\frac{\beta}{2}, \qquad \text{a.s.}$$

$$\liminf_{n\to\infty} \frac{\log W_{n,\beta}}{\log n} = -\frac{3\beta}{2}, \qquad \text{a.s.}$$

$$W_{n,\beta} = n^{-3\beta/2+o(1)}, \qquad \textit{in probability.}$$

The remark stated after Theorem 2, applies to Theorems 3 and 4 as well.

The paper has been put in http://arxiv.org/abs/math/0702799

Copolymers in Emulsion

Frank den Hollander
(joint work with Nicolas Pétrélis, Stu Whittington)

In this talk we consider a two-dimensional directed self-avoiding walk model of a random copolymer in a random emulsion. The copolymer is a random concatenation of monomers of two types, A and B, each occurring with density $\frac{1}{2}$. The emulsion is a random mixture of liquids of two types, A and B, organised in large square blocks occurring with density p and $1-p$, respectively, where $p \in (0,1)$. The copolymer in the emulsion has an energy that is minus α times the number of AA-matches minus β times the number of BB-matches, where without loss of generality the interaction parameters can be taken from the cone $\{(\alpha,\beta) \in R^2\colon\ \alpha \geq |\beta|\}$. To make the model mathematically tractable, we assume that the copolymer can only enter and exit a pair of neighbouring blocks at diagonally opposite corners.

In a recent paper with Stu Whittington, a variational expression was derived for the quenched free energy per monomer in the limit as the length n of the polymer tends to infinity and the blocks in the emulsion have size L_n such that $L_n \to \infty$ and $L_n/n \to 0$. Under this restriction, the free energy is self-averaging with respect to both types of randomness. It was found that in the supercritical percolation regime $p \geq p_c$, with p_c the critical probability for directed bond percolation on the square lattice, the free energy has a phase transition along a curve in the cone that is independent of p. At this critical curve, there is a transition from a phase where the polymer is fully delocalized into the A-blocks to a phase where it is partially localized near the interface in AB-blocks of which it diagonally crosses the A-block. In a recent preprint with Nicolas Pétrélis, it shown that: (1) the critical curve is strictly increasing; (2) the phase transition is second order; (3) the free energy is infinitely differentiable throughout the partially localized phase.

In the subcritical regime $p < p_c$, the phase diagram is more complex and depends on p. There are three critical curves, separating four phases, with two tricritical points. In one of the delocalized phases, the copolymer is fully delocalized into the A-blocks and into the B-blocks, but never inside a neighboring pair, while in the other delocalized phase it is fully delocalized into the A-blocks and into the B-blocks, sometimes inside a neighboring pair. In one of the localized phases, the copolymer is partially localized near the interface in blocks of which it diagonally crosses the B-block rather than the A-block, while in the other localized phase it is partially localized near the interface in both types of blocks. Very little is known so far about the fine details of the four critical curves. Some progress is underway in a forthcoming paper with Nicolas Pétrélis.

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Random walk in random environment with asymptotically zero perturbation

Mikhail V. Menshikov
(joint work with Andrew R. Wade)

1. Introduction

The problems of random walk in random environment and stochastic processes with asymptotically zero mean drifts have separately received considerable attention. We describe results for a model combining these two classical models: one-dimensional random walk in a perturbed random environment. This model exhibits behaviour that is significantly different to that of those previously studied systems.

We give criteria for recurrence and transience of the model, and also mention results on the 'speed' of the random walk (i.e., how far the particle is from the origin after a long time).

Given an infinite sequence $\omega = (p_0, p_1, p_2, \ldots)$ such that $p_i \in (0,1)$ for all $i \in \mathbf{Z}^+$, we consider $\Xi = (\eta_t; t \in \mathbf{Z}^+)$ the nearest-neighbour random walk on $\mathbf{Z}^+$ under probability measure P_ω defined as follows. Set $\eta_0 = 0$, and for $n \in \mathbf{N}$,

$$\begin{aligned} P_\omega[\eta_{t+1} = n-1 | \eta_t = n] &= p_n, \\ P_\omega[\eta_{t+1} = n+1 | \eta_t = n] &= 1 - p_n =: q_n, \end{aligned} \tag{1}$$

and $P_\omega[\eta_{t+1} = 0 | \eta_t = 0] = p_0$, $P_\omega[\eta_{t+1} = 1 | \eta_t = 0] = 1 - p_0 =: q_0$.

We call a sequence of jump probabilities ω an *environment*. For any such ω, Ξ is an irreducible, aperiodic Markov chain (under the 'quenched' measure P_ω).

Here, we take ω itself to be random — then Ξ is a *random walk in random environment* (RWRE). More precisely, $p_0, p_1, \ldots$ will be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We describe our particular model in the next section.

An important case in which the random environment is homogeneous and in some sense critical is the so-called *Sinai's regime* [6]. Here $(p_0, p_1, p_2, \ldots)$ is a sequence of i.i.d. random variables satisfying the condition $\mathbb{E}[\log(p_1/q_1)] = 0$, where $\mathbb{E}$ is expectation under $\mathbb{P}$. In this case, a result dating back to Solomon [7] says that Ξ is null-recurrent for $\mathbb{P}$-almost every ω. Solomon's result shows that Sinai's regime is critical in the sense that, for an i.i.d. random environment, Ξ is respectively ergodic (that is positive-recurrent, here) or transient as $\mathbb{E}[\log(p_1/q_1)] > 0$ or $\mathbb{E}[\log(p_1/q_1)] < 0$.

A notable property of the RWRE in Sinai's regime is its *speed* — roughly speaking η_t is of order $(\log t)^2$ for large t [6]. One way to state this more precisely is in terms of 'almost sure' behaviour, i.e. results that hold P-almost surely (a.s.) for $\mathbb{P}$-almost every (a.e.) ω. Sharp results of this type are given by Hu and Shi in [1].

The results we describe here are of two kinds: (i) qualitative characteristics: specifically, criteria for recurrence, transience and positive-recurrence (ergodicity, here), derived in [4] using the method of Lyapunov functions; and (ii) quantitative behaviour: specifically, results on speeds (more formally, almost sure bounds) derived in [5].

We study two particular cases of random environment. In the first, our environment will be a perturbation of the i.i.d. environment of Sinai's regime. In the second, our environment will be a random perturbation of the simple symmetric random walk.

2. Model and results

We now describe the particular RWRE model of the form of (1) that we study. Fix $\delta \in (0, 1/2)$. Let (ξ_i, Y_i), $i \in \mathbf{N}$, be a sequence of i.i.d. random vectors on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that Y_1 takes values in $[-1, 1]$ and the (technical) ellipticity condition $\mathbb{P}[\delta \leq \xi_1 \leq 1 - \delta] = 1$ holds. Note that we allow Y_1 and ξ_1 to be dependent.

We fix $\alpha > 0$. For a particular realization of the sequence (ξ_i, Y_i), $i \in \mathbf{N}$, we define $p_0 = q_0 = 1/2$ and the quantities p_n and q_n, $n = 1, 2, 3, \dots$ as follows:

$$\begin{aligned} p_n &:= \begin{cases} \xi_n + Y_n n^{-\alpha} & \text{if } (\delta/2) \leq \xi_n + Y_n n^{-\alpha} \leq 1 - (\delta/2) \\ \delta/2 & \text{if } \xi_n + Y_n n^{-\alpha} < (\delta/2) \\ 1 - (\delta/2) & \text{if } \xi_n + Y_n n^{-\alpha} > 1 - (\delta/2) \end{cases} \\ q_n &:= 1 - p_n. \end{aligned} \tag{2}$$

A particular realization of $(p_n; n \in \mathbf{N})$ specifies our random environment ω. We have that there exists $n_0 \in \mathbf{N}$ such that, for a.e. ω,

$$p_n = \xi_n + Y_n n^{-\alpha}, \; q_n = 1 - \xi_n - Y_n n^{-\alpha}, \; (n \geq n_0).$$

2.1. Perturbation of random walk in random environment in Sinai's regime.

Now we describe our first version of the model in (2). For $n \in \mathbf{N}$ set

$$\zeta_n := \log\left(\frac{\xi_n}{1 - \xi_n}\right), \; Z_n := \frac{Y_n}{\xi_n(1 - \xi_n)}.$$

With $\mathbb{E}$ denoting expectation under $\mathbb{P}$, suppose that $\mathbb{E}[\zeta_1] = 0$ and $\mathrm{Var}[\zeta_1] > 0$ (so our environment is truly random).

This model was introduced in more generality in [4]. In this case, the random environment described in (2) corresponds to a perturbation of Sinai's regime, in the sense that, in the limit as $n \to \infty$, we have $\mathbb{E}[\log(p_n/q_n)] \to 0$. Despite this, the behaviour of this model may be strikingly different to that of Sinai's RWRE, and is highly dependent on the nature of the perturbation.

Of separate interest are the two cases $\mathbb{E}[Z_1] = 0$ and $\mathbb{E}[Z_1] \neq 0$. The first result deals with the special case in which $Y_1/\xi_1 \stackrel{d}{=} -Y_1/(1 - \xi_1)$ (so that $\mathbb{E}[Z_1] = 0$).

Theorem 1. *Suppose* $Y_1/\xi_1 \stackrel{d}{=} -Y_1/(1 - \xi_1)$, $\mathbb{P}[Y_1 = 0] < 1$, $\mathbb{E}[\zeta_1] = 0$, *and* $\mathrm{Var}[\xi_1] > 0$. *Then* Ξ *is null-recurrent for a.e.* ω.

Our next result deal with the case $\mathbb{E}[Z_1] \neq 0$.

Theorem 2. *Suppose* $\mathbb{E}[Z_1] \neq 0$, $\mathbb{P}[Y_1 = 0] < 1$, $\mathbb{E}[\zeta_1] = 0$, *and* $\mathrm{Var}[\xi_1] > 0$. *For a.e.* ω, Ξ *is*

(i) *null-recurrent if* $\alpha \geq 1/2$;
(ii) *transient if* $\alpha < 1/2$ *and* $\mathbb{E}[Z_1] < 0$;
(iii) *ergodic if* $\alpha < 1/2$ *and* $\mathbb{E}[Z_1] > 0$.

This result quantifies the fact that in some sense a random environment is more stable than a homogeneous one, in that a much larger perturbation is required to disturb the null-recurrent situation than in the non-random environment case (see [2, 3]).

Theorem 2 is in fact a corollary to a much more refined result in [4], which deals with more general forms of the perturbation in (2).

In [5] almost sure upper and lower bounds for η_t are given (cf [1] for the RWRE in Sinai's regime). For example, the next result deals with the transient case when $\mathbb{E}[Z_1] < 0$ and $\alpha \in (0, 1/2)$, and gives a striking example of logarithmic transience.

Theorem 3. *Suppose* $\mathbb{E}[\zeta_1] = 0$, $\mathrm{Var}[\xi_1] \in (0, \infty)$, $\mathbb{E}[Z_1] < 0$, $\mathrm{Var}[Y_1] \in [0, \infty)$, *and* $\alpha \in (0, 1/2)$. *For a.e.* ω, *for any* $\varepsilon > 0$, *we have,* P_ω*-a.s., for all but finitely many* t,

$$(\log\log t)^{-(1/\alpha)-\varepsilon} < \frac{\eta_t(\omega)}{(\log t)^{1/\alpha}} < (\log\log t)^{(2/\alpha)+\varepsilon}.$$

2.2. **Simple random walk with random perturbation.** Our second model again fits into the framework of (2) above, but we now take $\mathbb{P}[\xi_1 = 1/2] = 1$ and $\mathrm{Var}[Y_1] > 0$. In this case we have $p_0 = q_0 = 1/2$ and for a.e. ω, for all $n \geq n_0$

$$p_n = \frac{1}{2} + Y_n n^{-\alpha}, \; q_n = \frac{1}{2} - Y_n n^{-\alpha}, \; (n \geq n_0).$$

Thus in the limit $n \to \infty$, we coincide with the symmetric SRW on $\mathbf{Z}^+$.

Theorem 4. *Suppose* $\mathbb{P}[\xi_1 = 1/2] = 1$ *and* $\mathrm{Var}[Y_1] > 0$.

(i) *If* $Y_1 \stackrel{d}{=} -Y_1$, *then* η_t *is null-recurrent for a.e.* ω.
(ii) *Suppose* $\mathbb{E}[Y_1] \neq 0$.
   (a) *If* $\beta \in (0, 1)$ *and* $\mathbb{E}[Y_1] > 0$ *then* Ξ *is ergodic for a.e.* ω.
   (b) *If* $\beta > 1$ *then* Ξ *is null-recurrent for a.e.* ω.
   (c) *If* $\beta \in (0, 1)$ *and* $\mathbb{E}[Y_1] < 0$ *then* Ξ *is transient for a.e.* ω.

Corresponding results on speeds appear in [5].

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Rate of growth of a transient cookie random walk

Arvind Singh
(joint work with Anne-Laure Basdevant)

The model of the excited random walk introduced by Benjamini and Wilson [3] in 2003 is a particular case of self-interacting random walk. It consists of a nearest neighbour random walk on $\mathbb{Z}^d$ which has a bias towards a specific direction upon its first visit to a site but moves like a symmetric random walk on subsequent visits. Benjamini and Wilson [3] proved that the walk is recurrent in dimension $d = 1$ but becomes transient in dimension $d \geq 2$. In the one-dimensional case, if multiple excitations are allowed (*i.e.* there is a bias, not only on the first visit but also on some subsequent visits), then the walk exhibits some interesting new features. It can for instance be recurrent or transient, depending on the strength of the bias.

We here consider the model of the 'cookie random walk' which is a particular case of the model of the multi-excited random walk described by Zerner [8] in 2005. Let us fix an integer M which represents the initial number of cookies per site and a vector $\bar{p} = (p_1, \ldots, p_M) \in [\frac{1}{2}, 1[^M$. We say that p_j represents the strength of the j^{th} cookie initially placed at each site of $\mathbb{Z}$. The cookie random walk $X = (X_n,\, n \in \mathbb{N})$ is a random walk on $\mathbb{Z}$ starting from the origin and 'eating' the cookies it finds along its path in the following way:

$$\mathbf{P}\{|X_{n+1} - X_n| = 1\} = 1,$$
$$\mathbf{P}\{X_{n+1} = X_n + 1 \mid X_1, \ldots, X_n\} = \begin{cases} p_j & \text{if } j = \sharp\{0 \leq i \leq n,\ X_i = X_n\} \leq M, \\ 1/2 & \text{otherwise.} \end{cases}$$

Thus, at each step, the walk eats the first cookie available at its present site if there remains any and then moves with a bias depending on the strength of the cookie it has just eaten. If there was no more cookie to be eaten, the walk just performs a symmetric random walk.

An important parameter for this model is the total displacement provided by the cookies initially placed at a site:

$$\alpha = \sum_{i=1}^{M} (2p_i - 1).$$

According to Zerner [8], the cookie random walk is recurrent if $\alpha \leq 1$ and becomes transient towards $+\infty$ when $\alpha > 1$. In particular, a cookie random walk can be transient with just two cookies per site. Zerner also proved that the limiting speed of the walk is well defined. Moreover, he showed that the speed is always zero when there are at most two cookies per site. On the other hand, Mountford, Pimentel and Valle [5] showed that it is possible to obtain a strictly positive speed if the initial number M of cookies per site is large enough.

The first result gives a criterion to decide whether or not the speed is zero:

Theorem 1. Let X be a $\bar{p}$-cookie random walk with limiting speed v. We have the equivalence

$$v > 0 \iff \alpha > 2.$$

Moreover, for each M, the speed v is a continuous function of $\bar{p}$ in the set of environment with at most M cookies per site.

In particular, a positive speed may be achieved with as few as three cookies per site. However, the calculation of the exact value of the speed in term of the initial cookie distribution $\bar{p}$ seems a challenging problem.

We also have the following result concerning the rate of transience of the walk in the sublinear regime:

Theorem 2. Let X be a transient $\bar{p}$-cookie random walk with zero speed (*i.e.* $1 < \alpha \leq 2$).

- If $1 < \alpha < 2$,

$$\frac{X_n}{n^{\alpha/2}} \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{M}_{\alpha/2}$$

  where $\mathcal{M}_{\alpha/2}$ is a Mittag-Leffler distribution with index $\alpha/2$.
- If $\alpha = 2$, there exists a constant $c > 0$ such that

$$\frac{\log n}{n} X_n \underset{n\to\infty}{\overset{\text{prob.}}{\longrightarrow}} c.$$

These results also hold with $\sup_{i\leq n} X_i$ and $\inf_{i\geq n} X_i$ in place of X_n.

The proofs of both theorems are based on precise studies of the hitting times of the walk. It is well known that, given a classical simple random walk, one can define a Galton-Watson process closely connected with its hitting times. In the same spirit, Kesten, Kozlov and Spitzer [4] constructed a branching process with immigration in random environment associated to a transient random walk in a random environment. More recently, Tóth used the same approach for the study of reinforced random walks. Here, we use again a similar method: we construct a positive recurrent Markov process Z closely related to the hitting times of the cookie random walk. It turns out that the resulting process is, in our setting, a branching process with random migration *i.e.* a branching process which allows (random) immigration and (random) emigration.

The study of the invariant probability of the process Z is achieved using the tools of probability generating functions and enables to prove Theorem 1. The proof of Theorem 2 relies on a precise estimate of the tail distribution of the total progeny of the branching process over an excursion away from 0. To this end, we use a martingale argument which may also be found of interest when dealing with general branching processes with migration.

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Central limit theorem for the excited random walk in dimensions $d \geq 2$

Alejandro F. Ramírez
(joint work with Jean Bérard)

Let $\{e_1, \ldots, e_d\}$ be the canonical unit vectors on the lattice $\mathbb{Z}^d$. The excited random walk with bias parameter $p \in (1/2, 1]$ on $\mathbb{Z}^d$ is defined as the discrete time random walk that jumps with probability $1/(2d)$ to its nearest neighbors when it is at a site that it visited previously, whereas otherwise it jumps with probability $1/(2d)$ to all nearest neighbor sites in a direction orthogonal to e_1, with probability p/d to the nearest neighbor site in the direction e_1 and probability $(1-p)/d$ to the nearest neighbor site in the direction $-e_1$.

The excited random walk was introduced by Benjamini and Wilson in 2003 [1]. They proved that it is transient in the direction e_1 in any dimension $d \geq 2$. Furthermore, they showed that a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot e_1}{n} > 0,$$

in dimensions $d \geq 4$. Subsequently, Kozma extended this result to dimensions $d = 3$ and $d = 2$ [3, 4]. Variations of the excited random walk, like for example the multi-excited random walk have also been studied (see Zerner [5]). Recently, den Hofstad and Holmes [2], proved using the lace expansion that when the bias parameter p is sufficiently small, a law of large numbers is satisfied for $d > 5$ and a central limit theorem for $d > 8$. Here we prove the following result.

Theorem 1. *Let $p \in (1/2, 1]$ and $d \geq 2$.*

(i) *There exists a $v(d,p) > 0$ such that a.s.*

$$\lim_{n\to\infty} \frac{X_n \cdot e_1}{n} = v.$$

(ii) *There exists a $\sigma(d,p) > 0$ such that*

$$n^{-1/2}(X_[nt] \cdot e_1 - [nt]v)$$

converges in law to a Brownian motion with variance σ^2.

The proof of this result is based on regeneration time techniques as used in the context of Random Walks on Random Environments. Indeed, we define a sequence $\{\kappa_n : n \geq 1\}$ of random times, which form a sequence of independent random variables. Furthermore $\{\kappa_n : n \geq 2\}$ forms a sequence of i.i.d. random variables. Similarly, $\{X_{\cdot\wedge\kappa_1}, X_{(\cdot+\kappa_1)\wedge\kappa_2} - X_{\kappa_1}, \ldots\}$, is a sequence of independent random variables i.i.d. except for the first term. Our definition of κ is the standard one

$$\kappa := \min\{n \geq 0 : \max_{0\leq k\leq n-1} X_k \cdot e_1 < X_n \cdot e_1 \leq \min_{k\geq n} X_k \cdot e_1\}.$$

The main difficulty of the proof of theorem 1, is showing that

$$E[\kappa^2] < \infty. \tag{1}$$

One of the key steps to establish (1) is the inclusion

$$\{\kappa > n\} \subset A_n \cup B_n \cup C_n,$$

valid for every natural n, where $A_n := \{X_n \cdot e_1 \leq n^{a_1}\}$, $B_n := \cup_{k=1}^{\lfloor n^{a_2}\rfloor}\{D_k < \infty\}$ and $C_n := \cup_{k=1}^{\lfloor n^{a_2}\rfloor}\{n^{a_3} < D_k < \infty\}$ and $a_2 + a_3 < a_1 < 3/4$. Here $D_1 = \min\{n \geq 1 : X_n \cdot e_1 < 1\}$ and for $k \geq 2$, D_k is the first time the random walk visits site $r_{k-1} - 1$, where r_{k-1} is the record of $X_n \cdot e_1$ between times 0 and D_{k-1}. To show that the probabilities of the events A_n and C_n are small, we use the concept of tan points (introduced by Benjamini and Wilson), which give a useful lower bound estimate on the range of the excited random walk. The idea is to couple the excited random walk with a simple symmetric random walk $\{Y_n : n \geq 0\}$ in such a way that the difference $(X_n - Y_n)\cdot e_1$ is non decreasing. In dimension $d = 2$, a *tan point* for $\{Y_n : n \geq 0\}$ is defined as any time $n \geq 0$ such that $Y_n \cdot e_1 > Y_k \cdot e_1$ for all $0 \leq k \leq n-1$ such that $Y_n \cdot e_2 = Y_k \cdot e_2$. I turns out that any tan point is necessarily a time when the excited random walk visits a site for the first time. This provides a lower bound for the range of the excited random walk in terms of the tan points. In fact, it can be shown that if J_n is the range of the excited random walk up to time n, then the probability of the event $\{J_n < n^{a_1}\}$ with $a_1 < 3/4$, decays like e^{-n^δ} for some $\delta > 0$.

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Problem session

Gady Kozma

We stated 4 problems originating from our Weizmann group's work around excited random walk.

(1) "Excited to the center". For this model we were able to show recurrence in all dimensions regardless of the exact details of how one interprets the name "excited to the center". The problem suggested to the audience was to find a shape theorem for the set of visited vertices, and it was conjectured that this set is a ball of radius $n^{1/(d+1)}$.

(2) "The H-walk". In this model, the walker is again on $\mathbb{Z}^d$ with a cookie at every site, but this time the drift he gets from eating a cookie alternates between left and right. In a formula,

$$\text{drift}_t = \begin{cases} 0 & \exists i < t \text{ s.t. } R(i) = R(t) \\ (-1)^{\#\text{range}(R[0,t])} \cdot e_1 & \text{otherwise} \end{cases}$$

We noted that this process is the sum of a martingale and an error bounded by 1. We asked: is it recurrent in two dimensions? Is it transient in 3? We noted that we do have a bound on the number of returns in 3 dimensions which is poly-log-log.

(3) "The Y-walk". In this model the walker decides whether to be excited to the left or to the right by examining the last vertical move and deciding to drift to the left if that last move was up and to the right if that last move was down. Questions are as for the H-walk.

(4) "Slugs". In this two dimensional model, there are two walkers walking simultaneously. Each one gets a drift to the right when it steps over the path of the other walker. We conjectured that the walks are diffusive and transient.

What is the difference between a square and a triangle?

Pierre Tarrès
(joint work with Vlada Limic)

Edge-Reinforced Random Walk (ERRW), introduced by Coppersmith and Diaconis [1] in 1986, is a process evolving in an environment constantly modified by its own behaviour: at each step, the probability to move along an edge is proportional to a function - called the weight function - of the number of visits to this edge. A similar notion of Vertex-Reinforced Random Walk [6] (VRRW) favours the more visits to vertices instead.

Sellke proved in 1994 that, if the weight function is reciprocally summable then on any graph of bounded degree *without odd cycles,* the (strongly) ERRW ends up visiting the same (random) edge back and forth. The Rubin construction he used towards the proof could not carry on to other graphs, even in the "simple" case of a triangle, thus introducing the "triangle conjecture" that the same behaviour should occur in general. A simple linear algebra argument [5] illustrates the difference in this respect between odd and even cycles.

Sellke's conjecture for nondecreasing weight functions was proved in a joint work with V. Limic [4]. However, our method requires a more detailed analysis of the behaviour of the walk.

The purpose of this talk is to give an introduction to this question, as well as a general overview of the subject and its techniques: martingales methods, Pólya urn models, a correspondence with random walks in random environment [2, 3, 7], Ray-Knight local time analysis [10]. The above variety of techniques reflects a large range of behaviours for reinforced random walks. It is worth noting in particular that, although VRRW and ERRW have analogous definitions, they show significantly different behaviours in the case of linear weight on the integer lattice: VRRW eventually gets stuck on five (random) consecutive sites almost surely [9], whereas the ERRW visits all sites of the lattice infinitely often, almost surely [1] .

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Recurrence of Edge-Reinforced Random Walk on a two-dimensional Graph

Franz Merkl, Silke Rolles

e consider linearly edge-reinforced random walk on a class of two-dimensional graphs with constant initial weights. The graphs are obtained from $\mathbb{Z}^2$ by replacing every edge by a sufficiently large, but fixed number of edges in series. We prove that linearly edge-reinforced random walk on these graphs is recurrent. Furthermore, we derive bounds for the probability that the edge-reinforced random walk hits the boundary of a large box before returning to its starting point. The details are given in [MR07].

About twenty years ago, Diaconis asked whether linearly edge-reinforced random walk on $\mathbb{Z}^2$ is recurrent. This turned out to be a hard problem which is unsolved up to the present day. However, we solve a variant of the problem posed by Diaconis. For a class of fully two-dimensional translationally symmetric graphs und sufficiently small constant initial weights, we show that edge-reinforced random walk visits every vertex infinitely often with probability one.

More specifically, given a natural number $r \in \mathbb{N}$, we consider the graph $G_r = (V_r, E_r)$ with vertex set

$$V_r = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 \in r\mathbb{Z} \text{ or } x_2 \in r\mathbb{Z}\}$$

and edge set

$$E_r = \{\{u, v\} \subset V_r : |u - v| = 1\}.$$

Here $|x|$ denotes the Euclidian norm of x. Note that the edges are *undirected.*

Let $0 := (0, 0)$. *Linearly edge-reinforced random walk* (ERRW) on G_r with constant initial weights $a > 0$ and starting point 0 is a stochastic process $(X_t)_{t\in\mathbb{N}_0}$ with law $P = P_{0,a}^{G_r}$ defined as follows: At every discrete time $t \in \mathbb{N}_0$, every edge $e \in E_r$ is assigned a strictly positive number $w_t(e)$ as a weight. Initially, all weights are equal to a:

$$w_0(e) = a \qquad \text{for all } e \in E_r.$$

The edge-reinforced random walker starts in the vertex 0 at time 0:

$$P[X_0 = 0] = 1.$$

At each discrete time $t \in \mathbb{N}_0$, the random walker jumps randomly from its current position X_t to a neighboring vertex v with probability proportional to the current weight of the connecting edge $\{X_t, v\}$:

$$P[X_{t+1} = v | X_0, X_1, \dots, X_t] = \frac{w_t(\{X_t, v\})}{w_t(X_t)} 1_{\{X_t, v\} \in E_r},$$

where $w_t(X_t) := \sum_{e \in E_r : e \ni X_t} w_t(e)$. The weight of the traversed edge is immediately increased by 1, whereas all other weights remain unchanged:

$$w_{t+1}(e) = w_t(e) + 1_{\{X_t, X_{t+1}\} = e} \qquad \text{for all } e \in E_r.$$

Thus, the weight of edge e at time t equals the initial weight increased by the number of times the reinforced random walker has traversed e up to time t:

$$w_t(e) = a + \sum_{s=0}^{t-1} 1_{\{X_s, X_{s+1}\} = e}.$$

We realize P as a probability measure on the set $\Sigma \subseteq V_r^{\mathbb{N}_0}$ of admissible paths in G_r, not necessarily starting in 0, endowed with the σ-field generated by the canonical projections $X_t : \Sigma \to V_r$, $t \in \mathbb{N}_0$.

We prove:

Theorem 1 [Recurrence] *For all $r \in \mathbb{N}$ with $r \geq 130$ and all $a \in (0, (r - 129)/512)$, linearly edge-reinforced random walk on G_r with constant initial weights $w_0 \equiv a$ visits all vertices infinitely often with probability one.*

In order to prove recurrence, we derive bounds for the probability that the edge-reinforced random walk hits the boundary of a large box before returning to its starting point. Let us introduce some notation before we state the result: For $A \subseteq V_r$, let

$$\tau_A := \inf\{t \geq 1 : X_t \in A\}$$

be the first time ≥ 1, the random walk visits the set A. If $A = \{v\}$ contains just one vertex, we simply write τ_v instead of $\tau_{\{v\}}$. Let

$$L_r := r\mathbb{Z}^2$$

be the set of "four-way-crossings" in the graph G_r. For $(v_1, v_2) \in V_r$, set $|(v_1, v_2)|_\infty := \max\{|v_1|, |v_2|\}$.

We prove:

Theorem 2 [Hitting probabilities] *For all $r \in \mathbb{N}$ with $r \geq 130$ and all initial weights $a \in (0, (r - 129)/512)$, there exist constants $l_0 = l_0(r, a) \in \mathbb{N}$ and $\xi = \xi(r, a) > 0$, such that the following hold:*

(a) *For all $\ell \in L_r$ with $|\ell|_\infty \geq l_0$, the probability that the edge-reinforced random walker hits ℓ before returning to its starting point satisfies*

$$P\left[\tau_\ell < \tau_0\right] \leq \left(\frac{r}{|\ell|_\infty}\right)^{1+\xi}.$$

(b) *As a consequence, for all $l \geq l_0$, the probability that the edge-reinforced random walker hits a vertex in the set $\mathcal{V}_l := \{v \in V_r : |v|_\infty = rl\}$ (or equivalently a vertex in the boundary of the box $V_r \cap [-rl, rl]^2$) before returning to 0 can be bounded as follows:*

$$P\left[\tau_{\mathcal{V}_l} < \tau_0\right] \leq 8l^{-\xi}.$$

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Waiting for the attracting edge to appear

Vlada Limic
(joint work with Codina Cotar)

The talk is based on a joint work [1].

Let $\mathcal{G}$ be a locally finite connected graph with the edge set $E(\mathcal{G})$ and the vertex set $V(\mathcal{G})$. Any two vertices u, v connected by an edge are called *adjacent*, in this case we write $u \sim v$ and denote by $\{u, v\} = \{v, u\}$ the edge connecting them. Finally, denote by $D(\mathcal{G}) = \sup_{v \in V(\mathcal{G})} \text{degree}(v)$ the degree of $\mathcal{G}$, where for any $v \in V(\mathcal{G})$ degree(v) equals the number of edges incident to v.

Let $(\ell_0^e,\, e \in E(\mathcal{G}))$ be given integers, and assume $\ell_0^e \geq 0$, $e \in E(\mathcal{G})$. Given a *reinforcement weight function* $w : \{0, 1, 2, \ldots, \} \mapsto (0, \infty)$, the edge-reinforced random walk (ERRW) on $\mathcal{G}$ makes nearest neighbor step transitions in $V(\mathcal{G})$. We will denote by I_n the (random) position of the edge reinforced random walk at time n. If $\mathcal{G}$ is a finite graph it seems natural from the point of notation to construct and study the edge reinforced random walk started at the *initial time* $t_0 := \sum_{e \in E(\mathcal{G})} \ell_0^e \geq 0$, a process starting at time 0 is obtained by a time shift. If $\mathcal{G}$ is an infinite graph, just set $t_0 := 0$. Then $I_{t_0} \in V(\mathcal{G})$ is the initial position, and $\{I_n, I_{n+1}\} \in E(\mathcal{G})$ for all $n \geq t_0$, almost surely. Moreover, the dynamics of the edge reinforced random walk is prescribed according to the rule:

$$P(I_{n+1} = v | \mathcal{F}_n) 1_{\{I_n = u\}} = \frac{w(X_n^{\{u,v\}})}{\sum_{y \sim u} w(X_n^{\{u,y\}})} 1_{\{I_n = u, u \sim v\}},$$

where for any $e \in E(\mathcal{G})$,

$$X_n^e = \ell_0^e + \sum_{i \leq n-1} 1_{\{\{I_i, I_{i+1}\} = e\}} \tag{1}$$

equals the *initial weight* ℓ_0^e incremented by the total number of traversals of edge e prior to time n. Note that t_0 is chosen so that, whenever $V(\mathcal{G}) < \infty$, $\sum_{e \in E(\mathcal{G})} X_k^e = k$ for all $k \geq t_0$, almost surely.

We denote by $\mathcal{G}_1$ be the random subgraph of $\mathcal{G}$ spanned by edges traversed at least once, i.e. edges $e \in E(\mathcal{G})$ such that

$$\sup_n X_n^e > \ell_0^e.$$

Pemantle [5] made a recent survey of stochastic reinforcement processes. Apart from the behavior analogous to recurrence or transience of Markov chains (see, for example, [6] or [4] or [5] theorems 5.2 and 5.6), ERRW may exhibit a very different asymptotic behavior as time increases. For example, it is easy to see, [7], [2] that the following assumption

$$\text{(A0)} \qquad \sum_k \frac{1}{w(k)} < \infty$$

is sufficient (and necessary if w is non-decreasing) for the event

$$\{\mathcal{G}_1 \text{ is a finite graph}\}$$

to have probability 1, whenever $D(\mathcal{G}) < \infty$.

In a recent work, Limic and Tarrès [3] obtained a general result conjectured by Sellke [7] and partially resolved by Limic [2]: for a fairly general class of reinforcement weights (in particular, whenever w is a non-decreasing function satisfying (A0)) on any graph of bounded degree the walk eventually keeps traversing a single (random) edge for all large times.

Our work assumes (A0) with w non-decreasing, and is devoted to the study of the tail behavior of the *time of attraction*

$$\text{(2)} \qquad T = \inf\{k \geq 0 : \forall n \geq k, \{I_n, I_{n+1}\} = \{I_{n+1}, I_{n+2}\}\}.$$

This random variable is an important statistic, useful for applications.

In particular, we obtain exact (up to multiplicative constant) asymptotics of $P(T > k)$, as $k \to \infty$, if the underlying graph has two edges. Let us denote the law of the ERRW on the graph with two edges by P^2. Next we show some extensions in the setting of finite and bounded degree infinite graphs. In particular, we obtain that for any $\mathcal{G}$ finite graph, $P(T > k)$ is up to a multiplicative constant of the same order as $P^2(T > k)$, where the multiplicative constant depends on w, ℓ_0^e, $e \in E(\mathcal{G})$ and, in the case of upper bound, exponentially on the size of $\mathcal{G}$. Therefore, we are able to obtain only partial results on the asymptotics of $P(T > k)$ on infinite graphs. A nice corollary of our analysis is that if the reinforcement weight has the form $W(k) = k^\rho$, $\rho > 1$, then (universally over finite graphs) the expected time to attraction is infinite if and only if $\rho \leq 1 + \frac{1+\sqrt{5}}{2}$.

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Trap Models for Spin Glass Dynamics

Gérard Ben Arous
(joint work with A. Bovier, J. Cerny)

We consider trap dynamics for (mean-field) p-spin glasses. We show a REM dynamical universality in the following sense. The clock process properly normalized converges to an α-stable subordinator as for the REM dynamics. This implies the same aging formula than for REM. These facts are valid in time scales $\exp(cN)$ for c smaller than a constant $c(p)$, and for $p \geq 3$ (i.e. not for the Sherrington Kirkpatrick model).

The important feature of the proof is to show that along the trajectory of a standard random walk on the hypercube $\{-1,1\}^{\mathbb{N}}$ with $\exp(cN)$ steps, the point process of the extreme values of energies is Poisson like for the REM, i.e. that the correlations are irrelevant. This reminds of the statics (equilibrum) REM conjectureof Bauke-Mertens, as proved by J. Chayes et al., A. Bovier - I. Kurkova. The same striking difference between SK model ($p = 2$) and p-spin models ($p \geq 3$) has been discovered recently in a joint work with A. Kuptsov.

Scaling limit for trap models in the complete graph

Luiz Renato Fontes
(joint work with Pierre Mathieu)

Generally speaking, what we mean by a trap model is a simple symmetric random walk on a regular graph $G = (V, E)$ in continuous time, where the jump rate at $x \in V$ is given by τ_x^{-1}, with

$$\tau := \{\tau_x,\, x \in V\} \text{ i.i.d. and} \tag{1}$$

such that $\mathbb{P}(\tau_x > 0) = 1$ and

$$\mathbb{P}(\tau_x > t) = L(t)/t^{\alpha}, \quad t > 0, \tag{2}$$

where $0 < \alpha < 1$ and $L : (0, \infty) \to (0, \infty)$ is slowly varying at ∞, so that τ_x is in the basin of attraction of a positive α-stable law.

They have been considered as models for processes exhibiting *localization* and/or *aging*. For example, let X be the trap model on $\mathbb{Z}$, starting at say the origin. Then one can show that [1]

$$\lim_{t\to\infty} \mathbb{E} \sum_k [\mathbb{P}(X_t = k|\tau)]^2 > 0, \tag{3}$$

and thus localization takes place (since the above sum is bounded above by $\max_k \mathbb{P}(X_t = k|\tau)$). The following is an aging result for X [1]

$$\lim_{t\to\infty} \mathbb{P}(X_t = X_{t+\theta t}) = R(\theta), \tag{4}$$

where R is a non-trivial function of θ. (This says that in order to have a reasonable chance to make a different observation of X after t, when X_t was observed, we must wait for a time of the order of t — notice that $R(0) = 1$ —; this ever longer times to make reasonably uncorrelated observations are characteristic of aging.) Both results above can be/are obtained from the scaling limit of X under the proper rescaling of space and time. Indeed, let us for simplicity of notation assume that $L(t) \to$ const as $t \to \infty$ (see (2)), and make

$$Z_t^{(\varepsilon)} = \varepsilon X_{\varepsilon^{-(1+\alpha)/\alpha} t},\ t \geq 0, \quad \rho^{(\varepsilon)} = \{\varepsilon^{1/\alpha}\tau_{\varepsilon^{-1}x},\ x \in \varepsilon\mathbb{Z}\}. \tag{5}$$

Then $(Z^{(\varepsilon)}, \rho^{(\varepsilon)})$ converge in distribution to the pair (Z, ρ), where ρ is the random measure whose distribution function is an α-stable process (with positive increments), and, given ρ, Z is a (one dimensional) diffusion with speed measure ρ. See [1] for more details, including for the general L case. The convergence is strong enough so that the limits in (3) and (4) can be expressed as $\mathbb{E}\sum_x[\mathbb{P}(Z_1 = x|\rho)]^2$ and $\mathbb{P}(Z_1 = Z_{1+\theta})$, respectively. (Notice that these are nonzero/nontrivial, since, although Z is a process with almost every path continuous, its single and double (deterministic) time distribution is (purely) atomic.) Similar aging and scaling limit results have been obtained in $\mathbb{Z}^d$, $d \geq 2$ [2, 3], and for a variant of the trap model in $\mathbb{Z}$ [4].

Trap models on large finite graphs, like the complete graph and the hypercube, have been considered in the study of aging. They can be seen as a simplified model of a dynamics for a spin glass at low temperature. Indeed, let us consider the simplest spin glass, the Random Energy Model (REM), and the set of its configurations, $C_N = \{-1, +1\}^N$, seen as hypercube, and the Random Hopping Times (RHT) dynamics for it. This is a simple symmetric random walk in C_N in continuous time with inverse jump rate at $\sigma \in C_N$ given by the the Gibbs factor of the REM, $e^{-\beta\sqrt{N}H(\sigma)}$, where $\{H(\sigma),\ \sigma \in C_N\}$ is a family i.i.d. standard Gaussian random variables, and $\beta > \beta_c = \sqrt{2\log 2}$, which indicates low temperature. This model looks a lot like the trap model in the hypercube. The difference is that in the latter model the analogue of the τ variables don't quite have the same marginal distribution. However, in times of the scale of their maxima, the (properly rescaled) REM Gibbs factors behave in the same way as the rescaled τ_i's, namely as an α-stable positive-increment process (where $\alpha = \beta_c/\beta$ for the RHT dynamics). Furthermore, both models in the hypercube, at diverging time scales, should be close to the respective models in the complete graph with $n = 2^N$ vertices. This is based in the assumption that the time spent by the processes between visits to the few vertices with large inverse jump rates (traps) is negligible in comparison to the time spent at the traps. Bouchaud and Dean [5] departed from this ansatz to study the trap model in the complete graph, and derive aging results for it. Then Ben Arous, Bovier and Gayrard [6] obtained the same aging results for the

RHT dynamics in the hypercube, thus establishing the ansatz as far as aging is concerned

Motivated by the above mentioned scaling limit approach to aging, we consider the trap model in the complete graph in the time scale of the deepest traps. Let us again assume $L(t) \to \text{const}$ as $t \to \infty$. Let us first (re)label the set of vertices of the complete graph with n vertices as $\{1, \dots, n\}$, such that $\tau_i = \tau_i^{(n)}$ is the i-th order statistic of τ in decreasing order. Let now

$$(6) \qquad Y_t^{(n)} = X_{n^{-1/\alpha}t},\ t \geq 0, \quad \gamma^{(n)} = \{n^{-1/\alpha}\tau_i^{(n)},\ 1 \leq i \leq n\}.$$

Then $(Y^{(n)}, \gamma^{(n)})$ converges in distribution to the pair (Y, γ), where $\gamma = \{\gamma_i,\ i \geq 1\}$ are the increments of an α-stable subordinator in $[0, 1]$ in decreasing order, and, given γ, Y is a $K(\gamma, 0)$ process [7], or K process, for short. The latter is a process in $\bar{\mathbb{N}}^* = \mathbb{N}^* \cup \{\infty\}$, where $\mathbb{N}^* = \{1, 2 \dots\}$, characterized by the following properties [7]: (i) Y is càdlàg and strongly Markovian; (ii) starting from any $i \in \mathbb{N}^*$, Y waits for an exponential time of mean γ_i, and then jumps to ∞; (iii) starting from ∞, for any finite $A \subset \mathbb{N}^*$, we have that $T_A < \infty$ almost surely, where $T_A = \inf\{t \geq 0 : Y(t) \in A\}$ is the entrance time of Y in A, and the law of $Y(T_A)$ is uniform on A; and (iv) the set of times spent by Y at ∞ is a Cantor set of vanishing Lebesgue measure. See [7] for more details, including a construction.

The $K(\gamma, 0)$ process belongs to a larger class of Markov processes in $\bar{\mathbb{N}}^*$, the $K(\gamma, c)$ processes, where $\gamma = \{\gamma_i,\ i \geq 1\}$, with $\sum_i \gamma_i < \infty$, and $c \geq 0$ are parameters. They also satisfy (i-iii) above, but not (iv) if $c > 0$; an example of such a case was proposed in [8]. See [7] for more details.

The properly rescaled trap model in the hypercube, as well as the RHT dynamics for the REM both in the hypercube and the complete graph, also converge to K processes [9]. The latter processes and variants thereof should show up in dynamics associated to other mean field spin glasses, like the Generalized Random Energy Model (GREM).

Y exhibits aging as follows. Y is ergodic, and thus at order 1 or larger times, it is close to or at equilibrium. Since aging is a far from equilibrium phenomenon, in order to move away from that, we take $\lim_{t \to 0}$ of an aging function similar to the one in (4), and get the following result [7]. For almost every γ

$$(7) \qquad \lim_{t \to 0} \mathbb{P}_\infty(Y_t = Y_{t+\theta t} | \gamma) = \Pi(\theta) := \frac{\sin(\pi\alpha)}{\pi} \int_{\frac{\theta}{1+\theta}}^1 s^{-\alpha}(1-s)^{\alpha-1}\, ds, \ \ \theta \geq 0.$$

This is the same limit obtained in [5] in a different *regime*: the n-limit is taken first (as above, where it is implicit) without rescaling time (above, time is rescaled in the order of the deepest traps); then the $\lim_{t\to\infty}$ is taken.

In [10], a yet different regime is considered, with the n limit taken together with time rescaled in an order short of the one of the deepest traps. This is obtained from the the scaling limit of the *clock process*, with the same rescaling, given by an α-stable subordinator.

The K process also has a clock process, and similarly as in the just mentioned result, it should also have a scaling limit, but for vanishing times as in (7), given by an α-stable subordinator. This would imply the aging result in (7). That aging result would also follow from a scaling limit of Y itself, for example, in the following form. Let $W_t^{(\varepsilon)} = \varepsilon^{-1}\gamma_{Y_{\varepsilon t}}$, $t \geq 0$. Then there's reason to conjecture that $W^{(\varepsilon)}$ converges in distribution as $\varepsilon \to 0$ to a nontrivial process W for almost every γ. W should be related in a simple form to an α-stable subordinator.

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Exploring near-critical percolation

Wendelin Werner

(joint work with Pierre Nolin)

We consider site percolation on the triangular planar lattice. Recall that this can be viewed as a random coloring of the hexagonal cells of a honeycomb lattice, where the color (black or white) of each cell is chosen independently: each of these cells has a probability p to be black and $1-p$ to be white, for some parameter p between 0 and 1. One is then interested in the connectivity properties of the set of black hexagons. They can be regrouped into connected components (or clusters). Percolation on this lattice presents a phase transition at $p = 1/2$: when $p < 1/2$ we observe (a.s.) an infinite cluster of white sites (subcritical regime), and when $p > 1/2$ an infinite cluster of black sites (supercritical regime). The intermediate regime, when $p = 1/2$, is called the critical regime.

A lot of progress has been made recently in the understanding of the large-scale behavior of critical percolation. In particular, Smirnov [5] proved the conformal invariance of the connection probabilities, which allowed to make the link with the Schramm-Loewner Evolution (SLE) with parameter 6 introduced by Schramm [4], and to use the SLE technology and computations of [2] to derive further properties of critical percolation, such as the value of certain critical exponents describing the asymptotic behavior of the probabilities of certain exceptional events (arm exponents), see for instance [6].

One precise relation to SLE goes as follows: Consider the large equilateral triangle T_N with side length N on the triangular grid such that the middle of the bottom part is the origin and the top point is the point at distance $\sqrt{3}N/2$ above the origin. We decide to color all cells on the boundary of the triangle, in white if their x-coordinate in positive and in black if their x-coordinate is negative, and we perform percolation in the inside of T_N. Then, we consider the interface γ^N between the set of black clusters attached to the left part of the triangle and the set of white clusters connected to the right part of the triangle. When $N \to \infty$ and $p = 1/2$, the law of γ^N/N converges (in an appropriate topology, see e.g. [1]) to that of the SLE(6) process from $(0, 0)$ to $(0, \sqrt{3}/2)$ in the equilateral triangle with unit side length.

For each p and N, we call $R(p, N)$ the probability that γ^N hits the right side of the triangle before the left side. Note that this can be expressed as a crossing probability (from the lower-left side to the right side). This is an increasing function of p, and $R(1/2, N) = 1/2$ because of symmetry.

Understanding the behavior of critical percolation allows also to derive some properties of percolation when the parameter p is very close to $1/2$ thanks to the scaling relations (or hyperscaling relations) that were first developed in the physics literature, and later rigorously derived in the case of percolation by Kesten [3]. For any $\varepsilon > 0$, one can define

$$p^*(N) = p^*(\varepsilon, N) = \inf\{p \ : \ R(p, N) > 1/2 + \varepsilon\}.$$

For this choice of $p = p^*(N)$, if one looks at the possible limiting behavior of γ^N/N, it is clear that it can not be exactly SLE(6) anymore, because it will hit the right side of the triangle before the left side with probability strictly larger than $1/2$. It is therefore natural to ask what can happen to the scaling limit of this curve in this regime, and to see how it is related (or not) to SLE(6): will its law be absolutely continuous with respect to the law of SLE(6) (just as the appropriate limit of biased random walk is absolutely continuous with respect to Brownian motion)?

One can define the so-called correlation length $L(p) = L(p, \varepsilon)$ in such a way that $p^*(\varepsilon, L(p)) \simeq p$. In other words, for $p > 1/2$,

$$L(p) = L(p, \varepsilon) = \sup\{N \ : \ R(p, N) < 1/2 + \varepsilon\}.$$

(recall that $R(p, N) \to 1$ as $N \to \infty$ because one is in the supercritical regime). Kesten has shown that it is possible to deduce from the exponents of critical percolation the behavior of $L(p)$ as $p \to 1/2$. Intuitively, it is in fact clear that the "four-arm exponent" will be essential i.e. one has to flip at least a "pivotal" site to increase the crossing probability. More precisely, combining Kesten's results with the exponents computed using SLE, one gets (see [6]) that

$$L(p) = (p - 1/2)^{-4/3+o(1)}$$

when $p \to 1/2+$, for any fixed choice of $\varepsilon > 0$.

Note that in order to get a non-trivial limit for γ^N/N (i.e. neither SLE(6) nor a process that hits the right-hand side before the left one with probability one), one has to take $p(N)$ in such a way that N is of the order of the correlation length $L(p)$ i.e. that for some ε i.e. n such a way that $p(N)$ and $p^*(\varepsilon, N)$ are very close for some fixed ε. Letting $p(N)$ go to the critical value in order to get non-trivial limits is often referred to as "finite-size scaling" and has been the subject of numerous and interesting works in the physics and mathematics community.

Russo-Seymour-Welsh type arguments show that when the family of laws of γ^N/N for $N \geq 2$ and $p(N) = p^*(\varepsilon, N)$ is relatively compact. Suppose now that γ is a (subsequential) limit. Then:

- The law of γ is singular with respect to that of SLE(6)
- It is still a random curve with dimension $7/4$.

The fact that the random curve is still of the same dimension (and more generally has the same critical exponents) follows from arguments similar to those of Kesten in [3]. One way to explain the fact that the law of γ is singular with respect to SLE(6) is the following. In the finite-size scaling regime, one sees on a macroscopic scale a difference between the law of the interface and that of the critical percolation interface (i.e. the non-critical interface is more to the "right" for instance). If one zooms in by a factor λ, one still sees a difference, but this difference tends to disappear, because one is not looking at a picture of size ca. $L(p)$ any more, but at a picture of size $L(p)/\lambda$. The question is whether this difference disappears sufficiently fast when $\lambda \to \infty$ or not. Note that one can show that (just as for the critical interface) the number of boxes of size N/λ visited by the path is of order $\lambda^{7/4}$ when λ is large (and N very large). Either this difference vanishes fast with λ and one is not able to almost surely detect a difference between the two macroscopic interfaces, or the difference between these two behaviors can be detected by averaging them out over the $\lambda^{7/4}$ parts of the path. In the end, one has to compare certain critical exponents to decide which scenario is correct and it turns out that for percolation, the second scenario holds. The "flavor" of super-critical percolation introduced by considering the correlation length is therefore still present in the scaling limit, and this regime can be considered as a truly intermediate regime between critical and super-critical percolation. The goal of the talk is to give an outline of the proof, and to explain what asymetry feature one is able to detect.

Seemingly, this surprises some theoretical physicists on this topic. Recall that one important aspect of the SLE approach to critical systems was precisely to show that critical conformally invariant models in the same "universality class" give rise to exactly the same curves in the scaling limit. For near-critical models that are not strictly conformally invariant such as the near-critical percolation here, this strong "universality" can fail to be true. If one encodes the curve via its Loewner driving function, then this driving function is probably much more complicated than Brownian motion. On the other hand, one can see that some important properties (such as the dimension of the curve) are shared by the critical interface and the near-critical interfaces in the scaling limit, so that the technology based on conformal invariance of the critical model still provide the correct description of the near critical interfaces in terms of exponents.

We emphasize that we do not really use any fine SLE technology to derive our results. They follow from general considerations based on Kesten's scaling ideas and the knowledge of the exact value of the critical exponents (the derivation of which however used SLE).

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On the disconnection of a discrete cylinder by a biased random walk

David Windisch

We consider a variation on the problem of "the termite in a wooden beam", introduced by Dembo and Sznitman [2] (see also [1], [3], [6], [7] for related work).

To this end, we define the *discrete cylinder* $E = \mathbb{T}_N^d \times \mathbb{Z}$, $d \geq 3$, where $\mathbb{T}_N^d$ denotes the d-dimensional integer torus $(\mathbb{Z}/N\mathbb{Z})^d$. The cylinder E is equipped with the Euclidean distance $|.|$ and the natural product graph structure, for which all vertices x_1, $x_2 \in E$ with $|x_1 - x_2| = 1$ are connected by an edge. The (discrete-time) *random walk with drift* $N^{-d\alpha}$ ($\alpha > 0$) is the Markov chain $(X_n)_{n\geq 0}$ on E with starting point $x \in E$ and transition probability

$$p_X(x_1, x_2) = \frac{1 + N^{-d\alpha}(\pi_{\mathbb{Z}}(x_2 - x_1))}{2d+2}\mathbf{1}_{\{|x_1 - x_2| = 1\}}, \quad x_1, x_2 \in E,$$

where $\pi_{\mathbb{Z}}$ denotes the projection from E onto its $\mathbb{Z}$-component. In particular, under P_0^0, X is the ordinary simple random walk on E. We say that a set $K \subseteq E$ *disconnects* E if $\mathbb{T}_N^d \times (-\infty, -M]$ and $\mathbb{T}_N^d \times [M, \infty)$ are contained in two distinct components of $E \setminus K$ for large $M \geq 1$. The central object of interest is the *disconnection time*

$$T_N^{disc} = \inf\{n \geq 0 : X([0, n]) \text{ disconnects } E\}.$$

We consider drifts of the form $N^{-d\alpha} = |\mathbb{T}_N^d|^{-\alpha}$, $\alpha > 0$. Our main result shows that the asymptotic behaviour of T_N^{disc} as $N \to \infty$ is the same as in the case without drift considered in [2] as long as $\alpha > 1$, and becomes exponential in N when $\alpha < 1$:

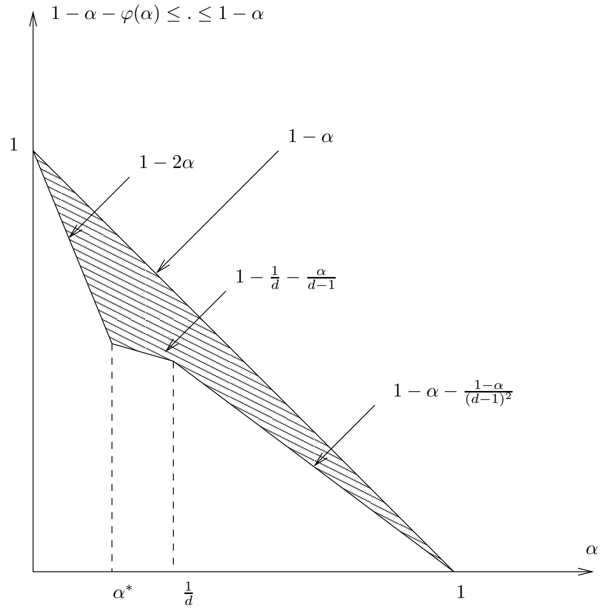

FIGURE 1. The shaded region lies between the exponents of the upper and lower bounds in Theorem 1 for $\alpha \in (0,1)$.

Theorem 1. $(d \geq 3, \alpha > 0, \epsilon > 0)$

$$\text{(1)} \qquad \begin{array}{lrcccl} \textit{For } \alpha > 1, & N^{2d-\epsilon} & \leq & T_N^{disc} & \leq & N^{2d+\epsilon}, \\ \textit{for } \alpha < 1, & \exp\{N^{d(1-\alpha-\varphi(\alpha))-\epsilon}\} & \leq & T_N^{disc} & \leq & \exp\{N^{d(1-\alpha)+\epsilon}\}, \end{array}$$

with probability tending to 1 *as* $N \to \infty$*, where the continuous function* $\varphi : (0,1) \to \big(0, \frac{1}{d-1}\big)$ *satisfies* $\lim_{\alpha\to 0} \varphi(\alpha) = \lim_{\alpha \to 1} \varphi(\alpha) = 0$ *(see Figure 1 for an illustration of the region between* $1-\alpha-\varphi(\alpha)$ *and* $1-\alpha$*).*

We give a brief outline of the ideas entering the proof of this result. The proof of the upper bounds on T_N^{disc} is based on the simple observation that the cylinder E is disconnected as soon as a slice of the form $\mathbb{T}_N^d \times \{z\} \subseteq E$ is completely covered by the walk. We thus show that the trajectory of the random walk X up to time $N^{2d+\epsilon}$ (for $\alpha > 1$), respectively $\exp\{N^{d(1-\alpha)+\epsilon}\}$ (for $\alpha < 1$), does cover such a slice with probability tending to 1 as $N \to \infty$. To this end, we exploit the Markovian structure of the locations of successive visits to the slice $\mathbb{T}_N^d \times \{z\}$ made by the random walk, and then apply a coupon-collector-type estimate on the cover time to the process recording these visits.

The derivation of the lower bounds is substantially more delicate, since one has to rule out the occurrence of any, and not just one particular kind of, interface in the trajectory of the random walk up to a certain time with high probability. We reduce the problem of finding a lower bound on T_N^{disc} to a large deviations problem concerning the disconnection of a certain finite subset of E by excursions of an unbiased simple random walk, and then derive estimates on this large deviations problem. Let us describe this problem in a little more detail. For any subsets K, $B \subseteq E$, B finite, we say that K $\frac{1}{3}$*-disconnects* B if K contains the relative boundary in B of a subset of B with relative volume between $\frac{1}{3}$ and $\frac{2}{3}$. The set whose disconnection concerns us is

$$B(\alpha) = \left[-\left[\frac{N}{4}\right], \left[\frac{N}{4}\right]\right]^d \times \left[-\left[\frac{N^{d\alpha\wedge 1}}{4}\right], \left[\frac{N^{d\alpha\wedge 1}}{4}\right]\right] \subseteq S_{2[N^{d\alpha\wedge 1}]}.$$

We define U as the first time when the trajectory of the random walk $\frac{1}{3}$-disconnects $B(\alpha)$, that is

$$U = \inf\left\{n \geq 0 : X([0,n]) \ \frac{1}{3}\text{-disconnects } B(\alpha)\right\}. \tag{2}$$

The random walk excursions featuring in the large deviations problem are excursions in and out of slices of the form

$$S_u = \mathbb{T}_N^d \times \big[-[u],[u]\big] \subseteq E \quad (u > 0).$$

Let us suppose there is a non-negative function f on $(0,\infty)^2$ with the following property: For $(\mathcal{R}_n)_{n\geq 1}$, $(\mathcal{D}_n)_{n\geq 1}$, the times of the successive returns to $S_{2[N^{d\alpha\wedge 1}]}$ and departures from $S_{4[N^{d\alpha\wedge 1}]}$ and the stopping time defined in (2) one has

$$\overline{\lim_{N\to\infty}} \frac{1}{N^\xi} \log \sup_{x\in S_{2[N^{d\alpha\wedge 1}]}} P_x^0\left[U \leq \mathcal{D}_{[N^\beta]}\right] < 0, \quad \text{for any } 0 < \xi < f(\alpha,\beta). \tag{3}$$

One can then show that, if the above function f additionally satisfies $f(\alpha,\beta) > 0$ for all $\alpha > 1$, $\beta \in (0, d-1)$, then the lower bound for $\alpha > 1$ in (1) holds, while the lower bound for $\alpha < 1$ holds with the exponent $\sup_{\beta>0}\big(\beta - (d\alpha-1)_+\big) \wedge f(\alpha,\beta)$. We then show that indeed (3) does hold for a function f such that the lower bounds of Theorem 1 follow (with a separate and rather straightforward argument for $\alpha < \frac{1}{d}$). The key techniques in the reduction of the lower bounds to the large deviations problem (3) are a geometric lemma in the spirit of [2], a Girsanov-type control to get rid of the drift and elementary estimates on one-dimensional biased random walk. In order to prove that (3) does hold for a certain function f, we use more geometric lemmas employing an isoperimetric inequality from [4]. These lemmas show that any trajectory $\frac{1}{3}$-disconnecting $B(\alpha)$ must have substantial presence in many small subcubes of $B(\alpha)$. The key control on an event of this form essentially follows from a tail estimate on the number of points inside the small subcubes visited by the random walk stopped when exiting a large set. In order to show this tail estimate, we apply Khaśminskii's Lemma (see [5]) to infer that if one divides the number of visited points by its expectation, one obtains a random variable whose exponential moment is uniformly bounded with N. The expected number of visited points can then be bounded with standard estimates on the Green function of the simple random walk.

An obvious question arising from Theorem 1 is whether one can prove the same result with $\varphi \equiv 0$ in (1). We show that this would follow directly from an estimate of the form (3) with a certain function f^*, larger than the f we currently have. In fact, the required function f^* can be shown to be the correct exponent associated to a large deviations problem (possibly) similar to (3), where one replaces the time U by U', defined as the first time when the trajectory of X covers $\mathbb{T}_N^d \times \{0\}$ (and thereby in particular $\frac{1}{3}$-disconnects $B(\alpha)$.

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Lamplighter random walks

Wolfgang Woess

Suppose that we have some (infinite, connected) graph where a lamp is located at each vertex, which may be in one of the two states "off" or "on". In the simplest model, a "lamplighter" performs a random walk along the graph; at each step, he can choose at random whether to move to a neighboring vertex or to change the state of the lamp at the current position (or he can combine different random choices of this type). Initially, all lamps are "off", so that after a finite number of steps, one observes (1) the actual position of the "lamplighter", and (2) the current configuration of the lamps that are "on" [=finite set of vertices]. All pairs (position, configuration) of this type constitute the state space of a Markov chain, and the successive random configurations are driven by a random walk on the base graph. [The underlying algebraic construction is that of the wreath product of groups.]

The main interest is in the interplay between geometrical properties of the underlying graph and the behaviour of the process.

In the talk, I have given an introductory survey, mentioning results by various authors [Pittet, Saloff-Coste, Erschler, and others], including hints at work of myself with Bartholdi, Brofferio, and Karlsson, respectively.

Random billiards and random chords in general domains

Serguei Popov

(joint work with Francis Comets, Gunter M. Schütz, Marina Vachkovskaia)

In [1], we consider a stochastic process that can be informally described as follows. A particle moves with constant speed inside some d-dimensional domain. This domain is supposed to be bounded, with a.e. continuously differentiable Lipschitz boundary. When the particle hits the domain boundary, it is reflected in some random direction, not depending on the incoming direction, and keeping the absolute value of its speed. The law of the reflection is supposed to be absolutely continuous and supported on the corresponding half-sphere. We denote by ξ_i, $i = 0, 1, 2, \dots$ the successive locations of the particle at the moments it hits the boundary, the process ξ is referred to as the random walk. Also, let X_t be the position of the particle at time t, and V_t be the corresponding direction (or vector speed). We refer to the process (X_t, V_t) as the stochastic billiard.

Let $K(x, y)$ be the transition density of the random walk, so that

$$\mathbf{P}[\xi_{n+1} \in A \mid \xi_n = x] = \int_A K(x, y).$$

When considering the specific case of the cosine law of reflection (i.e., the reflection density is proportional to the cosine of the angle with the normal vector), we write $\tilde{\xi}, \tilde{X}$ instead of ξ, X, and we call these processes *Knudsen random walk* (KRW) and *Knudsen stochastic billiard* (KSB); $\tilde{K}$ stands for the transition density of the Knudsen random walk.

Define $\hat{\mu}_0$ to be the "uniform" probability measure on $\partial\mathcal{D}$: $\hat{\mu}_0(A) = \frac{|A|}{|\partial\mathcal{D}|}$. It can be easily seen that $\tilde{K}(x, y)$ is symmetric, and so we immediately obtain that the KRW $\tilde{\xi}$ is reversible, with the reversible (and thus invariant) measure $\hat{\mu}_0$. For other reflection laws it is usually not easy to find the exact form of the invariant measure (except for some particular cases, see [2]), but nevertheless we prove that such a measure exists and is unique, the random walk converges to it exponentially fast.

Theorem 1. (i) *There exists a unique probability measure $\hat{\mu}$ on $\partial\mathcal{D}$ which is invariant for the random walk ξ_n. Moreover, there exists a function $\psi : \partial\mathcal{D} \to \mathbb{R}_+$ such that $\hat{\mu}(A) = \int_A \psi(x)\, dx$ which satisfies*

$$\psi(x) = \int_{\partial\mathcal{D}} \psi(y) K(y, x)\, dy. \tag{1}$$

Finally, the density ψ can be chosen in such a way that $\inf_{\partial\mathcal{D}} \psi > 0$.

(ii) *There exist positive constants β_0, β_1 (not depending on the initial distribution of ξ_0) such that*

$$\|\mathbf{P}[\xi_n \in \cdot] - \hat{\mu}\|_{\mathsf{v}} \le \beta_0 e^{-\beta_1 t}, \tag{2}$$

where $\|\cdot\|_{\mathsf{v}}$ is the total variation norm.

In particular, for the KRW, (2) holds with $\hat{\mu} = \hat{\mu}_0$.

Next, we show that the stochastic billiard converges exponentially fast to equilibrium, and we characterize the stationary measure (which is explicit for KSB). Let μ_0 be the uniform measure on $\mathcal{D}$ and ν_0 be the uniform measure on the $(d-1)$-dimensional sphere.

Theorem 2. *There exist a probability measure χ on $\mathcal{D} \times \mathbb{S}^{d-1}$ and positive constants β'_0, β'_1 (not depending on the initial distribution of position and direction) such that*

$$\|\mathbf{P}[X_t \in \cdot, V_t \in \cdot] - \chi\|_{\mathsf{v}} \le \beta'_0 e^{-\beta'_1 t}, \tag{3}$$

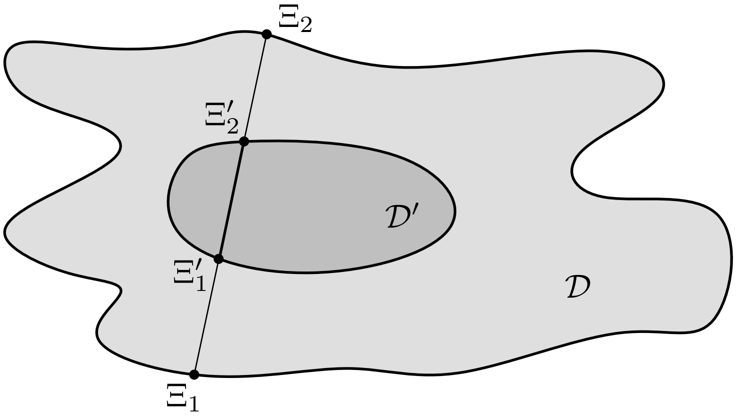

FIGURE 1. A random chord on $\mathcal{D}'$ induced by a random chord on $\mathcal{D}$

for all $t \geq 0$. The invariant measure χ is absolutely continuous with respect to $\mu_0 \otimes \nu_0$, and is given by

$$\chi(dx, dv) = \psi(z)\, \frac{\bar{\gamma}(U_z^{-1}v)}{\cos\phi_z(v)} dx\, dv,$$

where z is the point on the boundary which sees x in the direction v, and $\bar{\gamma}(U_z^{-1}v)$ is the density of the outgoing velocity, suitably rotated, see [1] *for details. In particular, the product measure $\mu_0 \otimes \nu_0$ is invariant for KSB $(\tilde{X}_t, \tilde{V}_t)$.*

For a domain $\mathcal{D}$, we construct the random chord as follows: take a point on $\partial\mathcal{D}$ uniformly at random, and draw a line from there using the cosine probability distribution. Formally:

Definition 3. *The random chord for a bounded domain $\mathcal{D}$ is a pair of random variables (Ξ_1, Ξ_2), in $\partial\mathcal{D}$, with the joint density $|\partial\mathcal{D}|^{-1}\tilde{K}(x, y)$.*

Let m be the mean chord length, we prove the following fact: the area (volume, ...) of the domain is always proportional to the product of its perimeter (surface area, ...) and m:

$$|\mathcal{D}| = \frac{m|\partial\mathcal{D}|}{\kappa_d}, \tag{4}$$

where $\kappa_d := \frac{\pi^{1/2}\Gamma(\frac{d+1}{2})d}{\Gamma(\frac{d}{2}+1)}$.

If $\mathcal{D}' \subset \mathcal{D}$ is convex, a chord of $\mathcal{D}$ which intersects $\mathcal{D}'$ defines a unique chord on $\mathcal{D}'$ by its intersection (see Figure 1). Let us generate independent random chords $(\Xi_1(i), \Xi_2(i))$ of $\mathcal{D}$, $i = 1, 2, \ldots$, till the chord hits the domain $\mathcal{D}'$, and then denote by (Ξ_1', Ξ_2') the intersection. We call (Ξ_1', Ξ_2') the *induced* chord on $\mathcal{D}'$. Like in the "acceptance-rejection" algorithm (or "hit or miss") for random variable simulation, we easily check that (Ξ_1', Ξ_2') has the same law as the endpoints of $[\Xi_1, \Xi_2] \cap \mathcal{D}'$ given that these sets intersect.

Theorem 4. *Let $\mathcal{D}' \subset \mathcal{D}$ be convex. Then, the chord induced on $\mathcal{D}'$ by the random chord of $\mathcal{D}$ is the random chord of $\mathcal{D}'$ in the sense of Definition 3.*

A similar result (with suitable modifications), can be proven for the case of nonconvex $\mathcal{D}'$ as well. Namely, the expected value of the measure giving unit weight to each chord induced on $\mathcal{D}'$ by the $\mathcal{D}$-random chord, given it intersects $\mathcal{D}'$, is the product of the expected number of induced chords given there is one at least, and the probability distribution for the (ordered) endpoints of the random chord of $\mathcal{D}'$. Also, if ι is the number of the induced chords, we have

$$\mathbf{E}[\iota|\ \iota \geq 1] = \frac{|\partial\mathcal{D}'|}{|\partial \mathrm{Conv}\mathcal{D}'|}\ , \quad \mathbf{E}[\iota] = \frac{|\partial\mathcal{D}'|}{|\partial\mathcal{D}|};$$

in particular, the number ι of induced chords is integrable, a property which does not seem easy to prove directly.

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