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Mini-Workshop: Arithmetik von Gruppenringen

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Abstract. The theory of group rings has many connections to other mathematical areas, such as number theory, K-theory, topology, representation theory, homological algebra and of course the theory of finite and infinite groups, and ring theory. This mini workshop focussed on the arithmetics of integral group rings, particularly their multiplicative structure and unit groups. The central topics were the recent developments in connection with the first Zassenhaus conjecture, constructive descriptions of units and subgroups of units, cohomological and representation theoretical methods, a complete description of the units of integral group rings of some classes of finite groups, and interactions with semigroup rings and orders.

Mathematics Subject Classification (2000): 16Sxx, 20Cxx.

Introduction by the Organisers

The mini workshop "Arithmetic of group rings" was attended by 16 participants from Belgium, Brazil, Canada, Germany, Hungary, Israel, Italy, Romania and Spain. The expertise was a good mixture between senior and young researchers. It was a very stimulating experience and the size of the group allowed excellent discussions amongst all participants. Very fruitful were the problem sessions, resulting in the problems listed at the end of this report.

The main highlights of the conference were:

- The complete calculation of the projective Schur subgroup of the Brauer group by Aljadeff and del Rio.
- Hertweck's solution of the first Zassenhaus conjecture for finite metacyclic groups.

- the description of special subgroups of the unit group of integral group rings, such as the hypercentre and the finite conjugacy centre, and the relation with respect to the normalizer of the trivial units.
- discussion of the present state of art via several survey talks presented and the problem sessions.

The group G determines its integral group ring $\mathbb{Z}G$ and its group $V(\mathbb{Z}G)$ of normalized units. Several talks addressed the interplay of the cohomological properties of these three objects. Further topics included twisted group rings, group rings over local rings, polynomial growth and identities, orders and semigroup rings, Lie structure, representation-theoretic and algorithmic methods.

The most important open problems suggested in the conference are:

- Unit groups of integral group rings $\mathbb{Z}G$.
  (1) Construction of subgroups of finite index (see also Problem 14 and 18)
  (2) The construction of specific subgroups by units of a given type (Problems 7 and 28).
  (3) Specific properties of the unit group especially when G is infinite (see Problems 6, 15, 18, 27, 29, 30).
- Torsion part of the unit group
  (1) The description of torsion units and torsion subgroups, in particular with respect to integral group rings of finite non-soluble groups and of infinite groups (see Problems 4, 9, 10, 13, 19, 20).
  (2) The first Zassenhaus conjecture (see also Problems 21, 22) .
  (3) The modular isomorphism problem, i.e. the question whether a finite p - group is determined by $\mathbb{F}_pG$ up to isomorphism (see Problems 8, 31).

Workshop: Mini-Workshop: Arithmetik von Gruppenringen

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Abstracts

Elementary abelian subgroup induction with applications to infinite groups

Eli Aljadeff

Elementary abelian subgroup induction plays a crucial role in cohomology and representation theory of finite groups. Roughly speaking, the results say that important cohomological properties hold for a group ring $R\Gamma$, Γ finite and R an arbitrary ring (with 1), if and only if they hold for RE where E runs over all elementary abelian subgroups of Γ (for instance a theorem of Chouinard says that a module M over a modular group algebra $R\Gamma$ is projective if and only if M is projective over RE where E runs over all elementary abelian subgroups of Γ). In general, similar statements are false if one replaces the family of elementary abelian subgroups by cyclics. Our main objective is to apply this theory to representations and cohomology of infinite groups.

In order to explain the main idea we recall Serre's theorem (1969) on cohomological dimension. Serre's theorem says that if Γ is a torsion free group and H a subgroup of finite index with finite cohomological dimension (say n) then Γ has also finite cohomological dimension. Note that once one knows that the dimension of Γ is finite it is not difficult to show that in fact the dimension of Γ is also n.

In 1976 Moore made the following conjecture which is a far reaching generalization of Serre's theorem. Let Γ be a torsion free group and H a subgroup of finite index. Let M be an $R\Gamma$ module. If M is projective as an RH module then M is projective as an $R\Gamma$ module.

Moore's conjecture was proved (see [3]) for a large families of groups, namely groups which belong to Kropholler's hierarchy, with the extra condition that the module M is finitely generated. This family of groups contains f.g. solvable groups and f.g. linear groups. On the other hand this family does not contain the R. Thompson group.

As mentioned above the main point of this lecture is to show how to apply "elementary abelian subgroup induction" in the theory of infinite groups. Consider a group ring $R\Gamma$ where Γ is torsion free. Let H be a subgroup of finite index and assume M is an $R\Gamma$-module, projective as an RH module. By taking the intersection of the conjugates of H we can assume that H is normal in Γ. This allows us to consider the crossed product $(RH) * (\Gamma/H)$, where Γ/H is a finite group. The idea is to show first, that Chouinard's theorem holds for crossed products i.e one can induce from the elementary abelian subgroups of Γ/H (see [1]). Then by taking a projective limit we show that if the profinite completion of Γ is torsion free (and indeed this is the case for the Thompson group), Moore's conjecture holds for the group Γ. This idea can be applied to other induction problems, namely (with the above notation) inducing from a subgroup of finite index H to the group Γ (see [2]).

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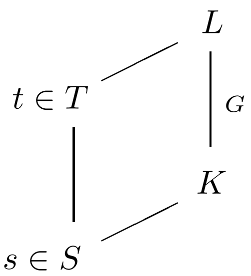
Remarks on twisted group rings

Matthias Künzer

(joint work with Harald Weber)

Let S be a discrete valuation ring of characteristic 0 with maximal ideal sS and $\operatorname{char} S/sS =: p \geq 3$. Let $T|S$ be a purely ramified extension of discrete valuation rings, let $tT \subseteq T$ be maximal, so that $T = S[t]$. Let $L = \operatorname{frac} T$, let $K := \operatorname{frac} S$, and assume $L|K$ to be galois with Galois group G.



Let $T \wr G := \{\sum_{g \in G} g x_g \,:\, x_g \in T\}$ be the twisted (aka skew) group ring, in which $gxg'x' = gg'x^{g'}x'$, where $g, g' \in G$ and $x, x' \in T$. Considering T as a $T \wr G$-module, we have an injective operation morphism $T \wr G \xrightarrow{\omega} \operatorname{End}_S T$. Writing matrices with respect to the S-linear basis $(t^0, \ldots, t^{|G|-1})$ of T, we even have

$$T \wr G \xrightarrow{\omega} \begin{pmatrix} S & \dots & \dots & S \\ sS & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ sS & \dots & sS & S \end{pmatrix} =: \Lambda \subseteq S^{|G| \times |G|} = \operatorname{End}_S T \,.$$

Now $(T \wr G)\omega = \Lambda$ if and only if $|G| \not\equiv_p 0$.

If $G = \mathcal{C}_p = \langle \rho \rangle$, $a := \mathrm{v}_t(t^\rho - t) \geq 2$ and $\mathrm{v}_s(p) \geq a - 1 + \lfloor \frac{a-1}{p} \rfloor$, then, writing

$$t' := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ s & & & 0 \end{pmatrix} \in \Lambda \,,$$

we obtain

$$(T \wr G)\omega = \{\lambda \in \Lambda \,:\, t'\lambda - \lambda t' \equiv_{t'^a \Lambda} 0 \,,\; t'^2\lambda - 2t'\lambda t' + \lambda t'^2 \equiv_{t'^{2a}\Lambda} 0 \,,\; \text{etc.}\} \,,$$

the congruences being derived from $t^\rho - t \equiv_{t^a} 0$, $(t^\rho - t)^2 \equiv_{t^{2a}} 0$, etc. As an application, we calculated the graded commutative ring $\operatorname{Ext}^*_{T \wr G}(T, T) \simeq \mathrm{H}^*(G, T; S)$ in this case.

We could not solve the case $G = \mathcal{C}_{p^2}$.

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On local group algebras

Marcos Soriano

(joint work with Martin Hertweck)

In this talk, we show how the isomorphism problem for small group rings can be solved [1], in the following sense: for a given finite p-group H, we determine precisely which central Frattini extensions of H give rise to isomorphic small group rings over the field with p elements.

Fix a finite p-group H, where p is an arbitrary rational prime. Let k denote the field with p elements. We consider central Frattini extensions with end term H, i.e., short exact sequences of finite p-groups

$$1 \to A \hookrightarrow G \to H \to 1\,,$$

where A is elementary abelian contained in the center of G and in the Frattini subgroup of G. Call the collection of all such sequences $\mathbb{F}(H)$. Note that any descending central series $E = E_1 \geq E_2 \geq \ldots \geq E_i \geq E_{i+1} \ldots$ of a finite p-group E with $E_2 = \Phi(E) = [E,E]E^p$, the Frattini subgroup of E, and elementary abelian sub-quotients E_i/E_{i+1}, allows us to describe E as being successively built from central Frattini extensions. Important examples of such central series are the lower p-central series and the Brauer-Jennings-Zassenhaus series.

Any central Frattini extension gives rise to a short exact sequence of algebras, the small group ring sequence (I_* denotes augmentation ideals):

$$0 \to \frac{I_A \cdot kG}{I_A \cdot I_G} \hookrightarrow \frac{kG}{I_A \cdot I_G} \to kH \to 0\,.$$

We might call two sequences $\mathbb{S}, \mathbb{S}'$ from $\mathbb{F}(H)$

- *isomorphic*, if there exists an isomorphism between the (group) short exact sequences $\mathbb{S}$ and $\mathbb{S}'$;
- *s-equivalent*, if there exists an isomorphism between the associated (algebra) short exact sequences of small group rings.

s-Equivalence is essentially isomorphism of small group rings, therefore, in order to solve the isomorphism problem for small group rings, one has to classify $\mathbb{F}(H)$ up to s-equivalence.

We show how to achieve this goal in terms of an action of the outer automorphism group $\mathrm{Out}(kH)$ on the kernel V of a certain 'universal' central Frattini extension of H. The restriction of the action to $\mathrm{Out}(H)$ yields precisely the isomorphism classes of extensions in $\mathbb{F}(H)$. An important technical tool for this classification is the concept of *obstruction spaces*, which has a cohomological origin and can be traced back at least to [2].

As a first application of these ideas, it is fairly easy to construct examples of central Frattini extensions with fixed end term and *non-isomorphic* middle groups giving rise to isomorphic small group rings. As a second application, one can give a new proof, within the context of small group rings, of a theorem showing that p-groups allowing a certain type of presentation are determined by their modular group algebras [3].

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Semigroup Rings

Eric Jespers

The aim of this paper is to present some of the recent work related to the unit group of integral semigroup rings. The emphasis is on the techniques used that often rely on the construction of concrete units and on a reduction to problems of group rings and linear groups over orders. Another important aspect is to state some open problems. Because semigroup theory is likely not so well known to the group and ring theorist, we also include the necessary back ground in semigroups and semigroup rings.

After an introduction to semigroup rings, we recall the definition of Bass cyclic units, bicyclic units and units. That these units determine a large part of the unit groups follows from the following result.

Theorem 1. [2, 3] *Let S be a finite semigroup. Assume that $\mathbb{Q}[S]$ does not have an homomorphic image that is a 2×2-matrix ring over either the rationals, an imaginary quadratic extension of the rationals or a non-commutative division ring. Further, assume that if G is a maximal subgroup of S so that $\mathbb{Q}[G]$ and $\mathbb{Q}[S]$ have an isomorphic non-commutative simple homomorphic image, then G does not have a non-abelian homomorphic image that is fixed-point free.*

Then the group generated by the following units is of finite index in $\mathcal{U}(\mathbb{Z}[S])$: the Bass cyclic units, the bicyclic units, and the units of the form $1 + x$, where x runs through a finite multiplicatively closed set of additive generators of the Jacobson radical $\mathcal{J}(\mathbb{Z}[S])$ of $\mathbb{Z}[S]$ (or of some additive subgroup of finite index).

Next we give a description of the unit group of some orders in some classical division algebras ([1]). In the following section we state some recent results on the normalizer (in the unit group) of the natural basis of an integral semigroup ring. It is shown that this group is strongly linked with the normalizer of the monomial matrices $Mon(\pm G)$ over a group G within $M_n(\mathbb{Z}G)$. For example the following can be shown.

Theorem 2. [4] *Let S be a finite Brandt semigroup and let M denote the union of all maximal subgroups of S, which are all isomorphic to a group G. If G is nilpotent, then $N(\pm M^0) = Mon(\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))(\pm G))$ and $N(\pm S) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}_0 S))Mon(\pm G)$.*

We end with mentioning extensive list of references, including many papers on the study of other special subgroups, such as the hypercentre and the finite conjugacy centre.

Some of the problems expanded on are the following:

(1) Let G and H be finite groups so that $M_n(\mathbb{Z}G) \cong M_m(\mathbb{Z}H)$. Does it follow that $\mathbb{Z}G \cong \mathbb{Z}H$? More general, determine invariants of the integral semigroup ring $\mathbb{Z}S$ of a finite semigroup S.
(2) Describe the unit group of $\mathbb{Z}Q_8 \times C_7$, where Q_8 is the quaternion group of order 8 and C_7 is the cyclic group of order 7.

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Group of units of integral group rings with good structure

Ángel del Río

All throughout G denotes a finite group and $\mathcal{U}(\mathbb{Z}G)$ is the group of units of the integral group ring $\mathbb{Z}G$. We consider the following approach to the study of $\mathcal{U}(\mathbb{Z}G)$. We fix a group theoretical condition and try to determine for which finite groups G, the condition is satisfied by $\mathcal{U}(\mathbb{Z}G)$ or by a subgroup of finite index of $\mathcal{U}(\mathbb{Z}G)$ (virtually satisfied).

It is clear that $\mathcal{U}(\mathbb{Z}G)$ is abelian if and only if G is abelian and, by a result of Higman, $\mathcal{U}(\mathbb{Z}G)$ is finite if and only if G is either abelian and its exponent divides either 4 or 6, or G is a Hamiltonian 2-group [2]. In the remaining cases, i.e. if $\mathcal{U}(\mathbb{Z}G)$ is neither abelian nor finite then $\mathcal{U}(\mathbb{Z}G)$ contains a non-abelian free group [1]. This indicates that to avoid trivial cases the condition fixed should not exclude the existence of non-abelian free groups.

Jespers [3] showed that there are only four finite groups for which $\mathcal{U}(\mathbb{Z}G)$ is virtually free non-abelian. Moreover this are precisely the groups for $\mathbb{Q}G$ is isomorphic to a direct product of fields, totally definite quaternion algebras over $\mathbb{Q}$ and <u>one</u> copy of $M_2(\mathbb{Q})$. This was extended by Jespers, Leal and del Río in a series of papers [4, 7, 6] where the finite groups G for which $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of free groups were classified. The finite groups satisfying this property are precisely those for which $\mathbb{Q}G$ is a direct product of fields, totally definite quaternion algebras (not necessarily over $\mathbb{Q}$) and (various) copies of $\mathbb{Q}G$.

The most general result known in this direction is due to Jespers, Leal, del Río, Ruiz and Zalesskii [5] who classified the finite groups G for which $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of free-by-free groups and characterized these groups in terms of the simple quotients of the rational group algebra. It turns out that these are precisely the groups of Kleinian type as defined in [9].

All the previous results were generalized in [8] for group of units over orders in number fields.

It is remarkable that one can characterize the groups included in the previous results in terms of virtual cohomological dimension as follows. Let $\mathbb{Q}G = \prod_{i=1}^{n} A_i$, with A_i a simple algebra. For each i, let R_i be an order in A_i and R_i^1 the group of reduced norm 1 elements of R_i. Then

- $\mathcal{U}(\mathbb{Z}G)$ is virtually abelian (i.e. abelian or finite) if and only if $\text{vcd}(R_i^1) = 0$ for each i.
- $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of free groups if and only if $\text{vcd}(R_i^1) \leq 1$ for each i.
- $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of free-by-free groups if and only if $\text{vcd}(R_i^1) \leq 2$ for each i.

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Zassenhaus conjecture for finite metacyclic groups

Martin Hertweck

There is a long-standing conjecture of Zassenhaus (see [5]) that runs as follows.

(ZC1) For a finite group G, every torsion unit in $\mathbb{Z}G$ is conjugate to an element of $\pm G$ by a unit of $\mathbb{Q}G$.

My contribution [4], as reported at the workshop, is as follows.

Theorem. *Suppose that $G = XA$ with $A \trianglelefteq G$, $X \leq G$ and A cyclic, X abelian. Then (ZC1) holds for G.*

Some basic tools and techniques for attacking (ZC1) for finite solvable groups were presented. Information about the partial augmentations of torsion units can be obtained from their interpretation as rational multiples of character values of the bimodules associated with the units, using character- and representation-theoretic methods. In recent investigations, a theorem of Al Weiss on p-permutation lattices [6] served as a starting point for proving rational conjugacy of certain units. A proof of Weiss' theorem [7] on the validity of the Zassenhaus conjectures for nilpotent groups was given in this context. One part of the above theorem arose out of an attempt to improve on one aspect, which is best formulated as a problem.

Problem (S. K. Sehgal)**.** Suppose G has a normal p-subgroup N, and that u is a torsion unit in $\mathbb{Z}G$ which maps to the identity under the map $\mathbb{Z}G \to \mathbb{Z}G/N$. Is it true that u is conjugate to an element of N by a unit of $\mathbb{Z}_pG$ (where $\mathbb{Z}_p$ stands for the p-adic integers)?

Rational conjugacy is guaranteed [2]. The answer is yes if G/N is abelian [2], or if G is as stated in the above theorem. Application of a result from [1] then allows to prove rational conjugacy to group elements for torsion units which map to the identity under the map $\mathbb{Z}G \to \mathbb{Z}G/A$. Requiring that A is covered by an abelian subgroup X, rather than assuming that G/A is abelian, allows to finally replace A by $\mathrm{C}_G(A)$. Handling of torsion units which do not map to the identity under the map $\mathbb{Z}G \to \mathbb{Z}G/A$ involves the Luthar–Passi method. (The Luthar–Passi method was extended in [3] in order to deal more efficiently with non-solvable groups.)

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Survey on free subgroup of units in integral group rings

Jairo Z. Gonçalves

Let $\mathbb{Z}G$ be the integral group ring of the group G over the ring of integers $\mathbb{Z}$, and let $U(\mathbb{Z}G)$ be its group of units. We make an historic retrospect from the early constructions of free subgroups of units in $U(\mathbb{Z}G)$, with Sehgal [1] and Hartley and Pickel [2], to the present with Gonçalves and Passman [4, 5], and Passman [6]. We also show how Tits' Criterion [3] was extended in different directions [5], allowing us to prove that powers of distinct types of units (e.g. bicyclic and Bass cyclic units) generate free subgroups in $U(\mathbb{Z}G)$ [4, 5, 6].

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On the unitary subgroups of group rings

Victor A. Bovdi

Let R be a commutative ring with unity. We denote by $U(R)$ the group of units of R. A (bijective) map $\diamond : R \to R$ is called an *involution* if for all $a, b \in R$ one has

$$(a+b)^\diamond = a^\diamond + b^\diamond, \qquad (ab)^\diamond = b^\diamond \cdot a^\diamond, \qquad a^{\diamond^2} = a.$$

Let KG be the group ring of a group G over the commutative ring K with unity, let σ be an antiautomorphism of order two of G and let $f : G \to U(K)$ be a homomorphism from G onto $U(K)$. For an element $x = \sum_{g\in G} \alpha_g g \in KG$ we define $x^\sigma = \sum_{g\in G} \alpha_g f(g)\sigma(g) \in KG$. Clearly $x \mapsto x^\sigma$ is an involution of KG if and only if

$$g\sigma(g) \in Kerf = \{h \in G \mid f(h) = 1\}$$

for all $g \in G$.

We assume that $x \mapsto x^\sigma$ is an involution of KG. The ring KG is called *σ-normal* if $xx^\sigma = x^\sigma x$ for each $x \in KG$. Note that the description of the "classical" normal group rings and twisted group rings were obtained in [1, 2, 3].

Let $U(KG)$ be the group of units of KG. An element $x \in U(KG)$ is called σ-unitary if $xx^\sigma = x^\sigma x = \varepsilon \in U(K)$. Denote by

$$U_\sigma(KG) = \{x \in U(KG) \mid xx^\sigma = x^\sigma x = \varepsilon \in U(K)\}$$

the σ-unitary subgroup of $U(KG)$.

In our talk we gave a description of the groups G, the rings K, the anti-automorphism σ of order two of the group G and the homomorphism $f : G \to U(K)$, such that one of the following conditions holds:

(i) KG is σ-normal (see [5]);
(ii) $U_\sigma(KG)$ is a normal subgroup of $U(KG)$ (see [4]);
(iii) each unit of $U(KG)$ is σ-unitary (i.e. $U_\sigma(KG) = U(KG)$, see [4]).

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Lie solvable group algebras and solvable unit group

Ernesto Spinelli

Let KG be the group algebra of a group G over a field K of characteristic p. In 1973 Passi, Passman and Sehgal proved that KG is Lie solvable if, and only if, either G is p-abelian or $p = 2$ and G contains a 2-abelian subgroup of index 2. Instead, for the modular case, the classification of when the unit group of KG, $\mathcal{U}(KG)$, is solvable was completed only recently by A. Bovdi (2005). All the results in this direction show a strong and interesting connection between the Lie structure of KG and the structure of its unit group. But, up to now, very little is known about the derived length $\mathrm{dl}(\mathcal{U}(KG))$ of $\mathcal{U}(KG)$ and the Lie derived length $dl_L(KG)$ of KG. Shalev (1992) provided a lower bound for $dl_L(KG)$, namely $\lceil \log_2(p+1) \rceil \le dl_L(KG)$, and two natural main questions arose from his paper:

(a) to classify KG such that $dl_L(KG) = \lceil \log_2(p+1) \rceil$;
(b) to establish, for p odd, if $dl_L(KG)$ is approximatively $\lceil \log_2(t(G')+1) \rceil$, where $t(G')$ is the nilpotency index of the augmentation ideal of KG'.

Whereas the second question remains *still open*, to the first one we have been able to give a complete answer in the following

Theorem 1. *Let KG be a non-commutative Lie solvable group algebra over a field K of characteristic $p > 0$. Let n be the positive integer such that $2^n \le p < 2^{n+1}$ and s, q (q odd) the non-negative integers such that $p - 1 = 2^s q$. Then $dl_L(KG) = \lceil \log_2(p+1) \rceil$ if, and only if, p and G satisfy one of the following conditions:*

(a) *$p = 2$, G' has exponent 2 and an order dividing 4 and G' is central;*
(b) *$p \ge 3$ and G' is central of order p;*
(c) *$5 \le p < 2^{n+2}/3$, G' has order p and $|G/C_G(G')| = 2^m$ with $m \le s$ a positive integer such that $p \le 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m} + 1, m)$,*

where, set $g(0,m) := 1$*, define recursively* $g(t,m) := g(t-1,m) \cdot 2^{m+1} + 1$*, and* $q_{n-m,m}$ *and* $\epsilon_{n-m,m}$ *are the quotient and the remainder of the Euclidean division of* $n-m-1$ *by* $m+1$*, respectively.*

The computation of $\mathrm{dl}(\mathcal{U}(KG))$ seems also more difficult. When G is finite and $p > 2$, Shalev (1991) proved that $\mathcal{U}(KG)$ is metabelian if, and only if, KG is Lie metabelian and M. Sahai (1996) showed that $\mathcal{U}(KG)$ is centrally metabelian if, and only if, KG is Lie centrally metabelian. Lately, by generalizing a result by Baginski (2002), we have showed that, if $p > 2$ and G is torsion and nilpotent with a cyclic p-group as commutator subgroup, then $\mathrm{dl}(\mathcal{U}(KG)) = dl_L(KG) = \lceil \log_2(|G'|+1) \rceil$. Finally, in the spirit of the first cited Shalev's paper, we have proved

Theorem 2. *Let* KG *be a non-commutative modular group algebra of a torsion nilpotent group* G *over a field* K *of characteristic* $p > 0$ *such that* $\mathcal{U}(KG)$ *is solvable. Then* $\mathrm{dl}(\mathcal{U}(KG)) \geq \lceil \log_2(p+1) \rceil$*. Moreover* $\mathrm{dl}(\mathcal{U}(KG)) = \lceil \log_2(p+1) \rceil$ *if, and only if,* $dl_L(KG) = \lceil \log_2(p+1) \rceil$*.*

Obviously, it would be interesting to know whether, for $p > 2$, the last Theorem holds when G is torsion, but non-nilpotent (it seems doable!) and when G is neither torsion nor nilpotent, which could be a more delicate task. But the main question (arising by all the results reviewed in the talk), which appears as a strong version of Problem n. 34 of *Topics in Group Rings* by Sehgal and Problem (b) by Shalev, is whether, for p odd, set $dl^L(KG)$ the strong Lie derived length of a Lie solvable group algebra KG,

$$dl_L(KG) = \mathrm{dl}(\mathcal{U}(KG)) = dl^L(KG).$$

Hartley's Conjecture and developments

Sudarshan K. Sehgal

Brian Hartley conjectured around 1980 that if the unit group $\mathcal{U}(KG)$ of the group algebra KG of a torsion group G over an infinite field K satisfies a group identity then KG satisfies a polynomial identity ($\mathcal{U} \in GI \Rightarrow KG \in PI$).

The conjecture was proved for semiprime case by Giambruno-Jespers-Valenti and completed by Giambruno-Sehgal-Valenti. The referee remarked thar the "conjecture is too weak". Consequently, a lot of activity issued. Passman proved that $\mathcal{U}(KG) \in GI$ if and only if $KG \in PI$ and G' is of bounded p-power exponent, when $\mathrm{char}K = p$, with the above conditions on G and K. Then more general groups and fields were considered. Liu and Passman gave a classification of torsion groups G so that $\mathcal{U}(KG) \in GI$ for arbitrary fields K. Then Giambruno-Sehgal-Valenti classified when $\mathcal{U}(KG) \in GI$ if K is infinite or G has an element of infinite order.

Then questions arose to consider smaller subsets of KG and certain special identities. Let us denote by

$$(KG)^+ = \{\gamma \in KG |\ \gamma^* = \gamma\}, \quad (KG)^- = \{\gamma \in KG |\ \gamma^* = -\gamma\},$$

where $*$ is the classical involution induced by $g \mapsto g^{-1}$, $g \in G$. These results are applications of the above classifications. A. Bovdi classified groups G so that $\mathcal{U}(KG)$ is solvable. Giambruno-Sehgal-Valenti classified groups G so that $\mathcal{U}^+(KG) \in GI$. Giambruno-Polcino-Sehgal classified when $(KG)^-$ is Lie nilpotent. The same was done for $(KG)^+$ by G. Lee. The groups with $\mathcal{U}^+(KG)$ solvable was handled by Lee-Sehgal-Spinelli. They also described when $(KG)^+$ is Lie solvable. More general involutions have been considered by Broche, Jespers, Dooms, Ruiz Marin and Polcino. They obtained many interesting results.

Group graded algebras and polynomial growth

Antonio Giambruno

Let F be a field of characteristic zero and let A be an F-algebra. In case F is algebraically closed (or simply contains a primitive nth root of one) there is a well understood duality between group gradings and group actions on A by finite abelian groups of order n. Through this duality it is possible to relate the G-graded polynomial identities of an algebra A with its G-identities, G a finite abelian group. As a consequence when F is algebraically closed, one can study the G-graded identities of A through the representation theory of the wreath product $G \wr S_n$, where S_n is the symmetric group on n symbols. In this setting one attaches to the algebra A a numerical sequence called the sequence of G-codimensions of A and, in case A satisfies an ordinary polynomial identity, such sequence is exponentially bounded ([4], [2]).

We give two different characterizations of the graded identities of A in case such sequence is polynomially bounded ([1]). A detailed study is done in case of superalgebras ($\mathbb{Z}_2$-gradings) ([3]).

For group algebras satisfying an ordinary polynomial identity, the ideal of identities of A coincides with that of $n \times n$ matrices for some n ([5]), and an interesting problem is that of studying the sequence of G-codimensions of a group algebra for some related finite grading group G.

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Normalizing units in group rings

Stanley O. Juriaans

Let G be a group. The normalizer of G in $\mathcal{U}_1(\mathbb{Z}G)$ is one of the subgroups that were studied by many researchers in the field in the past years. They are related to an important counter example due to M. Hertweck.

Let G be a group and let $\mathbb{Z}G$ be its integral group ring. We say that a unit is special if it is either a normalizing unit, a unit with a finite number of conjugates in $\mathcal{U}_1(\mathbb{Z}G)$ or belongs to the hypercenter of $\mathcal{U}_1(\mathbb{Z}G)$. Our approach to get a good understanding of such units is to study their support. We have the following.

Lemma 1. *Let G be a group, $u = \sum_{g\in G} u_g g \in \mathbb{Z}G$ be a special unit and let supp$(u) = \{g \in G \mid u_g \neq 0\}$ be its support. Then $\langle$supp$(u)\rangle$ is a polycyclic-by-finite group.*

From this we obtain:

Theorem 2. *Let G be a group and $u \in \mathbb{Z}G$ a special unit. Then there exists, $g \in G$ and $N \lhd G$ a finite normal subgroup, such that $g^{-1}u \in \mathbb{Z}N$.*

This result is the basis to study special units for other than finite groups. Most results known for finite groups can be generalized to arbitrary groups.

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Symmetric and skew-symmetric elements in group rings and its relation with some special units

Eric Jespers
(joint work with Manuel Ruiz Marín)

Let R be a commutative ring with identity, G is a group and φ is an involution on RG. The set of φ-symmetric elements in RG is

$$(RG)^+_\varphi = \{\alpha \in RG|\ \varphi(\alpha) = \alpha\}$$

and set of φ-antisymmetric elements in RG is

$$(RG)^-_\varphi = \{\alpha \in RG|\ \varphi(\alpha) = -\alpha\}.$$

Let

$$\mathcal{U}^+_\varphi(RG) = \{\alpha \in \mathcal{U}(RG)|\ \varphi(\alpha) = \alpha\}$$

be the set of symmetric units of RG and

$$\mathcal{U}_\varphi(RG) = \{\alpha \in \mathcal{U}(RG)|\ \varphi(\alpha)\alpha = \alpha\varphi(\alpha) = 1\}$$

the group of φ-unitary units.

Denote by $G_\varphi = \{g \in G | \varphi(g) = g\}$ and by $R_2 = \{r \in R | 2r = 0\}$ the set of elements in R of additive order 2. Then $(RG)^+_\varphi$ and $(RG)^-_\varphi$ are generated as an R-module by

$$\mathcal{S} = \{g + \varphi(g) | g \in G \setminus G_\varphi\} \cup G_\varphi$$

and

$$\mathcal{K} = \{g - \varphi(g) | g \in G\} \cup \{rg | g \in G_\varphi \text{ and } r \in R_2\}$$

respectively.

In this talk we gather some results concerning the commutativity of the sets $(RG)^+_\varphi$ and $(RG)^-_\varphi$ and how this affect the structure of the whole group ring RG. Also, we give a characterization of when the set of φ-symmetric units $\mathcal{U}^+_\varphi(RG)$ is a group and when the group of φ-unitary units $\mathcal{U}_\varphi(RG)$ satisfies a group identity. We consider several kinds of involutions, not only the classical involution.

The Brauer group of a field and some of its subgroups

Eli Aljadeff and Ángel del Río

The Brauer-Witt theorem describes the Schur group of a field k by means of Galois cohomology (see definition below). Recall that a Schur algebra over k is a k-central simple algebra which appears as a homomorphic image of a group algebra of the form kH where H is a finite group. Equivalently, a k-central simple algebra A is Schur over k if it is spanned by a finite group of units. A well known result of Brauer says that every group algebra kH and hence every Schur algebra over k is split by a cyclotomic extension of k. Let $S(k)$ be the subgroup of $Br(k)$ generated by (and in fact consisting of) Brauer classes that may be represented by Schur algebras over k. This is the Schur group of the field k. It follows from Brauer's result which was mentioned above that $S(k)$ is a subgroup (in a natural way) of the relative Brauer $H^2(G(k_{cyc}/k), k^*_{cyc})$. The Brauer-Witt theorem says that the elements of $S(k)$ "correspond" precisely to the cohomology classes in $H^2(G(k_{cyc}/k), k^*_{cyc})$ that have a representing 2-cocycle which takes finite values (i.e. roots of unity).

In 1978 Lorenz and Opolka introduced a projective analogue to the construction above [5]. They consider projective Schur algebras over k. These are k-central simple algebras which are spanned by a group of unit elements Γ which is finite modulo its centre. The projective Schur group of k (denoted by $PS(k)$) is the subgroup of $Br(k)$ generated by (and again, consisting of) Brauer classes that may be represented by a projective Schur algebra. One of the main reasons for making this generalization is that every symbol algebra is a projective Schur algebra in a natural way and also, as observed by Lorenz and Opolka, every element in $Br(k)$ where k is a global field may be represented by a projective Schur algebra. It follows (using a fundamental theorem of Merkurjev-Suslin and Lorenz and Opolka observation) that $PS(k) = Br(k)$ whenever k contains all roots of unity or alternatively k is a global field. Based on these examples it was conjectured that $PS(k) = Br(k)$ in general. But this turned out to be false as shown in [2]. The goal was then to describe $PS(k)$ by means of Galois cohomology. As in the Schur case, also here there is a natural way to construct projective Schur algebras over k. Recall that a radical (radical abelian) algebra over k is a crossed product $(K/k, G(K/k), \alpha)$ where K is a radical (radical abelian) extension (i.e. K is obtained from k by joining roots of elements of k) and the cocycle α gets values in K^* which are of finite order modulo k^*. It is easy to see that a radical algebra is spanned by a group of units which is finite modulo its center and hence is a projective Schur algebra in a natural way. The analogue of the Brauer-Witt conjecture in this case says that every element in $PS(k)$ may be represented by a radical and even radical abelian algebra. The conjecture was proved in [3] for fields of positive characteristics and in [4] for certain fields of characteristic zero namely fields with an henselian valuation, with a residue field k_0 which is a global field of characteristic zero. In a joint work of Aljadeff and del Río we proved the conjecture for arbitrary fields of characteristic zero [1]. In the two lectures (delivered by Aljadeff and del Río) we plan to give an overview of the problem and exhibit the main two ideas that led us to a complete proof of the conjecture.

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Wedderburn decomposition of group algebras and Schur groups

Gabriela Olteanu

We propose an explicit and effective computation way for the computation of the Wedderburn decomposition of group algebras of finite groups over fields of characteristic zero, relied mainly on an algorithmic proof of the Brauer–Witt Theorem [5]. The Brauer–Witt Theorem shows that questions on the Schur subgroup are reduced to a treatment of cyclotomic algebras. For G is a finite group, χ a complex irreducible character of G and F a field of characteristic zero, the Brauer–Witt Theorem states that the simple component of the group algebra FG associated to the character χ is Brauer equivalent to a cyclotomic algebra over $F(\chi)$. We look for a constructive approach of the theorem, in order to obtain a precise and constructive description of the cyclotomic algebras that appear in the theorem. The explicit knowledge of the Wedderburn decomposition has applications to different problems such as the study of the groups of units of group rings with coefficients of arithmetic type and of the Schur groups of abelian number fields.

The knowledge of the (local) Schur index of a Schur algebra provides in many cases useful information to be added at the description of the Wedderburn components of a semisimple group algebra. We study the Schur group of an abelian number field K and we compute the maximum local index of a Schur algebra over such fields K, calculating $p^{\beta_p(r)} = \max\{m_r(A) : [A] \in S(K)_p\}$, for every prime p with $\zeta_p \in K$ and r a rational prime number [2].

The classes of the Brauer group of a field K that contain cyclic cyclotomic algebras generate a subgroup $CC(K)$ of the Schur group $S(K)$. A cyclic cyclotomic algebra is a cyclic algebra $(K(\zeta)/K, \sigma, \xi)$, with ζ and ξ roots of unity, that is an algebra that has at the same time a representation as cyclic algebra and as cyclotomic algebra. These algebras combine properties of both cyclic and cyclotomic algebras and have the advantage of having a form that allows one to apply specific methods for both types of algebras. Every element of the Schur subgroup is represented, on one hand by a cyclotomic algebra (by the Brauer–Witt Theorem) and, on the other hand by a cyclic algebra (by a classical result from Class Field Theory given by the Brauer-Hasse-Noether-Albert Theorem). In general $CC(K) \neq S(K)$ and we try to estimate the gap between $CC(K)$ and $S(K)$ [1]. More precisely, we characterize when $CC(K)$ has finite index in the Schur group $S(K)$ in terms of the relative position of K in the lattice of cyclotomic extensions of the rationals.

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The normalizer of a finite unimodular group

Gabriele Nebe

This talk reports on the algorithm designed by Jürgen Opgenorth [2] to calculate the normalizer $N_{\mathrm{GL_n}(\mathbb{Z})}(G)$ of a finite subgroup $G \leq \mathrm{GL_n}(\mathbb{Z})$. This may be used to calculate the unit group of an $\mathbb{Z}$-order generated by a finite group, $\Lambda := \langle G \rangle_{\mathbb{Z}}$. If $n := \dim_{\mathbb{Z}}(\Lambda)$ then the regular representation Δ of Λ yields a finite subgroup $\Delta(G) \leq \mathrm{GL_n}(\mathbb{Z})$ such that

$$\Lambda^* = \mathrm{End}_\Lambda(\Lambda)^* = \mathrm{C}_{\mathrm{GL_n}(\mathbb{Z})}(\Delta(\mathrm{G}))$$

where the centralizer is of finite index in the normalizer. The program is available in CARAT [1]. The algorithm uses

$$\mathcal{F}(G) := \{F \in \mathbb{R}^{n\times n}_{sym} \mid gFg^{tr} = F \text{ for all } g \in G\}$$

the space of invariant forms of the finite subgroup $G \leq \mathrm{GL_n}(\mathbb{Z})$. The normalizer N of the Bravaisgroup of G (the stabiliser of all $F \in \mathcal{F}(G)$), which contains $N_{\mathrm{GL_n}(\mathbb{Z})}(G)$ of finite index, acts on the space $\mathcal{F}(G)$. It hence acts on the infinite graph Γ of G-perfect forms in $\mathcal{F}(G)$, where incidence is given by the Voronoi neighbor relation. Moreover Γ/N is finite and can be calculated efficiently as long as $\dim(\mathcal{F}(G))$ is not too big. This yields a finite generating set for N.

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Algorithms for definite quaternion algebras

Markus Kirschmer

Let $\mathcal{Q}$ be a definite quaternion algebra over a totally real number field K. Further let $\mathcal{M}$ be a maximal (or Eichler) $\mathbb{Z}_K$-order in $\mathcal{Q}$.

Definition: A $\mathbb{Z}_K$-lattice I is a finitely generated $\mathbb{Z}_K$-module in $\mathcal{Q}$ such that $KI = \mathcal{Q}$. Let $\mathcal{O}_r(I) := \{x \in \mathcal{Q} \mid Ix \subseteq I\}$ be the right order of I. Then I is called a right $\mathcal{M}$-ideal if $\mathcal{O}_r(I) = \mathcal{M}$. Similarly, one defines left and twosided $\mathcal{M}$-ideals.

We will describe algorithms to answer the following questions.

(1) Decide whether two right $\mathcal{M}$-ideals I and J are isomorphic (i.e. find $x \in \mathcal{Q}$ such that $xJ = I$).
(2) Decide whether two $\mathbb{Z}_K$-orders $\mathcal{O}$ and $\mathcal{O}'$ in $\mathcal{Q}$ are conjugate (i.e. find $x \in \mathcal{Q}$ such that $x\mathcal{O}x^{-1} = \mathcal{O}'$).
(3) Construct the class group of isomorphism classes of twosided $\mathcal{M}$-ideals.
(4) Compute representatives for all conjugacy classes of maximal orders in $\mathcal{Q}$.
(5) Find all right $\mathcal{M}$-ideal isomorphism classes.

The main tool will be the totally positive definite bilinear form

$$b\colon \mathcal{Q} \times \mathcal{Q} \to K,\ (x, y) \mapsto \mathrm{tr}(x\bar{y})\ .$$

For the first problem, we may assume $J = \mathcal{M}$. Hence it suffices to find $x \in I$ such that $\mathrm{nr}(x)\mathbb{Z}_K = \mathrm{nr}(I)$. Such an element is found by enumerating all elements in the $\mathbb{Z}$-lattice $\mathcal{M}$ of a certain length with respect to the positive definite form $\mathrm{Tr}_{K/\mathbb{Q}} \circ b$.

From [1] we know that

$$\begin{aligned}\mathrm{SO}(\mathcal{Q}, b) &:= \{\varphi \in \mathrm{End}_K(\mathcal{Q}) \mid \mathrm{nr}(x) = \mathrm{nr}(\varphi(x)) \text{ for all } x \in \mathcal{Q} \text{ and } \det(\varphi) = 1\} \\ &= \{y \mapsto edyd^{-1} \mid e, d \in \mathcal{Q}^*,\ \mathrm{nr}(e) = 1\}\ .\end{aligned}$$

Hence, the second problem asks for a $\mathbb{Z}_K$-isometry between the lattices $(\mathcal{O}, b)$ and $(\mathcal{O}', b)$ as addressed in [2].

The last two questions are equivalent. More precisely, if $\{\mathcal{M}_i \mid 1 \leq i \leq T\}$ represent the conjugacy classes of maximal orders in $\mathcal{Q}$ and $\{I_{i,j} \mid 1 \leq j \leq H_i\}$ represent the isomorphism classes of twosided $\mathcal{M}_i$-ideals, then $\{I_{i,j} \cdot (\mathcal{M}_i \cdot \mathcal{M}) \mid 1 \leq j \leq H_i,\ 1 \leq i \leq T\}$ represent all right $\mathcal{M}$-ideal classes.

Finally, the maximal orders $\mathcal{M}_i$ are found as left orders of right $\mathcal{M}_j$-ideals that are maximal amongst those contained in $\mathcal{M}_j$; where $\mathcal{M}_j$ is a maximal order that has already been obtained. The termination condition of the algorithm is given by Eichler's mass formula [1].

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Torsion units in integral group rings of finite insoluble groups

Wolfgang Kimmerle

The talk reports on results which have been obtained in the last years for integral group rings of non-soluble classes of finite groups. Further progress in this direction is the key for getting general results on torsion units.

1.General finite groups. The following answers a question of Marciniak (ICM Satellite Conference, Granada 2006).
Proposition. Let G be a finite group. Suppose that the Kleinian fourgroup $C_2 \times C_2$ is a subgroup of the normalized unit group $V(\mathbb{Z}G)$. Then G has a subgroup isomorphic to $C_2 \times C_2$.
Proof. Let G be a counterexample of minimal order. Then a Sylow 2 - subgroup of G has precisely one involution. By [1, Corollary2.8] we may assume that G has no normal subgroup of odd order. From well known results of Brauer and Suzuki it follows that G has precisely one involution. Now classical results of Cohn-Livingstone and Berman show that $V(\mathbb{Z}G)$ contains only one involution.
With different methods Hertweck has shown that the analogous result holds for subgroups of the type $C_p \times C_p, p$ an odd prime [2].

2.Finite insoluble groups. A survey on the known results on the Zassenhaus conjectures with respect to insoluble finite groups shows that in particular for quasi-simple groups very little is known. This motivates to consider weaker versions of these conjectures. The question whether the Gruenberg -Kegel graph (also called prime graph) of $V(\mathbb{Z}G)$ and G coincide [4] initiated within the last two years considerations of finite simple groups by many authors (V.Bovdi, Jespers, Konovalov, Linton, Siciliano). In the mean time this question has an affirmative answer for 12 of the sporadic simple groups and for the simple groups $PSL(2,p), p \geq 5$ a prime [3]. These results follow by applying the Luthar-Passi method. This method - including an extension due to Hertweck [3] - is shortly presented. Character - theoretical and arithmetical properties of finite simple groups seem at present the only available tools to deal with torsion units in integral group rings of finite quasi-simple groups.

3.Soluble extensions, Sylow subgroups. The situation is completely different for soluble extensions of nice groups. E.g. in the case when G is a Frobenius group it can even be shown that torsion units of $V(\mathbb{Z}G)$ have the same order as elements of G.
It follows from the results mentioned in the first part that under certain restrictions on Sylow p - subgroups of G each finite p - subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G. This is the case for arbitrary p when G has cyclic Sylow p - subgroups or for $p = 2$ and G has quaternion Sylow 2 - subgroups. Similar results are discussed for specific abelian Sylow 2 - subgroups.

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PROBLEMS

1. Let G be a finite group and let R be a ring with identity. Assume G acts on R by automorphisms. Let $R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$. Put

$$\mathrm{tr}_G \colon R \to R^G, \quad x \mapsto \sum_{\sigma \in G} \sigma(x).$$

It is easily seen that tr_G is surjective if and only if $1 \in \mathrm{Im}(\mathrm{tr}_G)$. Furthermore, if tr_G is surjective, then $\mathrm{tr}_H : R \to R^H$ is surjective for any subgroup H of G. A result of Aljadeff and Ginosard (based on Chouinard's Theorem) states that if tr_E is surjective onto R^E for all elementary abelian p-subgroups of G (p any prime), then tr_G is surjective onto R^G. Under these assumptions, let $r_E \in R$ so that $\mathrm{tr}_E(r_E) = 1$. By the previous, there exists an $r_G \in R$ so that $\mathrm{tr}_G(r_G) = 1$. By an observation of Shelah, it is known that there exists a formula (a polynomial in variables $g(r_E)$, $g \in G$ and E elementary abelian) to calculate r_G. Find such a formula. There are formulas for abelian groups G. Also for some non-abelian groups a formula is known (e.g., dihedral and quaternion groups). In general, it is enough to deal with p-groups, and one can even restrict to extra special and almost extra special groups G. Recall that if G is such group then it contains a normal cyclic subgroup C of order p such that G/C is an elementary abelian p-group.

E. Aljadeff

2. A ring R is said to have IBN (the invariant basis number property) if $R^n \cong R^m$ (an isomorphism of R-modules) implies that $n = m$. It is known that if a ring R has a ring epimorphic image (mapping the identity to the identity) that has IBN then so has R. Hence, for example, a group algebra KG of a group G over a field K has IBN. Let $t : G \to \mathrm{Aut}(K)$ be a group automorphism of a group G into the automorphism group $\mathrm{Aut}(K)$ of a field K. Does the skew group algebra K_tG has IBN? In case G has subexponential growth the answer is positive.

E. Aljadeff

3. Let G be a torsion free nilpotent group of finite cohomological dimension, say $\mathrm{cd}(G) = n$. Then $\mathrm{gl.dim}(\mathbb{C}G) = n$ and, for $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$, $1 \le \mathrm{gl.dim}(\mathbb{C}^\alpha G) \le n$. Also, if H is a subgroup of G with $\mathrm{cd}(H) = r$ and $\alpha_{|H} = 1$, then $\mathbb{C}H \subseteq \mathbb{C}^\alpha G$ and thus $\mathrm{gl.dim}(\mathbb{C}^\alpha G) \ge r$.

Question/Conjecture. If $\mathrm{gl.dim}(\mathbb{C}^\alpha G) = r$ then there exists a subgroup H of G so that $\alpha_{|H} = 1$ and $\mathrm{gl.dim}(\mathbb{C}H) = r$. The case $r = n$ has been answered positively.

E. Aljadeff

4. Let $\mathrm{t}(G)$ be the set of elements of finite order in a group G. Let $\mathrm{V_t}(\mathbb{Z}G)$ be the set of torsion units in the normalized unit group $\mathrm{V}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$. For an element $g \in \mathrm{t}(G)$, of order n, define $\widehat{g} = \sum_{i=0}^{n-1} g^i$, and let $\mathrm{B}_g = \mathrm{LB}_g \cup \mathrm{RB}_g$, where $\mathrm{LB}_g = \{g + (g-1)w\widehat{g} \mid g \in \mathrm{t}(G), w \in \mathbb{Z}G\}$ and $\mathrm{RB}_g = \{g + \widehat{g}w(g-1) \mid g \in \mathrm{t}(G), w \in \mathbb{Z}G\}$. Clearly, $\mathrm{B}_g \subseteq \mathrm{V_t}(\mathbb{Z}G)$. Furthermore, set $\mathrm{BV}(\mathbb{Z}G) = \{x^{-1}yx \mid x \in \mathrm{B}_g \text{and } y \in \mathrm{B}_h, \text{for } g, h \in \mathrm{t}(G)\} \subseteq \mathrm{V_t}(\mathbb{Z}G)$. For which groups G does $\mathrm{BV}(\mathbb{Z}G) = \mathrm{V_t}(\mathbb{Z}G)$ hold?

For $w_0 = a \in \mathrm{t}(G)$ and $b \in G$ define inductively $w_{i+1} = w_i + (w_i - 1)b\widehat{w_i}$, and set $S(a, b) = \{w_i \mid i = 0, 1, \ldots\}$. What can be said about the set $S(a, b)$? For example, what is the structure of the group generated by $S(a, b)$?

Comment. Each w_i is of the same order as a. If G is a finite group, then each w_i is conjugate to a by a unit in $\mathbb{Q}G$, since $\chi((a^j - 1)b\hat{a}) = 0$ for each irreducible character χ of G.

V. Bovdi

5. Let A be a finite dimensional K-algebra, K being a field, and let $\mathrm{J}(A)$ be its radical. A vector space basis B of A is said to be *filtered multiplicative* provided that $b_1, b_2 \in B$ implies $b_1b_2 = 0$ or $b_1b_2 \in B$, and $B \cap \mathrm{J}(A)$ is a basis of $\mathrm{J}(A)$. R. Bautista, P. Gabriel, A. V. Roĭter and L. Salmerón (*Invent. Math.* **81** (1985), no. 2, 217–285) proved that if K is algebraically closed and A is of finite representation type, then A has a filtered multiplicative basis.

Suppose G is a finite non-abelian p-group, and $\mathrm{char}(K) = p$. For $p > 2$, is it true that KG does not have a filtered multiplicative basis?

V. Bovdi

6. A discrete group G is *amenable* if it has a finitely-additive left-invariant probability measure. Let G be a group and $\mathrm{V}(\mathbb{Z}G)$ the group of augmentation one units of the integral group ring $\mathbb{Z}G$. Suppose x and y are nontrivial elements in $\mathrm{V}(\mathbb{Z}G)$. When is the group $\langle x, y\rangle$ non-amenable? Note that if a group contains a free (non-abelian) subgroup on two generators then it is non-amenable.

V. Bovdi

7. Let G be a finite group, and x in G an element of order n. For $i > 0$ with $(i, n) = 1$, and a multiple m of $\varphi(n)$, with φ denoting Euler's function, the element

$$u_{i,m}(x) = (1 + x + \cdots + x^{i-1})^m + \frac{1 - i^m}{n}(1 + x + \cdots + x^{n-1})$$

of $\mathbb{Z}G$ is a unit, called a *Bass cyclic unit*. Suppose that $n \notin \{1, 2, 3, 4, 6\}$, and that there is $y \in G$ with $xy \neq yx$. Do their exist natural numbers k, m, r and s so that $\langle u_{k,m}(x), u_{r,s}(yxy^{-1})\rangle \cong \langle u_{k,m}(x)\rangle * \langle u_{r,s}(yxy^{-1})\rangle$, a free product in the unit group of the integral group ring $\mathbb{Z}G$?

J. Gonçalves

8. Let G be a finite p-group and let k be the field with p elements. Suppose that G has a normal complement in the unit group $\mathrm{U}(kG)$ of the group algebra kG. (A *normal complement* for G in $\mathrm{U}(kG)$ is a normal subgroup N of $\mathrm{U}(kG)$ so that $\mathrm{U}(kG) = G \ltimes N$.) Let H be a group such that the k-algebras kG and kH are isomorphic. Are the groups G and H are isomorphic? The proof given by F. Röhl (*Proc. Amer. Math. Soc.* **111** (1991), no. 3, 611–618) contains a mistake. The same question with G of nilpotency class 2 (such a G has a normal complement, for then G is a circle group (R. Sandling, *Math. Z.* **140** (1974), 195–202).

M. Hertweck

9. Let G be a finite group. For an element $\sum_{g\in G} a_g g$ (all a_g in $\mathbb{Z}$) of the integral group ring $\mathbb{Z}G$, and any $x \in G$, set $\varepsilon_x(u) = \sum_{g\in C(x)} a_g$, where $C(x)$ denotes the G-conjugacy class of x. The $\varepsilon_x(u)$ are called the *partial augmentations* of u. The *augmentation* of u is the sum of all coefficients a_g.

Now let $G = \mathrm{GL}_n(k)$, the general linear group of degree n over a finite field k, and let $\mathrm{M}_n(k)$ be the $n \times n$ matrix ring over k. Let $\mathrm{V}(\mathbb{Z}G)$ denote the group of augmentation one units in the integral group ring $\mathbb{Z}G$. The embedding $G \subseteq \mathrm{M}_n(k)$ yields a ring homomorphism $\mathbb{Z}G \to \mathrm{M}_n(k)$ and thus we obtain a surjective group homomorphism $\pi\colon \mathrm{V}(\mathbb{Z}G) \to G$. For a torsion unit u in $\mathrm{V}(\mathbb{Z}G)$, is it true that $\varepsilon_{\pi(u)}(u)$ is nonzero?

M. Hertweck

10. Let G be a finite group, and ζ a primitive $|G|$-th complex root of unity. For each prime divisor p of $|G|$ and each central primitive idempotent e in the group ring $\mathbb{Z}_{(p)}[\zeta]G$ (here $\mathbb{Z}_{(p)}$ denotes localisation of $\mathbb{Z}$ at p), let $g_{p,e}$ be an element of G. Set $u_p = \sum_e e g_{p,e}$ (where the sum runs over all the primitive central idempotents of $\mathbb{Z}_{(p)}[\zeta]G$). Each u_p is a torsion unit in $\mathbb{Z}_{(p)}[\zeta]G$. Suppose that u_p and u_q are conjugate by a unit of $\mathbb{C}G$, for all p, q. Is some u_p (and hence all) conjugate to an element of G by a unit of $\mathbb{C}G$?

M. Hertweck

11. (Well-known problem). Let G and H be finite groups. Assume that $\mathrm{Z}(\mathbb{Z}H) \cong \mathrm{Z}(\mathbb{Z}G)$, an isomorphism of centers of integral group rings. Does this imply that G and H have equivalent character tables? In other words, under the isomorphism, are class sums mapped to class sums? For finite nilpotent groups, the answer is yes (M. Hertweck, *to appear in Proc. Amer. Math. Soc.*).

M. Hertweck

12. Denote by $\mathrm{M}_n(R)$ the $n \times n$ matrix ring over a ring R. Let G and H be finite groups. If $\mathrm{M}_n(\mathbb{Z}G)$ and $\mathrm{M}_n(\mathbb{Z}H)$ are isomorphic rings, does it follow that $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic rings?

E. Jespers

13. Let $G = \langle x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2}\rangle$. The group G is free abelian-by-finite, torsion free and supersolvable, see (Chapter 13 in D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, New York, 1977). Also, G is not a unique product group (S. D. Promislow, *Bull. London Math. Soc.* **20** (1988), no. 4, 302–304).

Let K be a field. For any proper subgroup H of G, the unit group $\mathrm{U}(KH)$ is trivial, that is, the units of KH are of the from λh, with $h \in H$ and $0 \neq \lambda \in K$. Is it true that $\mathrm{U}(KG)$ is trivial? (The old question whether KG can have nontrivial units if G is torsion free is due to Kaplansky.)

E. Jespers

14. Let Q_8 be the quaternion group of order 8 and C_7 the cyclic group of order 7. Find finitely many units in the integral group ring $\mathbb{Z}[Q_8 \times C_7]$ that generate a subgroup of finite index in the unit group of $\mathbb{Z}[Q_8 \times C_7]$.

The classical quaternion algebra $\mathrm{H}(\mathbb{Q}(\xi_7))$ (where ξ_7 is a primitive 7-th root of unity) is the only noncommutative simple epimorphic image of the rational group algebra $\mathbb{Q}[Q_8 \times C_7]$. The problem hence can be reduced to constructing finitely many units in $\mathrm{H}(\mathbb{Z}[\xi_7])$ that generate a subgroup of finite index in the unit group of $\mathrm{H}(\mathbb{Z}[\xi_7])$. Cf. (C. Corrales, E. Jespers, G. Leal and A. del Río, *Adv. Math.* **186** (2004), no. 2, 498–524).

E. Jespers

15. The *normalizer problem* (NP) for a group G asks whether the normalizer of G in the unit group of $\mathbb{Z}G$ consists of the "obvious" units only, that is, whether the normalizer is generated by G and the group of central units in $\mathbb{Z}G$. If this is true, we say that (NP) holds for G.

Let G be a group so that (NP) holds for all of its finite normal subgroups. Does (NP) holds for G? (this might be already implicit in Hertweck's papers). Is there a satisfactory classification of the groups for which (NP) holds?

S. O. Juriaans

16. The *finite conjugacy center* of a group G, denoted by $\Delta(G)$, is the subgroup of G consisting of the elements that have only finitely many conjugates.

Classify the groups G for which $\Delta(\mathrm{U}(\mathbb{Z}G))$ is non-central in $\mathrm{U}(\mathbb{Z}G)$.

Let A be a finite dimensional algebra over an algebraic number field and let Γ be a $\mathbb{Z}$-order in A. Suppose that the finite conjugacy centre $\Delta(\mathrm{U}(\Gamma))$ of the unit group $\mathrm{U}(\Gamma)$ of Γ is non-central in Γ. What can be said about A? Determine the Wedderburn–Malcev decomposition of A.

S. O. Juriaans

17. The *hypercenter* of a group G, denoted by $\mathrm{Z}_\infty(G)$, is the union of the terms of the upper central series of G.

For a group G, set $\mathcal{U} = \mathrm{U}(\mathbb{Z}G)$. Is $\mathrm{Z}_\infty(\mathcal{U}) \leq \Delta(\mathcal{U})$? Is $\Delta(\mathcal{U}) \leq \mathrm{N}_{\mathcal{U}}(G)$?

Let A be a finite dimensional simple $\mathbb{Q}$-algebra and Γ a $\mathbb{Z}$-order in A. Determine the finite conjugacy center of the unit group $\mathrm{U}(\Gamma)$ of Γ. The same problem for a Wedderburn component A of $\mathbb{Q}G$, where G is the counter-example to (NP) given by M. Hertweck. When Γ contains the projection $\bar{G}$ of G onto A, also determine the normalizer $\mathrm{N}_{\mathrm{U}(\Gamma)}(\bar{G})$.

S. O. Juriaans

18. (1) Classify the groups G for which $\mathrm{U}(\mathbb{Z}G)$ is a hyperbolic group in the sense of Gromov. A contribution to this problem is (S. O. Juriaans, I. B. S. Passi and D. Prasad, *Proc. Amer. Math. Soc.* **133** (2005), no. 2, 415–423).

(2) Determine groups G for which the unit group of $\mathbb{Z}G$ is finitely generated.

(3) Find a Dehn presentation of the group $\mathrm{SL}_1(\mathrm{H}(R))$, where R is the ring $\mathbb{Z}[(1+\sqrt{-7})/2]$, for which a presentation is given in (C. Corrales, E. Jespers, G. Leal and A. del Río, *Adv. Math.* **186** (2004), no. 2, 498–524). Furthermore, study the geometric properties of the compact manifold determined by this group.

S. O. Juriaans

19. Let $\mathcal{S}$ be the class of the finite groups which, whenever they are isomorphic, for some finite group G, to a subgroup of the normalized unit group $\mathrm{V}(\mathbb{Z}G)$, are necessarily isomorphic to a subgroup of G. Find members of $\mathcal{S}$.

Remark. Cyclic groups of prime power order are in $\mathcal{S}$ (J. A. Cohn and D. Livingstone, *Canad. J. Math.* **17** (1965), 583–593); $C_2 \times C_2$ is in $\mathcal{S}$ (W. Kimmerle, 2006, answering a question of Z. Marciniak); $C_p \times C_p$, for an odd prime p, is in $\mathcal{S}$ (M. Hertweck, *to appear in Comm. Algebra*). This suggests to study membership of p-groups, or of abelian groups, to $\mathcal{S}$. Not all groups belong to $\mathcal{S}$, as the counter example to the isomorphism problem shows (M. Hertweck, *Ann. of Math.* (2) **154** (2001), no. 1, 115–138).

W. Kimmerle

20. A *section* (or sub-quotient) of a group is a quotient of some subgroup of it. Let G be a finite group and denote by $\mathrm{V}(\mathbb{Z}G)$ the normalized unit group of its integral group ring $\mathbb{Z}G$. Is it true that each simple section of a finite subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a section of G?

Remark. This is true for soluble G, since then finite subgroups of $\mathrm{V}(\mathbb{Z}G)$ are known to be soluble. Also, any group basis of $\mathbb{Z}G$ has the same chief factors as G (W. Kimmerle, R. Lyons, R. Sandling and D. N. Teague, *Proc. London Math. Soc.* (3) **60** (1990), no. 1, 89–122).

W. Kimmerle

21. Let $\Pi(X)$ be the prime graph of a group X (also called the Gruenberg–Kegel graph of X). For a finite group G, is it true that $\Pi(\mathrm{V}(\mathbb{Z}G)) = \Pi(G)$? This means that whenever $\mathrm{V}(\mathbb{Z}G)$ contains an element of order pq, for distinct primes p and q, then also G should contain an element of order pq.

Remark. This holds when G is soluble or a Frobenius group. It has been verified for some simple groups using the Luthar–Passi method. See (W. Kimmerle, *in: Groups, rings and algebras, Contemp. Math., vol. 420*, Amer. Math. Soc., 2006, pp. 215–228). The first Zassenhaus conjecture for G makes a much stronger claim. In particular for simple groups, however, it seems to be reasonable to study the "prime graph question" first.

W. Kimmerle

22. Let H be a finite group. Is there a finite group G containing H such that each torsion unit in $\mathrm{V}(\mathbb{Z}H)$ is conjugate to an element of H by a unit of $\mathbb{Q}G$? If so, is it possible to take the conjugating units from $\mathbb{Q}H$?

Remark. At the meeting, M. Hertweck remarked that if one restricts attention to torsion units of prime order only, the answer to the first question is yes.

W. Kimmerle

23. Let G be a finite group, let p be a prime, let X be a simple $\mathbb{Q}G$-module, and let L be a $\mathbb{Z}_{(p)}G$-lattice in X. Write $E := \mathrm{End}_{\mathbb{Z}_{(p)}G}L$. We have an operation morphism $\mathbb{Z}_{(p)}G \to \mathrm{End}_E L$. Determine the index of its image as an abelian subgroup of $\mathrm{End}_E L$.

Considering the whole Wedderburn embedding instead of, as is done here, its projection to a single factor, the answer is known under some technical conditions; cf. Theorem 2.15 in (M. Künzer, *J. Group Theory* **7** (2004), 197–229).

M. Künzer

24. Let $p \geq 3$ be a prime, and ζ_{p^3} a primitive p^3-th root of unity. Let π be the norm of $\zeta_{p^3} - 1$ with respect to the subgroup $\mathcal{C}_{p-1}$ of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^3})|\mathbb{Q})$. This yields a purely ramified extension $\mathbb{Z}_{(p)}[\pi] \,|\, \mathbb{Z}_{(p)}$ with $\mathrm{Gal}(\mathbb{Q}(\pi)|\mathbb{Q}) = \mathcal{C}_{p^2}$. We have operation maps $\mathcal{C}_{p^2} \to \mathrm{End}_{\mathbb{Z}_{(p)}}\mathbb{Z}_{(p)}[\pi]$ and $\mathbb{Z}_{(p)}[\pi] \to \mathrm{End}_{\mathbb{Z}_{(p)}}\mathbb{Z}_{(p)}[\pi]$. Let R be the subring generated by the union of their images. Then R is isomorphic to the twisted (aka skew) group ring $\mathbb{Z}_{(p)}[\pi] \wr \mathcal{C}_{p^2}$. Describe R inside $\mathrm{End}_{\mathbb{Z}_{(p)}}\mathbb{Z}_{(p)}[\pi] \cong \mathbb{Z}_{(p)}^{p^2 \times p^2}$ by congruences of matrix entries—without direct reference to the (complicated) matrix entries of the elements of the image of $\mathcal{C}_{p^2}$ under its operation map.

This is known for ζ_{p^2} instead of ζ_{p^3}; cf. Theorem 1.19 and Example 5.4 in (M. Künzer, H. Weber, *Comm. Alg.* **33** (12) (2005), 4415–4455]). For an "upper bound" in the general case, see Lemma 1.5 in loc. cit.

M. Künzer

25. Let G be a finite group, and let R be the ring of integers in a finite extension field K of the p-adic numbers. Write $RG = \bigoplus_i P_i$ as a direct sum of indecomposable projective left RG-modules. What can be said about the structure of the endomorphism rings $E_i = \mathrm{End}_{RG}(P_i)$? Let $\epsilon_1, \ldots, \epsilon_s$ be the central primitive idempotents of KG. Then E_i is a subdirect product of the $\epsilon_j E_i$, which are local R-orders in a simple K-algebra. What can be said about these building blocks $\epsilon_j E_i$? Invariants of their conjugacy classes? Which local R-orders really can occur?

Some results describing p-adic group rings are presented in (G. Nebe, *Resenhas* **5** (2002), no. 4, 329–350, Around group rings (Jasper, AB, 2001)).

G. Nebe

26. Suppose $\mathcal{P}$ is a property of groups. Then a group is called *virtually* $\mathcal{P}$ if it has a subgroup of finite index which has property $\mathcal{P}$.

Let G be a finite group and write $\mathbb{Q}G = \bigoplus_{i=1}^{n} A_i$, a direct product of simple algebras. Let R_i be an order in A_i. By R_i^1 one denotes the units of reduced norm one. Denote by $\mathrm{vcd}(R_i^1)$ the virtual cohomological dimension of R_i^1. E. Jespers,

A. Pita, A. del Río, M. Ruiz and P. Zalesskii (*Adv. Math.* 212 (2007), no. 2, 692–722) proved the following for the unit group $\mathrm{U}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$.

$$\begin{aligned}
&\mathrm{U}(\mathbb{Z}G) \text{ is either abelian or finite} \\
&\qquad \Leftrightarrow \mathrm{vcd}(R_i^1) = 0 \text{ for all } 1 \le i \le n; \\
&\mathrm{U}(\mathbb{Z}G) \text{ is virtually a direct product of free groups} \\
&\qquad \Leftrightarrow \mathrm{vcd}(R_i^1) \le 1 \text{ for all } 1 \le i \le n; \\
&\mathrm{U}(\mathbb{Z}G) \text{ is virtually a direct product of free-by-free groups} \\
&\qquad \Leftrightarrow \mathrm{vcd}(R_i^1) \le 2 \text{ for all } 1 \le i \le n.
\end{aligned}$$

Can these results be extended to include the case when all $\mathrm{vcd}(R_i^1) \le 3$?

Á. del Río

27. A group G is said to be *subgroup separable* if for every finitely generated subgroup H of G and every $g \in G \setminus H$ there exists a subgroup N of finite index in G containing H but not g. When is the unit group $\mathrm{U}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ of a finite group G a subgroup separable group?

Á. del Río

28. Let G be a finite group. If $g, x \in G$ and g has order n, then the element $1 + (1 - g)x(1 + g + \cdots + g^{n-1})$ of the integral group ring $\mathbb{Z}G$ is easily seen to be a unit, and is called a *bicyclic unit*. Let $\mathcal{B}$ stand for the set of all bicyclic units in $\mathbb{Z}G$. If $b_1, b_2 \in \mathcal{B}$, then either $\langle b_1, b_2 \rangle$ is nilpotent or there exists a positive integer m so that $\langle b_1, b_2^m \rangle$ is a free non-cyclic group, see (J. Gonçalves, Á. del Río, Bicyclic units, Bass cyclic units and free groups, *to appear in J. Group Theory*, `arXiv:math/0612091v2 [math.RA]`). Pairs of bicyclic units of the second type exists if and only if G is non-Hamiltonian (Z. S. Marciniak, S. K. Sehgal, *Proc. Amer. Math. Soc.* **125** (1997), no. 4, 1005–1009). For a non-Hamiltonian group, define

$$m(G) = \min\{m \mid \langle b_1, b_2^m \rangle \text{ free for all } b_1, b_2 \in \mathcal{B} \text{ with } \langle b_1, b_2 \rangle \text{ not nilpotent}\}.$$

Is $\{m(G) \mid G \text{ a finite non-Hamiltonian group}\}$ bounded? It is enough to consider symmetric groups. We know that $m(S_3) = 1$ and $m(S_4) = 2$ (J. Gonçalves, Á. del Río, loc.cit.). Related is the following. A nonzero complex number z is said to be a *free point* if the group $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) \rangle$ is free. Set $m(z) = \min\{k \in \mathbb{N} \mid kz \text{ is a free point}\}$. Is the set

$$\{m(z) \mid z \text{ an algebraic integer}, z \neq 0\}$$

bounded? If this is true, the first question has a positive answer. (For known results on free points, see (J. Bamberg, *J. London Math. Soc.* (2) **62** (2000), no. 3, 795–801).)

J. Gonçalves, Á. del Río

29. For a finite group G, and a Dedekind domain R of characteristic zero in which no prime divisor of the order of G is invertible, K. W. Roggenkamp (*Arch. Math. (Basel)* **25** (1974), 125–128) proved that RG has no nontrivial idempotent ideals if and only if G is solvable. Determine what happens in case G is an infinite group.

For polycyclic-by-finite groups, this has been done by P. A. Linnell, G. Puninski and P. Smith (*J. Algebra* **305** (2006), no. 2, 845–858).

S. K. Sehgal

30. For a group G, and a commutative ring R, denote by $\Delta(G) = \Delta_R(G)$ the augmentation ideal of RG. Set $\delta_1(RG) = [\Delta(G), \Delta(G)]RG$, and define inductively $\delta_{n+1}(RG) = [\delta_n(RG), \delta_n(RG)]RG$ for all $n \in \mathbb{N}$. Let $G_{[n]}$ denote the n-th term of the derived series of G. Assuming G is a free group, is it true that $G \cap (1 + \delta_n(\mathbb{Z}G)) = G_{[n]}$? For free groups G, and a field F of positive characteristic, determine $G \cap (1 + \delta_n(FG))$. The answer will only depend on the characteristic (and not on the chosen field). Finally, investigate these problems for an arbitrary group G.

S. K. Sehgal

31. (M. Wursthorn). In a talk delivered at the workshop "Computational Representation Theory" which took place at IBFI Schloß Dagstuhl, Germany, May 1997, M. Wursthorn asked the following. Can any automorphism of the group algebra kG, where G is a finite p-group and k the field with p-elements, be written as the composition of an automorphism of G and a unipotent automorphism? Recall that an automorphism of kG is called *unipotent* if it induces the identity on $\Delta(G)/\Delta(G)^2$. Here, $\Delta(G)$ denotes the augmentation ideal of the group algebra kG.

Remark. A positive answer to this question would give a solution to the modular isomorphism problem, by Kimmerle's $G \times G$ trick.

M. Soriano

32. Let p be a prime and k the field with p elements. Count (estimate, or give bounds for) the number of isomorphism classes of local symmetric k-algebras of dimension p^n, for $n \in \mathbb{N}$. A finite dimensional k-algebra A is *symmetric* if there is a linear map $\lambda\colon A \to k$ whose kernel contains no left or right ideals different from zero, and satisfies $\lambda(ab) = \lambda(ba)$ for all $a, b \in A$.

M. Soriano

33. For a ring R, let $\mathrm{U}(R)$ be its group of units. If $\mathrm{U}(R)$ is solvable, let $\mathrm{dl}(\mathrm{U}(R))$ be its derived length. If R is Lie solvable, let $\mathrm{dl}_L(R)$ denote its Lie derived length. Let KG be a Lie solvable group algebra of a group G over a field K of odd characteristic. Does there exist a function $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{dl}(\mathrm{U}(KG)) \le f(\mathrm{dl}_L(KG))?$$

M. B. Smirnov (Vestsī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk (1983), no. 5, 20–23 (Russian)) proved for an arbitrary Lie solvable ring R without 2-torsion that $\mathrm{U}(R)$ is solvable and that

$$\mathrm{dl}(\mathrm{U}(R)) \le 4\mathrm{dl}_L(R) + 3$$

if $\mathrm{dl}_L(R) > 2$, and $\mathrm{dl}(\mathrm{U}(R)) \le 3$ otherwise. So it is really asked whether it is possible to improve Smirnov's bound when $R = KG$.

E. Spinelli

34. For a modular group algebra KG of a non-torsion group G over a field K of positive characteristic with solvable unit group $\mathrm{U}(KG)$, find lower bounds for the derived length $\mathrm{dl}(\mathrm{U}(KG))$.

E. Spinelli

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