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Algebraische Zahlentheorie

Organised by
Guido Kings, Regensburg
Mark Kisin, Chicago
Otmar Venjakob, Heidelberg

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Abstract. The conference brought together researchers from Europe, the US and Japan who reported on various recent developments in algebraic number theory and related fields. As at previous meetings, a clear theme was the prevalence of p-adic methods.

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Introduction by the Organisers

The conference brought together researchers from Europe, the US, and Japan who reported on various recent and ongoing developments in algebraic number theory and related fields. As at previous meetings, organized by Deninger, Schneider and Scholl, one of the clearest themes was the prevalence of p-adic methods across a range of areas. A notable difference with previous years was the number of younger people both as speakers and participants.

Colmez reported on his work relating unitary admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to local Galois representations. This realizes a program of Breuil and stands at the crossroads of p-adic Hodge theory, representations of p-adic reductive groups and explicit reciprocity laws, as well as having applications to modularity of global Galois representations. Related talks were given by Schneider who explained on going work with Vigneras, attempting to generalize some of Colmez' constructions to higher rank, as well as Orlik who discussed the construction of locally analytic representations from equivariant vector bundles on symmetric spaces.

L. Berger reported on an extension of his earlier work on classification of local Galois representations. Hartl explained how these ideas could be used to give a description of the image of the Rapoport-Zink period morphism. This was a satisfying complement to his talk at the previous meeting where he had sketched some of these ideas.

There were several talks related to Iwasawa theory and reciprocity laws. Zerbes reported on her work on reciprocity laws for higher dimensional local fields. Fukaya reported on joint work with Coates, Kato, Sujatha and Venjakob in non-abelian Iwasawa theory, and Ochiai discussed the Iwasawa theory of ordinary Hida families. The talk by Sharifi was also related to this area. It described a fascinating relation between Galois cohomology and modular symbols, which seems to be closely related to the Main conjecture of Iwasawa theory.

There were a number of talks dealing with congruences between automorphic forms, and applications. The most exciting of these was by Fujiwara who outlined how Taylor-Wiles systems could be used, in certain circumstances, to prove the Leopoldt conjecture for totally real fields. Sorensen discussed his work on level raising for GSp_4 and some applications to Selmer groups. T. Berger explained how to construct Galois representations attached to cusp forms on GL_2 over an imaginary field. These had been constructed by Taylor about 15 years ago, but were previously known to have the correct L-factors only at a set of primes of density 1. Berger also explained ongoing work on modularity lifting theorems in this situation. This would be an exciting advance since such theorems are currently available only over totally real fields.

There were two talks on polylogarithms. Blottiere explained his results on the Eisenstein classes on Hilbert modular varieties and applications to special values of L-functions. Bannai discussed the crystalline realization of the elliptic polylogarithm. Somewhat related to this was the talk of Huber on the p-adic Borel regulator.

Other talks were given by Yoshida who explained a computation of vanishing cycles on Shimura varieties, realizing the local Langlands and Jacquet-Langlands correspondences, Görtz who spoke on affine Deligne-Lusztig varieties, Saito who outlined his construction of the characteristic cycle of an l-adic sheaf, and Schmidt who discussed his work on integer rings of type $K(\pi, 1)$.

Workshop: Algebraische Zahlentheorie

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B-pairs and (φ, Γ)-modules

Laurent Berger

The goal of the talk was to present some of the results from my article [1]. Let K be a p-adic base field, for example some finite extension of $\mathbf{Q}_p$. One of the aims of p-adic Hodge theory is to describe some of the p-adic representations of $G_K = \mathrm{Gal}(\overline{K}/K)$, namely those which "come from geometry", in terms of some more amenable objects. The most satisfying result in this direction is Colmez-Fontaine's theorem which states that the functor $V \mapsto \mathrm{D}_{\mathrm{st}}(V)$ gives rise to an equivalence of categories between the category of semistable p-adic representations and the category of admissible filtered (φ, N)-modules.

If D is a filtered (φ, N)-module coming from the cohomology of a scheme X, then the underlying (φ, N)-module only depends on the special fiber of X (it is its log-crystalline cohomology) and the filtration only depends on the generic fiber of X (it is its de Rham cohomology). If D_1 and D_2 are two filtered (φ, N)-modules and $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$ then the (φ, N)-modules D_1 and D_2 are isomorphic if and only if $(\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D_1)^{N=0,\varphi=1}$ and $(\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D_2)^{N=0,\varphi=1}$ are isomorphic as $\mathbf{B}_{\mathrm{e}}$-representations of G_K. Similarly, the filtered modules $K \otimes_{K_0} D_1$ and $K \otimes_{K_0} D_2$ are isomorphic if and only if $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D_1)$ and $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D_2)$ are isomorphic as $\mathbf{B}_{\mathrm{dR}}^+$-representations of G_K.

The main idea of [1] is to separate the phenomena related to the special fiber from those related to the generic fiber by considering not just p-adic representations but *B-pairs* $W = (W_e, W_{dR}^+)$ where W_e is a $\mathbf{B}_{\mathrm{e}}$-representation of G_K and W_{dR}^+ is a $\mathbf{B}_{\mathrm{dR}}^+$-representation of G_K and $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} W_{dR}^+$. If V is a p-adic representation, then one associates to it $W(V) = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)$ and this defines a fully faithful functor from the category of p-adic representations to the category of B-pairs. One can extend the usual definitions of p-adic Hodge theory from p-adic representations to all B-pairs. For example, we say that a B-pair W is semistable if $\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$ is trivial and it is easy to see that the functor $D \mapsto W(D)$ which to a filtered (φ, N)-module D assigns the semistable B-pair $W(D) = ((\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=1}, \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D))$ is an equivalence of categories.

One of the main general purpose tools which we have for studying p-adic representations is the theory of (φ, Γ)-modules. There is an equivalence of categories between the category of p-adic representations and the category of étale (φ, Γ)-modules over the Robba ring. The main result of [1] is that one can associate to every B-pair W a (φ, Γ)-module $D(W)$ over the Robba ring and that the resulting functor is then an equivalence of categories.

The article [1] includes some other results which were not discussed in the lecture, among which: a description of isoclinic (φ, Γ)-modules, an answer to a question of Fontaine regarding $\mathbf{B}_{\mathrm{cris}}^{\varphi=1}$-representations, and a description of finite height (φ, Γ)-modules.

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Weight spectral sequence and non-abelian Lubin-Tate theory

Teruyoshi Yoshida

This is a continuation of my talk in the same workshop two years ago about my joint work with R. Taylor on the compatibilty of local and global Langlands correspondences. In order to compute the local monodoromy of the Galois representation attached to conjugate self-dual cuspidal automorphic representation of GL_n over CM field, we studied the semistable reduction of certain unitary Shimura varieties with Iwahori level structure ([TY]). There, the weight spectral sequence ([RZ], [S]) corresponding to the cuspidal automorphic reprsentation was shown to degenerate at E_1-terms by somewhat mysterious vanishing of dimensions expressed as binomial coefficients. This was done by forgetting the action of local Hecke algebra (affine Iwahori Hecke algebra), because we did not need it to deduce the degeneration of the weight spectral sequence. In this talk we determine this action completely, using a general intersection-theoretic formula to compute the action of algebraic correspondences on weight spectral sequences. This leads to an observation that the computation was entirely of local nature – it suggests that the same method will compute purely locally the Hecke action on the cohomology of Lubin-Tate spaces with Iwahori level structure, partially recovering the results of Boyer obtained by global methods ([B]).

First we explain our formula on the action of algebraic correspondences on weight spectral sequences. Let K be a complete discrete valuation field with a finite residue field k and the ring of integers $\mathcal{O}_K$. Let X be a proper strictly semistable scheme of relative dimension $n-1$ over $\mathcal{O}_K$. Then its special fiber $Y := X \times_{\mathcal{O}_K} k$ is written as $Y = \bigcup_{i\in\Delta} Y_i$ with $\Delta := \{1, ..., t\}$ and Y_i proper smooth over k, where Y_i and Y_j intersect transversally for $i \neq j$. Let $Y_I := \bigcap_{i\in I} Y_i$ for $I \subset \Delta$, which is proper smooth over k of dimension $n - |I|$ if not empty, and $Y^{(m)} := \coprod_{|I|=m} Y_I$ for $1 \leq m \leq n$. For a prime $\ell \neq \mathrm{char} k$, the weight spectral sequence reads

$$E_1^{i,j} := \bigoplus_{s\geq \max(0,-i)} H^{j-2s}\big(Y^{(i+2s+1)} \times_k \overline{k},\ \overline{\mathbb{Q}}_\ell(-s)\big) \Longrightarrow H^{i+j}(X \times_K \overline{K},\ \overline{\mathbb{Q}}_\ell).$$

Now let Γ be an algebraic correspondence on X (namely an n-dimensional cycle on $X \times_{\mathcal{O}_K} X$) such that two projection maps $\Gamma \to X$ are both finite. We are interested in the action $[\Gamma_K]^* := \mathrm{pr}_{1*} \circ ([\Gamma_K]\cup) \circ \mathrm{pr}_2^*$ of $\Gamma_K := \Gamma \times_{\mathcal{O}_K} K$ on $H^*(X \times_K \overline{K}, \overline{\mathbb{Q}}_\ell)$. Let $Y_{I,J} := Y_I \times_k Y_J$ for $I, J \subset \Delta$, and write $Y_{i,j} := Y_{\{i\},\{j\}}$. Let $(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$ be the smooth locus of the morphism $X \times_{\mathcal{O}_K} X \longrightarrow \mathrm{Spec}\mathcal{O}_K$, and let $Y_{i,j}^0 := Y_{i,j} \cap (X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$. Then $Y_{i,j}^0$ is a Cartier divisor of $(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$.

Theorem A. *There is a unique collection* $\{\Gamma_{I,J}\}$ *of cycles* $\Gamma_{I,J}$ *on* $Y_{I,J}$ *for all pairs* (I,J) *with* $|I|=|J|$, *satisfying the following two conditions.*

(i) $\Gamma_{i,j}$ *is the closure of the cycle* $\Gamma^0_{i,j} := Y^0_{i,j}\cdot\Gamma|_{(X\times_{\mathcal{O}_K}X)_{\rm sm}}$ *in* $Y_{i,j}$.

(ii) *When* $|I|=|J|+1=m$ *and* $I=\{i_1,\dots,i_m\}$, $J=\{j_1,\dots,j_{m-1}\}$ *are in increasing order, there is an equality:*

$$\sum_{h=1}^{m}(-1)^h\,Y_{I,J}\cdot\Gamma_{I\backslash\{i_h\},J}=\sum_{j\in\Delta\backslash J}(-1)^{h(j)}\,\Gamma_{I,J\cup\{j\}}$$

of $(n-m)$*-dimensional cycles on* $Y_{I,J}$, *where* $1\le h(j)\le m$ *is determined by* $j_{h(j)-1}<j<j_{h(j)}$ *(set* $j_m:=\infty$*).*

Then setting $\Gamma^{(m)} := \coprod_{|I|=|J|=m}\Gamma_{I,J}$ *as an* $(n-m)$*-dimensional cycle on* $Y^{(m)}\times_k Y^{(m)}$ *for* $1\le m\le n$, *the action* $\oplus[\Gamma^{(i+2s+1)}]^*$ *on* $E_1^{i,j}$ *is compatible with the action* $[\Gamma_K]^*$ *on* $H^{i+j}(X\times_K\overline{K},\overline{\mathbb{Q}}_\ell)$.

For the proof of this theorem, we build on the construction of [S], except that we eliminate the semistable resolution of $X\times_{\mathcal{O}_K}X$ from the description of the cycles $\Gamma_{I,J}$, in order to apply the formula to the Shimura varieties where the cycles have concrete moduli interpretation. For this we also need the cycles, not only cycle classes.

Now we introduce a class of Shimura varieties containg those studied in [HT]. Let F be a CM field, with complex conjugation c, of the form $F=EF^+$, where $F^+\subset F$ is the fixed field of c and $E/\mathbb{Q}$ is imaginary quadratic. Let B be a simple algebra with center F and $\dim_F B=n^2$, with a positive involution $*$ with $*|_F=c$ and an alternating form $\langle,\rangle : B\times B\to\mathbb{Q}$ such that $\langle bx,y\rangle=\langle x,b^*y\rangle$ for $\forall b\in B$. Let G be the $\mathbb{Q}$-similitude group of $(B,\langle,\rangle)$. Then $G_0:=\mathrm{Ker}(G\to\mathbb{Q}^\times)$ is the restriction of scalars from a unitary group over F^+. We choose $\langle,\rangle$ so that $G_0(\mathbb{R})\cong U(1,n-1)\times U(0,n)^{d-1}$, where $d:=[F^+:\mathbb{Q}]$. For each open compact subgroup $U\subset G(\mathbb{A}^\infty)$ small enough (where $\mathbb{A}^\infty:=\widehat{\mathbb{Z}}\otimes\mathbb{Q}$), we define the Shimura variety X_U/F as a moduli of isogeny classes of quadruples $(A,\lambda,i,\eta U)$ of an abelian variety A of dimension dn^2, a polarization $\lambda : A\to A^\vee$, a ring homomorphism $i : B\to\mathrm{End}(A)\otimes\mathbb{Q}$, satisfying the Kottwitz conditon on Lie$A$ corresponding to G (see [K]) and $\lambda\circ i(b)=i(b^*)^\vee\circ\lambda$ for $\forall b\in B$, and a right U-orbit ηU of $B\otimes\mathbb{A}^\infty$-isomorphisms $\eta : B\otimes\mathbb{A}^\infty\longrightarrow VA:=\left(\varprojlim_N A[N]\right)\otimes\mathbb{Q}$ which sends $\langle,\rangle$ to the λ-Weil pairing. Then X_U/F is a quasi-projective smooth variety of dimension $n-1$, which is projective if $d>1$ or if B is a division algebra. We choose a place v of F lying over a prime p which splits in E. Then $G(\mathbb{Q}_p)$ is a product of $GL_n(F_v)$ and other factors, so set $G(\mathbb{A}^\infty)=G(\mathbb{A}^{\infty,v})\times GL_n(F_v)$. We set $U=U^v\times\mathrm{Iw}_n$ where $U^v\subset G(\mathbb{A}^{\infty,v})$ and Iw_n is the open compact subgroup of $GL_n(\mathcal{O}_v)$ consisting of matrices which reduce to upper triangluar matrices modulo v. For $U_0:=U^v\times GL_n(\mathcal{O}_v)$ the X_{U_0} extends to a smooth scheme over $\mathcal{O}_v$ with a universal abelian scheme $\mathcal{A}/X_{U_0}$. Then $\mathcal{G}:=\mathrm{diag}(1,0,\dots,0)\mathcal{A}[v^\infty]$ is a 1-dimensional Barsotti-Tate $\mathcal{O}_v$-module of $\mathcal{O}_v$-height n, i.e. $\mathcal{G}[v]$ is a finite flat group scheme of degree $|k(v)|^n$, and the moduli of chain of n isogenies each of degree

$|k(v)|$ factoring $\mathcal{G} \to \mathcal{G}/\mathcal{G}[v]$ gives a regular strictly semistable model of X_U over $\mathcal{O}_v$, which is finite flat over X_{U_0}.

Now the special fiber $Y := X_U \otimes_{\mathcal{O}_v} k(v)$ is written as $Y = \bigcup_{1 \le i \le n} Y_i$, where Y_i, smooth over $k(v)$, is the locus where the i-th isogeny in the chain induces zero map on the Lie algebra. This moduli interpretation allows us to apply Theorem A to X_U, when it is proper, and the Hecke correspondences generating the local Hecke algebra $\mathcal{H}_n := \overline{\mathbb{Q}}_\ell\left[\mathrm{Iw}_n \backslash GL_n(F_v)/\mathrm{Iw}_n\right]$. It is generated by the generators $w_1, \ldots, w_{n-1}$ and $T_1^{\pm}, \ldots, T_n^{\pm}$ of the extended affine Weyl group, subject to certain relations (Bernstein-Zelevinsky presentation). It naturally contains the Iwahori Hecke algebra of Levi subgroups, say $\mathcal{H}_m \otimes \mathcal{H}_{n-m}$ of $GL_m \times GL_{n-m}$, which is generated by the above set of generators minus w_m, and makes $\mathcal{H}_n$ into a finite $\mathcal{H}_m \otimes \mathcal{H}_{n-m}$-algebra of dimension $\binom{n}{m}$. Now we refine the computation done in [TY]: we compute $H^*(Y^{(m)})$ as $\mathcal{H}_n$-module. By dividing it into open strata, we see that $H^*(Y^{(m)})$ (the alternating sum in the Grothendieck group) is the sum of $\mathcal{H}_n \otimes H_c^*(Y^0_{I_s})$ for $m \le s \le n$, where $I_s := \{1, \ldots, s\}$ and $Y^0_{I_s} := Y_{I_s} - \bigcup_{I_s \subset I \neq I_s} Y_I$ is the open stratum of Y_{I_s}, and the tensor product is over $\mathcal{H}_m \otimes \mathcal{H}_{s-m} \otimes \mathcal{H}_{n-s}$. Now, the action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m} \otimes \mathcal{H}_{n-s}$ on $H_c^*(Y^0_{I_s})$ is given as follows: the action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m}$ is computed locally by Theorem A, and the action of $\mathcal{H}_{n-s}$ is computed by counting of points on Igusa varieties via trace formula ([HT], this is where we need global assumptions). The action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m}$ is roughly given by $\mathrm{St}_m \otimes \mathrm{Tr}_{s-m}$, with some unramified twists corresponding to the Frobenius action, where St_n is a 1-dimensional $\mathcal{H}_n$-module given by $w_i \mapsto -1$ and $T_i \mapsto 1$, similarly Tr_n is a 1-dimensional $\mathcal{H}_n$-module given by $w_i \mapsto q := |k(v)|$ and $T_i \mapsto q^{i(n-i)}$, and $\otimes$ denotes the product corresponding to non-normalized induction. When we look at the E_1-term $H^*(Y^{(m)})$ of the weight spectral sequence after taking the limit with respect to U^v, making $H^*(Y^{(m)})$ into a $G(\mathbb{A}^{\infty,v}) \times \mathcal{H}_n \times \mathrm{Frob}^{\mathbb{Z}}$-module, its $\pi^{\infty,v}$-isotypic component, where $\pi = \pi^{\infty,v} \times \pi_v$ is a cuspidal automorphic respresentation of $G(\mathbb{A}^\infty)$, recovers $\pi_v^{\mathrm{Iw}_v}$ as $\mathcal{H}_n$-module and is pure of weight $n-m$ as $\mathrm{Frob}^{\mathbb{Z}}$-module. Two things are used: (1) the global result on the cohomology of $Y^0_{I_s}$ shows that as $\mathcal{H}_{n-s}$-module $H^*(Y^0_{I_s})[\pi^{\infty,v}]$ is essentially the (Iwahori invariants of) the Jacquet module of π_v to $GL_s \times GL_{n-s}$, and (2) the cancellation $\sum_{m=0}^{s}(-1)^m \mathrm{St}_m \otimes \mathrm{Tr}_{s-m} = 0$ in the Grothendieck group of $\mathcal{H}_s$-modules.

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Wild ramification and the characteristic cycle of an ℓ-adic sheaf

Takeshi Saito

We measure the wild ramification of an l-adic etale sheaf by introducing blow-ups of the self-product at the ramification locus in the diagonal.

Using the geometric construction, we define the characteristic cycle of an ℓ-adic sheaf as a cycle on the logarithmic cotangent bundle and prove that the intersection with the 0-section gives the characteristic class, under a certain condition.

1. Ramification along a divisor

Let k be a perfect field of characteristic $p > 0$, X be a smooth scheme of dimension d over k and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. We consider a smooth ℓ-adic sheaf $\mathcal{F}$ on U.

We construct a diagram

$$X \times X \longleftarrow (X \times X)^{\sim} \longleftarrow (X \times X)^{(R)},$$

where $R = r_1 D_1 + \cdots + r_m D_m$ is a linear combination of the irreducible components $D_1, \ldots, D_m$ of D with rational coefficients $r_i \geq 0, r_i \in \mathbb{Q}$. For simplicity in this note, we will assume $r_i > 0, r_i \in \mathbb{Z}$.

We define the log blow up $(X \times X)' \to X \times X$ to be the blow-up at $D_1 \times D_1, D_2 \times D_2, \ldots, D_m \times D_m$. We define the log product $(X \times X)^{\sim} \subset (X \times X)'$ to be the complement of the proper transforms of $D \times X$ and $X \times D$. The diagonal map $\delta : X \to X \times X$ is uniquely lifted to the log diagonal map $\tilde{\delta} : X \to (X \times X)^{\sim}$. The conormal sheaf $\mathcal{N}_{X/(X \times X)^{\sim}}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)$ of rank d.

We define $(X \times X)^{[R]} \to (X \times X)'$ to be the blow-up at the divisor $R \subset X$ in the log diagonal $X \subset (X \times X)'$. We define an open subscheme $(X \times X)^{(R)} \subset (X \times X)^{\sim} \times_{(X \times X)'} (X \times X)^{[R]}$ to be the complement of the proper transforms of the exceptional divisors of $(X \times X)^{\sim}$. The log diagonal map $\delta' : X \to (X \times X)'$ is uniquely lifted to a closed immersion $\delta^{(R)} : X \to (X \times X)^{(R)}$. The projections $(X \times X)^{(R)} \to X$ are smooth. The conormal sheaf $\mathcal{N}_{X/(X \times X)^{(R)}}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)(R)$.

We consider the commutative diagram

$$\begin{array}{ccc} U \times U & \xrightarrow{j^{(R)}} & (X \times X)^{(R)} \\ {\scriptstyle \delta_U} \uparrow & & \uparrow {\scriptstyle \delta^{(R)}} \\ U & \xrightarrow{j} & X \end{array}$$

of open immersions and the diagonal immersions.

Definition 1.1. *Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$. We define a smooth sheaf $\mathcal{H}$ on $U \times U$ by $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})$. Let $R = \sum_i r_i D_i \geq 0$ be an effective divisor with rational coefficients.*

We say that the log ramification of $\mathcal{F}$ along D is bounded by $R+$ if the identity $1 \in \mathrm{End}_U(\mathcal{F}) = \Gamma(U, \mathcal{E}nd_U(\mathcal{F})) = \Gamma(X, j_*\mathcal{E}nd_U(\mathcal{F}))$ is in the image of the base change map

$$\Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}) \longrightarrow \Gamma(X, j_*\mathcal{E}nd_U(\mathcal{F})) = \mathrm{End}_U(\mathcal{F}). \tag{1}$$

Definition 1 is related to the filtration by ramification groups in the following way. Let D_i be an irreducible component and K_i be the fraction field of the completion $\widehat{\mathcal{O}}_{X,\xi_i}$ of the local ring at the generic point ξ_i of D_i. We will often drop the index i in the sequel. The sheaf $\mathcal{F}$ defines an ℓ-adic representation $\mathcal{F}_{\bar{\eta}_i}$ of the absolute Galois group $G_{K_i} = \mathrm{Gal}(\overline{K}_i/K_i)$. The filtration $G^r_{K,\log} \subset G_K, r \in \mathbb{Q}, r > 0$ by the logarithmic ramification groups is defined. We put $G^{r+}_{K,\log} = \overline{\bigcup_{q>r} G^q_{K,\log}}$.

Lemma 1.2. *The following conditions are equivalent.*

(1) There exists an open neighborhood of ξ_i such that the log ramification of $\mathcal{F}$ along D is bounded by $R+$.

(2) The action of $G^{r_i+}_{K_i,\log}$ on $\mathcal{F}_{\bar{\eta}_i}$ is trivial.

The open subscheme $U \times U \subset (X \times X)^{(R)}$ is the complement of the inverse image $E = (X \times X)^{(R)} \times_X D$. The inverse image E is canonically identified with the vector bundle $\mathbf{V}(\Omega^1_X(\log D)(R)) \times_X D$.

Proposition 1.3. *Assume that the log ramification is bounded by $R+$. Then, for every geometric point $\bar{x}$ of D, the restriction $(j_*\mathcal{H})|_{E_{\bar{x}}}$ on the geometric fiber is isomorphic to the direct sum $\bigoplus_f \mathcal{L}_f^{\oplus n_f}$ where $\mathcal{L}_f$ is a smooth rank one sheaf defined by the Artin-Schreier equation $T^p - T = f$ and f denotes a linear form on the vector space $E_{\bar{x}}$.*

Proposition 1.3 has the following consequence. Let D_i be an irreducible component of D. The graded piece $\mathrm{Gr}^{r_i}_{\log} G_{K_i} = G^{r_i}_{K,\log}/G^{r_i+}_{K,\log}$ is abelian. The restriction of $\mathcal{F}_{\bar{\eta}_i}$ to $G^{r_i}_{K,\log}$ is decomposed into direct sum of characters $\bigoplus_\chi \chi^{n_\chi}$. The fiber $\Theta^{(r_i)}_{\log} = E^+ \times_{D^+} \xi_i$ at the generic point ξ_i is a vector space over the function field F_i of D_i. The restriction of $j_*\mathcal{H}$ on the geometric fiber $\Theta^{(r_i)}_{\log,\overline{F}_i}$ is decomposed as $\bigoplus_\chi \mathrm{End}_{I_i}(\mathcal{F}_{\bar{\eta}_i}) \otimes \mathcal{L}_\chi$ where $\mathcal{L}_\chi$ is a smooth rank one sheaf defined by the Artin-Schreier equation $T^p - T = f_\chi$ where $f_\chi = \mathrm{rsw}\ \chi$ is a linear form on $\Theta^{(r_i)}_{\log,\overline{F}_i}$ called the refined Swan character of χ.

Theorem 1.4. *The graded quotient $\mathrm{Gr}^r_{\log} G_K$ is annihilated by p and the map*

$$\mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\overline{F}_i}(\Theta^{(r)}_{\log}, \overline{F}_i) \tag{2}$$

sending a character χ to the refined Swan character $f_\chi = \mathrm{rsw}\ \chi$ is an injection.

2. Characteristic cycle

For a non-trivial character $\chi : \mathrm{Gr}^r_{\log} G_K \to \mathbb{F}_p$, the refined Swan character $\mathrm{rsw}\,\chi : \Theta^{(r)}_{\log} \to \overline{F}_i$ defines an $\overline{F}_i$-rational point $[\mathrm{rsw}\,\chi] : \mathrm{Spec}\,\overline{F}_i \to \mathbf{P}(\Omega^1_X(\log D)^*)$. We define a reduced closed subscheme $T_\chi \subset \mathbf{P}(\Omega^1_X(\log D)^*)$ to be the Zariski closure $\overline{\{[\mathrm{rsw}\,\chi](\mathrm{Spec}\,\overline{F}_i)\}}$ and let $L_\chi = \mathbf{V}(\mathcal{O}_{T_\chi}(1))$ be the pull-back to T_χ of the tautological sub line bundle $L \subset T^*X(\log D) \times_X \mathbf{P}(\Omega^1_X(\log D)^*)$. The inclusion $T_\chi \to \mathbf{P}(\Omega^1_X(\log D)^*)$ corresponds to a surjection $\Omega^1_X(\log D)^* \otimes \mathcal{O}_{T_\chi} \to \mathcal{O}_{T_\chi}(1)$ and hence defines a commutative diagram

$$\begin{array}{ccccc} L_\chi & \longrightarrow & T^*X(\log D) \times_X D_i & \longrightarrow & T^*X(\log D) = \mathbf{V}(\Omega^1_X(\log D)^*) \\ \downarrow & & \downarrow & & \downarrow \\ T_\chi & \xrightarrow{\pi_\chi} & D_i & \longrightarrow & X. \end{array} \tag{3}$$

We put $SS_\chi = \frac{1}{[T_\chi : D_i]} \pi_{\chi *}[L_\chi]$ in $Z_d(T^*X(\log D) \times_X D_i)_{\mathbb{Q}}$.

Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on $U = X \setminus D$ and $R = \sum_i r_i D_i$ be an effective divisor with rational coefficients $r_i \geq 0$. In the rest of talk, we assume that $\mathcal{F}$ satisfies the following conditions:

(R) The log ramification of $\mathcal{F}$ along D is bounded by $R+$.
(C) For each irreducible component D_i of D, the closure $\overline{S_{\mathcal{F}} \times F_i}$ is finite over D_i and the intersection $\overline{S_{\mathcal{F}} \times F_i} \cap D_i$ with the 0-section is empty.

The conditions imply $\mathcal{F}_{\bar\eta_i} = \mathcal{F}^{(r_i)}_{\bar\eta_i}$ for every irreducible component D_i of D.

Definition 2.1. *Let $\mathcal{F}$ be a smooth Λ-sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C).

For an irreducible component D_i of D with $r_i > 0$, let $\mathcal{F}_{\bar\eta_i} = \sum_\chi n_\chi \chi$ be the direct sum decomposition of the representation induced on $\mathrm{Gr}^{r_i}_{\log} G_{K_i}$. We define the characteristic cycle by

$$CC(\mathcal{F}) = (-1)^d \left(\mathrm{rank}\,\mathcal{F} \cdot [X] + \sum_{i, r_i > 0} r_i \cdot \sum_\chi n_\chi \cdot [SS_\chi] \right) \tag{4}$$

in $Z^d(T^*X(\log D))_{\mathbb{Q}}$.

Theorem 2.2. *Let X be a smooth scheme over k and D be a divisor with simple normal crossings. Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C).

Then we have

$$(CC(\mathcal{F}), X)_{T^*X(\log D)} = C(j_!\mathcal{F})$$

where the right hand side denotes the characteristic cycle of $j_!\mathcal{F}$. In particular, if X is proper, we have $(CC(\mathcal{F}), X)_{T^*X(\log D)} = \chi_c(U_{\bar k}, \mathcal{F})$.

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Iwasawa theory and non-abelian class field theory

Kazuhiro Fujiwara

After the breakthrough by A. Wiles ([4]), there has been a substantial progress in the class field theory for GL_2. At this meeting, the author has explained the project to understand Leopoldt's conjecture in algebraic number theory via the class field theory for GL_2.

For a totally real number field F, G_F denotes the absolute Galois group, and O_F denotes the integer ring. For a prime number p, Leopoldt's conjecture asserts that the kernel of the p-adic regulator map

$$O_F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{v|p} (O_{F_v}^\times)_p$$

has $\mathbb{Z}_p$-rank zero (where $(\cdot)_p$ denotes the pro-p completion). The conjecture is known if F is a subfield of an abelian extension of an imaginary quadratic number field by a method from transcendental number theory ([1]).

Here is the attempt from GL_2-class field theory viewpoint:

Theorem 1. *Assume $p \geq 3$, and assumption $A_{(F,p)}$ is satisfied. Then Leopoldt's conjecture for (F, p) is true.*

Assumption $A_{(F,p)}$, which will be explained later, is known to be true under mild conditions.

For the proof of the theorem, we make use of the nearly ordinary deformation rings of a two dimensional *reducible and indecomposable* representation.

Now we explain the assumption $A_{(F,\ p)}$. Take a p-adic field $E_\wp$ with the ring of the integers $o_\wp$ and the residue field $k_\wp$. A character of finite order $\chi : G_F \to o_\wp^\times$ is *nice* if

- χ is totally odd, of order prime to p.
- χ is unramified at $\forall v|p$, $\chi(\mathrm{Fr}_v) \neq 1$.
- $H_f^1(F,\ \bar{\chi}^{\pm 1}) = 0$ ($\Leftrightarrow$ the relative class number of F_χ/F is prime to p if χ is quadratic).

Here $\bar{\chi} = \chi \mod \wp$, and H_f^1 denotes the finite part. Then $A_{(F,\ p)}$ is described as follows:

Assumption $A_{(F,\ p)}$: there is *at least one* nice character χ.

Note that $A_{(F,p)}$ is satisfied if p is sufficiently large ([3]), or if $p = 3$ (and is conjectured to be true for $p \geq 3$).

We briefly explain the GL_2-set up. Assume $A_{(F,\ p)}$, and fix a nice character χ. Construct an indecomposable reducible $\bar{\rho} : G_F \to \mathrm{GL}_2(k_\wp)$ by the following conditions:

- $\bar{\rho}$ takes a form
$$0 \to 1 \to \bar{\rho} \to \bar{\chi} \to 0.$$
- $\bar{\rho}|_{I_{F_v}}$ is split except one finite place y s.t. $\chi(\mathrm{Fr}_y)^{-1} \equiv q_y \not\equiv 1 \mod p$.

$\bar{\rho}$ is unique up to isomorphisms. $\Sigma = \{v|p\} \cup \{\text{ramification set of } \bar{\rho}\}$.

To start the analysis of deformation rings of $\bar{\rho}$, we need the following modularity theorem.

Theorem 2. *Assume $[F : \mathbb{Q}] > 1$, and $q_y = \sharp k(y)$ is sufficiently large. Then $\bar{\rho}$ is (minimally) modular in the following sense:*

- *There exists a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ which is unramified outside $\Sigma \cup \{v|\infty\}$ and of parallel weight 2,*
- *π is nearly ordinary at $\forall v|p$,*
- *$\bar{\rho} \simeq \rho_{\pi,\wp} \mod \wp$.*

Using $\bar{\rho}$, we define a universal deformation ring $R_{\mathcal{D}_S}$ and Hida's nearly ordinary (cuspidal) Hecke algebra $T_{\mathcal{D}_S}$ depending on a finite set S of finite places of F.

After showing $R_{\mathcal{D}_S} = T_{\mathcal{D}_S}$ by the standard method (cf. [2]), we prove Theorem 1 by a closer analysis of Eisenstein ideal of $T_{\mathcal{D}_S}$: first by counting the exact number of generators for a well-chosen S, then by relating it to the tangent space of $R_{\mathcal{D}_S}$. A deeper understanding of the relation between the global and local tangent spaces is needed. In these arguments, we make essential uses of Taylor-Wiles systems.

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The appearance of modular symbols in Galois cohomology

Romyar T. Sharifi

Let p be an odd prime, and for any positive integer r, let $F_r = \mathbf{Q}(\zeta_{p^r})$ be the cyclotomic field of p^rth roots of unity. We are interested in comparing the following two sorts of objects for integers u and v prime to p:

1. cup product values $(1 - \zeta_{p^r}^u, 1 - \zeta_{p^r}^v)_r$ in $H^2_{\text{ét}}(\mathbf{Z}[\zeta_{p^r}, 1/p], \mathbf{Z}_p(2))$,
2. the projections $\xi_r(u : v)$ of Manin symbols to the ordinary part of the homology group $H_1(X_1(p^r); \mathbf{Z}_p)$ of the closed modular curve $X_1(p^r)$.

The relationship between these objects is explored in [S2], and we briefly describe a conjecture relating them here.

The aforementioned cup product values were studied by McCallum and the speaker, and we refer the reader to [McS, S1] for details. As for Manin symbols, briefly, we define $[u : v]_r$ to be the class in homology of $X_1(p^r)$ relative to the cusps of the geodesic from $\frac{-a}{cp^r}$ to $\frac{-b}{dp^r}$, where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

lies in $\mathrm{SL}_2(\mathbf{Z})$, $a \equiv u \bmod p^r$, and $b \equiv v \bmod p^r$ (compare with [Mn]). We project $[u : v]_r$ to $H_1(X_1(p^r); \mathbf{Q}_p)$ using the Manin-Drinfeld splitting, and further project to the ordinary part of said group, i.e., the part on which U_p acts invertibly. The resulting element $\xi_r(u : v)$ lies in the ordinary part of homology with $\mathbf{Z}_p$-coefficients.

We define an Eisenstein ideal of the weight 2 cuspidal $\mathbf{Z}_p$-Hecke algebra at level p^r by

$$I_r = (\{T_l - 1 - l\langle l\rangle,\, l \neq p \text{ prime}\} \cup \{U_p - 1\}).$$

To make things canonical, we consider a complex embedding ι of an algebraic closure of $\mathbf{Q}$ and use it to fix $\zeta_{p^r} = \iota^{-1}(e^{2\pi i/p^r})$. We put a Galois action on $H_1(X_1(p^r); \mathbf{Z}_p)$ using ι (arising, e.g., from the action on the abelianization of the étale fundamental group).

We then make the following conjecture [S2].

Conjecture 1. *There exists an isomorphism*

$$\nu_r \colon H^2_{\text{ét}}(\mathbf{Z}[\zeta_{p^r}, 1/p], \mathbf{Z}_p(2))^+ \to H_1(X_1(p^r); \mathbf{Z}_p)^+/I_r,$$

of $\mathbf{Z}_p[\mathrm{Gal}(F_r/\mathbf{Q})]$*-modules such that we have*

$$\nu_r((1 - \zeta_{p^r}^u, 1 - \zeta_{p^r}^v)_r^+) = \xi_r(u : v)^+ \pmod{I_r},$$

where we use $^+$ *to denote fixed parts under complex conjugation.*

Remark. The map ν_r is to be defined naturally from ι in such a way that the equality in the conjecture does not depend on its choice.

We can reword this in terms of Iwasawa theory and Hida theory. Let K denote the field of all p-power roots of unity. Let $\mathcal{U}_K$ denote the group of universal norm sequences in K, let $\mathfrak{X}_K$ denote the Galois group of the maximal abelian p-ramified pro-p extension of K, and let X_K denote its maximal abelian quotient. Using a coboundary that arises in a long exact sequence of inverse limits of cohomology groups, one obtains a reciprocity map

$$\Psi_K \colon \mathcal{U}_K \to X_K \otimes_{\mathbf{Z}_p} \mathfrak{X}_K$$

that interpolates certain inverse limits of cup products [S2]. We are particularly interested in the values of this on the universal norm sequence $1 - \zeta = (1 - \zeta_{p^r})$.

Let Λ denote the Iwasawa algebra $\mathbf{Z}_p[[\mathrm{Gal}(K/\mathbf{Q})]]$. We may define

$$\mathcal{L} = \varprojlim \sum_{\substack{j=1\\(j,p)=1}}^{p^r-1} U_p^{-r}\xi_r(j:1)\otimes[j] \in (\varprojlim H_1(X_1(p^r);\mathbf{Z}_p)^{\mathrm{ord}})\,\hat{\otimes}_{\mathbf{Z}_p}\Lambda,$$

where $\hat{\otimes}$ denotes completed tensor product and $[j]$ the group element attached to $j \in \mathbf{Z}_p^\times \cong \mathrm{Gal}(K/\mathbf{Q})$. This is essentially the p-adic L-function of Mazur and Kitagawa [K].

We may also define a map $\psi\colon \mathfrak{X}_K \to \Lambda$ given by

$$\psi(\sigma) = \varprojlim \sum_{\substack{j=1\\(j,p)=1}}^{p^r-1} \pi_{1-\zeta_{p^r}^j}(\sigma)[j],$$

where $\pi_{1-\zeta_{p^r}^j}\colon \mathfrak{X}_K \to \mathbf{Z}_p$ is the Kummer character attached to $1-\zeta_{p^r}^j$ using ζ.

We define $\mathcal{I}$ in Hida's ordinary $\mathbf{Z}_p$-Hecke algebra using the same generators as for I_r, and we define $\mathcal{Y}$ to be the resulting Eisenstein part of the inverse limit over r of the twist by $\mathbf{Z}_p(1)$ of the $H_1(X_1(p^r);\mathbf{Z}_p)$ with the above-mentioned Galois actions. (Equivalently, we may consider étale cohomology groups without the twist.)

We have the following equivalent form of Conjecture 1.

Conjecture 2. *There exists an isomorphism $\phi\colon X_K^- \to \mathcal{Y}^-/\mathcal{I}\mathcal{Y}^-$ of Λ-modules such that, letting*

$$\Xi = \phi\otimes\psi^-\colon X_K^-\otimes_{\mathbf{Z}_p}\mathfrak{X}_K^- \to \mathcal{Y}^-/\mathcal{I}\mathcal{Y}^-\otimes_{\mathbf{Z}_p}\Lambda^-,$$

we have

$$\Xi(\Psi_K(1-\zeta)^-) = \mathcal{L}^- \pmod{\mathcal{I}\mathcal{Y}^-\otimes_{\mathbf{Z}_p}\Lambda^-},$$

where we use $^-$ to denote (tensor products of) (-1)-parts under complex conjugation.

Remark. The twist by $\mathbf{Z}_p(1)$ of the map ϕ in Conjecture 2 will be the inverse limit of the ν_r in Conjecture 1.

We can also give a third form of this conjecture, which asserts a certain correspondence between cup products of limits of cyclotomic p-units in cohomology groups with various odd (p-adic) twists $H^1_{\text{ét}}(\mathbf{Z}[1/p],\mathbf{Z}_p(i))$ and specific values of the two-variable p-adic L-function given by specializing at a certain weight and character.

The difficulty in proving the conjecture is found in the construction of the correct map ϕ. We may construct one good candidate as follows. The action of Galois on $\mathcal{Y}$ provides a map $b\colon G_K \to \mathrm{Hom}(\mathcal{Y}^+,\mathcal{Y}^-)$ that is trivial on a decomposition group at p fixing $\mathcal{Y}^+$. It then induces a homomorphism

$$\bar{b}\colon X_K^- \to \mathrm{Hom}(\mathcal{Y}^+/\mathcal{I}\mathcal{Y}^+,\mathcal{Y}^-/\mathcal{I}\mathcal{Y}^-)$$

which is known to be an isomorphism under a fairly mild condition on Bernoulli numbers [O2, S2]. Using a certain twisted version of Poincaré duality [O1, S2], we may define a generator α of $\mathcal{Y}^+/\mathcal{I}\mathcal{Y}^+$ as a Hecke module, canonical up to ι. The map $\sigma \mapsto \bar{b}(\sigma)(\alpha)$ then provides one candidate for ϕ. As desired, the question of the validity of Conjecture 2 using this map for ϕ is independent of ι.

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Rings of integers of type $K(\pi, 1)$

Alexander Schmidt

Let Y be a connected locally noetherian scheme and let p be a prime number. We say that Y is a $K(\pi,1)$ for p if the higher homotopy groups of the p-completion $Y_{et}^{(p)}$ of its étale homotopy type Y_{et} vanish. In this talk we consider the case of an arithmetic curve, i.e. Y is an open subscheme of $\mathit{Spec}(\mathcal{O}_k)$, where k is a number field. Here the $K(\pi,1)$-property is linked to open questions in the theory of Galois groups with restricted ramification of number fields:

Let k be a number field, S a finite set of non-archimedean primes of k and p a prime number. For simplicity, we assume that p is odd or that k is totally imaginary. Let $k_S(p)$ denote the maximal p-extension of k unramified outside S and put $G_S(p) = \mathit{Gal}(k_S(p)|k)$. A systematic study of this group had been started by Šafarevič, and was continued by Koch, Kuzmin, Wingberg and many other people. See [NSW], VIII, §7 for basic properties of $G_S(p)$. In geometric terms (and omitting the base point) we have

$$G_S(p) \cong \pi_1\big((\mathit{Spec}(\mathcal{O}_k) \smallsetminus S)_{et}^{(p)}\big).$$

As is well known to the experts, if S contains the set S_p of primes dividing p, then $\mathit{Spec}(\mathcal{O}_k) \smallsetminus S$ is a $K(\pi,1)$ for p. In particular, if $S \supset S_p$, then $G_S(p)$ is of cohomological dimension less or equal to 2.

The group $G_S(p)$ is most mysterious in the *tame* case, i.e. if $S \cap S_p = \varnothing$. In this case, examples when $\mathit{Spec}(\mathcal{O}_k) \smallsetminus S$ is *not* a $K(\pi,1)$ are easily constructed. On the contrary, until recently not a single $K(\pi,1)$-example was known. The following properties of the group $G_S(p)$ were known so far

- $G_S(p)$ is a 'fab-group', i.e. the abelianization of each open subgroup is finite.

- $G_S(p)$ can be infinite (Golod-Šafarevič, 1964).
- $G_S(p)$ is a finitely presented pro-p-group (Koch, 1965).

A conjecture of Fontaine and Mazur ([FM], 1994) asserts that $G_S(p)$ has no infinite p-adic analytic quotients.

In 2005, Labute considered the case $k = \mathbb{Q}$ and found finite sets S of prime numbers (called strictly circular sets) with $p \notin S$ such that $G_S(p)$ has cohomological dimension 2. In [S1] the author showed that, in the examples given by Labute, $Spec(\mathbb{Z}) \smallsetminus S$ is a $K(\pi, 1)$ for p. We show that in the tame case rings of integers of type $K(\pi, 1)$ are cofinal in the following sense:

Theorem 1. *Let k be a number field and let p be a prime number such that*

$$(*) \qquad \zeta_p \notin k \quad \text{and} \quad p \nmid \#\mathrm{Cl}(k).$$

Let S be a finite set of primes of k with $S \cap S_p = \varnothing$. Let, furthermore, T be any set of primes of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $Spec(\mathcal{O}_k) \smallsetminus (S \cup T_1)$ is a $K(\pi, 1)$ for p.

We conjecture that condition $(*)$ can be removed from Theorem 1. Explicit examples of rings of integers of type $K(\pi, 1)$ can be found in [La], [S1] ($k = \mathbb{Q}$) and in [Vo] (k imaginary quadratic).

The $K(\pi, 1)$-property has strong consequences. We write $X = Spec(\mathcal{O}_k)$ and assume in all results below that $p \neq 2$ or k is totally imaginary, and that we are in the tame case $S \cap S_p = \varnothing$. Primes $\mathfrak{p} \in S$ with $\zeta_p \notin k_\mathfrak{p}$ are redundant in S in the sense that removing these primes from S does not change $(X \smallsetminus S)_{et}^{(p)}$. We therefore restrict our considerations to sets of primes whose norms are congruent to 1 modulo p. These are the results.

Proposition 2. *Let S be a finite non-empty set of primes of k whose norms are congruent to 1 modulo p. Then $X \smallsetminus S$ is p-contractible (i.e. $X \smallsetminus S$ is a $K(\pi, 1)$ for p and $G_S(p) = 1$) if and only if $S = \{\mathfrak{p}\}$ consists of a single prime and one of the following cases occurs.*

(a) $p = 2$, $k \neq \mathbb{Q}(\sqrt{-1})$ *is imaginary quadratic,* $2 \nmid h_k$ *and* $N(\mathfrak{p}) \not\equiv 1 \bmod 4$,
(b) $p = 2$, $k = \mathbb{Q}(\sqrt{-1})$ *and* $N(\mathfrak{p}) \not\equiv 1 \bmod 8$,
(c) $p = 3$, $k = (\mathbb{Q}\sqrt{-3})$ *and* $N(\mathfrak{p}) \not\equiv 1 \bmod 9$.

Theorem 3. *Let S be a finite non-empty set of primes of k whose norms are congruent to 1 modulo p. If $X \smallsetminus S$ is a $K(\pi, 1)$ for p and $G_S(p) \neq 1$, then the following holds.*

(i) $cd\, G_S(p) = 2$, $scd\, G_S(p) = 3$.
(ii) $G_S(p)$ *is a duality group.*

The dualizing module D of $G_S(p)$ is given by $D = tor_p C_S(k_S(p))$, i.e. it is the subgroup of p-torsion elements in the S-idèle class group of $k_S(p)$.

Remark: In the wild case $S \supset S_p$, where $X \smallsetminus S$ is always a $K(\pi, 1)$ for p, $G_S(p)$ is of cohomological dimension 1 or 2. The strict cohomological dimension is conjecturally equal to 2 (=Leopoldt's conjecture for each finite subextension of k in $k_S(p)$). In the wild case, $G_S(p)$ is often, but not always a duality group, cf. [NSW] Prop. 10.7.13.

Allowing ramification at a prime $\mathfrak{p}$ does not mean that the ramification is realized globally. Therefore it is a natural and interesting question how far we get locally at the primes in S when going up to $k_S(p)$. See [NSW] X, §3 for results in the wild case. In the tame case, we have the following

Theorem 4. *Let S be a finite non-empty set of primes of k whose norms are congruent to 1 modulo p. If $X \smallsetminus S$ is a $K(\pi,1)$ for p and $G_S(p) \neq 1$, then*

$$k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$$

for all primes $\mathfrak{p} \in S$, i.e. $k_S(p)$ realizes the maximal p-extension of the local field $k_{\mathfrak{p}}$.

The similar question for primes $\mathfrak{p} \notin S$ is open. We do not know whether or not a prime $\mathfrak{p} \notin S$ can split completely in $k_S(p)$ if $X \smallsetminus S$ is a $K(\pi,1)$ for p and $G_S(p) \neq 1$.

The next result addresses the question of enlarging the set S without destroying the $K(\pi,1)$-property.

Theorem 5. *Let $S \subset S'$ be finite non-empty sets of primes of k whose norms are congruent to 1 modulo p. Assume that $X \smallsetminus S$ is a $K(\pi,1)$ for p and that $G_S(p) \neq 1$. If each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in $k_S(p)$, then $X \smallsetminus S'$ is a $K(\pi,1)$ for p. Furthermore, in this case, the arithmetic form of Riemann's existence theorem holds: the natural homomorphism*

$$\mathop{\ast}_{\mathfrak{p} \in S' \setminus S(k_S(p))} T_{\mathfrak{p}}(k_{S'}(p)|k_S(p)) \longrightarrow Gal(k_{S'}(p)|k_S(p))$$

is an isomorphism, i.e. $Gal(k_{S'}(p)|k_S(p))$ is the free pro-p product of a bundle of inertia groups.

Proofs of the results above can be found in [S2]. The proof of Theorem 1 uses at an essential point Labute's results on mild pro-p-groups [La].

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On period spaces for p-divisible groups

Urs Hartl

In our talk we explained our results from [5] on the image of the Rapoport-Zink period morphism.

Fix a Barsotti-Tate group $\overline{X}_0$ over $\mathbb{F}_p^{\mathrm{alg}}$ of height h and dimension d. Let $W := W(\mathbb{F}_p^{\mathrm{alg}})$ be the ring of Witt vectors and let $K_0 := W[\frac{1}{p}]$. We consider Barsotti-Tate groups X over complete, rank one valued extensions $\mathcal{O}_K$ of W, $K := \mathrm{Frac}\mathcal{O}_K$, such that there exists an isogeny

$$\rho : X \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K \longrightarrow \overline{X}_0 \otimes_{\mathbb{F}_p^{\mathrm{alg}}} \mathcal{O}_K/p\mathcal{O}_K\,.$$

The theory of Grothendieck-Messing [7] associates to X an extension

$$0 \longrightarrow (\mathrm{Lie}X^\vee)^\vee_K \longrightarrow \mathbb{D}(X)_K \longrightarrow \mathrm{Lie}X_K \longrightarrow 0$$

where $\mathbb{D}(X)_K$ is the crystal of Grothendieck-Messing evaluated on K, and the isogeny ρ defines an isomorphism of crystals $\mathbb{D}(\rho)_K : \mathbb{D}(X)_K \xrightarrow{\sim} \mathbb{D}(\overline{X}_0)_K$. The K-subspace $\mathbb{D}(\rho)_K(\mathrm{Lie}X^\vee)^\vee_K$ defines a K-valued point in the Grassmannian $\mathcal{F} := \mathrm{Grass}(h-d, \mathbb{D}(\overline{X}_0)_{K_0})$ of $h-d$-dimensional subspaces of $\mathbb{D}(\overline{X}_0)_{K_0}$. In [4] Grothendieck posed the following

Problem. (A. Grothendieck, 1970)
Describe the subset of $\mathcal{F}$ formed by the points $\mathbb{D}(\rho)_K(\mathrm{Lie}X^\vee)^\vee_K$ for varying K, X, ρ.

A first solution to this problem was given by Rapoport-Zink [8] who constructed a rigid analytic period domain $\mathcal{F}^{\mathrm{rig}}_{wa}$ for Barsotti-Tate groups consisting of all weakly admissible filtrations on the isocrystal $\mathbb{D}(\overline{X}_0)_{K_0}$. In our talk we firstly showed that only in rare cases the Rapoport-Zink period domain is the correct answer, by exhibiting weakly admissible filtrations defined over infinite extensions K/K_0 which do not correspond to Barsotti-Tate groups X over $\mathcal{O}_K$. Secondly we described the correct solution of Grothendieck's problem as the open Berkovich subspace $\mathcal{F}_a$ of $\mathcal{F}$ consisting of those points for which the associated φ-module over the ("algebraic closure" of the) Robba ring is unit root. The space $\mathcal{F}_a$ is contained in the Berkovich subspace $\mathcal{F}_{wa}$ corresponding to the Rapoport-Zink period domain. The inclusion $\mathcal{F}_a \subset \mathcal{F}_{wa}$ induces an étale morphism of the associated rigid analytic spaces, which is a bijection on rigid analytic points by the theorems of Colmez-Fontaine [2], Breuil [1, Theorem 1.4] and Kisin [6]. The rational Tate module $T_pX_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of X gives rise to a local system of $\mathbb{Q}_p$-vector spaces on $\mathcal{F}_a$, whose associated space of $\mathbb{Z}_p$-lattices is the generic fiber of the Rapoport-Zink space, which parametrizes pairs (X, ρ) of Barsotti-Tate groups X over $\mathcal{O}_K$ and isogenies ρ as above.

These results are proved in [5]. They were independently obtained in a recent article by Faltings [3].

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Higher exponential maps and explicit reciprocity laws

Sarah L. Zerbes

1. Motivation

Let p be an odd prime, and let F be a finite extension of $\mathbb{Q}_p$ with absolute Galois group $\mathfrak{G}_F$. Let $(F_n)_{n\geq 1}$ be the cyclotomic tower, so $F_n = F(\mu_{p^n})$, and let $F_\infty = \bigcup_n F_n$. Define the Galois groups $H_F = \mathrm{Gal}(\bar{F}/F_\infty)$ and $\Gamma = \mathrm{Gal}(F_\infty/F)$, and note that Γ is isomorphic to an open subgroup of $\mathbb{Z}_p^*$ via the cyclotomic character. Let V be a de Rham representation of $\mathfrak{G}_G F$ with (ϕ,Γ)-module $D(V)$. Using the results of [6], which describes the $\mathfrak{G}_F$-cohomology of V in terms of $\mathbb{D}(V)$, one can construct a natural map $\iota : H^1(\Gamma, \mathbb{D}(V)^{\psi=1}) \to H^1(\mathfrak{G}_F, V)$ (c.f.Lemma I.5.2 in [4]. More precisely, Cherbonnier and Colmez show in [4] that for $n \gg 1$ we have a commutative diagram

$$\begin{array}{ccc} H^1(\Gamma, \mathbb{D}(V)^{\psi=1}) & \xrightarrow{\phi^{-n}} & H^1(\Gamma, (\mathbb{B}_{\mathrm{dR}} \otimes V)^{H_F}) \\ \downarrow \iota & & \downarrow \\ H^1(\mathfrak{G}_F, V) & \longrightarrow & H^1(\mathfrak{G}_F, \mathbb{B}_{\mathrm{dR}} \otimes V) \end{array}$$

Here, the map $H^1(K, V) \to H^1(\Gamma, \mathbb{B}_{\mathrm{dR}} \otimes V)$ is induced from the natural map $V \to \mathbb{B}_{\mathrm{dR}} \otimes V$.

In [5], Fontaine has constructed a short exact sequence of $\mathfrak{G}_F$-modules

$$0 \to V \to \mathbb{B}_{\max}^{\phi=1} \otimes_{\mathbb{Q}_p} V \to \mathbb{B}_{\mathrm{dR}}/\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V \to 0. \tag{1}$$

Taking $\mathfrak{G}_F$-cohomology of (1) gives a connection map $\delta : \mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V) \to H^1(\mathfrak{G}_F, V)$.

Definition (c.f. [2]). The Bloch-Kato exponential is the map $\exp_{V,F} : \mathbb{D}_{\mathrm{dR}}(V) \to H^1(\mathfrak{G}_F, V)$ obtained by composing δ with the natural quotiend map $\mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V)$.

One of the main results of [4] gives an explicit description of an element $y \in H^1(\Gamma, \mathbb{D}(V)^{\psi=1})$ in terms of the image of $\iota(y)$ under twists of the dual exponential map $\exp^*_{V,F}$.

2. Main results

In [9], we generalize the above results for higher dimensional local fields. Let K be a $(d+1)$-dimensional local field of mixed characteristic $(0,p)$ with residue field k_K, and let $X_1, \ldots, X_d$ be a p-basis of K. Let $K_n = K(\mu_{p^n}, X_1^{\frac{1}{p^n}}, \ldots, X_d^{\frac{1}{p^n}})$ for $n \geq 1$. Then $K_\infty = \bigcup_n K_n$ is a $(d+1)$-dimensional p-adic Lie extension of K whose Galois group G_K is isomorphic to $\mathbb{Z}_p^d(1) \rtimes \mathbb{Z}_p^*$. Define the Galois groups $\mathfrak{G}_K = \mathrm{Gal}(\bar{K}/K)$ and $H_K = \mathrm{Gal}(\bar{K}/K_\infty)$. Using the tower $(K_n)_{n\geq 1}$ - which is the analogue of the tower $(F_n)_{n\geq 1}$ in the 1-dimensional situation - Andreatta [1] and Scholl [8] have developed the theory of higher (ϕ, G_K)-modules: If V is a p-adic representation of $\mathfrak{G}_K$, then one associates to it a (ϕ, G_K)-module $\mathbf{D}_K(V)$, which is a finitely generated étale $\mathbf{A}_K$-module with continuous actions of ϕ and G_K. Kato [7] and Brinon [3] have constructed a higher dimensional analogue $\mathbf{B}_{\mathrm{dR}}$ of the ring $\mathbb{B}_{\mathrm{dR}}$, which is equipped with a connection $\nabla : \mathbf{B}_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}} \otimes_K \Omega^1_K$.

Theorem. Let V be a de Rham representation of $\mathfrak{G}_K$. Then for $n \gg 0$ and for all $1 \leq i \leq d$, we have a commutative diagram

$$\begin{array}{ccc} H^i(G_K, \mathbf{D}(V)^{\psi=1}) & \xrightarrow{\phi^{-n}} & H^i(G_K, (\mathbf{B}_{\mathrm{dR}}^{\nabla=0} \otimes V)^{H_K}) \\ \downarrow{\scriptstyle \iota^{(i)}} & & \downarrow{\scriptstyle \mathrm{inf}} \\ H^i(\mathfrak{G}_K, V) & \longrightarrow & H^i(\mathfrak{G}_K, \mathbf{B}_{\mathrm{dR}}^{\nabla=0} \otimes V) \end{array}$$

Here, the map $H^i(\mathfrak{G}_K, V) \to H^i(\mathfrak{G}_K, \mathbf{B}_{\mathrm{dR}}^{\nabla=0} \otimes V)$ is induced by the natural map $V \to \mathbf{B}_{\mathrm{dR}}^{\nabla=0} \otimes V$, and the map $\iota^{(i)}$ is defined using the description of the Galois cohomology groups $H^i(\mathfrak{G}_K, V)$ in terms of the (ϕ, G_K)-module $\mathbf{D}(V)$.

In the higher-dimensional situation, the short exact sequence (1) is replaced by the long exact sequence of $\mathfrak{G}_K$-modules

$$\begin{aligned} 0 &\to V \to \mathbf{B}_{\max}^{\phi=1} \otimes_{\mathbb{Q}_p} V \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V \\ &\to^{\nabla} V \otimes \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^{-1}\mathbf{B}_{\mathrm{dR}}^+ \otimes \Omega^1_K \to^{\nabla} V \otimes \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^{-2}\mathbf{B}_{\mathrm{dR}}^+ \otimes \Omega^2_K \to^{\nabla} \ldots, \end{aligned}$$

which gives rise to the spectral sequence $E_1^{m,n} \Rightarrow H^{m+n}(K, V)$, where

$$E_1^{m,n} = \begin{cases} H^m(K, \mathbf{B}_{\max}^{\phi=1} \otimes_{\mathbb{Q}_p} V) & \text{when } n = 0 \\ H^m(K, V \otimes \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-n}\mathbf{B}_{\mathrm{dR}}^+ \otimes \Omega_K^{n-1}) & \text{when } n \geq 1 \end{cases}$$

We use this spectral sequence to define higher Bloch-Kato exponentials $\exp_{(i),K,V}$ for $1 \leq i \leq d$, and we prove a higher-dimensional analogue of the explicit reciprocity law of Cherbonnier and Colmez:- If $y \in H^i(G_K, \mathbf{D}(V)^{\psi=1})$, then y has an explicit description in terms of the image of $\iota^{(i)}(y)$ under the higher dual exponential map $\exp^*_{(d+1-i),V,F}$.

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A *p*-adic Borel regulator

Annette Huber
(joint work with Guido Kings)

1. Motivation

Let F be a finite field extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$. The *Dirichlet regulator* is the map

$$\mathcal{L} : \mathcal{O}_F^* \to \bigoplus_{v|\infty} \mathbb{R}$$
$$u \mapsto (\log |u|_v)_v \quad .$$

It is used to prove:

(1) *Dirichlet's unit theorem:* The rank of $\mathcal{O}_F^*$ is $\#\{v|\infty\} - 1$.
(2) *The class number formula:*

$$\zeta_F(0)^* = -\frac{hR}{w}$$

where the left hand side means the leading coefficient of the Taylor expansion; h is the class number, w the number of roots of unity in F, and R is the regulator, ie. the determinant of $\mathcal{L}$.

This was generalized by Borel. He first defines higher regulator maps for $n \geq 1$

$$K_{2n-1}(\mathcal{O}_F) \to \bigoplus_{v|\infty} K_n(F_v) \xrightarrow{b_\infty} \bigoplus_{v|\infty} \mathbb{R} \quad .$$

He then proves:

(1) computation of the *rank* of $K_{2n-1}(\mathcal{O}_F)$ (see [1]);
(2) *formula* for $\zeta_F(1-n)^*$ up to a factor in $\mathbb{Q}^*$ (see [2]).

Our (long-term) aim is to correct his formula by a p-adic contribution and thus prove the conjecture of Bloch and Kato for number fields.

2. Borel's regulator

We recapitulate Borel's definition for $\mathbb{C}$. (The experts will realize that we simplify the truth somewhat.)

$$\begin{array}{ccc} K_{2n-1}(\mathbb{C}) & \xrightarrow{\quad b_\infty \quad} & \mathbb{C} \\ & {\scriptstyle \mathrm{Hur}} \searrow \quad \nearrow & \\ & H_{2n-1}(\mathrm{GL}(\mathbb{C}), \mathbb{Q}) & \end{array}$$

Defining the map b_∞ is equivalent to defining a system of elements in group cohomology $H^{2n-1}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C})$ for all N. Borel uses the van Est isomorphism between Lie algebra cohomology and continuous groups cohomology.

$$H^{2n-1}(\mathfrak{gl}_N, \mathbb{C}) \cong H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C}) \to H^{2n-1}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C})$$

Here Lie algebra cohomology of a Lie algebra $\mathfrak{g}$ can easily be defined as cohomology of the complex $\bigwedge^* \mathfrak{g}$ with differential induced by the Lie bracket. It is obviously finite dimensional and of course well known for $\mathfrak{g} = \mathfrak{gl}_N$. In fact, $H^*(\mathfrak{gl}_N, \mathbb{Q})$ is an exterior algebra on certain elements

$$p_n \in H^{2n-1}(\mathfrak{gl}_N, \mathbb{Q})$$

for $1 \leq n \leq N$ called primitive elements.

Definition 2.1. The Borel regulator b_∞ is defined as the image of p_n.

3. The p-adic version

Let p be a fixed prime. Let $K/\mathbb{Q}_p$ be a finite extension with ring of integers R. We want to define a p-adic regulator map

$$b_p : K_{2n-1}(R) \to K$$

or equivalently a system of elements in $H^{2n-1}(\mathrm{GL}_N(R), K)$. Note that $\mathrm{GL}_N(R)$ is a topological group and even a K-Lie group. Hence we can consider group cohomology with (locally) analytic cochains, i.e. maps which can be locally expressed by converging K-power series. We denote it $H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K)$. We use the map

$$\Phi : H^{2n-1}(\mathfrak{gl}_N, K) \to H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K)$$
$$f_1 \otimes \dots f_{2n-1} \mapsto (df_1)_e \wedge \cdots \wedge (df_{2n-1})_e$$

where $(df)_e$ is the differential of a locally analytic function f in the cotangent space at $e \in \mathrm{GL}_N(R)$.

Theorem 3.1. (1) *For $K = \mathbb{Q}_p$ and the subgroup $1 + pM_N(\mathbb{Z}_p) \subset \mathrm{GL}_N(\mathbb{Z}_p)$ this map agrees with Lazard's ([4]). In particular, it is an isomorphism in this case.*

(2) *It is an isomorphism for all K.*

Hence we can consider the composition

$$H^{2n-1}(\mathfrak{gl}_N, K) \cong H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \to H^{2n-1}(\mathrm{GL}_N(R), K)$$

Definition 3.2. The p-adic Borel regulator b_p is defined as the image of p_n.

4. The main result

Why is this the right map to consider? We see the following result as a good indication.

Theorem 4.1. *The p-adic Borel regulator agrees with Soulé's regulator under the Bloch-Kato exponential. More precisely, for $n \geq 1$*

$$H^{2n-1}(\mathrm{GL}_N(R), K) \xrightarrow{\exp_{\mathrm{BK}}} H^{2n-1}(\mathrm{GL}_N(R), H^1(K, \mathbb{Q}_p(n)))$$

maps b_p to the étale Chern class c_n.

This proves in particular that the étale Chern classes are continuous and even analytic.

We first give the reason *why* the Theorem actually holds: under the suspension map

$$H^{2n}_{\mathrm{DR}}(BG) \to H^{2n-1}_{\mathrm{DR}}(G) \cong H^{2n-1}(\mathfrak{gl}_N, K)$$

(with $G = \mathrm{GL}_{N,K}$) the universal Chern class in de Rham cohomology is mapped to p_n. This is nothing but the theory of characteristic classes and can in fact be seen as a definition of the Chern class.

In order to prove the theorem, we proceed as follows.

- Relate the de Rham Chern class to the étale Chern class via p-adic Hodge theory. Hence syntomic cohomology comes into play.
- Syntomic cohomology is viewed as a p-adic analogue of absolute Hodge cohomology.
- Now copy Beilinson's proof of the comparison isomorphism between the (infinite) Borel regulator and the Beilinson regulator, ie., the Chern class in absolute Hodge cohomology.

Some key points of the argument were explained in more detail in the talk. The complete argument is given in the preprint [3].

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On the Eisenstein classes of Hilbert-Blumenthal modular varieties

David Blottière

The sheaf theory of polylogarithms, first developed for the multiplicative group, provides an interpretation of special values of the zeta function in terms of Hodge theory (Part 1). Such a theory of polylogarithms exists also for complex abelian schemes (Part 2). For elliptic curves the objects of this theory have been intensely studied (see e.g. Results R1, R2 and R3 below). In the higher dimensional case, some results have been proven (see e.g. Results R1', R2'), but no link between this theory and some special values of L-functions was known. Specializing the geometric context to Hilbert-Blumenthal modular families of abelian varieties, we establish such a link (see Result R3') and obtain a geometric proof of the Klingen-Siegel Theorem (Part 3).

1. Review of the classical case

Beilinson's conjectures hold for Spec($\mathbb{Q}$) (Borel, Rapoport, ...) and one may interpret this result as follows. The subspace $\zeta(3)\mathbb{Q}$ of $\mathbb{R} = \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(3))$, where $\mathrm{MHS}_{\mathbb{R}}$ denotes the category of polarizable real mixed Hodge structures, compares extensions of motives and extensions of real mixed Hodge structures. Thus

$$(*) \qquad \zeta(3)\mathbb{Q} \text{ is a canonical subspace of } \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(3)).$$

The sheaf theory of polylogarithms for $\mathbb{G}_m$ (Beilinson, Deligne, Ramakrishnan) provides an explanation of the assertion $(*)$ using only Hodge theory. Let VMHS(X) be the category of admissible polarizable variations of rational mixed Hodge structures over X, for X a smooth complex algebraic variety. The objects of this theory are

- the logarithm (a pro-object of VMHS($\mathbb{G}_{m,\mathbb{C}}$)), denoted by $\mathcal{L}og$,
- the polylogarithm (an element of $\mathrm{Ext}^1_{\mathrm{VMHS}(\mathbb{G}_{m,\mathbb{C}}\setminus\{1\})}(\mathbb{Q}(0), \mathcal{L}og_{|\mathbb{G}_{m,\mathbb{C}}\setminus\{1\}})$,
- the Eisenstein classes (elements of $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathbb{Q}(k))$ for some $k \geq 0$, where MHS denotes the category of polarizable rational mixed Hodge structures).

All of these objects can be described explicitly, e.g. the polylogarithm corresponds to a pro-matrix in which appear all the multivalued functions Li_k, $k \geq 1$. It turns out that the Eisenstein classes are related to some special values of the zeta function and this provides the desired explanation of the assertion $(*)$ using only Hodge theory.

2. The abelian case

This sheaf theory of polylogarithms is defined in a more general geometric setting (cf. [13]) and gives some special elements (the Eisenstein classes) which should have remarkable properties. For instance, such a theory exists for complex abelian schemes.

Fix a smooth complex algebraic variety S and a complex abelian scheme $\pi\colon A \to S$ of pure relative dimension g. Let U be the complement of the zero section and let $\mathcal{H} := (R^1\pi_*\mathbb{Q})^\vee$ (polarizable variation of rational pure Hodge structures of weight -1 over S). For X a smooth complex algebraic variety, we denote by $\mathrm{MHM}(X)$ the category of algebraic mixed Hodge modules over X and we recall that one can see $\mathrm{VMHS}(X)$ as a full subcategory of $\mathrm{MHM}(X)$ in a canonical way. As in the case of the mupltiplicative group, one can define (see e.g. Sections 3–5 of [4])

- the logarithm (a pro-object of $\mathrm{VMHS}(A)$), denoted by $\mathcal{L}og$,
- the polylogarithm (an element of $\mathrm{Ext}^{2g-1}_{\mathrm{MHM}(U)}((\pi^*\mathcal{H})_{|U}, \mathcal{L}og_{|U}(g))$,
- the Eisenstein classes (elements of $\mathrm{Ext}^{2g-1}_{\mathrm{MHM}(S)}(\mathbb{Q}(0), (\mathrm{Sym}^k\mathcal{H})(g))$ for some $k \geq 0$).

For the elliptic case ($g = 1$), the definition and the study of these objects are due to Beilinson and Levin. For the universal elliptic curve over the modular curve, we have the following properties. We refer the reader to [1] for precise formulations and proofs (see also [11] for R3).

- R1 The Eisenstein classes have a motivic origin.
- R2 The polylogarithm is a 1-extension of admissible polarizable *variations* of rational mixed Hodge structures which can be explicitly described by a pro-matrix in which appear the Debye polylogarithms.
- R3 The residues of the Eisenstein classes at the ∞ cusp of the modular curve are related to some values of Bernoulli polynomials.

Later the definitions have been extended to any complex abelian schemes (this follows from the content of [13]) and the following results have been proven.

- R1' The Eisenstein classes have a motivic origin (see [6]).
- R2' The currents constructed by Levin in [9] provide an explicit description of the polylogarithm at the topological level. This result had been conjectured by Levin and is announced in the Note [2]. We refer to [4] for a proof (see the proof of Théorème 4.5 and Corollaire 4.7 in loc. cit.). We note that if the relative dimension of the abelian scheme is greater than 2, the polylogarithm is *not* an extension of admissible polarizable variations of rational mixed Hodge structures (cf. Theorem III-2.3 b) of [13]).

3. The Hilbert-Blumenthal case

If one specializes the geometric setting to Hilbert-Blumenthal modular families of abelian varieties, we show, using the result R2', the following generalization of the result R3.

R3' The Eisenstein classes degenerate at the ∞ cusp of the Baily-Borel compactification of the base in special values of an L-function associated to the underlying totally real number field. This result is stated in the Note [3] and the reader may consult [5] for a proof (see the proof of Théorème 5.2 in loc. cit.).

We mention that there exists a different proof of the result R3' (see [7]). Since the residues at the ∞ cusp are rational numbers, we can deduce from the result R3' the Klingen-Siegel Theorem. We note that our proof presents some analogy with the original one (cf. [8]). We also point out that there exist two other proofs due to Sczech [12] and Nori [10] which use rational cohomology classes to deduce the Theorem. Thus our proof has also some analogy with both of them.

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On the p-adic elliptic polylogarithm and the two-variable p-adic L-function for CM elliptic curves

Kenichi Bannai
(joint work with Shinichi Kobayashi and Takeshi Tsuji)

We explicitly calculate the p-adic realization of the elliptic polylogarithm for CM elliptic curves and relate it to special values of the two-variable p-adic L-function of the elliptic curve, when the elliptic curve has good ordinary reduction at $p \geq 5$. This extends previous results [1] which dealt with the one-variable case. Our result is based on my work with Shinichi Kobayashi [2] concerning a new method of constructing the two-variable p-adic L-function for CM elliptic curves, and we were able to give a very explicit description of the coherent module with connection underlying the elliptic polylogarithm sheaf. We expect that our method extends to when the elliptic curve has supersingular reduction at p, and we are hoping for similar results in this case.

0.1. **Background.** The classical polylogarithms are defined on the open unit disc $|t| < 1$ as the power series

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}.$$

This function may be extended as a multi-valued function on $\mathbb{C}$ as iterated integrals

$$\mathrm{Li}_{k+1}(s) := \int_0^s \mathrm{Li}_k(t) \frac{dt}{t} \quad (k \geq 0),$$

with $\mathrm{Li}_0(t) = t/(1-t)$. These functions were interpreted by Beilinson and Deligne (See for example [4]) as periods of a certain pro-variation of mixed Hodge structures, called the polylogarithm sheaf, on the projective line minus three points. Moreover, the construction of the Beilinson and Deligne works for any reasonable theory of mixed sheaves, including the conjectural category of mixed motivic sheaves. Hence one may consider various realizations of the polylogarithm sheaf, including the Hodge, the ℓ-adic, and the p-adic realizations.

Extending this construction, Beilinson and Levin [3] defined an analogous sheaf on an elliptic curve minus the identity. Let $\Gamma \subset \mathbb{C}$ be a lattice corresponding to the elliptic curve, A the fundamental volume of Γ divided by π, and $\chi_w(\gamma) = \exp((\gamma\overline{w} - \overline{\gamma}w)/A)$. Beilinson and Deligne explicitly describe the Hodge realization of this sheaf, and prove that the period of this sheaf is given by the Eisenstein-Kronecker-Lerch series, of the form

$$E_{a,b}(w) := \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{\overline{\gamma}^a}{\gamma^b} \chi_w(\gamma)$$

for integers $a < 0$ and $b \geq 0$. Note that the series converges only if $b > a + 2$, but one may give meaning to the function for any $a, b \geq 0$ by analytic continuation.

0.2. **p-adic case.** In our research, we constructed the p-adic elliptic polylogarithm as a (pro-) filtered overconvergent F-isocrystal on an elliptic curve minus the identity, using the construction of Beilinson and Levin. In order to describe the p-adic elliptic polylogarithm, we assume that the elliptic curve has complex multiplication in an imaginary quadratic field K. Assume for simplicity that the class number of K is one. By the theory of complex multiplication, there exists a model of this elliptic curve defined over K. Let $\Gamma \subset \mathbb{C}$ be the period lattice corresponding to this model. Then for z_0, $w_0 \in \Gamma \otimes \mathbb{Q}$, we define the Eisenstein-Kronecker number to be special values of Eisenstein-Kronecker-Lerch series, defined by the formula

$$e^*_{a,b}(z_0, w_0) := \sum_{\gamma \in \Gamma \setminus \{-z_0\}} \frac{(\overline{z}_0 + \overline{\gamma})^a}{(z_0 + \gamma)^b} \chi_{w_0}(\gamma).$$

Again, one can give meaning to these numbers for any integer a, b by analytic continuation. The classical theorem of Damerell asserts that $e^*_{a,b}(z_0, w_0)/A^a$ is algebraic if a, $b \geq 0$, hence we may interpret these numbers as p-adic numbers through a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. In order to obtain p-adic numbers for $a < 0$, we use p-adic interpolation. Suppose now that the elliptic curve has good ordinary reduction at $p \geq 5$. Since p is ordinary, p splits as $p = \mathfrak{p}\mathfrak{p}^*$ in K. Manin-Vishik, Katz, Yager, de Shalit and others defined a p-adic measure μ_{z_0,w_0} on $\mathbb{Z}_p \times \mathbb{Z}_p$ for z_0, $w_0 \notin \mathfrak{p}^{-\infty}\Gamma$ such that for any a, $b \geq 0$, we have

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,w_0}(x, y) = \frac{(-1)^{a+b} b!}{\Omega_{\mathfrak{p}}^{a+b} A^a} e^*_{a,b+1}(z_0, w_0)$$

where $\Omega_{\mathfrak{p}}$ is a certain p-adic period. A similar measure is constructed even when z_0 or $w_0 \in \Gamma$. This is precisely the measure used in the construction of the two-variable p-adic L-function associated to algebraic Hecke characters of K. We define the p-adic Eisenstein-Kronecker number by the formula

$$e^{(p)}_{a,b+1}(z_0, w_0) := \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,w_0}(x, y).$$

Note that the above numbers are defined even for $a < 0$. The importance of these numbers is that they are connected to special values of the two-variable p-adic L-function. Our main result is as follows.

Theorem 1. *Consider an elliptic curve with complex multiplication in an imaginary quadratic field, and assume that the elliptic curve has good ordinary reduction at the primes above p. Then the Frobenius structure of the p-adic elliptic polylogarithm sheaf restricted to a torsion point w_0 of order prime to $\mathfrak{p}$ is expressed by the p-adic Eisenstein-Kronecker numbers $e^{(p)}_{a,b}(0, w_0)$ for $a < 0$, $b \geq 0$.*

The above theorem is a p-adic analog of the result of Beilinson and Levin, for a single CM elliptic curve. Since the p-adic elliptic polylogarithm is the realization of a motivic sheaf, and $e^{(p)}_{a,b}(0, w_0)$ are numbers related to p-adic L-functions, the above result is a p-adic Beilinson conjecture type result.

In the supersingular case, similar p-adic distribution interpolating Eisenstein-Kroenecker numbers in one variable has been constructed by Boxall, Schneider-Teitelbaum, Fourquaux and Yamamoto, and we expect the p-adic elliptic polylogarithm to be related to special values of such distribution in a similar fashion. I am also currently working with Shinichi Kobayashi in an attempt to construct certain two-variable distribution interpolating Eisenstein-Kronecker numbers in the supersingular case.

0.3. **Main Tool.** The main tool used in the proof of the main theorem is the two-variable generating function of Eisenstein-Kronecker numbers. Let $\Gamma \subset \mathbb{C}$ be a lattice, and let $\theta(z)$ be the reduced theta function on $\mathbb{C}/\Gamma$ associated to the divisor (0), normalized so that $\theta'(0) = 1$. It may be given explicitly as $\theta(z) = \exp(-e_2^* z^2/2)\sigma(z)$, where $\sigma(z)$ is the Weierstrass sigma function and $e_2^* := e_{0,2}^*(0,0)$. We define the Kronecker theta function $\Theta(z,w)$ as follows.

$$\Theta(z,w) := \frac{\theta(z+w)}{\theta(z)\theta(w)}.$$

This function differs by an exponential factor from the two-variable Jacobi theta function studied by Zagier. The Kronecker theta function is a reduced theta function associated to the Poincaré bundle on the elliptic curve.

Theorem 2 (Kobayashi, B– [2]). *For any z_0, $w_0 \in \mathbb{C}$, let*

$$\Theta_{z_0,w_0}(z,w) := \exp\left[-\frac{z_0\overline{w}_0}{A}\right] \exp\left[-\frac{z\overline{w}_0 + w\overline{z}_0}{A}\right] \Theta(z+z_0, w+w_0).$$

Then we have

$$\Theta_{z_0,w_0}(z,w) = \frac{\delta_{z_0}}{z}\chi_{w_0}(z_0) + \frac{\delta_{w_0}}{w} + \sum_{a,b\geq 0} (-1)^{a+b} \frac{e_{a,b+1}^*(z_0,w_0)}{a!A^a} z^b w^a,$$

where $\delta_x = 1$ if $x \in \Gamma$ and $\delta_x = 0$ otherwise.

The exponential factors are algebraic translations which appear in Mumford's theory of algebraic theta functions. This function was used in [2] to give a new construction of the measure μ_{z_0,w_0}.

For application to the calculation of the elliptic polylogarithm, we use rational functions derived from $\Theta(z,w)$ to describe the elliptic polylogarithm sheaf, and relate it to special values of Eisenstein-Kronecker numbers.

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On the p-adic local Langlands correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$

Pierre Colmez

We described a functor $V \mapsto \Pi(V)$ attaching a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ to a 2-dimensional representation of $\mathrm{G}_{\mathbb{Q}_p}$. The construction goes through Fontaine's theory of (φ, Γ)-modules and gives a description locally analytic vectors of $\Pi(V)$.

Smooth representations and (φ, Γ)-modules in characteristic p

Peter Schneider
(joint work with Marie-France Vigneras)

The classical local Langlands correspondence (proved by Harris/Taylor and Henniart) establishes a distinguished bijection between n-dimensional discrete semisimple representations of the Weil-Deligne group of the nonarchimedean local field $\mathbb{Q}_p$ on the one hand and irreducible smooth representations of the group $GL_n(\mathbb{Q}_p)$ on the other hand. The Weil-Deligne group is a modification of the absolute Galois group of the field $\mathbb{Q}_p$ and its discrete representations are closely related to the ℓ-adic Galois representations where ℓ is any prime number different from p. If we consider p-adic Galois representations instead then the picture becomes much more complicated. On the other hand one can reduce it modulo p. By a theorem of Fontaine the category of p-adic Galois representations is equivalent to the category of etale (φ, Γ)-modules. So it seems a natural attempt to relate smooth representations of $GL_n(\mathbb{Q}_p)$ with torsion coefficients to etale (φ, Γ)-modules. In spectacular recent work Colmez has managed to do exactly this, and surprisingly even in a functorial way, in the special case of the group $GL_2(\mathbb{Q}_p)$.

In this talk I describe the general construction of a functor from the category of finitely presented smooth representations of $G(\mathbb{Q}_p)$ where G is any split reductive group over $\mathbb{Q}_p$ to the category of etale (φ, Γ)-modules but which are not required to be finitely generated. The crucial technique consists in introducing a much more general noncommutative analog of (φ, Γ)-modules.

Level-raising for GSp(4)

Claus M. Sorensen

In this talk we provide congruences between unstable and stable automorphic forms for the symplectic similitude group GSp(4). More precisely, we raise the level of certain CAP representations Π of Saito-Kurokawa type, arising from classical modular forms $f \in S_4(\Gamma_0(N))$ of square-free level and root number $\epsilon_f = -1$. We first transfer Π to a suitable inner form G such that $G(\mathbb{R})$ is compact modulo its center. This is achieved by viewing G as a similitude spin group of a definite quadratic form in five variables, and then θ-lifting the whole Waldspurger packet for $\widetilde{\mathrm{SL}}(2)$ determined by f. Thereby we obtain an automorphic representation π of G. For the inner form we prove a precise level-raising result, inspired by the work of Bellaiche and Clozel, and relying on computations of Schmidt. Thus we obtain a $\tilde{\pi}$ congruent to π, with a local component that is irreducibly induced from an unramified twist of the Steinberg representation of the Klingen Levi subgroup. To transfer $\tilde{\pi}$ back to GSp(4), we use Arthur's stable trace formula and the exhaustive work of Hales on Shalika germs and the fundamental lemma in this case. Since $\tilde{\pi}$ has a local component of the above type, all endoscopic error terms vanish. Indeed, by Weissauer, we only need to show that such a component does not participate in the θ-correspondence with any GO(4). This is an exercise in using Kudla's filtration of the Jacquet modules of the Weil representation. Thus we get a cuspidal automorphic representation $\tilde{\Pi}$ of GSp(4) congruent to Π, which is neither CAP nor endoscopic. In particular, its Galois representations are irreducible by work of Ramakrishnan. It is crucial for our application that we can arrange for $\tilde{\Pi}$ to have vectors fixed by the non-special maximal compact subgroups at all primes dividing N. Since G is necessarily ramified at some prime r, we have to show a non-special analogue of the fundamental lemma at r. Fortunately, by work of Kottwitz we can compare the involved orbital integrals to twisted orbital integrals over the unramified quadratic extension of $\mathbb{Q}_r$. The inner form G splits over this extension, and the comparison of the twisted orbital integrals can be done by hand. Finally we give an application of our main result to the Bloch-Kato conjecture. Assuming a conjecture on the rank of the monodromy operators at the primes dividing N, we construct a torsion class in the Selmer group of the motive $M_f(2)$.

Modular forms and Galois representations over imaginary quadratic fields

Tobias Berger
(joint work with Gergely Harcos, Krzysztof Klosin)

1. Associating Galois representations to cuspforms

Let F be an imaginary quadratic field with non-trivial automorphism c, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character ω. If π_∞ has Langlands parameter $W_{\mathbb{C}} = \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$ given by $z \mapsto \mathrm{diag}(z^{1-k}, \overline{z}^{1-k})$ for some integer $k \geq 2$, then by the Langlands philosophy π should give rise (for any prime number ℓ) to a continuous irreducible ℓ-adic representation $\rho = \rho_{\pi,\ell}$ of the Galois group $\mathrm{Gal}(\overline{F}/F)$ such that the associated L-functions agree. In other words, at each prime v of F the Frobenius polynomial of ρ at v agrees with the Hecke polynomial of π at v. Under the assumption that $\omega = \omega^c$ it is possible to relate π to holomorphic Siegel modular forms via theta lifts and deduce (using ℓ-adic cohomology on Siegel threefolds) some weak version of this predicted correspondence. In fact Taylor [10] managed to obtain the above equality of Frobenius and Hecke polynomials for all v outside a zero density set of places (under some technical assumptions that can be removed using the results of Friedberg and Hoffstein on the non-vanishing of certain central L-values).

I presented joint work with Gergely Harcos describing how the results of Laumon [5, 6] and Weissauer [12] on associating Galois representations to Siegel modular forms enable one to simplify Taylor's proof and conclude the statement for all v outside an explicit finite set:

Theorem 1. *Assume that $\omega = \omega^c$. Let S denote the set of places in F which divide ℓ or where $F/\mathbb{Q}$ or π or π^c is ramified. There exists a continuous irreducible representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that if v is a prime of F outside S then ρ is unramified at v and $L(s, \rho_v) = L(s, \pi_v)$.*

The proof of Theorem 1 can be briefly outlined as follows. The initial strategy is that of Taylor [10]. We can assume that π is neither a twist of a base change from $\mathbb{Q}$ nor a theta lift from a Grössencharakter of a quadratic extension of F, because the theorem is known in these cases. Using the deep results of [4] and [2] we construct a nonzero theta lift on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of the twist $\pi \otimes \mu$ for a *dense* set of quadratic idèle class characters μ of F. We call a set $\mathcal{M}$ of quadratic characters of F *dense* if it has the following property. If $\tilde{\mu}$ is a quadratic character of F and M is a finite set of rational primes then there is a character $\mu \in \mathcal{M}$ such that $\mu_v = \tilde{\mu}_v$ for all $v \in M$.

The irreducible constituent Π^μ of such a lift is generated by a vector-valued holomorphic semi-regular cusp form on the Siegel three-space. Using Hasse invariant forms and the theory of pseudo-representations developed by Wiles [13] and Taylor [8, 9], Taylor had shown that one can associate a 4-dimensional representation to Π^μ if one could associate 4-dimensional Galois representations to regular holomorphic Siegel cusp forms. This is now possible by work of Laumon [5, 6] and Weissauer [12]. We obtain therefore, for each μ in some dense set, a 4-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the same partial L-function as the one associated to Π^μ, and we prove that it is induced from some 2-dimensional representation ρ^μ of $\mathrm{Gal}(\overline{F}/F)$. By exploring global compatibility relations among the various ρ^μ we show that they can be replaced by quadratic twists $\rho \otimes \mu$ of a single 2-dimensional representation ρ of $\mathrm{Gal}(\overline{F}/F)$, and we verify that this ρ has the required property of Theorem 1.

2. Towards characterizing such Galois representations

I also reported on joint work in progress with Krzysztof Klosin studying deformations of a reducible residual representation of $\mathrm{Gal}(\overline{F}/F)$.

Let $p > 3$ be a rational prime which splits in $\mathcal{O}_F$. Fix embeddings $F \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ and let $\mathfrak{p}$ be the corresponding prime ideal of F over p. Fix a set Σ of primes of F containing $\{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Let G_Σ be the Galois group of the maximal extension of F unramified outside Σ. Consider an unramified Hecke character $\phi : F^* \backslash \mathbb{A}_F^* \to \mathbb{C}^*$ such that $\phi_\infty(z) = \frac{z}{\bar{z}}$ and $\phi^c = \phi^{-1}$. Write $\Psi : G_\Sigma \to \mathcal{O}^*$ for the p-adic Galois character associated to ϕ, where $\mathcal{O}$ is the ring of integers in some finite extension E of $\mathbb{Q}_p$. Let k be the residue field of $\mathcal{O}$. Put $\chi = \overline{\Psi} : G_\Sigma \to k^\times$.

Let

$$\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix} : G_\Sigma \to \mathrm{GL}_2(k)$$

be a continuous Galois representation with scalar centralizer. We impose conditions that ensure that ρ_0 is unique up to isomorphism.

We study deformations of ρ_0 with the goal of showing that under appropriate conditions all deformations are modular, i.e., arise from Galois representations attached to cuspforms of $\mathrm{GL}_2(\mathbb{A}_F)$.

As a first step we show that some modular deformations exist. Let $\mathbb{T}$ be the ordinary part of the $\mathcal{O}$-algebra generated by the Hecke operators $T_v, v \notin \Sigma$ acting on $S_2(K_f, \omega)$, the weight 2 cuspidal automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$ of a certain level K_f and character ω depending on ϕ and Σ. (To simplify the exposition we actually twist these forms in the following by an auxiliary Hecke character of infinity type z.) Define the Eisenstein ideal $\mathbb{I} \subset \mathbb{T}$ as the ideal generated by $\{T_v - (1 + \phi(\pi_v)) | v \notin \Sigma\}$.

Theorem 2 ([1, Theorem 6.3])**.**

$$\mathrm{val}_p \# \mathbb{T}/\mathbb{I} \geq \mathrm{val}_p \#(\mathcal{O}/L^{\mathrm{int}}(1, \phi)).$$

Corollary 3. *If $p \mid L^{\mathrm{int}}(1, \phi)$ then there exists an ordinary cuspidal automorphic representation π and a Galois representation $\rho : G_\Sigma \to \mathrm{GL}_2(\mathcal{O})$ equivalent to (a twist of) $\rho_{\pi,p}$ such that $\overline{\rho} = \rho_0$.*

Urban [11, Corollaire 2] proves that the Galois representation $\rho_{\pi,p}$ associated to an ordinary π is ordinary at $v \mid p$. Corollary 3 therefore provides us with irreducible ordinary deformations of ρ_0 and we obtain:

Corollary 4. *ρ_0 splits when restricted to $D_{\overline{\mathfrak{p}}}$.*

Definition 5. For T a finite set of places of F let $L_\Psi(T)$ be the maximal abelian pro-p extension of $F(\Psi)$ unramified outside T such that $\mathrm{Gal}(F(\Psi)/F)$ acts on $\mathrm{Gal}(L_\Psi(T)/F(\Psi))$ by Ψ^{-1}.

We deduce from a result of Greenberg [3]:

Proposition 6. (1) *$\mathrm{Gal}(L_\Psi(\Sigma \backslash \{\overline{\mathfrak{p}}\})/F(\Psi))$ is $\mathbb{Z}_p$-torsion.*
(2) *The $\mathbb{Z}_p$-rank of $\mathrm{Gal}(L_\Psi(\Sigma)/F(\Psi))$ is 1.*

Note that $\Psi|_{I_\mathfrak{p}} = \epsilon^{-1}|_{I_\mathfrak{p}}$ and $\Psi|_{I_{\overline{\mathfrak{p}}}} = \epsilon|_{I_{\overline{\mathfrak{p}}}}$, where ϵ is the p-adic cyclotomic character. This means that there exists no reducible ordinary deformation of ρ_0, since an ordinary representation of the form $\begin{pmatrix} 1 & * \\ 0 & \Psi \end{pmatrix}$ has to split when restricted to $I_{\overline{\mathfrak{p}}}$. However, we can define a reducible deformation of ρ_0 of the form

$$\rho^{\mathrm{Eis}} = \begin{pmatrix} 1 & * \\ 0 & \Psi \end{pmatrix}$$

which does not split when restricted to $I_{\overline{\mathfrak{p}}}$. It is *nearly ordinary* with respect to the Borel of upper-triangular matrices B, i.e., $\rho^{\mathrm{Eis}}(D_v) \subset B(\mathcal{O})$ for $v = \mathfrak{p}, \overline{\mathfrak{p}}$.

We are hoping to find suitably restrictive deformation conditions that are satisfied by both the irreducible ordinary deformations and the reducible deformation ρ^{Eis}. Then the method of [7] would give an $R = T$ theorem, i.e., prove the modularity of residually reducible Galois representations.

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Root numbers, Selmer groups, and non-commutative Iwasawa theory

Takako Fukaya[1]

(joint work with John Coates, Kazuya Kato, Ramdorai Sujatha)

Let F be a finite extension of $\mathbb{Q}$, A an abelian variety of dimension g defined over F, and p a prime number. We consider 'parity conjecture' for A and p. First we recall what the parity conjecture is.

Conjecture 1 (Birch and Swinnerton-Dyer). *The L-function* $L(A/F, s)$ *of* A *has an analytic continuation to* $s = 1$*, and*

$$\mathrm{ord}_{s=1} L(A/F, s) = \mathrm{rank}\, A(F). \tag{1}$$

Parity conjecture considers mod 2 of (1) using root numbers and p-Selmer coranks.

First we consider the left hand side of (1). Let $w(A/F) = \prod_v w_v(A/F)$, the root number, where v runs though all places of F and for each v, $w_v(A/F) \in \{\pm 1\}$

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is the local root number which is 1 for almost all v. The root number $w(A/F)$ appears in conjectural functional equation of $L(A/F, s)$, and is conjectured to satisfy (proved in some cases) $w(A/F) = (-1)^{\mathrm{ord}_{s=1} L(A/F,\, s)}$.

We consider the right hand side of (1). Let K be an algebraic extension of F. Put $S(A/K) = \mathrm{Ker}\,(H^1(K, A_{p^\infty}) \to \prod_v H^1(K_v, A(\bar{K_v})))$, the p-primary part of the Selmer group, where v runs over all places of K. From the definition it may be seen that $S(A/F)$ is cofinitely generated module over $\mathbb{Z}_p$, and we put $s(A/F) = \mathbb{Z}_p$-corank of $S(A/F)$. If the Tate-Shafarevich group $\text{Ш}_p(A/F)$ is finite, which is conjectured to be true always, we have $s(A/F) = \mathrm{rank}\, A(F)$.

Conjecture 2 ((p-)parity conjecture)**.** *We have*

$$w(A/F) = (-1)^{s(A/F)}.$$

The following is our main theorem on Conjecture 2.

Theorem 3 ([1])**.** *Conjecture* 2 *holds for* A *and* p *when the conditions* (i)–(iii) *are satisfied*: (i) *For the Galois module* $A[p]$ *of* p*-division points on* A*, there is a subgroup* C *of* $A[p]$ *of order* p^g*, stable under* $\mathrm{Gal}(\bar{F}/F)$*, and an isogeny* $\psi : A \to A^*$*, where* A^* *is the dual abelian variety, of degree prime to* p*, such that the dual isogeny* $\psi^* : A = (A^*)^* \xrightarrow{\psi^*} A^*$ *coincides with* ψ*, and also such that the Weil pairing* $\langle\ ,\ \rangle_{A,p}$ *annihilates* $C \times \psi(C)$; (ii) *Either* $p \geq 2g + 2$*, or* $p \geq g + 2$ *and* A *has semistable reduction at each finite place* v *of* F; (iii) *For each place* v *of* F *dividing* p*, either* A *is potentially ordinary at* v*, or* A *achieves semistable reduction over a finite abelian extension of* F_v*.*

By potentially ordinary at v, we mean that there is a finite extension L of F_v such that A has semistable reduction over L, and the connected component of the special fiber of the Néron model of $A \otimes_F L$ is an extension of an ordinary abelian variety by a torus. If A is an elliptic curve, A is potentially ordinary at v if and only if either A has potentially good ordinary reduction at v, or potentially multiplicative reduction at v.

Corollary 4 ([1])**.** *Assume* p *is an odd prime number, and that* E/F *is an elliptic curve admitting an* F*-isogeny of degree* p*. If* $p = 3$*, assume that* E *has semistable reduction at each finite place of* F*. If* $p > 3$*, suppose that for each prime* v *of* F *dividing* p*, either* E *has potentially good ordinary reduction at* v*, or* E *has potentially multiplicative reduction at* v*, or* E *achieves good supersingular reduction over a finite abelian extension of* F_v*. Then the parity conjecture* 2 *holds for* E *and* p*.*

Remark 5. *T. and V. Dokchitser proved a slightly weaker version of Corollary* 4 *by a similar method with ours. Nekovář, Kim, T. and V. Dokchitser, and other people have obtained results on Conjecture* 2*.*

The method of the proof of Theorem 3 is a generalization of the methods of Cassels, Fisher, Shuter, T. and V. Dokchitser, which reduce $s(A/F)$ to a local problem. We then compare those local factors of $s(A/F)$ with local root numbers $w_v(A/F)$ in local levels. Namely, Theorem 3 is reduced to the following results.

Theorem 6. *Let A/F be an abelian variety satisfying* (i) *of Theorem* 3*, with p an odd prime number. Then*

$$s(A/F) \equiv \sum_v h(v) \mod 2,$$

where the sum is taken over all places of F, and $h(v) = \mathrm{ord}_p\sharp(\mathrm{Coker}\,(\phi_v)/\mathrm{Ker}\,(\phi_v))$ with the homomorphism $\phi_v : A(F_v) \to A'(F_v)$ induced by the isogeny $\phi : A \to A' := A/C$.

Theorem 7. *Assume that A satisfies the hypotheses of Theorem* 3*. Let $h(v)$ be as in Theorem* 6*. For any place v of F, we have*

$$(-1)^{h(v)} = w_v(A/F)\chi_{C,v}(-1).$$

Here $\chi_{C,v} : F_v^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ is a character corresponding via class field theory to the determinant $\det(\alpha_{C,v}) : \mathrm{Gal}(F_v^{\mathrm{ab}}/F_v) \to (\mathbb{Z}/p\mathbb{Z})^\times$ of the homomorphism $\alpha_{C,v} : \mathrm{Gal}(\bar{F}_v/F_v) \to \mathrm{Aut}(C)$ which is the restriction to the decomposition group at v of the homomorphism $\alpha_C : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(C) \simeq GL_g(\mathbb{Z}/p\mathbb{Z})$ given by the action of $\mathrm{Gal}(\bar{F}/F)$ on C.

By the reciprocity law of global class field theory, we have $\prod_v \chi_{C,v}(-1) = 1$. Hence Theorem 3 follows from Theorems 6 and 7.

In what follows, we consider Artin twist version of the parity conjecture 2. We suppose $p \geq 5$. Let E/F be an elliptic curve, and put $F_\infty = F(E_{p^\infty})$, $G = \mathrm{Gal}(F_\infty/F)$, and $H = \mathrm{Gal}(F_\infty/F^{\mathrm{cyc}})$, here F^{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension of F. Let $\rho : G \to GL_{d_\rho}(\bar{\mathbb{Q}}_p)$ be an irreducible self-dual Artin representation of G. We denote by $w(E,\rho)$ the root number occurring in the conjectural functional equation of $L(E,\rho,s)$ ($\leftrightarrow L(E,\rho^*,2-s) = L(E,\rho,2-s)$). We write $s(E,\rho)$ for the number of the copies of ρ occurring in $X(E/K) \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p$, where K is any finite extension of F such that ρ factors through $\mathrm{Gal}(K/F)$ and $X(E/K)$ is the Pontryagin dual of $S(E/K)$.

Conjecture 8 (ρ-parity conjecture)**.**

$$w(E,\rho) = (-1)^{s(E,\rho)}.$$

We define u_G to be the order of the image of G under a composition $G \to GL_2(\mathbb{Z}_p) \to PGL_2(\mathbb{F}_p)$, where the first map is given by the action of G on the p-adic Tate module of E.

Now we assume (i) E admits an isogeny of degree p defined over F. Under this assumption, Rohrlich [2] has shown that there exist irreducible self-dual Artin representaions of G of dimension > 1 if and only if u_G is even. We assume for the rest of this abstract that u_G is even and the dimension d_ρ of ρ is > 1.

Concerning the root number $w(E,\rho)$, Rohrlich has shown the following.

Theorem 9 (Rohrlich [2])**.** *Assume* (i) *above, and* (ii) *E has potential good ordinary reduction at any place v of F above p. Then*

$$w(E,\rho) = (-1)^{u_G[F:\mathbb{Q}]/2 + \sum_{v:\textit{finite},\ \mathrm{ord}_v(j_E)\,<\,0} \langle \chi_v, \rho_v \rangle},$$

where j_E denotes the j-invariant of E, χ_v is a character of $\mathrm{Gal}(F_{\infty,v}/F_v)$ *such that $\chi_v = 1$ if E has split multiplicative reduction at v and a non-trivial quadratic character corresponding by class field theory to the quadratic extension of F_v at which E achieves split multiplicative redution otherwise. Moreover, ρ_v is the restriction of ρ to the decomposition group at v, $\langle \chi_v, \rho_v \rangle$ denotes the multiplicity of χ_v occurring in ρ_v.*

We study $s(E,\rho)$. Let $\mathfrak{M}_H(G)$ be the category of all finitely generated $\mathbb{Z}_p[[G]]$-modules M such that $M/M(p)$, where $M(p)$ is the p-primary part of M, is finitely generated over $\mathbb{Z}_p[[H]]$.

Theorem 10 ([1])**.** *Assume the conditions* (i)–(ii) *in Theorem* 9*. Assume furthermore* (iii) $X(E/F_\infty) \in \mathfrak{M}_H(G)$; (iv) *ρ is orthogonal. Then*

$$s(E,\rho) \equiv u_G[F:\mathbb{Q}]/2 + \sum_{v:\textit{finite},\ \mathrm{ord}_v(j_E)\,<\,0} \langle \chi_v, \rho_v \rangle \mod 2.$$

That is, under the hypotheses (i)–(iv)*, the ρ-parity conjecture* 8 *holds for E, p, and ρ.*

For the proof of Theorem 10, we use non-commutative Iwasawa theory.

Finally the author would like to express her sincere gratitude to the organizers of the conference for inviting her to the conference and giving her an opportunity to give a talk.

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Iwasawa Theory for Hida deformations with complex multiplication

Tadashi Ochiai

(joint work with Kartik Prasanna)

My subject of research is closely related to Greenberg's conjectural program on generalizing Iwasawa theory to families of Galois representations. In his article [G], he proposes the study of Iwasawa theory for families of nearly ordinary Galois representations $\widetilde{T}$ finite over deformation algebras $\widetilde{R}$. Though his plan is a tentative conjecture where the (conjectural) definition of the analytic p-adic L-functions for such $\widetilde{T}$ is still vague, he nevertheless indicates a fascinating direction of research.

In previous work on this subject, I focused on the first non-trivial example with which Greenberg's plan is concerned, namely two-variable Hida deformations. The articles [O1], [O2] and [O3] established Iwasawa theory for Hida deformations $\mathcal{T}$ *without* complex multiplication. A Hida deformation $\mathcal{T}$ is roughly associated to p-adic family of elliptic cuspforms f_k where the weight k of f_k varies in a p-adic parameter space. In the process one discovers new phenomena which do not arise in the usual cyclotomic Iwasawa theory for ordinary motives. For instance, one finds that a detailed study of complex and p-adic periods in families is essential to even cogently formulate the definition of analytic p-adic L-functions in general situations, a fact that does not seem to have been observed before. A consequence of this work is a formulation of Iwasawa theory for general nearly ordinary families of Galois representation that is more precise than before (cf. [O4]) especially with regards the analytic p-adic L-function.

In this work, based on this motivation, we study Iwasawa theory also for Hida deformations $\mathcal{T}$ *with* complex multiplication by an imaginary quadratic field K. For the associated family of f_k, $\mathcal{T}$ with complex multiplication the one where f_k is a lift of a grossencharacter ρ_k of weight $k-1$ on K for each k. Note that grossencharacters are modular forms on the group $GL(1)_{/K}$. From this point of view, Iwasawa theory has been studied previously by Katz, Coates-Wiles, Colmez, Yager, de Shalit, Rubin, Tilouine and other authors using tools from the theory of complex multiplication (eg. the Euler system of elliptic units and the evaluation of Eisenstein series at CM points). On the other hand, forgetting about the complex multiplication, f_k itself is a modular form on the group $GL(2)_{/\mathbb{Q}}$. From such another point of view, we have other tools such as modular symbols and the Beilinson-Kato Euler system, as is also taken in [O1], [O2] and [O3] (Note however that Beilinson-Kato elements cannot be used to bound the size of Selmer group in the CM case because the image of rank two Galois representation for a CM modular form is two small). Thus, there are two completely different approaches to Hida families with complex multiplication and it seems to us that the relation between the approach via $GL(1)_{/K}$ and that via $GL(2)_{/\mathbb{Q}}$ is not so obvious. The purpose of this article is to clarify some aspects of this relation.

More precisely, we will compare the algebraic and analytic p-adic L-functions as well as Iwasawa Main Conjecture for them from these two points of view. The situation is summarized in the following diagram which shows the relations of ideals in a two-variable Iwasawa algebra $A[[\widetilde{\Gamma}]]$ with $\widetilde{\Gamma} \cong \mathbb{Z}_p$ the Galois group of $\mathbb{Z}_p^2$-extension of K and A a finite extension of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$:

$$
(1) \qquad
\begin{array}{ccc}
(L_p^{\mathrm{alg}}(GL(1)_{/K})) & \overset{\text{(a)}}{=\!=\!=} & (L_p^{\mathrm{anal}}(GL(1)_{/K})) \\
\text{(c)}\,\| & & \vdots\,\text{(d)} \\
(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}})) & \underset{\text{(b)}}{\cdots\cdots} & (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}})).
\end{array}
$$

Two objects on the left are algebraic p-adic L-functions, which are defined to be the characteristic ideals of Selmer groups in each context. The ideal $(L_p^{\mathrm{alg}}(GL(1)_{/K}))$

is the characteristic ideal of the Galois group of certain infinite Galois extension over $\mathbb{Z}_p^2$-extension of K. The ideal $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}}))$ is the characteristic ideal of the Pontrjagin dual of the Selmer group $\mathrm{Sel}_{\mathcal{T}}$. Two objects on the right are analytic p-adic L-functions which interpolate critical values of L-functions. The analytic p-adic L-function $L_p^{\mathrm{anal}}(GL(1)_{/K})$ is constructed Katz, Yager, de Shalit, Tilouine. The analytic p-adic L-function $L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}})$ is constructed by Kitagawa, Greenberg, Panchishkin, Fukaya and the first author. However, for certain reasons, we have to take the one by Kitagawa which is the best candidate for the Iwasawa Main conjecture.

The relations in the diagram are explained below:

(a) The upper line is two-variable Iwasawa Main Conjecture from the point of view of $GL(1)_{/K}$. The equality is shown by Rubin (cf. [Ru1], [Ru2]) under fairly general conditions.

(b) The lower line is two-variable Iwasawa Main Conjecture from the point of view of $GL(2)_{/\mathbb{Q}}$ which was first formulated in [G, Chapter 4] and later refined by the first author in [O3, Conj. 2.4]. Note that the lower line makes sense for any Hida family, not just the CM ones. In fact, in [O3], we obtained the inequality $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}})) \supset (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ but only in the non-CM case. However, no equality was known between $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}}))$ and $(L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ for the CM case.

(c) In the algebraic side, it is not difficult to show $(L_p^{\mathrm{alg}}(GL(1)_{/K})) = (L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}}))$ by calculation of Galois cohomology.

Now there rests the part (d) the main theme in this paper. $(L_p^{\mathrm{anal}}(GL(1)_{/K}))$ is constructed via the theory of complex multiplication; $(L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ via the theory of modular symbols. These analytic p-adic L-functions are *a priori* different but we conjecture as follows:

Conjecture . *We have the equality of ideal* $(L_p^{\mathrm{anal}}(GL(1)_{/K})) = (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ *at* (d) *in the diagram.*

Note that, according to (a), (b) and (c), in most cases, Conjecture is equivalent to the Iwasawa main conjecture (b) for CM Hida deformations. The main result is the following theorem.

Main Theorem . *The above conjecture is true under certain assumptions, namely the irreducibility of the associated mod p representation and the vanishing of a certain μ-invariant.*

As seen from the diagram (1), we have an immediate corollary as follows.

Corollary . *Iwasawa Main Conjecture* $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}})) = (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ *at* (b) *in the diagram is true for CM Hida deformations satisfying assumptions in the theorem and the assumption required in Rubin's results* ([Ru1], [Ru2]).

Here we give some idea of the principal difficulties involved. Indeed, it may seem at first sight that Conjecture should be more or less immediate, since after all, the two p-adic L-functions interpolate the same set of L-values. However, there are two main obstructions to making such a conclusion. The first is that the periods that occur in the two interpolation formulae are not the same and need to be related to each other. Such "p-integral period relation" is uaually requires a hard work as is done in [Pr1] and [Pr2] recently. First, we established the requisite period relation up to p-adic units. But having done that, one is faced with a second difficulty which may be more formally described as follows. We explain briefly about the second difficulty as well as the argument of the proof. We are given two different elements $F(X_1, X_2)$ and $G(X_1, X_2)$ in $A[[X_1, X_2]]$ ($A[[X_1, X_2]]$ should be $A[[\widetilde{\Gamma}]]$ and $F(X_1, X_2)$, $G(X_1, X_2)$ should be $L_p^{\mathrm{anal}}(GL(1)_{/K})$ and $L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}})$ in the diagram (1)). A priori, we know no divisibility between $F(X_1, X_2)$ and $G(X_1, X_2)$. If we have established the period relation up to p-units above, Weierstrauss preparation shows that the elements $F(X_1, X_2)|_{X_2=(1+p)^n-1}, G(X_1, X_2)|_{X_2=(1+p)^n-1} \in A[[X_1]]$ is equal modulo multiplication by a unit u_n in A for each $n \geq 0$ (Note, however, that there seems to be no systematic choice of the constant u_n for varying n since there will be no canonical choice of modular symbol period). This is not sufficient to deduce the divisibility between $F(X_1, X_2)$ and $G(X_1, X_2)$ and we will can construct a counter example by an example similar to the one we explain below for the specialization of power series algebras in one-variable. Hence, we has developed an argument which allows us to deduce the desired divisibility under the assumption on the Iwasawa μ-invariant for $F(X_1, X_2)|_{X_2=(1+p)^n-1}$ or $G(X_1, X_2)|_{X_2=(1+p)^n-1} \in A[[X_1]]$. This proves our main theorem. The work is written in our paper [OP], which is available quite soon.

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Equivariant vector bundles on Drinfeld's upper half space

Sascha Orlik

Let K be a finite extension of $\mathbb{Q}_p$. We denote by

$$\mathcal{X} = \mathbb{P}^d_K \setminus \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H)$$

Drinfeld's upper half space of dimension $d \geq 1$ over K. Here H runs through all K-rational hyperplanes in K^{d+1}. Drinfeld [3] conjectured that the étale coverings of $\mathcal{X}$ realize the supercuspidal spectrum of the local Langlands correspondence for $G = \mathrm{GL}_{d+1}(K)$ by considering the ℓ-adic cohomology of these spaces. In [6] Schneider studies the cohomology of local systems on projective varieties which are uniformized by $\mathcal{X}$. For this purpose, he defines the notion of a p-adic holomorphic discrete series representation. These representations can be realized by the space of rigid analytic holomorphic sections $\mathcal{F}(\mathcal{X})$ of GL_{d+1}-equivariant vector bundles $\mathcal{F}$ on $\mathbb{P}^d_K$. They are reflexive K-Fréchet spaces with a continuous G-action. The strong dual $\mathcal{F}(\mathcal{X})'$ is a locally analytic G-representation in the sense of Schneider and Teitelbaum [8]. Those representations come up in the p-adic Langlands theory of Breuil and Schneider [2] as the locally analytic part of certain Banach space representations. On the other hand, Schneider and Stuhler computed in [9] the étale and the de Rham cohomology of $\mathcal{X}$. It turns out that the cohomology groups are duals of certain generalized Steinberg representations. It is desirable to have knowledge on the individual contributions of the de Rham complex, which are holomorphic discrete series representations. In the case of the canonical bundle, Schneider and Teitelbaum construct in [7] a G-equivariant decreasing filtration by closed subspaces

$$\Omega^d(\mathcal{X})^0 \supset \Omega^d(\mathcal{X})^1 \supset \cdots \supset \Omega^d(\mathcal{X})^{d-1} \supset \Omega^d(\mathcal{X})^d \supset \Omega^d(\mathcal{X})^{d+1} = \{0\}$$

on $\Omega^d(\mathcal{X})^0 = \Omega^d(\mathcal{X})$. The definition of the filtration involves the geometry of $\mathcal{X}$ being the complement of an hyperplane arrangement. Further they construct isomorphisms

$$I^{[j]} : (\Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1})' \xrightarrow{\sim} C^{an}(G, P_{\underline{j}}; V_j')^{\mathfrak{d}_{\underline{j}}=0}$$

of locally analytic G-representations. Here, $P_{\underline{j}}=P_{(j,d+1-j)} \subset G$ is the (lower) standard-parabolic subgroup attached to the decomposition $(j, d+1-j)$ of $d+1$. The right hand side is a locally analytic induced representation. The $P_{\underline{j}}$-representation V_j' is a locally algebraic representation. It is isomorphic to the tensor product $\mathrm{Sym}^j(K^{d+1-j}) \otimes \mathrm{St}_{d+1-j}$ of the irreducible algebraic GL_{d+1-j}-representation $\mathrm{Sym}^j(K^{d+1-j})$ and the Steinberg representation St_{d+1-j} of $\mathrm{GL}_{d+1-j}(K)$. The action of $\mathrm{GL}_j(K)$ is given by the inverse of the determinant character. The operation of the unipotent radical of $P_{\underline{j}}$ on V_j' is trivial. Finally, $\mathfrak{d}_{\underline{j}}$ denotes a system of differential equations which is here a submodule of a generalized Verma module. In particular, the case $j = 0$, i.e., the first subquotient of the above filtration is isomorphic to $H^d_{dR}(\mathcal{X})$ and yields the Steinberg representation of G.

Finally, we want to point out that a similar construction was given by Pohlkamp [5] for the structure sheaf $\Omega^0 = \mathcal{O}$ on $\mathbb{P}^d_K$.

In the talk we presented a decreasing G-equivariant filtration on $\mathcal{F}(\mathcal{X})$ for all G-equivariant vector bundles $\mathcal{F}$ on $\mathcal{X}$, which are induced by restriction of a homogeneous vector bundle on $\mathbb{P}^d_K \cong G/P_{(1,d)}$. Our approach is different from [7], [5]. We use local cohomology of coherent sheaves on rigid analytic varieties as a technical ingredient. In fact, $\mathcal{F}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{F})$ appears in an exact sequence

$$0 \to H^0(\mathbb{P}^d_K, \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{F}) \to H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F}) \to H^1(\mathbb{P}^d_K, \mathcal{F}) \to 0.$$

We consider the K-Fréchet space $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$, where $\mathcal{Y} \subset \mathbb{P}^d_K$ is the "closed" complement of $\mathcal{X}$ in $\mathbb{P}^d_K$. We use the acyclic resolution of the constant sheaf $\mathbb{Z}$ on $\mathcal{Y}^{ad}$ constructed in [4], where $\mathcal{Y}^{ad} \stackrel{i}{\hookrightarrow} (\mathbb{P}^d_K)^{ad}$ is the closed complement of the adic space $\mathcal{X}^{ad}$ in $(\mathbb{P}^d_K)^{ad}$. By applying the functor $\mathrm{Hom}(i_*(\ -\ ), \mathcal{F})$ to this complex, we get a spectral sequence converging to $H^1_{\mathcal{Y}^{ad}}((\mathbb{P}^d_K)^{ad}, \mathcal{F}^{ad}) = H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$. The canonical filtration on $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{F})$ coming from this spectral sequence gives rise to a decreasing filtration by closed K-subspaces

$$\mathcal{F}(\mathcal{X})^0 \supset \mathcal{F}(\mathcal{X})^1 \supset \cdots \supset \mathcal{F}(\mathcal{X})^{d-1} \supset \mathcal{F}(\mathcal{X})^d = H^0(\mathbb{P}^d, \mathcal{F})$$

on $\mathcal{F}(\mathcal{X})^0 = \mathcal{F}(\mathcal{X})$. Our first main theorem is:

Theorem 1: *Let $\mathcal{F}$ be a homogeneous vector bundle on $\mathbb{P}^d_K$. For $j = 0, \ldots, d-1$, there are extensions of locally analytic G-representations*

$$0 \to v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K, \mathcal{F})') \to (\mathcal{F}(\mathcal{X})^j / \mathcal{F}(X)^{j+1})' \to C^{an}(G, P_{\underline{j+1}}; U'_j)^{\mathfrak{d}_j = 0} \to 0.$$

Here the module $v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K, \mathcal{F})')$ is a generalized Steinberg representation with coefficients in the finite-dimensional algebraic G-module $H^{d-j}(\mathbb{P}^d_K, \mathcal{F})'$. The $P_{\underline{j+1}}$-representation U'_j is a tensor product $N'_j \otimes \mathrm{St}_{d-j}$ of an algebraic $P_{\underline{j+1}}$-representation N'_j and the Steinberg representation St_{d-j}. The symbol $\mathfrak{d}_j$ indicates again a system of differential equations depending on N_j. Indeed, the representation N_j is not uniquely determined. It is characterized by the property that it generates the kernel of the natural homomorphism $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{F})$ as a module with respect to the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra of G.

In the case where $\mathcal{F}$ arises from an irreducible representation of the Levi subgroup $L_{(1,d)}$, we can make our result more precise. Let $\lambda' = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ be a dominant integral weight of GL_d and a let $\lambda_0 \in \mathbb{Z}$. Set $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^{d+1}$. Denote by $\mathcal{F}_\lambda$ the homogeneous vector bundle on $\mathbb{P}^d_K$ such that its fibre in the base point is the irreducible algebraic $L_{(1,d)}$-representation corresponding to λ. Put $w_j := s_j \cdots s_1$, where $s_i \in W$ is the (standard) simple reflection in the Weyl group $W \cong S_{d+1}$ of G. By Bott [1] we know that there is at most one integer $i \geq 0$ with $H^i(\mathbb{P}^d_K, \mathcal{F}) \neq 0$. Denote this integer by i_0 if it exists. Otherwise, there is an $i_0 \leq d-1$ with $w_{i_0} * \lambda = w_{i_0+1} * \lambda$, where $*$ is the dot operator of W on the set of weights. For $j = 1, \dots, d$, we set

$$\mu_{j,\lambda} := \begin{cases} w_{j-1} * \lambda & : j \leq i_0 \\ w_j * \lambda & : j > i_0 \end{cases}.$$

Write $\mu_{j,\lambda} = (\mu', \mu'')$ with $\mu' \in \mathbb{Z}^j$ and $\mu'' \in \mathbb{Z}^{d-j+1}$. For $j = 1, \dots, d$, let

$$\Psi_{j,\lambda} = \bigcup_{k=0}^{|\mu''|} \Big\{ \big(\mu'' + (c_1, \dots, c_{d-j+1}), \mu' - (d_j, \dots, d_1)\big) \mid \textstyle\sum_l c_l = \sum_l d_l = k, \\ c_1 = 0 \text{ or } d_1 = 0,\ c_{l+1} \leq \mu''_l - \mu''_{l+1},\ l = 1, \dots, d-j, \\ d_{l+1} \leq \mu'_{j-l} - \mu'_{j-l+1},\ l = 1, \dots, j-1 \Big\}.$$

Here $|\mu''| = \mu''_1 - \mu''_{d-j+1}$. The elements in the finite set $\Psi_{j,\lambda}$ are dominant with respect to the Levi subgroup $L_{(d-j+1,j)}$ and (μ'', μ') is its highest weight. Hence, for $\mu \in \Psi_{j,\lambda}$, we may consider the irreducible algebraic $L_{(d-j+1,j)}$-representation V_μ attached to it.

Theorem 2: *Let $\mathcal{F} = \mathcal{F}_\lambda$ be the homogeneous vector bundle on $\mathbb{P}^d_K$ with respect to the dominant integral weight $\lambda \in \mathbb{Z}^{d+1}$ of $L_{(1,d)}$. Then we can choose N_j to be a quotient of $\bigoplus_{\mu \in \Psi_{d-j,\lambda}} V_\mu$.*

Our filtration coincides with the filtration of Schneider and Teitelbaum. More precisely, their filtration is related to ours by a shift, i.e., we have $\mathcal{F}(\mathcal{X})^i = \Omega^d(\mathcal{X})^{i+1}$ for $i \geq 1$. For $i = 0$, we get an extension

$$0 \to \Omega^d(\mathcal{X})^1/\Omega^d(\mathcal{X})^2 \to \mathcal{F}(\mathcal{X})^0/\mathcal{F}(\mathcal{X})^1 \to \Omega^d(\mathcal{X})^0/\Omega^d(\mathcal{X})^1 \to 0.$$

The dual sequence coincides with the corresponding one of Theorem 1.

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Affine Deligne-Lusztig varieties

Ulrich Görtz
(joint work with Thomas Haines, Robert Kottwitz, Daniel Reuman)

1. Introduction

The notion of (generalized) affine Deligne-Lusztig variety was defined in [14] by Rapoport; see also [15]. Affine Deligne-Lusztig varieties are analogs, in the setting of an affine root system, of usual Deligne-Lusztig varieties [2], [3], which are defined as follows: Let G be a connected reductive group over a finite field k, let $\overline{k}$ be an algebraic closure of k, and denote by σ the Frobenius on $\overline{k}$. Let $A \subseteq B \subseteq G$ be a maximal torus and a Borel subgroup. For an element $w \in W$ of the Weyl group, we define the Deligne-Lusztig variety associated with w by

$$X_w = \{g \in G/B;\ g^{-1}\sigma(g) \in BwB\}$$

(in [2], this variety is denoted by $X(w)$). Then X_w is a locally closed subvariety of G/B which is smooth of dimension $\ell(w)$ (the length of w as an element of W). The finite group $G(\mathbb{F}_q)$ acts on X_w and hence on its cohomology. Nowadays Deligne-Lusztig varieties play an important role in the representation theory of such groups. If we replace B with a parabolic subgroup P, we obtain the notion of generalized Deligne-Lusztig variety.

In the affine case, we fix a split reductive group G over a p-adic field (the *arithmetic case*), or over $k(\!(\epsilon)\!)$, where k, as before, is a finite field (the *function field case*). Let us assume for simplicity that we work in the function field case, and write $L = \overline{k}((\epsilon))$, $\mathfrak{o} = \overline{k}[[\epsilon]]$. Write $K = G(\mathfrak{o})$. Denote the Frobenius on L (acting on the coefficients) also by σ. The Cartan decomposition says that $G(L) = \coprod_{\mu \in X_*(A)_{\rm dom}} K\epsilon^\mu K$, where A is a fixed split maximal torus, and for a coweight μ, $\epsilon^\mu \in A(L) \subseteq G(L)$ is the image of $\epsilon \in L^\times = \mathbb{G}_m(L)$ under μ.

Definition. *Fix an element $b \in G(\overline{k}(\!(t)\!))$ and a dominant coweight $\mu \in X_*(A)_{\rm dom}$. The affine Deligne-Lusztig variety associated with b and μ is*

$$X_\mu(b) = \{g \in G(L)/K;\ g^{-1}b\sigma(g) \in K\epsilon^\mu K\}.$$

The affine Deligne-Lusztig variety $X_\mu(b)$ is a locally closed subset of the affine Grassmannian $G(L)/K$, an ind-scheme over $\overline{k}$, and hence carries a natural (reduced) scheme structure. It is locally of finite type over $\overline{k}$, but usually has infinitely many connected components. Again, there is an obvious variant, if we replace the maximal parahoric subgroup K with a general parahoric subgroup.

The affine case is more complicated than the previous one in several respects. For one thing, we have fixed in addition to the element w an element $b \in G(L)$. It is easy to see that $X_\mu(b)$ depends only on the σ-conjugacy class of b; since by Lang's theorem all elements of the group $G(\overline{k})$ are σ-conjugate, in the case of usual Deligne-Lusztig varieties this parameter does not matter. In the affine case, however, there are usually infinitely many different σ-conjugacy classes, and in particular $X_\mu(b)$ is empty for certain pairs (μ, b). In fact, the question which affine

Deligne-Lusztig varieties are non-empty, and which dimension the non-empty ones have, leads to difficult combinatorial questions about the Bruhat-Tits building of G and in the general case is insufficiently understood so far. Apart from these combinatorial difficulties, the geometric structure of affine Deligne-Lusztig varieties is more complicated than that of usual Deligne-Lusztig varieties. For instance, they are only locally of finite type and are not smooth in general.

Let us sketch the relation to Shimura varieties: The Newton strata in the special fiber are roughly (i. e. up to a finite morphism) the product of the central leaf (in the sense of Oort [13]) and a (truncated) Rapoport-Zink space $\mathcal{M}$, that is a certain moduli space of p-divisible groups. There is a natural bijection between the points of $\mathcal{M}$ and the corresponding affine Deligne-Lusztig variety. This establishes a connection to the work of Harris and Taylor [7], Fargues [4] and Mantovan [11], [12] on the local Langlands correspondence.

2. The hyperspecial case

For affine Deligne-Lusztig varieties for the hyperspecial parahoric subgroup K as above, Rapoport in [14] and Rapoport and Kottwitz [9] established criteria for the non-emptiness of affine Deligne-Lusztig varieties. Rapoport and Fontaine [5] discuss the relation to non-archimedean period domains. There is also a close relation to the converse of Mazur's inequality. The results of [9] concerning this converse have been generalized by Lucarelli [10] and by Wintenberger [19]; cf. also Kottwitz' article [8].

Theorem 1. *Assume that* $X_\mu(b) \neq \emptyset$. *Then* $\dim X_\mu(b) = \langle \rho, \mu - \nu \rangle - \frac{1}{2} \operatorname{def}(b)$.

Here, ν is the Newton vector of b, $\operatorname{def}(b)$ is the defect of b, i. e. the difference of the $k((\epsilon))$-ranks of G and of the σ-centralizer of b, and ρ denotes half the sum of the positive roots. This formula was conjectured in [15]. In [6], the proof of the theorem is reduced to the so-called superbasic case, which is proved in Viehmann's paper [18]. In some cases (when the affine Deligne-Lusztig varieties can be interpreted as moduli spaces of p-divisible groups), more information about the geometric structure has been obtained in [17]. Both papers of Viehmann use, and refine, techniques introduced by de Jong and Oort [1].

3. The Iwahori case

Now consider affine Deligne-Lusztig varieties for the standard Iwahori subgroup $I \subset K$. We have $G(L) = \coprod_{w \in \widetilde{W}} IwI$, where $\widetilde{W}$ is the extended affine Weyl group. We say that x is in the shrunken Weyl chambers, if $U_\alpha \cap xIx^{-1} \neq U_\alpha \cap I$ for all root subgroups U_α. Denote by $\eta_1 \colon \widetilde{W} \to W$ the projection to the finite Weyl group, by $\eta_2 \colon \widetilde{W} \to W$ the map sending x to the finite Weyl chamber it lies in, and by ℓ the length function on $\widetilde{W}$. There is ample computational evidence for the following conjecture; see [16], [6].

Conjecture 1. *Let b be basic, e. g. $b = 1$. Let $x \in \widetilde{W}$ lie in the shrunken Weyl chambers. Then $X_x(b)$ is non-empty if and only if $\eta_2(x)^{-1}\eta_1(x)\eta_2(x)$ is not contained in any proper parabolic subgroup of W. In this case,*

$$\dim X_x(b) = \frac{1}{2}\big(\ell(x) + \ell(\eta_2(x)^{-1}\eta_1(x)\eta_2(x)) - \mathrm{def}(b)\big)$$

By recent joint work of Haines, Kottwitz and myself, a conjectural criterion for non-emptiness for all $x \in \widetilde{W}$ can be given. Furthermore, we can prove one half of this criterion (namely that all affine Deligne-Lusztig varieties that are expected to be empty, are in fact empty), and correspondingly, one direction of the conjecture above.

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Participants

Prof. Dr. Kenichi Bannai
Graduate School of Mathematics
Nagoya University
Chikusa-Ku
Furo-cho
Nagoya 464-8602
JAPAN

Dipl.Math. Peter Barth
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Tobias Berger
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Laurent Berger
Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Dr. Werner Bley
FB 17 - Mathematik/Informatik -
Universität Kassel
Heinrich-Plett-Str. 40
34132 Kassel

Dr. David Blottiere
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn

Dr. Thanasis Bouganis
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Oliver Bültel
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. John H. Coates
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Pierre Colmez
Inst. de Mathematiques de Jussieu
Theorie des Nombres Case 247
Universite de Paris VI
4, Place Jussieu
F-75252 Paris

Dr. Daniel Delbourgo
School of Mathematical Sciences
University of Nottingham
University Park
GB-Nottingham NG7 2RD

Prof. Dr. Jean-Marc Fontaine
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Kazuhiro Fujiwara
Graduate School of Mathematics
Nagoya University
Chikusa-Ku
Furo-cho
Nagoya 464-8602
JAPAN

Prof. Dr. Takako Fukaya
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Priv.Doz. Dr. Ulrich Görtz
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Prof. Dr. Elmar Große-Klönne
Institut für Mathematik
Humboldt-Universität Berlin
Rudower Chaussee 25
12489 Berlin

Prof. Dr. Urs Hartl
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Annette Huber-Klawitter
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04109 Leipzig

Prof. Dr. Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Prof. Dr. Jennifer M. Johnson-Leung
Department of Mathematics
Brandeis University
Waltham , MA 02454-9110
USA

Moritz C. Kerz
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Guido Kings
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Mark Kisin
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Enno Nagel
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Volker Neumaier
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Wieslawa Niziol
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City , UT 84112-0090
USA

Dr. Tadashi Ochiai
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
99 Av. J.-B. Clement
F-93430 Villetaneuse

Dr. Sascha Orlik
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Prof. Dr. Damian Rössler
U.F.R. de Mathematiques
Case 7012
Universite de Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Takeshi Saito
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Alexander Schmidt
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Anthony J. Scholl
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Ehud de Shalit
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Romyar Sharifi
Department of Mathematics and
Statistics
McMaster University
1280 Main Street West
Hamilton , Ont. L8S 4K1
CANADA

Prof. Dr. Claus M. Sorensen
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Dr. Ramadorai Sujatha
Tata Institute of Fundamental
Research
School of Mathematics
Homi Bhabha Road, Colaba
400 005 Mumbai
INDIA

Georg Tamme
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Jeremy Teitelbaum
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 South Morgan
Chicago , IL 60607-7045
USA

Prof. Dr. Takeshi Tsuji
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Takahiro Tsushima
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Otmar Venjakob
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Annette Werner
Institut für Algebra und
Zahlentheorie
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart

Prof. Dr. Kay Wingberg
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Jean-Pierre Wintenberger
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Malte Witte
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Prof. Dr. Teruyoshi Yoshida
Dept. of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Prof. Dr. Sarah Zerbes
Department of Pure Mathematics
Imperial College London
South Kensington Campus
GB-London SW7 2AZ